A spectrum result on minimal blocking sets with respect to the planes of $\mathrm{PG}(3,q)$, q odd

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Abstract

This article presents a spectrum result on minimal blocking sets with respect to the planes of $\mathrm{PG}(3,q)$, q odd. We prove that for every integer k in an interval of, roughly, size $[q^2/4, 3q^2/4]$, there exists such a minimal blocking set of size k in $\mathrm{PG}(3,q)$, q odd. A similar result on the spectrum of minimal blocking sets with respect to the planes of $\mathrm{PG}(3,q)$, q even, was presented in [14]. Since minimal blocking sets with respect to the planes in $\mathrm{PG}(3,q)$ are tangency sets, they define maximal partial 1-systems on the Klein quadric $Q^+(5,q)$, so we get the same spectrum result for maximal partial 1-systems of lines on the Klein quadric $Q^+(5,q)$, q odd.

Key Words: minimal blocking sets, maximal partial 1-systems.

1 Introduction

A *blocking set* B with respect to the planes of $\mathrm{PG}(3,q)$ is a set of points intersecting every plane in at least one point. Such a blocking set is called *minimal* when no proper subset of B still is a blocking set. A blocking set B with respect to the planes of $\mathrm{PG}(3,q)$ is called *non-trivial* when it does not contain a line.

It was proven by Bruen and Thas [4] that a minimal blocking set of this type has at most size q^2+1, and that every minimal blocking set with respect to the planes of $\mathrm{PG}(3,q)$ of size q^2+1 is equal to an ovoid of $\mathrm{PG}(3,q)$, i.e., a set of q^2+1 points intersecting a plane in either one or $q+1$ points. For q odd, this implies the complete classification of the minimal blocking sets of size q^2+1 since Barlotti proved that every ovoid of $\mathrm{PG}(3,q)$, q odd, is equal to an elliptic quadric [1]. For q even, next to the elliptic quadric, there exists the Tits-ovoid in $\mathrm{PG}(3,q)$, $q=2^{2h+1}$, $h \geq 1$ [20].

Regarding large minimal blocking sets with respect to planes in $\mathrm{PG}(3,q)$, Metsch and Storme proved the non-existence of minimal blocking sets of size q^2-1, $q \geq 19$, and of size q^2 [10].

Attention has also been paid to the smallest minimal blocking sets with respect to the planes of $\mathrm{PG}(3,q)$. By Bose and Burton [2], the lines are the smallest minimal blocking sets with respect to the planes of $\mathrm{PG}(3,q)$. Bruen proved that the smallest non-trivial blocking sets with respect to the planes of $\mathrm{PG}(3,q)$ coincide with the smallest non-trivial blocking sets with respect to the lines of a plane $\mathrm{PG}(2,q)$ [3]. The following extensions to these results have been found.

In the following theorem, a *small* blocking set in $\mathrm{PG}(3,q)$ with respect to the planes of $\mathrm{PG}(3,q)$ is a blocking set of cardinality smaller than $3(q+1)/2$.

Theorem 1.1 (Sziklai, Szőnyi, and Weiner [16, 17, 19]) *Let B be a small minimal blocking set in $\mathrm{PG}(3,q)$, $q=p^h$, p prime, $h \geq 1$, with respect to the planes, then B intersects every plane in $1 \pmod p$ points. Let e be the maximal integer for which B intersects every plane in $1 \pmod{p^e}$ points, then e is a divisor of h.*

The preceding integer e is called the *exponent* of the small minimal blocking set B. The following theorem, which is based on results of [5, 17] in combination with Notation 3.3 and Proposition 3.5 of [19], states that the cardinality of a small minimal blocking set can only lie in a number of intervals of small size.

Theorem 1.2 *Let B be a small minimal blocking set in $\mathrm{PG}(3,q)$, $q=p^h$, p prime, $h \geq 1$, with respect to the planes. Then B intersects every plane in $1 \pmod{p^e}$ points. If e is the maximal integer for which B intersects every plane in $1 \pmod{p^e}$ points, then*

$$q+1+\frac{q}{p^e+2} \leq |B| \leq q+a_0\frac{q}{p^e}+a_1\frac{q}{p^{2e}}+\cdots+a_{h/e-2}p^e+1,$$

with a_n the n-th Motzkin number,

$$a_n = \frac{1}{n+1}\sum_{i=0}^{n+1}(-1)^i\binom{n+1}{i}\binom{2n+2-2i}{n-i}.$$

As an application of the exponent of a small minimal blocking set with respect to the planes of $\mathrm{PG}(3,q)$, we mention the following characterization result of Polverino and Storme [11, 12, 13].

Theorem 1.3 *Let B be a small minimal blocking set with respect to the planes of $\mathrm{PG}(3,q^3)$, $q=p^h$, p prime, $p\geq 7$, $h\geq 1$. Assume that B has an exponent larger than or equal to h, then B is one of the following minimal blocking sets:*

1. *a line,*
2. *a Baer subplane if q is a square,*
3. *a minimal planar blocking set of size q^3+q^2+1 projectively equivalent to the set $\{(1,x,x+x^q+x^{q^2})||x\in\mathbb{F}_{q^3}\}\cup\{(0,z,z+z^q+z^{q^2})||z\in\mathbb{F}_{q^3}\setminus\{0\}\}$,*
4. *a minimal planar blocking set of size q^3+q^2+q+1 projectively equivalent to the set $\{(1,x,x^q)||x\in\mathbb{F}_{q^3}\}\cup\{(0,z,z^q)||z\in\mathbb{F}_{q^3}\setminus\{0\}\}$,*
5. *a subgeometry $\mathrm{PG}(3,q)$.*

Next to studying large and small minimal blocking sets with respect to the planes of $\mathrm{PG}(3,q)$, *spectrum results* on minimal blocking sets with respect to the planes of $\mathrm{PG}(3,q)$ can be considered. A spectrum result gives a non-interrupted interval of values of k for which a minimal blocking set of size k with respect to the planes of $\mathrm{PG}(3,q)$ exists.

This has been studied by the authors for q even in [14]. In particular, the following results were obtained. In the following theorem, $\lfloor x\rfloor$ denotes the largest integer smaller than or equal to x.

Theorem 1.4 *For every integer k in the following intervals, there exists a minimal blocking set of size k with respect to the planes of $\mathrm{PG}(3,q)$, q even:*

- $q = 2^{4h}$:

$$k \in [\frac{q^2 + 194q + 10q\lfloor 48\log(q+1)\rfloor - 190}{10}, \frac{9q^2 - 69q + 440}{10}],$$

- $q = 2^{4h+1}$:

$$k \in [\frac{q^2 + 198q + 10q\lfloor 48\log(q+1)\rfloor - 230}{10}, \frac{9q^2 - 68q + 430}{10}],$$

- $q = 2^{4h+2}$:

$$k \in [\frac{q^2 + 196q + 10q\lfloor 48\log(q+1)\rfloor - 210}{10}, \frac{9q^2 - 66q + 410}{10}],$$

- $q = 2^{4h+3}$:

$$k \in [\frac{q^2 + 192q + 10q\lfloor 48\log(q+1)\rfloor - 170}{10}, \frac{9q^2 - 67q + 420}{10}].$$

The goal is to obtain a similar result for q odd. In Theorem 4.1, we prove that for every integer k in the following intervals, there exists a minimal blocking set of size k with respect to the planes of $\mathrm{PG}(3,q)$, q odd, $q \geq 47$:

1. $k \in [(q^2 + 30q - 47)/4 + 18(q-1)\log(q), (3q^2 - 18q + 71)/4]$, when $q \equiv 1 \pmod 4$,
2. $k \in [(q^2 + 28q - 37)/4 + 18(q-1)\log(q), (3q^2 - 12q + 57)/4]$, when $q \equiv 3 \pmod 4$.

In this way, a similar interval as for q even is obtained.

We wish to mention that also the following spectrum results on minimal blocking sets with respect to the planes of $\mathrm{PG}(3,q)$ have been found [9, 18]. In fact, they are spectrum results on minimal blocking sets with respect to the lines of a plane $\mathrm{PG}(2,q)$, but when this plane is embedded in $\mathrm{PG}(3,q)$, then an equivalent spectrum result on minimal blocking sets with respect to the planes of $\mathrm{PG}(3,q)$ is obtained.

Theorem 1.5 (Innamorati and Maturo [9]) *In* $\mathrm{PG}(2,q)$, $q \geq 4$, *for every integer* $k \in [2q-1, 3q-3]$, *there exists a minimal blocking set of size* k.

Theorem 1.6 (Szőnyi *et al* [18]) *In* $\mathrm{PG}(2,q)$ *,* q *square, for every integer* k *in the interval* $[4q\log q, q\sqrt{q}-q+2\sqrt{q}]$ *, a minimal blocking set of size* k *exists.*

To conclude the introduction, we mention that as a further application, we obtain an equivalent spectrum result on maximal partial 1-systems on the Klein quadric $Q^+(5,q)$, q odd.

2 The initial setting

We will use the ideas in the article of Szőnyi *et al* [18] for finding a spectrum result on minimal blocking sets with respect to the planes of $\mathrm{PG}(3,q)$, q odd. In particular, we will need the statement introduced by Füredi in [6, p. 190]:

Corollary 2.1 *For a bipartite graph with bipartition* $L \cup U$ *where the degree of the elements in* U *is at least* d *, there is a set* $L' \subseteq L$ *, for which* $|L'| \leq |L|\frac{1+\log(|U|)}{d}$ *, such that any element* $u \in U$ *is adjacent to at least one element of* L' *.*

The following setting is crucial for our purposes. We refer to Figure 1.

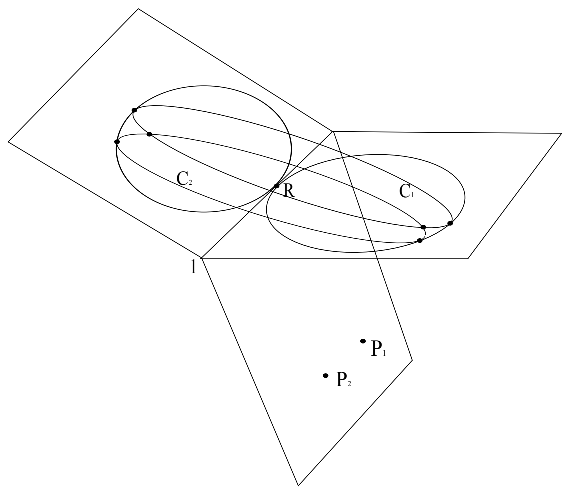

Figure 1: Conics of $Q^{-}(3,q)$ in planes through ℓ

Definition 2.2 Consider a plane π in $\mathrm{PG}\,(3,q)$ and a conic C in a plane π', with $\pi' \neq \pi$. We say that the plane π is *tangent* to the conic C if the line $\pi \cap \pi'$ is a tangent line to the conic C.

Consider the elliptic quadric $Q^{-}(3,q) : X_0^2 - dX_1^2 + X_2X_3 = 0$, d a non-square, in $\mathrm{PG}(3,q)$, q odd. Consider the point $R = (0,0,0,1)$ of $Q^{-}(3,q)$, then its tangent plane is $T_R(Q^{-}(3,q)) : X_2 = 0$. Consider the tangent line $\ell : X_0 = X_2 = 0$ to $Q^{-}(3,q)$ passing through R. Then ℓ lies in the secant planes $X_0 = 0$ and $X_0 = X_2$.

There are exactly q planes tangent to the conics $(X_0 = 0) \cap Q^{-}(3,q)$ and $(X_0 = X_2) \cap Q^{-}(3,q)$, in points of $Q^{-}(3,q)$ different from R.

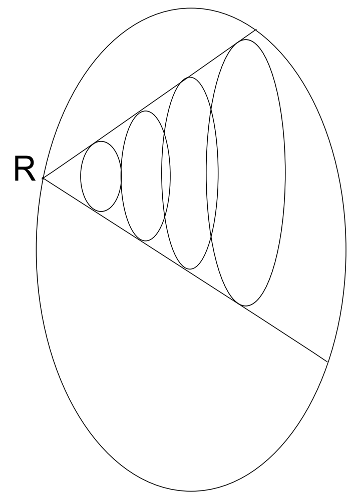

Figure 2: Group of q conics of $Q^-(3,q)$ tangent to $(X_0=0)\cap Q^-(3,q)$ and $(X_0=X_2)\cap Q^-(3,q)$

One of these planes is the plane $X_0-2dX_1+dX_2+X_3=0$ intersecting $Q^-(3,q)$ in the points $(0,1,1,d)$ and $(1,1,1,d-1)$ of $X_0=0$ and $X_0=X_2$. The other planes tangent to the conics $(X_0=0)\cap Q^-(3,q)$ and $(X_0=X_2)\cap Q^-(3,q)$, in a point of $Q^-(3,q)$ different from R, can be obtained by applying one of the transformations

$$\alpha_c:\begin{pmatrix}x_0\\x_1\\x_2\\x_3\end{pmatrix}\mapsto\begin{pmatrix}1&0&0&0\\0&1&c&0\\0&0&1&0\\0&2cd&dc^2&1\end{pmatrix}\begin{pmatrix}x_0\\x_1\\x_2\\x_3\end{pmatrix},$$

for $c\in\mathbb{F}_q$.

Note that the transformations α_c form an elementary abelian group of order q fixing $Q^-(3,q),R,$ and all planes passing through ℓ.

Lemma 2.3 *These q planes which form the orbit of the plane $X_0-2dX_1+dX_2+X_3=0$ under the transformations α_c, $c\in\mathbb{F}_q$, are the only planes tangent to the conics $Q^-(3,q)\cap(X_0=0)$ and $Q^-(3,q)\cap(X_0=X_2)$, in points different from R. The q conics of $Q^-(3,q)$ in these planes are intersected by the same $(q+3)/2$ planes through ℓ. Two of them, $X_0=0$ and $X_0=X_2$, contain exactly one point of each of those q conics, and the other $(q-1)/2$ planes through ℓ contain exactly two points of each of those q conics.*

Every point, different from R, in $Q^-(3,q)\cap(X_0=0)$ and in $Q^-(3,q)\cap(X_0=X_2)$ lies in exactly one of those q conics, and the other points of $Q^-(3,q)$, lying in at least one of those q conics, lie in exactly two of those q conics.

Proof. We first prove that there are exactly q such conics. Each such conic C is uniquely defined by its intersection point with the conic $Q^-(3,q)\cap(X_0=0)$. For let P be this tangent point, then the plane of C contains the tangent line to $Q^-(3,q)\cap(X_0=0)$ in P; it then also contains the intersection point P' of this tangent line with ℓ. This point P' lies on the tangent line ℓ to the conic $Q^-(3,q)\cap(X_0=X_2)$ and on one other tangent line ℓ' to the conic $Q^-(3,q)\cap(X_0=X_2)$. This line ℓ' then determines the plane of C completely.

There are exactly $(q-1)q/2$ points of $Q^-(3,q)\setminus\{R\}$ in the $(q-1)/2$ planes through ℓ intersecting these q conics in two points. Let π be one of the $(q-1)/2$ planes through ℓ intersecting these q conics in two points. The q points, different from R, in $Q^-(3,q)\cap\pi$, form one orbit under the group of transformations α_c, $c\in\mathbb{F}_q$. Assume that the conic C of $Q^-(3,q)$ in the plane $X_0-2dX_1+dX_2+X_3=0$ contains the points P and $\alpha_c(P)$ of $Q^-(3,q)\cap\pi$. Then $\alpha_{c'}(P)$ and $\alpha_{c'+c}(P)$ belong to $\alpha_{c'}(P)$.

But then $\alpha_c(P)$ belongs to $\alpha_c(C)$ and P belongs to $\alpha_{-c}(P)$. So every point P belongs to exactly two of those conics tangent to $X_0=0$ and $X_0=X_2$ in points of $Q^-(3,q)\setminus\{R\}$.

This then accounts for the total $2(q-1)q/2=(q-1)q$ incidences of the q conics of $Q^-(3,q)$ tangent to $X_0=0$ and $X_0=X_2$ in the planes through ℓ different from $X_0=0$ and $X_0=X_2$. □

The polar points of the q conic planes to $Q^-(3,q)$ tangent to the conics $(X_0=0)\cap Q^-(3,q)$ and $(X_0=X_2)\cap Q^-(3,q)$, in points different from R, lie in the plane $2X_0 = X_2$, in which they are the points, different from R , of the conic $\{(1/2, 1+c, 1, d(c+1)^2) || c \in \mathbb{F}_q\} \cup \{R\}$.

We will also need to consider the conic which is the intersection $(2X_0 = X_2) \cap Q^-(3,q)$. This is the conic of the points $\{(1/2, c, 1, dc^2 - 1/4) || c \in \mathbb{F}_q\} \cup \{R\}$.

Lemma 2.4 *A conic of* $Q^-(3,q)$ *, tangent to the conics* $(X_0 = 0) \cap Q^-(3,q)$ *and* $(X_0 = X_2) \cap Q^-(3,q)$ *, in points different from* R *, shares two points with the plane* $2X_0 = X_2$ *if and only if* $q \equiv 3 \pmod 4$ *.*

Proof. By using the elementary abelian group of the transformations α_c , $c \in \mathbb{F}_q$, it is sufficient to check the intersection of the line

$$\begin{cases} X_0 - 2dX_1 + dX_2 + X_3 &= 0 \\ 2X_0 &= X_2 \end{cases}$$

with $Q^-(3,q)$.

This leads to the quadratic equation $X_2^2(-1-4d) + 8dX_1X_2 - 4dX_1^2 = 0$ having discriminant $-16d$. This is a square if and only if $q \equiv 3 \pmod 4$. □

The following result is obvious, but we state it explicitly since we will make use of the point $(1,0,0,-1)$ in the construction of the minimal blocking sets with respect to the planes of $\mathrm{PG}(3,q)$, q odd.

Lemma 2.5 *The* q *planes tangent to the conics* $(X_0 = 0) \cap Q^-(3,q)$ *and* $(X_0 = X_2) \cap Q^-(3,q)$ *, in points different from* R *, all pass through the point* $(1,0,0,-1)$ *.*

This point $(1,0,0,-1)$ *is the polar point of the plane* $2X_0 = X_2$ *with respect to* $Q^-(3,q)$ *.*

Proof. The point $(1,0,0,-1)$ lies in the plane $X_0 - 2dX_1 + dX_2 + X_3 = 0$. Since all transformations α_c , $c \in \mathbb{F}_q$, fix $(1,0,0,-1)$, this point lies in all these q planes tangent to the conics $(X_0 = 0) \cap Q^-(3,q)$ and $(X_0 = X_2) \cap Q^-(3,q)$, in points different from R . □

3 Construction

From the above section, we know that there are exactly q planes tangent to the conics $(X_0 = 0) \cap Q^-(3,q)$ and $(X_0 = X_2) \cap Q^-(3,q)$, in points different from R. Of these, we select two conics C_1 and C_2 in such a way that they intersect in two points, not in the plane $2X_0 = X_2$, and that the polar points of their planes are not incident with the plane of the other conic. We first prove that this indeed is possible.

Lemma 3.1 *Consider a conic C_1 of $Q^-(3,q)$ tangent to the conics $(X_0 = 0)\cap Q^-(3,q)$ and $(X_0 = X_2)\cap Q^-(3,q)$, in points different from R. Then if $q \equiv 1 \pmod 4$, C_1 intersects the $q-1$ other conics of $Q^-(3,q)$ tangent to the conics $(X_0 = 0)\cap Q^-(3,q)$ and $(X_0 = X_2)\cap Q^-(3,q)$, in points different from R, in zero or two points, and if $q \equiv 3 \pmod 4$, C_1 intersects two of the $q-1$ other conics of $Q^-(3,q)$ tangent to the conics $(X_0 = 0)\cap Q^-(3,q)$ and $(X_0 = X_2)\cap Q^-(3,q)$, in points different from R, in one point, and the $q-3$ other conics of $Q^-(3,q)$ tangent to the conics $(X_0 = 0)\cap Q^-(3,q)$ and $(X_0 = X_2)\cap Q^-(3,q)$, in points different from R, in zero or two points.*

Proof. Let C_1 be the conic of $Q^-(3,q)$ in the plane $X_0 - 2dX_1 + dX_2 + X_3 = 0$. Applying the elementary abelian group acting in one orbit on the q conics of $Q^-(3,q)$ tangent to the conics $(X_0 = 0) \cap Q^-(3,q)$ and $(X_0 = X_2) \cap Q^-(3,q)$, in points different from R, the other conics lie in the planes $X_0 + (-2d + 2cd)X_1 + (-2cd + d + dc^2)X_2 + X_3 = 0$.

To find the intersection with $Q^-(3,q)$ of the intersection line of the planes $X_0 - 2dX_1 + dX_2 + X_3 = 0$ and $X_0 + (-2d + 2cd)X_1 + (-2cd + d + dc^2)X_2 + X_3 = 0$, with $c \neq 0$, the quadratic equation

$$(4d^2c^2 - 8d^2c - dc^2 + 4d^2 + 4cd - 4d)X_2^2 + (8cd - 8d + 4)X_2X_3 + 4X_3^2 = 0,$$

needs to be solved.

The discriminant of this quadratic equation is equal to $4 + 4dc^2$ and is zero if and only if $c^2 = -1/d$. Since d is a non-square, this has two solutions in c if and only if $q \equiv 3 \pmod 4$. □

We now use the results of Lemma 3.1 to select two conics C_1 and C_2 of $Q^-(3,q)$ tangent to the conics $(X_0 = 0)\cap Q^-(3,q)$ and $(X_0 = X_2)\cap Q^-(3,q)$ in points different from R. These two conics C_1 and C_2 will be used in the construction method which will lead to the non-interrupted interval for the sizes k of the minimal blocking sets with respect to the planes of $\mathrm{PG}(3,q)$ (Corollary 3.2 and Theorem 4.1). In particular, we will select these two conics C_1 and C_2 in such a way that they share two distinct points. This will give us the freedom of a new parameter u which can vary from 0 to 2; helping us to find the non-interrupted spectrum of Theorem 4.1.

Namely, if one selects C_1, one of the q conics of $Q^-(3,q)$ tangent to the conics $(X_0 = 0) \cap Q^-(3,q)$ and $(X_0 = X_2) \cap Q^-(3,q)$ in points different from R, there are always at least $(q-3)/2$ other conics of $Q^-(3,q)$ tangent to the conics $(X_0 = 0) \cap Q^-(3,q)$ and $(X_0 = X_2) \cap Q^-(3,q)$ in points different from R, which intersect C_1 in two distinct points. Now the polar points of the q planes tangent to the conics $(X_0 = 0) \cap Q^-(3,q)$ and $(X_0 = X_2) \cap Q^-(3,q)$ are in the plane $2X_0 = X_2$ and C_1 shares two points with this plane when $q \equiv 3 \pmod 4$. We impose that the two intersection points of C_1 and C_2 do not lie in the plane $2X_0 = X_2$. The motivation is as follows: to get a non-interrupted spectrum, we need to let vary a parameter u, where $0 \leq u \leq 2$ (see (1)). The parameter u is the number of points in $C_1 \cap C_2$ that are not deleted when constructing the new blocking set. So sometimes, they both will not be deleted ($u = 2$), sometimes only one of them will be deleted ($u = 1$), and sometimes both of them will be deleted ($u = 0$). But we always delete the points of $Q^-(3,q)$ in the plane $2X_0 = X_2$. So, to be able to let vary u from 0 to 2, we must make sure that none of the points of $C_1 \cap C_2$ lies in the plane $2X_0 = X_2$. The plane of C_1 intersects the plane $2X_0 = X_2$ in a line containing at most two points of $Q^-(3,q)$. If this is the case, they lie on a second conic of $Q^-(3,q)$ tangent to X_0 and $X_0 = X_2$, so we need to exclude at most two possibilities for C_2. We also impose that the polar point of C_1 does not lie in the plane of C_2, and vice versa. These polar points lie on a conic in $2X_0 = X_2$. So we exclude at most two other possibilities for C_2. For q large enough, we still have at least $\frac{q-11}{2}$ choices for C_2.

We would like to use Corollary 2.1 in order to obtain a spectrum of minimal blocking sets with respect to the planes of $\mathrm{PG}(3,q)$, for q odd. Therefore we need to introduce variables s and r, where s is the number of conics in planes through the tangent line ℓ which are not replaced by their polar point where r of these planes intersect C_1 and C_2. Thus $q-s$ conics in planes through the line ℓ are replaced by their polar points on the line $X_1 = X_2 = 0$.

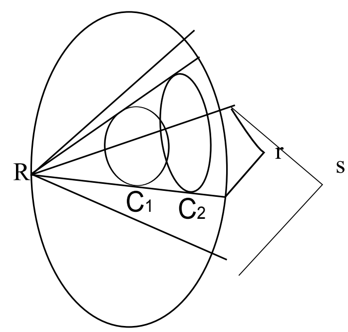

Figure 3: We leave s conics of $Q^-(3,q)$ in the planes through ℓ in the blocking set of which r intersect C_1 and C_2

For the bipartite graph we need to construct in order to be able to use Corollary 2.1, we form sets U and L with respect to the tangent line ℓ.

The elements of L are the conics in planes through ℓ except $X_0 = 0$, $X_0 = X_2$, and except those conics in planes through ℓ intersecting the q conics of $Q^-(3,q)$ tangent to $(X_0 = 0) \cap Q^-(3,q)$ and $(X_0 = X_2) \cap Q^-(3,q)$ in points different from R. So $|L| = \frac{q-3}{2}$.

For the elements of the set U, we use the conics of $Q^-(3,q)$ except those in a plane containing ℓ and the q conics of $Q^-(3,q)$ tangent to $(X_0 = 0) \cap Q^-(3,q)$ and $(X_0 = X_2) \cap Q^-(3,q)$ in points different from R, thus $|U| \leq q^3 - q < q^3$. A lower bound on the degree is given in [15] by $d \geq \frac{q-6-3\sqrt{q}}{4}$. But since we always delete the conic of $Q^-(3,q)$ in the plane $2X_0 = X_2$, and this conic belongs to L when $q \equiv 1 \pmod 4$, we decrease the lower bound on d to $d \geq \frac{q-10-3\sqrt{q}}{4}$. From Corollary 2.1, we get an upper bound on $|L'|$:

$$\begin{aligned} |L'| &\leq \frac{q-3}{2} \cdot \frac{1+\log(q^3)}{\frac{1}{4}(q-10-3\sqrt{q})} \\ &\leq 2 \cdot (1+3\log(q)) \cdot \frac{q-3}{q-10-3\sqrt{q}}. \end{aligned}$$

This imposes a further condition on q. For $q \geq 47$, $(q-3)/(q-10-3\sqrt{q}) \leq 3$ and we get $|L'| \leq 6 + 18\log(q)$.

The result of Füredi now states that there exists, within the set of $(q-3)/2$ conics of L, a set L' of at most $6+18\log(q)$ conics such that every conic of $Q^-(3,q)$ in U intersects at least one of the conics of L'. There are $s-r$ conics in L. In terms of the cardinalities of the minimal blocking sets, this implies the following condition on the parameters s and r:

$$s - r \geq 6 + 18\log(q).$$

We impose this condition for the following reason: we will not delete the conics of $Q^-(3,q)$ in the set L' in the construction of the new set B of which we will show that it is a minimal blocking set with respect to the planes of PG$(3,q)$. Then every plane of PG$(3,q)$ intersecting $Q^-(3,q)$ in a conic of the set U intersects at least one of the conics in the set L' in a point. This point is not deleted from $Q^-(3,q)$ to construct the new set B (of which we will show that it is a minimal blocking set with respect to the planes of PG$(3,q)$), thus showing that all the planes intersecting the elliptic quadric $Q^-(3,q)$ in a conic of the set U are blocked by a point of the newly constructed set B, and thus implying that only a small number of planes of PG$(3,q)$ still need to be verified whether they are blocked by the newly constructed set B (see also the proof of Theorem 3.3).

Altogether, we get the following construction of minimal blocking sets with respect to the planes of PG$(3,q)$, q odd, which will give a non-interrupted interval of sizes k of minimal blocking sets.

Corollary 3.2 *We construct a new minimal blocking set B with respect to the planes of* PG$(3,q)$, *q odd: First we replace $q-s$ conics of $Q^-(3,q)$ in* *planes through ℓ by their polar points, assuming that r of the s remaining conics in planes through ℓ intersect the tangent conics C_1 and C_2. We always delete the conic of $Q^-(3,q)$ in the plane $2X_0 = X_2$ and replace it by its polar point $(1,0,0,-1)$. We add back the point R. Then we remove C_1 and C_2, and replace both by their polar points P_1 and P_2. The set B has cardinality $k = (s+1)q - s - 4r + u'$, with $3 \leq u' \leq 9$. We prove this as follows.*

The s non-deleted conics of $Q^-(3,q)$ in planes through ℓ, together with the $q-s$ polar points of the $q-s$ deleted conics of $Q^-(3,q)$ in planes through ℓ, give a set of $1+(s+1)q-s$ points. We assume that r of the s non-deleted conics in planes through ℓ intersect C_1 and C_2. Assume that two of those r conics are $X_0 = 0$ and $X_0 = X_2$ only sharing one point with C_1 and C_2. Assume that u, with $0 \leq u \leq 2$, of the two intersection points of C_1 and C_2 lie in one of those r conics. Then these r conics in planes through ℓ contain $(r-2)\cdot 2\cdot 2 + 2\cdot 2 - u$ points of C_1 and C_2. Then, when we delete C_1 and C_2, we delete another $4r-4-u$ points from $Q^-(3,q)$ and add back two polar points. So the new cardinality is

$$1 + (s+1)q - s - (4r-4-u) + 2 = (s+1)q - s - 4r + u + 7, \quad (1)$$

with $0 \leq u \leq 2$.

But we can also let the plane $X_0 = 0$ contain one of the deleted conics, then we get sizes $(s+1)q - s - 4r + u + 5$, with $0 \leq u \leq 2$, or we can also let the planes $X_0 = 0$ and $X_0 = X_2$ contain one of the deleted conics, then we get sizes $(s+1)q - s - 4r + u + 3$, with $0 \leq u \leq 2$. This all leads to sizes $k = (s+1)q - s - 4r + u'$, with $3 \leq u' \leq 9$.

We also impose the following constraints:

1. $4 \leq r \leq \frac{q-7}{2}$,
2. *if* $s \geq \frac{q-1}{2}$, *then* $r \geq s - \frac{q-3}{2}$,
3. $s - r \geq 6 + 18\log(q)$.

The restrictions follow from the construction above and the application of Corollary 2.1 in the construction. For instance, the condition $r \geq 4$ follows from the fact that, depending on the cardinality desired, the two planes through ℓ containing the two intersection points of C_1 and C_2, and the two planes $X_0 = 0$ and $X_0 = X_2$ are deleted or non-deleted. To make sure that these four planes can be non-deleted, we impose $r \geq 4$. But we always delete the conic of $Q^-(3,q)$ in the plane $2X_0 = X_2$, and this conic intersects C_1 and C_2 when $q \equiv 3 \pmod 4$, so when also the two conics of $Q^-(3,q)$ in the planes through ℓ containing the two intersection points of C_1 and C_2, and the two conics of $Q^-(3,q)$ in the planes $X_0 = 0$ and $X_0 = X_2$ are deleted, then $r \leq (q-7)/2$, so we also impose $r \leq (q-7)/2$.

Theorem 3.3 *The set B is a minimal blocking set with respect to the planes of* $\mathrm{PG}(3,q)$, *q odd.*

Proof. Part 1. We first prove that B effectively is a blocking set.

Consider a tangent plane π to the elliptic quadric $Q^-(3,q)$. This tangent plane π either still contains its tangent point R of $Q^-(3,q)$ when R belongs to B, or in case R does not belong to B, then π contains the polar point of the deleted conic C of $Q^-(3,q)$ to which R belongs.

Consider a secant plane π to $Q^-(3,q)$. If π intersects $Q^-(3,q)$ in a conic which is deleted from $Q^-(3,q)$ in the construction of B, then either π passes through R or passes through $(1,0,0,-1)$, and these points belong to B. If the conic $\pi \cap Q^-(3,q)$ is not deleted from $Q^-(3,q)$ in the construction of B, we only discuss planes π not passing through R since $R \in B$. If the conic $\pi \cap Q^-(3,q)$ is not intersected by the same $(q+3)/2$ planes through $\ell : X_0 = X_2 = 0$ as the conics C_1 and C_2, then by the definition of the set L', the conic $\pi \cap Q^-(3,q)$ shares at least one point with one of the conics in L', and their points belong to B. If the conic $\pi \cap Q^-(3,q)$ is intersected by the same $(q+3)/2$ planes through $\ell : X_0 = X_2 = 0$, then it is one of the q conics of $Q^-(3,q)$ tangent to the conics of $Q^-(3,q)$ in $X_0 = 0$ and $X_0 = X_2$. In this case, the plane π passes through $(1,0,0,-1)$, and this point belongs to B.

We have discussed all cases: every plane of $\mathrm{PG}(3,q)$ contains at least one point of B.

Part 2. We now show that B is a minimal blocking set.

We first show the necessity of the point $(1,0,0,-1)$. We selected the two conics C_1 and C_2 such that their planes do not contain the corresponding polar points P_1 and P_2. So, the only point of B that they contain is $(1,0,0,-1)$. This shows the necessity of $(1,0,0,-1)$.

We now show the necessity of a point T of $B \cap Q^-(3,q)$, with $T \neq R$. Then T lies in a plane π through ℓ in which the conic $C = \pi \cap Q^-(3,q)$ is not deleted in the construction of B. Its tangent plane π_T to $Q^-(3,q)$ intersects the line $X_1 = X_2 = 0$ in the polar point $\tilde{T}$ of C. But since C is not deleted, $\tilde{T} \notin B$. Also, P_1 and P_2 do not lie in π_T, or else $T \in C_1$ or $T \in C_2$, but then $T \notin B$. Hence, $\pi_T \cap B = \{T\}$, so T is necessary.

The point R is also required in B. Since $r \leq (q-7)/2$, we delete at least five conics in planes through ℓ intersecting C_1 and C_2. For at least one of those planes, R is the only point of B in that plane, so R is necessary. This concludes the necessity of the points of $B \cap Q^-(3,q)$.

We now discuss the necessity of a point T on $X_1 = X_2 = 0$, being the polar point of a deleted conic C of $Q^-(3,q)$ in a plane through ℓ.

This point T lies in q tangent planes to $Q^-(3,q)$ in the points of $C \setminus \{R\}$. The only points of B that possibly could belong to these q tangent planes are P_1 and P_2. If they all contain either P_1 or P_2, then, for instance, P_1 belongs to at least $q/2$ of those tangent planes. Consider the line TP_1 and its intersection S with the plane $T^\perp$. Then S would belong to at least $q/2$ tangent lines to C in $T^\perp$. This implies $q/2 \leq 2$. Note that this argument also works for the point $T = (1,0,0,-1)$ which is the polar point of the deleted conic of $Q^-(3,q)$ in the plane $2X_0 = X_2$.

Finally, we discuss the necessity of the points P_1 and P_2 in B. The point P_1 is the polar point of the conic C_1. Of the s conics in planes through ℓ that are still belonging to B, r of them intersect C_1 and C_2. Consider a tangent plane π, passing through P_1, to $Q^-(3,q)$ in the point P. Suppose that π intersects $X_1 = X_2 = 0$ in T. Then $T \in T_P(Q^-(3,q))$ if and only if $P \in T^\perp$, where $T^\perp$ is a plane through ℓ. If T corresponds to one of the r non-deleted conics of $Q^-(3,q)$ through ℓ intersecting C_1 and C_2, then $T \notin B$. So this tangent plane contains in this case, besides P_1, at most the point P_2. But if this is the case, then $P \in C_1 \cap C_2$. So this occurs for only two points of C_1. Since we imposed $r \geq 4$, there exists a point $P \in C_1 \setminus C_2$. So P_1 is necessary for B.

We have discussed all the points of B; we have shown that B is a minimal blocking set. □

4 Calculation of the interval

We know from Corollary 3.2 how to construct a blocking set B of size k and proved in Theorem 3.3 that B is a minimal blocking set with respect to the planes of $\mathrm{PG}(3,q)$, q odd. We proceed as follows to find a non-interrupted interval of values of k for which a minimal blocking set B of size k exists in $\mathrm{PG}(3,q)$, q odd.

For a given pair (s,r), we can construct minimal blocking sets of sizes $(s+1)q-s-4r+3,\ldots,(s+1)q-s-4r+9$. For a given s, the larger r, the smaller the size of the minimal blocking set. To get a large non-interrupted interval of values of k for which a minimal blocking set of size k in $\mathrm{PG}(3,q)$, q odd, exists, we must make sure that for a given value s, the smallest value for the size k in the interval of sizes arising from the different values for r for this given value of s, is smaller than or equal to the largest value for the size k in the interval of sizes arising from the different values for r for the next value $s'=s-1$.

We first discuss the maximum possible value for the size k of a minimal blocking set in the non-interrupted interval that can be obtained by our arguments.

The largest possible value for r that is allowed is $r=(q-7)/2$. Then the smallest value for the size of the minimal blocking set is $(s+1)q-s-4r+3=(s+1)q-s-2q+17$.

The largest value for the size of the minimal blocking set, for an allowed pair of parameters (s',r') is $(s'+1)q-s'-4r'+9$. For $s'=s-1$, this is the value $sq-s-4r'+10$. We investigate when the following condition

$$sq-s-4r'+10 \geq (s+1)q-s-2q+17$$

is valid to make sure that the intervals for the sizes k of the minimal blocking sets corresponding to the parameters s and $s-1$ overlap.

This condition implies that $r' \leq (q-7)/4$.

So we must be able to use the value $r'=(q-7)/4$ for $s'=s-1$.

When $q \equiv 1 \pmod 4$, we always delete the conic in the plane $2X_0=X_2$ which is skew to C_1 and C_2. We examined $s=(q-3)/2-1+(q-9)/4$, and the values smaller than and larger than this value of s. This showed that $k=(3q^2-18q+71)/4$ is the maximum value for the non-interrupted interval of sizes of k for which a minimal blocking set is constructed. This value of k is obtained for $(s,r) = ((q-3)/2 - 1 + (q-5)/4, (q-5)/4)$. For $q \equiv 3 \pmod 4$, we tested the value of $s = (q-3)/2 + (q-7)/4$, and the smaller and larger values of s, and found that $k = (3q^2 - 12q + 57)/4$ is the maximum value of the non-interrupted interval. This value of k is obtained for $(s,r) = ((q-3)/2 + (q-3)/4, (q-3)/4)$.

Now we discuss the minimum possible value for the size k of a minimal blocking set in the non-interrupted interval that can be obtained by our arguments. We know that $s - r \geq 6 + 18\log(q)$. We let $s = r' + 6 + 18\log(q)$, so for a given value s, necessarily $4 \leq r \leq r'$. For a given value s, the largest value for the size k is obtained for $r = 4$, and is equal to $(s+1)q - s - 4r + 9$. For $s = r' + 6 + 18\log(q)$, this gives the value $r'q + 7q - r' - 13 + 18(q-1)\log(q)$.

For r equal to r', which is the maximum allowed value for r when $s = r' + 6 + 18\log(q)$, the smallest value of k for the given parameter $s = r' + 6 + 18\log(q)$ is equal to $(s+1)q - s - 4r' + 3$, which reduces to $r'q + 7q - 5r' - 3 + 18(q-1)\log(q)$.

For $q \equiv 1 \pmod 4$, we looked at the value $s = (q+7)/4 + 6 + 18\log(q)$, and the values larger than and smaller than s. This showed that the smallest value of the non-interrupted interval is $k = (q^2 + 30q - 47)/4 + 18(q-1)\log(q)$. This value is obtained for $(s,r) = ((q+7)/4 + 6 + 18\log(q), (q+7)/4)$. For $q \equiv 3 \pmod 4$, we inspected the value $s = (q+5)/4 + 6 + 18\log(q)$, and the values larger than and smaller than s. This showed that the smallest value of the non-interrupted interval is $k = (q^2 + 28q - 37)/4 + 18(q-1)\log(q)$. This value is obtained for $(s,r) = ((q+5)/4 + 6 + 18\log(q), (q+5)/4)$.

We now summarize the results on the interval in the next theorem.

Theorem 4.1 *There exists a minimal blocking set B with respect to the planes of $\mathrm{PG}(3,q)$, q odd, $q \geq 47$, for every integer k in the following intervals*

1. $k \in [(q^2 + 30q - 47)/4 + 18(q-1)\log(q), (3q^2 - 18q + 71)/4]$, *when* $q \equiv 1 \pmod 4$,
2. $k \in [(q^2 + 28q - 37)/4 + 18(q-1)\log(q), (3q^2 - 12q + 57)/4]$, *when*

$q \equiv 3 \pmod 4$.

5 Application

Another application of our spectrum result is a spectrum result on maximal partial 1-systems of the Klein quadric $Q^+(5,q)$ [7, Section 15.4].

Definition 5.1 A *1-system* $\mathcal{M}$ on $Q^+(5,q)$ is a set of q^2+1 lines $\ell_1, \ldots, \ell_{q^2+1}$ on $Q^+(5,q)$ such that $\ell_i^\perp \cap \ell_j = \emptyset$, for all $i, j \in \{1, \ldots, q^2+1\}$, $i \neq j$.

A *partial 1-system* $\mathcal{M}$ on $Q^+(5,q)$ is a set of $s \leq q^2+1$ lines $\ell_1, \ldots, \ell_s$ on $Q^+(5,q)$ such that $\ell_i^\perp \cap \ell_j = \emptyset$, for all $i, j \in \{1, \ldots, s\}$, $i \neq j$.

A line of the Klein quadric lies in two planes of the Klein quadric. The above definition of a 1-system is equivalent to the definition that a 1-system $\mathcal{M}$ on $Q^+(5,q)$ is a set of q^2+1 lines $\ell_1, \ldots, \ell_{q^2+1}$ on $Q^+(5,q)$ such that every line ℓ_j is skew to the two planes of the Klein quadric through any line ℓ_i, for all $i, j \in \{1, \ldots, q^2+1\}$, $i \neq j$.

A similar observation can be made regarding the definition of a partial 1-system.

Via the Klein correspondence, points of the Klein quadric correspond to lines of $\mathrm{PG}(3,q)$, and lines of the Klein quadric correspond to planar pencils of $\mathrm{PG}(3,q)$, i.e., they correspond to the lines of $\mathrm{PG}(3,q)$ through a point R in a plane Π passing through R.

A *tangency set* $\mathcal{T}$ of $\mathrm{PG}(3,q)$ is a set of points of $\mathrm{PG}(3,q)$, such that for every point $R \in \mathcal{T}$, there is a plane Π_R intersecting $\mathcal{T}$ only in R. It is proven in [10] that a tangency set in $\mathrm{PG}(3,q)$ is equivalent to a partial 1-system on the Klein quadric.

A minimal blocking set B w.r.t. the planes of $\mathrm{PG}(3,q)$ is an example of a tangency set; thus we can apply the results of Theorem 4.1.

Corollary 5.2 *For every value k belonging to one of the intervals of Theorem 4.1, there exists a maximal partial 1-system of size k on the Klein quadric $Q^+(5,q)$, q odd.*

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