Constructing 3D mappings onto the unit sphere with the hypercomplex Szegö kernel

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Abstract

In classical complex analysis the Szegö kernel method provides a constructive method to construct conformal maps from a given simply-connected domain $G \subset \mathbb{C}$ onto the unit disc. In this paper we revisit this method in the three-dimensional case. We investigate whether it is possible to construct 3D mappings from some elementary domains into the three dimensional unit ball by using the hypercomplex Szegö kernel. In the cases of rectangular domains, L-shaped domains, cylinders and the symmetric double cone the proposed method leads surprisingly to qualitatively very good results. In the case of the cylinder we get even better results than those obtained by the hypercomplex Bergman method that was very recently proposed by several authors.

We round of with also giving an explicit example of a domain, namely the T-piece, where the method does not lead to the desired result. This shows that one has to adapt the methods in accordance to different classes of domains.

Keywords: numerical conformal mapping, hypercomplex Szegö kernel, Clifford analysis, 3D mapping problems

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1 Introduction

In classical complex analysis the famous Riemann mapping theorem tells us that one can map any simply-connected domain $G \subset \mathbb{C}$ conformally onto the

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unit disc $D = \{z \in \mathbb{C} \mid |z| < 1\}$. This allows one to treat for instance difficult aerodynamic problems equivalently in the simpler setting of the unit disc in which the calculations simplify significantly. The theory of Bergman and Hardy spaces provides a numerical method to approximate the mapping function in terms of an orthogonal function series. See for instance [1, 2, 17] and elsewhere. The Bergman space $B^2(G,\mathbb{C})$ is the space of functions that are L^2 integrable over a domain $G \subset \mathbb{C}$ and holomorphic in its inside. It is endowed with the scalar product $\langle f, g\rangle := \int_G \overline{f(z)}g(z)dxdy$. The Hardy space $H^2(\partial G,\mathbb{C})$ is the closure of the space of functions that are L^2 over the boundary of such a domain G, holomorphic in its inside with continuous extension to the boundary. This one is endowed with the scalar product where the integration is correspondingly extended over the boundary of the domain.

Both function spaces are Hilbert spaces with a continuous point evaluation. Hence, they possess a uniquely defined reproducing kernel. In the first case it is called the Bergman kernel $B(z,w)$ and in the latter one the Szegö kernel $S_G(z,w)$. The kernels satisfy

$$g(z) = \int_G B_G(z,w)g(w)dw \quad \forall g \in B^2(G,\mathbb{C}) \tag{1}$$

$$g(z) = \int_{\partial G} S_G(z,w)g(w)dw \quad \forall g \in H^2(\partial G,\mathbb{C}) \tag{2}$$

respectively. In contrast to the Cauchy kernel, the Bergman and the Szegö kernel depend on the domain. For each domain one has a different Bergman and Szegö kernel.

In this paper we focus exclusively on extending the Szegö kernel method (SKM). The classical SKM is a method for approximating the conformal map f which maps G onto D in such a way that $f(0) = 0$ and $f'(0) > 0$. As a consequence of the reproducing property (2) and the transformation formula one obtains in the case where $0 \in G$ the well-known explicit relation

$$f(z) = \frac{2\pi}{S_G(0,0)} \int_0^z S^2(0,z)dz, \tag{3}$$

where the line integral is extended over any path from 0 to z. More precisely the SKM involves the following calculation steps

1. Choose a basis $(g_j)_{j=1}^{+\infty}$ for the space $H^2(\partial G,\mathbb{C})$.
2. Orthonormalize the subset $(g_j)_{j=1}^{N}$ by the Gram-Schmidt algorithm to obtain an orthonormal set $(h_j)_{j=1}^{N}$.
3. Approximate the Szegö kernel $S_G(0,\cdot)$ by the Fourier sum

$$S_G^N(0,z) = \sum_{j=1}^{N} h_j(z)\langle h_j, S_G(0,\cdot)\rangle = \sum_{j=1}^{N} h_j(z)\overline{h_j(0)}.$$

4. Approximate the mapping function f by

$$f_N(z) = \frac{2\pi}{S_G(0,0)} \int\limits_0^z {S_G^N}^2(0,z)dz.$$

In view of many questions from aerodynamics and fluid dynamics one is interested in three dimensional analogous constructions of mappings from a given domain, such as for instance an airplane wing, into the three dimensional unit ball. This would allow us to analogously do many calculations in the simpler setting of the unit ball and would reduce the number of expensive wind channel experiments. Unfortunately, there is no direct analogue of the Riemann mapping theorem for dimensions $n \geq 3$. In fact, due to the famous theorem of Liouville [22] (see also [10]), the only conformal mappings in $\mathbb{R}^n$ are Möbuis transformations. However, in the setting of quaternions, it was possible to introduce direct analogues of the classical Bergman and Hardy spaces for the three and four-dimensional case, cf. [7, 15, 8, 28, 27] and elsewhere.
In this paper we investigate whether it is possible to construct 3D mappings from some elementary domains into the three-dimensional unit ball by adapting the classical method in the way using the quaternionic Szegö kernel instead. For rectangular bounded domains, L-shaped domains, cylinders and the double cone this methods leads surprisingly to qualitatively very good results. In the case of the cylinder we obtain better results than those that were previously obtained in works of S. Bock et al., B. Boone and J. Rüsges with the use of the quaternionic Bergman kernel method, cf. [6, 4, 5, 26]. We round off with one example, namely the T-piece, where the method does not lead to the desired result. This exhibits the need of further investigation in this direction which still offers a rich spectrum of possibilities for fine tuning.

2 Quaternionic Hardy spaces and Szegö kernels

2.1 Basics on quaternions and quaternionic analysis

For details on quaternions and quaternionic analysis we refer the interested reader for instance to [8, 16, 21, 20]. By $\{1, e_1, e_2, e_3\}$ we denote the basis elements of the four dimensional vector space $\mathbb{R}\oplus\mathbb{R}^3 \cong \mathbb{R}^4$. This can be endowed with a product according to the multiplication rules

$$e_1^2 = e_2^2 = e_3^2 = -1, \quad e_1e_2 = -e_2e_1 = e_3.$$

This multiplication operation extends the vector space to an algebra. This is called the algebra of real quaternions, denoted by $\mathbb{H}$. Notice that the product is non-commutative. In what follows we identify each vector $x = (x_0, x_1, x_2, x_3)^T \in \mathbb{R}^4$ with the quaternion $z = x_0 + x_1e_1 + x_2e_2 + x_3e_3 \in \mathbb{H}$. The conjugate of z is defined by $\overline{z} = x_0 - x_1e_1 - x_2e_2 - x_3e_3$. The Euclidean norm of the quaternion z has the form $|z| = \sqrt{z\overline{z}} = \sqrt{\sum_{i=0}^{3} x_i^2}$. Any $z \in \mathbb{H}\backslash\{0\}$ is invertible and

$z^{-1} = \frac{\overline{z}}{|z|^2}$. The quaternions form a skew-field. In what follows we identity the three-dimensional vector space $\mathbb{R}^3$ with the subset of quaternions that are of the form $z = x_0 + x_1 e_1 + x_2 e_2$.

A meaningful generalization of the concept of holomorphic functions in $\mathbb{C}$ to the three-dimensional case can be introduced by extending the Riemann approach. Let G be a domain in $\mathbb{R} \oplus \mathbb{R}^2 \cong \mathbb{R}^3$. Consider $\mathbb{H}$-valued functions of the form $f : \mathbb{R}^3 \to \mathbb{H}$, $f(z) = f_0(z)+f_1(z)e_1+f_2(z)e_2+f_3(z)e_3$, where $z = (x_0, x_1, x_2)$ is a shortened quaternion. The three-dimensional analogue of the Cauchy-Riemann operator is defined as $D := \frac{\partial}{\partial x_0} + \frac{\partial}{\partial x_1} e_1 + \frac{\partial}{\partial x_2} e_2$. Following [8, 20] and others, a function $f : G \to \mathbb{H}$ is called left (right) quaternionic holomorphic if $Df(z) = 0$ (resp. $fD = 0$) for all $z \in G$. In view of the non-commutativity one needs to distinguish between left or right quaternionic holomorphy. However, as shown in [8, 20] and elsewhere, for both sets of functions and analogous function theory can be established. We hence restrict ourselves to focus on the left quaternionic holomorphic case. In fact, as shown in the above mentioned works, many classical theorems from complex analysis carry over to the higher dimensional context using this approach. In particular, every function f that is (left) quaternionic holomorphic in a neighborhood of the closure $\overline{G}$ of a domain $G \subset \mathbb{R}^3$ satisfies a generalized Cauchy integral formula of the form

$$f(z) = \frac{1}{4\pi} \int_{\partial G} \frac{\overline{z - w}}{|z - w|^3} \, d\sigma(w) \, f(w), \tag{4}$$

where $d\sigma(w) = dw_1 \wedge dw_2 - e_1 dw_0 \wedge dw_2 + e_2 dw_0 \wedge dw_1$ is the oriented surface measure.

2.2 The three-dimensional Szegö kernel method

In all that follows let G be a domain in $\mathbb{R}^3$. Then, following e.g. [7, 8, 19] the three-dimensional analogue of the classical Hardy space in the quaternionic setting can be introduced as follows:

Definition 1. *Let $G \subset \mathbb{R}^3$ be a domain and let ∂G be the set of its boundary points. The closure of the set*

$$\mathcal{A}^2(\partial G, \mathbb{H}) := \{f \in C^1(\overline{G}) \cap L^2(\partial G); \mid Df(z) = 0 \; \forall z \in G\}$$

endowed with the quaternion valued scalar product defined by

$$\langle f, g \rangle := \int_{\partial\Omega} \overline{f(z)} g(z) dS(z),$$

where $dS(z) = |d\sigma(z)|$ is the scalar valued surface measure, is called the quaternionic Hardy space of left monogenic functions in G. This one will be denoted by $H^2(\partial G, \mathbb{H})$.

Strictly speaking, $H^2(G,\mathbb{H})$ only forms a right $\mathbb{H}$-module in view of the non-commutativity of the quaternions. By means of the generalized Cauchy integral formula for left quaternionic holomorphic functions given in (4) one can prove in close analogy to the complex case that this function space has a continuous point evaluation. See for instance [7, 15, 8] for details.
It hence possesses a uniquely defined reproducing kernel, called the quaternionic Szegö kernel. Due to the lack of a direct analogue of the Riemann mapping theorem, it is very difficult to compute closed formulas for the Szegö kernel. Closed formulas for the unit ball, the half-space, the rectangular strip domain $0 < x_0 < d$ and for the infinite cylinder are resp. given in [7, 10, 13, 14, 25]. In the case where G is a bounded domain that contains the origin in its inside, the kernel can be approximated by applying the Gram-Schmidt algorithm on the set of the Fueter polynomials. For convenience we recall their definition in the three-dimensional case (cf. e.g. [8, 20, 23, 24]):

Definition 2. *(*cf. e.g. [8], p. 68, [20] pp.113)
Let $z = x_0 + x_1 e_1 + x_2 \in \mathbb{R}^3$ and $\mathcal{Z}_i := x_i - x_0 e_i$ for $i = 1, 2$. Further, for $k \in \mathbb{N}$ let $(l_1, \ldots, l_k) \in \{1, 2\}^k$. The Fueter polynomials then are defined by

$$\begin{aligned} p_0(z) &:= 1 \\ p_{l_1,\ldots,l_k}(z) &:= \frac{1}{k!}\sum_{\pi \in \mathcal{S}_k} \mathcal{Z}_{\pi(l_1)} \ldots \mathcal{Z}_{\pi(l_k)}, \end{aligned}$$

where $\mathcal{S}_k$ stands for the symmetric group of permutations on k elements.

These polynomials form a basis for $H^2(\partial G,\mathbb{H})$ if G is a bounded domain containing the origin. This is a consequence of the Taylor expansion theorem for monogenic functions, cf. e.g. [8] p. 73, [20] p. 183.
The orthonormalization process of Gram-Schmidt applied to the set of the Fueter polynomials then produces an orthonormal set $(h_j)_j$ of $H^2(\partial G,\mathbb{H})$. Notice that the set of left monogenic functions only forms a right-$\mathbb{H}$-module. The coefficients produced by the Gram-Schmidt algorithm thus appear at the right hand side, i.e. the n-th step of the procedure has the form

$$\tilde{h}_n := p_n - \sum_{j=1}^{n-1} h_j \langle h_j, p_n \rangle, \quad h_n := \frac{1}{\langle \tilde{h}_n, \tilde{h}_n \rangle} \tilde{h}_n,$$

where p_n is the n-the Fueter polynomial that is to be orthonormalized.
Note that the use of the Fueter polynomials up to degree N corresponds to a total of $n := \frac{(N+1)(N+2)}{2}$ functions. The Szegö kernel is then approximated by the finite Fourier sum

$$S_G^N(0,z) = \sum_{j=1}^{(N+1)(N+2)/2} h_j(z)\overline{h_j(0)}, \qquad N = 0, 1, \ldots$$

Then we compute the line integral

$$f_N(z) = \int_0^z {S_G^N}^2(0,u)du, \quad N = 0, 1, \ldots \tag{5}$$

In the classical two dimensional complex case, the complex analogue of the function series $(f_N)_N$ converges to the function that maps the given domain G conformally onto the unit disc. In the higher dimensional case we cannot expect f to be conformal in the classical sense of Gauss in general, as we know from Liouville's theorem that the set of conformal maps in the sense of Gauss coincides with the set of Möbius transformations.

Remarks and assumptions for the calculations:

1. In the three-dimensional case, the line integral (5) is not independent from the choice of the path. Here, we choose the direct line connection from 0 to z as integration path. In view of the non-commutativity this still leads to two different choices of integration, namely

$$f := z \mapsto \int_0^1 z S_G^2 G(tz, 0)dt \tag{6}$$

resp.

$$f := z \mapsto \int_0^1 S_G^2(tz, 0)z dt. \tag{7}$$

In this paper we perform the calculations for both variants. For the numerical evaluation the first variant is used only.

2. We assume without restriction that the origin is an interior point of the domains considered here. Suppose that $S_G(\cdot,\cdot)$ is the Szegö kernel of a domain G. Then the Szegö kernel of the translated domain $G + d$, $d \in \mathbb{H}$ is given by $S_G(\cdot - d, \cdot - d)$. The calculations that we performed lead to the conjecture that symmetry with respect to the origin leads to a positive effect on the quality of the results.

3. The hypercomplex integral is computed componentwise. One computes the corresponding four real-valued integrals of the real components of the quaternionic expression.

4. In this paper we restrict to consider domains in $\mathbb{R}^3$. Although the Fueter polynomials are only $\mathbb{R}^3$ valued, as shown for instance in [23, 20], the orthonormalized functions h_j take in general values in the four-dimensional space $\mathbb{H}$. To obtain a map to $\mathbb{R}^3$ we here simply cut off the e_3-component.

5. As known, the usual Gram-Schmidt algorithm has the crucial disadvantage that it is numerically very unstable. Even small round-off errors can cause

significant effects on the final result. To overcome this problem we perform all calculations with the MAPLE program which calculates symbolically and hence exactly. However, this implies very long computation times.

3 Numerical experiments

In this section we present explicit numerical experiments for some elementary domains. These include the regular cube as a typical example of a rectangular domain, an L-piece, a closed cylinder and the regular double pyramid. We shall see that the proposed method works pretty well for these domains. At the end we also present the T-piece. For the T-piece however the method does not lead to the desired result. All result from calculations with the program MAPLE, thus from symbolically computations. The pictures in the left row provide the mappings that result from applying the integration variant (6), i.e. where z is multiplied from the left-hand side from the square of the Szegö kernel. Correspondingly, the pictures in the middle row refer to the calculations based on the integration variant (7). The picture in the right row refer to the integration over the arithmetic mean value of both integration variants, i.e. to constructing the mapping function by taking

$$\frac{1}{2}\Big(\int_0^1 zS_G^2G(tz,0)dt + \int_0^1 S_G^2G(tz,0)zdt\Big).$$

The value N indicates that in the approximation all Fueter polynomials up to total degree N are involved. The pictures show the evolution of the mapping with growing degree of polynomials that are involved.

3.1 The unit cube

The unit cube $[-0.5, 0.5]^3$ has been treated with the Bergman kernel method in all the works [4, 5, 6, 26]. As the pictures show, also the Szegö kernel method that we propose in this paper surprisingly leads to results of a similar quality:

Figure 1: N=2

Figure 2: N=6

Figure 3: N=12

Figure 4: N=22

In the following table we analyze the quality of the mapping numerically. The function has been evaluated on a grid of test points lying on the surface of the domain. Then we computed the minimum and maximum values, the mean value, the median and the standard deviation to the mean value of the image points.

Cube					
N	r_{min}	r_{max}	r_{median}	r_{mean}	σ_{mean}
2	0.50000000	0.86602540	0.64342831	0.64525227	0.08356600
4	0.98720516	1.15243593	1.04434276	1.04987762	0.03502725
6	1.11372639	1.22799009	1.1547270	1.16339697	0.03182012
8	1.11373570	1.23203983	1.15316478	1.16293096	0.03305621
10	1.11293754	1.23597561	1.15202359	1.16251405	0.03365807
12	1.11735315	1.24909031	1.15588925	1.16462883	0.03385515
14	1.11731079	1.25824223	1.15610186	1.16536819	0.03409364
16	1.11799870	1.25606978	1.15637633	1.16588101	0.03397282
18	1.11795488	1.25377415	1.15737486	1.16600946	0.03394483
20	1.11799870	1.256069781	1.15637633	1.16588101	0.03397282
22	1.11832249	1.25748121	1.15704597	1.16630522	0.03390004

One observes that the variance of the radii decreases rapidly up to $N = 6$. After that one observes a stagnation of the variance. The same effect appears when applying the Bergman kernel method, as analyzed in detail in [26]. Nevertheless, both approaches lead to a very ball-like domain in the case of the unit cube.

The stagnation effect of the variance of the radii has not been observed in the preceding work of [4, 5]. The reason is that the stagnation effect appears in the BKM for the cube only up from $N = 14$. The authors from [4, 5] however have stopped at the step $N = 12$ because of the heavy long calculation times. With the MAPLE versions that were used on the existing computer servers in 2004 it was not practicable to get exact results beyond $N = 12$. The conclusion in [4, 5] that the proposed algorithm really converges to a perfect ball thus turned out to be drawn to quick. One gets a very ball-like domain, but there is still a perturbation effect that additionally needs to be compensated. This topic will be discussed in Section 3.6 of this paper.

Cube : e_3 component					
N	$e3_{min}$	$e3_{max}$	$e3_{median}$	$e3_{mean}$	σ_{mean}
2	0.00000e+00	0.00000e+00	0.00000e+00	0.00000000e+00	0.0000e+00
4	0.00000e+00	0.00000e+00	0.00000e+00	0.00000000e+00	0.0000e+00
6	0.00000e+00	0.00000e+00	0.00000e+00	0.00000000e+00	0.0000e+00
8	-0.00228758	0.00228758	0.00000e+00	0.22427271e-24	0.00105679
10	-0.00230644	0.00230644	0.00000e+00	0.64077918e-24	0.00106629
12	-0.00192833	0.00192833	0.00000e+00	-0.17941817e-23	0.00071375
14	-0.00180835	0.00180835	0.00000e+00	0.32551582e-23	0.00070496
16	-0.00144594	0.00144594	0.00000e+00	0.11085479e-23	0.00073729
18	-0.01117734	0.01117734	0.00000e+00	-0.18582596e-23	0.00268926
20	-0.00166455	0.00166455	0.00000e+00	0.23388440e-23	0.00074334
22	-0.00161783	0.00161783	0.00000e+00	0.21658336e-23	0.00074106

In the construction of the 3D image the e_3-component has been cut off. This table list the corresponding statistical values for the e_3-component and exhibits the influence of the cut-off effect of this component. We observe that the values of the e_3 component of f_N are rather small in the case of the unit cube. The variance of the e_3-component is rather small, too. Up to $N = 4$ the variance of the e_3-component decreases faster with the proposed Szegö kernel method than with the Bergman kernel method evaluated in [26]. It also reaches its best value (N=6) faster than the Bergman kernel method (N=10). However, up from $N = 8$ the variance of the e_3-component is twice as much as for the Bergman kernel method. This could be one plausible explanation why the variance of the radii from the previous table is twice as much as for the Bergman kernel method. In the thesis [19] other examples cuboids have been treated as well. For all these domains, the method produced similar results.

3.2 L-shaped domains

More general rectangular domains than the cuboid are domains that are composed by several cuboids. These include for instance L-pieces, T-pieces or U-pieces. To compare the Bergman kernel method with the Szegö kernel method we here treat in this paper the same L-piece that has been considered in the earlier works [4, 26].
Since the underlying L-piece results from cutting a cuboid out off a cube, it is rather logical that the pictures obtained by applying the integration variant (6) have a very similar form than for the cube. The only difference is that one observes a deeper grave at one of the sides instead of the usual round surface part. An application of the integration variant (7) surprisingly leads to very similar results. There are only insignificant differences visible at the upper and the top and the ground face. Up from $N = 8$ the results and the images are very similar to those that were obtained in [4, 26] with the Bergman kernel method. However, when applying the Bergman kernel method, one still observes some irregular peaks at the surfaces of the image. These peaks do not appear with the

Szegö kernel method. For the L-piece the Szegö kernel method hence produces a better result than the Bergman kernel method. However, fine-tuning is still necessary here, too. One still observes a stagnation effect of the variance of the radii for increasing N. However, its range is only in the scale of 10^{-3}.

Figure 5: N=1

Figure 6: N=1

Figure 7: N=5

L-shaped domain					
N	r_{min}	r_{max}	r_{median}	r_{mean}	σ_{mean}
1	1.29110504	4.22985007	1.90557024	2.02486247	0.51739487
2	1.59809997	5.07927861	2.17863892	2.27370955	0.47476290
3	1.63736788	5.03931703	2.16433255	2.32549655	0.48629799
4	2.54886031	4.77818610	3.31881835	3.37611382	0.36107106
5	2.74479081	5.71129134	3.64954531	3.72384741	0.57386032
6	2.93080994	5.54651383	3.82458952	3.97682430	0.60970685
7	3.06404575	5.60116956	3.92206356	4.13410487	0.67945785
8	3.08726704	5.68528393	3.90940097	4.15092371	0.68781810
9	3.11336417	5.81721474	3.92411841	4.16962584	0.68747531
10	3.16333834	5.81188175	3.94874348	4.19816488	0.67524324
11	3.17983624	5.79778564	3.96058365	4.20895869	0.67041921
12	3.20757014	5.87300760	3.97151572	4.22188708	0.66420943
13	3.22909373	5.84779945	3.97877892	4.234227911	0.66087301
14	3.24375778	5.89087350	3.98668491	4.24203212	0.65819242

Figure 8: N=5

Figure 9: N=10

L-shaped domain : e_3 component					
N	$e_{3_{min}}$	$e_{3_{max}}$	$e_{3_{median}}$	$e_{3_{mean}}$	σ_{mean}
1	-0.32751953	0.32751953	-0.01370106	-0.02501915	0.11399155
2	-0.12887333	0.12887333	0.00143468	0.00404507	0.03701626
3	-0.20132596	0.20132596	0.0000e+00	-0.00916962	0.07296013
4	-0.14821086	0.14821086	-0.00098458	-0.78166246e-5	0.03471432
5	-0.12802317	0.12799876	-0.00208522	-0.00797864	0.05012145
6	-0.13911314	0.13911314	-0.00185279	-0.00836915	0.04140280
7	-0.06966210	0.06958879	-0.00060571	-0.00407754	0.02304851
8	-0.04925492	0.04922581	-0.00043222	-0.00245824	0.01367536
9	-0.04917340	0.04911626	-0.00016990	-0.00113724	0.01332052
10	-0.02759811	0.027598110	-0.00015535	-0.00074930	0.00868300
11	-0.03498017	0.03498017	-0.00018160	-0.00103293	0.00946875
12	-0.02392399	0.02392399	-0.00013241	-0.00057233	0.00730533
13	-0.02596903	0.02596903	-0.00030332	-0.00063560	0.00664760
14	-0.02544119	0.02548315	-0.00036848	-0.00065572	0.00712399

3.3 Cylinders

In this subsection we treat two typical cylindrical domains of finite height. First, we take the regular cylinder $D \times [-1/2, 1/2]$ where $D := \{(x_0, x_1)^T \mid x_0^2 + x_1^2 \leq 1\}$ is the two dimensional unit disc. As one could expect, the computations turn out to cost more time than for the unit cube. For small values of N the effort however still remains acceptable. Up from $N = 12$ the effort however increases significantly.

When applying the integration variant (6) the sequence of images converges rapidly to a very ball-like domain with increasing N. One observes that the resulting images show around the circle of the equator a uniform "Einkerbung" which seems to remain for growing N. Outside this area one does not observe any deviation from the ball form.

When however applying the integration variant (7) then the image sequence converges to a different kind of domain which is very asymmetric. This one is still a closed domain but does not show any similarity to a ball. The arithmetic mean of both integration variants gives an image which evidently turns out

Figure 10: N=10

Figure 11: N=14

Figure 12: N=14

to be a mixture case out of two different figures looking a like an asymmetric hamburger. This underlines how much influence the choice of the placement of the factors in the quaternionic product has on the resulting image.
When applying the integration variant (6), then the variance of the radii decreases very fast already in the first four approximation steps $N = 4$. Up from $N = 4$ one observes again a stagnation. The variance of the e_3-component is negligible small. In view of the calculation accuracy which improves with growing N one can assume that the e_3 component vanishes; hence we get indeed a mapping into $\mathbb{R}^3$.
A comparison to the results obtained for this cylinder in [26] with the Bergman kernel method shows that the Szegö kernel method proposed here leads to a better result for the cylinder. This is also observed in the following second example.
Take now the less symmetric cylinder $D \times [-3/2, 3/2]$. Again when applying the integration variant (6), then we obtain a very ball-like domain. Here we get even better results than for the cylinder treated before. The deformation effect that we observed around the equator at the first cylinder does not appear hear. One only observes a decreasing deformation in the region of the north pole. Concerning the behavior of the e_3 component we can make the same comment as for the other cylinder. We can assume that the e_3 component vanishes with growing N so that $\lim_{N\to+\infty} f_N$ indeed turns out to be a function that takes only values in $\mathbb{R}^3$.
Here for this concrete example the Szegö kernel method turns out to be much more efficient than the Bergman kernel method. A quantitative comparison to the results obtain with the Bergman kernel method in [26] shows that the variance of the radii is 20 times smaller with the Szegö kernel method than with the Bergman kernel method.

Figure 13: N=1

Figure 14: N=4

Figure 15: N=7

Cylinder [1,1,3]					
N	r_{min}	r_{max}	r_{median}	r_{mean}	σ_{mean}
1	0.00158314	0.00285405	0.00244214	0.00238152	0.00034455
2	0.00242315	0.00327263	0.00282067	0.00280683	0.00028537
3	0.00242315	0.00327263	0.00282067	0.00280683	0.00028537
4	0.00360382	0.00423581	0.00362364	0.00376276	0.00023215
5	0.00360382	0.00423581	0.00362364	0.00376276	0.00023215
6	0.00341783	0.00438405	0.00355619	0.00368638	0.00031369
7	0.00341783	0.00438405	0.00355619	0.00368638	0.00031369
8	0.00349747	0.00447397	0.00353591	0.00370332	0.00031115
9	0.00349747	0.00447397	0.00353591	0.00370332	0.00031115
10	0.00344938	0.00454236	0.00356252	0.00370536	0.00032172

Cylinder [1,1,3] : e_3 component					
N	$e_{3_{min}}$	$e_{3_{max}}$	$e_{3_{median}}$	$e_{3_{mean}}$	σ_{mean}
1	0.00e+00	0.00e+00	0.0000e+00	0.00000000e+00	0.00000000e+00
2	-0.2e-22	0.2e-22	0.0000e+00	-0.17968217e-25	0.36127734e-23
3	-0.2e-22	0.2e-22	0.0000e+00	-0.17968217e-25	0.36127734e-23
4	-0.3e-22	0.3e-22	0.0000e+00	-0.84454696e-25	0.46042072e-23
5	-0.3e-22	0.3e-22	0.0000e+00	-0.84454696e-25	0.46042072e-23
6	-0.4e-22	0.4e-22	0.0000e+00	0.24664872e-26	0.70084603e-23
7	-0.4e-22	0.4e-22	0.0000e+00	-0.60478405e-25	0.70586156e-23
8	-0.5e-22	0.5e-22	0.0000e+00	0.49709419e-25	0.79834586e-23
9	-0.4e-22	0.5e-22	0.0000e+00	0.16597078e-24	0.82536710e-23
10	-0.63e-22	0.6e-22	0.0000e+00	0.11360774e-24	0.10152898e-22

3.4 The double-cone

Our investigation shows that the symmetry of the domain has a strong influence on the calculation effort of the procedure and on the quality of the approximation of the ball. We shall see that we will get excellent results for the double-cone which is a very symmetric domain around the origin. In the thesis [19] the single cone has also been treated. Indeed, for the single-cone which is far less symmetric the approximation turned out to be much worse.

Figure 16: N=10

When applying the integration variant (6) then the image of the double cone gets a very ball-like domain with increasing values for N. The quality of the convergence to a ball is much better than for all the other examples. Furthermore, the variance of the radii decreases constantly with increasing N. There is no stagnation effect of the variance. The sequence of the maps f_N actually leads to a ball. One observes similarities to the images that result from applying the integration variant (6) to the unit cylinder. One observes a uniform "Einkerbung" around the equator and small deformations at the north and south pole. The images that result from applying the integration variant (7) are less symmetric. As the table shows the values of the e_3-component are in the range of 10^{-20}; so we actually have a mapping into $\mathbb{R}^3$.

Figure 17: N=1

Figure 18: N=2

Figure 19: N=4

Double-cone					
N	r_{min}	r_{max}	r_{median}	r_{mean}	σ_{mean}
1	0.70710678	1.0000e+00	0.79699435	0.81537185	0.09221557
2	0.78535246	1.57451220	0.86944714	0.96767296	0.21897766
3	0.78535246	1.57451220	0.86944714	0.96767296	0.21897766
4	1.06335843	1.17979150	1.13107275	1.12867477	0.03868012
5	1.06335843	1.17979150	1.13107275	1.12867477	0.03868012
6	1.06977691	1.16577452	1.13531226	1.12857269	0.03142416

Double-cone : e_3 component					
N	$e_{3_{min}}$	$e_{3_{max}}$	$e_{3_{median}}$	$e_{3_{mean}}$	σ_{mean}
1	0.00e+00	0.00e+00	0.0000e+00	0.0000e+00	0.00000000e+00
2	-0.1e-20	0.1e-20	0.0000e+00	0.0000e+00	0.35488437e-21
3	-0.1e-20	0.1e-20	0.0000e+00	0.0000e+00	0.35488437e-21
4	-.5e-20	0.5e-20	0.0000e+00	0.0000e+00	0.89365513e-21
5	-.5e-20	0.5e-20	0.0000e+00	0.0000e+00	0.89365513e-21
6	-0.19e-24	0.24e-19	0.0000e+00	-0.82160707e-23	0.19427223e-20

3.5 A negative example

In the preceding subsection we treated a number of different elementary domains. In all the cases the Szegö kernel method applied in the proposed way,

Figure 20: N=6

provided a mapping to a ball-like domain. As the following example shows, the proposed method does not always lead to the desired result. Let us consider the T-piece with the coordinates ...

Figure 21: N=1

Figure 22: N=5

As the table shows, the tendency of the variance of the radii is increasing. Also the e_3-component increases tendentiously with increasing values for N.

T-shaped domain					
N	r_{min}	r_{max}	r_{median}	r_{mean}	σ_{mean}
1	0.62869011	3.25638346	1.45243092	1.55209673	0.66278507
2	1.58746150	3.88501251	3.12257072	3.01030995	0.48166481
3	1.99716411	5.10222146	3.47785357	3.54189457	0.62226441
4	2.96976280	5.94358776	4.90110743	4.87076052	0.49241734
5	3.21222302	6.63945740	5.10698776	5.13929779	0.59966766
6	3.93782517	7.00930407	5.87131361	5.87962579	0.53224723
7	4.24044045	7.82022191	6.20906818	6.18624177	0.66360582
8	4.68281732	7.94880591	6.68282069	6.63691241	0.64621133
9	5.16737501	8.88266468	7.11249434	7.10468791	0.74201834
10	5.42091318	9.26735105	7.38437372	7.37597136	0.79198273
11	5.70569596	9.78585978	7.56275106	7.63099778	0.85313835
12	6.15720875	10.29145546	7.96898281	8.07221013	0.92017379
13	6.21257453	10.66292250	8.04832502	8.12272326	0.93784211
14	6.60881968	11.36873232	8.28063366	8.49797189	1.03307091

Figure 23: N=10

Figure 24: N=14

T-shaped domain : e_3 component					
N	$e_{3_{min}}$	$e_{3_{max}}$	$e_{3_{median}}$	$e_{3_{mean}}$	σ_{mean}
1	-0.16472993	0.16472993	0.0000e+00	0.16660258e-22	0.06311044
2	-0.05310026	0.05310026	0.0000e+00	0.53120594e-22	0.02087962
3	-0.80036860	0.80036860	0.0000e+00	0.62155581e-22	0.20528667
4	-0.21511797	0.21511797	0.0000e+00	-0.81058567e-22	0.07179865
5	-0.54296123	0.54296123	0.0000e+00	0.13815199e-20	0.13915914
6	-0.60924710	0.60924710	0.0000e+00	0.34666154e-21	0.18505258
7	-0.62790810	0.62790810	0.0000e+00	-0.12623349e-21	0.14794378
8	-1.69379604	1.69379604	0.0000e+00	0.78175060e-21	0.32218857
9	-1.57521761	1.57521761	0.0000e+00	0.19864154e-21	0.34464221
10	-1.90113518	1.90113518	0.0000e+00	0.98679994e-21	0.38067378
11	-2.67010729	2.67010729	0.0000e+00	0.13840830e-20	0.52133534
12	-2.42997746	2.42997746	0.0000e+00	-0.70485710e-21	0.51423994
13	-2.38352815	2.38352815	0.0000e+00	-0.41971036e-20	0.49548429
14	-3.06866824	3.06866824	0.0000e+00	0.82019735e-21	0.64679373

3.6 Discussion

In the cases of rectangular domains, cylinders and the double cone the proposed Szegö kernel method lead to very ball-like domains. In particular, we obtained very good results for symmetric cylinders and the symmetric double cone where we observed a tendentiously decreasing variance of radii with increasing N as well as negligible small values for the e_3-components. In the case of rectangular domains, one obtains very ball-like domains; however one observes a stagnation effect in the variance of the radii up from a certain value of N. This indicates that there is a perturbation effect that still needs to be compensated. On the one hand, in the case of rectangular domains, the variance of the e_3-component shows the same behavior. It does not seem to converge to zero, although its values are small. Notice that cutting of the e_3-component is only one possibility among many other possibilities to get a three-dimensional object. It is definitely not the most canonical one. Furthermore, we observed a significant difference between applying the integration variant (6) and the variant (7). Actually, the integration variant (7) lead in all cases a worse result. This however, does not mean that the integration variant (6) does already provide us with the best choice. A further possibility of fine tuning consists of choosing the integration path from $z = 0$ to $z = z_0$. Notice that in the hypercomplex case the integral is not path independent. In all the examples treated before we choose the direct

path from $z = 0$ to z_0. However, as the following calculation shows one can get better results for the unit cube when instead choosing the path .... .
TO BE INSERTED
Here, we have another possibility of fine tuning.
Finally, we observed that the quality of convergence to a ball seems to be influenced by the symmetry of the original domain. The more symmetric is the original domain with respect to the origin, the better result we obtained. In fact, this makes it important to re-consider the starting point of the integration. Instead of choosing a priori always the origin, a different choice might be more appropriate. In fact, it is plausible that we can increase the quality of the convergence to a ball by applying fine-tuning. it cannot be a coincidence that one obtains very ball like domains for so different kinds of domains. Indeed, we claim that f is a conformal map in the sense of Gauss on each boundary part of the domain. Indeed this is logical, because the Riemann mapping theorem is still valid on a number of classes of 2-manifolds in $\mathbb{R}^3$. This might be one possible explanation why we still get for so many domains images to ball-like domains.
However, as the results of the T-piece show, that we cannot expect the method to work universally for all kind of domains. We expect that the geometry of the domain enters significantly in the scheme. Summarizing, the result indicate that still a lot of research with promising results can be performed on the basis of the generalized Szegö kernel method.

References

[1] S. Bell. *The Cauchy Transform, Potential Theory and Conformal Mapping.* CRC Press, Boca Raton, 1992.

[2] S. Bergman. *The kernel function and conformal mapping.* Mathematical Surveys, 2nd ed. Vol. 5. American Mathematical Society, Providence RI, 1970.

[3] Bernstein, S. and L.S. Lanzani: *Szegö projections for Hardy spaces of monogenic functions and applications.* Int. Journal Math. Math. Sci. 29 (2002) No. 10, 613 – 624.

[4] S. Bock, M.I. Falcão and K. Gürlebeck. Applications of the Bergman kernel functions. In: *Internationales Kolloquium über Anwendungen der Informatik und Mathematik in Architektur und Bauwesen*, Bauhaus Universittät Weimar.

[5] S. Bock, M.I. Falcão, K. Gürlebeck and H. Malonek. A 3-dimensional Bergman kernel method with applications to rectangular domains. *Journal of Compuational and Applied Mathematics* **189** (2006), 67–79.

[6] B. Boone. *Bergmankern en conforme afbeelding.* Diplomarbeit, Lehrstuhl II für Mathematik RWTH Aachen, Aachen, 1992.

[7] Brackx, F. and R. Delanghe: *Hypercomplex function theory and Hilbert modules with reproducing kernel.* Proc. London Math. Soc. 37 (1978), 545 – 576.

[8] F. Brackx, R. Delanghe and F. Sommen. *Clifford Analysis.* Pitman Res. Notes **76**, Boston-London-Melbourne, 1982.

[9] Calderbank, D.: *Dirac operators and Clifford analysis on manifolds with boundary.* Max Planck Institute for Mathematics Bonn, Preprint number 96 – 131, 1996.

[10] J. Cnops. *An Introduction to Dirac operators on manifolds.* Birkhäuser, Basel, 2002.

[11] Constales, D.: *The relative position of L^2 domains in complex and Clifford analysis.* Ph.D. Thesis, Ghent State University, 1989-1990.

[12] Constales, D.: *The Bergman and Szegö kernels for separately monogenic functions.* Zeitschrift für Analysis und ihre Anwendungen 9 (1990) No. 2, 97 – 103.

[13] Constales, D. and R.S. Kraußhar: *Szegö and Polymonogenic Bergman Kernels for half-space and strip domains, and Single-periodic Functions in Clifford Analysis.* Complex Variables 47 (2002) No. 4, 349 – 360.

[14] Constales, D. and R.S. Kraußhar: Hilbert Spaces of Solutions to Polynomial Dirac equations, Fourier Transforms and Reproducing Kernel Functions for Cylindrical Domains. *Zeitschrift für Analysis und Ihre Anwendungen* **24**, No. 3 (2005), 611–636

[15] Delanghe, R.: *On Hilbert modules with reproducing kernel.* Funct. theor. Meth. part. Differ. Equat., Proc. int. Symp., Darmstadt 1976, Lect. Notes Math. 561 (1976), 158 – 170.

[16] Delanghe, R., Sommen, F. and V. Souček: *Clifford Algebra and Spinor Valued Functions*, Dordrecht-Boston-London: Kluwer 1992.

[17] W. Fischer and I. Lieb. *Ausgewählte Kapitel der Funktionentheorie.* Vieweg, 1988.

[18] Gradshteyn, I. and I.M. Ryzhik: *Table of Integrals, Series and Products.* New York: Academic Press 1980.

[19] D. Grob. *Berechnungen zur Konstruktion von Abbildungen in die Einheirskugel mit dem Szegökern im Hyperkomplexen.* Diplomarbeit, Lehrstuhl II für Mathematik RWTH Aachen, Aachen, 2007.

[20] K. Gürlebeck, K. Habetha, and W. Sprößig. *Funktionentheorie in der Ebene und im Raum.* Birkhäuser Verlag, Basel, 2006

[21] K. Gürlebeck and W. Sprössig. *Quaternionic and Clifford Calculus for Physicists and Engineers*. John Wiley & Sons, Chichester-New York, 1997.

[22] J. Liouville. Extension au cas de trois dimensions de la question du tracé géographique. *Application de l' analyse à la géometrie*, G. Monge, Paris (1850), 609-616

[23] H. Malonek. *Zum Holomorphiebegriff in höheren Dimensionen*. Habilitation Thesis, Halle, 1987.

[24] H. Malonek. Power series representation for monogenic functions in $\mathbb{R}^{m+1}$ based on a permutational product. *Complex Variables* **15** (1990), 181–191.

[25] Peetre, J. and P. Sjölin: *Three-line theorems and Clifford analysis.* Complex Variables 19 (1992) No. 3, 151 – 163.

[26] J. Rüsges. *Bergmankerne und Abbildungen auf die Einheitskugel*, Diplomarbeit, Lehrstuhl II für Mathematik RWTH Aachen, Aachen, 2007.

[27] Shapiro, M. and N. Vasilevski: *On the Bergman kernel function in the Clifford analysis. In: Clifford algebras and their applications in mathematical physics*, Fund. Theor. Phys. 55, Dortrecht: Kluwer 1993, 183 – 192.

[28] Shapiro, M. and N. Vasilevski: *On the Bergmann kernel function in hyperholomorphic analysis.* Acta Appl. Math. 46 (1997) No. 1, 1 – 27.