The valuations of the near octagon $\mathbb{G}_4$

B. De Bruyn and P. Vandecasteele

Ghent University, Department of Pure Mathematics and Computer Algebra, Krijgslaan 281 (S22), B-9000 Gent, Belgium, E-mail: `bdb@cage.ugent.be`, `pvdecast@cage.ugent.be`

Abstract

In [4] it was shown that the dual polar space $DH(2n-1,4)$, $n \geq 2$, has a sub near-$2n$-gon $\mathbb{G}_n$ with a large automorphism group. In this paper, we classify the valuations of the near octagon $\mathbb{G}_4$. We show that each such valuation is either classical, the extension of a non-classical valuation of a $\mathbb{G}_3$-hex or is associated with a valuation of Fano-type of an $\mathbb{H}_3$-hex. In order to describe the latter type of valuation we must study the structure of $\mathbb{G}_4$ with respect to an $\mathbb{H}_3$-hex. This study also allows us to construct new hyperplanes of $\mathbb{G}_4$. We also show that each valuation of $\mathbb{G}_4$ is induced by a (classical) valuation of the dual polar space $DH(7,4)$.

Keywords: near polygon, generalized quadrangle, dual polar space, valuation, hyperplane.
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1 Introduction

1.1 Basic definitions

Let $\mathcal{S}$ be a *dense near* $2n$*-gon*, i.e. $\mathcal{S}$ is a partial linear space which satisfies the following properties:

(i) For every point p and every line L, there exists a unique point $\pi_L(p)$ on L nearest to p. Here, distances $\mathrm{d}(\cdot,\cdot)$ are measured in the collinearity graph of $\mathcal{S}$.

(ii) Every line of $\mathcal{S}$ is incident with at least three points.

(iii) Every two points of $\mathcal{S}$ at distance 2 from each other have at least two common neighbours.

(iv) The maximal distance between two points of $\mathcal{S}$ is equal to n.

A dense near 0-gon is a point, a dense near 2-gon is a line and a dense near quadrangle is a generalized quadrangle (Payne and Thas [17]).

For every point y of $\mathcal{S}$ and every non-empty set X of points, we define $\mathrm{d}(y, X) := \min\{\mathrm{d}(y, x) \,|\, x \in X\}$. If X is a non-empty set of points of $\mathcal{S}$, then for every $i \in \mathbb{N}$, $\Gamma_i(X)$ denotes the set of points y of $\mathcal{S}$ at distance i from X. If X is a singleton $\{x\}$, we will also write $\Gamma_i(x)$ instead of $\Gamma_i(X)$.

One of the following two cases occurs for two lines K and L of $\mathcal{S}$ (see e.g. [5, Theorem 1.3]): (i) there exist unique points $k^* \in K$ and $l^* \in L$ such that $\mathrm{d}(k, l) = \mathrm{d}(k, k^*) + \mathrm{d}(k^*, l^*) + \mathrm{d}(l^*, l)$ for all $k \in K$ and $l \in L$; (ii) the map $K \rightarrow L; x \mapsto \pi_L(x)$ is a bijection and its inverse is equal to the map $L \rightarrow K; y \mapsto \pi_K(y)$. If the latter case occurs, then K and L are called *parallel*.

By Theorem 4 of Brouwer and Wilbrink [2], every two points x and y of $\mathcal{S}$ at distance $\delta \in \{0, \ldots, n\}$ from each other are contained in a unique convex subspace $\langle x, y \rangle$ of diameter δ. These convex subspaces are called *quads*, respectively *hexes*, if $\delta = 2$, respectively $\delta = 3$. The lines and quads through a given point x of $\mathcal{S}$ define a linear space which is called the *local space at* x. If $X_1, X_2, \ldots, X_k$ are non-empty sets of points, then $\langle X_1, X_2, \ldots, X_k \rangle$ denotes the smallest convex subspace containing $X_1 \cup X_2 \cup \cdots \cup X_k$. Clearly, $\langle X_1, X_2, \ldots, X_k \rangle$ is the intersection of all convex subspaces containing $X_1 \cup X_2 \cup \cdots \cup X_k$.

A point x of $\mathcal{S}$ is called *classical* with respect to a convex subspace F of $\mathcal{S}$ if there exists a (necessarily unique) point $\pi_F(x)$ in F such that $\mathrm{d}(x, y) = \mathrm{d}(x, \pi_F(x)) + \mathrm{d}(\pi_F(x), y)$ for every point y of F. Every point of $\Gamma_1(F)$ is classical with respect to F. A convex subspace F of $\mathcal{S}$ is called *classical* (in $\mathcal{S}$) if every point of $\mathcal{S}$ is classical with respect to F. Every line of $\mathcal{S}$ is classical. If every quad of $\mathcal{S}$ is classical, then $\mathcal{S}$ is a so-called *dual polar space* (Cameron [3]). The near polygon $\mathcal{S}$ is then isomorphic to a geometry Δ whose points and lines are the maximal and next-to-maximal singular subspaces of a given polar space Π (natural incidence). A proper convex subspace F of $\mathcal{S}$ is called *big* (in $\mathcal{S}$) if every point of $\mathcal{S}$ has distance at most 1 from F. If F is big, then F is also classical. If F is big and if every line of $\mathcal{S}$ is incident with precisely 3 points, then we can define a *reflection* $\mathcal{R}_F$ *about* F which is an automorphism of $\mathcal{S}$: if $x \in F$, then we define $\mathcal{R}_F(x) := x$; if $x \notin F$, then $\mathcal{R}_F(x)$ is the unique point on the line $x\pi_F(x)$ different from x and $\pi_F(x)$.

Near polygons were introduced by Shult and Yanushka [18]. We refer to (Chapter 2 of) De Bruyn [5] for more background information on (dense) near polygons.

A function f from the point-set of $\mathcal{S}$ to $\mathbb{N}$ is called a *valuation* of $\mathcal{S}$ if it satisfies the following properties:

(V1) $f^{-1}(0) \neq \emptyset$;

(V2) every line L of $\mathcal{S}$ contains a unique point x_L such that $f(x) = f(x_L)+1$ for every point x of L different from x_L;

(V3) every point x of $\mathcal{S}$ is contained in a (necessarily unique) convex subspace F_x such that the following properties are satisfied for every $y \in F_x$:

(i) $f(y) \leq f(x)$;

(ii) if z is a point collinear with y such that $f(z) = f(y) - 1$, then $z \in F_x$.

Valuations of dense near polygons were introduced in De Bruyn and Vandecasteele [11]. For many classes of dense near polygons, see [10], it can be shown that property (V3) is a consequence of property (V2).

If f is a valuation of $\mathcal{S}$, then we denote by O_f the set of points with value 0. A quad Q of $\mathcal{S}$ is called *special (with respect to f)* if it contains two distinct points of O_f, or equivalently (see [11]), if it intersects O_f in an ovoid of Q. We denote by G_f the partial linear space with points the elements of O_f and with lines the special quads (natural incidence).

Proposition 1.1 (Proposition 2.12 of [11]) *Let $\mathcal{S}$ be a dense near polygon and let $F = (\mathcal{P}', \mathcal{L}', \mathrm{I}')$ be a (not necessarily convex) subpolygon of $\mathcal{S}$ for which the following holds:* (1) *F is a dense near polygon;* (2) *F is a subspace of $\mathcal{S}$;* (3) *if x and y are two points of F, then $d_F(x, y) = d_{\mathcal{S}}(x, y)$. Let f denote a valuation of $\mathcal{S}$ and put $m := \min\{f(x) \,|\, x \in \mathcal{P}'\}$. Then the map $f_F : \mathcal{P}' \to \mathbb{N}; x \mapsto f(x) - m$ is a valuation of F.*

Definition. The valuation f_F of F defined in Proposition 1.1 is called the valuation of F *induced* by f.

Examples. Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a dense near $2n$-gon, $n \geq 2$.

(1) For every point x of $\mathcal{S}$, the map $\mathcal{P} \to \mathbb{N}; y \mapsto \mathrm{d}(x, y)$ is a valuation of $\mathcal{S}$ which we call a *classical valuation*.

(2) Suppose O is an *ovoid* of $\mathcal{S}$, i.e. a set of points meeting each line in a unique point. For every point x of $\mathcal{S}$, we define $f_O(x) = 0$ if $x \in O$ and $f_O(x) = 1$ otherwise. Then f_O is a valuation of $\mathcal{S}$ which we call an *ovoidal valuation*.

(3) Let x be a point of $\mathcal{S}$ and let O be a set of points at distance n from x having a unique point in common with every line at distance $n-1$ from x. For every point y of $\mathcal{S}$, we define $f(y) = \mathrm{d}(x, y)$ if $\mathrm{d}(x, y) \leq n - 1$,

$f(y) = n - 2$ if $y \in O$ and $f(y) = n - 1$ otherwise. Then f is a valuation of $\mathcal{S}$ which we call a *semi-classical valuation*.

(4) Suppose $F = (\mathcal{P}', \mathcal{L}', \mathrm{I}')$ is a convex subspace of $\mathcal{S}$ which is classical in $\mathcal{S}$. Suppose that $f' : \mathcal{P}' \to \mathbb{N}$ is a valuation of F. Then the map $f : \mathcal{P} \to \mathbb{N}; x \mapsto f(x) := \mathrm{d}(x, \pi_F(x)) + f'(\pi_F(x))$ is a valuation of $\mathcal{S}$. We call f the *extension* of f'.

In the literature, valuations have been used for the following important applications: (i) classification of dense near polygons ([9], [16]); (ii) constructions of new hyperplanes of dense near polygons, in particular of dual polar spaces ([8], [12]); (iii) classification of certain hyperplanes of dense near polygons, in particular of dual polar spaces ([6]); (iv) study of isometric full embeddings between dense near polygons, in particular between dual polar spaces ([7], [14], [15]).

We will now define two classes of dense near polygons which will be important throughout this paper.

(I) Let X be a set of size $2n+2$, $n \geq 2$, and let $\mathbb{H}_n = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be the following point-line geometry:

(i) $\mathcal{P}$ is the set of all partitions of X in $n+1$ subsets of size 2;

(ii) $\mathcal{L}$ is the set of all partitions of X in $n-1$ subsets of size 2 and one subset of size 4;

(iii) a point $p \in \mathcal{P}$ is incident with a line $L \in \mathcal{L}$ if and only if the partition determined by the point p is a refinement of the partition determined by L.

By Brouwer, Cohen, Hall and Wilbrink [1], see also De Bruyn [5, Section 6.2], $\mathbb{H}_n$ is a dense near $2n$-gon with three points per line. The near polygon $\mathbb{H}_n$ has $\frac{(2n+2)!}{2^{n+1}\cdot(n+1)!}$ points and each point is incident with $\binom{n+1}{2}$ lines. Every quad of $\mathbb{H}_n$ is isomorphic to either the (3×3)-grid or the generalized quadrangle $W(2)$. Every $W(2)$-quad is classical in $\mathbb{H}_n$. By De Bruyn [10, Corollary 1.4], a map $f : \mathcal{P} \to \mathbb{N}$ is a valuation of $\mathbb{H}_n$ if and only if it satisfies properties (V1) and (V2).

The near hexagon $\mathbb{H}_3$ will be of interest in this paper. Every $W(2)$-quad of $\mathbb{H}_3$ is big. Every local space of $\mathbb{H}_3$ is isomorphic to the Fano-plane in which a point has been removed. Hence, every point of $\mathbb{H}_3$ is contained in three grid-quads and these grid-quads partition the set of lines through x. If x is a point of $\mathbb{H}_3$ at distance 2 from a grid-quad Q, then $\Gamma_2(x) \cap Q$ is an ovoid of Q. Moreover, the three quads through x which meet Q are grids.

(II) Let $H(2n-1, 4)$, $n \geq 2$, denote the Hermitian variety $X_0^3 + X_1^3 + \cdots + X_{2n-1}^3 = 0$ of $\mathrm{PG}(2n-1, 4)$ (with respect to a given reference system). The number of nonzero coordinates (with respect to the same reference system) of a point p of $\mathrm{PG}(2n-1,4)$ is called the *weight* of p. With the Hermitian variety $H(2n-1,4)$, there is associated a dual polar space which is denoted by $DH(2n-1,4)$. The points and lines of $DH(2n-1,4)$ are the maximal and next-to-maximal subspaces of $H(2n-1,4)$ (natural incidence). Let $\mathbb{G}_n = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be the following subgeometry of $DH(2n-1,4)$:

(i) $\mathcal{P}$ is the set of all maximal subspaces of $H(2n-1,4)$ containing n points with weight 2;

(ii) $\mathcal{L}$ is the set of all $(n-2)$-dimensional subspaces of $H(2n-1,4)$ containing at least $n-2$ points of weight 2;

(iii) incidence is reverse containment.

By De Bruyn [4], see also De Bruyn [5, 6.3], $\mathbb{G}_n$ is a dense near $2n$-gon with three points on each line and its above-defined embedding in $DH(2n-1,4)$ is isometric, i.e. preserves distances. The near polygon $\mathbb{G}_n$ has $\frac{3^n \cdot (2n)!}{2^n \cdot n!}$ points and each point of $\mathbb{G}_n$ is contained in precisely $\frac{n(3n-1)}{2}$ lines. Every quad of $\mathbb{G}_n$ is isomorphic to either the (3×3)-grid, $W(2)$ or the generalized quadrangle $Q(5,2)$. Every $Q(5,2)$-quad is classical in $\mathbb{G}_n$. By De Bruyn [10, Corollary 1.4], a map $f : \mathcal{P} \to \mathbb{N}$ is a valuation of $\mathbb{G}_n$ if and only if it satisfies properties (V1) and (V2).

1.2 The main result

The near octagon $\mathbb{G}_4$ has hexes isomorphic to $\mathbb{G}_3$ and $\mathbb{H}_3$. Every $\mathbb{G}_3$-hex F is big in $\mathbb{G}_4$ and hence every valuation f of F will give rise to a valuation of $\mathbb{G}_4$, namely the extension of f. No $\mathbb{H}_3$-hex is big in F. We will later show (Propositions 5.1 and 6.10) that if f is a valuation of an $\mathbb{H}_3$-hex F such that G_f is a Fano-plane, then there exists a unique valuation $\overline{f}$ of $\mathbb{G}_4$ such that $O_{\overline{f}} = O_f$. We will call $\overline{f}$ a valuation of *Fano-type* of $\mathbb{G}_4$. In this paper, we classify all valuations of $\mathbb{G}_4$. We will show the following.

Theorem 1.2 (Section 6) *If f is a valuation of $\mathbb{G}_4$, then f is one of the following:*

(1) *f is a classical valuation of $\mathbb{G}_4$;*

(2) *f is the extension of a non-classical valuation of a $\mathbb{G}_3$-hex of $\mathbb{G}_4$;*

(3) *f is a valuation of Fano-type of $\mathbb{G}_4$.*

Each of these valuations is induced by a unique (classical) valuation of $DH(7,4)$.

Notice that all valuations of $DH(7,4)$ are classical by Theorem 6.8 of De Bruyn [5]. In order to describe the valuations of Fano-type of $\mathbb{G}_4$ (see Section 5), we must study the structure of $\mathbb{G}_4$ with respect to an $\mathbb{H}_3$-hex (Section 4).

This study allows us to construct a class of hyperplanes of $\mathbb{G}_4$ (Proposition 4.14).

2 The valuations of the near hexagons $\mathbb{G}_3$, $\mathbb{H}_3$, $Q(5,2) \times \mathbb{L}_3$ and $W(2) \times \mathbb{L}_3$

The valuations of the near hexagons $\mathbb{G}_3$, $\mathbb{H}_3$, $Q(5,2) \times \mathbb{L}_3$ and $W(2) \times \mathbb{L}_3$ were determined in De Bruyn and Vandecasteele [13].

There are two types of valuations in $\mathbb{G}_3$: the classical valuations and the non-classical valuations. In the following lemma, we collect some known facts about non-classical valuations of $\mathbb{G}_3$.

Lemma 2.1 ([13]) *Suppose f is a non-classical valuation of $\mathbb{G}_3$. Then:*

(*i*) *G_f is isomorphic to $\overline{W(2)}$, the linear space obtained from the generalized quadrangle $W(2)$ by adding its ovoids as extra lines.*

(*ii*) *$|O_f| = 15$ and every two distinct points of O_f lie at distance 2 from each other.*

(*iii*) *Every point with value 1 is contained in a unique special quad.*

(*iv*) *Every $Q(5,2)$-quad Q of $\mathbb{G}_3$ contains a unique point with value 0. Moreover, $f(y) = d(y, Q \cap O_f)$ for every point y of Q.*

(*v*) *Every point x of O_f is contained in three special grid-quads and two special $W(2)$-quads. These five quads determine a partition of the set of lines through x.*

If f is a valuation of $\mathbb{H}_3$, then any two distinct points of O_f lie at distance 2 from each other. There are four types of valuations in the near hexagon $\mathbb{H}_3$: the classical valuations, the extensions of the ovoidal valuations of the $W(2)$-quads (*valuations of extended type*), the valuations f for which G_f is a line of size 3 (*valuations of grid-type*) and the valuations f for which G_f is a Fano-plane (*valuations of Fano-type*). In the following two lemmas, we collect some known facts about valuations of grid-type and Fano-type.

Lemma 2.2 ([13]) *Let f be a valuation of grid-type of $\mathbb{H}_3$. Then O_f is an ovoid of a grid-quad Q of $\mathbb{H}_3$. If $d(x, O_f) \leq 2$, then $f(x) = d(x, O_f)$. If $d(x, O_f) = 3$, then $f(x) = 1$.*

Lemma 2.3 ([13]) *Let f be a valuation of Fano-type of $\mathbb{H}_3$. Then:*

(*i*) *Every $W(2)$-quad R contains a unique point of O_f and $f(y) = d(y, O_f \cap R)$ for every $y \in R$.*

(*ii*) *Every grid-quad intersects* O_f *in either the empty set or an ovoid of the grid-quad. If a grid-quad* Q *is disjoint from* O_f*, then* Q *intersects the set of points with value* 1 *in an ovoid of* Q*.*

(*iii*) *For every* $x \in O_f$*, the three grid-quads through* x *are special.*

(*iv*) *Every point with value* 1 *is contained in a unique special quad.*

Lemma 2.4 *Let* f *be a valuation of Fano-type of* $\mathbb{H}_3$*. Let* Q *be a* $W(2)$*-quad of* $\mathbb{H}_3$ *and let* G_2 *and* G_3 *be two grid-quads of* $\mathbb{H}_3$ *such that* (*i*) Q*,* G_2 *and* G_3 *are mutually disjoint, and* (*ii*) $\mathcal{R}_Q(G_2) = G_3$*. Put* $G_1 := \pi_Q(G_2) = \pi_Q(G_3)$*. Then one of the following cases occurs:*

(1) *There exists precisely one* $i \in \{2, 3\}$ *such that* $|G_i \cap O_f| = 3$ *and* $|G_{5-i} \cap O_f| = 0$*. Moreover, the unique point in* $O_f \cap Q$ *is not contained in* G_1*.*

(2) $|G_2 \cap O_f| = |G_3 \cap O_f| = 0$ *and the unique point in* $O_f \cap Q$ *is contained in* G_1*.*

Proof. Let x^* denote the unique point of $O_f \cap Q$. Recall that $f(y) = \mathrm{d}(y, x^*)$ for every $y \in Q$. We distinguish two cases.

(1) Suppose x^* is not contained in G_1. Put $\Gamma_1(x^*) \cap G_1 = \{x_1, x_2, x_3\}$ and let L_i, $i \in \{1, 2, 3\}$, denote the unique line through x_i meeting G_2 and G_3. Since $x^* \notin G_1$, we have $\mathrm{d}(x^*, G_2) = \mathrm{d}(x^*, G_3) = 2$. Hence, each of the three quads through x^* meeting G_2 (G_3) is a grid. So, $\langle x^* x_1, L_1\rangle$, $\langle x^* x_2, L_2\rangle$ and $\langle x^* x_3, L_3\rangle$ are the three grid-quads through x^* meeting G_2 (G_3) in a point. By Lemma 2.3(iii) these three grid-quads are special with respect to the valuation f (recall $x^* \in O_f$). Hence, $|L_1 \cap O_f| = 1$. Choose $i \in \{2, 3\}$ such that $G_i \cap L_1 \cap O_f \neq \emptyset$. Then again by Lemma 2.3(iii), $|G_i \cap O_f| = 3$. Since every point of $G_1 \setminus \{x_1, x_2, x_3\}$ has value 2, no point of $(G_2 \cup G_3) \setminus (L_1 \cup L_2 \cup L_3)$ belongs to O_f by property (V2) in the definition of valuation. It follows that $G_i \cap O_f = (G_i \cap L_1) \cup (G_i \cap L_2) \cup (G_i \cap L_3)$. For every $j \in \{1, 2, 3\}$, $L_j \cap G_i$ has value 0 and $L_j \cap Q$ has value 1. Hence, $L_j \cap G_{5-i}$ has value 1 by property (V2). Together with $(G_{5-i} \setminus (L_1 \cup L_2 \cup L_3)) \cap O_f = \emptyset$, this implies that $G_{5-i} \cap O_f = \emptyset$.

(2) Suppose that the unique point x^* in $O_f \cap Q$ is contained in G_1. Suppose y^* is a point of $O_f \cap G_2$. Then $\mathrm{d}(x^*, y^*) = 2$. Hence, the unique point z^* of G_2 collinear with x^* is also collinear with y^*. It follows that $\langle x^*, y^*\rangle$ and G_2 are two special grid-quads meeting in the line $y^* z^*$, a contradiction. Hence, $G_2 \cap O_f = \emptyset$. In a similar way, one shows that $G_3 \cap O_f = \emptyset$. ■

The near hexagon $Q(5, 2) \times \mathbb{L}_3$ is obtained by taking three isomorphic copies of the generalized quadrangle $Q(5, 2)$ and joining the corresponding points to form lines of size 3. There are two types of valuations in $Q(5, 2) \times \mathbb{L}_3$:

the classical valuations and the extensions of the ovoidal valuations of the grid-quads.

The near hexagon $W(2) \times \mathbb{L}_3$ is obtained by taking three isomorphic copies of the generalized quadrangle $W(2)$ and joining the corresponding points to form lines of size 3. There are four types of valuations in $W(2) \times \mathbb{L}_3$: the classical valuations, the extensions of the ovoidal valuations of the grid-quads, the extensions of the ovoidal valuations of the $W(2)$-quads and the semi-classical valuations.

3 Properties of the near octagon $\mathbb{G}_4$

We start with some properties of the near $2n$-gon $\mathbb{G}_n$, $n \geq 3$, whose proofs can be found in the book [5]. Let U denote the set of points of weight 1 and 2 of $\mathrm{PG}(n-1,4)$ (with respect to a certain reference system) and let $\mathcal{L}_U$ denote the linear space induced on the set U by the lines of $\mathrm{PG}(n-1,4)$. Then every local space of $\mathbb{G}_n$ is isomorphic to $\mathcal{L}_U$. Every quad of $\mathbb{G}_n$, $n \geq 3$, is isomorphic to either the (3×3)-grid, $W(2)$ or $Q(5,2)$. The near polygon $\mathbb{G}_n$, $n \geq 3$, has two types of lines:

(i) SPECIAL LINES: these are lines which are not contained in a $W(2)$-quad.

(ii) ORDINARY LINES: these are lines which are contained in at least one $W(2)$-quad.

There are two possible grid-quads in $\mathbb{G}_n$, $n \geq 3$.

(i) GRID-QUADS OF TYPE I: these grid-quads contain three ordinary and three special lines; the lines of each type partition the point set of the grid.

(ii) GRID-QUADS OF TYPE II: these grid-quads contain six ordinary lines.

If $n = 3$, then every grid-quad is of type I. If $n \geq 4$, then both types of grid-quads occur.

The automorphism group of $\mathbb{G}_n$, $n \geq 3$, acts transitively on the set of special lines, the set of ordinary lines, the set of $Q(5,2)$-quads, the set of $W(2)$-quads, the set of grid-quads of type I and the set of grid-quads of type II.

In the following lemma, we collect some properties of the near octagon $\mathbb{G}_4$.

Lemma 3.1 (1) *The near octagon* $\mathbb{G}_4$ *has* 8505 *points, each line of* $\mathbb{G}_4$ *contains* 3 *points and each point of* $\mathbb{G}_4$ *is contained in* 22 *lines.*

(2) *Every quad of* $\mathbb{G}_4$ *is isomorphic to either the* (3×3)*-grid,* $W(2)$ *or* $Q(5,2)$*. Every* $Q(5,2)$*-quad is classical in* $\mathbb{G}_4$*.*

(3) *Every hex of* $\mathbb{G}_4$ *is isomorphic to either* $\mathbb{G}_3$*,* $\mathbb{H}_3$*,* $W(2)\times\mathbb{L}_3$ *or* $Q(5,2)\times\mathbb{L}_3$*. Every* $\mathbb{G}_3$*-hex is big in* $\mathbb{G}_4$*.*

(4) *If* x *is a point of* $\mathbb{G}_4$*, then every* $Q(5,2)$*-quad through* x *contains precisely two special lines through* x*. Conversely, every two distinct special lines through* x *are contained in a unique* $Q(5,2)$*-quad.*

(5) *If* x *is a point of* $\mathbb{G}_4$*, then every* $\mathbb{G}_3$*-hex through* x *contains precisely three special lines through* x*. Conversely, every three distinct special lines through* x *are contained in a unique* $\mathbb{G}_3$*-hex.*

(6) *Every point is contained in* 4 *special lines,* 18 *ordinary lines,* 36 *grid-quads of type I,* 27 *grid-quads of type II,* 36 $W(2)$*-quads,* 6 $Q(5,2)$*-quads,* 4 $\mathbb{G}_3$*-hexes,* 18 $Q(5,2)\times\mathbb{L}_3$*-hexes,* 36 $W(2)\times\mathbb{L}_3$*-hexes and* 27 $\mathbb{H}_3$*-hexes.*

(7) *Every special line is contained in* 9 *grid-quads of type I,* 0 *grid-quads of type II,* 0 $W(2)$*-quads,* 3 $Q(5,2)$*-quads,* 0 $\mathbb{H}_3$*-hexes,* 3 $\mathbb{G}_3$*-hexes,* 9 $Q(5,2)\times\mathbb{L}_3$*-hexes and* 9 $W(2)\times\mathbb{L}_3$*-hexes.*

(8) *Every ordinary line is contained in* 2 *grid-quads of type I,* 3 *grid-quads of type II,* 6 $W(2)$*-quads,* 1 $Q(5,2)$*-quad,* 9 $\mathbb{H}_3$*-hexes,* 2 $\mathbb{G}_3$*-hexes,* 4 $Q(5,2)\times\mathbb{L}_3$*-hexes and* 6 $W(2)\times\mathbb{L}_3$*-hexes.*

(9) *Every* $W(2)$*-quad is contained in precisely* 1 $\mathbb{G}_3$*-hex,* 1 $W(2)\times\mathbb{L}_3$*-hex,* 0 $Q(5,2)\times\mathbb{L}_3$*-hexes and* 3 $\mathbb{H}_3$*-hexes.*

(10) *Every* $Q(5,2)$*-quad is contained in precisely* 2 $\mathbb{G}_3$*-hexes,* 3 $Q(5,2)\times\mathbb{L}_3$*-hexes,* 0 $W(2)\times\mathbb{L}_3$*-hexes and* 0 $\mathbb{H}_3$*-hexes.*

(11) *Every grid-quad of type I is contained in* 1 $\mathbb{G}_3$*-hex,* 0 $\mathbb{H}_3$*-hexes,* 1 $Q(5,2)\times\mathbb{L}_3$*-hex and* 3 $W(2)\times\mathbb{L}_3$*-hexes.*

(12) *Every grid-quad of type II is contained in* 0 $\mathbb{G}_3$*-hexes,* 3 $\mathbb{H}_3$*-hexes,* 2 $Q(5,2)\times\mathbb{L}_3$*-hexes and* 0 $W(2)\times\mathbb{L}_3$*-hexes.*

(13) *Suppose the point* x *of* $\mathbb{G}_4$ *is contained in a* $Q(5,2)$*-quad* Q *and a hex* H*, then* $Q\cap H$ *is either* Q *or a line of* Q*.*

(14) *Suppose the point* x *of* $\mathbb{G}_4$ *is contained in a* $\mathbb{G}_3$*-hex* H *and an* $\mathbb{H}_3$*-hex* H'*. Then* $H\cap H'$ *is a* $W(2)$*-quad.*

Proof. Claims (1), (2), (3) (as well as parts of Claims (4), (5), (6), (7) and (8)) were proved in De Bruyn [5, Section 6.3] in a more general context, namely that of the near $2n$-gon $\mathbb{G}_n$, $n\geq 3$. Claims (4)–(14) readily follow from information on the local spaces which we will now provide.

Let x be an arbitrary point of $\mathbb{G}_4$. Then the local space of $\mathbb{G}_4$ at the point x is isomorphic to $\mathcal{L}_U$ where U is the set of all points of weight 1 or 2 of $\mathrm{PG}(3,4)$ with respect to a certain reference system $(\bar{e}_1,\bar{e}_2,\bar{e}_3,\bar{e}_4)$ of $V(4,4)$. A convex subspace F through x corresponds to a certain subspace of $\mathcal{L}_U$ and hence to a certain set X_F of points of $\mathrm{PG}(3,4)$. If F_1 and F_2 are two convex subspaces through x, then $F_1\subset F_2$ if and only if $X_{F_1}\subset X_{F_2}$. We discuss all the possibilities for the lines, quads and hexes.

(i) If F is a special line, then $X_F = \{\langle \bar{e}_i \rangle\}$ for some $i \in \{1, 2, 3, 4\}$.

(ii) If F is an ordinary line, then $X_F = \{\langle \bar{e}_i + \lambda \bar{e}_j \rangle\}$ for two distinct $i, j \in \{1, 2, 3, 4\}$ and a $\lambda \in \mathbb{F}_4^* := \mathbb{F}_4 \setminus \{0\}$.

(iii) If F is a $Q(5, 2)$-quad, then $X_F = \{\langle \bar{e}_j \rangle, \langle \bar{e}_i + \lambda \bar{e}_j \rangle \mid \lambda \in \mathbb{F}_4\}$ for two distinct $i, j \in \{1, 2, 3, 4\}$.

(iv) If F is a $W(2)$-quad, then $X_F = \{\langle \bar{e}_i + \lambda \bar{e}_j \rangle, \langle \bar{e}_i + \mu \bar{e}_k \rangle, \langle \lambda \bar{e}_j + \mu \bar{e}_k \rangle\}$ for three mutually distinct $i, j, k \in \{1, 2, 3, 4\}$ and some $\lambda, \mu \in \mathbb{F}_4^*$.

(v) If F is a grid-quad of type I, then $X_F = \{\langle \bar{e}_i \rangle, \langle \bar{e}_j + \lambda \bar{e}_k \rangle\}$ for three mutually distinct $i, j, k \in \{1, 2, 3, 4\}$ and some $\lambda \in \mathbb{F}_4^*$.

(vi) If F is a grid-quad of type II, then $X_F = \{\langle \bar{e}_i + \lambda \bar{e}_j \rangle, \langle \bar{e}_k + \mu \bar{e}_l \rangle\}$ for some $\lambda, \mu \in \mathbb{F}_4^*$ and some i, j, k, l satisfying $\{i, j, k, l\} = \{1, 2, 3, 4\}$.

(vii) If F is a $\mathbb{G}_3$-hex, then $X_F = \langle \bar{e}_i, \bar{e}_j, \bar{e}_k \rangle \cap U$ for three mutually distinct $i, j, k \in \{1, 2, 3, 4\}$.

(viii) If F is an $\mathbb{H}_3$-hex, then $X_F = \alpha \cap U$ where α is a plane of $\mathrm{PG}(3, 4)$ disjoint from $\{\langle \bar{e}_1 \rangle, \langle \bar{e}_2 \rangle, \langle \bar{e}_3 \rangle, \langle \bar{e}_4 \rangle\}$. So, $|X_F| = 6$ and X_F contains a unique point of each of the lines $\langle \bar{e}_i, \bar{e}_j \rangle$, $i, j \in \{1, 2, 3, 4\}$ with $i \neq j$.

(ix) If $F \cong Q(5, 2) \times \mathbb{L}_3$, then $X_F = \{\langle \bar{e}_i + \lambda \bar{e}_j \rangle\} \cup \{\langle \bar{e}_l \rangle, \langle \bar{e}_k + \mu \bar{e}_l \rangle \mid \mu \in \mathbb{F}_4\}$ for some $\lambda \in \mathbb{F}_4^*$ and some i, j, k, l satisfying $\{i, j, k, l\} = \{1, 2, 3, 4\}$.

(x) If $F \cong W(2) \times \mathbb{L}_3$, then $X_F = \{\langle \bar{e}_i + \lambda \bar{e}_j \rangle, \langle \bar{e}_i + \mu \bar{e}_k \rangle, \langle \lambda \bar{e}_j + \mu \bar{e}_k \rangle, \langle \bar{e}_l \rangle\}$ for some $\lambda, \mu \in \mathbb{F}_4^*$ and some i, j, k, l satisfying $\{i, j, k, l\} = \{1, 2, 3, 4\}$. ■

4 Structure of $\mathbb{G}_4$ with respect to an $\mathbb{H}_3$-hex

In this section, H denotes a given $\mathbb{H}_3$-hex of $\mathbb{G}_4$.

Lemma 4.1 *Let $x \in \Gamma_2(H)$ and Q a quad of H such that $\Gamma_2(x) \cap Q$ is an ovoid of Q. Then:*

(1) $\langle x, Q \rangle$ is a hex of $\mathbb{G}_4$;

(2) if Q is a $W(2)$-quad, then $\langle x, Q \rangle \cong \mathbb{G}_3$;

(3) if Q is a grid-quad, then Q is a grid-quad of type II and $\langle x, Q \rangle \cong \mathbb{H}_3$.

Proof. (1) Let x_1 and x_2 be two distinct points of $\Gamma_2(x) \cap Q$ and let x_3 be a common neighbour of x_1 and x_2. Then $x_3 \in Q \setminus \Gamma_2(x)$ has distance 3 from x and $\langle x, x_3 \rangle$ is a hex. Now, x_1 and x_2 are two points on a geodesic path from x_3 to x. Hence, $\langle x, x_1, x_2 \rangle \subseteq \langle x, x_3 \rangle$. On the other hand, since x_3 is a common neighbour of x_1 and x_2, we also have $\langle x, x_3 \rangle \subseteq \langle x, x_1, x_2 \rangle$. Hence, $\langle x, x_1, x_2 \rangle = \langle x, x_3 \rangle$. Since x_1 and x_2 are two points of Q at distance 2 from each other, $Q = \langle x_1, x_2 \rangle$. It follows that $\langle x, Q \rangle = \langle x, x_1, x_2 \rangle = \langle x, x_3 \rangle$ is a hex.

(2) Since $x \in \Gamma_2(Q)$, the $W(2)$-quad Q is not big in the hex $\langle x, Q\rangle$. Among the near hexagons which can occur as hex in $\mathbb{G}_4$, only $\mathbb{G}_3$ has non-big $W(2)$-quads (recall Lemma 3.1(3)). It follows that $\langle x, Q\rangle \cong \mathbb{G}_3$.

(3) The grid-quad Q is contained in the $\mathbb{H}_3$-hex H. Hence, by Lemma 3.1(11), Q is a grid-quad of type II. Since $x \in \Gamma_2(Q)$, the grid-quad Q of type II is not big in the hex $\langle x, Q\rangle$. Among the near hexagons which can occur as hex in $\mathbb{G}_4$, only $\mathbb{G}_3$ and $\mathbb{H}_3$ have non-big grid-quads. By Lemma 3.1(12), a $\mathbb{G}_3$-hex cannot contain grid-quads of type II. Hence, $\langle x, Q\rangle \cong \mathbb{H}_3$. ∎

Remark. If (x, Q) is a point-quad pair of a dense near hexagon such that $\mathrm{d}(x, Q) = 2$, then $\Gamma_2(x) \cap Q$ is an ovoid of Q since every line of Q contains a unique point nearest to (and hence at distance 2 from) x.

Proposition 4.2 *It holds that* $|H| = 105$*,* $|\Gamma_1(H)| = 3360$*,* $|\Gamma_2(H)| = 5040$ *and* $|\Gamma_i(H)| = 0$ *for every* $i \geq 3$*. If* $x \in \Gamma_2(H)$*, then there are two possibilities:*

(*a*) $\Gamma_2(x) \cap H$ *is an ovoid of a* $W(2)$*-quad* Q *of* H *and* $\langle x, Q\rangle \cong \mathbb{G}_3$*;*

(*b*) $\Gamma_2(x) \cap H$ *is an ovoid of a grid-quad of type II of* H *and* $\langle x, Q\rangle \cong \mathbb{H}_3$*.*

Proof. Suppose $y \in \Gamma_i(H)$ with $i \geq 3$. For every line L of H, we have $\mathrm{d}(y, L) \leq 3$ since L contains a unique point nearest to y. Hence $i = 3$ and $|\Gamma_3(y) \cap L| = 1$ for every line L of H. It follows that $\Gamma_3(y) \cap H$ is an ovoid of H. But this is impossible since H has no ovoids by [13, Lemma 5.5]. Hence, $|\Gamma_i(H)| = 0$ for every $i \geq 3$. Clearly, $|H| = 105$, $|\Gamma_1(H)| = |H| \cdot (22 - 6) \cdot 2 = 3360$ and $|\Gamma_2(H)| = 8505 - |H| - |\Gamma_1(H)| = 5040$.

Suppose $x \in \Gamma_2(H)$. Applying Proposition 1.1 to the classical valuation f of $\mathbb{G}_4$ with $O_f = \{x\}$, we find that the map $g : H \to \mathbb{N}; y \mapsto \mathrm{d}(x, y) - 2$ is a valuation of H. The valuation g is not classical since each of its values is at most 2. (A classical valuation of a dense near hexagon has maximal value equal to 3.) By Section 2, there are three possibilities:

(a) $O_g = \Gamma_2(x) \cap H$ is an ovoid in a $W(2)$-quad Q of H;

(b) $O_g = \Gamma_2(x) \cap H$ is an ovoid in a grid-quad Q of H;

(c) $O_g = \Gamma_2(x) \cap H$ is a set of 7 points and G_g is a Fano-plane.

If case (a) occurs, then $\langle x, Q\rangle \cong \mathbb{G}_3$ by Lemma 4.1(2). If case (b) occurs, then Q is a grid-quad of type II and $\langle x, Q\rangle \cong \mathbb{H}_3$ by Lemma 4.1(3).

We will now prove that case (c) cannot occur. Suppose the contrary. Let u denote an arbitrary point of O_g and let Q_1, Q_2 and Q_3 denote the three grid-quads of H through u. These grid-quads are special with respect to g by Lemma 2.3(iii). Hence, $\Gamma_2(x) \cap Q_i$ is an ovoid of Q_i for every $i \in \{1,2,3\}$. By Lemma 4.1(3), the grid-quads Q_1, Q_2 and Q_3 have type II and $\langle x, Q_1\rangle \cong \langle x, Q_2\rangle \cong \langle x, Q_3\rangle \cong \mathbb{H}_3$. In the near hexagon $\langle x, Q_1\rangle \cong \mathbb{H}_3$, the quad $\langle x, u\rangle$ is one of the three quads through x which meet Q_1. It follows that $\langle x, Q\rangle$ is a grid-quad. By Lemma 3.1(11), $\langle x, u\rangle$ is a grid-quad of type II. By Lemma 3.1(7), every line of $\langle x, u\rangle$ is an ordinary line. Let L be one of the two (ordinary) lines of $\langle x, u\rangle$ through u. By Lemma 3.1(8), L is contained in a unique $Q(5,2)$-quad Q. By Lemma 3.1(13), $Q \cap H$ is a line L'. Since Q_1, Q_2 and Q_3 determine a partition of the lines of H through u, we have $L' \subseteq Q_i$ for precisely one $i \in \{1,2,3\}$. Now, the $\mathbb{H}_3$-hex $\langle x, Q_i\rangle$ contains L' and $L \subseteq \langle x, u\rangle$. So, the $Q(5,2)$-quad $Q = \langle L, L'\rangle$ would be contained in the $\mathbb{H}_3$-hex $\langle x, Q_i\rangle$, clearly a contradiction, since $\mathbb{H}_3$ has only grid-quads and $W(2)$-quads. ■

Definition. A point x of $\Gamma_2(H)$ is said to be of type (a), respectively (b), if case (a), respectively case (b), of Proposition 4.2 occurs.

Lemma 4.3 *Let H' be a hex meeting H in a quad Q. Then* $\Gamma_2(H) \cap H' = \Gamma_2(Q) \cap H'$.

Proof. Suppose $x \in \Gamma_2(H) \cap H'$. Then x has distance at least 2 from Q. Since x and Q are contained in H', every point of Q has distance at most 3 from x. Hence, for every line L of Q, $\mathrm{d}(x, L) \leq 2$ since L contains a unique point nearest to x. It follows that $x \in \Gamma_2(Q) \cap H'$.

Conversely, suppose that $x \in \Gamma_2(Q) \cap H'$. Then $x \notin H$ since $H \cap H' = Q$. Suppose $x \in \Gamma_1(H)$. Then x is classical with respect to H and $\mathrm{d}(x, y) = 1 + \mathrm{d}(\pi_H(x), y)$ for every point $y \in H$. It follows that the point $\pi_H(x)$ is collinear with every point of the ovoid $\Gamma_2(x) \cap Q$ of Q. This implies that $\pi_H(x) \in Q$. But this is in contradiction with $\pi_H(x) \sim x \in \Gamma_2(Q)$. It follows that $x \in \Gamma_2(H) \cap H'$. ■

Lemma 4.4 *In* $\Gamma_2(H)$*, there are* 3360 *points of type* (*a*) *and* 1680 *points of type* (*b*).

Proof. In a given $\mathbb{G}_3$-hex, there are 120 points at distance 2 from any of its $W(2)$-quads. There are 28 $W(2)$-quads in H and each such quad is contained in a unique $\mathbb{G}_3$-hex by Lemma 3.1(9). Lemma 4.3 now implies that the total number of points of type (a) in $\Gamma_2(H)$ is equal to $28 \cdot 1 \cdot 120 = 3360$.

In a given $\mathbb{H}_3$-hex, there are 24 points at distance 2 from any of its grid-quads. Now, there are 35 grid-quads (of type II) in H and each of these grid-quads is contained in precisely 2 $\mathbb{H}_3$-hexes distinct from H (see Lemma

3.1(11)+(12)). Lemma 4.3 now implies that the number of points of type (b) in $\Gamma_2(H)$ is equal to $35 \cdot 2 \cdot 24 = 1680$.

(CHECK: The total number of points of $\Gamma_2(H)$ is indeed equal to $3360 + 1680 = 5040$ as shown in Proposition 4.2). ■

Lemma 4.5 (Chapter 7 of [5]) *Suppose one of the following cases occurs: (i) Q is a grid-quad of $\mathcal{S} \cong \mathbb{H}_3$; (ii) Q is a $W(2)$-quad of $\mathcal{S} \cong \mathbb{G}_3$. Let x be a point of $\mathcal{S}$ at distance 2 from Q. Then every line of $\mathcal{S}$ through x has a unique point in common with $\Gamma_1(Q)$.*

Let S denote the set of lines of $\mathbb{G}_4$ contained in $\Gamma_2(H)$.

Lemma 4.6 *Let x be a point of $\Gamma_2(H)$ and let Q be the quad $\langle \Gamma_2(x) \cap H \rangle$. Then the lines through x contained in S are precisely the lines through x not contained in the hex $\langle x, Q \rangle$. If x has type (a), then precisely 10 lines through x are contained in S. If x has type (b), then precisely 16 lines through x are contained in S.*

Proof. If x is a point of type (a), then $Q \cong W(2)$ and $\langle x, Q \rangle \cong \mathbb{G}_3$. If x is a point of type (b), then Q is a grid-quad and $\langle x, Q \rangle \cong \mathbb{H}_3$. By Lemmas 4.3 and 4.5, every line through x contained in $\langle x, Q \rangle$ contains a point of $\Gamma_1(H)$. Conversely, suppose that L is a line through x containing a point $y \in \Gamma_1(H)$. Then y is classical with respect to H and the point $\pi_H(y)$ lies at distance 2 from x. Hence, $\pi_H(y) \in Q$ and $L \subseteq \langle x, \pi_H(y) \rangle \subseteq \langle x, Q \rangle$.

So, the number of lines through x contained in S is equal to the number of lines through x not contained in the hex $\langle x, Q \rangle$. If x is a point of type (a), then x is contained in $22 - 12 = 10$ lines of S. If x is a point of type (b), then x is contained in $22 - 6 = 16$ lines of S. ■

From Lemmas 4.4 and 4.6, we readily obtain:

Corollary 4.7 $|S| = \frac{1}{3}[3360 \cdot 10 + 1680 \cdot 16] = 20160.$

Lemma 4.8 *Let $L = \{x_1, x_2, x_3\}$ be a line of S. For every $i \in \{1, 2, 3\}$, put $Q_i := \langle \Gamma_2(x_i) \cap H \rangle$ and $H_i := \langle x_i, Q_i \rangle$. Then H_1, H_2 and H_3 are mutually disjoint hexes.*

Proof. By symmetry, it suffices to show that $H_1 \cap H_2 = \emptyset$. Suppose to the contrary that u is a point of $H_1 \cap H_2$. Every point on a shortest path between $u \in H_1 \cap H_2$ and $x_1 \in H_1$ belongs to H_1. If $x_1 \notin H_2$, then since x_1 is classical with respect to H_2, the point $x_2 = \pi_{H_2}(x_1)$ lies on such a shortest path. Hence, $x_1 \in H_2$ or $x_2 \in H_1$. So, the line $x_1 x_2$ is contained in H_1 or H_2.

Lemma 4.5 then implies that L contains a point of $\Gamma_1(H)$. This contradicts the fact that $L \in S$. ■

Lemma 4.9 *Let $L = \{x_1, x_2, x_3\}$ be a line of S, put $Q_i = \langle \Gamma_2(x_i) \cap H \rangle$ and $H_i = \langle x_i, Q_i \rangle$. If x_1 is of type (a), then x_2 and x_3 have the same type and $\mathcal{R}_{H_1}(H_2) = H_3$.*

Proof. By Proposition 4.2, $Q_1 \cong W(2)$ and $H_1 \cong \mathbb{G}_3$. So, H_1 is big in $\mathbb{G}_4$. By Lemma 4.8, H_1 and H_2 are mutually disjoint. Let H_3' be the reflection of H_2 about H_1 (in the near octagon $\mathbb{G}_4$) and let Q_3' denote the reflection of Q_2 about Q_1 (in the near hexagon H). Then $Q_3' \cong Q_2$, $H_3' \cong H_2$ and $Q_3' \subset H_3$. Since x_2 is a point of H_2 at distance 2 from the quad Q_2 of H_2, $x_3 = \mathcal{R}_{H_1}(x_2)$ is a point of $H_3' = \mathcal{R}_{H_1}(H_2)$ at distance 2 from $Q_3' = \mathcal{R}_{H_1}(Q_2)$. So, $\Gamma_2(x_3) \cap Q_3'$ is an ovoid of Q_3'. This implies that $Q_3 = Q_3'$ and $H_3 = H_3'$. Since $H_3' \cong H_2$, x_3 is of the same type as x_2. ■

Lemma 4.10 *Every point x of type (a) of $\Gamma_2(H)$ is contained in precisely 6 lines of S which only contain points of type (a).*

Proof. Put $Q := \langle \Gamma_2(x) \cap H \rangle$.

Let $\{x, x_1, x_2\}$ be a line of S through x which only contains points of type (a) and let $Q_i = \langle \Gamma_2(x_i) \cap H \rangle$, $i \in \{1, 2\}$. Then by Lemmas 4.8 and 4.9, the $W(2)$-quads Q, Q_1 and Q_2 are mutually disjoint and Q_2 is the reflection of Q_1 about Q (in the near hexagon H).

Let Q' be a $W(2)$-quad of H disjoint from Q and let H' denote the unique $\mathbb{G}_3$-hex through Q' (recall Lemma 3.1(9)). We prove that $\langle x, Q \rangle \cap H' = \emptyset$. Suppose to the contrary that $\langle x, Q \rangle \cap H'$ contains a point u. If $u \in H$, then $u \in Q = \langle x, Q \rangle \cap H$ and $u \in Q' = H' \cap H$, a contradiction. If $u \in \Gamma_1(H)$, then $u \notin \Gamma_2(Q) \cup \Gamma_2(Q')$ by Lemma 4.3 and hence $\pi_H(u) \in Q \cap Q'$, a contradiction. If $u \in \Gamma_2(H)$, then $u \in \Gamma_2(Q) \cap \Gamma_2(Q')$ and hence $\Gamma_2(u) \cap H \subseteq Q \cap Q'$, again a contradiction. So, the big $\mathbb{G}_3$-hexes $\langle x, Q \rangle$ and H' are disjoint. Hence, the line $x\pi_{H'}(x)$ belongs to S by Lemma 4.6. Since x and $\pi_{H'}(x)$ are points of type (a), also the third point of $x\pi_{H'}(x)$ has type (a) by Lemma 4.9. So, the $W(2)$-quad Q' determines a line of S through x which only consists of points of type (a). If we denote by $Q'' \cong W(2)$ the reflection of Q' about Q (in H) and by H'' the unique $\mathbb{G}_3$-hex through Q'', then $H'' = \mathcal{R}_{H'}(\langle x, Q \rangle)$ and $x\pi_{H'}(x) = x\pi_{H''}(x)$. So, the $W(2)$-quads Q' and Q'' determine the same line of S through x.

Since there are 12 $W(2)$-quads in H disjoint with Q, it follows from the above discussion that there are $\frac{12}{2} = 6$ lines of S through x containing only points of type (a). ■

From Lemmas 4.4 and 4.10, we readily obtain:

Corollary 4.11 *There are* $\frac{3360 \cdot 6}{3} = 6720$ *lines of* S *containing precisely three points of type* (a)*.*

Lemma 4.12 *There are* 13440 *lines of* S *containing one point of type* (a) *and two points of type* (b)*.*

Proof. Let x be one of the 3360 points of type (a). By Lemmas 4.6, 4.9 and 4.10, x is contained in $10 - 6 = 4$ lines of S which contain a unique point of type (a). Hence, the required number is equal to $3360 \cdot 4 = 13440$. ■

By Corollary 4.7, Corollary 4.11 and Lemma 4.12, we obtain:

Corollary 4.13 *There are two types of lines in* S*:*

(1) *Lines of* S *only containing points of type* (a)*.*

(2) *Lines of* S *containing a unique point of type* (a) *and two points of type* (b)*.*

Recall that a *hyperplane* of a point-line geometry is a proper subspace meeting each line.

Proposition 4.14 *Let* X *denote the set of points of* $\mathbb{G}_4$ *consisting of the points of* H*, the points of* $\Gamma_1(H)$ *and the points of type* (a) *of* $\Gamma_2(H)$*. Then* X *is a hyperplane of* $\mathbb{G}_4$*.*

Proof. We need to prove that every line L containing a point x of type (b) of $\Gamma_2(H)$ intersects X in a unique point. Put $Q := \langle \Gamma_2(x) \cap H \rangle$. Then Q is a grid-quad of type II and $\langle x, Q \rangle \cong \mathbb{H}_3$.

If L is not contained in $\langle x, Q \rangle$, then $L \in S$ by Lemma 4.6. Corollary 4.13 then implies that $|L \cap X| = 1$.

If L is contained in $\langle x, Q \rangle$, then L contains a unique point y of $\Gamma_1(Q)$ by Lemma 4.5. Let $z \in \Gamma_2(Q)$ denote the third point on the line L. By Lemma 4.3 applied to the hex $H' = \langle x, Q \rangle$, $z \in \Gamma_2(H)$. Since $z \in \Gamma_2(Q)$ and Q are contained in the hex $\langle x, Q \rangle$, $\Gamma_2(z) \cap Q$ is an ovoid of Q. It follows that $\Gamma_2(z) \cap H = \Gamma_2(z) \cap Q$. Since Q is a grid, z is of point of type (b) and y is the unique point of X contained in L. ■

5 A new class of valuations of $\mathbb{G}_4$

Let H denote a hex of $\mathbb{G}_4$ isomorphic to $\mathbb{H}_3$ and let f denote a valuation of Fano-type of H. Recall that every point of $\Gamma_1(H)$ is classical with respect to H. Lemma 2.3(i)+(ii) allows us to define the following function $\overline{f}$ from the point-set of $\mathbb{G}_4$ to $\mathbb{N}$:

(i) If $x \in H$, then we define $\overline{f}(x) := f(x)$.

(ii) If $x \in \Gamma_1(H)$, then we define $\overline{f}(x) := 1 + f(\pi_H(x))$.

(iii) If x is a point of type (a) of $\Gamma_2(H)$, then $\overline{f}(x) := \mathrm{d}(x, x^*)$, where x^* is the unique point of O_f contained in the $W(2)$-quad $\langle \Gamma_2(x) \cap H \rangle$.

(iv) Let x be a point of type (b) of $\Gamma_2(H)$ such that $|O_f \cap Q| = 3$, where Q is the unique grid-quad of H containing $\Gamma_2(x) \cap H$. Then $\overline{f}(x) := 2$ if $\Gamma_2(x) \cap (O_f \cap Q) \neq \emptyset$ and $\overline{f}(x) := 1$ otherwise.

(v) Let x be a point of type (b) of $\Gamma_2(H)$ such that $|O_f \cap Q| = 0$ where Q is the unique grid-quad of H containing $\Gamma_2(x) \cap H$. Let X denote the ovoid of Q consisting of all points with f-value 1. We define $\overline{f}(x) := 3$ if $\Gamma_2(x) \cap X \neq \emptyset$ and $\overline{f}(x) := 2$ otherwise.

Proposition 5.1 *The map $\overline{f}$ is a valuation of $\mathbb{G}_4$.*

Proof. Recall that a function from the point-set of $\mathbb{G}_4$ to $\mathbb{N}$ is a valuation of $\mathbb{G}_4$ if and only if it satisfies properties (V1) and (V2). Clearly, $\overline{f}$ satisfies property (V1). It remains to show that $\overline{f}$ also satisfies property (V2). Let L be an arbitrary line of $\mathbb{G}_4$. We can distinguish 6 possibilities by corollary 4.13:

(1) L is contained in H. Then L satisfies property (V2) with respect to $\overline{f}$ since L satisfies property (V2) with respect to f.

(2) L intersects H in a unique point x_L. Then $\overline{f}(x) = f(x_L) + 1 = \overline{f}(x_L) + 1$ for every point x of $L \setminus \{x_L\}$. So, L satisfies property (V2).

(3) $L \subseteq \Gamma_1(H)$. Then $\pi_H(L) := \{\pi_H(x) \,|\, x \in L\}$ is a line of H parallel with L. For every point x of L, $\overline{f}(x) = f(\pi_H(x)) + 1$. Since $\pi_H(L)$ satisfies property (V2) with respect to f, L satisfies property (V2) with respect to $\overline{f}$.

(4) $L \cap \Gamma_1(H) \neq \emptyset$ and $L \cap \Gamma_2(H) \neq \emptyset$. Let x denote an arbitrary point of $L \cap \Gamma_2(H)$ and let Q denote the unique quad of H containing $\Gamma_2(x) \cap H$. Then $\langle x, Q \rangle$ is a hex containing L.

From the definition of $\overline{f}$, we see that there exists a constant $\epsilon \in \{-1, 0\}$ such that the map $u \mapsto \overline{f}(u) + \epsilon$ defines a valuation f' of $\langle x, Q \rangle$. If x is a point of type (a), then $\epsilon = 0$ and f' is a classical valuation of $\langle x, Q \rangle \cong \mathbb{G}_3$ by Lemma 2.3(i). If x is a point of type (b), then $\epsilon = 0$ if $|O_f \cap Q| = 3$ and $\epsilon = -1$ if $|O_f \cap Q| = 0$. Moreover, by Lemma 2.2, f' is a valuation of grid-type of $\langle x, Q \rangle \cong \mathbb{H}_3$.

By the previous paragraph, the line $L \subseteq \langle x, Q \rangle$ satisfies property (V2) with respect to $\overline{f}$.

(5) $L \subseteq \Gamma_2(H)$ and every point of L is of type (a). Put $L = \{x_1, x_2, x_3\}$ and let Q_i, $i \in \{1, 2, 3\}$, denote the unique $W(2)$-quad of H containing

$O_i = \Gamma_2(x_i) \cap H$. The set O_i is an ovoid of Q_i. Put $H_i := \langle x_i, Q_i \rangle$, $i \in \{1, 2, 3\}$. By Lemmas 4.8 and 4.9, H_1, H_2 and H_3 are three mutually disjoint $\mathbb{G}_3$-hexes, $\mathcal{R}_{H_1}(H_2) = H_3$ and $\mathcal{R}_{Q_1}(Q_2) = Q_3$ (reflection about the big $W(2)$-quad Q_1 in the $\mathbb{H}_3$-hex H). So, every line meeting Q_1 and Q_2 also meets Q_3. We have $\mathcal{R}_{H_1}(O_2) = \mathcal{R}_{H_1}(\Gamma_2(x_2) \cap Q_2) = \Gamma_2(x_3) \cap Q_3 = O_3$. In a similar way, one can prove that $O_3 = \mathcal{R}_{H_2}(O_1)$. It follows that $O_1 \cup O_2 \cup O_3$ is the union of 5 lines which meet Q_1, Q_2 and Q_3. Let u_i^*, $i \in \{1, 2, 3\}$, denote the unique point of Q_i with f-value 0 (recall Lemma 2.3(i)). Since every two points of O_f lie at distance 2 from each other, $\mathrm{d}(u_1^*, u_2^*) = 2$. Since u_1^* is classical with respect to Q_2, the unique point v of Q_2 collinear with u_1^* is collinear with u_2^*. Let w denote the point of the line $u_1^* v$ distinct from u_1^* and v. The quad $\langle u_1^*, u_2^* \rangle$ contains the line $u_1^* v$ and hence contains the point $w \in Q_3$. Since the local space of H at the point w is a Fano plane minus a point, the quads $\langle u_1^*, u_2^* \rangle$ and Q_3 meet in a line. Since $u_1^*, u_2^* \in O_f$, the quad $\langle u_1^*, u_2^* \rangle$ of H is special with respect to f. So, $\langle u_1^*, u_2^* \rangle$ is a grid and the line $\langle u_1^*, u_2^* \rangle \cap Q_3$ contains a unique point of O_f which necessarily coincides with u_3^*. The points u_1^*, $\pi_{Q_1}(u_2^*)$ and $\pi_{Q_1}(u_3^*)$ of Q_1 form a line of Q_1 which intersects O_1 in a unique point. It follows that $O_1 \cup O_2 \cup O_3$ has a unique point u^* in common with $\{u_1^*, u_2^*, u_3^*\}$. If $i \in \{1, 2, 3\}$ such that $u^* = u_i^*$, then $\overline{f}(x_i) = 2$ and $\overline{f}(x_j) = 3$ for all $j \in \{1, 2, 3\} \setminus \{i\}$. This proves that L satisfies property (V2).

(6) $L \subseteq \Gamma_2(H)$, L contains a unique point x_1 of type (a) and two points x_2 and x_3 of type (b). Let Q_1 denote the unique $W(2)$-quad of H containing all points of $\Gamma_2(x_1) \cap H$ and put $H_1 := \langle x_1, Q_1 \rangle$. Let G_i, $i \in \{2, 3\}$, denote the grid-quad of H containing all points of $\Gamma_2(x_i) \cap H$ and put $H_i := \langle x_i, G_i \rangle$. Then $H_1 \cong \mathbb{G}_3$ and $H_2 \cong H_3 \cong \mathbb{H}_3$. Moreover, by Lemmas 4.8 and 4.9, H_1, H_2 and H_3 are mutually disjoint, $\mathcal{R}_{H_1}(H_2) = H_3$ and $\mathcal{R}_{Q_1}(G_2) = G_3$. Put $G_1 := \pi_{Q_1}(G_2) = \pi_{Q_1}(G_3)$. We have $\mathcal{R}_{H_1}(\Gamma_2(x_2) \cap G_2) = \Gamma_2(x_3) \cap G_3$. Moreover, $\pi_{Q_1}(\Gamma_2(x_2) \cap G_2) = \pi_{Q_1}(\Gamma_2(x_3) \cap G_3) = \Gamma_2(x_1) \cap G_1$ since every line connecting a point of $\Gamma_2(x_2) \cap G_2 \subseteq \Gamma_3(x_1)$ and $\Gamma_2(x_3) \cap G_3 \subseteq \Gamma_3(x_1)$ contains a unique point nearest to x_1. We distinguish four possibilities (cf. Lemma 2.4):

(i) $|G_2 \cap O_f| = |G_3 \cap O_f| = 0$, the unique point x^* in $O_f \cap Q_1$ is contained in G_1 and $\mathrm{d}(x^*, x_1) = 2$. Then the unique line through x^* meeting G_2 and G_3 intersects G_2 and G_3 in points with f-value 1 belonging respectively to $\Gamma_2(x_2)$ and $\Gamma_2(x_3)$. It follows that $\overline{f}(x_1) = 2$ and $\overline{f}(x_2) = \overline{f}(x_3) = 3$. So, L satisfies property (V2).

(ii) $|G_2 \cap O_f| = |G_3 \cap O_f| = 0$, the unique point x^* in $O_f \cap Q_1$ is contained in G_1 and $\mathrm{d}(x^*, x_1) = 3$. Hence, the ovoid $\Gamma_2(x_1) \cap G_1$ of G_1 contains two points with f-value 1 and one point with f-value 2 (recall Lemma 2.3(i)). Since each of the three lines meeting $\Gamma_2(x_1) \cap G_1$, $\Gamma_2(x_2) \cap G_2$ and $\Gamma_2(x_3) \cap G_3$

contains a unique point with smallest f-value, there exists an $i \in \{2,3\}$ such that (a) the ovoid $\Gamma_2(x_i) \cap G_i$ contains two points with f-value 2 and 1 point with f-value 1, and (b) the ovoid $\Gamma_2(x_{5-i}) \cap G_{5-i}$ contains three points with f-value 2. It follows that $\overline{f}(x_1) = 3$, $\overline{f}(x_i) = 3$ and $\overline{f}(x_{5-i}) = 2$. So, L satisfies property (V2).

(iii) There exists an $i \in \{2,3\}$ such that $|G_i \cap O_f| = 3$ and $|G_{5-i} \cap O_f| = 0$. Moreover, we assume that $\mathrm{d}(x_1, x^*) = 2$, where x^* is the unique point in $O_f \cap Q_1$. (Recall $x^* \not\in G_1$.) Since $\{x^*\} \cup (\Gamma_2(x_1) \cap G_1)$ is contained in the ovoid $\Gamma_2(x_1) \cap Q_1$ of Q_1, no point of $\Gamma_2(x_1) \cap G_1$ is collinear with x^*. So, $\Gamma_2(x_1) \cap G_1$ only contains points with f-value 2 (recall Lemma 2.3(i)). Since every line meeting $\Gamma_2(x_1) \cap G_1$, $\Gamma_2(x_2) \cap G_2$ and $\Gamma_2(x_3) \cap G_3$ has a unique point with smallest f-value and G_i contains only points with f-value 0 or 1, $\Gamma_2(x_i) \cap G_i$ only contains points with f-value 1 and $\Gamma_2(x_{5-i}) \cap G_{5-i}$ only contains points with f-value 2. It follows that $\overline{f}(x_1) = 2$, $\overline{f}(x_i) = 1$ and $\overline{f}(x_{5-i}) = 2$. This proves that L satisfies property (V2) with respect to $\overline{f}$.

(iv) There exists an $i \in \{2,3\}$ such that $|G_i \cap O_f| = 3$ and $|G_{5-i} \cap O_f| = 0$. Moreover, we assume that $\mathrm{d}(x_1, x^*) = 3$ where x^* is the unique point in $O_f \cap Q_1$. (Recall $x^* \not\in G_1$.) Then $\Gamma_2(x_1) \cap G_1 \subseteq \Gamma_2(x_1) \cap Q_1$ contains at least one point with f-value 1 (collinear with x^*). The unique line through each such point meeting G_2 and G_3 contains a unique point with smallest f-value. Hence, $\Gamma_2(x_i) \cap G_i$ contains at least one point with f-value 0 and $\Gamma_2(x_{5-i}) \cap G_{5-i}$ contains at least one point with f-value 1 (recall that every point of G_i has f-value 0 or 1). It follows that $\overline{f}(x_1) = 3$, $\overline{f}(x_i) = 2$ and $\overline{f}(x_{5-i}) = 3$. This proves that L satisfies property (V2). ∎

The valuation $\overline{f}$ of $\mathbb{G}_4$ defined above is called a *valuation of Fano-type* of $\mathbb{G}_4$.

6 The classification of the valuations of $\mathbb{G}_4$

6.1 Some lemmas

During the classification of the valuations of $\mathbb{G}_4$, we will need the following three properties which hold for valuations of general near polygons:

Lemma 6.1 ([11]) *Let f be a valuation of a dense near $2n$-gon $\mathcal{S}$.*

(i) f is a classical valuation if and only if there exists a point with value n.

(ii) If $d(x, O_f) \leq 2$, then $f(x) = d(x, O_f)$.

(iii) No two distinct special quads intersect in a line.

Now, suppose that f is a valuation of $\mathbb{G}_4$.

Lemma 6.2 *If $x, y \in O_f$, then $d(x, y)$ is even.*

Proof. By Property (V2), $d(x, y) \neq 1$. Suppose $d(x, y) = 3$. Let H denote the unique hex through x and y. If f' denotes the valuation of H induced by f (recall Proposition 1.1), then $O_{f'}$ contains two points at distance 3 from each other. This is impossible since none of the near hexagons $\mathbb{G}_3$, $W(2) \times \mathbb{L}_3$, $Q(5, 2) \times \mathbb{L}_3$, $\mathbb{H}_3$ has such valuations. ∎

Lemma 6.3 *If there exists a $\mathbb{G}_3$-hex H such that $|H \cap O_f| = 15$, then $O_f = H \cap O_f$.*

Proof. Since $|H \cap O_f| = 15$, the valuation f' of $H \cong \mathbb{G}_3$ induced by f is non-classical. Suppose $x \in O_f \setminus H$. Since H is big in $\mathbb{G}_4$, x is classical with respect to H. The point $\pi_H(x)$ has value 1 and hence is contained in a unique quad Q of H which is special with respect to f' (recall Lemma 2.1(iii)). If y is a point of $Q \cap O_f$ at distance 2 from $\pi_H(x)$, then $d(x, y) = 3$, contradicting Lemma 6.2. ∎

Lemma 6.4 *If x and y are two different points of O_f, then $d(x, y) = 2$.*

Proof. Suppose the contrary. Then $d(x, y) = 4$ by Lemma 6.2. Let H denote an arbitrary $\mathbb{G}_3$-hex through x. Since $y \in O_f \setminus H$, the valuation induced in H is classical by Lemma 6.3 (recall that $|O_g| = 15$ for every non-classical valuation g of $\mathbb{G}_3$). Hence, $f(\pi_H(y)) = d(x, \pi_H(y)) = 3$. On the other hand, since $d(\pi_H(y), y) = 1$ and $f(y) = 0$, it holds that $f(\pi_H(y)) = 1$, a contradiction. ∎

Lemma 6.5 *One of the following cases occurs:*

(A) $|O_f| = 1$;

(B) There exists a unique $\mathbb{G}_3$-hex H such that $O_f \subseteq H$ and $|H \cap O_f| = 15$;

(C) $|O_f| \geq 2$ and every special quad is a grid-quad of type II.

Proof. Suppose $|O_f| \geq 2$ and let x_1 and x_2 denote two distinct points of O_f. Then $d(x_1, x_2) = 2$ by Lemma 6.4. Let Q denote the unique special quad through x_1 and x_2. Then Q is not isomorphic to $Q(5, 2)$ since this generalized quadrangle has no ovoids (Payne and Thas [17]). If Q is a $W(2)$-quad or a grid-quad of type I, then Q is contained in a unique $\mathbb{G}_3$-hex H, see Lemma 3.1(9)+(11). Since $Q \cap O_f \subseteq H \cap O_f$, the valuation of H induced by f is non-classical and hence $|H \cap O_f| = 15$ by Lemma 2.1. By Lemma 6.3, it then follows that $O_f = H \cap O_f$. The lemma is now clear. ∎

6.2 Treatment of case (A) of Lemma 6.5

Proposition 6.6 *If f is a valuation of $\mathbb{G}_4$ such that $|O_f| = 1$, then f is a classical valuation.*

Proof. Put $O_f = \{x\}$ and let H denote an arbitrary $\mathbb{G}_3$-hex through x. By Lemma 3.1(5)+(6), there exists a unique special line L through x not contained in H. Let x' denote an arbitrary point of $L \setminus \{x\}$. By Lemmas 3.1(5)+(6), there exists a unique $\mathbb{G}_3$-hex H' through x' not containing the special line L. We will show that the valuation f' of H' induced by f is classical. Suppose the contrary and let Q denote a grid-quad of H' through $x' \in O_{f'}$ which is special with respect to f'. (Such a grid-quad exists by Lemma 2.1(v).) By Lemma 3.1(12), Q is a grid-quad of type I. By Lemma 3.1(11), Q is contained in a unique $Q(5,2) \times \mathbb{L}_3$-hex H''. By Lemma 3.1(4)+(5), H'' has two special lines though x'. One of these lines is contained in the grid-quad Q of type I. The other special line L' cannot be contained in H' since otherwise $H'' = \langle Q, L\rangle = H'$, which is clearly absurd. Since there is only 1 special line through x' not contained in the $\mathbb{G}_3$-hex H' (Lemmas 3.1(5)+(6)), we must have $L' = L$. Now, the valuation of $H'' = \langle L, Q\rangle$ induced by f contains a unique point with value 0 (namely x) and a point with value 1 at distance 3 from it (which is contained in $\Gamma_2(x') \cap Q$). But $Q(5,2) \times \mathbb{L}_3$ does not have valuations of this type. So, we have a contradiction. It follows that the valuation induced in H' is classical. This implies that every point of H' at distance 3 from x' has value 4. By Lemma 6.1(i), it then follows that f is classical. ■

6.3 Treatment of case (B) of Lemma 6.5

Proposition 6.7 *If f is a valuation of $\mathbb{G}_4$ such that O_f is a set of 15 points in a $\mathbb{G}_3$-hex H of $\mathbb{G}_4$, then f is the extension of a non-classical valuation of $\mathbb{G}_3$.*

Proof. Let f' denote the valuation of H induced by f. Then f' is a non-classical valuation of H with $O_{f'} = O_f$. Hence, $f(x) = f'(x)$ for every point $x \in H$. Now, let x be an arbitrary point of $\mathbb{G}_4$ not contained in H. Recall that the $\mathbb{G}_3$-hex H is big in $\mathbb{H}_4$. So, x is collinear with the point $\pi_H(x)$ of H. Let Q denote an arbitrary $Q(5,2)$-quad of H through $\pi_H(x)$. Among the near hexagons which can occur as hex in $\mathbb{G}_4$, only $\mathbb{G}_3$ and $Q(5,2) \times \mathbb{L}_3$ have $Q(5,2)$-quads. It follows that the hex $\langle x, Q\rangle = \langle x\pi_H(x), Q\rangle$ is isomorphic to $\mathbb{G}_3$ or $Q(5,2) \times \mathbb{L}_3$. The hex $\langle x, Q\rangle$ contains a unique point of O_f, namely the unique point of O_f in Q (recall Lemma 2.1(iv)). Now, all valuations of the near hexagons $\mathbb{G}_3$ and $Q(5,2) \times \mathbb{L}_3$ which contain a unique point with value

0 are classical. In particular, the valuation induced in $\langle x, Q\rangle$ by f is classical. Hence, $f(x) = \mathrm{d}(x, O_f \cap Q) = 1 + \mathrm{d}(\pi_H(x), O_f \cap Q) = 1 + f'(\pi_H(x))$, where the latter equality follows from Lemma 2.1(iv). This proves that f is the extension of f'. ■

6.4 Treatment of case (C) of Lemma 6.5

In this subsection, we suppose that f is a valuation of $\mathbb{G}_4$ such that $|O_f| \geq 2$ and such that every special quad is a grid-quad of type II. By Lemma 6.4, every two distinct points of O_f are contained in a unique special quad. Since a special grid-quad contains three points of O_f, we have $|O_f| \geq 3$.

Lemma 6.8 *It holds that* $|O_f| > 3$.

Proof. Suppose to the contrary that $|O_f| = 3$. Let Q denote the unique special grid-quad of type II and put $\{x_1, x_2, x_3\} = Q \cap O_f$. By Lemma 3.1(12), there exists a $Q(5,2) \times \mathbb{L}_3$-hex F through Q. This hex contains precisely 2 special lines through x_1 by Lemmas 3.1(4)+(5). So, F has an ordinary line L through x_1 not contained in Q. Let $y \in L \setminus \{x_1\}$. By Lemmas 3.1(6)+(8), there exists a $\mathbb{G}_3$-hex H' through y not containing the line L. Let f' denote the valuation of H' induced by f. Since $\pi_{H'}(\{x_1, x_2, x_3\}) \subseteq O_{f'}$, f' is non-classical. By Lemma 2.1(v), there exists a $W(2)$-quad Q' of H' through y which is special with respect to f'. Now, by Lemma 3.1(9), Q' is contained in 1 $\mathbb{G}_3$-hex (namely H'), 1 $W(2) \times \mathbb{L}_3$-hex (namely the hex $\langle Q', M\rangle$ where M is the unique special line through y not contained in H') and three $\mathbb{H}_3$-hexes. Hence, $\langle L, Q'\rangle$ is isomorphic to $\mathbb{H}_3$. This implies that $\langle L, Q'\rangle$ does not contain Q since $\langle L, Q\rangle \cong Q(5,2) \times \mathbb{L}_3$. It follows that the valuation of $\langle L, Q'\rangle \cong \mathbb{H}_3$ induced by f contains a unique point with value 0 (namely x_1) and a point with value 1 at distance 3 from it (which is contained in $\Gamma_2(y) \cap Q'$). This is impossible, since $\mathbb{H}_3$ does not have such valuations. ■

Lemma 6.9 *O_f is a set of 7 points in an $\mathbb{H}_3$-hex of $\mathbb{G}_4$.*

Proof. Let x denote an arbitrary point of O_f. By Lemmas 6.4 and 6.8, there are two distinct special grid-quads G_1 and G_2 (of type II) through x. By Lemma 6.1(iii), $G_1 \cap G_2 = \{x\}$. Let u_1 be an arbitrary point of $(O_f \cap G_1) \setminus \{x\}$. By Lemma 6.4, u_1 has distance 2 from every point of $O_f \cap G_2$. If $\mathrm{d}(u_1, G_2) = 1$, then u_1 is classical with respect to G_2 and all points of $O_f \cap G_2$ would be collinear with $\pi_{G_2}(u_1)$, clearly a contradiction. Hence, $\mathrm{d}(u_1, G_2) = 2$. Since every line of G_2 contains a unique point nearest to u_1, we have $G_2 \setminus O_f \subseteq \Gamma_3(u_1)$. Now, let u_2 be an arbitrary point of $G_2 \setminus O_f$. Then $\langle u_1, u_2\rangle$ is a hex. Since $O_f \cap G_2 \subseteq \Gamma_2(u_1)$, there are two distinct points

v_1 and v_2 of $O_f \cap G_2$ collinear with u_2 which are on a geodesic path from u_2 to u_1. Hence, $G_2 = \langle v_1, v_2 \rangle \subseteq \langle u_1, u_2 \rangle$. In particular, $x \in \langle u_1, u_2 \rangle$. Since $x, u_1 \in \langle u_1, u_2 \rangle$, we have $G_1 = \langle x, u_1 \rangle \subseteq \langle u_1, u_2 \rangle$. So, $H := \langle G_1, G_2 \rangle$ is a hex. By Lemma 3.1(12), H is isomorphic to either $\mathbb{H}_3$ or $Q(5,2) \times \mathbb{L}_3$. (Recall that G_1 and G_2 are grids of type II). Now, in the near hexagon $Q(5,2) \times \mathbb{L}_3$ any two distinct grid-quads through the same point meet each other in a line. Since $G_1 \cap G_2 = \{x\}$, we necessarily have $H \cong \mathbb{H}_3$. Since $|G_1 \cap O_f| = |G_2 \cap O_f| = 3$, the valuation f_H of H induced by f must be of Fano-type. Hence, $|O_f \cap H| = 7$.

We show that $\Gamma_1(H) \cap O_f = \emptyset$. Suppose to the contrary that y is a point of $\Gamma_1(H) \cap O_f$. Then y is classical with respect to H. Since $f(y) = 0$, $f(\pi_H(y)) = 1$ and hence by Lemma 2.3(iv) $\pi_H(y)$ is contained in a unique quad Q of H which is special with respect to f_H. Any point of $Q \cap O_{f_H} = Q \cap O_f$ at distance 2 from $\pi_H(y)$ lies at distance 3 from y, contradicting Lemma 6.4. Hence, $\Gamma_1(H) \cap O_f = \emptyset$.

We show that $f(y) \geq 2$ for every point y of type (a) of $\Gamma_2(H)$. Let Q denote the $W(2)$-quad of H containing all points of $\Gamma_2(y) \cap H$ and let H' be the $\mathbb{G}_3$-hex $\langle y, Q \rangle$. Let u denote the unique point of $O_f \cap Q$ (recall Lemma 2.3(i)) and let L be a line of Q through u. If the valuation $f_{H'}$ of H' induced by f is not classical, then by Lemma 2.1(v) there exists a quad of H' through L which is special with respect to $f_{H'}$. This implies that there is a point of $O_{f_{H'}} \subseteq O_f$ contained in $\Gamma_1(H)$, a contradiction. Hence, $f_{H'}$ is a classical valuation of H'. It follows that $f(y) = f_{H'}(y) = \mathrm{d}(y, u) \geq 2$.

We show that $f(y) \geq 1$ for every point y of type (b) of $\Gamma_2(H)$. By Lemma 4.6 there exists a line $L \in S$ through y and this line contains a unique point u of type (a) by Corollary 4.13. Since $f(u) \geq 2$, we have $f(y) \geq 1$. ■

Let H denote the unique $\mathbb{H}_3$-hex of $\mathbb{G}_4$ containing all points of O_f and let f' be the valuation of H induced by f. By Lemma 6.9, f' is a valuation of Fano-type of H.

Proposition 6.10 *The valuation f is obtained from f' in the way as described in Section 5.*

Proof. Let x denote an arbitrary point of $\mathbb{G}_4$.

If $x \in H$, then $\mathrm{d}(x, O_f) \leq 2$ and hence $f(x) = \mathrm{d}(x, O_f) = \mathrm{d}(x, O_{f'}) = f'(x)$ by Lemma 6.1(ii).

If $x \in \Gamma_1(H)$ such that $\mathrm{d}(\pi_H(x), O_f) \leq 1$, then $\mathrm{d}(x, O_f) \leq 2$ and hence $f(x) = \mathrm{d}(x, O_f) = 1 + \mathrm{d}(\pi_H(x), O_f) = 1 + f'(\pi_H(x))$ by Lemma 6.1(ii).

Let $x \in \Gamma_1(H)$ such that $\mathrm{d}(\pi_H(x), O_f) = 2$, or equivalently, such that $f'(\pi_H(x)) = 2$. Let H' denote an arbitrary $\mathbb{G}_3$-hex through the line $x\pi_H(x)$.

Then $H' \cap H$ is a $W(2)$-quad Q by Lemma 3.1(14). The hex H' contains a unique point y with f-value 0, namely the unique point of O_f in Q (recall Lemma 2.3(i)). Hence, the valuation induced in H' is classical. Since $\mathrm{d}(\pi_H(x), O_f) = 2$, we have $\mathrm{d}(\pi_H(x), y) = 2$. It follows that $f(x) = \mathrm{d}(x, y) = 1 + \mathrm{d}(\pi_H(x), y) = 3 = 1 + f'(\pi_H(x))$.

Let x denote a point of type (a) of $\Gamma_2(H)$. Let Q denote the $W(2)$-quad of H containing all points of $\Gamma_2(x) \cap H$ and let x^* denote the unique point of O_f in Q. The hex $\langle x, Q\rangle$ is isomorphic to $\mathbb{G}_3$ and contains a unique point of O_f, namely x^*. Hence, the valuation induced in $\langle x, Q\rangle$ is classical. It follows that $f(x) = \mathrm{d}(x, x^*)$.

Let x denote a point of type (b) of $\Gamma_2(H)$ such that $|O_f \cap Q| = 3$, where Q is the unique grid-quad of H containing $\Gamma_2(x) \cap H$. The hex $\langle x, Q\rangle$ is isomorphic to $\mathbb{H}_3$ and the valuation of $\langle x, Q\rangle$ induced by f is of grid-type. It follows from Lemma 2.2 that $f(x) = 2$ if $\Gamma_2(x) \cap O_f \cap Q \neq \emptyset$ and $f(x) = 1$ otherwise.

Let x denote a point of type (b) of $\Gamma_2(H)$ such that $|O_f \cap Q| = 0$, where Q is the unique grid-quad of H containing $\Gamma_2(x) \cap H$. By Lemma 2.3(ii), the points with f-value 1 determine an ovoid of Q. So, the grid-quad Q is special with respect to the valuation f' of $\langle x, Q\rangle \cong \mathbb{H}_3$ induced by f. This implies that the valuation f' is either of grid-type or of Fano-type. We will show that the latter possibility cannot occur.

Suppose that f' is a valuation of Fano-type. Let u denote one of the three points of Q with f-value 1. By Lemma 2.3(iv), there exists a point $v \notin Q$ of O_f collinear with u. Let $G \neq Q$ denote a grid-quad of $\langle x, Q\rangle$ through u (which is special with respect to f'). Then $G \cap Q = \{u\}$. Let w be a point of $G \cap \Gamma_2(u)$. If $w \in \Gamma_1(Q)$, then w is classical with respect to Q, $\pi_Q(w)$ would be a common neighbour of u and w, and the quad $G = \langle u, w\rangle$ would contain the line $u\pi_Q(w)$ of Q, a contradiction. So, $w \in \Gamma_2(Q)$. By Lemma 4.3 applied to the hexes $\langle x, Q\rangle$ and H, we see that $w \in \Gamma_2(H)$. So, there exists a unique hex through w meeting H in a quad and this hex coincides with $\langle x, Q\rangle$. This implies that the hex $\langle vu, G\rangle \neq \langle x, Q\rangle$ intersects H in the line uv. It follows that the valuation induced in $\langle vu, G\rangle$ contains a unique point with value 0 (namely v) and a point with value 1 at distance 3 from it (which is contained in $\Gamma_2(u) \cap G$). Among the near hexagons which can occur as hex in $\mathbb{G}_4$, only $W(2) \times \mathbb{L}_3$ has such valuations. So, $\langle vu, G\rangle \cong W(2) \times \mathbb{L}_3$ and the valuation induced in $\langle vu, G\rangle$ is semi-classical. But in a $W(2) \times \mathbb{L}_3$-hex, every grid-quad is of type I, while the grid-quad G has type II since it is contained in the $\mathbb{H}_3$-hex $\langle x, Q\rangle$ (recall Lemma 3.1(11)+(12)). So, we have a contradiction and the valuation f' must be of grid-type.

By Lemma 2.2 it now follows $f(x) = 3$ if $\Gamma_2(x) \cap Q$ has a point with f'-value 1 and $f(x) = 2$ otherwise.

This proves the proposition. ■

6.5 A lemma

Recall that by Section 1.1, the near polygon $\mathbb{G}_n$ can be isometrically embedded into the dual polar space $DH(2n-1, 4)$.

Lemma 6.11 *Let F be a hex of $\mathbb{G}_4$ and let f be a valuation of F. Suppose that one of the following cases occurs: (i) $F \cong \mathbb{H}_3$ and f is a valuation of Fano-type of F; (ii) $F \cong \mathbb{G}_3$ and f is a non-classical valuation of F. Suppose also that $\mathbb{G}_4$ is isometrically embedded into the dual polar space $DH(7, 4)$. Then there exists a unique point $x \in DH(7, 4) \setminus \mathbb{G}_4$ such that $O_f \subseteq \Gamma_1(x)$.*

Proof. For every convex subspace A of $\mathbb{G}_4$, there exists a unique convex subspace $\overline{A}$ of $DH(7, 4)$ containing A and having the same diameter as A. If A has diameter δ and if x_1 and x_2 are two points of A at distance δ from each other, then $\overline{A}$ is the unique convex subspace of $DH(7, 4)$ containing x_1 and x_2.

Let Q be a quad of F which is special with respect to f. We moreover assume that Q is a $W(2)$-quad if we are in case (ii) of the lemma. Put $Q \cap O_f = \{x_1, x_2, \dots, x_k\}$, where $k = 3$ (case (i)) or $k = 5$ (case (ii)). Let y be an arbitrary point of $O_f \setminus Q$. Then $\mathrm{d}(y, x_i) = 2$ for every $i \in \{1, \dots, k\}$. If $\mathrm{d}(y, Q) = 1$, then y is classical with respect to Q and all points of the ovoid $Q \cap O_f = \{x_1, \dots, x_k\}$ of Q would be collinear with $\pi_Q(y)$, clearly a contradiction. Hence, $\mathrm{d}(y, Q) = 2$. Since every point of $\overline{Q} \setminus Q$ is collinear with some point of Q, we have $y \notin \overline{Q}$. Since the quad $\overline{Q}$ of $DH(7, 4)$ is big in the hex $\overline{F}$ of $DH(7, 4)$, this implies that $\mathrm{d}(y, \overline{Q}) = 1$. Since $\mathrm{d}(y, x_i) = 2$ and y is classical with respect to $\overline{Q}$, we have $\mathrm{d}(\pi_{\overline{Q}}(y), x_i) = 1$ for every $i \in \{1, \dots, k\}$. For every $i \in \{1, \dots, k\}$, let Q_i denote the unique quad of $\mathbb{G}_4$ through y and x_i. Since $y, x_i \in Q_i \cap O_f$, Q_i is special with respect to f. So, Q_i is either a (3×3)-grid or a $W(2)$-quad and there exists a unique ovoid O_i of Q_i containing y and x_i. Now, the k quads $Q_1, \dots, Q_k$ are all the quads through y which are special with respect to f. Since any two distinct points of O_f lie at distance 2 from each other, we necessarily have $O_f = O_1 \cup O_2 \cup \cdots \cup O_k$.

We prove that $\pi_{\overline{Q}}(y) \notin \mathbb{G}_4$. Suppose to the contrary that $\pi_{\overline{Q}}(y) \in \mathbb{G}_4$. Since $\pi_{\overline{Q}}(y)$ is collinear with the points $x_1, \dots, x_k$, we would then have that $\pi_{\overline{Q}}(y) \in Q$. This is impossible since $\mathrm{d}(y, Q) = 2$. Hence, $\pi_{\overline{Q}}(y) \notin \mathbb{G}_4$.

Since $\pi_{\overline{Q}}(y)$ is collinear with the points y and x_i, $i \in \{1, \ldots, k\}$, $\pi_{\overline{Q}}(y)$ is contained in $\overline{Q_i}$. So, $\Gamma_1(\pi_{\overline{Q}}(y)) \cap Q_i$ is an ovoid of Q_i containing y and x_i. It follows that $\Gamma_1(\pi_{\overline{Q}}(y)) \cap Q_i = O_i$. Hence, $O_f = O_1 \cup O_2 \cup \cdots \cup O_k \subseteq \Gamma_1(\pi_{\overline{Q}}(y))$.

Conversely, suppose z is a point of $DH(7,4) \setminus \mathbb{G}_4$ such that $O_f \subseteq \Gamma_1(z)$. Since z is collinear with the points $x_1, \ldots, x_k$, we have $z \in \overline{Q}$. Since z is collinear with y. We necessarily have $z = \pi_{\overline{Q}}(y)$. ∎

6.6 The valuations of $\mathbb{G}_4$ are induced by valuations of $DH(7,4)$

Let the near octagon $\mathbb{G}_4$ be isometrically embedded in $DH(7,4)$. For every point x of $DH(7,4)$, the classical valuation g_x of $DH(7,4)$ with $O_{g_x} = \{x\}$ induces a valuation f_x of $\mathbb{G}_4$. It holds that $\max\{f_x(u) \mid u \in \mathbb{G}_4\} = 4 - \mathrm{d}(x, \mathbb{G}_4)$ in view of the following result which holds for general dense near polygons.

Lemma 6.12 (Proposition 2.2 of [14]) *Let $\mathcal{S}$ be a dense near $2n$-gon and let $F = (\mathcal{P}', \mathcal{L}', \mathrm{I}')$ be a dense near $2n$-gon which is fully and isometrically embedded in $\mathcal{S}$. Let x be a point of $\mathcal{S}$ and let f_x denote the valuation of F induced by the classical valuation g_x of $\mathcal{S}$ with $O_{g_x} = \{x\}$, then $d(x, F) = n - M$, where M is the maximal value attained by f_x.*

If $x \in \mathbb{G}_4$, then f_x is a classical valuation of $\mathbb{G}_4$ and $O_{f_x} = \{x\}$. If $x \notin \mathbb{G}_4$, then f_x is not classical and hence is either the extension of a non-classical valuation of a $\mathbb{G}_3$-hex or is a valuation of Fano-type.

Proposition 6.13 *Let f be a valuation of $\mathbb{G}_4$. Then there exists a unique point x of $DH(7,4)$ such that $f = f_x$.*

Proof. Obviously, the proposition holds if f is classical. The required point x is then the unique point contained in O_f.

Suppose now that f is non-classical. By the classification of the valuations of $\mathbb{G}_4$, we then know that $F := \langle O_f \rangle$ is either an $\mathbb{H}_3$-hex or a $\mathbb{G}_3$-hex of $\mathbb{G}_4$. Moreover, if f' denotes the valuation of F induced by f, then $O_{f'} = O_f$, f' is a valuation of Fano-type of F if $F \cong \mathbb{H}_3$ and f' is a non-classical valuation of F if $F \cong \mathbb{G}_3$. By Lemma 6.11, there exists a unique point $x^* \in DH(7,4) \setminus \mathbb{G}_4$ such that $O_{f'} \subseteq \Gamma_1(x^*)$. Then $O_f \subseteq O_{f_{x^*}}$. Hence, $O_f = O_{f_{x^*}}$ and $f = f_{x^*}$ by the classification of the valuations of $\mathbb{G}_4$.

Conversely, suppose that $f = f_x$ for some point x of $DH(7,4)$. Since f is non-classical, its maximal value is equal to 3. Lemma 6.12 then implies that $\mathrm{d}(x, \mathbb{G}_4) = 1$. We have $O_f = \Gamma_1(x) \cap \mathbb{G}_4$. Since $O_f \subseteq \Gamma_1(x)$, Lemma 6.11 implies that $x = x^*$. ∎

By Proposition 6.13, the number of valuations of $\mathbb{G}_4$ is equal to the number of points of $DH(7,4)$. The number of classical valuations of $\mathbb{G}_4$ is equal to the number of points of $\mathbb{G}_4$, i.e., equal to 8505. The number of valuations of $\mathbb{G}_4$ which are extensions of non-classical valuations in $\mathbb{G}_3$-hexes is equal to (# $\mathbb{G}_3$-hexes) $\times$ (# non-classical valuations in a $\mathbb{G}_3$-hex) $= 84 \cdot 486 = 40824$. The number of valuations of Fano-type of $\mathbb{G}_4$ is equal to (# $\mathbb{H}_3$-hexes) $\times$ (# valuations of Fano-type in an $\mathbb{H}_3$-hex) $= 2178 \cdot 30 = 65610$. The number 8505+40824+65610=114939 is indeed equal to the total number of points of $DH(7,4)$.

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