L^p-L^q BOUNDEDNESS OF FOURIER MULTIPLIERS ON FUNDAMENTAL DOMAINS OF LATTICES IN $\mathbb{R}^d$

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ABSTRACT. In this paper we study the L^p-L^q boundedness of Fourier multipliers on the fundamental domain of a lattice in $\mathbb{R}^d$ for $1 < p, q < \infty$ under the classical Hörmander condition. First, we introduce Fourier analysis on lattices and have a look at possible generalisations. We then prove the Hausdorff-Young inequality, Paley's inequality and the Hausdorff-Young-Paley inequality in the context of lattices. This amounts to a quantitative version of the L^p-L^q boundedness of Fourier multipliers. We will show that this delivers also some L^p-estimates by using some standard Lebesgue space embeddings, which come from the finite measure of the fundamental domain. Moreover, the Paley inequality allows us to prove the Hardy-Littlewood inequality. As an application we treat some Sobolev embedding results, which also indicate the sharpness of our main inequalities.

CONTENTS

1. INTRODUCTION

The main goal of this paper is to prove the L^p-L^q boundedness of Fourier multipliers on a fundamental domain Ω of a lattice L in $\mathbb{R}^d$. The Fourier analysis on lattices is very similar to the toroidal case, which corresponds to the theory of the usual Fourier series. In fact, lattices and their fundamental domains can be viewed as linear deformations of the Euclidean lattice $\mathbb{Z}^d$ and the torus $\mathbb{T}^d$ (which can be

viewed as a fundamental domain of $\mathbb{Z}^d$), respectively. However, we should be careful with this interpretation, since we identify $\mathbb{T}^d$ here with $[0,1)^d$. If we consider the usual embedding of $\mathbb{T}^d$ in $\mathbb{R}^{d+1}$, then we should rather think of Ω as a different parametrisation domain of the embedded torus given by a linear change of variables. Since we start with the basics, this paper may be suitable for anyone new in this field, especially because this paper is essentially self-contained. It may also be interesting to consider this theory from the viewpoint of the more general theory of pseudo-differential operators and Fourier analysis on groups.

This may be the first text on this particular topic. However, pseudo-differential operators on the lattice $\mathbb{Z}^d$ and on the torus $\mathbb{T}^d$ have already been studied in [6] and [24, Chapter 4], respectively. There are a lot of results available on the L^p-L^q boundedness of Fourier multipliers. A fundamental article on Fourier multipliers on the Euclidean spaces is [15] by Lars Hörmander. The results of this article are extended to Fourier multipliers on SU(2) in [1], on compact homogeneous manifolds in [2], on Riemannian symmetric spaces of the noncompact type in [4], on smooth manifolds in [7], on compact Lie groups in [25], on locally compact groups in [3], on compact hypergroups in [17], to Fourier multipliers associated with the anharmonic oscillator in [8, 9] and to Fourier multipliers associated with a generalised (k, a)-Fourier transform in [16], just to mention a few of them.

1.1. **Organisation of this paper.** In Section 2 we explore Fourier analysis on a lattice L in $\mathbb{R}^d$. Our first task is to ensure that we have some form of Fourier analysis on a fundamental domain Ω of L. This will be the content of Theorem 2.1 of Fuglede, which roughly relates Fourier analysis on a subset $\Omega \subseteq \mathbb{R}^d$ with tilings of $\mathbb{R}^d$ with the lattice L. The notion of the dual lattice $L^\perp$ shows up here, as this occurs in the explicit description of the orthonormal basis of $L^2(\Omega)$ which enables Fourier analysis on Ω. Next, we introduce some function spaces, and we construct the Fourier transform $\mathcal{F}_\Omega$ of a periodic smooth function $f \in C^\infty_{\mathrm{per}}(\Omega)$ and look at some properties of it. At the end of this section we invite the reader to extend our theory to the case of spectral sets and to lattices living in proper subspaces of $\mathbb{R}^d$, for which lattices of type A_d can be a guiding example.

In Section 3 we prove some important inequalities that will help to prove the L^p-L^q boundedness of Fourier multipliers on Ω. A general result of Hilbert space theory yields Plancherel's identity (3.1) in our context of lattices. We obtain the Hausdorff-Young inequality for $\mathcal{F}_\Omega$ by interpolating Plancherel's formula and the estimate $\|\widehat{f}\|_{\ell^\infty(L^\perp)} \le \|f\|_{L^1(\Omega)}$, where we denote $\mathcal{F}_\Omega f(\kappa)$ by $\widehat{f}(\kappa)$.

Subsections 3.2 and 3.4 are devoted to a generalisation of some classical results in [15] by Hörmander. The first result that we mention is Paley's inequality.

Theorem 1.1 ([15, Theorem 1.10]). *Let $\varphi > 0$ be a measurable function on $\mathbb{R}^d$ such that*

$$m\{\xi \in \mathbb{R}^d : \varphi(\xi) \ge s\} \le C/s \quad \text{for all} \quad s > 0,$$

where $C > 0$ is some constant and m is the Lebesgue measure on $\mathbb{R}^d$. Then for any $1 < p \le 2$ we have

$$\left(\int_{\mathbb{R}^d} \left| \frac{\widehat{f}(\xi)}{\varphi(\xi)} \right|^p \varphi(\xi)^2 \, \mathrm{d}\xi \right)^{\frac{1}{p}} \le C_p \|f\|_{L^p(\mathbb{R}^d)} \quad \text{for all} \quad f \in L^p(\mathbb{R}^d),$$

where $C_p > 0$ only depends on p and C. Here $\widehat{f}(\xi)$ denotes the usual Euclidean Fourier transform of f.

In our situation we consider the measure space $L^\perp$ endowed with the counting measure. It is natural to consider this measure, because the discrete version of integration is summation. The translation of Theorem 1.1 in our setting then suggests that

$$\left(\sum_{\kappa \in L^\perp} |\widehat{f}(\kappa)|^p \, \varphi(\kappa)^{2-p} \right)^{\frac{1}{p}} \lesssim M_\varphi^{\frac{2-p}{p}} \|f\|_{L^p(\Omega)} \quad \text{for all} \quad f \in L^p(\Omega)$$

under the condition that φ is a positive function on $L^\perp$ satisfying

$$M_\varphi := \sup_{s>0} s \sum_{\substack{\kappa \in L^\perp \\ \varphi(\kappa) \geq s}} 1 < \infty.$$

Here the notation $\lesssim$ means that the (weak) inequality holds if we multiply the right-hand side with a certain positive constant depending only on the exponents of the relevant Lebesgue spaces, so in this case the constant only depends on p. The factor $M_\varphi^{\frac{2-p}{p}}$ is written explicitly since this embodies the dependence on φ. This Paley-type inequality is proven in Theorem 3.2 in Subsection 3.2.

In Subsection 3.4 we discuss the analogue of Hörmander's classical version of the Hausdorff-Young-Paley inequality, which generalises both the Hausdorff-Young inequality and Paley's inequality.

Theorem 1.2 ([15, Corollary 1.6]). *Assume φ satisfies the condition of Theorem 1.1. Let $1 < p \leq 2$ and $1 < p \leq b \leq p' < \infty$, where p' denotes the conjugate exponent of p, i.e. $\frac{1}{p} + \frac{1}{p'} = 1$. Then for all $f \in L^p(\mathbb{R}^d)$ we have*

$$\left(\int_{\mathbb{R}^d} \left| \widehat{f}(\xi) \, \varphi(\xi)^{\frac{1}{b} - \frac{1}{p'}} \right|^b \, \mathrm{d}\xi \right)^{\frac{1}{b}} \leq C_p \|f\|_{L^p(\mathbb{R}^d)}.$$

Note that this inequality reduces to the Hausdorff-Young inequality for $b = p'$ and to Paley's inequality for $b = p$. We can interpret the Hausdorff-Young-Paley inequality in our context in the same way as for Paley's inequality. The resulting hypothesis is that

$$\left(\sum_{\kappa \in L^\perp} \left| \widehat{f}(\kappa) \, \varphi(\kappa)^{\frac{1}{b} - \frac{1}{p'}} \right|^b \right)^{\frac{1}{b}} \lesssim M_\varphi^{\frac{1}{b} - \frac{1}{p'}} \|f\|_{L^p(\Omega)},$$

where φ is a positive function on $L^\perp$ satisfying

$$M_\varphi := \sup_{s>0} s \sum_{\substack{\kappa \in L^\perp \\ \varphi(\kappa) \geq s}} 1 < \infty.$$

Again, we included the factor $M_\varphi^{\frac{1}{b} - \frac{1}{p'}}$ in the right-hand side of the inequality as it contains the dependence on φ. We will show that this assertion is true in Theorem 3.5.

In Subsection 3.3 we prove the Hardy-Littlewood inequality as a consequence of the Paley inequality. The original result of Hardy and Littlewood appeared in [14].

Theorem 1.3 ([14, Theorem 5]). *Let $1 < p \leq 2$. For every $f \in L^p(\mathbb{T})$ we have*

$$\sum_{m=-\infty}^{+\infty} |\widehat{f}(m)|^p \, (|m| + 1)^{p-2} \lesssim \|f\|_{L^p(\mathbb{T})}^p. \tag{1.1}$$

We can view $\mathbb{T}$ as the fundamental domain of the one-dimensional lattice $\mathbb{Z}$, which enables us to interpret this inequality in our setting, where we allow more general weight functions than $(|m|+1)^{p-2}$ in (1.1). Such a weight function φ needs to decay sufficiently fast, which we express by asking that

$$\sum_{\kappa \in L^{\perp}} \frac{1}{\varphi(\kappa)^{\beta}} < \infty \quad \text{for some} \quad \beta > 0.$$

Under this condition we prove in Theorem 3.3 that for every $f \in L^p(\Omega)$ we have

$$\left(\sum_{\kappa \in L^{\perp}} |\widehat{f}(\kappa)|^p \, \varphi(\kappa)^{\beta(p-2)} \right)^{\frac{1}{p}} \lesssim_{p,\varphi} \|f\|_{L^p(\Omega)}.$$

The hidden constant in this inequality indeed depends on both p and φ, but the dependence on φ can be eliminated by including a similar constant as in the Paley inequality.

Next, in Section 4 we prove the L^p-L^q boundedness of Fourier multipliers on Ω for $1 < p, q < \infty$. Hörmander proved in the Euclidean case [15, Theorem 1.11] that a Fourier multiplier with symbol $\sigma(\xi)$ has a bounded L^p-L^q extension as a consequence of the Hausdorff-Young-Paley inequality, provided that $1 < b < \infty$, $1 < p \leq 2 \leq q < \infty$ with $\frac{1}{p} + \frac{1}{q} = \frac{1}{b}$ and $\sigma(\xi)$ is a measurable function satisfying

$$m\{\xi \in \mathbb{R}^d : |\sigma(\xi)| \geq s\} \leq \frac{C}{s^b} \quad \text{for all} \quad s > 0,$$

where $C > 0$ is some constant. We will prove a similar result in Theorems 4.4 and 4.5. The fact that Ω has finite measure delivers some Lebesgue space embeddings in Lemma 4.6. This will allow us to obtain also L^p-L^q boundedness results for the remaining cases in Theorem 4.7 and Remark 4.8. Moreover, we will show in the case $1 < q \leq 2 \leq p < \infty$ that the weakest Hörmander-type condition is sufficient. We point out that these results give sufficient conditions for L^p-boundedness of Fourier multipliers on Ω.

Finally, we discuss the application of the main results on L^p-L^q boundedness of Fourier multipliers on Ω to Sobolev embedding results in Section 5. In particular, we define the Laplacian on Ω there as a Fourier multiplier to be able to construct the Sobolev spaces. To this end we need to introduce functions of Fourier multipliers, and we will provide a general estimate for this type of Fourier multipliers under suitable conditions on the function. This will allow us to deduce continuous embeddings $L_s^p(\Omega) \hookrightarrow L_t^q(\Omega)$ and $L_s^p(\Omega) \hookrightarrow L^q(\Omega)$ under suitable conditions on the parameters p, q, s and t. At the end of this section we remark that these Sobolev embeddings indicate the sharpness of our main inequalities.

1.2. **Notation and conventions.** We follow the convention that $0 \in \mathbb{N}$.

Throughout this paper L stands for a lattice in $\mathbb{R}^d$ and Ω denotes a fundamental domain of L. These concepts are introduced in Section 2.

For vectors $x, y \in \mathbb{R}^d$ we write $x \cdot y = \sum_{j=1}^d x_j y_j$ for the Euclidean inner product and $|x| = \sqrt{x \cdot x}$ for the Euclidean norm.

Let X and Y be normed vector spaces. We denote by $\mathcal{L}(X, Y)$ the normed vector space of all bounded linear mappings from X to Y.

If we consider the Lebesgue space $\ell^p(A)$ for some set A and some $1 \leq p \leq \infty$, it is always tacitly assumed that the measure on A is the counting measure. When we sum over the empty set, this is by definition equal to 0.

We denote the conjugate exponent of a real number $1 \leq p \leq \infty$ by p', i.e. $\frac{1}{p} + \frac{1}{p'} = 1$.

Let $f \in L^p(X, \mu)$ and $g \in L^q(Y, \nu)$. We write $\|f\|_{L^p(X,\mu)} \lesssim \|g\|_{L^q(Y,\nu)}$ if there exists some constant $C_{p,q} > 0$ depending only on the exponents of the relevant Lebesgue spaces such that $\|f\|_{L^p(X,\mu)} \leq C_{p,q}\|g\|_{L^q(Y,\nu)}$. If we want to make this explicit or include other parameters, then we write the parameters as indices, e.g. $\|f\|_{L^p(X,\mu)} \lesssim_{p,\varphi} \|g\|_{L^q(Y,\nu)}$. This notation will also be used in related contexts, such as for norm inequalities for some $A \in \mathcal{L}(L^p(X, \mu), L^q(Y, \nu))$.

2. Fourier analysis on fundamental domains of lattices in $\mathbb{R}^d$

In this section we present the basics of Fourier analysis on fundamental domains of lattices in $\mathbb{R}^d$. We refer to [19] and [12] for more details on this topic.

A lattice L in $\mathbb{R}^d$ is a discrete subgroup of $\mathbb{R}^d$ spanned by d linearly independent vectors, which means

$$L = \{k_1 a_1 + k_2 a_2 + ... + k_d a_d \mid k_i \in \mathbb{Z}\}$$

for some linearly independent column vectors $a_1, \ldots, a_d$ in $\mathbb{R}^d$. The $d \times d$-matrix A with $a_1, \ldots, a_d$ as column vectors is called the generator matrix of L as

$$L = A\mathbb{Z}^d = \{Ak \mid k \in \mathbb{Z}^d\}.$$

In this case we sometimes denote L by L_A.

A *fundamental set* of a lattice L in $\mathbb{R}^d$ is a set $\Omega \subseteq \mathbb{R}^d$ such that $\Omega + L = \mathbb{R}^d$ as a direct sum, which means that Ω contains one representative for every coset of $\mathbb{R}^d/L$. We can also write down this condition by means of the characteristic function χ_Ω of Ω as

$$\sum_{\lambda \in L} \chi_\Omega(x + \lambda) = 1 \quad \text{for all } x \in \mathbb{R}^d.$$

This expresses that the translated copies of Ω do not overlap and do not leave gaps. A *fundamental domain* of L is a measurable fundamental set of L. Every lattice $L = L_A$ has a natural fundamental domain, namely the parallelotope

$$\Omega_P := \left\{ \sum_{i=1}^{d} t_i a_i \mid t = (t_1, \ldots, t_d) \in [0, 1)^d \right\}, \tag{2.1}$$

which is bounded. Remark that a fundamental domain of a lattice L is clearly not unique.

Now consider any fundamental domain Ω of L. We can move every set of the countable decomposition $\{\Omega \cap (\Omega_P + \lambda) \mid \lambda \in L\}$ of Ω to Ω_P, and this results in Ω_P since both Ω and Ω_P are fundamental domains of L. Because of the countable additivity and translation invariance of the Lebesgue measure, we easily see that Ω and Ω_P have the same measure. In particular, since Ω_P has positive finite measure, this is also the case for every fundamental domain of L.

We are also interested in periodic arrangements of subsets of $\mathbb{R}^d$ determined by the points of a lattice which (almost) fill $\mathbb{R}^d$ under a less restrictive condition than for fundamental domains. This can be expressed by the notion of a tiling. We say that a set $\Omega \subseteq \mathbb{R}^d$ tiles $\mathbb{R}^d$ with the lattice L if

$$\sum_{\lambda \in L} \chi_\Omega(x + \lambda) = 1 \quad \text{for almost all } x \in \mathbb{R}^d.$$

This expresses that the union of all translated copies of the set Ω almost fills the whole space $\mathbb{R}^d$ with almost no overlaps. We also write this as $\Omega + L \approx \mathbb{R}^d$.

In order to examine Fourier analysis on a fundamental domain Ω of a lattice L in $\mathbb{R}^d$ we need to construct an appropriate orthonormal basis for $L^2(\Omega)$. We note that we endow $L^2(\Omega)$ with the normalized inner product, namely

$$\langle f, g \rangle_{L^2(\Omega)} := \frac{1}{|\Omega|} \int_\Omega f(x)\, \overline{g(x)}\, \mathrm{d}x.$$

Here $|\Omega|$ denotes the Lebesgue measure of Ω.

The construction of our orthonormal basis for $L^2(\Omega)$ makes use of the concept of a dual lattice. Given a lattice L in $\mathbb{R}^d$, the dual lattice $L^\perp$ is defined by

$$L^\perp := \{\kappa \in \mathbb{R}^d \mid \forall \lambda \in L : \kappa \cdot \lambda \in \mathbb{Z}\},$$

where $\kappa \cdot \lambda$ denotes the usual Euclidean inner product of the vectors $\kappa, \lambda \in \mathbb{R}^d$. The generator matrix of $L_A^\perp$ is $A^\perp := A^{-\mathrm{T}}$, where A^T denotes the transpose of the matrix A and $A^{-\mathrm{T}} := (A^{-1})^\mathrm{T}$. Note that $(L^\perp)^\perp = L$ for any lattice L.

Surprisingly, it turns out that Fourier analysis on a fundamental domain of a lattice and tilings with this lattice are very closely related as demonstrated by the *Fuglede theorem*.

Theorem 2.1 (Fuglede, [12, Section 6]). *Let $\Omega \subseteq \mathbb{R}^d$ be a measurable set with $0 < |\Omega| < \infty$ and let L be a lattice in $\mathbb{R}^d$. Then $\Omega + L \approx \mathbb{R}^d$ if and only if $\{e^{2\pi i \kappa \cdot x} \mid \kappa \in L^\perp\}$ is an orthonormal basis for $L^2(\Omega)$.*

The Fuglede theorem thus provides a natural orthonormal basis to work with. This theorem works for any fundamental domain of a lattice L as it tiles $\mathbb{R}^d$ with L. Moreover, this is in a certain sense the only type of sets that satisfy the tiling condition. By the lemma in [12, Section 6] every set $\Omega \subseteq \mathbb{R}^d$ satisfying $\Omega + L \approx \mathbb{R}^d$ differs from a fundamental domain only by a null set.

As a consequence of the Fuglede theorem, any function $f \in L^2(\Omega)$ can be expanded into a Fourier series, namely

$$f(x) = \sum_{\kappa \in L^\perp} \widehat{f}(\kappa)\, e^{2\pi i \kappa \cdot x}, \quad \text{where} \quad \widehat{f}(\kappa) = \frac{1}{|\Omega|} \int_\Omega f(x)\, e^{-2\pi i \kappa \cdot x}\, \mathrm{d}x. \tag{2.2}$$

In summary, choosing a lattice or, equivalently, a generator matrix and fixing a certain fundamental domain of this lattice, the Fuglede theorem enables Fourier analysis on this fundamental domain through exponential functions related to the dual lattice.

2.1. **The Fourier transform on a fundamental domain.** We now define the Fourier transform on a fundamental domain Ω of a lattice L in $\mathbb{R}^d$ and discuss some properties of it. Formula (2.2) suggests that we can define the Fourier transform $\mathcal{F}_\Omega$ on $L^2(\Omega)$ by

$$\mathcal{F}_\Omega f(\kappa) := \widehat{f}(\kappa) = \frac{1}{|\Omega|} \int_\Omega f(x)\, e^{-2\pi i \kappa \cdot x}\, \mathrm{d}x \quad \text{for} \quad \kappa \in L^\perp. \tag{2.3}$$

As in the Euclidean case we would like to find a dense subspace of $L^2(\Omega)$ where the Fourier transform has very useful properties. For the Euclidean Fourier transform this space is the class of Schwartz functions $\mathcal{S}(\mathbb{R}^d)$. We will base our findings on the *toroidal Fourier transform*, which is actually a special case of Fourier transforms on fundamental domains of lattices in $\mathbb{R}^d$. See [24, Chapter 3] for a discussion of toroidal Fourier transforms.

Of course, in the setting of Fourier analysis on fundamental domains of lattices in $\mathbb{R}^d$ we consider different function spaces than in the Euclidean setting. Consider a lattice L with fundamental domain Ω. Now note that $\mathbb{R}^d/L$ is homeomorphic to

$\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ so that Ω is compact for the initial topology. This initial topology is defined as the weakest topology that makes the projection mapping $x \in \Omega \mapsto x+L \in \mathbb{R}^d/L$ continuous, and one can easily check that $\Omega \cong \mathbb{R}^d/L$ with this topology. In fact, we can view this initial topology as an L-periodised Euclidean topology. As a consequence, every complex-valued function on Ω has compact support. Hence, it is a natural choice to consider the Fourier transform on the function space $C_{\mathrm{per}}^\infty(\Omega)$, which is constructed as follows.

Definition 2.2. A function $f : \mathbb{R}^d \to \mathbb{C}$ satisfying $f(x+\lambda) = f(x)$ for all $x \in \mathbb{R}^d$ and $\lambda \in L$ is called L-periodic. The space $C_{\mathrm{per}}^m(\Omega)$ consists of all m-times continuously differentiable L-periodic functions f considered to be defined on Ω, and $C_{\mathrm{per}}^\infty(\Omega) := \bigcap_{m \in \mathbb{N}} C_{\mathrm{per}}^m(\Omega)$. The topology on $C_{\mathrm{per}}^\infty(\Omega)$ is the topology of uniform convergence on compact sets of the functions and all their derivatives.

Remark 2.3. Since $\Omega \cong \mathbb{R}^d/L$, it is natural to consider function spaces on $\mathbb{R}^d/L$, which can be identified with periodic functions. Indeed, the function space $C_{\mathrm{per}}^\infty(\Omega)$ corresponds to $C^\infty(\mathbb{R}^d/L)$, but we write $C_{\mathrm{per}}^\infty(\Omega)$ instead of $C^\infty(\Omega)$ or $C^\infty(\mathbb{R}^d/L)$ to avoid confusion. Notice that this difference is not really relevant for the Lebesgue spaces, as the periodic Lebesgue space $L_{\mathrm{per}}^p(\Omega)$ is isometrically isomorphic to $L^p(\Omega)$, where the isomorphism is given by the restriction $f \in L_{\mathrm{per}}^p(\Omega) \mapsto f|_\Omega \in L^p(\Omega)$.

We now define the analogue of the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ on a lattice L. Since L is a discrete space, the requirement of smoothness is removed.

Definition 2.4. Let L be a lattice. We say that $f : L \to \mathbb{C}$ belongs to $\mathbf{s}(L)$ if for all $N \in \mathbb{N}$ there exists a constant $C_N > 0$ such that for all $\lambda \in L$ we have

$$|f(\lambda)| \leq C_N(1 + |\lambda|^2)^{-N/2}.$$

We endow $\mathbf{s}(L)$ with the topology generated by the countable family of seminorms $p_j(f) := \sup_{\lambda \in L} |f(\lambda)| \, (1 + |\lambda|^2)^{j/2}$ for $j \in \mathbb{N}$.

These function spaces appear in the theory of the toroidal Fourier transform. The torus $\mathbb{T}^d$ is namely the quotient $\mathbb{R}^d/\mathbb{Z}^d$ of the space $\mathbb{R}^d$ with the lattice $\mathbb{Z}^d$ and can be viewed as the fundamental domain $[0,1)^d$ of $\mathbb{Z}^d$ (with the initial topology). The toroidal Fourier transform has the following useful property, which can be directly compared to the Euclidean case.

Theorem 2.5 ([24, page 301])**.** *Let*

$$\mathcal{F}_{\mathbb{T}^d} : C_{\mathrm{per}}^\infty(\mathbb{T}^d) \to \mathbf{s}(\mathbb{Z}^d) : f \mapsto \widehat{f}(\xi) := \int_{\mathbb{T}^d} f(x) \, e^{-2\pi i x \cdot \xi} \, \mathrm{d}x$$

be the toroidal Fourier transform. Then $\mathcal{F}_{\mathbb{T}^d}$ is an isomorphism from $C_{\mathrm{per}}^\infty(\mathbb{T}^d)$ to $\mathbf{s}(\mathbb{Z}^d)$, which means that it is a linear homeomorphism. Its inverse is given by

$$\mathcal{F}_{\mathbb{T}^d}^{-1} : \mathbf{s}(\mathbb{Z}^d) \to C_{\mathrm{per}}^\infty(\mathbb{T}^d) : f \mapsto \check{f}(x) = \sum_{\xi \in \mathbb{Z}^d} f(\xi) \, e^{2\pi i x \cdot \xi}.$$

We can relate the Fourier transform on a fundamental domain Ω of a lattice L in $\mathbb{R}^d$ to this toroidal case. Since $A : \mathbb{T}^d \to A\mathbb{T}^d = \Omega$ is a homeomorphism, we find an isomorphism $\alpha : C_{\mathrm{per}}^\infty(\Omega) \to C_{\mathrm{per}}^\infty(\mathbb{T}^d)$ with $\alpha[f](x) := f(Ax)$ for every $x \in \mathbb{T}^d$. Similarly, we see that $\beta : \mathbf{s}(\mathbb{Z}^d) \to \mathbf{s}(L^\perp)$ with $\beta[f](x) := f((A^\perp)^{-1}x) = f(A^{\mathrm{T}}x)$ for every $x \in L^\perp$ is also an isomorphism. Note that the isomorphisms α and β actually just represent a change of variables.

The Fourier transform on the fundamental domain Ω is related to the toroidal Fourier transform via the isomorphisms α and β as represented in the following diagram:

$$\mathcal{F}_\Omega : C^\infty_{\mathrm{per}}(\Omega) \xrightarrow{\alpha} C^\infty_{\mathrm{per}}(\mathbb{T}^d) \xrightarrow{\mathcal{F}_{\mathbb{T}^d}} \mathfrak{s}(\mathbb{Z}^d) \xrightarrow{\beta} \mathfrak{s}(L^\perp).$$

Since $\mathcal{F}_\Omega = \beta \circ \mathcal{F}_{\mathbb{T}^d} \circ \alpha$ is a composition of isomorphisms, Theorem 2.5 gives the following result.

Theorem 2.6. *Let*

$$\mathcal{F}_\Omega : C^\infty_{\mathrm{per}}(\Omega) \to \mathfrak{s}(L^\perp) : f \mapsto \widehat{f}(\kappa) := \frac{1}{|\Omega|} \int_\Omega f(x)\, e^{-2\pi i \kappa \cdot x}\, \mathrm{d}x$$

be the Fourier transform on a fundamental domain Ω of a lattice L in $\mathbb{R}^d$. Then $\mathcal{F}_\Omega$ is an isomorphism from $C^\infty_{\mathrm{per}}(\Omega)$ to $\mathfrak{s}(L^\perp)$. Its inverse is given by

$$\mathcal{F}^{-1}_\Omega : \mathfrak{s}(L^\perp) \to C^\infty_{\mathrm{per}}(\Omega) : f \mapsto \check{f}(x) = \sum_{\kappa \in L^\perp} f(\kappa)\, e^{2\pi i \kappa \cdot x}.$$

Later we will see that both $\mathcal{F}_\Omega$ and $\mathcal{F}^{-1}_\Omega$ have bounded $L^p(\Omega)$-$\ell^{p'}(L^\perp)$ extensions for every $1 \le p \le 2$ as a consequence of the Hausdorff-Young inequalities in Theorem 3.1, where $\frac{1}{p} + \frac{1}{p'} = 1$.

Note that $C^\infty_{\mathrm{per}}(\Omega)$ is dense in $L^p(\Omega)$ for $1 \le p < \infty$ since this is generally known for $\Omega \subseteq \mathbb{R}^d$. It is also true that $\mathfrak{s}(L^\perp)$ is dense in $\ell^p(L^\perp)$ for $1 \le p < \infty$, because we can approximate an arbitrary $f \in \ell^p(L^\perp)$ by a sequence of finitely supported functions on $L^\perp$ and these belong to $\mathfrak{s}(L^\perp)$. Hence, the bounded $L^p(\Omega)$-$\ell^{p'}(L^\perp)$ extensions of $\mathcal{F}_\Omega$ and $\mathcal{F}^{-1}_\Omega$ are unique.

2.2. **Spectral sets and the Fuglede conjecture.** We now briefly discuss the Fuglede theorem outside the context of lattices. A measurable set $\Omega \subseteq \mathbb{R}^d$ with $0 < |\Omega| < \infty$ is called a *spectral set* if there exists a set $\Lambda \subseteq \mathbb{R}^d$ such that $\{e^{2\pi i \lambda \cdot x} \mid \lambda \in \Lambda\}$ is an orthonormal basis for $L^2(\Omega)$. In this case the set Λ is called an *exponent set* for Ω. Note that every exponent set is countably infinite.

Further, a measurable set $\Omega \subseteq \mathbb{R}^d$ with $0 < |\Omega| < \infty$ is called a *direct summand* if there exists a set $\Gamma \subseteq \mathbb{R}^d$ such that (after modifying Ω with a null set) $\Omega + \Gamma = \mathbb{R}^d$ as a direct sum. This set Γ is called a *translation set* for Ω.

It is asserted in [12, page 119] that exponent sets and translation sets are always discrete, closed and total, which means that these sets are not contained in a proper subspace of $\mathbb{R}^d$. As a consequence every translation set is also countably infinite. Remark that the Fuglede theorem tells us that a measurable set $\Omega \subseteq \mathbb{R}^d$ is a spectral set admitting an exponent *subgroup* $\Lambda \subseteq \mathbb{R}^d$ if and only if Ω is a direct summand admitting a translation *subgroup* $\Gamma \subseteq \mathbb{R}^d$.

In [12] Fuglede conjectures that, more generally, a measurable set $\Omega \subseteq \mathbb{R}^d$ with $0 < |\Omega| < \infty$ is a spectral set if and only if Ω is a direct summand. Theorem 2.1 states that this conjecture is true for fundamental domains of lattices in $\mathbb{R}^d$. This conjecture has also been shown in [18] to be true for any compact convex set $\Omega \subseteq \mathbb{R}^d$ with non-empty interior, but counterexamples have been found in $\mathbb{R}^d$ for $d \ge 3$. However, the conjecture remains unknown for $d \in \{1, 2\}$.

We can introduce a Fourier transform on a spectral set Ω in the same way as in Subsection 2.1. We also have an obvious candidate for the inverse Fourier transform. However, we do not have some isomorphisms that link this Fourier transform to a canonical Fourier transform as before. So the question arises whether we can prove that this Fourier transform is invertible and whether it is an isomorphism on a suitable function space on Ω, such as $L^2(\Omega)$. This would provide a way to

generalise the results in this paper to any spectral set $\Omega \subseteq \mathbb{R}^d$. However, it may seem improbable that we can invert this Fourier transform, because we might loose control over the corresponding Dirichlet kernel if we do not have more structural information about the exponent set Λ. Otherwise we could ask whether there is another approach that works for all spectral sets. We leave this question open.

2.3. **Lattices of type** A_d. Sometimes a lattice L has a simpler description in a higher-dimensional space. One can try to extend the theory in this paper to lattices living in proper subspaces. We will give an example of such a lattice, which appears in [19]. We study this particular kind of lattices through homogeneous coordinates.

Let $d \geq 1$. If we identify $\mathbb{R}^d$ with the hyperplane

$$\mathbb{R}_H^{d+1} := \{(t_1, t_2, \ldots, t_{d+1}) \in \mathbb{R}^{d+1} \mid t_1 + t_2 + \cdots + t_{d+1} = 0\}$$

in $\mathbb{R}^{d+1}$, then the lattice of type A_d is defined as

$$\mathbb{Z}_H^{d+1} := \mathbb{Z}^{d+1} \cap \mathbb{R}_H^{d+1} = \{(k_1, k_2, \ldots, k_{d+1}) \in \mathbb{Z}^{d+1} \mid k_1 + k_2 + \cdots + k_{d+1} = 0\}.$$

Note that this lattice lives in a d-dimensional subspace of $\mathbb{R}^{d+1}$, so it is described by homogeneous coordinates $\mathbf{t}$, for which we will use bold letters, and $A_d := \mathbb{Z}_H^{d+1}$.

We consider the fundamental domain of the lattice A_d tiling $\mathbb{R}_H^{d+1}$ given by

$$\Omega_H := \{\mathbf{t} \in \mathbb{R}_H^{d+1} \mid -1 < t_i - t_j \leq 1, \forall 1 \leq i < j \leq d+1\}.$$

Note that the strict inequality ensures that the translated copies of Ω_H do not overlap. As an example, for $d = 2$ this fundamental domain Ω_H is a regular hexagon.

Note that we need slightly different but similar definitions for the generator matrix of the lattice A_d and related concepts because this lattice lives in a proper subspace of $\mathbb{R}^{d+1}$. The natural choice for the generator matrix of the lattice A_d is the $(d+1) \times d$ matrix

$$A := \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ -1 & -1 & \cdots & -1 & -1 \end{pmatrix},$$

since $A_d = A\mathbb{Z}^{d+1}$ with this choice. The dual lattice $A_d^\perp$ is defined to be generated by the matrix

$$A^\perp := A(A^{\mathrm{T}}A)^{-1} = \frac{1}{d+1} \begin{pmatrix} d & -1 & \cdots & -1 & -1 \\ -1 & d & \cdots & -1 & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & \cdots & d & -1 \\ -1 & -1 & \cdots & -1 & d \\ -1 & -1 & \cdots & -1 & -1 \end{pmatrix}.$$

We now give an explicit description of the dual lattice $A_d^\perp$. Consider

$$\mathbb{H} := \{\mathbf{k} \in A_d \mid k_1 \equiv k_2 \equiv \cdots \equiv k_{d+1} \pmod{d+1}\}.$$

To every element $A^\perp j$ with $j \in \mathbb{Z}^d$ of the dual lattice $A_d^\perp$ we relate the vector $\mathbf{k} = (d+1)A^\perp j$, which belongs to A_d because the integers are closed under linear combinations. Moreover, we can easily compute that $\mathbf{k} \in \mathbb{H}$. Conversely, we immediately see that $j = A^{\mathrm{T}}\mathbf{k}/(d+1)$, so it follows that $j \in \mathbb{Z}^d$ for all $\mathbf{k} \in \mathbb{H}$. In short,

this means that

$$A_d^\perp = \left\{ A^\perp j \mid j \in \mathbb{Z}^d \right\} = \left\{ \frac{\mathbf{k}}{d+1} \mid \mathbf{k} \in \mathbb{H} \right\}.$$

The Fuglede theorem tells us that $\{\phi_\mathbf{j} \mid \mathbf{j} \in \mathbb{H}\}$ is an orthonormal basis of $L^2(\Omega_H)$, where

$$\phi_\mathbf{j}(\mathbf{t}) := e^{\frac{2\pi i}{d+1} \mathbf{j} \cdot \mathbf{t}} \quad \text{for} \quad \mathbf{j} \in \mathbb{H} \quad \text{and} \quad \mathbf{t} \in \mathbb{R}_H^{d+1}.$$

In particular, this means that

$$\langle \phi_\mathbf{j}, \phi_\mathbf{k} \rangle := \frac{1}{|\Omega_H|} \int_{\Omega_H} \phi_\mathbf{j}(\mathbf{t}) \, \overline{\phi_\mathbf{k}(\mathbf{t})} \, \mathrm{d}\mathbf{t} = \delta_{\mathbf{jk}},$$

where $\delta_{\mathbf{jk}} = 1$ if $\mathbf{j} = \mathbf{k}$ and 0 otherwise. We note that $|\Omega_H| = \sqrt{\det(A^\mathrm{T} A)} = \sqrt{d+1}$.

Let us call a function f on $\mathbb{R}_H^{d+1}$ H-periodic if it is periodic with respect to the lattice A_d, which means that $f(\mathbf{t} + \mathbf{k}) = f(\mathbf{t})$ for all $\mathbf{k} \in A_d$. Note that the functions $\phi_\mathbf{j}$ are H-periodic. Further, we find that every H-periodic $L^2(\Omega_H)$-function f has a Fourier series expansion:

$$f(\mathbf{t}) = \sum_{\mathbf{k} \in \mathbb{H}} \widehat{f}_\mathbf{k} \phi_\mathbf{k}(\mathbf{t}), \quad \text{where} \quad \widehat{f}_\mathbf{k} := \frac{1}{\sqrt{d+1}} \int_{\Omega_H} f(\mathbf{t}) \, \phi_{-\mathbf{k}}(\mathbf{t}) \, \mathrm{d}\mathbf{t}.$$

3. Some fundamental inequalities

In this section we prove some classical inequalities in the setting of fundamental domains of lattices in $\mathbb{R}^d$, namely the Hausdorff-Young inequality, Paley's inequality and the Hausdorff-Young-Paley inequality. We will also treat the Hardy-Littlewood inequality, but this is not necessary for the further developments. We begin by deriving the Plancherel formula in our setting.

Let L be a lattice in $\mathbb{R}^d$ with fundamental domain Ω. By a well-known general fact about Hilbert spaces [22, Theorem 4.18] applied to $L^2(\Omega)$ with orthonormal basis $\{e^{2\pi i \kappa \cdot x} \mid \kappa \in L^\perp\}$ we find for $f \in L^2(\Omega)$ the following *Plancherel formula*:

$$\|f\|_{L^2(\Omega)}^2 = \frac{1}{|\Omega|} \int_\Omega |f(x)|^2 \, \mathrm{d}x = \sum_{\kappa \in L^\perp} |\widehat{f}(\kappa)|^2 = \|\widehat{f}\|_{\ell^2(L^\perp)}^2, \tag{3.1}$$

where $\widehat{f}(\kappa)$ is the Fourier transform of f as in equation (2.3).

3.1. Hausdorff-Young inequality.
The first inequality that we will consider in this section, is the Hausdorff-Young inequality. It is fundamental since it shows that the Fourier transform and its inverse have bounded L^p-$L^{p'}$ extensions. The proof proceeds as in the Euclidean case [24, Corollary 1.3.14].

Theorem 3.1 (Hausdorff-Young inequality). *Let $1 \leq p \leq 2$ and $\frac{1}{p} + \frac{1}{p'} = 1$. If $f \in L^p(\Omega)$, then $\widehat{f} \in \ell^{p'}(L^\perp)$ and*

$$\|\widehat{f}\|_{\ell^{p'}(L^\perp)} \leq \|f\|_{L^p(\Omega)}. \tag{3.2}$$

Similarly, if $f \in \ell^p(L^\perp)$, then $\check{f} \in L^{p'}(\Omega)$ and

$$\|\check{f}\|_{L^{p'}(\Omega)} \leq \|f\|_{\ell^p(L^\perp)}. \tag{3.3}$$

Proof. For all $f \in L^1(\Omega)$ we have

$$\|\widehat{f}\|_{\ell^\infty(L^\perp)} = \sup_{\kappa \in L^\perp} \frac{1}{|\Omega|} \left| \int_\Omega f(x) \, e^{-2\pi i \kappa \cdot x} \, \mathrm{d}x \right| \leq \|f\|_{L^1(\Omega)}.$$

We obtain the desired result by a straightforward application of the Riesz-Thorin interpolation theorem [13, Theorem 1.3.4] using this estimate and the Plancherel formula (3.1).

The second inequality follows similarly by applying Riesz-Thorin interpolation to the estimates

$$\|\check{f}\|_{L^{\infty}(\Omega)} = \operatorname{ess\,sup}_{x\in\Omega}\left|\sum_{\kappa\in L^{\perp}} f(\kappa)\, e^{2\pi i\kappa\cdot x}\right| \leq \sum_{\kappa\in L^{\perp}} |f(\kappa)| = \|f\|_{\ell^{1}(L^{\perp})}$$

and

$$\|\check{f}\|_{L^{2}(\Omega)}^{2} = \frac{1}{|\Omega|}\int_{\Omega}\left|\sum_{\kappa\in L^{\perp}} f(\kappa)\, e^{2\pi i\kappa\cdot x}\right|^{2}\,\mathrm{d}x \leq \frac{1}{|\Omega|}\int_{\Omega}\sum_{\kappa\in L^{\perp}}|f(\kappa)|^{2}\,\mathrm{d}x = \|f\|_{\ell^{2}(L^{\perp})}^{2}.$$

This completes the proof. $\qquad\square$

3.2. **Paley's inequality.** We now consider a Paley-type inequality, which can be seen as a weighted version of the Plancherel formula (3.1) for $L^{p}(\Omega)$ with $1 < p \leq 2$. Our proof strategy is based on the proof of [7, Theorem 4.2].

Theorem 3.2 (Paley's inequality). *Let $1 < p \leq 2$. If $\varphi(\kappa)$ is a positive function on $L^{\perp}$ such that*

$$M_{\varphi} := \sup_{s>0} s \sum_{\substack{\kappa\in L^{\perp} \\ \varphi(\kappa)\geq s}} 1 < \infty,$$

then for every $f \in L^{p}(\Omega)$ we have

$$\left(\sum_{\kappa\in L^{\perp}}|\widehat{f}(\kappa)|^{p}\,\varphi(\kappa)^{2-p}\right)^{\frac{1}{p}} \lesssim M_{\varphi}^{\frac{2-p}{p}}\|f\|_{L^{p}(\Omega)}. \qquad (3.4)$$

Proof. Note that the case $p = 2$ is fulfilled by Plancherel's formula (3.1). So now we suppose $1 < p < 2$. We define a measure μ on $L^{\perp}$ by $\mu\{\kappa\} = \varphi(\kappa)^{2}$ for every $\kappa \in L^{\perp}$. Consider the (sub)linear operator A defined by

$$Af(\kappa) = \frac{\widehat{f}(\kappa)}{\varphi(\kappa)}.$$

Now, our goal is to show that A is of weak type $(1,1)$ and $(2,2)$ so that the result follows from Marcinkiewicz' interpolation theorem [13, Theorem 1.3.2]. More specifically, we will see for every $s > 0$ that

$$\mu\{\kappa \in L^{\perp} : |Af(\kappa)| \geq s\} \leq \left(\frac{\|f\|_{L^{2}(\Omega)}}{s}\right)^{2}$$

and

$$\mu\{\kappa \in L^{\perp} : |Af(\kappa)| \geq s\} \leq 2M_{\varphi}\frac{\|f\|_{L^{1}(\Omega)}}{s}.$$

Because of the Plancherel formula (3.1) we obtain for every $f \in L^2(\Omega)$ that

$$s^2 \mu\{\kappa \in L^\perp : |Af(\kappa)| \geq s\} = s^2 \sum_{\substack{\kappa \in L^\perp \\ |Af(\kappa)| \geq s}} \varphi(\kappa)^2$$

$$\leq \sum_{\kappa \in L^\perp} |Af(\kappa)|^2 \varphi(\kappa)^2$$

$$= \sum_{\kappa \in L^\perp} |\widehat{f}(\kappa)|^2$$

$$= \|f\|_{L^2(\Omega)}^2.$$

This proves that A is of weak type $(2, 2)$.

Now note that $|Af(\kappa)| = |\widehat{f}(\kappa)|/\varphi(\kappa) \leq \|f\|_{L^1(\Omega)}/\varphi(\kappa)$ by the Hausdorff-Young inequality (3.2). Hence for $f \in L^1(\Omega)$ we have

$$\mu\{\kappa \in L^\perp : |Af(\kappa)| \geq s\} \leq \mu\left\{\kappa \in L^\perp : \varphi(\kappa) \leq \frac{\|f\|_{L^1(\Omega)}}{s}\right\} = \sum_{\substack{\kappa \in L^\perp \\ \varphi(\kappa) \leq \sigma}} \varphi(\kappa)^2,$$

where we set $\sigma := \|f\|_{L^1(\Omega)}/s$. Next we find

$$\sum_{\substack{\kappa \in L^\perp \\ \varphi(\kappa) \leq \sigma}} \varphi(\kappa)^2 = \sum_{\substack{\kappa \in L^\perp \\ \varphi(\kappa) \leq \sigma}} \int_0^{\varphi(\kappa)^2} \mathrm{d}\tau \leq \int_0^{\sigma^2} \sum_{\substack{\kappa \in L^\perp \\ \tau^{1/2} \leq \varphi(\kappa) \leq \sigma}} 1 \, \mathrm{d}\tau,$$

where we used Fubini's theorem and enlarged the 'domain of double integration', viewing summation as an integral with respect to the counting measure. Finally, the substitution $t = \sqrt{\tau}$ gives

$$\int_0^{\sigma^2} \sum_{\substack{\kappa \in L^\perp \\ \tau^{1/2} \leq \varphi(\kappa) \leq \sigma}} 1 \, \mathrm{d}\tau = 2 \int_0^\sigma t \sum_{\substack{\kappa \in L^\perp \\ t \leq \varphi(\kappa) \leq \sigma}} 1 \, \mathrm{d}t \leq 2 \int_0^\sigma t \sum_{\substack{\kappa \in L^\perp \\ t \leq \varphi(\kappa)}} 1 \, \mathrm{d}t \leq 2 M_\varphi \sigma.$$

We thus found for every $f \in L^2(\Omega)$ that

$$\mu\{\kappa \in L^\perp : |Af(\kappa)| \geq s\} \leq 2 M_\varphi \frac{\|f\|_{L^1(\Omega)}}{s},$$

which means that A is of weak type $(1, 1)$.

Remark that the Marcinkiewicz' interpolation theorem [13, Theorem 1.3.2] also ensures that the hidden constant in inequality (3.4) only depends on p, since we eliminated the dependence on φ by explicitly writing the factor $M_\varphi^{\frac{2-p}{p}}$. $\square$

3.3. **Hardy-Littlewood inequality.** As an intermezzo we include the celebrated Hardy-Littlewood inequality since it follows by a simple application of the Paley inequality. It expresses that we have a weighted version of Plancherel's formula (3.1) for $L^p(\Omega)$ with $1 < p \leq 2$ if the weight decays sufficiently fast. To this end we apply a similar reasoning as in the proof of [17, Theorem 3.5].

Theorem 3.3 (Hardy-Littlewood inequality)**.** *Let $1 < p \leq 2$, and let $\varphi(\kappa)$ be a positive function on $L^\perp$ growing sufficiently fast in the sense that*

$$\sum_{\kappa \in L^\perp} \frac{1}{\varphi(\kappa)^\beta} < \infty \quad \text{for some} \quad \beta > 0.$$

Then for every $f \in L^p(\Omega)$ it holds that

$$\left(\sum_{\kappa \in L^\perp} |\widehat{f}(\kappa)|^p \, \varphi(\kappa)^{\beta(p-2)} \right)^{\frac{1}{p}} \lesssim_{p,\varphi} \|f\|_{L^p(\Omega)}.$$

Proof. Put

$$C := \sum_{\kappa \in L^\perp} \frac{1}{\varphi(\kappa)^\beta} < \infty.$$

Then we find for any $s > 0$ that

$$C \geq \sum_{\substack{\kappa \in L^\perp \\ \varphi(\kappa)^\beta \leq \frac{1}{s}}} \frac{1}{\varphi(\kappa)^\beta} \geq s \sum_{\substack{\kappa \in L^\perp \\ \varphi(\kappa)^\beta \leq \frac{1}{s}}} 1 = s \sum_{\substack{\kappa \in L^\perp \\ \frac{1}{\varphi(\kappa)^\beta} \geq s}} 1$$

so that

$$\sup_{s > 0} s \sum_{\substack{\kappa \in L^\perp \\ \frac{1}{\varphi(\kappa)^\beta} \geq s}} 1 \leq C < \infty.$$

Hence, by Paley's inequality (3.4), applied to the function $1/\varphi(\kappa)^\beta$, we obtain for every $f \in L^p(\Omega)$ that

$$\left(\sum_{\kappa \in L^\perp} |\widehat{f}(\kappa)|^p \, \varphi(\kappa)^{\beta(p-2)} \right)^{\frac{1}{p}} \lesssim_{p,\varphi} \|f\|_{L^p(\Omega)}.$$

It is clear that the hidden constant in the Hardy-Littlewood inequality indeed also depends on φ since we did not include the factor $M_{1/\varphi(\kappa)^\beta}^{\frac{2-p}{p}}$ from Paley's inequality (3.4). $\qquad\square$

3.4. **Hausdorff-Young-Paley inequality.** We conclude this section with the classical Hausdorff-Young-Paley inequality. In [15] Lars Hörmander applied the Euclidean version of this inequality to prove the L^p-L^q boundedness of Fourier multipliers. It will also play a crucial role in our arguments in the next section.

Before stating the Hausdorff-Young-Paley inequality we mention the following important interpolation result, which we will use in our proof.

Theorem 3.4 (Stein-Weiss, [5, Corollary 5.5.4]). *Consider two measure spaces (X, μ) and (Y, ν). Let $1 \leq p_0, p_1, q_0, q_1 < \infty$. Assume that the linear operator A defined on $L^{p_0}(X, w_0 \, \mathrm{d}\mu) + L^{p_1}(X, w_1 \, \mathrm{d}\mu)$ satisfies*

$$\|Af\|_{L^{q_0}(Y, \widetilde{w}_0 \, \mathrm{d}\nu)} \leq M_0 \|f\|_{L^{p_0}(X, w_0 \, \mathrm{d}\mu)} \quad \text{for all} \quad f \in L^{p_0}(X, w_0 \, \mathrm{d}\mu)$$

and

$$\|Af\|_{L^{q_1}(Y, \widetilde{w}_1 \, \mathrm{d}\nu)} \leq M_1 \|f\|_{L^{p_1}(X, w_1 \, \mathrm{d}\mu)} \quad \text{for all} \quad f \in L^{p_1}(X, w_1 \, \mathrm{d}\mu)$$

for some $M_0 > 0$ and $M_1 > 0$, where w_j and $\widetilde{w}_j$ are weight functions for $j \in \{0, 1\}$, i.e. positive measurable functions. Let $0 \leq \theta \leq 1$ be arbitrary, and let

$$\frac{1}{p} = \frac{1 - \theta}{p_0} + \frac{\theta}{p_1} \quad \text{and} \quad \frac{1}{q} = \frac{1 - \theta}{q_0} + \frac{\theta}{q_1}.$$

Then A extends to a bounded linear operator $A : L^p(X, w \, \mathrm{d}\mu) \to L^q(Y, \widetilde{w} \, \mathrm{d}\nu)$ satisfying

$$\|Af\|_{L^q(Y, \widetilde{w} \, \mathrm{d}\nu)} \leq M_0^{1-\theta} M_1^\theta \|f\|_{L^p(X, w \, \mathrm{d}\mu)} \quad \text{for all} \quad f \in L^p(X, w \, \mathrm{d}\mu),$$

where

$$w = w_0^{\frac{p(1-\theta)}{p_0}} w_1^{\frac{p\theta}{p_1}} \quad and \quad \widetilde{w} = \widetilde{w}_0^{\frac{q(1-\theta)}{q_0}} \widetilde{w}_1^{\frac{q\theta}{q_1}}.$$

We are now ready to state and prove the Hausdorff-Young-Paley inequality, which is in a sense an interpolated version of the Hausdorff-Young inequality (3.2) and Paley's inequality (3.4). We follow the same line of reasoning as in the proof of [7, Theorem 4.6].

Theorem 3.5 (Hausdorff-Young-Paley inequality). *Let $1 < p \leq 2$, and let $1 < p \leq b \leq p' < \infty$ with $\frac{1}{p} + \frac{1}{p'} = 1$. Suppose that $\varphi(\kappa)$ is a positive function on $L^\perp$ satisfying*

$$M_\varphi := \sup_{s>0} s \sum_{\substack{\kappa \in L^\perp \\ \varphi(\kappa) \geq s}} 1 < \infty.$$

Then for every $f \in L^p(\Omega)$ we have

$$\left(\sum_{\kappa \in L^\perp} \left| \widehat{f}(\kappa)\, \varphi(\kappa)^{\frac{1}{b} - \frac{1}{p'}} \right|^b \right)^{\frac{1}{b}} \lesssim M_\varphi^{\frac{1}{b} - \frac{1}{p'}} \|f\|_{L^p(\Omega)}. \tag{3.5}$$

Proof. Plancherel's identity (3.1) treats the case $p = 2$, so assume that $p < 2$. Paley's inequality (3.4) ensures that the linear operator $\mathcal{F}_\Omega : L^p(\Omega) \to \ell^p(L^\perp, w_0)$ with $\mathcal{F}_\Omega f(\kappa) = \widehat{f}(\kappa)$ is bounded, where $w_0(\kappa) = \varphi(\kappa)^{2-p}$. Next, we note that $\mathcal{F}_\Omega : L^p(\Omega) \to \ell^{p'}(L^\perp)$ is also bounded because of the Hausdorff-Young inequality (3.2). Now, choose $0 \leq \theta \leq 1$ such that $\frac{1}{b} = \frac{1-\theta}{p} + \frac{\theta}{p'}$, so $\theta = \left(\frac{1}{p} - \frac{1}{b} \right) / \left(\frac{1}{p} - \frac{1}{p'} \right)$. Then Theorem 3.4 tells us that $\mathcal{F}_\Omega : L^p(\Omega) \to \ell^b(L^\perp, w)$ is bounded, where

$$w(\kappa) = w_0(\kappa)^{\frac{b(1-\theta)}{p}} = \left(\varphi(\kappa)^{\frac{1}{b} - \frac{1}{p'}} \right)^b.$$

Here we note that Theorem 3.4 ensures that the hidden constant in the Hausdorff-Young-Paley inequality only depends on p and b, since the dependence on $\varphi(\kappa)$ is contained in the factor $M_\varphi^{\frac{1}{b} - \frac{1}{p'}}$. $\qquad \square$

4. L^p-L^q BOUNDEDNESS OF FOURIER MULTIPLIERS FOR $1 < p, q < \infty$

We are finally ready to accomplish our main goal, namely establishing the L^p-L^q boundedness of Fourier multipliers on fundamental domains of lattices in $\mathbb{R}^d$. In this section we prove this for $1 < p, q < \infty$ under a Hörmander-type condition, which is inspired by Lars Hörmander's result [15, Theorem 1.11].

We start by defining Fourier multipliers, which intuitively correspond to multiplication by a function in the (multidimensional) 'frequency space'.

Definition 4.1. An operator $A : C_{\text{per}}^\infty(\Omega) \to C_{\text{per}}^\infty(\Omega)$ is called a *Fourier multiplier* with symbol $\sigma : L^\perp \to \mathbb{C}$ if for all $f \in C_{\text{per}}^\infty(\Omega)$ and $\kappa \in L^\perp$ it holds that

$$\widehat{Af}(\kappa) = \sigma(\kappa)\, \widehat{f}(\kappa). \tag{4.1}$$

Example 4.2. A typical example of a Fourier multiplier is any linear partial differential operator with constant coefficients, i.e. $A = \sum_{|\alpha| \leq k} a_\alpha \partial^\alpha$ for some $k \in \mathbb{N}$, as the Fourier transform converts derivatives into polynomials. Here we used for $\alpha \in \mathbb{N}^d$ the multi-index notations $\partial^\alpha = \partial_{x_1}^{\alpha_1} \ldots \partial_{x_d}^{\alpha_d}$ and $|\alpha| = \alpha_1 + \cdots + \alpha_d$.

Indeed, let us consider the parallelotopic fundamental domain Ω_P as given by (2.1). Then the formula for integration by parts [10, page 712] for $f, g \in C^\infty_{\text{per}}(\Omega_P)$ becomes

$$\int_{\Omega_P} f(x)\,\partial_{x_j} g(x)\,\mathrm{d}x = \int_{\partial\Omega_P} f(x)\,g(x)\,n_j(x)\,\mathrm{d}S - \int_{\Omega_P} \partial_{x_j} f(x)\,g(x)\,\mathrm{d}x$$

$$= -\int_{\Omega_P} \partial_{x_j} f(x)\,g(x)\,\mathrm{d}x,$$

where the normal $n(x)$ in the point $x \in \partial\Omega_P$ changes sign on opposite faces of the parallelepiped $\partial\Omega_P$, while the other functions stay the same on corresponding points on opposite faces because of their L-periodicity, which implies that the boundary term is indeed zero. This corresponds to the fact that the torus $\Omega \cong \mathbb{R}^d/L$ has no boundary.

Since the Fourier transform does not depend on the choice of fundamental domain, we can choose $\Omega = \Omega_P$ so that we find

$$\widehat{\partial^\alpha f}(\kappa) = \int_{\Omega_P} \partial^\alpha f(x)\,e^{-2\pi i\kappa\cdot x}\,\mathrm{d}x = (-1)^{|\alpha|}\int_{\Omega_P} f(x)\,\partial^\alpha e^{-2\pi i\kappa\cdot x}\,\mathrm{d}x = (2\pi i\kappa)^\alpha \widehat{f}(\kappa),$$

where $\kappa^\alpha := \kappa_1^{\alpha_1}\ldots\kappa_d^{\alpha_d}$. Indeed, if $P(\partial) = \sum_{|\alpha|\le k} a_\alpha \partial^\alpha$ is a linear partial differential operator with constant coefficients, then its symbol is $P(2\pi i\kappa)$.

Remark 4.3. Since $\mathcal{F}_\Omega : C^\infty_{\text{per}}(\Omega) \to s(L^\perp)$ is invertible, a Fourier multiplier A satisfies the relation

$$Af(x) = \mathcal{F}_\Omega^{-1}[\sigma(\kappa)\,\widehat{f}(\kappa)](x) = \sum_{\kappa\in L^\perp} \sigma(\kappa)\,\widehat{f}(\kappa)\,e^{2\pi i\kappa\cdot x}. \tag{4.2}$$

This is a special case of a pseudo-differential operator on Ω, which also allows symbols $\sigma(x, \kappa)$ that depend on both x and κ. Remark that every Fourier multiplier is a linear operator as a consequence of (4.2). Moreover, a straightforward verification using (4.1) shows that any Fourier multiplier $A : C^\infty_{\text{per}}(\Omega) \to C^\infty_{\text{per}}(\Omega)$ is continuous if the symbol has at most polynomial growth. One can easily verify that the condition on the symbol in the following theorems always restrict the symbol to be bounded, so we always consider continuous Fourier multipliers.

We start by treating the case $p = q = 2$ separately. In Remark 4.9 we will give sufficient conditions for L^p-boundedness for all $1 < p < \infty$.

Theorem 4.4. *Let $A : C^\infty_{\text{per}}(\Omega) \to C^\infty_{\text{per}}(\Omega)$ be a Fourier multiplier with bounded symbol $\sigma \in \ell^\infty(L^\perp)$. Then we have*

$$\|A\|_{\mathcal{L}(L^2(\Omega), L^2(\Omega))} := \sup_{f\neq 0} \frac{\|Af\|_{L^2(\Omega)}}{\|f\|_{L^2(\Omega)}} = \|\sigma\|_{\ell^\infty(L^\perp)}.$$

In particular, A can be extended to a bounded linear operator from $L^2(\Omega)$ to $L^2(\Omega)$.

Proof. The Plancherel formula (3.1) yields

$$\|Af\|_{L^2(\Omega)} = \|\widehat{Af}\|_{\ell^2(L^\perp)} = \left(\sum_{\kappa\in L^\perp} |\sigma(\kappa)\,\widehat{f}(\kappa)|^2\right)^{\frac{1}{2}} \le \|\sigma\|_{\ell^\infty(L^\perp)}\|\widehat{f}\|_{\ell^2(L^\perp)}$$

$$= \|\sigma\|_{\ell^\infty(L^\perp)}\|f\|_{L^2(\Omega)}.$$

This proves that $\|A\|_{\mathcal{L}(L^2(\Omega), L^2(\Omega))} \le \|\sigma\|_{\ell^\infty(L^\perp)}$.

Conversely, choose for every $n \in \mathbb{N}$ some $\kappa_n \in L^\perp$ such that

$$\|\sigma\|_{\ell^\infty(L^\perp)} - |\sigma(\kappa_n)| \leq \frac{1}{n}$$

and set $f_n(x) := e^{2\pi i \kappa_n \cdot x}$. Then we have

$$\widehat{f_n}(\kappa) = \frac{1}{|\Omega|} \int_\Omega e^{2\pi i \kappa_n \cdot x} e^{-2\pi i \kappa \cdot x} \, \mathrm{d}x = \delta_{\kappa,\kappa_n}$$

so that Plancherel's formula yields

$$\|A f_n\|_{L^2(\Omega)} = \|\widehat{A f_n}\|_{\ell^2(L^\perp)} = \left(\sum_{\kappa \in L^\perp} |\sigma(\kappa) \, \widehat{f}(\kappa)|^2 \right)^{\frac{1}{2}} = |\sigma(\kappa_n)|.$$

Hence, we find

$$\left(\|\sigma\|_{\ell^\infty(L^\perp)} - \frac{1}{n} \right) \|f_n\|_{L^2(\Omega)} \leq |\sigma(\kappa_n)| = \|A f_n\|_{L^2(\Omega)}$$

so that

$$\|\sigma\|_{\ell^\infty(L^\perp)} - \frac{1}{n} \leq \|A\|_{\mathcal{L}(L^2(\Omega), L^2(\Omega))}$$

for all $n \in \mathbb{N}$. We conclude that $\|\sigma\|_{\ell^\infty(L^\perp)} \leq \|A\|_{\mathcal{L}(L^2(\Omega), L^2(\Omega))}$. $\square$

Next, we examine the case that Hörmander considered in [15, Theorem 1.11]. Our proof is similar to that of [7, Theorem 4.8].

Theorem 4.5. *Let $1 < p \leq 2 \leq q < \infty$ with p and q not both equal to 2, and let $A : C^\infty_{\mathrm{per}}(\Omega) \to C^\infty_{\mathrm{per}}(\Omega)$ be a Fourier multiplier with symbol σ satisfying*

$$\sup_{s>0} s \left(\sum_{\substack{\kappa \in L^\perp \\ |\sigma(\kappa)| \geq s}} 1 \right)^{\frac{1}{p} - \frac{1}{q}} < \infty. \tag{4.3}$$

Then

$$\|A\|_{\mathcal{L}(L^p(\Omega), L^q(\Omega))} := \sup_{f \neq 0} \frac{\|Af\|_{L^q(\Omega)}}{\|f\|_{L^p(\Omega)}} \lesssim \sup_{s>0} s \left(\sum_{\substack{\kappa \in L^\perp \\ |\sigma(\kappa)| \geq s}} 1 \right)^{\frac{1}{p} - \frac{1}{q}}.$$

In particular, A can be extended to a bounded linear operator from $L^p(\Omega)$ to $L^q(\Omega)$.

Proof. First, we assume that $p \leq q'$. Note that $q' \leq 2$. Hence the Hausdorff-Young inequality (3.3) implies that

$$\|Af\|_{L^q(\Omega)} = \|\mathcal{F}_\Omega^{-1}(\widehat{Af})\|_{L^q(\Omega)} \leq \|\widehat{Af}\|_{\ell^{q'}(L^\perp)} = \left(\sum_{\kappa \in L^\perp} |\sigma(\kappa) \, \widehat{f}(\kappa)|^{q'} \right)^{\frac{1}{q'}}.$$

Set $\frac{1}{r} = \frac{1}{p} - \frac{1}{q} = \frac{1}{q'} - \frac{1}{p'}$. An application of the Hausdorff-Young-Paley inequality (3.5) with $b := q$ and $\varphi(\kappa) = |\sigma(\kappa)|^r$, where we note that $1 < p \leq q' \leq p' < \infty$,

yields

$$\|Af\|_{L^q(\Omega)} \leq \left(\sum_{\kappa \in L^\perp} |\sigma(\kappa) \widehat{f}(\kappa)|^{q'} \right)^{\frac{1}{q'}} = \left(\sum_{\kappa \in L^\perp} |\widehat{f}(\kappa) \varphi(\kappa)^{\frac{1}{b} - \frac{1}{p'}}|^{q'} \right)^{\frac{1}{q'}}$$

$$\lesssim \left(\sup_{s>0} s \sum_{\substack{\kappa \in L^\perp \\ \varphi(\kappa) \geq s}} 1 \right)^{\frac{1}{r}} \|f\|_{L^p(\Omega)}.$$

Hence we find

$$\|A\|_{\mathcal{L}(L^p(\Omega), L^q(\Omega))} \lesssim \left(\sup_{s>0} s \sum_{\substack{\kappa \in L^\perp \\ |\sigma(\kappa)|^r \geq s}} 1 \right)^{\frac{1}{r}} = \left(\sup_{s>0} s^r \sum_{\substack{\kappa \in L^\perp \\ |\sigma(\kappa)| \geq s}} 1 \right)^{\frac{1}{r}}$$

$$= \sup_{s>0} s \left(\sum_{\substack{\kappa \in L^\perp \\ |\sigma(\kappa)| \geq s}} 1 \right)^{\frac{1}{r}},$$

which is finite by assumption. This completes the proof for the case $p \leq q'$.

Now we consider the case $q' \leq p$, which is equivalent to $p' \leq q = (q')'$. It is well-known [24, Theorem C.4.50] that the dual space of the Banach space $L^p(\Omega)$ is $L^{p'}(\Omega)$. General Banach space theory [23, Theorem 4.10] tells us that the adjoint operator A^* satisfies $\|A^*\|_{\mathcal{L}(L^{q'}(\Omega), L^{p'}(\Omega))} = \|A\|_{\mathcal{L}(L^p(\Omega), L^q(\Omega))}$.

We now prove that A^* is a Fourier multiplier with symbol $\overline{\sigma}$. A general property of an orthonormal basis of a Hilbert space is that it satisfies *Parseval's identity* [22, Theorem 4.18]. Considering the orthonormal basis $\{e^{2\pi i \kappa \cdot x} \mid \kappa \in L^\perp\}$ of $L^2(\Omega)$ we get for all $f, g \in L^2(\Omega)$ that

$$\langle f, g \rangle_{L^2(\Omega)} = \sum_{\kappa \in L^\perp} \widehat{f}(\kappa) \widehat{g}(\kappa).$$

Hence, on the one hand we find for all $f, g \in C_{\mathrm{per}}^\infty(\Omega) \subseteq L^2(\Omega)$ that

$$\langle Af, g \rangle_{L^2(\Omega)} = \sum_{\kappa \in L^\perp} \widehat{Af}(\kappa) \overline{\widehat{g}(\kappa)} = \sum_{\kappa \in L^\perp} \sigma(\kappa) \widehat{f}(\kappa) \overline{\widehat{g}(\kappa)} = \sum_{\kappa \in L^\perp} \widehat{f}(\kappa) \overline{\overline{\sigma(\kappa)} \widehat{g}(\kappa)},$$

while on the other hand we have

$$\langle Af, g \rangle_{L^2(\Omega)} = \langle f, A^*g \rangle = \sum_{\kappa \in L^\perp} \widehat{f}(\kappa) \overline{\widehat{A^*g}(\kappa)}.$$

We can choose $\widehat{f}(\kappa) = \delta_{\kappa, \kappa_0}$ for any $\kappa_0 \in L^\perp$, where $\delta_{\kappa, \kappa_0}$ is a Kronecker delta, since such an f belongs to $\mathscr{s}(L^\perp)$ and thus can be inverted by the Fourier transform. Hence, we find for all $\kappa \in L^\perp$ that

$$\widehat{A^*g}(\kappa) = \overline{\sigma(\kappa)} \widehat{g}(\kappa),$$

which shows indeed that A^* is a Fourier multiplier with symbol $\overline{\sigma}$.

Of course, $|\sigma(\kappa)| = |\overline{\sigma(\kappa)}|$ for all $\kappa \in L^\perp$ so that $\overline{\sigma}$ also fulfills the growth condition (4.3). Thus, if we apply the first case to the Fourier multiplier A^* with $1 < q' \leq 2 <$

$p' < \infty$, we obtain

$$\|A\|_{\mathcal{L}(L^p(\Omega), L^q(\Omega))} = \|A^*\|_{\mathcal{L}(L^{q'}(\Omega), L^{p'}(\Omega))} \lesssim \sup_{s>0} s \left(\sum_{\substack{\kappa \in L^\perp \\ |\sigma(\kappa)| \geq s}} 1 \right)^{\frac{1}{q'} - \frac{1}{p'}}$$

$$= \sup_{s>0} s \left(\sum_{\substack{\kappa \in L^\perp \\ |\sigma(\kappa)| \geq s}} 1 \right)^{\frac{1}{p} - \frac{1}{q}},$$

where we note again that $\frac{1}{p} - \frac{1}{q} = \frac{1}{q'} - \frac{1}{p'}$. $\qquad\square$

Hörmander proved in [15, Theorem 1.12] that we can only derive L^p-L^q boundedness from a Hörmander-type condition (4.3) in the setting of the Euclidean space $\mathbb{R}^d$ if $p \leq 2 \leq q$. Moreover, it is shown in [15, Theorem 1.1] that the only Fourier multiplier that is bounded from $L^p(\mathbb{R}^d)$ to $L^q(\mathbb{R}^d)$ with $p > q$ is the zero operator. The situation on fundamental domains of lattices is drastically different because we now dispose of *continuous embeddings* between Lebesgue spaces due to the finite measure of the fundamental domain Ω. The proof of this fact is standard, but we include it for the convenience of the reader.

Lemma 4.6. *Let $1 \leq p, q \leq \infty$ with $q \leq p$. Then we have the continuous embedding*
$$L^p(\Omega) \hookrightarrow L^q(\Omega).$$

Proof. Choose $\widetilde{q} \geq 1$ such that $\frac{1}{p} + \frac{1}{\widetilde{q}} = \frac{1}{q}$, namely $\widetilde{q} = \frac{pq}{p-q}$. Then, for every $f \in L^p(\Omega)$ we have
$$\|f\|_{L^q(\Omega)} \leq \|f\|_{L^p(\Omega)} \|1\|_{L^{\widetilde{q}}(\Omega)} = |\Omega|^{\frac{1}{\widetilde{q}}} \|f\|_{L^p(\Omega)} \lesssim \|f\|_{L^p(\Omega)}$$
by (a simple generalisation of) Hölder's inequality. $\qquad\square$

These embeddings of Lebesgue spaces on Ω allow us to prove L^p-L^q boundedness of Fourier multipliers in a broader range for p and q than the case $1 < p \leq 2 \leq q < \infty$ that Hörmander considered. This idea and our proof originate from [7, Corollary 4.9].

Theorem 4.7. *Let $1 < p, q < \infty$, and let $A : C^\infty_{\mathrm{per}}(\Omega) \to C^\infty_{\mathrm{per}}(\Omega)$ be a Fourier multiplier with symbol σ. Assuming that the upper bound in each case is finite, we get the following inequalities, where we interpret this upper bound as in Theorem 4.4 if the exponent is zero.*

(1) *If $1 < p, q \leq 2$, then*

$$\|A\|_{\mathcal{L}(L^p(\Omega), L^q(\Omega))} \lesssim \sup_{s>0} s \left(\sum_{\substack{\kappa \in L^\perp \\ |\sigma(\kappa)| \geq s}} 1 \right)^{\frac{1}{p} - \frac{1}{2}}.$$

(2) *If $2 \leq p, q < \infty$, then*

$$\|A\|_{\mathcal{L}(L^p(\Omega), L^q(\Omega))} \lesssim \sup_{s>0} s \left(\sum_{\substack{\kappa \in L^\perp \\ |\sigma(\kappa)| \geq s}} 1 \right)^{\frac{1}{q'} - \frac{1}{2}}.$$

In particular, A can be extended to a bounded linear operator from $L^p(\Omega)$ to $L^q(\Omega)$.

Proof. Consider first the case $1 < p, q \leq 2$. We assume that $q \neq 2$, because the other case is already treated in the previous theorems. By Lemma 4.6 we get the continuous embedding $L^2(\Omega) \hookrightarrow L^q(\Omega)$ so that

$$\|A\|_{\mathcal{L}(L^p(\Omega), L^q(\Omega))} = \sup_{f \neq 0} \frac{\|Af\|_{L^q(\Omega)}}{\|f\|_{L^p(\Omega)}} \lesssim \sup_{f \neq 0} \frac{\|Af\|_{L^2(\Omega)}}{\|f\|_{L^p(\Omega)}} = \|A\|_{\mathcal{L}(L^p(\Omega), L^2(\Omega))}.$$

Theorem 4.5 yields

$$\|A\|_{\mathcal{L}(L^p(\Omega), L^q(\Omega))} \lesssim \|A\|_{\mathcal{L}(L^p(\Omega), L^2(\Omega))} \lesssim \sup_{s > 0} s \left(\sum_{\substack{\kappa \in L^\perp \\ |\sigma(\kappa)| \geq s}} 1 \right)^{\frac{1}{p} - \frac{1}{2}}.$$

For the other case, namely $2 \leq p, q < \infty$, we have $1 < p', q' \leq 2$ so that the first case gives that

$$\|A\|_{\mathcal{L}(L^p(\Omega), L^q(\Omega))} = \|A^*\|_{\mathcal{L}(L^{q'}(\Omega), L^{p'}(\Omega))} \lesssim \sup_{s > 0} s \left(\sum_{\substack{\kappa \in L^\perp \\ |\sigma(\kappa)| \geq s}} 1 \right)^{\frac{1}{q'} - \frac{1}{2}},$$

where we have used the same properties of the adjoint operator A^* as in the proof of Theorem 4.5. □

Remark 4.8. In fact, we can cover all cases where $1 < p, q < \infty$. The only remaining case is $1 < q \leq 2 \leq p < \infty$. Suppose $A : C_{\text{per}}^\infty(\Omega) \to C_{\text{per}}^\infty(\Omega)$ is a Fourier multiplier with symbol σ. By Lemma 4.6 we have the continuous embeddings $L^p(\Omega) \hookrightarrow L^2(\Omega)$ and $L^2(\Omega) \hookrightarrow L^q(\Omega)$ so that

$$\|A\|_{\mathcal{L}(L^p(\Omega), L^q(\Omega))} = \sup_{f \neq 0} \frac{\|Af\|_{L^q(\Omega)}}{\|f\|_{L^p(\Omega)}} \lesssim \sup_{f \neq 0} \frac{\|Af\|_{L^2(\Omega)}}{\|f\|_{L^2(\Omega)}} = \|\sigma\|_{\ell^\infty(L^\perp)}$$

if σ is bounded because of Theorem 4.4. Thus, every Fourier multiplier on Ω with a bounded symbol can be extended to a bounded linear operator from $L^p(\Omega)$ to $L^q(\Omega)$ if $1 < q \leq 2 \leq p < \infty$. Remark that we could choose other embeddings that would yield other conditions of the type (4.3) on the symbol, but our condition on boundedness is the weakest one, since it can easily be verified that every Hörmander-type condition (4.3) implies that the symbol is bounded.

Remark 4.9. Note that Theorems 4.4 and 4.7 give sufficient conditions for L^p-boundedness. We summarise these results here. Let $A : C_{\text{per}}^\infty(\Omega) \to C_{\text{per}}^\infty(\Omega)$ be a Fourier multiplier with symbol σ.

(1) Consider the case $p = 2$. Then A has a bounded L^2-extension if and only if its symbol σ is bounded. In this case we have $\|A\|_{\mathcal{L}(L^2(\Omega), L^2(\Omega))} = \|\sigma\|_{\ell^\infty(L^\perp)}$.

(2) If $1 < p < 2$, then

$$\|A\|_{\mathcal{L}(L^p(\Omega), L^p(\Omega))} \lesssim \sup_{s > 0} s \left(\sum_{\substack{\kappa \in L^\perp \\ |\sigma(\kappa)| \geq s}} 1 \right)^{\frac{1}{p} - \frac{1}{2}}.$$

(3) If $2 < p < \infty$, then

$$\|A\|_{\mathcal{L}(L^p(\Omega), L^p(\Omega))} \lesssim \sup_{s>0} s \left(\sum_{\substack{\kappa \in L^\perp \\ |\sigma(\kappa)| \geq s}} 1 \right)^{\frac{1}{p'} - \frac{1}{2}} = \sup_{s>0} s \left(\sum_{\substack{\kappa \in L^\perp \\ |\sigma(\kappa)| \geq s}} 1 \right)^{\frac{1}{2} - \frac{1}{p}}.$$

Remark in each case that if the upper bound on the norm of the Fourier multiplier A is finite, then A extends to a bounded linear operator on $L^p(\Omega)$.

5. Application to Sobolev space embeddings

As an application of the L^p-L^q boundedness theorems for Fourier multipliers on a fundamental domain Ω of a lattice L, we present results on Sobolev space embeddings. Moreover, these embeddings will indicate the sharpness of the main inequalities in this paper.

First, we need to construct a Laplacian on Ω in order to define Sobolev spaces on Ω. It is reasonable to construct it as a Fourier multiplier with a symbol that looks the same as in the Euclidean case.

Definition 5.1. We define the Laplacian $\Delta_\Omega : C^\infty_{\mathrm{per}}(\Omega) \to C^\infty_{\mathrm{per}}(\Omega)$ as the Fourier multiplier on Ω with symbol $-4\pi^2 |\kappa|^2$.

Remark 5.2. Notice that this Laplacian actually acts on smooth periodic functions in the way that we would expect. Indeed, if we denote the usual Laplacian by $\Delta : C^\infty_{\mathrm{per}}(\Omega) \to C^\infty_{\mathrm{per}}(\Omega) : f \mapsto \sum_{j=1}^d \partial^2_{x_j} f$, then we find

$$\widehat{\Delta_\Omega f}(\kappa) = \frac{1}{|\Omega|} \int_\Omega f(x) \left(-4\pi |\kappa|^2\right) e^{2\pi i \kappa \cdot x} \, \mathrm{d}x = \frac{1}{|\Omega|} \int_\Omega f(x) \, \Delta e^{2\pi i \kappa \cdot x} \, \mathrm{d}x$$

$$= \frac{1}{|\Omega|} \int_\Omega \Delta f(x) \, e^{2\pi i \kappa \cdot x} \, \mathrm{d}x = \widehat{\Delta f}(\kappa)$$

so that indeed $\Delta_\Omega f = \Delta f$. Note that the boundary terms are indeed zero as remarked in Example 4.2.

In the definition of Sobolev spaces we want to consider operators of the form $(I - \Delta_\Omega)^{s/2}$ with $s \geq 0$, so we need to define how functions act on Fourier multipliers.

Definition 5.3. Let $\varphi : \mathbb{C} \to \mathbb{C}$ be a function, and let A be a Fourier multiplier on Ω with symbol σ. Then we define $\varphi(A)$ as the Fourier multiplier with symbol $\varphi(\sigma(\kappa))$.

Definition 5.4. Let $1 \leq p \leq \infty$ and $s \geq 0$. The Sobolev space $L^p_s(\Omega)$ consists of all (periodic) functions $f \in L^p(\Omega)$ such that

$$(I - \Delta_\Omega)^{s/2} f \in L^p(\Omega),$$

where I is the identity operator. We endow this space with the *Sobolev norm*

$$\|f\|_{L^p_s(\Omega)} = \|(I - \Delta_\Omega)^{s/2} f\|_{L^p(\Omega)}.$$

Remark 5.5. The Sobolev space $L^p_s(\Omega)$ is sometimes referred to as an *inhomogeneous* Sobolev space or *(inhomogeneous) Bessel potential space*.

Remark 5.6. Note that we have the continuous embedding $L^p_s(\Omega) \hookrightarrow L^q_s(\Omega)$ whenever $q \leq p$. This is an immediate consequence of the Lebesgue space embeddings in Lemma 4.6.

Remark 5.7. Note that if A and B are Fourier multipliers with symbols a and b respectively, then their composition AB is the Fourier multiplier with symbol ab. This follows easily from

$$AB = \mathcal{F}_\Omega^{-1}\, a\, \mathcal{F}_\Omega\, \mathcal{F}_\Omega^{-1}\, b\, \mathcal{F}_\Omega = \mathcal{F}_\Omega^{-1}\, ab\, \mathcal{F}_\Omega.$$

Note that this is no longer true in general for pseudo-differential operators; that is, when a and b may also depend on the variable $x \in \Omega$. In particular, this means that the Fourier multiplier $(I - \Delta_\Omega)^{s/2}$ is invertible since it has a strictly positive symbol.

In order to obtain our Sobolev embedding results, we need an estimate on the L^p-L^q operator norm of the Fourier multiplier $(I - \Delta_\Omega)^{s/2}$. The following theorem examines this question for general functions of the negative Laplacian on Ω under some suitable conditions.

Theorem 5.8. *Let $1 < p \le 2 \le q < \infty$, and let φ be any continuous decreasing function on $[0, \infty)$ such that $\varphi(0) > 0$ and $\lim_{u \to \infty} \varphi(u) = 0$. Then*

$$\|\varphi(-\Delta_\Omega)\|_{\mathcal{L}(L^p(\Omega), L^q(\Omega))} \lesssim \sup_{u>0} \varphi(u)\, (1 + u)^{\frac{d}{2}\left(\frac{1}{p} - \frac{1}{q}\right)}.$$

Proof. Recall that Δ_Ω has symbol $-4\pi^2\, |\kappa|^2$ so that $\varphi(-\Delta_\Omega)$ is the Fourier multiplier with symbol $\varphi(4\pi^2\, |\kappa|^2) =: \phi(|\kappa|^2)$. If $p = q = 2$ then the theorem follows directly from Theorem 4.4 because the range of φ is contained in $[0, \varphi(0)]$.

So, suppose $p \ne q$. By Theorem 4.5 it follows that

$$\|\varphi(-\Delta_\Omega)\|_{\mathcal{L}(L^p(\Omega), L^q(\Omega))} \lesssim \sup_{s>0} s \left(\sum_{\substack{\kappa \in L^\perp \\ \phi(|\kappa|^2) \ge s}} 1 \right)^{\frac{1}{p} - \frac{1}{q}} = \sup_{0 < s \le \phi(0)} s \left(\sum_{\substack{\kappa \in L^\perp \\ \phi(|\kappa|^2) \ge s}} 1 \right)^{\frac{1}{p} - \frac{1}{q}}$$

because ϕ is decreasing. Because of the continuity of ϕ, every $s \in (0, \phi(0)]$ can be written as $s = \phi(u)$ for some $u \in [0, \infty)$. Hence we get

$$\|\varphi(-\Delta_\Omega)\|_{\mathcal{L}(L^p(\Omega), L^q(\Omega))} \lesssim \sup_{u \ge 0} \phi(u) \left(\sum_{\substack{\kappa \in L^\perp \\ \phi(|\kappa|^2) \ge \phi(u)}} 1 \right)^{\frac{1}{p} - \frac{1}{q}} = \sup_{u \ge 0} \phi(u) \left(\sum_{\substack{\kappa \in L^\perp \\ |\kappa|^2 \le u}} 1 \right)^{\frac{1}{p} - \frac{1}{q}}$$

by the decreasing character of ϕ.

Our goal now is to find an estimate for the number of points of $L^\perp$ with $|\kappa|^2 < u$. Clearly, since $|\kappa| \le \|\kappa\|_\infty \sqrt{d}$ with $\|\kappa\|_\infty := \max_{1 \le l \le d} |\kappa_l|$, we can enlarge this set by looking at the points of $L^\perp$ for which

$$\kappa \in \left[-\sqrt{\frac{u}{d}}, \sqrt{\frac{u}{d}} \right]^d.$$

Because there is a bijective correspondence between $\mathbb{Z}^d$ and $L^\perp$ given by $\kappa(j) = A^\perp j$ for $j \in \mathbb{Z}^d$, the number of points of $L^\perp$ in $\left[-\sqrt{u/d}, \sqrt{u/d} \right]^d$ is equal to the number of points of $\mathbb{Z}^d$ in $\left(A^\perp \right)^{-1} \left[-\sqrt{u/d}, \sqrt{u/d} \right]^d$. The idea is to consider the point of the latter domain with maximal distance $\rho(u)$ from the origin so that the domain is contained in $[-\rho(u), \rho(u)]^d$ because $\|\kappa\|_\infty \le |\kappa|$. Denote the column vectors of

$(A^\perp)^{-1} = A^{\mathrm{T}}$ by $a_1^{\mathrm{T}}, \ldots, a_d^{\mathrm{T}}$. Then this maximal distance $\rho(u)$ is at most

$$\sum_{j=1}^{d} |a_j^{\mathrm{T}}| \sqrt{\frac{u}{d}},$$

which is the length when traversing every column vector a_j^{T} with 'step length' $\sqrt{u/d}$. Let $C := \sum_{j=1}^{d} |a_j^{\mathrm{T}}|$. Then the number of points of $\mathbb{Z}^d$ in $(A^\perp)^{-1} \left[-\sqrt{u/d}, \sqrt{u/d} \right]^d$ is bounded by

$$(1 + 2\rho(u))^d \le \left(1 + 2C\sqrt{\frac{u}{d}} \right)^d \lesssim (1 + u)^{\frac{d}{2}}.$$

Thus, we obtain the estimate

$$\|\varphi(-\Delta_\Omega)\|_{\mathcal{L}(L^p(\Omega), L^q(\Omega))} \lesssim \sup_{u>0} \phi(u) (1 + u)^{\frac{d}{2}\left(\frac{1}{p} - \frac{1}{q}\right)} = \sup_{u>0} \varphi(4\pi^2 u) (1 + u)^{\frac{d}{2}\left(\frac{1}{p} - \frac{1}{q}\right)}.$$

Finally, the substitution $v = 4\pi^2 u$ and a trivial estimate indeed lead to the desired result. $\qquad\square$

We now dispose of all the tools to establish some Sobolev space embedding results. We will first focus on the case $1 < p \le 2 \le q < \infty$.

Theorem 5.9. *Let $1 < p \le 2 \le q < \infty$. Then we have the continuous embedding*

$$L_s^p(\Omega) \hookrightarrow L_t^q(\Omega)$$

for all $s - t \ge d\left(\frac{1}{p} - \frac{1}{q}\right)$ with $s, t \ge 0$.

Proof. An application of Theorem 5.8 with $\varphi(u) := \frac{1}{(1+u)^{(s-t)/2}}$ gives

$$\|(I - \Delta_\Omega)^{(t-s)/2}\|_{\mathcal{L}(L^p(\Omega), L^q(\Omega))} \lesssim \sup_{u>0} (1 + u)^{\frac{d}{2}\left(\frac{1}{p} - \frac{1}{q}\right) - \frac{s-t}{2}},$$

which is finite precisely when $s - t \ge d\left(\frac{1}{p} - \frac{1}{q}\right)$. In this case we find for every $f \in L_s^p(\Omega)$ that

$$\|f\|_{L_t^q(\Omega)} \lesssim \|(I - \Delta_\Omega)^{(t-s)/2}\|_{\mathcal{L}(L^p(\Omega), L^q(\Omega))} \|(I - \Delta_\Omega)^{s/2} f\|_{L^p(\Omega)} \lesssim \|f\|_{L_s^p(\Omega)},$$

which concludes the proof of this theorem. $\qquad\square$

Remark 5.10. We notice that for $p = q = 2$ we have the continuous embedding $L_s^2(\Omega) \hookrightarrow L_t^2(\Omega)$ for all $s \ge t$.

With a completely similar argument as for Theorem 5.9 with an application of Theorem 4.7 and Remark 4.8 instead of Theorem 4.5 we obtain the following results.

Theorem 5.11. *Let $1 < p, q < \infty$ and $s, t \ge 0$. We have the continuous embedding*

$$L_s^p(\Omega) \hookrightarrow L_t^q(\Omega)$$

if any of the following conditions is satisfied:

(1) $1 < p, q \le 2$ *and* $s - t \ge d\left(\frac{1}{p} - \frac{1}{2}\right)$,
(2) $2 \le p, q < \infty$ *and* $s - t \ge d\left(\frac{1}{q'} - \frac{1}{2}\right)$,
(3) $1 < q \le 2 \le p < \infty$ *and* $s \ge t$.

If we set $t = 0$ in Theorems 5.9 and 5.11 then we get the following corollary about embeddings of Sobolev spaces in Lebesgue spaces.

Corollary 5.12. *Let $1 < p, q < \infty$ and $s, t \geq 0$. We have the continuous embedding*

$$L_s^p(\Omega) \hookrightarrow L^q(\Omega)$$

if any of the following conditions is satisfied:

(1) $1 < p \leq 2 \leq q < \infty$ and $s \geq d\left(\frac{1}{p} - \frac{1}{q}\right)$,

(2) $1 < p, q \leq 2$ and $s \geq d\left(\frac{1}{p} - \frac{1}{2}\right)$,

(3) $2 \leq p, q < \infty$ and $s \geq d\left(\frac{1}{q'} - \frac{1}{2}\right)$,

(4) $1 < q \leq 2 \leq p < \infty$ and $s \geq 0$ is arbitrary.

Remark 5.13. The Paley-inequality, the Hausdorff-Young-Paley inequality, the Hardy-Littlewood inequality and the $L^p(\Omega)$-$L^q(\Omega)$ estimate for Fourier multipliers have sharp exponents. This is known for the torus [20, 21, 26, 14] and one can generalise these arguments to arbitrary fundamental domains of lattices. Alternatively, we can remark that these inequalities are locally the same as the Sobolev embedding, which is known to be sharp. See for example [11] for a proof in the general setting of graded Lie groups.

Acknowledgements

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