Unbounded arithmetic

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1 Introduction

When comparing the different axiomatizations of bounded arithmetic and Peano arithmetic, it becomes clear that there are similarities between the fragments of these theories. In particular, it is tempting to draw an analogy between the hierarchies of bounded arithmetic and Peano arithmetic. However, one cannot deny that there are essential and deeply rooted differences and the most one can claim is a weak analogy between these hierarchies. The following quote by Kaye (see [?kaye1337]) expresses this argument in an elegant way.

> Many authors have emphasized the analogies between the fragments Σ_n^b-IND of $I\Delta_0+(\forall x)(x^{\log x}$ exists) and the fragments $I\Sigma_n$ of Peano arithmetic. Sometimes this is helpful, but often one feels that the bounded hierarchy of theories is of a rather different nature and new techniques must be developed to answer the key questions concerning them.

In this paper, we propose a (conjectured) hierarchy for Peano arithmetic which is much closer to that of bounded arithmetic than the existing one. In light of this close relation, techniques developed to establish properties of the new hierarchy should carry over naturally to the (conjectured) hierarchy of bounded arithmetic. As the famous P vs. NP problem is related to the collapse of the hierarchy of bounded arithmetic, the new hierarchy may prove particularly useful in solving this famous problem.

2 Preliminaries

We assume that the reader is familiar with the fundamental notions concerning bounded arithmetic and Peano arithmetic. For details, we refer to the first two chapters of [?buss]. For completeness, we mention the Hardy hierarchy and some of its essential properties. Let α be an ordinal and let λ be a limit ordinal and λ_n its n-th predecessor.

$$\begin{aligned} H_0(x) &:= x, \\ H_{\alpha+1}(x) &:= H_\alpha(x+1), \\ H_\lambda(x) &:= H_{\lambda_x}(x). \end{aligned}$$

The well-known Ackermann function $A(x)$ corresponds to $H_{\omega^\omega}(x)$. For a given function $H_\alpha(x)$, the inverse $H_\alpha^{-1}(x)$ is defined as $(\mu m \leq x)(H_\alpha(m) \geq x)$. In general, the function $H_\alpha^{-1}(x)$ is of much lower complexity than $H_\alpha(x)$. Indeed, it is well-known that $A(x)$ is not primitive recursive and that $A^{-1}(x)$ is. For brevity, we sometimes write $|x|_\alpha$ instead of $H_\alpha^{-1}(x)$.

3 Two fundamental differences

In this section, we point out two fundamental differences between bounded arithmetic and Peano arithmetic. In section **??**, we attempt to overcome these differences.

3.1 The logarithmic function

In bounded arithmetic, the log function is defined as $|x| := \lceil \log_2(x+1) \rceil$. Because the inverse of log, i.e. the exponential function, is not total in bounded arithmetic, the log does not have its 'usual' properties. The following theorem illustrates this claim.

Theorem 1 *The theory of bounded arithmetic does not prove that the* log *function is unbounded, i.e.* $S_2 \nvdash (\forall x)(\exists y)(|y| > x)$.

Proof. Assume S_2 proves $(\forall x)(\exists y)(|y| > x)$. By Parikh's theorem, there is a term t such that S_2 proves $(\forall x)(\exists y \leq t(x))(|y| > x)$. As $|x|$ is weakly increasing, there follows $(\forall x)(|t(x)| > x)$. However, this implies that $t(x)$ grows as fast as the exponential function, which is impossible.

By completeness, there is a model of S_2 in which $|x|$ is bounded. At the very least, this theorem shows that one should be careful with 'visual' proofs. Indeed, even most mathematicians would claim that it is clear from the graph of $\log x$ that this function is unbounded. However, by itself, the previous theorem is not a big revelation. Indeed, the same theorem (and proof) holds for PRA and $A^{-1}(x)$ instead of S_2 and $|x|$. It is easy to verify that the function $H_{\varepsilon_0}^{-1}(x)$ has the same property for Peano arithmetic.

So far, we showed that the log function has unusual properties in bounded arithmetic, but there seem to be similarly 'strange' functions in Peano arithmetic. However, the axioms of Peano arithmetic do not involve $H_{\varepsilon_0}^{-1}(x)$, whereas the log function is used explicitly in the axiomatization of bounded arithmetic. Indeed, consider the following axiom schema.

Axiom schema 2 (Φ-LIND) *For every* $\varphi \in \Phi$, *we have*

$$\big[\varphi(0) \wedge (\forall n)(\varphi(n) \to \varphi(n+1))\big] \to (\forall n)\varphi(|n|).$$

This axiom schema is called 'length induction'. The theory S_2^i of bounded arithmetic consists of the basic theory BASIC plus the Σ_n^b-LIND schema. Furthermore, the theory T_2^i consists of BASIC plus the Σ_n^b-induction schema and the (conjectured) hierarchy of bounded arithmetic is as follows, for $i \geq 2$,

$$S_2^1 \subseteq T_2^1 \subseteq \cdots \subseteq S_2^i \subseteq T_2^i \subseteq S_2^{i+1} \subseteq T_2^{i+1} \subseteq \cdots \subseteq S_2 = T_2. \tag{1}$$

Thus, the log function appears in a non-trivial way in the axiomatization of bounded arithmetic, although it has unusual properties (see theorem **??**). By contrast, the function $H_{\varepsilon_0}^{-1}(x)$ does not appear in the axioms of Peano arithmetic.

Finally, it is worth mentioning that in the presence of the exponential function, which is available in $I\Sigma_1$, Σ_n-IND and Σ_n-LIND coincide. Thus, at first glance there is no analogue of the length induction axioms for Peano arithmetic. In section **??**, we shall fill this gap.

3.2 The 'smash' function

In bounded arithmetic, Nelson's 'smash' function $x\#y := 2^{|x|\cdot|y|}$ plays an important role. The presence of this function guarantees that Gödel numbering can be done elegantly, that sharply bounded quantifiers can be pushed into bounded quantifiers and that there is a natural correspondence between the polynomial time hierarchy and the hierarchy of bounded arithmetical formulas (see [`?buss`, p. 100] for details).

However, the smash function is not Σ_1-definable in $I\Delta_0$. Thus, it is added to $I\Delta_0$, either by the axiom Ω_1 which defines the function $\omega_1(x,y) = x^{|y|}$, or through the axioms BASIC which guarantee that $x\#y = 2^{|x|\cdot|y|}$. The natural counterparts for this function in PRA and Peano arithmetic are

$$x\%y := A\big[A^{-1}(x).A^{-1}(y)\big] \text{ and } x@y := H_{\varepsilon_0}\big[H_{\varepsilon_0}^{-1}(x).H_{\varepsilon_0}^{-1}(y)\big].$$

It is easily verified that $x\%y$ is not primitive recursive and that $x@y$ is not provably total in Peano arithmetic. It should be noted that the latter function has recently been considered in [`?simmons`] in the context of 'Ackermannian degrees'.

4 A new hierarchy for Peano arithmetic

In this section we introduce a new (conjectured) hierarchy of Peano arithmetic, inspired by the (conjectured) hierarchy of bounded arithmetic. Thus, we refer to these theories as 'unbounded arithmetic'. The following axiom schema plays a central role.

Axiom schema 3 (Φ-$\mathbb{LIND}$) *For every $\varphi \in \Phi$, we have*

$$\big[\varphi(0) \wedge (\forall n)(\varphi(n) \to \varphi(n+1))\big] \to (\forall n)\varphi(|n|_{\varepsilon_0}).$$

Thus, we have introduced the function $H_{\varepsilon_0}^{-1}(x)$ explicitly and the Σ_n-$\mathbb{LIND}$ axioms are the natural counterpart for the Σ_n^b-LIND axioms of bounded arithmetic.

However, the theory $Q + \Sigma_i$-$\mathbb{LIND}$ is not a good counterpart for S_2^i. Indeed, recall that S_2^i consists of the axiom schema Σ_i^b-LIND plus the axiom set BASIC. The latter makes sure that $x\#y = 2^{|x|.|y|}$ is available. Thus, the natural counterpart of the smash function in Peano arithmetic, namely $x@y = H_{\varepsilon_0}(|x|_{\varepsilon_0}.|y|_{\varepsilon_0})$, is missing from $Q + \Sigma_n$-$\mathbb{LIND}$. Thus, we define $\mathbb{BASIC}$ as Robinson's theory Q plus the statement that $x@y = H_{\varepsilon_0}(|x|_{\varepsilon_0}.|y|_{\varepsilon_0})$ is total. Next, we define $\mathbb{S}_2^i$ as the theory $\mathbb{BASIC}$ plus Σ_i-$\mathbb{LIND}$ and $\mathbb{T}_2^i$ as the theory $\mathbb{BASIC}$ plus Σ_i-IND. Finally, we define $\mathbb{T}_2$ (respectively $\mathbb{S}_2$) as the union of all theories $\mathbb{T}_2^i$ (respectively $\mathbb{S}_2^i$). It is immediate that $\mathbb{T}_2$ is very close to Peano Arithmetic. We have partial proofs for the following theorem.

Theorem 4 *For $i \geq 2$, we have*

$$\mathbb{S}_2^1 \subseteq \mathbb{T}_2^1 \subseteq \cdots \subseteq \mathbb{S}_2^i \subseteq \mathbb{T}_2^i \subseteq \mathbb{S}_2^{i+1} \subseteq \mathbb{T}_2^{i+1} \subseteq \cdots \subseteq \mathbb{S}_2 = \mathbb{T}_2 = \mathrm{PA} + \mathbb{BASIC}.$$

The ubiquity of fast growing functions in PA allows us to give an alternative hierarchy. The following axiom schema is fundamental.

Axiom schema 5 (Φ-$\mathcal{LIND}$) *For every $\varphi \in \Phi$, we have*

$$\big[\varphi(0) \wedge (\forall n)(\varphi(n) \to \varphi(n+1))\big] \to (\forall n)\varphi\big(|n|_{(\varepsilon_0)_n}\big).$$

As in the previous, we define $\mathcal{BASIC}_n$ as Robinson's theory Q plus the statement that $H_{(\varepsilon_0)_n}(|x|_{(\varepsilon_0)_n}.|y|_{(\varepsilon_0)_n})$ is total. Next, we define $\mathcal{S}_2^i$ as the theory $\mathcal{BASIC}_i$ plus Σ_i-$\mathcal{LIND}$ and $\mathcal{T}_2^i$ as the theory $\mathcal{BASIC}_i$ plus Σ_i-$\mathcal{LIND}$. Finally, we define $\mathcal{T}_2$ (respectively $\mathcal{S}_2$) as the union of all theories $\mathcal{T}_2^i$ (respectively $\mathcal{S}_2^i$). It is immediate that $\mathcal{T}_2$ is essentially Peano Arithmetic. We have partial proofs for the following theorem.

Theorem 6 *For $i \geq 2$, we have*

$$\mathcal{S}_2^1 \subseteq \mathcal{T}_2^1 \subseteq \cdots \subseteq \mathcal{S}_2^i \subseteq \mathcal{T}_2^i \subseteq \mathcal{S}_2^{i+1} \subseteq \mathcal{T}_2^{i+1} \subseteq \cdots \subseteq \mathcal{S}_2 = \mathcal{T}_2 = \mathrm{PA}.$$

Incidentally, if we replace the ordinal ε_0 in schema **??** with an ordinal parameter α, then $\alpha = \varepsilon_0$ corresponds to $\mathbb{LIND}$, $\alpha = (\varepsilon_0)_n$ to $\mathcal{LIND}$ and $\alpha = \omega^2$ essentially to LIND. Thus, all the above length induction schemas are 'branches of the same tree'.

5 Some time functions

The attentive reader has noted that the length induction axioms of bounded arithmetic is not the only place where the log-function is used explicitly. Indeed, the latter function is also used explicitly in the definition of the polynomial time functions. In this section, we introduce two additional function classes which play the role of the polynomial time functions in our two new hierarchies of Peano Arithmetic.

The class FP of the polynomial time functions is obtained by closing a certain set of initial functions under projection, composition and a restricted version of primitive recursion, called 'limited iteration'. Essentially,

primitive recursion is allowed as long as the resulting function $f(z, \boldsymbol{x})$ 'does not grow too fast'. In particular, f has to satisfy the following growth condition:

$$|f(z, \boldsymbol{x})| \leq p(|z|, |\boldsymbol{x}|), \text{ for all } z, \boldsymbol{x},$$

where p is some polynomial.

Analogously, the class $\mathbb{FP}$ is defined by closing the same initial functions under projection, composition and a restricted version of double recursion. In particular, double recursion is allowed if the resulting function $f(z, \boldsymbol{x})$ satisfies

$$|f(z, \boldsymbol{x})|_{\varepsilon_0} \leq h\big(|z|_{\varepsilon_0}, |\boldsymbol{x}|_{\varepsilon_0}\big),$$

where h is some primitive recursive function.

Analogously, the class $\mathcal{FP}$ is defined by closing the same initial functions under projection, composition and a restricted version of double recursion. In particular, double recursion is allowed if the resulting function $f(z, \boldsymbol{x})$ satisfies

$$A^{-1}\big[f(z, \boldsymbol{x})\big] \leq h\big(A^{-1}(z), A^{-1}(\boldsymbol{x})\big),$$

where h is some primitive recursive function.

The functions in $\mathcal{FP}$ and $\mathbb{FP}$ may be called 'primitive recursive time' functions. The class $\mathbb{FP}$ is closely related to the total functions of $\mathbb{S}^1_2$ and the class $\mathcal{FP}$ is closely related to the total functions of $\mathcal{S}^1_2$.

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