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CONTINUOUS QUIVERS OF TYPE A (IV) CONTINUOUS MUTATION AND GEOMETRIC MODELS OF E-CLUSTERS

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Abstract. This if the final paper in the series *Continuous Quivers of Type A*. In this part, we generalize existing geometric models of type A cluster structures for the new **E**-clusters introduced in part (III). We also introduce an isomorphism of cluster theories and a weak equivalence of cluster theories. Examples of both are given. We use these geometric models and isomorphisms of cluster theories to begin classifying continuous type A cluster theories. We also introduce a continuous generalization of mutation. This encompasses mutation and (infinite) sequences of mutation. Then we link continuous mutation to our earlier geometric models. Finally, we introduce the space of mutations which generalizes the exchange graph of a cluster structure, and show that paths in this space are continuous mutations.

Introduction

History. Cluster algebras were first introduced by Fomin and Zelevinksy in [11]. In particle physics they can be used to study scattering diagrams (see work of Golden, Goncharov, Spradlin, Vergud, and Volovicha in [13]). The structure of cluster algebras was first categoricalized independently by two teams in 2006: Buan, Marsh, Reineke, Reiten, and Todorov in [7] and Caldero, Chapaton, and Schiffler in [9]. The first team's construction provided a way to construct a cluster category from the category of finitely generated representations of a Dynkin quiver and the second team's construction related the category to a geometric model. This geometric model on a polygon was extended to the infinity-gon by Holm and Jørgensen and the completed infinity-gon by Baur and Graz in [15] and [5], respectively. In [10], Fomin, Shapiro, and Thurston expanded on [9] and studies the relationship between triangulated surfaces and cluster algebras. We refer the reader to Amiot's [1, Chapter 4.1] for the state of the art at the time of writing. A continuous construction, both categorically and geometrically, was introduced by Igusa and Todorov in [19]. Structures relating to clusters are still actively studied ([2, 26, 22, 23]). In particular, continuous structures were studied by Arkani-Hamed, He, Salvatori, and Thomas in [3] and by Kulkarni, Matherne, Mousavand and the author in [21].

In Part (I) of this series Igusa, Todorov, and the author introduced continuous quivers of type A, denoted $A_{\mathbb{R},S}$, which generalize quivers of type A_n [16]. Results about decomposition of pointwise finite-dimensional representations of such a quiver and the category of finitely-generated representations (denoted $\mathrm{rep}_{\Bbbk}(A_{\mathbb{R},S})$) were proven. In Part (II) the author generalized the Auslander–Reiten quiver for

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finitely-generated representations of an A_n quiver and its bounded derived category to the Auslander–Reiten space for $\text{rep}_{\Bbbk}(A_{\mathbb{R},S})$ and its bounded derived category, denoted $\mathcal{D}^b(A_{\mathbb{R},S})$ [24]. Results were proven about constructions of extensions in $\text{rep}_{\Bbbk}(A_{\mathbb{R},S})$ and distinguished triangles in $\mathcal{D}^b(A_{\mathbb{R},S})$ in relation to the Auslander–Reiten space. In Part (III) Igusa, Todorov, and the author used Parts (I) and (II) to classify which continuous quivers of type A are derived equivalent, construct the new continuous cluster category (denoted $\mathcal{C}(A_{\mathbb{R},S})$) with **E**-clusters (Definition 1.2.6), and generalize the notion of cluster structures to cluster theories [17]. It was shown that each element in an **E**-cluster has none or one choice of mutation and the result of mutation yielded another **E**-cluster. It was also shown that some type A cluster theories (recovered from existing cluster structures) can be embedded in this new cluster theory.

Contributions. The final part of this series begins with a review of the relevant parts of the previous works. Then, we define an isomorphism of cluster theories and a weak equivalence of cluster theories (Definition 1.3.6). In Sections 2.1 and 2.2, we construct geometric models of **E**-cluster theories from part (III) of this series [17]. We obtain an additive category $\mathcal{C}_{\overline{\mathbb{R},S}}$ and a pairwise compatibility condition $\mathbf{N}_{\overline{\mathbb{R},S}}$ (Definition 2.2.14) on its indecomposables that induces the cluster theory $\mathscr{T}_{\mathbf{N}_{\overline{\mathbb{R},S}}}(\mathcal{C}_{\overline{\mathbb{R},S}})$ (Theorem A below). The purpose of the geometric models is to generalize triangulations of polygons and ideal triangulations of the hyperbolic plane, which encode several existing type A cluster structures [9, 19]. In particular, we want a connection to the cluster theory $\mathscr{T}_{\mathbf{E}}(\mathcal{C}(A_{\mathbb{R},S}))$ from [17]. We prove that the geometric models are "correct" in Theorem A and then use them to prove Theorem B.

Theorem A (Theorems 2.1.12 and 2.2.17)**.** *Let $A_{\mathbb{R},S}$ be a continuous quiver of type A. The pairwise compatibility condition $\mathbf{N}_{\overline{\mathbb{R},S}}$ induces the $\mathbf{N}_{\overline{\mathbb{R},S}}$-cluster theory of $\mathcal{C}_{\overline{\mathbb{R},S}}$ and there is an isomorphism of cluster theories $(F,\eta) : \mathscr{T}_{\mathbf{N}_{\overline{\mathbb{R},S}}}(\mathcal{C}_{\overline{\mathbb{R},S}}) \to \mathscr{T}_{\mathbf{E}}(\mathcal{C}(A_{\mathbb{R},S}))$.*

Theorem B (Corollary 2.3.7)**.** *Let $A_{\mathbb{R},S}$ and $A_{\mathbb{R},R}$ be continuous quivers of type A such that one of the following is true: (i) $|S| = |R|$ and $|S| < \infty$, or (ii) S and R are both bounded on exactly one side, or (iii) both S and R are indexed by $\mathbb{Z}$. Then $\mathscr{T}_{\mathbf{E}}(\mathcal{C}(A_{\mathbb{R},S})) \cong \mathscr{T}_{\mathbf{E}}(\mathcal{C}(A_{\mathbb{R},R}))$.*

In Section 2.4 we use the geometric models to show how one may visualize **E**-mutations. Some of these pictures are different from the usual "swap diagonals on a quadrilateral" that appears for triangulations of polygons and ideal triangulations of the hyperbolic plane.

In Section 3 we define a continuous generalization of mutation (Definition 3.1.2) with two key motivations. The first is to unify various ways of describing a sequence of mutations (possibly infinite as in [5]). In Part (III), Igusa, Todorov, and the author show that the indecomposable objects that were projective in $\text{rep}_{\Bbbk}(A_{\mathbb{R}})$ form an **E**-cluster but many of the elements are not **E**-mutable [17, Examples 4.3.2 and 4.4.1]. The second motivation for continuous mutation is to work around this obstruction so that we may mutate the cluster of projectives into the cluster of injectives as one usually does for type A_n. In Section 3.4 we show how mutations and continuous mutations can be interpreted with these geometric models.

We use continuous mutation to define mutation paths (Definition 3.3.2) and generalize the exchange graph of a cluster structure to the space of mutations for a cluster theory (Definition 3.5.2). For a cluster theory $\mathscr{T}_{\mathbf{P}}(\mathcal{C})$, we denote its space of mutations by $\mathbf{P}(\mathcal{C})$.

Theorem C (Propositions 3.5.3, 3.5.5, and 3.5.6)**.** *Let $\mathscr{T}_{\mathbf{P}}(\mathcal{C})$ be a cluster theory and $\mathbf{P}(\mathcal{C})$ its space of mutations. Then $\mathbf{P}(\mathcal{C})$ is a topological space. Moreover, $\mathbf{P}(\mathcal{C})$ is non-Hausdroff if and only if $\mathscr{T}_{\mathbf{P}}(\mathcal{C})$ contains at least one nontrivial $\mathbf{P}$-mutation. Furthermore, each path begins and ends at a $\mathbf{P}$-cluster, up to homotopy.*

In Definition 3.5.7 we define what it means for one cluster to be (strongly) reachable from another. We then show we have achieved the goal of working around the afore-mentioned obstruction of having non-mutable elements.

Theorem D (Theorem 3.5.8)**.** *Consider the $\mathbf{E}$-cluster theory of $\mathcal{C}(A_{\mathbb{R},S})$ where $A_{\mathbb{R},S}$ has the straight descending orientation. The cluster of injectives is strongly reachable from the cluster of projectives.*

Future Work. The exchange graph of an A_n cluster structure is well-understood but the space of mutations for $\mathbf{E}$-clusters poses difficult question due to continuous mutations (Section 3.5). However, preliminary calculations suggest the techniques to prove Theorem D may be generalized to all continuous quivers $A_{\mathbb{R},S}$ where $|S| < \infty$.

It is not yet clear which $\mathbf{E}$-cluster theories for continuous type A quivers are equivalent. Some theories are shown to be isomorphic (see Propositions 2.3.4 and 2.3.5) but the whole classification is still open (Section 2.3).

The next question to ask is, "What about continuous types other than A?" The next steps are continuous types $\widetilde{A}$ and D. Each present their own complications to our constructions. If one performs a similar constructions for continuous type D then the resulting cluster theory should be similar to Igusa and Todorov's construction in [18]. Preliminary work by the C. Paquette, E. Yıldırım, and the author show that continuous representations of type D decompose similarly to representations of a D_n quver. Also, Hanson and the author have proven that representations of continuous type $\widetilde{A}$ decompose analogously to representations of $\widetilde{A}_n$ [14].

Conventions. Here we state conventions used throughout the paper. We have a fixed field $\Bbbk$ throughout. When we say a "skeletally small, KRS, and additive" category we mean a "skeletally small Krull–Remak–Schmidt additive" category.

Let $a < b \in \mathbb{R} \cup \{\pm\infty\}$. By the notation $|a, b|$ we mean an interval subset of $\mathbb{R}$ whose endpoints are a and b. The $|$'s indicate that the inclusion of a or b is not known or not relevant. Additionally, by $\overline{\mathbb{R}}$ we denote the set $\mathbb{R} \cup \{\pm\infty\}$.

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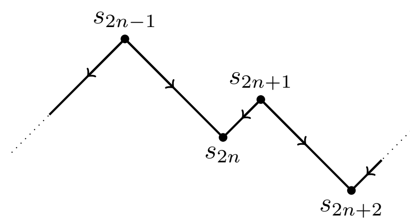


FIGURE 1. An example of a continuous quiver of type A.

1. Prerequisites from This Series

In this section we revisit the most relevant definitions and theorems from parts (I) and (III) in this series, divided into Sections 1.1 and 1.2, respectively. In Section 1.3, state some definitions about general cluster theories and prove a lemma that we use in Section 2.

1.1. **Continuous Quivers of Type A and Their Representations.** In this section we state relevant definitions and theorems from part (I) of this series. In particular, we provide a definition of a continuous quiver of type A, its representations, and its indecomposables. The reader may use the picture in Figure 1 for intuition when reading the definition of a continuous quiver of type A.

Definition 1.1.1. *A continuous quiver of type A, denoted by $A_{\mathbb{R},S}$, is a triple $(\mathbb{R}, S, \preceq)$, where:*

(1) (a) *$S \subset \mathbb{R}$ is a discrete subset, possibly empty, with no accumulation points.*
   (b) *Order on $S \cup \{\pm\infty\}$ is induced by the order of $\mathbb{R}$, and $-\infty < s < +\infty$ for $\forall s \in S$.*
   (c) *Elements of $S \cup \{\pm\infty\}$ are indexed by a subset of $\mathbb{Z} \cup \{\pm\infty\}$ so that s_n denotes the element of $S \cup \{\pm\infty\}$ with index n. The indexing must adhere to the following two conditions:*
      i1 *There exists $s_0 \in S \cup \{\pm\infty\}$.*
      i2 *If $m \leq n \in \mathbb{Z} \cup \{\pm\infty\}$ and $s_m, s_n \in S \cup \{\pm\infty\}$ then for all $p \in \mathbb{Z} \cup \{\pm\infty\}$ such that $m \leq p \leq n$ the element s_p is in $S \cup \{\pm\infty\}$.*

(2) *New partial order $\preceq$ on $\mathbb{R}$, which we call the orientation of $A_{\mathbb{R}}$, is defined as:*
   p1 *The $\preceq$ order between consecutive elements of $S \cup \{\pm\infty\}$ does not change.*
   p2 *Order reverses at each element of S.*
   p3 *If n is even s_n is a sink.*
   p3' *If n is odd s_n is a source.*

Definition 1.1.2. *Let $A_{\mathbb{R},S} = (\mathbb{R}, S \preceq)$ be a continuous quiver of type A. A representation V of $A_{\mathbb{R},S}$ is the following data:*

- *A vector space $V(x)$ for each $x \in \mathbb{R}$.*
- *For every pair $y \preceq x$ in $A_{\mathbb{R},S}$ a linear map $V(x,y) : V(x) \to V(y)$ such that if $z \preceq y \preceq x$ then $V(x,z) = V(y,z) \circ V(x,y)$.*

We say V is pointwise finite-dimensional if $\dim V(x) < \infty$ for all $x \in \mathbb{R}$.

Definition 1.1.3. *Let $A_{\mathbb{R},S}$ be a continuous quiver of type A and $I \subseteq \mathbb{R}$ be an interval. We denote by M_I the representation of $A_{\mathbb{R},S}$ where*

$$M_I(x) = \begin{cases} \Bbbk & x \in I \\ 0 & \text{otherwise} \end{cases} \qquad\qquad M_I(x,y) = \begin{cases} 1_\Bbbk & y \preceq x \in I \\ 0 & \text{otherwise.} \end{cases}$$

We call M_I an <u>*interval indecomposable*</u>.

We require the two following results from [16] (the first recovers a result from [6]).

Theorem 1.1.4 (Theorems 2.3.2 and 2.4.13 in [16])**.** *Let $A_{\mathbb{R},S}$ be a continuous quiver of type A. For any interval $I \subseteq \mathbb{R}$, the representation M_I of $A_{\mathbb{R},S}$ is indecomposable. Any indecomposable pointwise finite-dimensional representation of $A_{\mathbb{R},S}$ is isomorphic to M_I for some interval I. Finally, any pointwise finite-dimensional representation V of $A_{\mathbb{R},S}$ is the direct sum of interval indecomposables.*

Theorem 1.1.5 (Theorem 2.1.6 and Remark 2.4.16 in [16])**.** *Let P be a projective indecomposable in the category of pointwise finite-dimensional representations of a continuous quiver $A_{\mathbb{R},S}$. Then there exists $a \in \mathbb{R} \cup \{\pm\infty\}$ such that P is isomorphic to one of P_a, $P_{(a}$, or $P_{a)}$, given by:*

$$P_a(x) = \begin{cases} \Bbbk & x \preceq a \\ 0 & \text{otherwise} \end{cases} \qquad P_a(x,y) = \begin{cases} 1_\Bbbk & y \preceq x \preceq a \\ 0 & \text{otherwise} \end{cases}$$

$$P_{(a}(x) = \begin{cases} \Bbbk & x \preceq a \text{ and } x > a \text{ in } \mathbb{R} \\ 0 & \text{otherwise} \end{cases} \qquad P_{(a}(x,y) = \begin{cases} 1_\Bbbk & y \preceq x \preceq a \text{ and } x,y > a \\ 0 & \text{otherwise} \end{cases}$$

$$P_{a)}(x) = \begin{cases} \Bbbk & x \preceq a \text{ and } x < a \text{ in } \mathbb{R} \\ 0 & \text{otherwise} \end{cases} \qquad P_{a)}(x,y) = \begin{cases} 1_\Bbbk & y \preceq x \preceq a \text{ and } x,y < a \\ 0 & \text{otherwise} \end{cases}$$

These allow us to define the category of finitely-generated representations:

Definition 1.1.6. *Let $A_{\mathbb{R},S}$ be a continuous quiver of type A. By $\operatorname{rep}_\Bbbk(A_{\mathbb{R},S})$ we denote the full subcategory of pointwise finite-dimensional representations whose objects are finitely generated by the indecomposable projectives in Theorem 1.1.5.*

By [16, Theorem 3.0.1], the category $\operatorname{rep}_\Bbbk(A_{\mathbb{R},S})$ is Krull–Remak–Schmidt with global dimension 1.

1.2. **Cluster Theories and Embeddings.** In this section we state the cluster theories content we need from Part (III) [17]. However, we first need just one result from Part (II).

Proposition 1.2.1 (Proposition 5.1.2 in [24])**.** *Let $A_{\mathbb{R},S}$ be a continuous quiver of type A. Then $\mathcal{D}^b(A_{\mathbb{R},S})$ is a Krull–Remak–Schmidt category. The indecomposable objects are shifts of indecomposables in the category* $\operatorname{rep}_\Bbbk(A_{\mathbb{R},S})$.

Theorem 1.2.2 (Theorem A in [17])**.** *Let $A_{\mathbb{R},S}$ and $A_{\mathbb{R},R}$ be continuous quivers of type A. Then $\mathcal{D}^b(A_{\mathbb{R},S})$ is triangulated equivalent to $\mathcal{D}^b(A_{\mathbb{R},R})$ if and only if (i) S and R are both finite, or (ii) S and R both bounded on exactly one side, or (iii) S and R are both indexed by $\mathbb{Z}$.*

Definition 1.2.3 (Definitions 3.1.1 and 3.1.2 in [17])**.** *Let $\mathcal{D}^b(A_{\mathbb{R},S})^{(2)}$ be the triangulated category whose objects are pairs (X,Y) where X and Y are objects in $\mathcal{D}^b(A_{\mathbb{R},S})$. Hom spaces are given by* $\operatorname{Hom}_{\mathcal{D}^b(A_{\mathbb{R},S})^{(2)}}((X_1,X_2),(Y_1,Y_2)) = \operatorname{Hom}_{\mathcal{D}^b(A_{\mathbb{R},S})}(X_1\oplus$

$X_2, Y_1 \oplus Y_2)$. *The shift functor in* $\mathcal{D}^b(A_{\mathbb{R},S})^{(2)}$ *is given by* $(X,Y)[1] = (X[1], Y[1])$. *The shift of a morphism in* $\mathcal{D}^b(A_{\mathbb{R},S})^{(2)}$ *is given by*

$$\begin{bmatrix} f & g \\ h & k \end{bmatrix}[1] = \begin{bmatrix} f[1] & -g[1] \\ -h[1] & k[1] \end{bmatrix}.$$

Distinguished triangles in $\mathcal{D}^b(A_{\mathbb{R},S})^{(2)}$ *are of the form*

$$(X_1, X_2) \xrightarrow{[f_1, f_2]} (Y_1, Y_2) \xrightarrow{g} (Z_1, Z_2) \xrightarrow{\begin{bmatrix} h_1 \\ h_2 \end{bmatrix}} (X_1[1], X_2[1]$$

where

$$X_1 \oplus X_2 \xrightarrow[{[f_1, f_2]}]{} Y_1 \oplus Y_2 \xrightarrow[g]{} Z_1 \oplus Z_2 \xrightarrow[{\begin{bmatrix} -h_1 \\ h_2 \end{bmatrix}}]{} X_1[1] \oplus X_2[1]$$

is distinguished in $\mathcal{D}^b(A_{\mathbb{R},S})$. *The category* $\mathcal{D}^b(A_{\mathbb{R},S})^{(2)}$ *is triangulated equivalent to* $\mathcal{D}^b(A_{\mathbb{R},S})$.

There is a triangulated self equivalence F *on* $\mathcal{D}^b(A_{\mathbb{R},S})^{(2)}$ *given by* $F(X,Y) = (Y[1], X[1])$, *called the* *almost shift*. *The category* $\mathcal{C}(A_{\mathbb{R},S})$ *is the orbit category of* $\mathcal{D}^b(A_{\mathbb{R},S})^{(2)}$ *under almost-shift and, as in* [19, 12], *is a triangulated category.*

Importantly, the isomorphism classes of indecomposables in $\mathcal{C}(A_{\mathbb{R},S})$ are the same as if we had taken the orbit of $\mathcal{D}^b(A_{\mathbb{R},S})$ by shift. That is, $V \cong V[1]$ for all indecomposables V in $\mathcal{C}(A_{\mathbb{R},S})$. The doubling process ensures $\mathcal{C}(A_{\mathbb{R},S})$ is a triangulated category. Thus, we have distinguished triangles in $\mathcal{C}(A_{\mathbb{R},S})$ of the form $Q_V \to P_V \to V \to Q_V$ where $Q_V \to P_V \to V \to 0$ is the minimal projective resolution of V in $\operatorname{rep}_{\Bbbk}(A_{\mathbb{R},S})$. Furthermore, for indecomposables V and W in $\mathcal{C}(A_{\mathbb{R},S})$, either $\operatorname{Hom}_{\mathcal{C}(A_{\mathbb{R},S})}(V,W) \cong \Bbbk$ or $\operatorname{Hom}_{\mathcal{C}(A_{\mathbb{R},S})}(V,W) = 0$ [17, Proposition 3.1.2]. The authors of [17] then defined g-vectors following Jørgensen and Yakimov in [20].

Definition 1.2.4. *Let* V *be an indecomposable in* $\mathcal{C}(A_{\mathbb{R},S})$. *The* *g-vector* *of* V *is the element* $[P_V] - [Q_V]$ *in* $K_0^{split}(\mathcal{C}(A_{\mathbb{R},S}))$ *where* $Q_V \to P_V \to V \to 0$ *is the minimal projective resolution of* V *in* $\operatorname{rep}_{\Bbbk}(A_{\mathbb{R},S})$.

The Euler form below is used to define **E**-compatibility and **E**-clusters.

Definition 1.2.5. *For* $[A] = \sum_i m_i[A_i]$ *and* $[B] = \sum_j n_j[B_j]$ *in* $K_0^{split}(\mathcal{C}(A_{\mathbb{R},S}))$, *define:*

$$\langle [A], [B] \rangle := \sum_i \sum_j \langle m_i[A_i], n_j[B_j] \rangle = \sum_i \sum_j (m_i \cdot n_j \cdot \dim_{\Bbbk} \operatorname{Hom}_{\mathcal{C}(A_{\mathbb{R},S})}(A_i, B_j)).$$

Definition 1.2.6.

- *Let* V *and* W *be two indecomposables in* $\mathcal{C}(A_{\mathbb{R},S})$ *with g-vectors* $[P_V] - [Q_V]$ *and* $[P_W] - [Q_W]$. *We say* $\{V, W\}$ *is* ***E**-compatible* *if*

$$\langle [P_V] - [Q_V], [P_W] - [Q_W] \rangle \geq 0 \qquad \text{and} \qquad \langle [P_W] - [Q_W], [P_V] - [Q_V] \rangle \geq 0.$$

- *A set T is called $\mathbf{E}$-compatible if for every $V, W \in T$ the set $\{V, W\}$ is $\mathbf{E}$-compatible. If T is maximally $\mathbf{E}$-compatible then we call T an $\mathbf{E}$-cluster.*
- *Let T be an $\mathbf{E}$-cluster and $V \in T$ such that there exists $W \notin T$ where $\{V, W\}$ is not $\mathbf{E}$-compatible but $(T \setminus \{V\}) \cup \{W\}$ is $\mathbf{E}$-compatible. Then we say V is $\mathbf{E}$-mutable. The bijection (see Theorem 1.2.8) $T \to (T \setminus \{V\}) \cup \{W\})$ given by $V \mapsto W$ and $X \mapsto X$ if $X \neq V$ is called an $\mathbf{E}$-mutation or $\mathbf{E}$-mutation at V.*

The following is used in Section 2.2.

Proposition 1.2.7 (Proposition 4.2.4 in [17])**.** *Let V and W be indecomposables in $\mathcal{C}(A_{\mathbb{R},S})$ and let $\widetilde{V}$ and $\widetilde{W}$ be the respective indecomposables in $\operatorname{rep}_{\Bbbk}(A_{\mathbb{R},S})$. Then V and W are not $\mathbf{E}$-compatible if and only if there exists an extension $\widetilde{V} \hookrightarrow E \twoheadrightarrow \widetilde{W}$ or $\widetilde{W} \hookrightarrow E \twoheadrightarrow \widetilde{V}$ such that $E \ncong \widetilde{V} \oplus \widetilde{W}$.*

The words $\mathbf{E}$-cluster and $\mathbf{E}$-mutation are justified with the following theorem.

Theorem 1.2.8 (Theorem 4.3.8 in [17])**.** *Let T be an $\mathbf{E}$-cluster and $V \in T$ be $\mathbf{E}$-mutable with choice W. Then $(T \setminus \{V\}) \cup \{W\}$ is an $\mathbf{E}$-cluster and any other choice W' for V is isomorphic to W.*

The key difference between $\mathbf{E}$-clusters and the usual cluster structures (such as those in [8]) is that not all V in an $\mathbf{E}$-cluster T need be mutable. The authors only require there be none or one choice. This is generalized to the abstract notion of $\mathbf{P}$-compatibility, $\mathbf{P}$-clusters,and $\mathbf{P}$-cluster theories.

1.3. **General Cluster Theories.** In this section we recall general facts about cluster theories from [17]. We then add the notions of a weak equivalence and an isomorphism of cluster theories (Definition 1.3.6) and Lemma 1.3.8, which we need in Section 2.

Definition 1.3.1 (Definition 4.1.1 in [17])**.** *Let $\mathcal{C}$ be a skeletally small, KRS, and additive category and $\mathbf{P}$ be a pairwise compatibility condition on its (isomorphism classes of) indecomposable objects. Suppose that for each (isomorphism class of) indecomposable X in a maximally $\mathbf{P}$-compatible set T there exists none or one (isomorphism class of) indecomposable Y such that $\{X, Y\}$ is not $\mathbf{P}$-compatible but $(T \setminus \{X\}) \cup \{Y\}$ is $\mathbf{P}$-compatible. Then*

- *We call the maximally $\mathbf{P}$-compatible sets $\mathbf{P}$-clusters.*
- *We call a function of the form $\mu : T \to (T \setminus \{X\}) \cup \{Y\}$ such that $\mu Z = Z$ when $Z \neq X$ and $\mu X = Y$ a $\mathbf{P}$-mutation or $\mathbf{P}$-mutation at X.*
- *If there exists a $\mathbf{P}$-mutation $\mu : T \to (T \setminus \{X\}) \cup \{Y\}$, where $X \ncong Y$, we say $X \in T$ is $\mathbf{P}$-mutable.*
- *The subcategory $\mathscr{T}_{\mathbf{P}}(\mathcal{C})$ of $\mathcal{S}et$ whose objects are $\mathbf{P}$-clusters and whose morphisms are generated by $\mathbf{P}$-mutations (and identity functions) is called the $\mathbf{P}$-cluster theory of $\mathcal{C}$.*
- *The functor $I_{\mathbf{P},\mathcal{C}} : \mathscr{T}_{\mathbf{P}}(\mathcal{C}) \to \mathcal{S}et$ is the inclusion of the subcategory.*

Remark 1.3.2. *We note three things immediately about Definition 1.3.1.*

- *The set $(T \setminus \{X\}) \cup \{Y\}$ must be maximally $\mathbf{P}$-compatible, so this does not need to be checked in practice.*
- *Since $\mathbf{P}$-clusters contain isomorphism classes of indecomposables as elements and $\mathcal{C}$ is skeletally small, the category $\mathscr{T}_{\mathbf{P}}(\mathcal{C})$ is small.*

- *Finally, the pairwise compatibility condition* $\mathbf{P}$ *determines the cluster theory. Thus we say that* $\mathbf{P}$ *induces the cluster theory.*

Definition 1.3.3 (Definition 4.1.4 in [17])**.** *Let* $\mathcal{C}$ *be a skeletally small, KRS, and additive category and* $\mathbf{P}$ *a pairwise compatibility condition such that* $\mathbf{P}$ *induces the* $\mathbf{P}$*-cluster theory of* $\mathcal{C}$*. If, for every* $\mathbf{P}$*-cluster* T *and* $X \in T$*, there is a* $\mathbf{P}$*-mutation at* X *then we call* $\mathscr{T}_{\mathbf{P}}(\mathcal{C})$ *the* <u>*tilting* $\mathbf{P}$*-cluster theory*</u>*.*

Remark 1.3.4. *Every cluster structure in the sense of* [7, 8] *yields a tilting cluster theory.*

Definition 1.3.5 (Definition 4.1.9 in [17])**.** *Let* $\mathcal{C}$ *and* $\mathcal{D}$ *be two skeletally small, KRS, and additive categories with respective pairwise compatibility conditions* $\mathbf{P}$ *and* $\mathbf{Q}$*. Suppose these compatibility conditions induce the* $\mathbf{P}$*-cluster theory and* $\mathbf{Q}$*-cluster theory of* $\mathcal{C}$ *and* $\mathcal{D}$*, respectively.*

Suppose there exists a functor $F : \mathscr{T}_{\mathbf{P}}(\mathcal{C}) \to \mathscr{T}_{\mathbf{Q}}(\mathcal{D})$ *such that* F *is an injection on objects and an injection from* $\mathbf{P}$*-mutations to* $\mathbf{Q}$*-mutations. Suppose also there is a natural transformation* $\eta : I_{\mathbf{P},\mathcal{C}} \to I_{\mathbf{Q},\mathcal{D}} \circ F$ *whose morphisms* $\eta_T : I_{\mathbf{P},\mathcal{C}}(T) \to I_{\mathbf{Q},\mathcal{D}} \circ F(T)$ *are all injections. Then we call* $(F, \eta) : \mathscr{T}_{\mathbf{P}}(\mathcal{C}) \to \mathscr{T}_{\mathbf{Q}}(\mathcal{D})$ *an* <u>*embedding of cluster theories*</u>*.*

Recall that an isomorphism of categories $F : \mathcal{C} \to \mathcal{D}$ has an inverse functor $G : \mathcal{D} \to \mathcal{C}$ such that $GF = 1_{\mathcal{C}}$ and $FG = 1_{\mathcal{D}}$; the compositions are *equal* to the identity.

Definition 1.3.6. *Let* $\mathcal{C}$ *and* $\mathcal{D}$ *be a skeletally small, KRS, and additive categories. Let* $\mathbf{P}$ *and* $\mathbf{Q}$ *be pairwise compatibility conditions in* $\mathcal{C}$ *and* $\mathcal{D}$ *such that they, respectively, induce the cluster theories* $\mathscr{T}_{\mathbf{P}}(\mathcal{C})$ *and* $\mathscr{T}_{\mathbf{Q}}(\mathcal{D})$*. A* <u>*weak equivalence of cluster theories*</u> *is an embedding of cluster theories* $(F, \eta) : \mathscr{T}_{\mathbf{P}}(\mathcal{C}) \to \mathscr{T}_{\mathbf{Q}}(\mathcal{D})$ *such that* F *is an isomorphism of categories. We instead say* (F, η) *is an* <u>*isomorphism of cluster theories*</u> *if additionally each* η_T *is an isomorphism.*

Remark 1.3.7. *An isomorphism of categories is ordinarily a very stringent requirement. However, since every cluster theory is a groupoid the only control we have over comparing the "size" of each category is to insist they be identically the same via an isomorphism on objects. And, since clusters in a cluster theory are sets of* isomorphism classes of *objects in* $\mathcal{C}$ *and* $\mathcal{D}$*, respectively, we are already accounting for the type of equivalence with which we are familiar.*

We use the following lemma in Sections 2.1 and 2.2.

Lemma 1.3.8. *Let* $\mathcal{C}$ *and* $\mathcal{D}$ *be skeletally small, KRS, and additive categories. Let* $\mathbf{P}$ *be a pairwise compatibility condition in* $\mathcal{C}$ *such that* $\mathbf{P}$ *induces the cluster theory* $\mathscr{T}_{\mathbf{P}}(\mathcal{C})$ *and let* $\mathbf{Q}$ *be a pairwise compatibility condition in* $\mathcal{D}$*. Suppose*

- *there is a bijection* $\Phi : \mathit{Ind}(\mathcal{C}) \to \mathit{Ind}(\mathcal{D})$ *and*
- *for indecomposables* A *and* B *in* $\mathcal{C}$*,* $\{A, B\}$ *is* $\mathbf{P}$*-compatible if and only if* $\{\Phi(A), \Phi(B)\}$ *is* $\mathbf{Q}$*-compatible.*

Then $\mathbf{Q}$ *induces the* $\mathbf{Q}$*-cluster theory of* $\mathcal{D}$ *and* Φ *induces an isomorphism of cluster theories* $(F, \eta) : \mathscr{T}_{\mathbf{Q}}(\mathcal{D}) \to \mathscr{T}_{\mathbf{P}}(\mathcal{C})$*.*

Proof. Let T be a maximally $\mathbf{Q}$-compatible set of $\mathcal{D}$-indecomposables and let $F(T) = \{\Phi^{-1}(A) \mid A \in T\}$. First we show $F(T)$ is an $\mathbf{P}$-cluster. Suppose $\{X\} \cup F(T)$ is

P-compatible. Then $\{\Phi(X)\} \cup T$ is **Q**-compatible. However, T is maximally **Q**-compatible and so $\Phi(X) \in T$ and $X \in F(T)$.

Suppose there is $A \in T$ and $B \notin T$ such that $(T \setminus \{A\}) \cup \{B\}$ is **Q**-compatible. Then $\{A, B\}$ is not **Q**-compatible since T is maximally **Q**-compatible. So $\{\Phi^{-1}(A), \Phi^{-1}(B)\}$ is not **P**-compatible but $(F(T) \setminus \{\Phi^{-1}(A)\}) \cup \{\Phi^{-1}(B)\}$ is **P**-compatible. This is a **P**-mutation and so $(F(T) \setminus \{\Phi^{-1}(A)\}) \cup \{\Phi^{-1}(B)\}$ is a **P**-cluster. Then by a similar argument to beginning of this proof, $(T \setminus \{A\}) \cup \{B\}$ is maximally **Q**-compatible. Suppose there is $C \notin T$ such that $(T \setminus \{A\}) \cup \{C\}$ is **Q**-compatible. Again, $\{A, C\}$ is not **Q**-compatible and $(T \setminus \{A\}) \cup \{C\}$ is maximally **Q**-compatible. However, this means $\Phi^{-1}(B) = \Phi^{-1}(C)$ and so $C = B$. Therefore, **Q** induces the cluster theory $\mathscr{T}_{\mathbf{Q}}(\mathcal{D})$.

We have already shown F is a functor. Suppose $T \neq T'$. Then $T \cap T' \subsetneq T$ and $T \cap T' \subsetneq T'$ Using Φ^{-1} we see $F(T) \cap F(T') \subsetneq F(T)$ and $F(T) \cap F(T') \subsetneq F(T')$ which means $F(T) \neq F(T')$. Suppose L is a **P**-cluster. Then $\{\Phi(X) \mid X \in L\}$ is a **Q**-cluster by a similar argument to that at the beginning of the proof. Therefore, F is an isomorphism of categories. Finally, for each **Q**-cluster T, we define $\eta_T : T \to F(T)$ by $A \mapsto \Phi^{-1}(A)$. These are isomorphisms, as desired. $\square$

2. Geometric Models of E-clusters

In this section we construct geometric models of **E**-clusters. In Section 2.1 we address the straight descending orientation of $A_{\mathbb{R}}$ and in Section 2.2 we address the rest of the orientations. See [25] for a more general version of this technique. We discuss the classification of cluster theories of continuous type A in Section 2.3.

2.1. Straight orientaion: $A_{\mathbb{R}}$.

In this section we construct a geometric model of the cluster theory $\mathscr{T}_{\mathbf{E}}(\mathcal{C}(A_{\mathbb{R}}))$ when $A_{\mathbb{R}}$ has the straight descending orientation. That is, $A_{\mathbb{R}} = A_{\mathbb{R},S}$ where $S = \emptyset$, $s_0 = -\infty$, and $s_1 = +\infty$. With this orientaiton there is a single frozen indecomposable in every **E**-cluster (Definition 1.2.6): $P_{+\infty} = M_{(-\infty,+\infty)}$. The geometric model of **E**-clusters of this orientation is a generalization of the models in [15, 5]. The generic arc in [5] is similar to the projective $P_{+\infty}$.

Remark 2.1.1. *We will use the words "macroscopic" and "microscopic" in order to differentiate between two types of interactions between endpoints of arcs. We use "macroscopic" when we are talking about interactions involving endpoints $\overline{a}$ and $\overline{b}$ where we know $\overline{a} < \overline{b}$, or vice versa. We also use "macroscopic" when generally talking about points $\overline{a}$, $\overline{b}$, etc., that have distinct real values a,b, etc. We use "microscopic" when we are talking about interactions involving endpoints (a, ε) and (a, ε'), for some $a \in \mathbb{R}$.*

Recall our convention for intervals $|a, b|$ in $\overline{\mathbb{R}}$ on page 3. It is straightforward to check that for $M_{|a,b|}$ and $M_{|c,d|}$ where a, b, c, d are all distinct the set $\{M_{|a,b|}, M_{|c,d|}\}$ is not **E**-compatible if and only if $a < c < b < d$ or $c < a < d < b$. If $a < c < b < d$ we can draw the crossing arcs from a to b and from c to d, for the "macroscopic" perspective, in Figure 2, both of which are always **E**-compatible with $P_{+\infty} = M_{(-\infty,+\infty)}$.

However, on the "microscopic" scale, things are different. Because we allow all types of intervals, we need two possible arc endpoints per $x \in \mathbb{R}$, but only one endpoint at each $-\infty$ and $+\infty$.

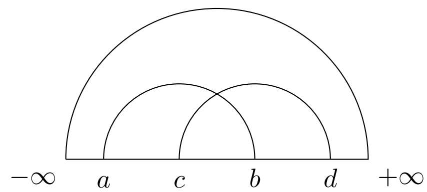


FIGURE 2. Schematic of crossing on the "macroscopic" scale (Remark 2.1.1).

Definition 2.1.2. *Let $A_{\mathbb{R}}$ have the straight descending orientation. In the set $\{-,+\}$ we consider $- < +$ and denote an arbitrary element by ε, ε', etc. We give the set $\mathcal{E} := (\mathbb{R} \times \{-,+\}) \cup \{\pm\infty\}$ the total ordering where*

- $-\infty < (x, \pm) < +\infty$ *for all* $x \in \mathbb{R}$ *and*
- $(x, \varepsilon) < (y, \varepsilon')$ *if either* $x < y$ *or* $x = y$ *and* $\varepsilon < \varepsilon'$.

For ease of notation we write $(-\infty, +)$ for $-\infty$ and $(+\infty, -)$ for $+\infty$. We also write $\overline{a}$ to mean (a, ε) for arbitrary $\varepsilon \in \{-,+\}$.

Definition 2.1.3. *Let $\overline{a}, \overline{b} \in \mathcal{E}$ such that $\overline{a} < \overline{b}$. Then we call the ordered pair $(\overline{a}, \overline{b})$ an arc. Let $\mathcal{A} = \{(\overline{a}, \overline{b}) \in \mathcal{E} \times \mathcal{E} \mid \overline{a} < \overline{b}\}$. We call $\mathcal{A}$ the set of arcs.*

Definition 2.1.4. *Let $A_{\mathbb{R}}$ have the straight descending orientation. Let $M_{|a,b|}$ be the indecomposable in $\mathcal{C}(A_{\mathbb{R}})$ that is the image of the indecomposable with the same name in $\mathrm{rep}_{\Bbbk}(A_{\mathbb{R}})$. We define $\Phi : Ind(\mathcal{C}(A_{\mathbb{R}})) \to \mathcal{A}$ by $\Phi(M_{|a,b|}) = (\overline{a}, \overline{b})$ where*

- $\overline{a} = (a, -)$ *if* $a \in |a, b|$ *and* $\overline{a} = (a, +)$ *if* $a \notin |a, b|$, *and*
- $\overline{b} = (b, -)$ *if* $b \notin |a, b|$ *and* $\overline{b} = (b, +)$ *if* $b \in |a, b|$.

This defines $\Phi : Ind(\mathcal{C}(A_{\mathbb{R}})) \to \mathcal{A}$. Note $\Phi(M_{[a,a]}) = ((a, -), (a, +))$.

Remark 2.1.5. *Immediately from Definition 2.1.4 we have, for $a \neq b$ in $\overline{\mathbb{R}}$,*

$$\Phi(M_{[a,b)}) = ((a,-),(b,-)) \qquad \Phi(M_{(a,b)}) = ((a,+),(b,-))$$
$$\Phi(M_{[a,b]}) = ((a,-),(b,+)) \qquad \Phi(M_{(a,b]}) = ((a,+),(b,+))$$
$$\Phi(M_{[a,a]}) = ((a,-),(a,+)).$$

We now define the crossing function.

Definition 2.1.6. *Let $A_{\mathbb{R}}$ have the straight descending orientation. Define a crossing function $\mathfrak{c} : \mathcal{A} \times \mathcal{A} \to \{0, 1\}$*

$$\mathfrak{c}(\alpha, \beta) = \begin{cases} 1 & \overline{a} < \overline{c} \leq \overline{b} < \overline{d} \text{ or } \overline{c} < \overline{a} \leq \overline{d} < \overline{b} \\ 0 & \text{otherwise}, \end{cases} \tag{2.1.6}$$

where $\alpha = (\overline{a}, \overline{b})$ and $\beta = (\overline{c}, \overline{d})$. If $\alpha \neq \beta$ and $\mathfrak{c}(\alpha, \beta) = 1$ we say α and β cross. Otherwise, we say α and β do not cross.

Remark 2.1.7. *Notice the difference from the usual convention regarding $\mathfrak{c}$ in Definition 2.1.6, equation 2.1.6. If two arcs meet from opposing sides we still consider them to cross. This only happens on the "microscopic" scale. I.e., for $a < b < d$, $(\overline{a}, (b, -))$ and $((b, +), \overline{d})$ do not cross but any other combination of $+$ and $-$ for $\overline{b}$ cross (see Figure 3).*

Proposition 2.1.8. *The map $\Phi : Ind(\mathcal{C}(A_{\mathbb{R}})) \to \mathcal{A}$ given by $\Phi(M_{|a,b|}) = (\overline{a}, \overline{b})$ in Definition 2.1.4 is a bijection.*

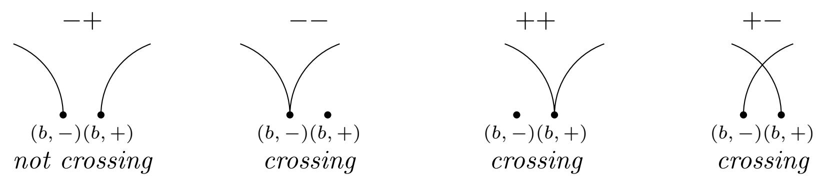


FIGURE 3. Possibilities for crossing and not crossing on the "microscopic" scale (Remark 2.1.1). Here, we are depicting possible crossing for $\alpha = (\overline{a}, \overline{b})$ and $\beta = (\overline{c}, \overline{d})$ when $(b,-) = (c,-)$ and $(b,+) = (c,+)$.

Proof. Suppose $M_{|a,b|} \not\cong M_{|c,d|}$. Then $|a,b| \neq |c,d|$ and so one of the endpoints of the intervals must differ. I.e., even if $a = c$ and $b = d$ then $a \notin |a,b|$ or $a \notin |c,d|$ or $b \notin |a,b|$ or $b \notin |c,d|$. Then endpoints of the arcs associated to $M_{|a,b|}$ and $M_{|c,d|}$ are different. Let $\alpha = (\overline{a}, \overline{b})$ be an arc. Then $\alpha = \Phi(M_{|a,b|})$ where $a \in |a,b|$ if and only if $\bar{a} = (a,-)$ and $b \in |a,b|$ if and only if $\overline{b} = (b,+)$. Therefore Φ is both injective and surjective and so bijective. □

Lemma 2.1.9. *Let $M_{|a,b|} \neq M_{|c,d|}$ be indecomposables in $\mathcal{C}(A_{\mathbb{R}})$, $\alpha = \Phi(M_{|a,b|})$, and $\beta = \Phi(M_{|c,d|})$. Then $\{M_{|a,b|}, M_{|c,d|}\}$ is $\mathbf{E}$-compatible if and only if $\mathfrak{c}(\alpha,\beta) = 0$.*

Proof. Suppose $\{M_{|a,b|}, M_{|c,d|}\}$ is not $\mathbf{E}$-compatible. As we have discussed, if a, b, c, d are all distinct then $a < c < b < d$ or $c < a < d < b$. In either case it follows that α and β cross. Suppose $a = c$. Since the g-vectors of $M_{|a,b|}$ and $M_{|c,d|}$ are not $\mathbf{E}$-compatible, (Definition 1.2.6), we must have $a \notin |a,b|$ and $c \in |c,d|$ or vice versa.

Without loss of generality suppose $a \notin |a,b|$ and $c \in |c,d|$. Then either $d < b$ or if $d = b$ then $d \notin |c,d|$ and $b \in |a,b|$. In either case $\mathfrak{c}(\alpha,\beta) \neq 0$. We can perform a similar argument starting with $b = d$ and see that $\mathfrak{c}(\alpha,\beta) \neq 0$.

Now suppose $(\alpha,\beta) \neq 0$. Then $\overline{a} < \overline{c} \leq \overline{b} < \overline{d}$ or $\overline{c} < \overline{a} \leq \overline{d} < \overline{b}$. Without loss of generality assume the first. Then if $a = c$, $a \in |a,b|$ but $c \notin |c,d|$. Similarly if $b = d$ then $b \notin |a,b|$ and $d \in |c,d|$. In all cases we see that the g-vectors of $M_{|a,b|}$ and $M_{|c,d|}$ are not $\mathbf{E}$-compatible and so the set $\{M_{|a,b|}, M_{|c,d|}\}$ is not $\mathbf{E}$-compatible. □

Definition 2.1.10. *Let $\mathcal{C}_{\overline{\mathbb{R}}}$ be the additive category whose indecomposable objects are arcs, i.e. the elements of $\mathcal{A}$. We define $\mathrm{Hom}_{\mathcal{C}_{\overline{\mathbb{R}}}}(\alpha,\beta)$ and composition $\alpha \xrightarrow{f} \beta \xrightarrow{g} \gamma$, for f and g nonzero, by*

$$\mathrm{Hom}_{\mathcal{C}_{\overline{\mathbb{R}}}}(\alpha,\beta) = \begin{cases} \Bbbk & \mathfrak{c}(\alpha,\beta) = 1 \\ 0 & \textit{otherwise.} \end{cases} \qquad g \circ f = \begin{cases} g \cdot f \in \Bbbk & \alpha = \beta \textit{ or } \beta = \gamma \\ 0 & \textit{otherwise.} \end{cases}$$

Extending bilinearly, we have a skeletally small, KRS, and additive category. We call $\mathcal{C}_{\overline{\mathbb{R}}}$ a <u>category of arcs</u>.

For $\alpha \neq \beta$, we define $\{\alpha,\beta\}$ to be $\mathbf{N}_{\overline{\mathbb{R}}}$-compatible if and only if $\mathfrak{c}(\alpha,\beta) = 0$.

Corollary 2.1.11 (to Lemma 2.1.9)**.** *Let $M_{|a,b|}$ and $M_{|c,d|}$ be in Ind$(\mathcal{C}(A_{\mathbb{R}}))$, $\alpha = \Phi(M_{|a,b|})$, and $\beta = \Phi(M_{|c,d|})$. Then the pair of arcs $\{\alpha,\beta\}$ is $\mathbf{N}_{\overline{\mathbb{R}}}$-compatible if and only if the pair of objects $\{M_{|a,b|}, M_{|c,d|}\}$ is $\mathbf{E}$-compatible.*

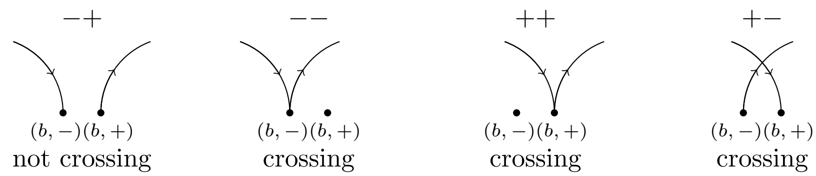


FIGURE 4. Four of the possibilities for crossing and not crossing on the "microscopic" scale (Remark 2.1.1) for all orientations of $A_{\mathbb{R},S}$, using directed arcs. We are again depicting possible crossing for $\alpha = (\overline{a}, \overline{b})$ and $\beta = (\overline{c}, \overline{d})$ when $(b, -) = (c, -)$ and $(b, +) = (c, +)$, similar to Figure 3. In this case $\overline{b}, \overline{c} \in \mathcal{E}^{\uparrow}$ or $\overline{b}, \overline{c} \in \mathcal{E}^{\downarrow}$.

Proof. The Corollary is immediately true if $\alpha = \beta$ or $M_{|a,b|} = M_{|c,d|}$. Suppose $\alpha \neq \beta$. Then $\{\alpha, \beta\}$ is $\mathbf{N}_{\overline{\mathbb{R}}}$-compatible if and only if $\mathfrak{c}(\alpha, \beta) = 0$. Now apply Lemma 2.1.9. $\square$

Theorem 2.1.12. *The pairwise compatibility condition $\mathbf{N}_{\overline{\mathbb{R}}}$ induces the $\mathbf{N}_{\overline{\mathbb{R}}}$-cluster theory of the category of arcs $\mathcal{C}_{\overline{\mathbb{R}}}$ and Φ induces the isomorphism of cluster theories $(F, \eta) : \mathscr{T}_{\mathbf{N}_{\overline{\mathbb{R}}}}(\mathcal{C}_{\overline{\mathbb{R}}}) \to \mathscr{T}_{\mathbf{E}}(\mathcal{C}(A_{\mathbb{R}}))$.*

Proof. We have shown there is a bijection $\Phi : \mathrm{Ind}(\mathcal{C}(A_{\mathbb{R}})) \to \mathrm{Ind}(\mathcal{C}_{\overline{\mathbb{R}}})$ (Proposition 2.1.8) and the pair of objects $\{M_{|a,b|}, M_{|c,d|}\}$ is $\mathbf{E}$-compatible if and only if the pair of corresponding arcs $\{\Phi(M_{|a,b|}), \Phi(M_{|c,d|})\}$ is $\mathbf{N}_{\overline{\mathbb{R}}}$-compatible (Corollary 2.1.11). By Lemma 1.3.8, $\mathbf{N}_{\overline{\mathbb{R}}}$ induces the cluster theory $\mathscr{T}_{\mathbf{N}_{\overline{\mathbb{R}}}}(\mathcal{C}_{\overline{\mathbb{R}}})$ and we have the isomorphism of cluster theories given by $F(T) := \{\Phi^{-1}(\alpha) \mid \alpha \in T\}$ and $\eta_T(\alpha) := \Phi^{-1}(\alpha)$. $\square$

Remark 2.1.13. *If we remove the arc $((-\infty, +), (+\infty, -))$ from our geometric model, we still have a weak equivalence of cluster theories.*

2.2. **Other orientations.** We now construct a geometric model of the cluster theory $\mathscr{T}_{\mathbf{E}}(\mathcal{C}(A_{\mathbb{R},S}))$ for orientations of $A_{\mathbb{R},S}$ other than the straight orientation. Then $S \neq \emptyset$. We take inspiration from the model of representations in [4]. In the case of straight $A_{\mathbb{R},S}$, we think of all the arcs as being directed: originating at the lower point and ending at the upper point. We update our pictures from Definition 2.1.6 to those in Figure 4. When $A_{\mathbb{R},S}$ has the straight descending orientation, these are the only possibilities.

Now suppose $A_{\mathbb{R},S}$ has some orientation other than straight descending or straight ascending. Then $S \neq \emptyset$. We construct a set of endpoints $\mathcal{E}$ as the union of two sets: $\mathcal{E}^{\downarrow}$ and $\mathcal{E}^{\uparrow}$. Recall in the definition of a continuous quiver of type A (Definition 1.1.1) that sinks have even index, s_{2n}, and sources have odd index, s_{2n+1}. Recall also that if the sinks and sources of $A_{\mathbb{R},S}$ are bounded below then $-\infty$ is assigned the next available index below and similarly for $+\infty$ when the sinks and sources are bounded above. When the sinks and sources are not bounded below (above) we assign the index $-\infty$ to $-\infty$, i.e $s_{-\infty} = -\infty$ ($+\infty$ to $+\infty$, i.e. $s_{+\infty} = +\infty$).

Recall $- < +$ in $\{-, +\}$ and we write ε to mean an arbitrary element in $\{-, +\}$, i.e. $\varepsilon = -$ or $\varepsilon = +$.

Definition 2.2.1. *The sets $\mathcal{E}^{\downarrow}$ and $\mathcal{E}^{\uparrow}$ are defined as follows, where each s_n in the notation is a sink or source in $A_{\mathbb{R},S}$ or one of $\pm\infty$, where appropriate.*

$$\begin{aligned}\mathcal{E}^{\downarrow} :=& \Big(\{x \in \mathbb{R} \mid \exists \text{ a sink and source } s_{2m} < x < s_{2m+1}\}\\ &\cup \{\lfloor s_{2n-1}, s_{2n} \rfloor \mid s_{2n} \in S\}\Big) \times \{-,+\}\\ \mathcal{E}^{\uparrow} :=& \Big(\{x \in \mathbb{R} \mid \exists \text{ a source and sink } s_{2m-1} < x < s_{2m}\}\\ &\cup \{\lceil s_{2n}, s_{2n+1} \rceil \mid s_{2n} \in S\}\Big) \times \{-,+\}\end{aligned}$$

Note that elements of the form $\lfloor s_{2n-1}, s_{2n} \rfloor$ and $\lceil s_{2n}, s_{2n+1} \rceil$ do not stand for intervals of $\mathbb{R}$; they are special elements indexed by consecutive pairs of elements of $\overline{S}$. Note also that there are no elements in $\mathcal{E}$ of the form (s_m, ε), where $s_m \in \overline{S}$. We use $[$ and $]$, i.e. $[[s_m, s_{m+1}]]$, when we do not know if the lower element of the pair in $\overline{S}$ has even or odd index. We write $\bar{a}$ to mean an element (a, ε) when ε is unknown or understood from context. In this case a may be in $\mathbb{R}$ or equal to some $[s_m, s_{m+1}]$. We define a total order on $\mathcal{E} := \mathcal{E}^{\downarrow} \cup \mathcal{E}^{\uparrow}$ in the following way.

- *We say $(x, \varepsilon) < (y, \varepsilon')$ if $x < y$ or $x = y$ and $\varepsilon < \varepsilon'$.*
- *We say $([s_m, s_{m+1}], \varepsilon) < ([s_n, s_{n+1}], \varepsilon')$ if $s_m < s_n$ or $s_m = s_n$ and $\varepsilon < \varepsilon'$.*
- *We say $(x, \varepsilon) < ([s_m, s_{m+1}], \varepsilon')$ if $x < s_m$ or if $s_m < x < s_{m+1}$ and $\varepsilon' = +$.*
- *We say $([s_m, s_{m+1}], \varepsilon) < (y, \varepsilon')$ if $s_{m+1} < y$ or if $s_m < y < s_{m+1}$ and $\varepsilon = -$.*

The set $\mathcal{E}^{\downarrow}$ has a maximal (respectively minimal) element if and only if $+\infty$ has an even index (respectively $-\infty$ has an odd index). Dually, $\mathcal{E}^{\uparrow}$ has a maximal (respectively minimal) element if and only if $+\infty$ has an odd index (respectively $-\infty$ has an even index).

Definition 2.2.2. *Let $A_{\mathbb{R},S}$ be a continuous quiver of type A. An arc is a pair $(\overline{a}, \overline{b})$ where $\overline{a}, \overline{b} \in \mathcal{E}$ and $\overline{a} < \overline{b}$. Denote $\mathcal{A} = \{(\overline{a}, \overline{b}) \in \mathcal{E} \times \mathcal{E} \mid \overline{a} < \overline{b}\}$, which we call the set of arcs. For an arc $(\overline{a}, \overline{b}) \in \mathcal{A}$, we call $\overline{a}$ and $\overline{b}$ the endpoints.*

Example 2.2.3. *Let $A_{\mathbb{R},S}$ have sinks $s_{-2} = -2, s_0 = 0, s_2 = 2$ and sources $s_{-1} = -1, s_1 = 1$. Then $-\infty = s_{-3}$ and $+\infty = s_3$. The set $\mathcal{E}^{\downarrow}$ has minimum element $(\lfloor -\infty, -2 \rfloor, -)$ and $\mathcal{E}^{\uparrow}$ has maximum element $(\lceil 2, +\infty \rceil, +)$. In Figure 5 we draw $\mathcal{E}^{\downarrow}$ and $\mathcal{E}^{\uparrow}$ using piece-wise linear curves in the plane and draw arcs on the "macroscopic" scale (Remark 2.1.1) as lines between two points in $\mathcal{E}$. For example, let $\alpha = (\overline{a}, \overline{b})$ where $s_{-2} < a < s_{-1}$ and $s_1 < b < s_2$. Since $\overline{a} < \overline{b}$, we draw α oriented from $\overline{a}$ to $\overline{b}$.*

Recall that if the sinks and sources of $A_{\mathbb{R},S}$ are unbounded below (respectively above) then no indecomposable in $\text{rep}_{\Bbbk}(A_{\mathbb{R},S})$ may have $-\infty$ as a lower endpoint (respectively $+\infty$ as an upper endpoint) of its support. Thus if we have $M_{|a,b|}$ and $a = -\infty$ (respectively $b = +\infty$) then we know the sinks and sources of $A_{\mathbb{R},S}$ are bounded below (respectively above).

In the definition below we us $\overline{x}$ and $\overline{y}$ instead of $\overline{a}$ and $\overline{b}$ because the value x of (x, ε) in $\mathcal{E}$ might not be a real number; i.e. $x = [s_m, s_{m+1}]$ for some sink or source s_m and source or sink s_{m+1}.

Definition 2.2.4. *We now define $\Phi : Ind(\mathcal{C}(A_{\mathbb{R},S})) \to \mathcal{A}$. Let $M_{|a,b|}$ be an indecomposable in $\mathcal{C}(A_{\mathbb{R},S})$. We define $\Phi(M_{|a,b|}) := (\overline{x}, \overline{y})$ where $\overline{x}$ and $\overline{y}$ are define below. We define $\overline{x}$ first:*

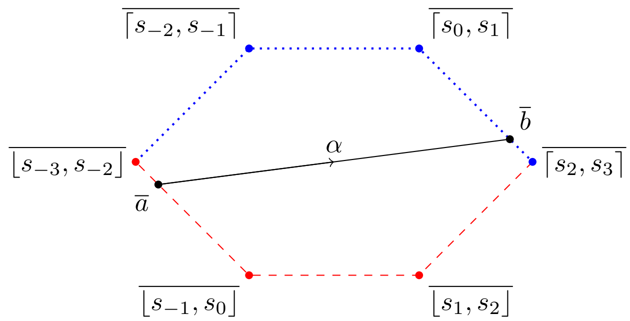


FIGURE 5. We now set the conventions for drawing $\mathcal{E}$ and arcs for $A_{\mathbb{R},S}$, which is defined in Example 2.2.3. We draw on the "macroscopic" scale only (Remark 2.1.1). The blue, dotted lines depict $\mathcal{E}^{\uparrow}$. The red, dashed lines depict $\mathcal{E}^{\downarrow}$. We have marked the points in $\mathcal{E}$ that are a sink and source pair in order to give the reader their bearings in the figure. We draw arcs and their endpoints in solid black with an arrow mid-arc to depict the orientation. The arc α is drawn from $\overline{a}$ to $\overline{b}$ because $\overline{a} < \overline{\lfloor s_{-1}, s_0 \rfloor} < \overline{\lceil s_0, s_1 \rceil} < \overline{b}$.

- *If $a \in \mathbb{R}$ is neither a sink nor a source then $\overline{x} = (a, \varepsilon)$ where $\varepsilon = -$ if and only if $a \in |a, b|$.*
- *If $a = -\infty = s_m$ then $\overline{x} = ([s_m, s_{m+1}], -)$.*
- *If $-\infty < a = s_m$ and $a \in |a, b|$ then $\overline{x} = ([s_m, s_{m+1}], -)$.*
- *If $-\infty < a = s_m$ and $a \notin |a, b|$ then $\overline{x} = ([s_{m-1}, s_m], +)$.*

Now, $\overline{y}$.

- *If $b \in \mathbb{R}$ is neither a sink nor a source then $\overline{y} = (b, \varepsilon)$ where $\varepsilon = +$ if and only if $b \in |a, b|$.*
- *If $b = +\infty = s_n$ then $\overline{y} = ([s_{n-1}, s_n], +)$.*
- *If $+\infty > b = s_n$ and $b \in |a, b|$ then $\overline{y} = ([s_{n-1}, s_n], +)$.*
- *If $+\infty > b = s_n$ and $b \notin |a, b|$ then $\overline{y} = ([s_n, s_{n+1}], -)$.*

Proposition 2.2.5. *The function $\Phi : Ind(\mathcal{C}(A_{\mathbb{R},S})) \to \mathcal{A}$ defined as $\Phi(M_{|a,b|}) = (\overline{x}, \overline{y})$ in Definition 2.2.4 is a bijection.*

Proof. Let $M_{|a,b|} \ncong M_{|c,d|}$ be indecomposables in $\mathcal{C}(A_{\mathbb{R},S})$. Let $(\overline{x}, \overline{y}) = \Phi(M_{|a,b|})$ and $(\overline{z}, \overline{w}) = \Phi(M_{|c,d|})$. Using the definition it is straightforward to check that if $a \neq c$ or $b \neq d$ then $(\overline{x}, \overline{y}) \neq (\overline{z}, \overline{w})$. Now suppose $a = c$ and $b = d$. Since $M_{|a,b|} \ncong M_{|c,d|}$ the endpoints of $|a, b|$ and $|c, d|$ must differ by at least one point. By symmetry and possibly reversing the roles of $M_{|a,b|}$ and $M_{|c,d|}$, assume $a \in |a, b|$ and $c \notin |a, b|$. Then $\overline{x} \neq \overline{z}$ and so $(\overline{x}, \overline{y}) \neq (\overline{z}, \overline{w})$. Thus, Φ is injective.

Let $\alpha = (\overline{x}, \overline{y})$ be an arc in $\mathcal{A}$. We now construct an interval $|a, b|$ such that $\Phi(M_{|a,b|}) = \alpha$. If $x \in \mathbb{R}$ then x is neither a sink nor a source and we let $a = x$ and $a \in |a, b|$ if and only if $\varepsilon = -$. If $y \in \mathbb{R}$ then y is neither a sink nor a source and we let $b = y$ and $b \in |a, b|$ if and only if $\varepsilon' = +$.

Suppose $x = [s_m, s_{m+1}]$. If $\varepsilon = +$, then either $y \in \mathbb{R}$ is greater than s_{m+1} or $y = [s_n, s_{n+1}]$ where $n > m$. In this case we let $a = s_{m+1}$ and $a \notin |a, b|$. If $\varepsilon = -$, then either $y \in \mathbb{R}$ is greater than s_m or $y = [s_n, s_{n+1}]$ where $n \geq m$; if $n = m$ then $\overline{y} = ([s_{m-1}, s_m], +)$. In this case if $s_m = -\infty$ then we let $a = -\infty$ and note

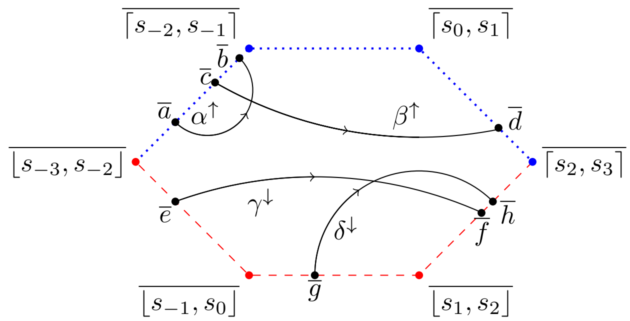


FIGURE 6. We continue the conventions in Figure 5 for $A_{\mathbb{R},S}$ defined in Example 2.2.3. In this figure we depict arcs in Example 2.2.7. The arcs marked with $\uparrow$ have endpoints in $\mathcal{E}^\uparrow$ and the arcs marked with $\downarrow$ have endpoints in $\mathcal{E}^\downarrow$. We see that $\gamma^\downarrow$ and $delta^\downarrow$) cross and that $\alpha^\uparrow$ and $\beta^\uparrow$ cross. However, $\gamma^\downarrow$ does not cross $\beta^\uparrow$, which is why we draw $\gamma^\downarrow$ and $\beta^\uparrow$ so 'flat'.

$a \notin |a, b|$. If $s_m > -\infty$ then we let $a = s_m$ and $a \in |a, b|$. We perform the dual constructions for $\overline{y}$ and b as well.

Now we have $a \leq b$ and the requirements for $|a, b|$ to contain either a or b. We need to check the case where $a = b$ to ensure that, in this case, $a, b \in |a, b|$ by our construction. If $a = b \in \mathbb{R}$ is neither a sink nor a source then $\alpha = \{(a, -), (a, +)\}$ and so $|a, b| = [a, a]$. If $a = b \in \mathbb{R}$ is a sink or a source let $s_n = a = b$. Then we have that $|a, b| = \{s_n\}$. Thus Φ is surjective and so bijective. □

Our rules for crossing are more complicated than before. The cases are: straightforward (Definition 2.2.6), the "macroscopic" (Definition 2.2.8), and the "microscopic" (Definition 2.2.11). See Remark 2.1.1 for more on "macroscopic" versus "microscopic."

Definition 2.2.6 (straightforward case)**.** *Let α and β be arcs with endpoints in $\mathcal{E}$.*

- *If both α and β have endpoints in $\mathcal{E}^\downarrow$ then we follow Definition 2.1.6.*
- *If both α and β have endpoints in $\mathcal{E}^\uparrow$ then we follow Definition 2.1.6.*
- *If α has endpoints in $\mathcal{E}^\downarrow$ and β has endpoints in $\mathcal{E}^\uparrow$ then we say α and β do not cross.*

Example 2.2.7 (Example of Definition 2.2.6)**.** *Let $A_{\mathbb{R},S}$ have sinks $s_{-2} = -2, s_0 = 0, s_2 = 2$ and sources $s_{-1} = -1, s_1 = 1$ with $-\infty = s_{-3}$ and $+\infty = s_3$ as in Example 2.2.3. Let $\overline{a} < \overline{c} < \overline{b} < \overline{d}$ be in $\mathcal{E}^\uparrow$ and let $\overline{e} < \overline{g} < \overline{f} < \overline{h}$ be in $\mathcal{E}^\downarrow$. Next, let*

$$\alpha^\uparrow = (\overline{a}, \overline{b}) \qquad \beta^\uparrow = (\overline{c}, \overline{d}) \qquad \gamma^\downarrow = (\overline{e}, \overline{f}) \qquad \delta^\downarrow = (\overline{g}, \overline{h}).$$

Then $\alpha^\uparrow$ and $\beta^\uparrow$ cross, $\gamma^\downarrow$ and $\delta^\downarrow$ cross, but arcs with $\uparrow$ do not cross arcs with $\downarrow$. These crossings and not crossings are depicted in Figure 6.

Definition 2.2.8 ("macroscopic" case)**.** *Let α and β be arcs in $\mathcal{A}$. Suppose $\alpha = (\overline{a}, \overline{b})$ has one endpoint in $\mathcal{E}^\downarrow$ and the other in $\mathcal{E}^\uparrow$. Let $\beta = (\overline{c}, \overline{d})$. We assume $\overline{a}$, $\overline{b}$, $\overline{c}$, and $\overline{d}$ are all distinct.*

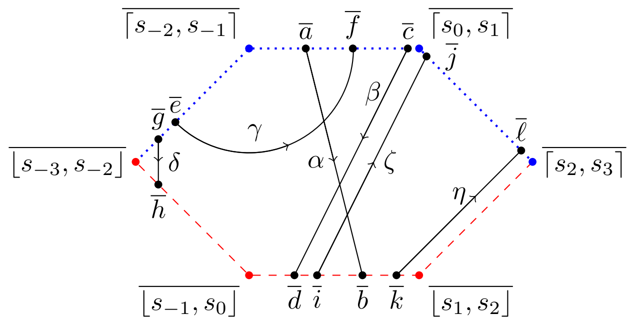


FIGURE 7. We continue the conventions in Figure 5 for $A_{\mathbb{R},S}$ defined in Example 2.2.3 and consider Example 2.2.10. We have δ and η, examples of arcs with endpoints in both $\mathcal{E}^{\downarrow}$ and $\mathcal{E}^{\uparrow}$ that do not cross α. We see δ goes from $\mathcal{E}^{\uparrow}$ to $\mathcal{E}^{\downarrow}$ and η goes from $\mathcal{E}^{\downarrow}$ to $\mathcal{E}^{\uparrow}$. We have β and ζ, which do cross α (Definition 2.2.8, (3) and (4), respectively). We see β goes from $\mathcal{E}^{\uparrow}$ to $\mathcal{E}^{\downarrow}$ and ζ goes from $\mathcal{E}^{\downarrow}$ to $\mathcal{E}^{\uparrow}$. Finally, we have γ, with both endpoints in $\mathcal{E}^{\uparrow}$, which crosses α but not β (Definition 2.2.8(2)).

(1) *Suppose $\overline{c}, \overline{d} \in \mathcal{E}^{\downarrow}$. If $\overline{c} < \overline{x} < \overline{d}$, where $\{\overline{x}\} = (\{\overline{a}, \overline{b}\} \cap \mathcal{E}^{\downarrow})$, we say α and β cross.*
(2) *Suppose $\overline{c}, \overline{d} \in \mathcal{E}^{\uparrow}$. If $\overline{c} < \overline{x} < \overline{d}$, where $\{\overline{x}\} = (\{\overline{a}, \overline{b}\} \cap \mathcal{E}^{\uparrow})$, we say α and β cross.*
(3) *Suppose either (i) $\overline{a}, \overline{c} \in \mathcal{E}^{\downarrow}$ and $\overline{b}, \overline{d} \in \mathcal{E}^{\uparrow}$ or (ii) $\overline{a}, \overline{c} \in \mathcal{E}^{\uparrow}$ and $\overline{b}, \overline{d} \in \mathcal{E}^{\downarrow}$. If $\overline{a} < \overline{c} < \overline{d} < \overline{b}$ or $\overline{c} < \overline{a} < \overline{b} < \overline{d}$ we say α and β cross.*
(4) *Suppose either (i) $\overline{a}, \overline{d} \in \mathcal{E}^{\downarrow}$ and $\overline{b}, \overline{c} \in \mathcal{E}^{\uparrow}$ or (ii) $\overline{a}, \overline{d} \in \mathcal{E}^{\uparrow}$ and $\overline{b}, \overline{c} \in \mathcal{E}^{\downarrow}$. If $\overline{a} < \overline{d}$ and $\overline{c} < \overline{b}$ we say α and β cross.*

Remark 2.2.9. *Notice that $\overline{a} < \overline{d}$ and $\overline{c} < \overline{b}$ are enough for the straight orientation or Definition 2.2.8(4) but not enough for Definition 2.2.8(3). In this case, if $\overline{c} < \overline{a} < \overline{d} < \overline{b}$, for example, then α and β* do not *cross. However, if α and β cross, we must have $\overline{a} < \overline{d}$ and $\overline{c} < \overline{b}$. See Example 2.2.10 and Figure 7.*

Example 2.2.10 (Example of Definition 2.2.8)**.** *Let $A_{\mathbb{R},S}$ have sinks $s_{-2} = -2, s_0 = 0, s_2 = 2$ and sources $s_{-1} = -1, s_1 = 1$ with $-\infty = s_{-3}$ and $+\infty = s_3$ as in Example 2.2.3. Let $\overline{a}, \overline{c}, \overline{e}, \overline{f}, \overline{g}\overline{j}, \overline{l} \in \mathcal{E}^{\uparrow}$ and $\overline{b}, \overline{d}, \overline{h}, \overline{i}, \overline{k} \in \mathcal{E}^{\downarrow}$ such that*

$$\overline{g} < \overline{e} < (\lfloor s_{-3}, s_{-2} \rfloor, +) < \overline{h} < (\lceil s_{-2}, s_{-1} \rceil, +) < \overline{a} < \overline{f} < \overline{c}$$

and

$$\overline{c} < (\lfloor s_{-1}, s_0 \rfloor, +) < \overline{d} < \overline{i} < \overline{b} < \overline{k} < (\lceil s_0, s_1 \rceil, +) < \overline{j} < \overline{l}.$$

Let

$$\alpha = (\overline{a}, \overline{b}) \qquad \beta = (\overline{c}, \overline{d}) \qquad \gamma = (\overline{e}, \overline{f} \qquad \delta = (\overline{g}, \overline{h}) \qquad \zeta = (\overline{i}, \overline{j}) \qquad \eta = (\overline{k}, \overline{\ell}).$$

Then, according to Definition 2.2.8, α crosses β, γ, and ζ, but not δ or η. See Figure 7 for a depiction and explanation of these crossings and not crossings.

The only case not covered by Definitions 2.2.6 and 2.2.8 is when two arcs share an endpoint.

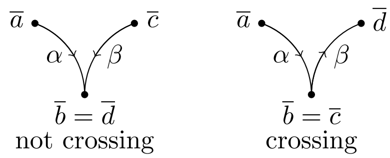


FIGURE 8. A depiction of Definition 2.2.11, which is on the "microscopic" scale (Remark 2.1.1). Case $\overline{a}=\overline{d}$ is equivalent to case $\overline{b}=\overline{c}$. Case $\overline{a}=\overline{c}$ is equivalent to case $\overline{b}=\overline{d}$.

Definition 2.2.11 ("microscopic" case)**.** *Let $\alpha=(\overline{a},\overline{b})$ and $\beta=(\overline{c},\overline{d})$ be in $\mathcal{A}$. We have four cases: $\overline{a}=\overline{c}$, $\overline{a}=\overline{d}$, $\overline{b}=\overline{c}$, and $\overline{b}=\overline{d}$. (If two equalities hold at once we have $\alpha=\beta$.)*

- *If $\overline{a}=\overline{d}$ or $\overline{b}=\overline{c}$ then we say α and β cross.*
- *If $\overline{a}=\overline{c}$ or $\overline{b}=\overline{d}$ then we say α and β do not cross.*

See Figure 8 for a visual depiction of this rule.

We now define the crossing function.

Definition 2.2.12. *Define the crossing function $\mathfrak{c}:\mathcal{A}\times\mathcal{A}\to\{0,1\}$ by*

$$\mathfrak{c}(\alpha,\beta)=\begin{cases}1 & (\alpha=\beta) \text{ or } (\alpha \text{ and } \beta \text{ cross by Definitions 2.2.6, 2.2.8, and 2.2.11})\\ 0 & (\alpha\neq\beta) \text{ and } (\alpha \text{ and } \beta \text{ do not cross by Definitions 2.2.6, 2.2.8, and 2.2.11}).\end{cases}$$

For $\alpha\neq\beta$, if $\mathfrak{c}(\alpha,\beta)=1$ we say α and β cross. Otherwise, we say α and β do not cross.

We are now ready to prove the following lemma.

Lemma 2.2.13. *Let $M_{|a,b|}\neq M_{|c,d|}$ be indecomposables in $\mathcal{C}(A_{\mathbb{R},S})$. Then $\{M_{|a,b|},M_{|c,d|}\}$ is $\mathbf{E}$-compatible if and only if $\mathfrak{c}(\Phi(M_{|a,b|}),\Phi(M_{|c,d|}))=0$.*

Proof. Setup. Let $\alpha=(\overline{x},\overline{y})=\Phi(M_{|a,b|})$ and $\beta=(\overline{z},\overline{w})=\Phi(M_{|c,d|})$ as in Definition 2.2.4 and note that by Proposition 2.2.5, $\alpha\neq\beta$. We note that Definitions 2.2.6, 2.2.8, and 2.2.11 cover all possible combinations of endpoints for α and β. We show that if $\mathfrak{c}(\alpha,\beta)=1$ then $\{M_{|a,b|},M_{|c,d|}\}$ is not $\mathbf{E}$-compatible and if $\mathfrak{c}(\alpha,\beta)=0$ then $\{M_{|a,b|},M_{|c,d|}\}$ is $\mathbf{E}$-compatible. We follow the order in which the definitions were stated.

Definition 2.2.6. If the endpoints of α and β are all contained in $\mathcal{E}^{\downarrow}$ or all contained in $\mathcal{E}^{\uparrow}$ then our if and only if statement follows from arguments similar to those in the proof of Lemma 2.1.9. Without loss of generality, suppose α has endpoints in $\mathcal{E}^{\downarrow}$ and β has endpoints in $\mathcal{E}^{\uparrow}$. Then $\mathfrak{c}(\alpha,\beta)=0$. If $a=s_m$ then a is a source and if $b=s_n$ then b is a sink. Dual statements for c and d are true as well. Using Definition 1.2.6 and Proposition 1.2.7 we see that $\{M_{|a,b|},M_{|c,d|}\}$ is $\mathbf{E}$-compatible.

Definition 2.2.8. Suppose α has both endpoints in $\mathcal{E}^{\downarrow}$ and β has one endpoint each in $\mathcal{E}^{\downarrow}$ and $\mathcal{E}^{\uparrow}$. For now we assume all four endpoints of α and β are distinct. Suppose $\overline{x}<\overline{y}$, $\overline{z}\in\mathcal{E}^{\downarrow}$, and $\overline{w}\in\mathcal{E}^{\uparrow}$. If $\overline{x}<\overline{z}<\overline{y}$ then $\mathfrak{c}(\alpha,\beta)=1$ and one verifies there exists a distinguished triangle

$$M_{|a,b|}\to M_{|a,d|}\oplus M_{|c,b|}\to M_{|c,d|}\to$$

in $\mathcal{C}(A_{\mathbb{R},S})$. By Proposition 1.2.7, $\{M_{|a,b|}, M_{|c,d|}\}$ is not **E**-compatible. If $\overline{x} < \overline{z} < \overline{y}$ then $\mathfrak{c}(\alpha,\beta) = 1$ and one verifies there exists a distinguished triangle

$$M_{|c,d|} \to M_{|c,b|} \oplus M_{|a,d|} \to M_{|a,b|} \to$$

in $\mathcal{C}(A_{\mathbb{R},S})$ and by the same proposition $\{M_{|a,b|}, M_{|c,d|}\}$ is not **E**-compatible. If $\overline{z} < \overline{x}$ or $\overline{y} < \overline{z}$ we know $\mathfrak{c}(\alpha,\beta) = 0$ and it is straightforward to check that the g-vectors of $M_{|a,b|}$ and $M_{|c,d|}$ are **E**-compatible. Thus, $\{M_{|a,b|}, M_{|c,d|}\}$ is **E**-compatible.

Now we check when α and β each have one endpoint in $\mathcal{E}^{\downarrow}$ and the other in $\mathcal{E}^{\uparrow}$. Suppose $\mathfrak{c}(\alpha,\beta) = 1$. For Definition 2.2.8(3), and without loss of generality, let $\overline{x}, \overline{z} \in \mathcal{E}^{\downarrow}$ and $\overline{y}, \overline{w} \in \mathcal{E}^{\uparrow}$. Up to symmetry, we have $\overline{x} < \overline{z} < \overline{y} < \overline{w}$ and so $a < c < b < d$ in $\mathbb{R}$. One then verifies there exists a distinguished triangle

$$M_{|a,b|} \to M_{|a,d|} \oplus M_{|c,b|} \to M_{|c,d|} \to$$

in $\mathcal{C}(A_{\mathbb{R},S})$. Again using Proposition 1.2.7 we see $\{M_{|a,b|}, M_{|c,d|}\}$ is not **E**-compatible.

For Definition 2.2.8(4), and without loss of generality, let $\overline{x}, \overline{w} \in \mathcal{E}^{\downarrow}$ and $\overline{y}, \overline{z} \in \mathcal{E}^{\uparrow}$. Then $\overline{x} < \overline{w}$ and $\overline{z} < \overline{y}$ and one verifies there exists a distinguished triangle

$$M_{|c,d|} \to M_{|c,b|} \oplus M_{|a,d} \to M_{|a,b|} \to$$

in $\mathcal{C}(A_{\mathbb{R},S})$. Again, $\{M_{|a,b|}, M_{|c,d|}\}$ is not **E**-compatible.

Now suppose $\mathfrak{c}(\alpha,\beta) = 0$. If $\overline{x} > \overline{w}$ or $\overline{z} > \overline{y}$, one verifies the g-vectors of $M_{|a,b|}$ and $M_{|c,d|}$ are **E**-compatible. If $\overline{x} < \overline{w}$ and $\overline{z} < \overline{y}$ then, up to symmetry $\overline{x}, \overline{z} \in \mathcal{E}^{\downarrow}$ and $\overline{y}, \overline{w} \in \mathcal{E}^{\uparrow}$. This means that $\overline{x} < \overline{z}$ and $\overline{yw}$ or that $\overline{z} < \overline{x}$ and $\overline{w} < \overline{y}$. Again one my check the g-vectors to see that $M_{|a,b|}$ and $M_{|c,d|}$ are **E**-compatible.

Definition 2.2.11. Now we assume α and β share an endpoint.

If $\overline{x} = \overline{z}$, then a straightforward calculation shows the g-vectors of $M_{|a,b|}$ and $M_{|c,d|}$ are **E**-compatible. Symmetrically, if $\overline{y} = \overline{w}$, then $\{M_{|a,b|}, M_{|c,d|}\}$ is **E**-compatible.

Next suppose $\overline{x} = \overline{w} = (e, \varepsilon)$, for $e \in \mathbb{R}$. Then $M_{|a,b|} = M_{|e,b|}$ and $M_{|c,d|} = M_{|c,e|}$. In particular, $e \in |e,b|$ if and only if $e \notin |c,e|$. Then one verifies the following is a distinguished triangle in $\mathcal{C}(A_{\mathbb{R}})$:

$$M_{|c,d|} \to M_{|c,b|} \to M_{|a,b|} \to .$$

By Proposition 1.2.7 again we see $\{M_{|a,b|}, M_{|c,d|}\}$ is not **E**-compatible.

Finally, suppose $\overline{x} = \overline{w} = ([s_n, s_{n+1}], \varepsilon)$ and note that $\overline{y} \neq \overline{z}$. If $\varepsilon = -$ then $|a,b| = [s_n, b|$ and $|c,d| = |c, s_n)$. If $\varepsilon = +$ then $|a,b| = (s_{n+1}, b|$ and $|c,d| = |c, s_{n+1}]$. In either case, one verifies we have the following distinguished triangle in $\mathcal{C}(A_{\mathbb{R},S})$:

$$M_{|c,d|} \to M_{|c,b|} \to M_{|a,b|} \to .$$

By Proposition 1.2.7 again we see $\{M_{|a,b|}, M_{|c,d|}\}$ is not **E**-compatible.

Conclusion. For each of Definitions 2.2.6, 2.2.8, and 2.2.11 we have shown (i) if $\mathfrak{c}(\alpha,\beta) = 1$ then $\{M_{|a,b|}, M_{|c,d|}\}$ is not **E**-compatible and (ii) if $\mathfrak{c}(\alpha,\beta) = 0$ then $\{M_{|a,b|}, M_{|c,d|}\}$ is **E**-compatible. □

Definition 2.2.14. *Let $\mathcal{C}_{\overline{\mathbb{R},S}}$ be an additive category whose indecomposable objects are $\mathcal{A}$. Define Hom spaces and composition of morphisms in the same way as in Definition 2.1.10. This also yields a skeletally small, KRS, and additive category. We say $\{\alpha,\beta\} \subset \mathcal{A}$ is $\mathbf{N}_{\overline{\mathbb{R},S}}$-compatible if and only if $\alpha = \beta$ or $\mathfrak{c}(\alpha,\beta) = 0$.*

Remark 2.2.15. *Notice $\mathbf{N}_{\overline{\mathbb{R},S}}$-compatible is equivalent to* Hom-*orthogonal, not Ext-orthogonal.*

Corollary 2.2.16 (to Lemma 2.2.13)**.** *Let $M_{|a,b|}$ and $M_{|c,d|}$ be indecomposables in $\mathcal{C}(A_{\mathbb{R}})$. Then $\{\Phi(M_{|a,b|}), \Phi(M_{|c,d|})\}$ is $\mathbf{N}_{\overline{\mathbb{R}}}$-compatible if and only if $\{M_{|a,b|}, M_{|c,d|}\}$ is $\mathbf{E}$-compatible.*

Theorem 2.2.17. *Let $A_{\mathbb{R},S}$ be a continuous quiver of type A. The pairwise compatibility condition $\mathbf{N}_{\overline{\mathbb{R},S}}$ induces the $\mathbf{N}_{\overline{\mathbb{R},S}}$-cluster theory of $\mathcal{C}_{\overline{\mathbb{R},S}}$ and Φ induces an isomorphism of cluster theories $(F,\eta) : \mathscr{T}_{\mathbf{N}_{\overline{\mathbb{R},S}}}(\mathcal{C}_{\overline{\mathbb{R},S}}) \to \mathscr{T}_{\mathbf{E}}(\mathcal{C}(A_{\mathbb{R},S}))$.*

Proof. By Proposition 2.2.5 and Definition 2.2.14 we have a bijection $\Phi : \mathrm{Ind}(\mathcal{C}(A_{\mathbb{R},S})) \to \mathrm{Ind}(\mathcal{C}_{\overline{\mathbb{R},S}})$. The set $\{M_{|a,b|}, M_{|c,d|}\}$ is $\mathbf{E}$-compatible if and only if $\{\Phi(M_{|a,b|}), \Phi(M_{|c,d|})\}$ is $\mathbf{N}_{\overline{\mathbb{R},S}}$-compatible, by Corollary 2.2.16. Thus, by Lemma 1.3.8, $\mathbf{N}_{\overline{\mathbb{R},S}}$ induces the cluster theory $\mathscr{T}_{\mathbf{N}_{\overline{\mathbb{R},S}}}(\mathcal{C}_{\overline{\mathbb{R},S}})$ and we have the isomorphism of cluster theories given by $F(T) := \{\Phi^{-1}(\alpha) \mid \alpha \in T\}$ and $\eta_T(\alpha) := \Phi^{-1}(\alpha)$. □

2.3. **On the Classification of Cluster Theories of Continuous Type** A**.** In this section we identity some cluster theories of continuous type A which are isomorphic. We show there are at least four isomorphism classes of such cluster theories. The following notations are useful.

Notation 2.3.1. *Let $\mathscr{T}_{\mathbf{P}}(\mathcal{C})$ and $\mathscr{T}_{\mathbf{Q}}(\mathcal{D})$ be two cluster theories. If there is an isomorphism of cluster theories $(F,\eta) : \mathscr{T}_{\mathbf{P}}(\mathcal{C}) \to \mathscr{T}_{\mathbf{Q}}(\mathcal{D})$ then we say $\mathscr{T}_{\mathbf{P}}(\mathcal{C})$ is isomorphic to $\mathscr{T}_{\mathbf{Q}}(\mathcal{D})$ and write $\mathscr{T}_{\mathbf{P}}(\mathcal{C}) \cong \mathscr{T}_{\mathbf{Q}}(\mathcal{D})$.*

Notation 2.3.2. *Let $A_{\mathbb{R},S}$ be a continuous quiver of type A.*

- *By $(A_{\mathbb{R},S})^{-1}$ we denote the continuous quiver $A_{\mathbb{R},R}$ where, if $-\infty \neq s_0$, each source r_n in $A_{\mathbb{R},R}$ is equal to a sink s_{n-1} and similarly for sinks in R. If $-\infty = s_0$ in $A_{\mathbb{R},S}$, then each source r_n in $A_{\mathbb{R},R}$ is instead equal to a sink s_{n+1} in $A_{\mathbb{R},S}$ and similarly for sinks in R. For example if $s_{-1} = -\infty$, $s_0 = 0$, and $s_1 = +\infty$, then $r_0 = -\infty$, $r_1 = 0$, and $r_2 = +\infty$. Another example, if $s_0 = -\infty$ and $s_1 = +\infty$, then $r_0 = +\infty$ and $r_1 = -\infty$.*
- *By $-(A_{\mathbb{R},S})$ we denote the continuous quiver $A_{\mathbb{R},R}$ where each sink r_{2n} in $A_{\mathbb{R},R}$ is equal to the sink $-s_{-2n}$ in $A_{\mathbb{R},S}$ and similarly for sources.*

Remark 2.3.3. *Notice that $-(-(A_{\mathbb{R},S})) = A_{\mathbb{R},S}$. Furthermore, if $-\infty = s_0$ or $-\infty = s_{-1}$ then $((A_{\mathbb{R},S})^{-1})^{-1}) = A_{\mathbb{R},S}$. If $-\infty \neq s_0$ and $-\infty \neq s_1$, then we still have $\mathrm{rep}_{\Bbbk}(A_{\mathbb{R},S})$ is equivalent to $\mathrm{rep}_{\Bbbk}(((A_{\mathbb{R},S})^{-1})^{-1})$ as $\Bbbk$-linear abelian categories.*

Finally, we see $(-(A_{\mathbb{R},S}))^{-1} = ((-((A_{\mathbb{R},S})^{-1}))^{-1})^{-1}$ and so $\mathrm{rep}_{\Bbbk}((-(A_{\mathbb{R},S}))^{-1})$ is equivalent to $\mathrm{rep}_{\Bbbk}(-((A_{\mathbb{R},S})^{-1}))$ as abelian categories.

Proposition 2.3.4. *Let $A_{\mathbb{R},S}$ be a continuous quiver of type A and $A_{\mathbb{R},R} = (A_{\mathbb{R},S})^{-1}$. Then $\mathscr{T}_{\mathbf{N}_{\overline{\mathbb{R},S}}}(\mathcal{C}_{\overline{\mathbb{R},S}}) \cong \mathscr{T}_{\mathbf{N}_{\overline{\mathbb{R},R}}}(\mathcal{C}_{\overline{\mathbb{R},R}})$.*

Proof. Denote the sets of endpoints and arcs for $A_{\mathbb{R},S}$ by $\mathcal{E}_S$ and $\mathcal{A}_S$, respectively. Denote the sets of endpoints and arcs for $A_{\mathbb{R},R}$ by $\mathcal{E}_R$ and $\mathcal{A}_R$, respectively. Let the respective crossing functions be $\mathfrak{c}_S$ and $\mathfrak{c}_R$. There is an order preserving bijections $g : \mathcal{E}^{\downarrow}_S \xrightarrow{\cong} \mathcal{E}^{\uparrow}_R$ and $h : \mathcal{E}^{\uparrow}_S \xrightarrow{\cong} \mathcal{E}^{\downarrow}_R$. Let $f : \mathcal{E}_S \xrightarrow{\cong} \mathcal{E}_R$ be the bijection that is g on $\mathcal{E}^{\downarrow}_S$ and h on $\mathcal{E}^{\uparrow}_S$.

Notice Definitions 2.1.6, 2.2.8, and 2.2.11 are symmetric with respect to $\mathcal{E}^{\uparrow}$ and $\mathcal{E}^{\downarrow}$, except at sinks and sources. Let $\alpha = (\overline{x}, \overline{y})$ and $\beta = (\overline{z}, \overline{w})$ be in $\mathcal{A}_S$. Let

$\gamma = (f(\overline{x}), f(\overline{y}))$ and $\delta = (f(\overline{z}), f(\overline{w}))$. Then $\mathfrak{c}_S(\alpha, \beta) = 1$ if and only if $\mathfrak{c}_R(\gamma, \delta) = 1$. Now apply Lemma 1.3.8. $\square$

Proposition 2.3.5. *Let $A_{\mathbb{R},S}$ be a continuous quiver of type A and $A_{\mathbb{R},R} = -(A_{\mathbb{R},S})$. Then $\mathscr{T}_{\mathbf{N}_{\overline{\mathbb{R},S}}}(\mathcal{C}_{\overline{\mathbb{R},S}}) \cong \mathscr{T}_{\mathbf{N}_{\overline{\mathbb{R},R}}}(\mathcal{C}_{\overline{\mathbb{R},R}})$.*

Proof. Let $\mathcal{E}_S$, $\mathcal{A}_S$, $\mathfrak{c}_s$, $\mathcal{E}_R$, $\mathcal{A}_R$, and $\mathfrak{c}_R$ be as in the proof of Proposition 2.3.4. Then we have order *reversing* bijections $g : \mathcal{E}_S^{\downarrow} \stackrel{\cong}{\to} \mathcal{E}_R^{\downarrow}$ and $h : \mathcal{E}_S^{\uparrow} \to \mathcal{E}_R^{\uparrow}$. Let $f : \mathcal{E}_S \stackrel{\cong}{\to} \mathcal{E}_R$ be the bijection that is g on $\mathcal{E}_S^{\downarrow}$ and h on $\mathcal{E}_S^{\uparrow}$. Now proceed by a similar argument to Proposition 2.3.4. $\square$

Theorem 2.3.6. *Let $A_{\mathbb{R},S}$ be a continuous quiver of type A. Then there is a diagram of isomorphisms of cluster theories:*

$$\begin{tikzcd}
\mathscr{T}_{\mathbf{E}}(\mathcal{C}(A_{\mathbb{R},S})) \arrow[r, leftrightarrow, "\cong"] \arrow[d, leftrightarrow, "\cong"'] & \mathscr{T}_{\mathbf{E}}(\mathcal{C}((A_{\mathbb{R},S})^{-1})) \arrow[d, leftrightarrow, "\cong"] \\
\mathscr{T}_{\mathbf{E}}(\mathcal{C}(-(A_{\mathbb{R},S}))) \arrow[r, leftrightarrow, "\cong"'] & \mathscr{T}_{\mathbf{E}}(\mathcal{C}(-((A_{\mathbb{R},S})^{-1}))).
\end{tikzcd}$$

Proof. Apply Propositions 2.3.4 and 2.3.5 and Remark 2.3.3. $\square$

Corollary 2.3.7. *Let $A_{\mathbb{R},S}$ and $A_{\mathbb{R},R}$ be continuous quivers of type A such that one of the following is true: (i) $|S| = |R|$ and $|S| < \infty$, (ii) S and R are both bounded on exactly one side, or (iii) both S and R are indexed by $\mathbb{Z}$. Then $\mathscr{T}_{\mathbf{E}}(\mathcal{C}(A_{\mathbb{R},S})) \cong \mathscr{T}_{\mathbf{E}}(\mathcal{C}(A_{\mathbb{R},R}))$.*

Proof. In all cases, we can construct an order preserving or reversing bijection $S \stackrel{\cong}{\to} R$ such that, in the indexing, either odds are taken to evens or odds are taken to odds. Then $\mathrm{rep}_{\Bbbk}(A_{\mathbb{R},R})$ is equivalent to one of $\mathrm{rep}_{\Bbbk}(A_{\mathbb{R},S})$, $\mathrm{rep}_{\Bbbk}((A_{\mathbb{R},S})^{-1})$, $\mathrm{rep}_{\Bbbk}(-A_{\mathbb{R},S})$, or $\mathrm{rep}_{\Bbbk}(-((A_{\mathbb{R},S})^{-1})$. Thus, $\mathscr{T}_{\mathbf{E}}(\mathcal{C}(A_{\mathbb{R},R}))$ is equivalent to one of $\mathscr{T}_{\mathbf{E}}(\mathcal{C}(A_{\mathbb{R},S}))$, $\mathscr{T}_{\mathbf{E}}(\mathcal{C}((A_{\mathbb{R},S})^{-1}))$, $\mathscr{T}_{\mathbf{E}}(\mathcal{C}(-(A_{\mathbb{R},S})))$, or $\mathscr{T}_{\mathbf{E}}(\mathcal{C}(-((A_{\mathbb{R},S})^{-1})))$. Therefore, by Theorem 2.3.6, $\mathscr{T}_{\mathbf{E}}(\mathcal{C}(A_{\mathbb{R},S})) \cong \mathscr{T}_{\mathbf{E}}(\mathcal{C}(A_{\mathbb{R},R}))$. $\square$

The classification of cluster theories in Corollary 2.3.7 is nearly the classification of derived categories in Theorem 1.2.2.

We have two remaining isomorphsisms of cluster theories we would like:

(1) Any isomorphism between $\mathscr{T}_{\mathbf{N}_{\overline{\mathbb{R},S}}}(\mathcal{C}_{\overline{\mathbb{R},S}})$ and $\mathscr{T}_{\mathbf{N}_{\overline{\mathbb{R},R}}}(\mathcal{C}_{\overline{\mathbb{R},R}})$ where $A_{\mathbb{R},S}$ has an even number ≥ 2 of sinks and sources in $\mathbb{R}$ and $A_{\mathbb{R},R}$ has an odd number of sinks and sources in $\mathbb{R}$.
(2) An isomorphism between $\mathscr{T}_{\mathbf{N}_{\overline{\mathbb{R}}}}(\mathcal{C}_{\overline{\mathbb{R}}})$ and $\mathscr{T}_{\mathbf{N}_{\overline{\mathbb{R},S}}}(\mathcal{C}_{\overline{\mathbb{R},S}})$ where $A_{\mathbb{R}}$ has no sinks or sources in $\mathbb{R}$ and $A_{\mathbb{R},S}$ has an even number ≥ 2 of sinks and sources in $\mathbb{R}$.

We immediately share the unfortunate news:

Proposition 2.3.8. *Let $A_{\mathbb{R}}$ be a continuous quiver of type A with straight descending or straight ascending orientation. Let $A_{\mathbb{R},S}$ be a continuous quiver of type A with at least one sink or source in $\mathbb{R}$. Then there is no isomorphism of cluster theories $\mathscr{T}_{\mathbf{N}_{\overline{\mathbb{R}}}}(\mathcal{C}_{\overline{\mathbb{R}}}) \to \mathscr{T}_{\mathbf{N}_{\overline{\mathbb{R},S}}}(\mathcal{C}_{\overline{\mathbb{R},S}})$.*

Proof. The arc α corresponding to the indecomposable $M_{(-\infty,+\infty)}$ in $\mathcal{C}(A_{\mathbb{R}})$ is in every $\mathbf{N}_{\overline{\mathbb{R}}}$-cluster of $\mathscr{T}_{\mathbf{N}_{\overline{\mathbb{R}}}}(\mathcal{C}_{\overline{\mathbb{R}}})$. The arcs corresponding to the projectives from

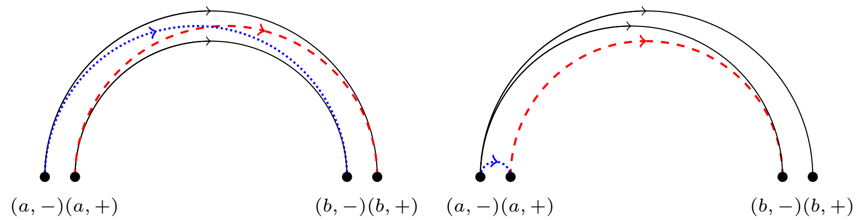


FIGURE 9. Depictions of mutation on the "microscopic" scale (Remark 2.1.1), where we replace the red (dashed) arc with the blue (dotted) arc. First we mutate at $((a,+),(b,+))$ and then mutate at $((a,+),(b,-))$. Notice the arc orientations pointing from the lower element to the upper element. However, we only *need* these orientations in the right picture.

$\mathrm{rep}_{\Bbbk}(A_{\mathbb{R},S})$ form an $\mathbf{N}_{\overline{\mathbb{R},S}}$-cluster; this is similarly true for the arcs corresponding to the injectives from $\mathrm{rep}_{\Bbbk}(A_{\mathbb{R},S})$. However, there are not projective-injective objects in $\mathrm{rep}_{\Bbbk}(A_{\mathbb{R},S})$ and so these two clusters share no elements. Therefore, there cannot be such an isomorphism of cluster theories. $\square$

This leaves us with at least four isomorphism classes of cluster theories of continuous type A: (i) no sinks or sources in $\mathbb{R}$, (ii) finitely-many sinks and sources in $\mathbb{R}$, (iii) half-bounded sinks and sources in $\mathbb{R}$, and (iv) unbounded sinks and sources in $\mathbb{R}$. However, it is not clear whether (ii) is just one class, separate classes for even and odd numbers, or a separate class for all numbers.

Open Questions:

- Does there exist a weak equivalence of cluster theories
$$\mathscr{T}_{\mathbf{N}_{\overline{\mathbb{R}}}}(\mathcal{C}_{\overline{\mathbb{R}}}) \to \mathscr{T}_{\mathbf{N}_{\overline{\mathbb{R},S}}}(\mathcal{C}_{\overline{\mathbb{R},S}}) \qquad \text{or} \qquad \mathscr{T}_{\mathbf{N}_{\overline{\mathbb{R},S}}}(\mathcal{C}_{\overline{\mathbb{R},S}}) \to \mathscr{T}_{\mathbf{N}_{\overline{\mathbb{R}}}}(\mathcal{C}_{\overline{\mathbb{R}}}),$$
where $A_{\mathbb{R}}$ has no sinks or sources in $\mathbb{R}$ and $A_{\mathbb{R},S}$ has an even number ≥ 2 of sinks and sources in $\mathbb{R}$?
- Does there exist an isomorphism of cluster theories or weak equivalence of cluster theories
$$\mathscr{T}_{\mathbf{N}_{\overline{\mathbb{R},S}}}(\mathcal{C}_{\overline{\mathbb{R},S}}) \to \mathscr{T}_{\mathbf{N}_{\overline{\mathbb{R},R}}}(\mathcal{C}_{\overline{\mathbb{R},R}}) \qquad \text{or} \qquad \mathscr{T}_{\mathbf{N}_{\overline{\mathbb{R},R}}}(\mathcal{C}_{\overline{\mathbb{R},R}}) \to \mathscr{T}_{\mathbf{N}_{\overline{\mathbb{R},S}}}(\mathcal{C}_{\overline{\mathbb{R},S}}),$$
where $A_{\mathbb{R},S}$ has an odd number n of sinks and sources in $\mathbb{R}$ and $A_{\mathbb{R},R}$ has $n+1$ sinks and sources in $\mathbb{R}$?

2.4. **Connection to E-Mutations.** Let $A_{\mathbb{R},S}$ be a continuous quiver of type A. In this section we use geometric models to draw an $\mathbf{N}_{\overline{\mathbb{R},S}}$-mutation corresponding to an $\mathbf{E}$-mutation. Because of our definitions on crossing, mutation is not as clearly described as swapping diagonals of a quadrilateral. However, we can make similar descriptions. Let us begin with the "microscopic" scale (Remark 2.1.1). Let $A_{\mathbb{R},S}$ be a continuous quiver of type A with at least one sink or source in $\mathbb{R}$. Let $a < b \in \mathbb{R}$ such that neither a nor b is a sink or source and $(a,\varepsilon),(b,\varepsilon) \in \mathcal{E}^{\downarrow}$, for any $\varepsilon \in \{-,+\}$.

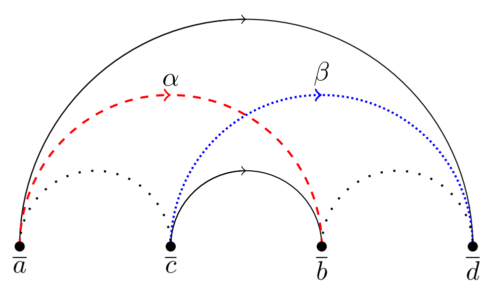
Let T be an $\mathbf{N}_{\overline{\mathbb{R},S}}$-cluster such that $((a,-),(b,+)),((a,+),(b,+)),((a,+),(b,-)) \in T$. These correspond to the indecomposables $M_{[a,b]}$, $M_{(a,b]}$, and $M_{(a,b)}$, respectively, in $\mathcal{C}(A_{\mathbb{R},S})$. We can mutate at $((a,+),(b,+))$ to obtain $(T \setminus \{((a,+),(b,+))\}) \cup \{((a,-),(b,-))\}$. The picture one should have in mind is Figure 9.

We now move to the "macroscopic" scale (Remark 2.1.1). In $\mathcal{C}(A_{\mathbb{R},S})$, we know that if $\{M_{|a,b|}M_{|c,d|}\}$ is not **E**-compatible then, up to reversing the roles of the indecomposables, we have the following distinguished triangle in $\mathcal{C}(A_{\mathbb{R},S})$:

$$M_{|a,b|} \to M_{|a,d|} \oplus M_{|c,b|} \to M_{|c,d|} \to$$

where one of $M_{|a,d|}$ or $M_{|c,b|}$ may be 0. Now suppose we **E**-mutate in some cluster at $M_{|a,b|}$ and obtain $M_{|c,d|}$; this is the top picture in Figure 10. If the middle object in the distinguished triangle is not an indecomposable, then, in the geometric model, we have two of the four sides of the quadrilateral we see in triangulations of polygons.

However, we do not know if we have the indecomposables corresponding to the two dotted arcs that complete the quadrilateral. The dotted arcs may be incompatible with $M_{|a,b|}$ and/or $M_{|c,d|}$. For example, if $b \in |a,b|$ then there is no arc with $(b,-)$ as a lower endpoint that is compatible with either of the arcs corresponding to $M_{|a,b|}$ and $M_{|c,b|}$. In the case where one of $M_{|c,b|}$ or $M_{|a,d|}$ is 0, we instead have the bottom picture in Figure 10. If some of the endpoints are in $\mathcal{E}^{\downarrow}$ and others in $\mathcal{E}^{\uparrow}$ then we instead draw pictures such as those in Figure 11.



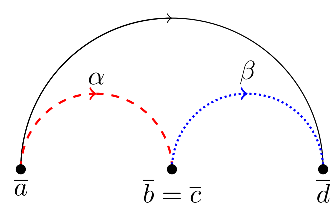


FIGURE 10. Depictions of mutation on the "macroscopic" scale (Remark 2.1.1) where we replace red (dashed) arcs $\alpha = (\overline{a}, \overline{b})$ with blue (dotted) arcs $\beta = (\overline{c}, \overline{d})$. The loosely dotted arcs are sometimes, but not always, present. This depends on the infinitesimal differences in the other arcs.

3. Continuous Mutation and Mutation Paths

This section is dedicated to the definition of a continuous mutation and the basic properties of continuous mutations. These generalize the familiar notion of mutation in a cluster structure. We define this new type of mutation for all cluster theories (Definition 1.3.1) though we use type A cluster theories for our examples. Notably, for a cluter theory $\mathscr{T}_{\mathbf{P}}(\mathcal{C})$, any **P**-mutation can be thought of as a continuous **P**-mutation (see Example 3.2.1). In Section 3.4, we show how to interpret continuous mutations via geometric models (Sections 2.1 and 2.2) using continuous type A as an example. The final subsection of this section is dedicated to the space of mutations (Definition 3.5.2), which generalizes the exchange graph of a cluster structure. We pose questions related specifically to the **E**-cluster theory of an arbitrary continuous quiver of type A at the end.

For Section 3 we fix $\mathcal{C}$ a skeletally small, KRS, and additive category and **P** a pairwise compatibility condition on the indecomposables in $\mathcal{C}$ such that **P** induces the **P**-cluster theory of $\mathcal{C}$ (Definition 1.3.1).

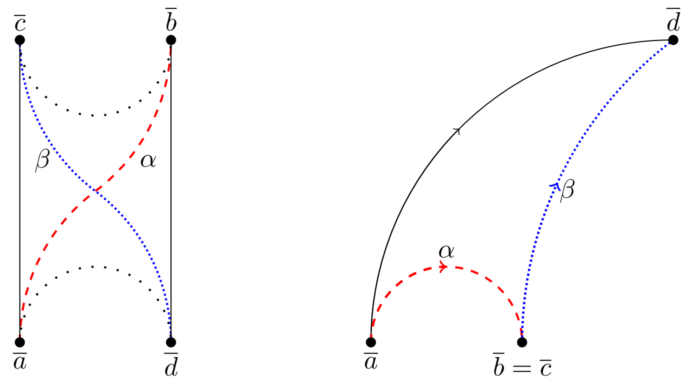


FIGURE 11. Depiction of "macroscopic" (Remark 2.1.1) mutation where the arcs have endpoints mixed in $\mathcal{E}^{\uparrow}$ and $\mathcal{E}^{\downarrow}$. We mutate the red (dashed) arc $\alpha = (\overline{a}, \overline{b})$ to the blue (dotted) arc $\beta = (\overline{c}, \overline{d})$. The orientations of the arcs in the left picture do not matter (i.e, we could have $\beta = (\overline{d}, \overline{c})$) but the orientations are necessary in the right picture. As in Figure 10, the loosely dotted arcs may or may not be present.

3.1. **Continuous Mutation.** In this section we define continuous mutation. In order to better interpret continuous mutations, we need to define a trivial mutation.

Definition 3.1.1. *A trivial $\mathbf{P}$-mutation is an identity function $id_T : T \to T$, for any $\mathbf{P}$-cluster T.*

Definition 3.1.2. *Let T and T' be $\mathbf{P}$-clusters and $\mu : T \to T'$ a bijection. We call μ a continuous $\mathbf{P}$-mutation if it satisfies the following four properties.*

- *There is a set $S \subset T$ such that $\mu X = X$ if and only if $X \notin S$.*
- *Let $S' = \mu(S)$. For all $\mu X \in S'$, (i) $\mu X \notin T$ and (ii) $\{X, \mu X\}$ is not $\mathbf{P}$-compatible.*
- *There exist injections $f_\mu : S \to [0,1]$ and $g_\mu : S' \to [0,1]$ such that $(g_\mu \circ \mu)|_S = (f_\mu)|_S$.*
- *For any subinterval $J \subset [0,1]$, where $0 \in J$ and $1 \notin J$, the following is a $\mathbf{P}$-cluster:*
$$\big(T \setminus f_\mu^{-1}(J)\big) \cup g_\mu^{-1}(J) = \big(T' \setminus g_\mu^{-1}(\bar{J})\big) \cup f_\mu{-}1(\bar{J}),$$
where $\bar{J} = [0,1] \setminus J$.

We need to justify the word 'mutation.' We do this with Propositions 3.1.3 and 3.1.4. The first states that every continuous mutation can be reversed. The second states that we may consider a continuous mutation a collection of mutations, one each at time t, for all $t \in [0,1]$.

The proof of the following proposition is a straightforward application of the definition.

Proposition 3.1.3. *Let $\mu : T \to T'$ be a continuous $\mathbf{P}$-mutation. Then $\mu^{-1} : T' \to T$ is also a continuous $\mathbf{P}$-mutation.*

Proposition 3.1.4. *Let $\mu : T \to T'$ be a continuous* **P***-mutation. For every $t \in [0,1]$, the following bijection $(T \setminus f_\mu^{-1}([0,t))) \cup g_\mu^{-1}([0,t)) \to (T \setminus f_\mu^{-1}([0,t])) \cup g_\mu^{-1}([0,t])$ is a* **P***-mutation:*

$$X \mapsto \begin{cases} X & X \neq f^{-1}(t) \\ g^{-1}(t) & X = f^{-1}(t). \end{cases}$$

Proof. In the case $(T \setminus f_\mu^{-1}([0,t))) \cup g_\mu^{-1}([0,t)) = (T \setminus f_\mu^{-1}([0,t])) \cup g_\mu^{-1}([0,t])$ we have a trivial **P**-mutation. Suppose $(T \setminus f_\mu^{-1}([0,t))) \cup g_\mu^{-1}([0,t)) \neq (T \setminus f_\mu^{-1}([0,t])) \cup g_\mu^{-1}([0,t])$. Since f_μ and g_μ are injections, $f_\mu^{-1}([0,t))$ differs from $f_\mu^{-1}([0,t])$ by at most one element and by assumption they differ by at least one element; thus differing by exactly one element. This is similarly true for $g_\mu^{-1}([0,t))$ and $g_\mu^{-1}([0,t])$. By definition, $\mu(f_\mu^{-1}(t)) = g_\mu^{-1}(t)$ and $\{f_\mu^{-1}(t), g_\mu^{-1}(t)\}$ are not **P**-compatible. Therefore, we have a **P**-mutation. □

We conclude with this final definition that is useful in asking questions about the classification of **E**-clusters in $\mathcal{C}(A_{\mathbb{R},S})$ (Definition 1.2.6) at the end of Section 3.5.2.

Definition 3.1.5. *Let $Z = \{1, \ldots, n\}$ or $Z = \mathbb{Z}_{>0}$. For each $i \in Z$ let μ_i be a continuous* **P***-mutation such that the target of μ_i is the source of μ_{i+1} when $i, i+1 \in Z$. We call $\{\mu_i\}_{i \in Z}$ a sequence of continuous* **P***-mutations. If each μ_i mutates only one element of T_i we may also say that $\{\mu_i\}$ is a sequence of* **P***-mutations.*

3.2. **Examples.** In this section we highlight two existing examples of continuous mutations that do not feel so continuous followed by a new example. The first (Example 3.2.1) shows that a mutation, in the traditional sense, can be thought of as a continuous mutation. The second (Example 3.2.3) describes an infinite sequence of mutations. While these both exist in the literature, the contribution is that continuous mutation unifies the way to describe these types of mutations. We conclude with Proposition 3.2.5, which, as far as the author knows, does not exist anywhere in the literature.

Example 3.2.1. *Let $\mathcal{C}$ be a skeletally small, KRS, and additive category with pairwise compatibility condition* **P** *on indecomposables such that* **P** *induces the* **P***-cluster theory of $\mathcal{C}$. Let $\mu : T \to (T \setminus \{X\}) \cup \{Y\}$ be a* **P***-mutation. Furthermore, let $S = \{X\}$, $S' = \{Y\}$, and $T' = (T \setminus \{X\}) \cup \{Y\}$. Finally, let $f : \{X\} \to [0,1]$ and $g : \{Y\} \to [0,1]$ each send X and Y to $\frac{1}{2}$, respectively. This meets the requirements for the definition of a continuous mutation.*

The second example is based on the completed infinity-gon from [5].

Definition 3.2.2. *Let $\mathcal{E} = \mathbb{Z} \cup \{-\infty, +\infty\}$ with the usual total ordering. Let*

$$\mathcal{A} = \{(i,j) \in \mathcal{E} \times \mathcal{E} \mid \exists k \in \mathcal{E} \text{ s.t } i < k < j\} \setminus \{(-\infty, +\infty)\}.$$

Define the crossing function $\mathfrak{c} : \mathcal{A} \times \mathcal{A} \to \{0,1\}$ by

$$\mathfrak{c}((i,j),(i',j')) = \begin{cases} 1 & ((i,j) = (i',j')) \text{ or } (i < i' < j < j') \text{ or } (i' < i < j < j') \\ 0 & \text{otherwise.} \end{cases}$$

We define $\mathcal{C}(A_{\overline{\infty}})$ to be the additive category whose indecomposable objects are $\mathcal{A}$. Define Hom spaces and composition in the same way as in Definition 2.1.10. We again obtain a skeletally small, KRS, and additive category. For $\alpha \neq \beta$, we say $\{\alpha, \beta\}$ is $\mathbf{N}_{\overline{\infty}}$-compatible if and only if $\mathfrak{c}(\alpha, \beta) = 0$.

Baur and Graz proved in [5] that $\mathbf{N}_{\overline{\infty}}$ induces the $\mathbf{N}_{\overline{\infty}}$-cluster theory of $\mathcal{C}(A_{\overline{\infty}})$.

Baur and Graz define a T-admissible sequence of arcs $\{\alpha_i\}$ is one where α_1 is $\mathbf{N}_{\overline{\infty}}$-mutable in $T_1 = T$ and each T_i for $i > 1$ is obtained by mutating α_{i-1} which must be mutable in T_{i-1}. Note this sequence may be infinite so long as there is a first arc in the sequence. Baur and Graz note that mutating along a T-admissible sequence does not always result in an $\mathbf{N}_{\overline{\infty}}$-cluster. I.e., the colimit of such a sequence of mutations may not be an $\mathbf{N}_{\overline{\infty}}$-cluster.

Example 3.2.3. *Let T be an $\mathbf{N}_{\overline{\infty}}$-cluster in $\mathcal{C}(A_{\overline{\infty}})$ and $\{\alpha_i\}$ a T-admissible sequence of arcs. Since each $\mathbf{N}_{\overline{\infty}}$-mutation $\mu_i : T_i \to T_{i+1}$ is also a continuous $\mathbf{N}_{\overline{\infty}}$-mutation any admissible sequence of arcs yields a sequence of continuous $\mathbf{N}_{\overline{\infty}}$-mutations.*

Now suppose $\{\alpha_i\} \subset T$ and the result of mutating along $\{\alpha_i\}$ yields an $\mathbf{N}_{\overline{\infty}}$-cluster T'. Then we let $S = \{\alpha_i\}$ and let $f : S \to [0,1]$ be given by $\alpha_i \mapsto 1 - \frac{1}{i+1}$. Let $S' = \{\mu_i(\alpha_i)\}$ and let $g : S' \to [0,1]$ be given by $\mu_i(\alpha_i) \mapsto 1 - \frac{1}{i+1}$. We now have a continuous $\mathbf{N}_{\overline{\infty}}$-mutation.

In general, a T-admissible sequence of arcs can be "grouped" into intervals of arcs which each belong to the first cluster of the group. This yields a sequence of $\mathbf{N}_{\overline{\infty}}$-mutations in a somewhat minimal way. Of course, this does not work if $\{\alpha_i\} \subset T$ and mutation along $\{\alpha_i\}$ does not result in an $\mathbf{N}_{\overline{\infty}}$-cluster.

Remark 3.2.4. *Let $\mathcal{C}$ be a skeletally small, KRS, and additive category with pairwise compatibility condition $\mathbf{P}$ on indecomposables such that $\mathbf{P}$ induces the $\mathbf{P}$-cluster theory of $\mathcal{C}$. As seen in Example 3.2.3 it might be possible to construct a sequence of (continuous) $\mathbf{P}$-mutations that does not yield a $\mathbf{P}$-cluster. The authors of* [5] *provide a way to complete their compatible sets for their cluster theory.*

Proposition 3.2.5. *Let $A_{\mathbb{R}}$ have the straight descending orientation, $\mathcal{P}roj$ be the $\mathbf{E}$-cluster containing all the projectives from $\operatorname{rep}_{\Bbbk}(A_{\mathbb{R}})$, and $\mathcal{I}nj$ be the $\mathbf{E}$-cluster containing the injectives from $\operatorname{rep}_{\Bbbk}(A_{\mathbb{R}})$. There is a sequence of two continuous mutations $\{\mu_1, \mu_2\}$ from $\mathcal{P}roj$ to $\mathcal{I}nj$.*

Proof. Recall that every indecomposable in $\mathcal{C}(A_{\mathbb{R}})$ comes from an indecomposable M_I in $\operatorname{rep}_{\Bbbk}(A_{\mathbb{R}})$ (Definition 1.1.3, Theorem 1.1.5, Proposition 1.2.1, and [17, Proposition 3.1.4]). Recall also that $|a,b|$ means the inclusion of a or b is either indeterminate or clear from context (see Conventions on 3) and Theorem 1.1.4). Note that $\mathcal{P}roj \cap \mathcal{I}nj = \{M_{(-\infty,+\infty)}\}$.

We construct two continuous $\mathbf{E}$-mutations to mutate $\mathcal{P}roj$ to $\mathcal{I}nj$. First, let $S_1 = \mathcal{P}roj$ and define $f_1 : \mathcal{P}roj \to [0,1]$ in two parts. For $M_{(-\infty,x)}$ and $M_{(-\infty,x]}$ in $\mathcal{P}roj$, we let

$$f_1\left(M_{(-\infty,x)}\right) = \frac{1}{2} - \left(\frac{\tan^{-1} x}{2\pi} + \frac{1}{4}\right) \qquad f_1\left(M_{(-\infty,x]}\right) = 1 - \left(\frac{\tan^{-1} x}{2\pi} + \frac{3}{4}\right).$$

The "middle" $\mathbf{E}$-cluster is

$$T_2 := \left\{M_{(-\infty,+\infty)}\right\} \cup \left\{M_{[x,+\infty)}, M_{[x,x]} \mid x \in \mathbb{R}\right\}.$$

We then define $g_1 : T_2 \to [0,1]$ to match with f_1:

$$g_1\left(M_{[x,x]}\right) = \frac{1}{2} - \left(\frac{\tan^{-1} x}{2\pi} + \frac{1}{4}\right) \qquad g_1\left(M_{[x,+\infty)}\right) = 1 - \left(\frac{\tan^{-1} x}{2\pi} + \frac{3}{4}\right).$$

Both f_1 and g_1 are injections and we may define $\mu_1(M) = g^{-1}(f(M))$ and obtain the continuous $\mathbf{E}$-mutation $\mu_1 : \mathcal{P}roj \to T_2$.

Now let $S_2 = \{M_{[x,x]} \mid x \in \mathbb{R}\} \subset T_2$ and $S_2' = \{M_{(x,+\infty)} \mid x \in \mathbb{R}\} \subset \mathcal{I}nj$. We define $f_2 : T_2 \to [0,1]$ and $g_2 : \mathcal{I}nj \to [0,1]$ by

$$f_2\left(M_{[x,x]}\right) = \frac{\tan^{-1} x}{\pi} + \frac{1}{2} = g_2\left(M_{(x,+\infty)}\right).$$

We define $\mu_2(M)$ to be M if $M \notin S_2$ and $g^{-1}(f(M))$ if $M \in S_2$. This gives the continuous **E**-mutation $\mu_2 : T_2 \to \mathcal{I}nj$. Thus we have a sequence of continuous **E**-mutations $\{\mu_1, \mu_2\}$ to mutate the projectives into the injectives. □

3.3. **Mutation Paths.** In this section we define mutation paths, which should be thought of as a generalization of a sequence of mutations. At first we formally define a long sequence of continuous mutations (Definition 3.3.1) and then move on to mutation paths in general (Definition 3.3.2). Note also that a continuous mutation is an example of a mutation path (Example 3.3.5) just as a mutation is an example of a continuous mutation.

A mutation path should be thought of as a generalization of a path of mutations in the exchange graph of a cluster structure. This is formalized in Section 3.5. As before, our definitions are for any cluster theory but our interest is in **E**-cluster theories of $A_{\mathbb{R}}$ quivers.

Definition 3.3.1. *Let*

$$\overline{\mu} = \{{}_i\mu \mid {}_iT_0 \to {}_iT_1\}_{i\in\mathbb{Z}}$$

be a collection of continuous mutations such that ${}_iT_1 = {}_{i+1}T_0$. *This yields a diagram in* $\mathcal{S}ets$*:*

$$\cdots \xrightarrow{{}_{i-1}\mu} {}_{i-1}T_1 = {}_iT_0 \xrightarrow{{}_i\mu} {}_iT_1 = {}_{i+1}T_0 \xrightarrow{{}_{i+1}\mu} {}_{i+1}T_1 = {}_{i+2}T_0 \xrightarrow{{}_{i+2}\mu} \cdots$$

If this diagram has a limit and colimit in $\mathcal{S}et$ *with both objects in* $\mathscr{T}_{\mathbf{P}}(\mathcal{C})$*, we call* $\overline{\mu}$ *a* *long sequence of continuous mutations*. *We call the limit and colimit the* *source* *and* *target* *of* $\overline{\mu}$*, respectively.*

Definition 3.3.2. *Define a category* $\mathcal{I}$ *whose objects are pairs* $(x,i) \in [0,1]\times\{0,1\}$. *Consider* $[0,1]$ *and* $\{0,1\}$ *with their respective usual total ordering. Morphisms in* $\mathcal{I}$ *are defined by*

$$\mathrm{Hom}_{\mathcal{I}}\left((s,i),(t,j)\right) := \begin{cases} \{*\} & s < t \text{ or } (s = t \text{ and } i \leq j) \\ \emptyset & \text{otherwise.} \end{cases}$$

Let $\overline{\mu} : \mathcal{I} \to \mathcal{S}ets$ *be a functor such that* $\overline{\mu}(*) : \overline{\mu}(s,0) \to \overline{\mu}(s,1)$ *is a (possibly trivial)* **P***-mutation in* $\mathscr{T}_{\mathbf{P}}(\mathcal{C})$. *Then we call* $\overline{\mu}$ *a* **P***-mutation path*.

Remark 3.3.3. *The reader may notice that the target of the functor is not* $\mathscr{T}_{\mathbf{P}}(\mathcal{C})$*, but just* $\mathcal{S}ets$. *This is because we have not defined* $\mathscr{T}_{\mathbf{P}}(\mathcal{C})$ *(in Definition 1.3.1) to be closed under any kind of transfinite composition. However, transfinite composition is indeed sometimes defined in* $\mathcal{S}ets$. *For example, if every set in a diagram has the same cardinality and every morphism is a bijection, the transfinite composition is well-defined (and in this case is also a bijection). We only ensure the smallest morphisms* $(s,0) \to (s,1)$ *are in* $\mathscr{T}_{\mathbf{P}}(\mathcal{C})$.

Proposition 3.3.4. *Let* $\overline{\mu} : \mathcal{I} \to \mathcal{S}ets$ *be* **P***-mutation path.*

Let $\overline{\mu}^{-1} : \mathcal{I} \to \mathcal{S}ets$ *be a functor given by*

$$\overline{\mu}^{-1}(s,i) := \overline{\mu}(1-s, 1-i)$$
$$\overline{\mu}^{-1}((s_i) \to (t,j)) := \overline{\mu}((1-t, 1-j) \to (1-s, 1-i)).$$

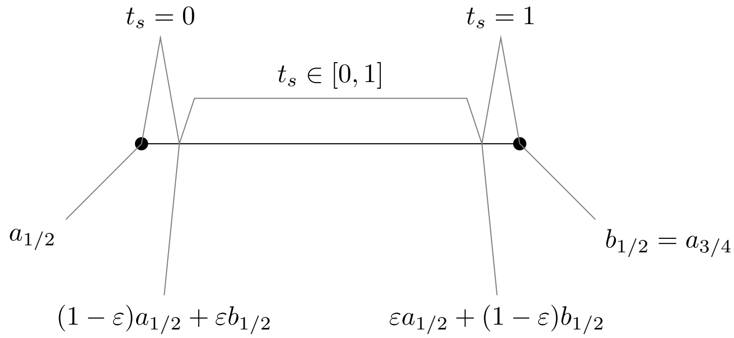


FIGURE 12. Schematic of t_s for $s \in [\frac{1}{2}, \frac{3}{4}]$.

Then $\overline{\mu}^{-1}$ is also a $\mathbf{P}$-mutation path.

Proof. Since $\mathscr{T}_{\mathbf{P}}(\mathcal{C})$ is a groupoid inside $\mathcal{S}ets$ the definition of $\overline{\mu}^{-1}$ amounts to reversing the order of the objects and taking the inverse morphism between each pair of objects in the image. $\square$

Example 3.3.5. *Let $\mu : T \to T'$ be a continuous $\mathbf{P}$-mutation. Let $\bar{\mu} : \mathcal{I} \to \mathcal{S}ets$ be defined in the following way. On objects,*

$$\overline{\mu}(s,0) = (T \setminus f^{-1}[0,s)) \cup g^{-1}[0,s)$$
$$\overline{\mu}(s,1) = (T \setminus f^{-1}[0,s]) \cup g^{-1}[0,s].$$

By Proposition 3.1.4, for each $s \in [0,1]$, μ defines a $\mathbf{P}$-mutation $\overline{\mu}(s,0) \to \overline{\mu}(s,1)$. Define $\overline{\mu} : \overline{\mu}(s,0) \to \overline{\mu}(s,1)$ to be precisely this $\mathbf{P}$-mutation. Thus each continuous $\mathbf{P}$-mutation is a $\mathbf{P}$-mutation path.

Below we construct some variables i_s, a_s, b_s, and t_s for each $s \in [0,1]$. We use these to show how a long sequence of continuous mutations can be considered as a mutation path.

Construction 3.3.6. *Let $\overline{\mu}$ be a long sequence of continuous mutations and fix $0 < \varepsilon << 1$. For each $s \in (0,1)$, there exists $i \in \mathbb{Z}$ such that*

$$\left(\frac{\tan^{-1} i}{\pi} + \frac{1}{2}\right) \leq s < \left(\frac{\tan^{-1}(i+1)}{\pi} + \frac{1}{2}\right).$$

Note that since the right inequality is strict, there is a unique such i for each $s \in (0,1)$. Denote it by i_s. Let

$$a_s := \left(\frac{\tan^{-1} i_s}{\pi} + \frac{1}{2}\right) \qquad\qquad b_s := \left(\frac{\tan^{-1}(i_s+1)}{\pi} + \frac{1}{2}\right).$$

Note that if $i_s = i_{s'}$ for s and s' then $a_s = a_{s'}$ and $b_s = b_{s'}$.

We now define t_s; the reader is encouraged to reference Figure 12.

$$t_s := \begin{cases} 0 & s \in [a_s,\ (1-\varepsilon)a_s + \varepsilon b_s] \\ (s - (1-\varepsilon)a_s - \varepsilon b_s)/((1-2\varepsilon)(b_s - a_s)) & s \in [(1-\varepsilon)a_s + \varepsilon b_s,\ \varepsilon a_s + (1-\varepsilon)b_s] \\ 1 & s \in (\varepsilon a_s + (1-\varepsilon)b_s,\ b_s) \end{cases}$$

Proposition 3.3.7. *Let $\overline{\mu}$ be a (long) sequence of continuous mutations. Then $\overline{\mu}$ is also a mutation path.*

Proof. We may consider $\overline{\mu}$ as a functor $\mathcal{I} \to \mathcal{S}ets$ in the following way. We now make our assignment on objects:

$$(s,0) \mapsto \begin{cases} {}_iT_0 = {}_{i-1}T_1 & s \in [a_s, (1-\varepsilon)a_s + \varepsilon b_s) \\ ({}_{i_s}T \setminus {}_if^{-1}[0,t_s)) \cup {}_ig^{-1}[0,t_s) & s \in [(1-\varepsilon)a_s + \varepsilon b_s, \varepsilon a_s + (1-\varepsilon)b_s] \\ {}_iT_1 = {}_{i+1}T_0 & s \in (\varepsilon a_s + (1-\varepsilon)b_s, b_s) \end{cases}$$

$$(s,1) \mapsto \begin{cases} {}_iT_0 = {}_{i-1}T_1 & s \in [a_s, (1-\varepsilon)a_s + \varepsilon b_s) \\ ({}_{i_s}T \setminus {}_if^{-1}[0,t_s]) \cup {}_ig^{-1}[0,t_s] & s \in [(1-\varepsilon)a_s + \varepsilon b_s, \varepsilon a_s + (1-\varepsilon)b_s] \\ {}_iT_1 = {}_{i+1}T_0 & s \in (\varepsilon a_s + (1-\varepsilon)b_s, b_s). \end{cases}$$

When $s \in [(1-\varepsilon)a_s + \varepsilon b_s, \varepsilon a_s + (1-\varepsilon)b_s]$ we see by Proposition 3.1.4 that the morphism $* : (s,0) \to (s,1)$ is sent to a (possibly trivial) $\mathbf{P}$-mutation. When $s \in [a_s, (1-\varepsilon)a_s + \varepsilon b_s) \cup (\varepsilon a_s + (1-\varepsilon)b_s, b_s)$ the morphism $* : (s_0) \to (s_1)$ is sent to the trivial $\mathbf{P}$-mutation on $\bar{\mu}(s,0)$. This defines a mutation path. □

Remark 3.3.8. *The ε "padding" in Construction 3.3.6 is necessary to prove Proposition 3.3.7. If we did not have the "padding" we would attempt to assign two $\mathbf{P}$-mutations, or their composition, to morphisms such as $* : (\frac{3}{4},0) \to (\frac{3}{4},1)$.*

Let $\overline{\mu}$ be a long sequence of continuous $\mathbf{P}$-mutations. We see in Proposition 3.3.7 that for a fixed ε the inverse path $\overline{\mu}^{-1}$ *agrees with the inverse* sequence $\{{}_{-i}\mu\}_{i\in\mathbb{Z}}$. *Thus when working with a long sequence of continuous mutations we need not be specific about which inverse we take as long as an ε has been chosen.*

Definition 3.3.9. *Let $\overline{\mu}_1, \overline{\mu}_2 : \mathcal{I} \to \mathcal{S}ets$ be two $\mathbf{P}$-mutation paths and suppose $\overline{\mu}_1(1,0) = \overline{\mu}_2(0,0)$ and $\overline{\mu}_1(1,1) = \overline{\mu}_2(0,1)$.*

We define the composition of $\mathbf{P}$-mutation paths, denoted $\overline{\mu}_1 \cdot \overline{\mu}_2$ in the following way:

$$\overline{\mu}_1 \cdot \overline{\mu}_2(s,i) := \begin{cases} \overline{\mu}_1(2s,i) & 0 \le s \le \frac{1}{2} \\ \overline{\mu}_2(2s-1,i) & \frac{1}{2} \le s \le 1. \end{cases}$$

$$\overline{\mu}_1 \cdot \overline{\mu}_2((s,0) \to (s,1)) := \begin{cases} \overline{\mu}_1((2s,0) \to (2s,1)) & 0 \le s \le \frac{1}{2} \\ \overline{\mu}_2((2s-1,0) \to (2s-1,1)) & \frac{1}{2} \le s \le 1. \end{cases}$$

Proposition 3.3.10. *Let $\overline{\mu}_1$ and $\overline{\mu}_2$ be $\mathbf{P}$-mutation paths such that*

$$\overline{\mu}_1(1,0) = \overline{\mu}_2(0,0) \qquad \text{and} \qquad \overline{\mu}_1(1,1) = \overline{\mu}_2(0,1).$$

Then $\overline{\mu}_1 \cdot \overline{\mu}_2$ is a $\mathbf{P}$-mutation path.

Proof. By assumption the definitions agree at $\frac{1}{2}$. For $0 \le s < \frac{1}{2}$ and $\frac{1}{2} < t \le 1$, the morphism $\overline{\mu}_1 \cdot \overline{\mu}_2 * : \overline{\mu}_1 \cdot \overline{\mu}_2(s,i) \to \overline{\mu}_1 \cdot \overline{\mu}_2(t,j)$ is the composition

$$\overline{\mu}_1 \cdot \overline{\mu}_2(s,i) \to \overline{\mu}_1 \cdot \overline{\mu}_2\left(\frac{1}{2},0\right) \to \overline{\mu}_1 \cdot \overline{\mu}_2\left(\frac{1}{2},1\right) \to \overline{\mu}_1 \cdot \overline{\mu}_2(t,j).$$

□

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Remark 3.3.11. *The composition of two long sequences of continuous mutations as in Definition 3.3.9 is not a long sequence of continuous mutations as in Proposition 3.3.7.*

3.4. **Connection to $\mathbf{N}_{\overline{\mathbb{R},S}}$-mutations.** The more interesting pictures of $\mathbf{N}_{\overline{\mathbb{R},S}}$ mutations (Section 2.2) are those of continuous mutations. In this section we use our geometric models to show how one may picture a continuous **E**-mutation by drawing the corresponding continuous $\mathbf{N}_{\overline{\mathbb{R},S}}$-mutation. In particular, those continuous mutations that cannot be described as any type of sequence of mutations, which is discrete. Consider $A_{\mathbb{R}}$ with straight descending orientation. Let T be $\{P_x, M_{[x,x]} \mid x \in \mathbb{R}\} \cup \{P_{+\infty}\}$ and $\phi : \mathbb{R} \to (0,1)$ be some order *reversing* bijection. Let $f : \{P_x \mid x \in \mathbb{R}\} \to [0,1]$ be given by $P_x \mapsto \phi(x)$. Let $g : \{I_x \mid x \in \mathbb{R}\} \to [0,1]$ be given by $I_x \mapsto \phi(x)$ and let $T' = \{I_x, M_{[x,x]} \mid x \in \mathbb{R}\} \cup \{P_{+\infty}\}$. Then we have a continuous mutation $T \to T'$.

We would like to show what this looks like in terms of arcs. Of course, we cannot depict each of the mutations at time t for all $t \in (0,1)$, as we do not have uncountably-many pages. However, we can think of the process as an animation and take a few select frames so that we have the general idea. In Figure 13, we only show 6 frames. One could make a proper animation at a sufficiently high frame rate to get the full effect.

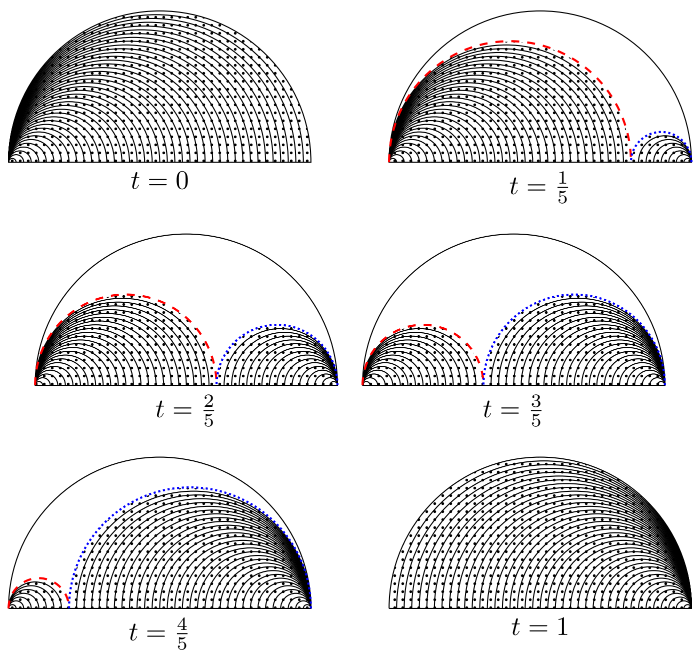

FIGURE 13. Six frames depicting a continuous $\mathbf{N}_{\overline{\mathbb{R}}}$-mutation. (All arcs have orientation left to right.) In the frames between $t = 0$ and $t = 1$ we mutate the red (dashed) arc to the blue (dotted) arc. The first and sixth frames are T and T', respectively. The other four frames are at time $\frac{i}{5}$ for $i \in \{1, 2, 3, 4\}$. We include ≈ 40 arcs of the uncountably many in the loosely-dotted region in the same way one includes level curves in a topographical map.

3.5. **Space of Mutations.** In this section we define the space of mutations (Definition 3.5.2) which generalizes the exchange graph of a cluster structure. The intent is to view mutation paths (Definition 3.3.2) as paths in a topological space just as a sequence of mutations of a cluster structure forms a path in the exchange graph. The majority of this section is for cluster theories in general. However, its purpose is to study **E**-clusters in the future and so we return our attention to **E**-clusters at the end of the section.

Since $\mathcal{C}$ is skeletally small, KRS, and additive, we see $\mathscr{T}_{\mathbf{P}}(\mathcal{C})$ is small. In particular, the class of morphisms in $\mathscr{T}_{\mathbf{P}}(\mathcal{C})$ is a set. Denote the set of mutations by $(\mathscr{T}_{\mathbf{P}}(\mathcal{C}))_1$.

Notation 3.5.1. *Let $\overline{\mu}$ be a $\mathbf{P}$-mutation path and denote by $p_{\overline{\mu}}$ the induced function from $[0,1]$ to $(\mathscr{T}_{\mathbf{P}}(\mathcal{C}))_1$.*

Definition 3.5.2. *We define the set $\mathbf{P}(\mathcal{C}) \subset (\mathscr{T}_{\mathbf{P}}(\mathcal{C}))_1$ to be the set containing all (trivial) $\mathbf{P}$-mutations.*

We give the set of $\mathbf{P}$-mutations a topology in the following way. Consider $[0,1]$ with the usual topology. A set $U \subset \mathbf{P}(\mathcal{C})$ is called open if, for all $p_{\overline{\mu}} : [0,1] \to \mathbf{P}(\mathcal{C})$ induced by a $\mathbf{P}$-mutation $\overline{\mu}$, $p_{\overline{\mu}}^{-1}(U)$ is open in $[0,1]$. We call $\mathbf{P}(\mathcal{C})$ the space of $\mathbf{P}$-mutations.

Proposition 3.5.3. *Then the open sets in Definition 3.5.2 form a topology on $\mathbf{P}(\mathcal{C})$.*

Proof. Trivially, both $\emptyset$ and $\mathbf{P}(\mathcal{C})$ are open. Suppose $p_{\overline{\mu}} : [0,1] \to \mathbf{P}(\mathcal{C})$ is induced by a $\mathbf{P}$-mutation path $\overline{\mu}$. Let $\{U_1, \dots, U_n\}$ be open in $\mathbf{P}(\mathcal{C})$. Since $\displaystyle\bigcap_{i=1}^{n} p_{\overline{\mu}}^{-1}(U_i) = p_{\overline{\mu}}^{-1}\left(\bigcap_{i=1}^{n} U_i\right)$, we see that $\bigcap_{i=1}^{n} U_i$ is open in $\mathbf{P}(\mathcal{C})$. Now consider a collection $\{U_\alpha\}$ of open sets in $\mathbf{P}(\mathcal{C})$. Since $\displaystyle\bigcup_{\alpha} p_{\overline{\mu}}^{-1}(U_\alpha) = p_{\overline{\mu}}^{-1}\left(\bigcup_{\alpha} U_\alpha\right)$, we see that $\bigcup_\alpha U_\alpha$ is open in $\mathbf{P}(\mathcal{C})$. This concludes the proof. □

Remark 3.5.4. *We consider a $\mathbf{P}$-cluster T to be the trivial mutation $T \to T$ in $\mathbf{P}(\mathcal{C})$. We wish to consider paths that start and end at clusters rather than at mutations (see Proposition 3.5.6).*

Proposition 3.5.5. *The space of $\mathbf{P}$-mutations is non-Hausdorff if and only if $\mathscr{T}_{\mathbf{P}}(\mathcal{C})$ contains at least one nontrivial mutation.*

Proof. First suppose there is a nontrivial $\mathbf{P}$-mutation $\mu : T \to (T \setminus \{X\}) \cup \{Y\}$ in $\mathscr{T}_{\mathbf{P}}(\mathcal{C})$. Let $\overline{\mu}$ be the $\mathbf{P}$-mutation path that induces the path $p_{\overline{\mu}}$ given by

$$p_{\overline{\mu}}(t) = \begin{cases} T & t < 1 \\ \mu & t = 1 \end{cases}$$

Let U be an open set that contains μ. If $T \notin U$ then $p_{\overline{\mu}}^{-1}(U)$ is not open. This would be a contradiction and so $T \in U$. Thus, for any $\mathbf{P}$-mutation $\mu : T \to T'$ and open set U containing μ, we have $T, T' \in U$ as well. Therefore, $\mathbf{P}(\mathcal{C})$ is not Hausdorff.

How suppose $\mathscr{T}_{\mathbf{P}}(\mathcal{C})$ contains only trivial $\mathbf{P}$-mutations, which are identity functions. Then $\mathscr{T}_{\mathbf{P}}(\mathcal{C})$ is a discrete category. Therefore, $\mathbf{P}(\mathcal{C})$ has the discrete topology and is thus Hausdorff □

Proposition 3.5.6. *Let $p : [0,1] \to \mathbf{P}(\mathcal{C})$ be a path in $\mathbf{P}(\mathcal{C})$. Then there is a path $q : [0,1] \to \mathbf{P}(\mathcal{C})$ whose endpoints are clusters (see Remark 3.5.4) such that p and q are homotopic.*

Proof. Let $p : [0,1] \to \mathbf{P}(\mathcal{C})$ be a path in $\mathbf{P}(\mathcal{C})$, let T_0 be the source of $p(0)$, and let T_1 the target of $p(1)$.

For any $0 < \varepsilon << \frac{1}{2}$, let $q_\varepsilon : [0,1] \to \mathbf{P}(\mathcal{C})$ be the path given by:

$$q_\varepsilon(t) = \begin{cases} T_0 & \text{if } t < \varepsilon \\ T_1 & \text{if } (1-\varepsilon) < t \\ p\left((t-\frac{1}{2})(1-2\varepsilon)+\frac{1}{2}\right) & \text{if } \varepsilon \leq t \leq (1-\varepsilon) \end{cases}$$

We see that q_ε is homotopic to the composition of three paths. The first is constant at T_0 except the last point is $p(0)$. The second is p. The third is constant at T_1 except the first point is $p(1)$. In particular, the first and third path are induced by $\mathbf{P}$-mutation paths. Thus, q_ε is indeed a path. Let $q_0 = p$.

Fix a $0 < \varepsilon << \frac{1}{2}$. Let $H : [0,1] \times [0,1] \to \mathbf{P}(\mathcal{C})$ be given by:

$$H(t,s) := q_{s\varepsilon}(t).$$

Let U be open in $\mathbf{P}(\mathcal{C})$. If the inverse image of U does not contain $p(0)$ or $p(1)$ then $H^{-1}(U)$ is open in $[0,1] \times [0,1]$.

Now suppose U contains $p(0)$. By the proof of Proposition 3.5.5 we see that $T_0 \in U$ as well. Similarly, if $p(1) \in U$ then $T_1 \in U$. Therefore, if U is open in $\mathbf{P}(\mathcal{C})$ then $H^{-1}(U)$ is open in $[0,1] \times [0,1]$, completing the proof. □

Definition 3.5.7. *Let T_1 and T_2 be $\mathbf{P}$-clusters of $\mathcal{C}$.*

1. *We say T_2 is <u>reachable</u> from T_1 if there is a path $p : [0,1] \to \mathbf{P}(\mathcal{C})$ such that $p(0) = T_1$ and $p(1) = T_2$.*
2. *We say T_2 is <u>strongly reachable</u> from T_1 if there is a $\mathbf{P}$-mutation path $\overline{\mu}$ that (i) comes from a long sequence of continuous $\mathbf{P}$-mutations and (ii) induces a path $p_{\overline{\mu}} : [0,1] \to \mathbf{P}(\mathcal{C})$ such that $p_{\overline{\mu}}(0) = T_1$ and $p_{\overline{\mu}}(1) = T_2$.*

Theorem 3.5.8. *Let $A_\mathbb{R}$ be the continuous quiver of type A with straight descending orientation. The cluster of injectives, $\mathcal{I}nj$ is strongly reachable from the cluster of projectives, $\mathcal{P}roj$.*

Proof. In Proposition 3.2.5 we see there is a sequence of $\mathbf{E}$-mutations $\{\mu_1, \mu_2\}$ to mutate $\mathcal{P}roj$ to $\mathcal{I}nj$. Choose some $0 < \varepsilon << \frac{1}{2}$ and note that a sequence of $\mathbf{E}$-mutations is also a long sequence of $\mathbf{E}$-mutations. Then, as in Proposition 3.3.7, we have a $\mathbf{E}$-mutation path $\overline{\mu}$ with source $\mathcal{P}roj$ and target $\mathcal{I}nj$. □

Open Questions. Let $A_{\mathbb{R},S}$ be a continuous quiver of type A and $\mathscr{T}_\mathbf{E}(\mathcal{C}(A_{\mathbb{R},S}))$ the $\mathbf{E}$-cluster theory of $\mathcal{C}(A_{\mathbb{R},S})$ (Definition 1.2.6).

- Is the space $\mathbf{E}(\mathcal{C}(A_{\mathbb{R},S}))$ path connected?
- If $\mathbf{E}(\mathcal{C}(A_{\mathbb{R},S}))$ is not path-connected, what do its path components look like? What does the path component containing the cluster of projectives look like?
- If $\mathbf{E}(\mathcal{C}(A_{\mathbb{R},S}))$ is path-connected, are there clusters T and T' that are reachable but not strongly reachable (Definition 3.5.7) from one another.
- If $\mathbf{E}(\mathcal{C}(A_{\mathbb{R},S}))$ is path connected, which clusters are strongly reachable from the cluster of projectives?

Declarations

The author has no competing interests to declare that are relevant to the content of this article.

The author did not collect or produce any data. Thus there is no data to make available.

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