Cohomological Invariants of Structurable Algebras

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Thesis submitted to the

Faculty of Sciences

Ghent University

for the degree of

Doctor of Science: Mathematics

September 2022

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Acknowledgements

I am sincerely grateful to my supervisor, Tom De Medts, for entrusting me with the project, getting me started on the right trajectory, asking good questions, and appreciating the results along the way.

In the category of mathematical support, I have Alberto Elduque, Skip Garibaldi, and Jari Desmet to thank for reviewing my draft manuscripts, Victor Petrov, for suggesting to work together and for teaching me so much, and Philippe Gille, for generously assisting at a point where it really mattered, and for inviting me to Lyon.

My gratitude extends to the remaining members of the examining committee, Wouter Castryck, Anneleen De Schepper, Jean-Pierre Tignol, Marnix Van Daele, and Hendrik Van Maldeghem for their thoughtful comments and valuable presence.

Not for the first time, I acknowledge Juana Sanchez Ortega, who introduced me to mathematics research as my supervisor at the University of Cape Town, and encouraged me to come to Ghent.

To my office mates, Michiel Van Couwenberghe, Jens Bossaert, Jeroen Meulewaeter, Paulien Jansen, Jari Desmet, Dibyayoti Jena, and Ahmed Umer Ashraf, thanks for being mentors and friends. I also thank Sam Perez for his very capable administration and welcoming approach.

My family, Wil, Ursula, and Matthew Rigby had an enormous impact on everything from the start. Their support has been the wind in my sails. Marit de Haan, your role defies any description – your closeness, connection, and cheerfulness changed everything for the better, and I dare to say you understand more than anyone what this thesis actually means. Thank you to Willem de Haan, Marianne Kootstra, and Nienke de Haan for being like family too. And thank you to Freddy Caemaert for welcoming me on my very first day in Ghent.

My friends, thanks for being here to celebrate this milestone, and for all the good times that took the edge off work. A special mention goes to Johannes Roth who has been a real ally through all of this, to Serkan Yıldız who created the photograph on the cover, and to Wojciech Stadejek who brewed a West Coast IPA for the occasion of my public defence.

I am grateful to the FWO for financially supporting this research project.

Simon Rigby
Ghent, 2022

Publications

Some of the work in this thesis has been published or circulated in the following documents:

[R1] S. W. Rigby, *The tensor product of two octonion algebras and its structure group*, Mini-Workshop: Rank One Groups and Exceptional Algebraic Groups (T. De Medts, B. Mühlherr, and A. Stavrova, eds.), Oberwolfach Reports, no. 52/2019, Mathematisches Forschungsinstitut Oberwolfach, 2019, 3254–3256.

[PR] V. Petrov and S. W. Rigby, *The Allison–Faulkner construction of* E_8, Canadian Mathematical Bulletin **65** (2022), no. 3, 686–701.

[R2] S. W. Rigby, *Galois cohomology of algebraic groups acting on the tensor product of two composition algebras*, arXiv preprint (2021), no. arXiv:2112.09677v1, 1–81.

The report [R1] includes a few results from Chapter V. The paper [PR] has some overlap with Chapter IV. The preprint [R2] has substantial overlap with Chapters III–VI and VIII, and is intended for publication.

The following publications on other mathematical topics of interest to the author were written during the term of this project, but are not related to the topic of this thesis:

[R3] S. W. Rigby, *Tensor products of Steinberg algebras*, Journal of the Australian Mathematical Society **111** (2021), no. 1, 111–126.

[RV] S. W. Rigby and T. Van Den Hove, *A classification of ideals in Steinberg and Leavitt path algebras over arbitrary rings*, Journal of Algebra **588** (2021), 200–249.

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Contents

Introduction

Structurable algebras were first introduced for the purpose of constructing exceptional Lie algebras over arbitrary fields. (It only works if the characteristic of the underlying field k is not 2 or 3.) Roughly speaking, one adds up various copies of the algebra itself, together with some of its subspaces and spaces of linear operators, and with a clever definition this becomes a Lie algebra. Alternative and Jordan algebras are classical examples of structurable algebras. The tensor product of two octonion algebras, which we call a bioctonion algebra, provides a more exotic example.

The topic of this project was inspired by the success that cohomological invariants have had in other areas of algebra, such as Jordan algebras, central simple algebras (with involutions), and quadratic form theory. A cohomological invariant is a function that assigns to an algebraic object (for example, an algebra or a quadratic form) a unique element of a Galois cohomology group. A cohomological invariant must be compatible with base change, so that extending scalars on the algebraic side commutes with extending scalars on the cohomological side.

Applications of cohomological invariants are numerous. The first and most basic application is to tell whether two objects are isomorphic or not. Depending on the invariant, it may do this job more or less successfully. For example, the invariant δ that assigns the symbol $(a)\cdot(b) \in H^2(k, \boldsymbol{\mu}_2)$ to a quaternion algebra $(a, b)_k$ is injective and so it classifies quaternion algebras up to isomorphism.

Another application is that the invariant could detect some of the object's properties. A famous example is the Serre–Rost invariant of exceptional Jordan algebras (Albert algebras). This invariant sends an Albert algebra J to an element $g_3(J) \in H^3(k, \mathbb{Z}/3\mathbb{Z})$ who knows if J is a division algebra: $g_3(J) \neq 0$ if and only if J is a division algebra.

A third application, which we encounter once in this project (33.13), is that invariants can sometimes prove the existence of something we cannot see directly. If we know that an invariant (of a type of algebra, say) is not identically zero, then there must be some algebra on which it is nonzero, even if the invariant vanishes on all the known examples.

At this point it is worth mentioning the right terminology. We are mostly interested in cohomological invariants of algebraic groups, which are natural maps $H^1(k, G) \to H^d(k, C)$ where G is an algebraic group and C is a Galois module like $\boldsymbol{\mu}_p$ or $\mathbb{Z}/p\mathbb{Z}$. The set $H^1(k, G) = H^1(\mathcal{G}al(k^{\text{sep}}/k), G(k^{\text{sep}}))$ is understood in the sense of Serre's nonabelian cohomology.

This notion relates to the previous examples if G is an automorphism group. For example, F_4 is the automorphism group of a (split) Albert algebra, so we say that g_3 is a cohomological invariant of F_4. Similarly, δ is a cohomological invariant of $\mathbf{PGL}_2$. Reversely, a cohomological invariant of $\mathbf{O}_n$ can be understood as a function from the set of isometry classes of quadratic forms on k^n to a Galois cohomology group, because $\mathbf{O}_n$ is the automorphism group of an n-dimensional quadratic form.

The hope was that some interesting new invariants could be found by studying some of the exceptional structurable algebras and, rather optimistically, that this might lead to new invariants of exceptional groups. This turns out to be difficult. Groups of type E_6, E_7, and E_8 have few large subgroups, so there are limitations in the range of Lie algebras (or groups) of type E_r that you can actually construct out of basic building blocks, whether the building blocks are structurable algebras or something else entirely.

In other words, the constructions we have are not surjective. If new invariants are discovered in this way, they are at best defined on a subset of $H^1(k, E_r)$, like the set of Lie algebras of type E_r over k with a prescribed grading. Nevertheless, it can be interesting to see a glimpse of an invariant even if you are not sure whether it is defined on all of $H^1(k, E_r)$.

At a certain point during the project, we also became interested in classifying all the invariants of some groups. Once the realisation hit that cohomological invariants are very hard to find, the thought crossed my mind that perhaps it would be easier in some cases to prove that there are no more. The set of cohomological invariants of G with coefficients in $\mathbb{Z}/2\mathbb{Z}$ is an abelian group, and also a module for the cohomology ring $H^\bullet(k, \mathbb{Z}/2\mathbb{Z})$. Sometimes it is possible to calculate this group $\mathrm{Inv}(G, 2)$ and give a basis for it, or a set of generators. This is the subject of the final chapter of the thesis.

There were some other minor objectives to the research, like building up knowledge of the exceptional structurable algebras. For example, we did not have any criteria to determine if a bioctonion algebra is a division algebra, let alone a cohomological criterion. (More correctly, we knew that a certain 14-dimensional Albert quadratic form is anisotropic if and only if the algebra is division, but only had a proof in characteristic 0.) Another goal of the project was to study some of the existing invariants, like Rost invariants, to understand how they are expressed in the structurable algebras and whether they can be used for any applications.

Data

This thesis contains a lot of data: on algebraic groups, Lie algebras, structurable algebras, and cohomological invariants. In the final results, we try to organise this data and report it as systematically as possible, often in the form of tables. The focus of the project is on invariants – not just cohomological invariants, but also numerical invariants like degree and dimension, (Dynkin) diagrams, and occasionally quadratic forms or root systems.

We try to encode this data in a unique or at least a standard way. For example, a split semisimple group is written as an almost-direct product of some simple subgroups. A reductive group is specified by its centre, its semisimple part, and their intersection. A quadratic form is written as an element of the Witt ring, rather than as a function. This is all done to make it easier for us, as readers, to recognise and compare objects of the same type even if they are represented on different spaces.

Description	Parameters	$(A,-)$	$\Theta(A,-)$		
(a) Associative					
Orthogonal involutions	$n \geq 1$	$\mathbb{M}_n(k)$	n^2,	$\frac{1}{2}n(n-1)$,	n
Symplectic involutions	$n \geq 2$ even	$\mathbb{M}_{n/2}(\mathbb{H})$	n^2,	$\frac{1}{2}n(n+1)$,	n
Unitary involutions	$n \geq 2$ even	$\mathbb{M}_{n/2}(\mathbb{B})$	$\frac{1}{2}n^2$,	$\frac{1}{4}n^2$,	n
(b) Special Jordan					
Quadratic form type	$n \geq 3$	$\mathcal{J}Spin_{n-1}(k)$	n,	0,	2
Orthogonal type	$n \geq 3$	$\mathcal{H}_n(k)$	$\frac{1}{2}n(n+1)$,	0,	n
Sympletic type	$n \geq 3$	$\mathcal{H}_n(\mathbb{H})$	$n(2n-1)$,	0,	n
Unitary type	$n \geq 3$	$\mathcal{H}_n(\mathbb{B})$	n^2,	0,	n
(c) Hermitian type					
Orthogonal type	$n \geq 2$, $d \geq 1$ $(n,d) \neq (2,2)$	$S(\mathbb{M}_n(k), k^{nd})$	$n^2 + nd$,	$\frac{1}{2}n(n-1)$,	$2n$
Symplectic type	$n \geq 2$ even, $d \geq 1$	$S(\mathbb{M}_{n/2}(\mathbb{H}), k^{2nd})$	$n^2 + 2nd$,	$\frac{1}{2}n(n+1)$,	n
Unitary type	$n \geq 2$ even, $d \geq 1$ $(n,d) \neq (2,1)$	$S(\mathbb{M}_{n/2}(\mathbb{B}), k^{nd})$	$\frac{1}{2}n^2 + nd$,	$\frac{1}{4}n^2$,	n

Table 1: Infinite families of central simple structurable algebras over a separably closed field of characteristic $\neq 2, 3, 5$, and their Θ-invariants, $\Theta(A,-) = (\dim A, \dim \operatorname{Skew}(A,-), \deg(A,-))$. Algebras are listed under (a) if they are associative, (b) if they are special Jordan algebras, and (c) if they are structurable algebras of hermitian forms, not isomorphic to quartic Cayley algebras, and not already under (a) or (b).

Outline

Chapter I is all about (central simple) structurable algebras. We clarify some aspects of the existing classification theorem by working over a separably closed field. The results are quite dry and the methods are basic, but it is a useful exercise to have done. The classification of central simple structurable algebras appears in Tables 1 and 2.

Chapter II studies the Tits–Kantor–Koecher (TKK) construction of $\mathbb{Z}$-graded Lie algebras over arbitrary fields of characteristic not 2 or 3. Labelled Dynkin diagrams are the preferred invariant to describe a $\mathbb{Z}$-grading on a simple (algebraic) Lie algebra, and the results based on various inputs are presented in Table 4. The Allison–Faulkner (AF) construction is also studied in this chapter; it is a generalisation of the TKK construction with a different grading.

In Chapter III, we calculate the automorphism groups of exceptional structurable algebras and some of the classical ones too. The effort is made here to work rationally. We also calculate the derivation algebras, and the split and quasi-split forms of the semisimple structure groups. The split forms of these automorphism and structure groups are presented in Table 5.

Description	$(A,-)$	$\Theta(A,-)$		
(d) Exceptional Jordan				
Albert algebra	$\mathcal{H}_3(\mathbb{O})$	27,	0,	3
(e) Tensor product algebras				
Octonion algebra	$\mathbb{O}$	8,	7,	2
$(8,2)$-product	$\mathbb{O}\otimes\mathbb{B}$	16,	8,	4
$(8,4)$-product	$\mathbb{O}\otimes\mathbb{H}$	32,	10,	4
$(8,8)$-product	$\mathbb{O}\otimes\mathbb{O}$	64,	14,	8
Smirnov algebra	$T(\mathbb{O})$	35,	7,	7
(f) Skew-dimension one				
Quartic 2×2 matrices	$M(k)$	4,	1,	4
Quartic Cayley algebra	$M(k^3)$	8,	1,	4
Green algebra	$M(\mathcal{H}_3(k))$	14,	1,	4
Blue algebra	$M(\mathcal{H}_3(\mathbb{B}))$	20,	1,	4
Red algebra	$M(\mathcal{H}_3(\mathbb{H}))$	32,	1,	4
Brown algebra	$M(\mathcal{H}_3(\mathbb{O}))$	56,	1,	4

Table 2: Exceptional central simple structurable algebras over a separably closed field of characteristic $\neq 2,3,5$, and their Θ-invariants, $\Theta(A,-) = (\dim A, \dim \mathrm{Skew}(A,-), \deg(A,-))$. Contains all the algebras not in Table 1.

Chapter IV is about Galois cohomology and cohomological invariants, with a focus on applications to structurable algebras and quadratic forms. Most of this chapter is foundational, to create the right setting for later results.

Chapter V is on the theory of bicomposition algebras, particularly bioctonion algebras. A calculation of the full (reductive) structure groups turns out to be informative and leads to plenty of applications, like a criterion for bioctonion division algebras and a new proof of Rost's theorem on 14-dimensional quadratic forms. There is an interesting diversion at the end about matrix-factorising a certain large octic polynomial.

Chapter VI is the first time we seriously deal with cohomological invariants of structurable algebras. The focus here is again on bioctonion algebras, whose invariants can be described very concretely, have some applications, and express themselves in the Lie algebras too.

Chapter VII is about the exceptional algebras of skew-dimension one, especially Brown algebras. We study all of the main constructions and calculate their trace forms and Rost invariants. The Rost invariant determines certain properties of nondivision Brown algebras, and also exposes some limitations in the constructions available to us. Some examples are included at the end.

Chapter VIII exists entirely to prove one main theorem: the classification of cohomological invariants of $\mathbf{Spin}_{14}$, the simply connected cover of the special orthogonal group $\mathbf{O}^+_{14}$. Along the way, we classify the cohomological invariants of several other groups and classes of quadratic forms. The results are summarised in Table 3.

F	Restr.	$\mathrm{Inv}(F,2)$	Generators	Ref.
$\mathbf{PGO}_4$-torsors	none	$H(k)^{\oplus 4}$	$\{1, y_1, y_2, y_4\}$	35.9
$(G_2 \times G_2) \rtimes S_2$-torsors	none	$H(k)^{\oplus 4}$	$\{1, b_1, b_3, b_6\}$	35.9
PI^3_{12}	none	$H(k)^{\oplus 2} \oplus J_1(k)$	$\{1, z_3\} \cup \{h \cdot z_5 : h \in J_1(k)\}$	37.9
I^3_{12}	none	$H(k)^{\oplus 3} \oplus J_1(k)$	$\{1, z_3, z_5\} \cup \{z^h : h \in J_1(k)\}$	37.8
$\mathbf{Spin}_{12}$-torsors	none	isomorphic to $\mathrm{Inv}(I^3_{12})$		37.8
PI^3_{14}	none	$H(k)^{\oplus 2} \oplus J_2(k)$	$\{1, a_3\} \cup \{h \cdot a_6 : h \in J_2(k)\}$	38.10
I^3_{14}	$\sqrt{-1} \in k$	$H(k)^{\oplus 4}$	$\{1, a_3, a_6, a_7\}$	38.15
$\mathbf{Spin}_{14}$-torsors	$\sqrt{-1} \in k$	isomorphic to $\mathrm{Inv}(I^3_{14})$		38.15

Table 3: Structure of various groups $\mathrm{Inv}(F,2)$ of mod 2 cohomological invariants where F is a functor $\mathsf{Fields}_{/k} \to \mathsf{Sets}$ and k is a field of characteristic not 2 or 3, subject to a possible restriction indicated in the second column. Definitions of $H(k)$ and $J_m(k)$ are in 15.1 and 16.2. Generators of $\mathrm{Inv}(F,2)$ as an $H(k)$-module are also listed.

Concluding remarks

It is clear that some structurable algebras are more interesting than others. And for most questions in algebraic group theory there are more suitable methods. However, structurable algebras do make an alternative set of tools available. For understanding subgroups of exceptional groups and gradings on exceptional Lie algebras, they can provide an alternative to root system methods.

A lack of structure theory, like we have for Albert algebras, is a limitation. For some of the exceptional structurable algebras, I believe it would be useful to know more about subalgebras and their centralisers, stabilisers in the automorphism and structure groups, and possible Skolem–Noether type extension theorems.

Probably, some of the main results even in this thesis could have been achieved without using structurable algebras. But the algebras were part of the process all along and led to many questions and ideas without which I might not have known what to look for.

I am particularly glad to have finalised the classification of $\mathbf{Spin}_{14}$'s invariants mod 2, especially because it might be the last of the even Spin groups whose invariants we can hope to classify at this time. Based on essential dimension and representation-theoretic evidence, the situation becomes devastatingly more complex past the frontier $n = 14$. For all we know, the groups $\mathbf{Spin}_{2m}$ $(m \geq 8)$ might have lots of invariants in very high degrees. I still know very little about the invariants of $\mathbf{Spin}_{13}$, and I hope this will also be resolved sometime in the future.

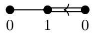
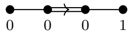
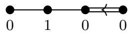
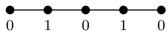
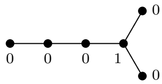
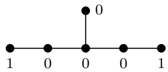
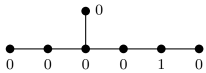
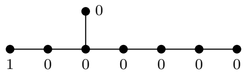
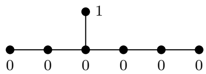
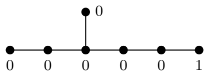
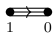
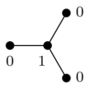
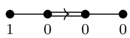
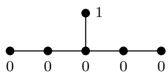
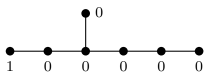
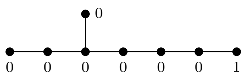
Algebra $(A,-)$	Type of $K(A,-)$	Labelled Dynkin diagram
Quaternion algebra	C_3	
Octonion algebra	F_4	
Octonion, nonstandard	C_4	
$(4,2)$-product algebra	A_5	
$(4,4)$-product algebra	D_6	
$(8,2)$-product	E_6	
$(8,4)$-product	E_7	
$(8,8)$-product	E_8	
Smirnov algebra	E_7	
Albert algebra	E_7	
Quartic 2×2 matrices	G_2	
Quartic Cayley	D_4	
Green algebra	F_4	
Blue algebra	E_6	
Red algebra	E_7	
Brown algebra	E_8	

Table 4: Labelled Dynkin diagrams of 5-graded simple Lie algebras $K(A,-)$, where $(A,-)$ is a central simple structurable algebra that is either exceptional, a $(4,m)$-product algebra, or an octonion algebra with nonstandard involution.

Algebra $(A,-)$	$\mathbf{Aut}(A,-)$	$(\mathbf{Str}(A,-)^\circ)^{\mathrm{der}}$	$\mathbf{Aut}(K(A,-))$
Quaternion algebra	$\mathbf{PGL}_2$	$\dfrac{\mathbf{SL}_2\times\mathbf{SL}_2}{\boldsymbol{\mu}_2}$	$\mathbf{PGSp}_6$
Octonion algebra	G_2	$\mathbf{Spin}_7$	F_4
Octonion, nonstandard	$\dfrac{\mathbf{SL}_2\times\mathbf{SL}_2}{\boldsymbol{\mu}_2}$	$\dfrac{\mathbf{SL}_2\times\mathbf{Sp}_4}{\boldsymbol{\mu}_2}$	$\mathbf{PGSp}_8$
$(4,2)$-product	$\mathbf{PGL}_2\times\mathbb{Z}/2\mathbb{Z}$	$\dfrac{\mathbf{SL}_2\times\mathbf{SL}_2\times\mathbf{SL}_2}{\boldsymbol{\mu}_2}$	$\mathbf{PGL}_6\rtimes\mathbb{Z}/2\mathbb{Z}$
$(4,4)$-product$^{(1)}$	$(\mathbf{PGL}_2\times\mathbf{PGL}_2)\rtimes\mathbb{Z}/2\mathbb{Z}$	$\dfrac{\mathbf{SL}_4\times\mathbf{O}_4^+}{\boldsymbol{\mu}_2}$	$\mathbf{PGO}_{12}^+\rtimes\mathbb{Z}/2\mathbb{Z}$
$(8,2)$-product	$G_2\times\mathbb{Z}/2\mathbb{Z}$	$\mathbf{Spin}_8$	$E_6^{\mathrm{ad}}\rtimes\mathbb{Z}/2\mathbb{Z}$
$(8,4)$-product	$G_2\times\mathbf{PGL}_2$	$\dfrac{\mathbf{Spin}_{10}\times\mathbf{SL}_2}{\boldsymbol{\mu}_2}$	E_7^{ad}
$(8,8)$-product$^{(1)}$	$(G_2\times G_2)\rtimes\mathbb{Z}/2\mathbb{Z}$	$\mathbf{Spin}_{14}$	E_8
Smirnov algebra	G_2	$\mathbf{SL}_7$	E_7^{ad}
Albert algebra	F_4	E_6^{sc}	E_7^{ad}
Quartic 2×2 matrices	$\boldsymbol{\mu}_3\rtimes\mathbb{Z}/2\mathbb{Z}$	$\mathbf{SL}_2$	G_2
Quartic Cayley$^{(2)}$	$(\mathbf{G}_m\times\mathbf{G}_m)\rtimes(S_3\times\mathbb{Z}/2\mathbb{Z})$	$\dfrac{\mathbf{SL}_2\times\mathbf{SL}_2\times\mathbf{SL}_2}{\boldsymbol{\mu}_2\times\boldsymbol{\mu}_2}$	$\mathbf{PGO}_8^+\rtimes S_3$
Green algebra$^{(3)}$	$\mathbf{SL}_3\rtimes\mathbb{Z}/2\mathbb{Z}$	$\mathbf{Sp}_6$	F_4
Blue algebra$^{(4)}$	$\dfrac{\mathbf{SL}_3\times\mathbf{SL}_3}{\boldsymbol{\mu}_3}\rtimes(\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z})$	$\dfrac{\mathbf{SL}_6}{\boldsymbol{\mu}_3}$	$E_6^{\mathrm{ad}}\rtimes\mathbb{Z}/2\mathbb{Z}$
Red algebra$^{(5)}$	$\dfrac{\mathbf{SL}_6}{\boldsymbol{\mu}_2}\rtimes\mathbb{Z}/2\mathbb{Z}$	$\mathbf{HSpin}_{12}$	E_7^{ad}
Brown algebra$^{(6)}$	$E_6^{\mathrm{sc}}\rtimes\mathbb{Z}/2\mathbb{Z}$	E_7^{sc}	E_8

Table 5: Exceptional central simple structurable algebras and the split forms of their automorphism groups, their semisimple structure groups, and the automorphism groups of their TKK Lie algebras.

Notes: All quotients are by diagonally-embedded central subgroups, except: $^{(2)}$ The central subgroup $\boldsymbol{\mu}_2^2\subset\mathbf{SL}_2^3$ is the one generated by $(-1,-1,1)$ and $(1,-1,-1)$. All semidirect products are nontrivial (i.e., not direct). In col. 3, the groups $\mathbb{Z}/2\mathbb{Z}$ and S_3 act by diagram automorphisms. In col. 1, the splitting is determined by: $^{(1)}$ $\mathbb{Z}/2\mathbb{Z}$ exchanges the maximal simple subgroups. $^{(2)}$ S_3 acts on $\mathbf{G}_m\times\mathbf{G}_m$ by $(1\ 2\ 3)\cdot(x,y)=(x^{-1}y^{-1},x)$ and $(1\ 2)\cdot(x,y)=(y,x)$ and $\mathbb{Z}/2\mathbb{Z}$ acts by $(x,y)\mapsto(x^{-1},y^{-1})$. $^{(3)}$ $\mathbb{Z}/2\mathbb{Z}$ acts on $\mathbf{SL}_3$ by $x\mapsto\tau(x)^{-1}$ for a hyperbolic orthogonal involution τ (fixing a subgroup $\mathbf{O}_3$). $^{(4)}$ $1\times\mathbb{Z}/2\mathbb{Z}$ acts on $(\mathbf{SL}_3\times\mathbf{SL}_3)/\boldsymbol{\mu}_3$ by $(X,Y)\mapsto(Y,X)$ and $\mathbb{Z}/2\mathbb{Z}\times 1$ acts by $(X,Y)\mapsto(\tau(Y)^{-1},\tau(X)^{-1})$ for an orthogonal involution τ. $^{(5)}$ $\mathbb{Z}/2\mathbb{Z}$ acts on $\mathbf{SL}_6/\boldsymbol{\mu}_2$ by $x\mapsto\sigma(x)^{-1}$ for a symplectic involution σ (fixing a subgroup $\mathbf{PGSp}_6$). $^{(6)}$ $\mathbb{Z}/2\mathbb{Z}$ acts by the canonical diagram automorphism (fixing a subgroup F_4).

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Chapter I

Structurable algebras

Since we are dealing with many different kinds of algebras, there are a number of general definitions and concepts needing a precise introduction; this is how the chapter begins in §1. The theory of structurable algebras, initiated by Allison in [3], has well-developed notions of invertibility, isotopy, norms, and traces [10, 14, 150]. These notions are discussed in §2, along with several examples that later become some of the main subjects of the thesis. In §3, we revisit the classification of central simple structurable algebras to make some of its details more precise and get more mileage out of it.

We assume throughout that k is a field of characteristic not 2 or 3, and write k^s and k^a for a separable or algebraic closure of k, respectively. We write $\Gamma_k = \mathcal{G}al(k^s/k)$ for the absolute Galois group.

1. Basics of nonassociative algebra

All algebras are assumed to be finite-dimensional but they are not assumed to be associative. We write commutators as $[x, y] = xy - yx$ and associators as $[x, y, z] = (xy)z - x(yz)$. The nucleus and centre of an algebra A are

$$\begin{aligned} \mathrm{Nuc}(A) &= \{a \in A : [a, A, A] = [A, a, A] = [A, A, a] = 0\}, \\ Z(A) &= \{a \in \mathrm{Nuc}(A) : [a, A] = 0\}. \end{aligned}$$

If $n \in \mathrm{Nuc}(A)$, the inner automorphism implemented by n is denoted by $\mathrm{Int}(n) \in \mathrm{Aut}(A)$, $\mathrm{Int}(n)(x) = nxn^{-1}$, and the inner derivation by $\mathrm{ad}_n \in \mathrm{Der}(A)$, $\mathrm{ad}_n(x) = nx - xn$.

The *centroid* of a k-algebra A is the centre of the subalgebra of $\mathrm{End}_k A$ generated by left- and right-multiplication operators $\{L_a, R_a : a \in A\}$. We say that A is *central* if its centroid is $k\,\mathrm{id}$, and *simple* if $A^2 \neq 0$ and A has no two-sided ideals besides $\{0\}$ and A.

Tensor products are usually taken over k unless specified otherwise. For a commutative unital k-algebra R, we define the R-algebra $A_R = A \otimes R$ (external tensor product). The internal tensor product of k-algebras is characterised as follows: for subalgebras $B, C \subset A$, $A = B \otimes C$ if and only if $A = BC$, $[B, C] = 0$, and $\dim A = (\dim B)(\dim C)$; see [94, Proposition 4.8]. One can easily check that $Z(A \otimes B) = Z(A) \otimes Z(B)$ and $\mathrm{Nuc}(A \otimes B) = \mathrm{Nuc}(A) \otimes \mathrm{Nuc}(B)$ if A and B are unital algebras.

1.1. *Associative and alternative algebras.* A k-algebra A is *associative* if $A = \mathrm{Nuc}(A)$, *power-associative* if the subalgebra generated by any element is associative, *strictly power-associative* if A_K is power-associative for every field extension K/k, and *alternative* if for all $x, y \in A$,

$$[x, y, x] = [x, x, y] = [y, x, x] = 0.$$

We say A is *separable* if for every field extension K/k, A_K is a direct product of simple ideals. An *étale algebra* over k is a separable commutative associative algebra or, equivalently, a direct product of separable field extensions of k. A central simple associative algebra A always has dimension d^2 for some $d \geq 1$, and we say that d is the *degree* of A. The *index* of A is the degree of the central division algebra D such that $A \simeq M_n(D)$.

1.2. *Generic norms and traces.* Let A be a unital and strictly power-associative k-algebra, with a basis $\{a_1, \ldots, a_n\}$. Let $K = k(t_1, \ldots, t_n)$ be the function field in n indeterminates. The generic element $v = \sum a_i t_i \in A_K$ has a unique monic polynomial of minimal degree

$$m_v(\lambda) = \lambda^d - \sigma_1(v)\lambda^{d-1} + \cdots + (-1)^d \sigma_d(v)$$

such that $m_v(v) = 0$. Each $\sigma_i(v)$ is a degree i homogeneous polynomial in $k[t_1, \ldots, t_n]$, and specialisation gives a value $\sigma_i(a) \in k$ for every $a \in A$. We call d the *generic degree* of A, $\sigma_1 = t_A$ the *generic trace* of A, and $\sigma_d = N_A$ the *generic norm* of A [90]. We have $N_A(1) = 1$ and $t_A(1) = d$. If $A = B \times C$ is a direct product of subalgebras B, C, then $t_A(b, c) = t_B(b) + t_C(c)$ and $N_A(b, c) = N_B(b)N_C(c)$.

The generic norm and trace of a central simple associative algebra A are equal to the reduced norm and trace, respectively denoted by Nrd_A and Trd_A [101, p. 5].

1.3. *Involutions.* An involution on a k-algebra is an anti-automorphism of order 2. If $(A, -)$ is an algebra with involution, since $\frac{1}{2} \in k$, there is a decomposition into linear subspaces $A = \mathrm{Skew}(A, -) \oplus \mathrm{Herm}(A, -)$, where

$$\mathrm{Skew}(A, -) = \{a \in A : \bar{a} = -a\}$$
$$\mathrm{Herm}(A, -) = \{a \in A : \bar{a} = a\}.$$

The *skew-dimension* of $(A, -)$ is the dimension of $\mathrm{Skew}(A, -)$.

It is clear that $\mathrm{Nuc}(A)$ and $Z(A)$ are stabilised by the involution. We define $\mathrm{Nuc}(A, -) = \mathrm{Nuc}(A) \cap \mathrm{Herm}(A, -)$ and $Z(A, -) = Z(A) \cap \mathrm{Herm}(A, -)$. A unital k-algebra with involution $(A, -)$ is *central* if $Z(A, -) = k1$ and *simple* if the only two-sided ideals stabilised by the involution are $\{0\}$ and A.

We refer to [101] for all matters regarding associative central simple algebras with involution. If (A, σ) is associative central simple, we say σ is of the *first kind* if $Z(A) = k$. In this case, A is a central simple algebra, say of degree d. Standard terminology is that σ is an *orthogonal involution* if the skew-dimension is $\frac{1}{2}d(d-1)$, and it is a *symplectic involution* if the skew-dimension is $\frac{1}{2}d(d+1)$ [101, (2.5)–(2.6)]. If $Z(A) \neq k$ then $Z(A)$ is a quadratic étale extension of A and σ is of the *second kind*, also known as a *unitary involution.* We define the degree of a unitary involution as in [101, p. 21].

1.4. *Quadratic forms.* Standard references for the theory of quadratic and bilinear forms include [53, 106, 151]. If (V, q) is a quadratic space, we write

$$q(x, y) = q(x + y) - q(x) - q(y)$$

for the associated bilinear form. For a bilinear form b, we denote by ad_b the adjoint involution on $\operatorname{End} V$, so that $b(F(v), w) = b(v, \mathrm{ad}_b(F)(w))$ for $v, w \in V$, $F \in \operatorname{End} V$.

We denote by $\mathbb{H} = \langle 1, -1\rangle$ the hyperbolic quadratic form. For $n \in \mathbb{N}$, we write $nq = q \perp \cdots \perp q$ (n times). For n-Pfister forms, we use the notation $\langle\!\langle c_1, \ldots, c_n\rangle\!\rangle = \bigotimes_{i=1}^n \langle 1, -c_i\rangle$. If ϕ is a Pfister form, we write ϕ' for its pure part, i.e. the unique quadratic form such that $\phi \simeq \langle 1\rangle \perp \phi'$. Two quadratic forms are called similar if one is isometric to a scalar multiple of the other.

1.5. *Hermitian forms.* A hermitian space over a k-algebra with involution (E, σ) is a unital left E-module W and a k-bilinear mapping $h : W \times W \to E$ such that

$$h(ew_1, w_2) = eh(w_1, w_2), \qquad h(w_1, w_2) = \sigma(h(w_2, w_1))$$

for all $e \in E$ and $w_1, w_2 \in W$. An isometry of hermitian spaces $(W, h) \to (W', h')$ is an E-linear bijection $\phi : W \to W'$ such that $h_2(\phi(w_1), \phi(w_2)) = h_1(w_1, w_2)$.

A hermitian $d \times d$ matrix M with entries in E determines a hermitian form on E^d by the formula

$$h(a, b) = \begin{pmatrix} a_1 & \cdots & a_d \end{pmatrix} M \begin{pmatrix} \sigma(b_1) & \ldots & \sigma(b_d) \end{pmatrix}^t \qquad \text{for all } a, b \in E^d.$$

If (D, σ) is a division algebra with involution, every hermitian form on D^d is isometric to a hermitian form represented by a diagonal matrix [85, Theorem 8].

More generally, for integers n, d, let $E = M_n(D)$ and let $W = M_{n\times d}(D)$, the set of $n \times d$-matrices with entries in D. Then W is naturally a left E-module. A hermitian matrix $M \in M_d(D)$ determines a hermitian form h on W, namely

$$h(a, b) = aM\sigma(b)^t \qquad \text{for all } a, b \in M_{n\times d}(D),$$

where $\sigma(b)$ is the matrix b with σ applied entrywise. Every nonsingular hermitian space (W, h) over E is of this form [107, Proposition 3.1]; this is just an explicit version of hermitian Morita theory for finite-dimensional central simple algebras. If M is a diagonal matrix with diagonal entries $(a_1, \ldots, a_d)$, then we write $h = \langle a_1, \ldots, a_d\rangle$.

1.6. *Composition algebras.* A composition algebra over k is a unital k-algebra C on which there exists a nondegenerate quadratic form $n : C \to k$ such that

$$n(xy) = n(x)n(y)$$

for all $x, y \in C$. If such an n exists, it is unique and it is the generic norm of C. It is also a Pfister form. Composition algebras exist only in dimensions 1 (fields), 2 (quadratic étale algebras), 4 (quaternion algebras), and 8 (octonion algebras). They are always alternative and have generic degree 2. The generic trace of C is

$$t(x) = n(x, 1).$$

A composition algebra also has a *standard involution*:

$$x \mapsto \bar{x} = t(x) - x.$$

We refer to [87] and [164] for background on composition algebras. We write C_0 for the kernel of the trace map $t(x) = n(x, 1)$; equivalently, $C_0 = \mathrm{Skew}(C, -)$. This subspace equipped with the commutator product is a central simple Malcev algebra denoted by C_0^- (see 1.9).

The standard involution on C is the unique involution with skew-dimension equal to $\dim C - 1$. Any other involution of the first kind on C is called *nonstandard*. An involution on a quaternion algebra is nonstandard if and only if it is an orthogonal involution (with skew-dimension 1). An involution on an octonion algebra is nonstandard if and only if it has skew-dimension 3: more information on this in 8.1.

1.7. *Jordan algebras.* A Jordan algebra over a field of characteristic not 2 is a commutative unital algebra satisfying the identity

$$((aa)b)a = (aa)(ba).$$

McCrimmon's book [115] is a comprehensive guide to Jordan algebras of finite and infinite dimension. If A is an associative algebra, it becomes a Jordan algebra A^+ under the bullet product $a \bullet b = \frac{1}{2}(ab + ba)$. A Jordan algebra is called *special* if it is a subalgebra of A^+ for some associative algebra A, and otherwise it is called *exceptional*.

A Jordan algebra J has a bilinear trace form $T_J : J \times J \to k$, defined as $T_J(x, y) = t(xy)$ where $t : J \to k$ is the generic trace of J. We say that $T_J(x) = t(x^2)$ is the *quadratic trace form* of J.

1.8. Examples. We list some more examples of Jordan algebras

(i) If A is associative and has an involution σ, $\mathcal{H}(A, \sigma) = \mathrm{Herm}(A, \sigma)$ is a special Jordan subalgebra of A^+.

(ii) If $n \leq 3$ and C is an octonion algebra, the algebra $\mathcal{H}_n(C)$ of $n \times n$ hermitian matrices with entries C is a Jordan algebra under the bullet product.

(iii) If (V, q) is a nondegenerate quadratic space, there is a Jordan algebra structure on $J(V, q) = k \oplus V$ given by the product

$$(\lambda, v)(\mu, w) = (\lambda\mu + \tfrac{1}{2}q(v, w), \lambda w + \mu v)$$

for all $\lambda, \mu \in k$ and $v, w \in V$. If $\dim V \geq 3$, then $J(V, q)$ is central simple.

All of these examples are special Jordan algebras, except for $\mathcal{H}_3(C)$. Exceptional simple Jordan algebras are all exactly 27-dimensional and become isomorphic to an $\mathcal{H}_3(C)$ over some field extension [115]. Such algebras are called *Albert algebras*.

More examples of (reduced) Albert algebras can be found by taking a diagonal matrix Γ with diagonal entries $(\gamma_1, \gamma_2, \gamma_3)$ and setting

$$J = \mathcal{H}_3(C, \gamma) = \mathcal{H}(M_3(C), *_\Gamma)$$

where $*_\Gamma$ is the involution $X \mapsto \Gamma^{-1}\bar{X}^t\Gamma$ on $M_3(C)$. To produce Albert division algebras, more sophisticated constructions are needed – see [101, §38–39] and [113].

1.9. *Malcev algebras.* A Malcev algebra is an anticommutative algebra S that satisfies the identity

$$(xy)(xz) = ((xy)z)x + ((yz)x)x + ((zx)x)y$$

for all $x, y, z \in S$. Malcev algebras are a natural generalisation of Lie algebras, just like alternative algebras are a natural generalisation of associative algebras. Some of the main results in the subject over the last half-century are neatly exposed in the editors' comments at the end of [105].

We say that a k-algebra S is an *exceptional simple Malcev algebra* if it is a central simple Malcev algebra that is not a Lie algebra.

1.10. *Algebraic groups.* We take the functorial view of algebraic groups, in the style of [122] and [101], so by algebraic group we mean an affine algebraic group scheme. In other words, an algebraic group G is a representable functor $R \rightsquigarrow G(R)$ from the category of finitely-generated commutative k-algebras to the category of groups.

The notation G° refers to the connected component of the identity, $\pi_0(G) = G/G^\circ$ is called the group of components, and G^{der} refers to the derived subgroup of G. The notation $C_G(H)$ stands for the (scheme-theoretic) centraliser of a subgroup H of G, and $Z(G)$ stands for the centre of G. If λ is a homomorphism taking values in G, we write $C_G(\lambda)$ for the centraliser of the image of λ in G.

We make use of several well-known algebraic groups and their standard notations, like the multiplicative and additive groups $\mathbf{G}_m$ and $\mathbf{G}_a$, and the automorphism group scheme $\mathbf{Aut}(A)$ of a k-algebra A. In this chapter, there is limited content on algebraic groups, but these become much more prominent in subsequent chapters. We do count on some background knowledge of root systems.

2. Introduction to structurable algebras

Let R be a commutative ring in which 2 and 3 are invertible. Given a unital R-algebra with involution $(A, -)$, we define linear endomorphisms $V_{x,y} \in \operatorname{End}_R A$:

$$V_{x,y}z = \{x, y, z\} = (x\bar{y})z + (z\bar{y})x - (z\bar{x})y \qquad \text{for } x, y, z \in A.$$

2.1. Definition. The R-algebra $(A, -)$ is called a *structurable algebra* if the identity

$$[V_{x,y}, V_{z,w}] = V_{\{x,y,z\},w} - V_{z,\{y,x,w\}} \tag{2.1.1}$$

holds in $\operatorname{End}_R A$, for all $x, y, z, w \in A$.

If $(A, -)$ is a structurable algebra, the algebra S^- with underlying vector space $\operatorname{Skew}(A, -)$ and product $(s, t) \mapsto [s, t]$ is a Malcev algebra [3, Proposition 18]. Of course, S^- is a Lie algebra if A is associative.

In addition to $V_{x,y}$ defined above, various operators $L_x, R_x, T_x, U_x, U_{x,y}, D_{x,y} \in \operatorname{End}_R(A)$ are defined as follows:

$$\begin{aligned} L_x(z) &= xz, & R_x(z) &= zx \\ T_x(z) &= V_{x,1}(z), & D_{x,y}(z) &= \tfrac{1}{3}\big[[x,y] + [\bar{x},\bar{y}], z\big] + [z,y,x] - [z,\bar{x},\bar{y}] \\ U_{x,y}(z) &= V_{x,z}(y), & U_x &= U_{x,x} \end{aligned}$$

for all $x, y, z \in A$. Given a subspace $B \subset A$, we write $V_{B,B}$ for the linear span of $\{V_{x,y} \colon x, y \in B\}$, $L_B L_B$ for the linear span of $\{L_x L_y \colon x, y \in B\}$, and similarly for T_B,

$D_{B,B}$, and so on. We refer to [3] for various identities satisfied by these operators, and other properties of structurable algebras.

By the defining identity (2.1.1), $V_{A,A}$ is a Lie subalgebra of $\mathfrak{gl}(A)$. It is called the *inner structure Lie algebra*. By [3, p. 135, Remark (iii)], the linear map $A \to T_A$, $a \mapsto T_a$, is bijective. By [3, p. 139] there is a direct sum decomposition of vector spaces

$$V_{A,A} = T_A \oplus D_{A,A}. \tag{2.1.2}$$

The subspace $D_{A,A}$ is an ideal of $\mathrm{Der}(A,-)$ called the Lie algebra of *inner derivations*. The subspace $L_S L_S$ is an ideal of $V_{A,A}$ [3, Corollary 5 (vii)].

2.2. *First examples of structurable algebras.* Any unital alternative algebra with any involution whatsoever is a structurable algebra [149, p. 411]. Any Jordan algebra equipped with the identity involution is a structurable algebra, because it satisfies the defining 5-linear identity (2.1.1); see [115, Formulas 5.2.3 (FFV)']. Conversely, any commutative algebra which is structurable with respect to the identity involution is a Jordan algebra.

A tensor product $C_1 \otimes C_2$ of a pair of unital composition algebras is a structurable algebra, if it is equipped with the tensor product of the canonical involutions on its factor algebras C_1, C_2 [3, §8 (iii)]. These are called (m_1, m_2)*-product algebras*, where $m_i = \dim C_i$.

If $(A,-)$ is a structurable algebra and $I \subset A$ is an ideal such that $\bar{I} = I$, then A/I is structurable too. If $B \subset A$ is a subalgebra such that $\overline{B} = B$ and B contains an idempotent u such that $ub = bu = u$ for all $b \in B$, then $(B,-)$ is structurable too.

2.3. *The quartic* 2×2 *matrix algebra.* Define an algebra structure on the 4-dimensional vector space

$$M(k) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a,b,c,d \in k \right\}$$

by giving it the product

$$\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} = \begin{pmatrix} a_1a_2 + 3b_1c_2 & a_1b_2 + b_1d_2 + 2c_1c_2 \\ c_1a_2 + d_1c_2 + 2b_1b_2 & d_1d_2 + 3c_1b_2 \end{pmatrix}$$

and involution

$$\overline{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} = \begin{pmatrix} d & b \\ c & a \end{pmatrix}.$$

This algebra is central simple and structurable of skew-dimension one [51, Remark 3.3]. It is related to the "rather peculiar algebra" mentioned in [124, Example 25.3].

2.4. *The Smirnov algebra.* The tensor square $C \otimes C$ of an octonion algebra C contains a 36-dimensional subalgebra $\mathrm{Sym}^2(C)$ of symmetric tensors, which decomposes uniquely as a direct product of two central simple structurable algebras [11],

$$\mathrm{Sym}^2(C) \simeq k \times T(C).$$

The 35-dimensional simple subalgebra $T(C)$, with the canonical involution it inherits from $C \otimes C$, is called a *Smirnov algebra*.

The Smirnov algebra was actually the last-discovered example of a central simple structurable algebra: it was missed by Allison for many years, and first sighted by

Smirnov [159]. (Evidently, the examples here are not ordered chronologically, but rather by ease of description.)

2.5. *Structurable algebras of hermitian forms.* If (E,σ) is any associative algebra with involution and $\mu \in Z(E,\sigma)$ is invertible, then the algebra $(E \oplus E, *)$ obtained by the *classical* Cayley–Dickson construction [115, II. §2.5] is a structurable algebra if one gives it the (nonstandard) involution $* : (e,e') \mapsto (\sigma(e), e')$. This example and Example 1.8 (iii) have a common generalisation:

Let (E,σ) be a unital associative algebra with involution, and let W be a unital left E-module equipped with a hermitian form $h : E \times E \to k$. Define on the vector space $E \oplus W$ the structure of an algebra with involution, by

$$(e_1,w_1)(e_2,w_2) = (e_1e_2 + h(w_2,w_1), e_2w_1 + \sigma(e_1)w_2), \qquad \overline{(e,w)} = (\sigma(e), w).$$

for all $e, e_1, e_2 \in E$ and $w, w_1, w_2 \in W$. This algebra, denoted by $S(E,\sigma,W,h) = (E\oplus W, -)$, is a structurable algebra. It is central simple if and only if (E,σ) is central simple and either $W = \{0\}$ or h is nonsingular [3, Lemma 23, proof of Theorem 25]. It is clear that $S(E,\sigma,W,h) \simeq S(E,\sigma,W',h')$ if (W,h) and (W',h') are isometric hermitian spaces.

2.6. *Matrix structurable algebras of skew-dimension one.* Yet another kind of structurable algebra can be constructed from the norm and trace of a cubic Jordan algebra. Let J be a separable Jordan algebra of generic degree 3 with generic norm $N : J \to k$ and bilinear trace $T : J \times J \to k$, $T(x,y) = t(xy)$. Since T is nondegenerate [90, Theorem 8], there is a unique quadratic mapping $\sharp : J \to J$ such that

$$N(x + \lambda y) = N(x) + \lambda T(x^\sharp, y) + \lambda^2 T(y^\sharp, x) + \lambda^3 N(y) \tag{2.6.1}$$

holds in the function field $k(\lambda)$, for all $x, y \in A$. Its linearisation $\times : J \times J \to J$ is defined by $x \times y = (x+y)^\sharp - x^\sharp - y^\sharp$.

Now let $\eta \in k^\times$ and define on the vector space

$$\begin{pmatrix} k & J \\ J & k \end{pmatrix} = \left\{ \begin{pmatrix} \alpha & j \\ j' & \beta \end{pmatrix} : \alpha, \beta \in k, j, j' \in J \right\}$$

the structure of an algebra with involution by giving it the product

$$\begin{pmatrix} \alpha_1 & j_1 \\ j_1' & \beta_1 \end{pmatrix} \begin{pmatrix} \alpha_2 & j_2 \\ j_2' & \beta_2 \end{pmatrix} = \begin{pmatrix} \alpha_1\alpha_2 + \eta T(j_1, j_2') & \alpha_1 j_2 + \beta_2 j_1 + \eta(j_1' \times j_2') \\ \alpha_2 j_1' + \beta_1 j_2' + j_1 \times j_2 & \beta_1\beta_2 + \eta T(j_2, j_1') \end{pmatrix}$$

and the involution

$$\overline{\begin{pmatrix} \alpha & j \\ j' & \beta \end{pmatrix}} = \begin{pmatrix} \beta & j \\ j' & \alpha \end{pmatrix}.$$

This algebra, which is denoted by $M(J,\eta)$, is a central simple structurable algebra of skew-dimension one [7, Proposition 1.10]. If $J \simeq J'$ and $\eta = \eta'$ modulo $k^{\times 3}$, it is straightforward to show that $M(J,\eta) \simeq M(J',\eta')$; see the proof of [55, Lemma 2.8 (2)]. In particular, if η has a cube root then $M(J,\eta) \simeq M(J,1)$. We shall write $M(J) = M(J,1)$ for short.

There are some variations on this construction. If one relaxes the assumption that J has generic degree 3 and takes instead $J = k$, $N(x) = x^3$, $T(x,y) = 3xy$, and

$x^\sharp = x^2$, for all $x, y \in J$, then $M(k)$ is the quartic 2×2 matrix algebra described in 2.3. If one takes $J = k$, $N(x) = x^\sharp = 0$, and $T(x, y) = xy$ for all $x, y \in J$, then the algebra defined above is isomorphic to $M_2(k)$ with its hyperbolic orthogonal involution.

The 56-dimensional algebra $M(\mathbb{A})$, where $\mathbb{A}$ is an Albert algebra, was first discovered by Brown and reported in [38]. Since then, it has been dubbed the Brown algebra. In keeping with this theme, we shall call $M(J)$ a *green*, *blue*, *red*, or *Brown* algebra according as J is a simple cubic Jordan algebra of dimension 6, 9, 15, or 27. The algebra $M(k^3)$ is called a *quartic Cayley algebra*.

2.7. *Skew-alternativity.* Structurable algebras have a weak form of alternativity, called skew-alternativity:

$$[s, x, y] = -[x, s, y] = [x, y, s]$$

for all $x, y \in A$ and $s \in \mathrm{Skew}(A, -)$ [3, Proposition 1]. The operator versions of this identity are:

$$\begin{aligned}
[R_y, L_s] &= R_y R_s - R_{sy} = R_s R_y - R_{ys} \\
[R_s, L_x] &= L_{xs} - L_x L_s = L_{sx} - L_s L_x.
\end{aligned}$$

and these imply the good-looking identities:

$$L_{[s,x]} = [L_s, L_x] \qquad\qquad R_{[s,y]} = [R_y, R_s]. \tag{2.7.1}$$

Other consequences of skew-alternativity are that

$$L_{s^2} = L_s^2 \qquad\qquad R_{s^2} = R_s^2 \qquad\qquad R_s L_s = L_s R_s$$

for all $s \in \mathrm{Skew}(A, -)$, so $s(xs) = (sx)s$ for all $x \in A$, and furthermore that

$$L_{sts} = L_s L_t L_s \tag{2.7.2}$$

for all $s, t \in \mathrm{Skew}(A, -)$ (see [31, (2.5)–(2.6)]).

2.8. *Conjugate inverses and structurable division algebras.* Suppose $x \in A$ for a structurable R-algebra $(A, -)$. Then x is called *conjugate-invertible* if there exists a $y \in A$ such that $V_{x,y} = \mathrm{id}$. We say that y is the *conjugate inverse* of x. The conjugate inverse of any element is unique if it exists, so we write it as $y = \hat{x}$. In fact, if x is conjugate-invertible then U_x is invertible and $\hat{x} = U_x^{-1}(x)$. It follows that if $(B, -)$ is a subalgebra of $(A, -)$ and $b \in B$, then b is conjugate-invertible in $(A, -)$ if and only if it is conjugate-invertible in $(B, -)$. For more details on conjugate-invertibility, we refer to [14]. We say that $(A, -)$ is a structurable *division algebra* if every nonzero element is conjugate-invertible.

2.9. *Isotopes.* The theory of isotopy for structurable algebras is due to Allison and Hein [14]. It is a weaker equivalence relation than isomorphism, and a generalisation of isotopy for Jordan algebras, which is an older idea due to Jacobson [91, I. §12].

Let $(A, -)$ and $(B, -)$ be structurable R-algebras. An R-linear bijection $\alpha : A \to B$ is called an *isotopy* if there exists another R-linear bijection $\beta : A \to B$ such that $\alpha V_{x,y} \alpha^{-1} = V_{\alpha(x),\beta(y)}$ for all $x, y \in A$. The map β is uniquely determined if it exists, so we write it as $\beta = \hat{\alpha}$. Two structurable algebras are called *isotopic* if there exists an isotopy between them.

Given a conjugate-invertible element $u \in A$, there is a way to define an algebra with involution $(A^{\langle u \rangle}, -^{\langle u \rangle})$ in such a way that $A^{\langle u \rangle} = A$ as vector spaces, the identity in $A^{\langle u \rangle}$ is $\hat{u}$, and $(A^{\langle u \rangle}, -^{\langle u \rangle})$ is isotopic to $(A, -)$. The algebras so defined are called isotopes of $(A, -)$. Further details about isotopes, including a precise definition, can be found in [14]. A linear map $\alpha : A \to B$ is an isotopy if and only there is some conjugate-invertible $u \in A$ such that the composition of linear maps

$$A^{\langle u \rangle} \xrightarrow{\text{id}} A \xrightarrow{\alpha} B$$

is an isomorphism of algebras with involution [14, Proposition 8.5].

2.10. *Left and right multiplication operators.* If $(A, -)$ is a structurable R-algebra, we define $A^* \subset A \setminus \{0\}$ to be the set of conjugate-invertible elements in A, and $S^* = \mathrm{Skew}(A, -) \cap A^*$. We note some important facts from [14, §11] about skew invertible elements, namely that

$$S^* = \{s \in S \colon L_s \in \mathrm{GL}(A)\}$$

and L_s is an isotopy for all $s \in S^*$, with

$$\hat{L}_s = L_{\hat{s}} = -L_s^{-1}.$$

Similar statements hold for left multiplication by nuclear elements and right multiplication by nuclear similitudes. Lacking a reference, we prove these statements below.

2.11. Lemma. *If $(A, -)$ is a structurable algebra over k and $n \in \mathrm{Nuc}(A)$, then the following notions of invertibility are equivalent:*

(1) *$xn = nx = 1$ has a solution in A,*

(2) *$L_n \in \mathrm{GL}(A)$,*

(3) *$R_n \in \mathrm{GL}(A)$,*

(4) *n is conjugate-invertible.*

Assuming these conditions are met,

$$\hat{n} = L_{\bar{n}}^{-1}(1)$$

and L_n is an isotopy with

$$\hat{L}_n = L_{\bar{n}}^{-1}.$$

In contrast, R_n is an isotopy if and only if $n\bar{n} \in Z(A)$, and in this case

$$\hat{R}_n = R_{\bar{n}}^{-1}.$$

Proof. We first show that (1)–(3) are equivalent. Assuming (1), for all $b \in A$ we have a solution to $ny = b$ because $n(xb) = (nx)b = b$, so (2) holds. Assuming (2), let $x = L_n^{-1}(1)$. Then $nx = L_n(x) = 1$ and since $n \in \mathrm{Nuc}(A)$ this implies

$\mathrm{id} = L_{nx} = L_n L_x$, so $L_x = L_n^{-1}$. Consequently $xn = L_x L_n(1) = L_n^{-1} L_n(1) = 1$, so (1) holds. Symmetrically, we can prove that (1) is equivalent to (3).

Assuming (1)–(3), it is clear that not only n but also $\bar{n}$ is an invertible nuclear element, and we may write $\bar{n}^{-1} = \overline{n^{-1}}$ without causing confusion. It is straightforward to show that $V_{nx,\bar{n}^{-1}y}(nz) = nV_{x,y}(z)$ and therefore L_n is an isotopy with $\hat{L}_n = L_{\bar{n}^{-1}} = L_{\bar{n}}^{-1}$. From this it follows that $V_{n,\bar{n}^{-1}} = L_n V_{1,1} L_n^{-1} = L_n \,\mathrm{id}\, L_n^{-1} = \mathrm{id}$, so (4) holds and $\hat{n} = \bar{n}^{-1} = L_{\bar{n}}^{-1}(1)$. Assuming only (4), $U_n \in \mathrm{GL}(A)$ by [14, §6], so $A = U_n(A) \subset nA \subset A$. Then L_n is surjective and (2) holds because A is finite-dimensional.

Lastly, R_n is an isotopy if and only if $\mathrm{Int}(n) = L_n R_n^{-1}$ is too, and $\mathrm{Int}(n)$ is an isotopy if and only if it preserves the involution [14, Corollary 8.6], or equivalently if $n\bar{n} \in Z(A)$. It is easy to show that $V_{xn,y\bar{n}^{-1}}(zn) = V_{x,y}(z)n$ using the fact that $n\bar{n} \in Z(A)$, so $\hat{R}_n = R_{\bar{n}^{-1}} = R_{\bar{n}}^{-1}$. □

2.12. *Norms of structurable algebras.* Since many structurable algebras are not power-associative, we cannot get far with generic norms. Allison and Faulkner [10] defined norms on some structurable algebras over infinite fields. I suggest the following definition, which agrees with theirs but is more general:

2.13. Definition. A *norm* of a structurable k-algebra $(A, -)$ is a homogeneous polynomial function $N \in k[A]$, such that:

(i) N is normalised in the sense that $N(1) = 1$,

(ii) N detects invertibility in the sense that for all field extensions K/k and all $x \in A_K$, $N(x) \neq 0$ if and only if x is conjugate-invertible, and

(iii) N is of minimal degree with respect to (i) and (ii).

If the set of conjugate-invertible elements in $A \otimes_k k^a$ is Zariski-dense, then a norm exists and it is unique. If $(A, -)$ is Jordan or alternative, the norm exists and equals the generic norm. The norm also exists and is unique if $(A, -)$ is central simple or if $\mathrm{char}(k) = 0$. These facts and further details about norms on structurable algebras can be found in [10].

When the norm of $(A, -)$ exists and is unique, we write it as N_A. We define the *degree* of $(A, -)$ as

$$\deg_k(A, -) = \deg N_A.$$

In some familiar examples, particularly when N_A coincides with the generic norm, N_A depends only on the algebra structure of A and not on its involution. However, we caution that in general N_A depends not only on the algebra structure but also on the involution, because the notion of conjugate-inversion itself depends on the involution. (In other words, the norm actually deserves the notation $N_{(A,-)}$ but we settle on N_A for aesthetic reasons.)

The generic norm of any central simple Jordan or alternative algebra is an irreducible polynomial [90, Theorem 7]. It seems likely that N_A is irreducible for any central simple structurable algebra $(A, -)$, but I do not know of a proof.

2.14. *Trace forms and other invariant bilinear forms.* Allison used trace forms extensively in his classification of simple structurable algebras [3]. The *bilinear trace* of A is the symmetric bilinear form $T_A : A \times A \to k$ defined by

$$T_A(x, y) = \operatorname{tr}(L_{x\bar{y}+y\bar{x}}).$$

We call $T_A(x) = T_A(x, x)$ the *quadratic trace form*. (Technically, if $(A, -) = (J, \mathrm{id})$ for a Jordan algebra J, this definition of T_J differs by a scalar from the T_J defined in 1.7. If the choice of scalar matters then we try to make it clear in context.)

2.15. Definition. Let $(A, -)$ be a structurable algebra. An *invariant bilinear form* on A is a symmetric bilinear form $b : A \times A \to k$ such that for all $x, y, z \in A$,

$$b(\bar{x}, \bar{y}) = b(x, y) \tag{2.15.1}$$

$$b(zx, y) = b(x, \bar{z}y). \tag{2.15.2}$$

Clearly, a symmetric bilinear form b satisfies (2.15.1) if and only if $\operatorname{Herm}(A, -)$ is orthogonal to $\operatorname{Skew}(A, -)$. It is also clear that (2.15.1) and (2.15.2) together imply

$$b(xz, y) = b(x, y\bar{z}). \tag{2.15.3}$$

The following lemma is important, and references are scattered, so a full proof is given below.

2.16. Lemma. *Let $(A, -)$ be a structurable algebra over a field k with* $\operatorname{char}(k) \neq 2, 3$.

(i) *The bilinear trace T_A is invariant.*

(ii) *If $(A, -)$ is simple and b is any invariant bilinear form on A, then b is either nondegenerate or zero.*

(iii) *If $(A, -)$ is central simple and b, b' are invariant bilinear forms on A such that $b \neq 0$, then b' is a scalar multiple of b.*

Proof. (i) For $s \in \operatorname{Skew}(A, -)$ and $h \in \operatorname{Herm}(A, -)$, we have

$$T_A(s, h) = \operatorname{tr}(L_{[s,h]}) = \operatorname{tr}([L_s, L_h]) = 0$$

by (2.7.1). This implies (2.15.1). To prove (2.15.2), the most economical proof is probably still the original one from [3, Theorem 17]. First observe that $L_{x\bar{y}+y\bar{x}} = V_{x,y} + V_{y,x}$ by definition of the V-operators. By the main identity (2.1.1),

$$[T_z, V_{x,y}] = V_{T_z(x),y} - V_{x,T_{\bar{z}}(y)},$$

where $T_z = V_{z,1}$. Taking traces and repeating with a change of variables yields that

$$0 = \operatorname{tr}(V_{T_z(x),y} - V_{x,T_{\bar{z}}(x)} + V_{y,T_z(x)} - V_{T_{\bar{z}}(y),x}) = T_A(T_z(x), y) - T_A(x, T_{\bar{z}}(y)). \tag{2.16.1}$$

If $h \in \operatorname{Herm}(A, -)$ then $T_h = L_h$, so $0 = T_A(hx, y) - T_A(x, \bar{h}x)$. On the other hand, if $z = s \in \operatorname{Skew}(A, -)$ then $T_s = L_s + 2R_s$, and also $\overline{T_s(\bar{x})} = -R_s - 2L_s$. Now (2.16.1) shows that

$$T_A(3L_s(x), y) = T_A(T_s(x) + 2\overline{T_s(\bar{x})}, y)) = T(x, T_{\bar{s}}(y)) + 2T_A(\overline{T_s(\bar{x})}, y).$$

Applying (2.15.1) twice, the last term becomes

$$2T_A(T_s(\bar{x}), \bar{y}) = 2T_A(\bar{x}, T_{\bar{s}}(\bar{y})) = 2T_A(x, \overline{T_{\bar{s}}(\bar{y})}).$$

Since $T_{\bar{s}}(y) + 2\overline{T_{\bar{s}}(\bar{y})} = 3L_{\bar{s}}$ we have

$$T_A(3L_s(x), y) = T(x, 3L_{\bar{s}}(y))$$

and dividing by 3 proves (2.15.2) holds for $z = s$.

(ii) The identity (2.15.1) implies the radical $\mathrm{rad}(b) = \{r \in A \colon b(r, A) = 0\}$ is stabilised by the involution, and (2.15.2) implies it is a left ideal. Combining the two identities gets you so $\mathrm{rad}(b)$ is a right ideal too. So if $(A, -)$ is simple then $\mathrm{rad}(b)$ is either $\{0\}$ or all of A.

(iii) For this claim, there is a very neat proof tucked away in an endnote of [150]. If b' is zero then the claim is obvious, so we can assume going forward that both forms are nonzero. By (ii), b and b' are nondegenerate, so there is a unique $F \in \mathrm{GL}(A)$ such that

$$b(F(x), y) = b'(x, y)$$

for all $x, y \in A$. Now $b(zF(x), y) = b(F(x), \bar{z}y) = b'(x, \bar{z}y) = b'(zx, y) = b'(F(zx), y)$ for all $x, y, z \in A$, which implies $L_z F = F L_z$ for all $z \in A$. Similarly, (2.15.3) implies $R_z F = F R_z$ for all $z \in A$. Finally,

$$b(\overline{F(x)}, y) = b(F(x), \bar{y}) = b'(x, \bar{y}) = b'(\bar{x}, y) = b(F(\bar{x}), y)$$

for all $x, y \in A$, so F commutes with the involution. But since $(A, -)$ is central simple over k, the operators $\{L_a, R_a \colon a \in A\}$ together with the involution generate all of $\mathrm{End}_k(A)$ [89, X. Theorem 4], which implies $F \in Z(\mathrm{End}_k(A)) = k\,\mathrm{id}$. Hence $F = c\,\mathrm{id}$ for some $c \in k^\times$, and $b'(x, y) = cb(x, y)$ for all $x, y \in A$. □

2.17. *Algebraic groups acting on a structurable algebra.* Let $(A, -)$ be a structurable algebra over k with norm N_A. There are several important algebraic groups acting on A:

$$\mathbf{Aut}(A, -) \quad \subset \quad \mathbf{Str}(A, -) \quad \subset \quad \mathbf{Sim}(N_A) \quad \subset \quad \mathbf{GL}(A).$$

The automorphism group $\mathbf{Aut}(A, -) \subset \mathbf{Aut}(A)$ is the group scheme of automorphisms of A that commute with the involution. Its Lie algebra $\mathrm{Der}(A, -) = \mathrm{Lie}(\mathbf{Aut}(A, -))$ is the algebra of derivations that commute with the involution.

The *structure group* $\mathbf{Str}(A, -)$ is the group whose R-points are isotopies from $(A_R, -)$ to itself. We can identify $\mathbf{Aut}(A, -)$ as the subgroup of $\mathbf{Str}(A, -)$ that fixes $1 \in A$ [14, Corollary 8.6].

The group $\mathbf{Sim}(N_A)$ of *norm-similitudes* of A is the group whose R-points are all $\beta \in \mathrm{GL}(A_R)$ such that there exists $\mu_R(\beta) \in R^\times$ with $N_A(\beta(x)) = \mu_R(\beta) N_A(x)$ for all $x \in A_R$. It is proved in [10, Proposition 4.7] that $\mathbf{Str}(A, -)$ is a subgroup of $\mathbf{Sim}(N_A)$. The scalar $\mu_R(\beta)$ is called the multiplier of β, and the map $\mu : \mathbf{Sim}(N_A) \to \mathbf{G}_m$ is a homomorphism. The kernel of μ is the norm-preserving group of A, denoted by $\mathbf{Iso}(N_A)$.

3. Classification of structurable algebras

A classification of simple structurable algebras over fields of characteristic 0 appeared in Allison's original paper from 1978 [3, Theorem 25]. It was not complete, however, because it omitted the 35-dimensional algebras that later became known as Smirnov algebras. Allison's classification was corrected by Smirnov and extended to fields of any characteristic besides 2, 3, and 5 [160, Theorem 3.8]. The Allison–Smirnov classification of structurable algebras says the following:

3.1. Theorem (Allison–Smirnov [3, 160])**.** *Every central simple structurable algebra over a field of characteristic not* 2, 3, *or* 5 *is at least one of the following kinds:*

1. *An associative algebra.*

2. *A Jordan algebra.*

3. *The structurable algebra of a hermitian form.*

4. *A structurable algebra of skew-dimension one.*

5. *A form of a tensor product of composition algebras.*

6. *A Smirnov algebra.*

This is a foundational theorem in the theory of structurable algebras and 5-graded Lie algebras. Unfortunately, it is not a very systematic or organised classification: it refers to some algebraic conditions (associativity, Jordan-ness), a construction from another sort of object (hermitian forms), a numerical invariant (skew-dimension), and then it has two exceptions appended at the end.

The classification also contains duplicates: many algebras fit into more than one of the above classes. For example, a quaternion algebra with orthogonal involution belongs to classes 1, 3, and 4 simultaneously:

3.2. Lemma. *Let E/k be a quadratic étale extension with $\mathrm{Aut}_k(E) = \{1, \iota\}$. If $p \in k$ and $h : E \times E \to k$ is the hermitian form $h(x, y) = px\iota(y)$, then $S(E, \iota, E, h)$ is isomorphic to the quaternion algebra $(E/k, p)$ with an orthogonal involution.*

Proof. It is an easy exercise with the generators and relations to show that

$$S(E, \iota, E, h) \simeq (E/k, p)$$

as algebras; see [176, p. 46]. The involution is orthogonal because the skew-dimension is one [101, Proposition 2.6]. □

Another instance of duplication is the following:

3.3. Lemma. *Let (Q, τ) be a quaternion algebra with orthogonal involution, and let $h : Q \times Q \to Q$ be a hermitian form. Then $S(Q, \tau, Q, h)$ becomes isomorphic to $M(k^3)$ after some separable extension of k*

Proof. The algebra $S(Q, \tau, Q, h)$ has dimension 8, skew-dimension 1, and degree 4, so the claim follows from [7, Theorem 9.1]. For an explicit isomorphism in the case where $Q = M_2(k)$ and τ is adjoint to the hyperbolic form, see [4, p. 1872]. □

What makes matters even more complicated is that there exist quartic Cayley algebras that are not the structurable algebra of any hermitian form [31, Proposition 6.4.2]. Besides this, there are various other reasons to consider quartic Cayley algebras as exceptional – see for instance 3.10 and Theorem 8.2.

3.4. *Towards a descent-based classification.* In this section, the classification of central simple structurable algebras will be reorganised and rewritten. The goal is a complete, duplicate-free, countably infinite list of central simple structurable algebras over a separably closed field $k = k^s$ of characteristic not 2, 3, or 5.

The advantage of this approach is that it breaks the classification programme into two distinct steps. The first step is classifying (in a systematic way) all the central simple structurable algebras over k^s. The second step is studying the *twisted forms* of a given algebra $(A, -)$ over k^s, i.e., the k-algebras $(A', -)$ such that $(A'_{k^s}, -) \simeq (A, -)$.

The study of twisted forms is more challenging: it is not always possible to understand, classify, parametrise, or even describe all the k-defined forms of a structurable algebra. Of course, this is an age-old problem even for associative central simple algebras. The extent to which this can be done depends on the base field and the type of algebra (or from another point of view, the Galois cohomology of its automorphism group).

The list of central simple structurable algebras over a separably closed field (of characteristic not 2, 3, or 5) is independent of the field, and each algebra in the list is in fact defined over $\mathbb{Z}$ (in at least one way) and therefore over every field. Informally, we may call these the "split structurable algebras". For any field k, the *absolute type* of a central simple structurable k-algebra $(A, -)$ is the isomorphism class of its extension to k^s.

I will divide the absolute types into some "classical" infinite families, and a finite list of "exceptional" types. This situation is comparable to the classification of simple Jordan algebras, simple Lie algebras, or simple algebraic groups. It is the way that people have been thinking about structurable algebras for a long time, but the distinction between classical and exceptional types has been somewhat blurry.

3.5. *Numerical invariants.* Lacking a precise and user-friendly classification of the absolute types, it is easy for misconceptions to emerge, for instance that Brown algebras are the only 56-dimensional structurable algebras with skew-dimension one (this misconception appears in both [55] and [47]). On the contrary, over an algebraically closed field there are exactly three nonisomorphic 56-dimensional structurable algebras with skew-dimension one: two of them are structurable algebras of hermitian forms over $k \times k$ and $M_2(k)$ respectively, and the third one is the Brown algebra.

In order to alleviate this kind of confusion and make sure that there are no duplicate entries (or accidental isomorphisms) in the list of split structurable algebras, some numerical invariants will be defined shortly. One of these invariants

$$\Psi : \{ \text{ central simple structurable algebras over } k^s \text{ } \} \longrightarrow \mathbb{N}^4$$

defined in 3.13 is strong enough to separate nonisotopic algebras. We shall use this invariant to prove a bonus theorem: isotopic structurable algebras over an arbitrary field become isomorphic over a finite separable extension. In addition, the invariant Ψ is indispensible for working out the type of the TKK Lie algebra obtained from a central simple structurable algebra.

3.6. *The Θ-invariant.* We first define a numerical invariant Θ taking values in $\mathbb{N}^3$, and show that it comes close to classifying the absolute type of any central simple structurable algebra, failing only in five easily-identifiable cases.

Let $(A, -)$ be a central simple structurable algebra over k, and define

$$\Theta(A,-) = \big(\dim_k A,\ \dim_k \mathrm{Skew}(A,-),\ \deg_k(A,-)\big) \in \mathbb{N}^3.$$

3.7. Lemma. *The Θ-invariant is stable under scalar extension and isotopy in the sense that:*

(i) $\Theta(A_K,-) = \Theta(A,-)$ *for all field extensions K/k and all structurable algebras $(A,-)$ over k.*

(ii) *If $(A,-)$ and $(A',-)$ are isotopic structurable algebras over k, then $\Theta(A,-) = \Theta(A',-)$.*

Proof. (i) Clearly dimension is stable under scalar extension. It is easy to verify that the norm of A_K agrees with the scalar extension of N_A to K, and therefore has the same degree.

(ii) Suppose $\alpha : (A,-) \to (A',-)$ is an isotopy of structurable algebras over k. Then $\dim_k A = \dim_k A'$ by definition, and $\dim_k \mathrm{Skew}\,(A,-) = \dim_k \mathrm{Skew}\,(A',-)$ by [14, Corollary 12.2]. According to [14, Proposition 8.2], α maps conjugate-invertible elements to conjugate-invertible elements, and it is bijective, so $N_{A'} \circ \alpha$ is a scalar multiple of N_A. In particular, $N_{A'}$ and N_A have the same degree. □

3.8. *Notation for some standard algebras.* We shall use the following notation for *split* composition algebras, called split binarions, quaternions, and octonions respectively:

$\mathbb{B}$ for $k \times k$ with the exchange involution $(x, y) \mapsto (y, x)$,

$\mathbb{H}$ for $M_2(k)$ with its symplectic involution $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$,

$\mathbb{O}$ for the algebra of Zorn matrices [89, p. 142] with its canonical involution.

The algebra $\mathcal{J}Spin_n(k) = J(k^n, n\langle 1\rangle)$ is the $(n+1)$-dimensional Jordan algebra of the nondegenerate n-dimensional quadratic form $q = \langle 1, \ldots, 1\rangle$ [115, p. 59].

If (A, σ) is an associative k-algebra with involution, let $\mathbb{M}_n(A, \sigma)$ be the algebra of $n \times n$ matrices with entries in A, equipped with the conjugate-transpose involution $(a_{ij}) \mapsto (\sigma(a)_{ji})$.

Let $\mathcal{H}_n(A, \sigma) = \mathrm{Herm}(\mathbb{M}_n(A, \sigma))$ be the Jordan algebra of $n \times n$ hermitian matrices with entries in A, equipped with the identity involution. We define $\mathcal{H}_3(\mathbb{O})$ similarly.

For structurable algebras of hermitian forms, we use the abbreviated notation $S(E, \sigma, k^{nd})$ when $E = M_n(k)$ is split and $h = \langle 1, \ldots, 1\rangle$ is the standard diagonal hermitian form on $k^{nd} \simeq M_{n\times d}(k)$ (see 1.5). Likewise, if $E \simeq M_n(k) \times M_n(k)$ with the exchange involution $\sigma : (x, y) \mapsto (y^t, x^t)$ then any nonsingular hermitian form h on $k^{2nd} = M_{n\times d}(k) \times M_{n\times d}(k)$ is hyperbolic so we can omit h and use the simpler notation $S(E, \sigma, k^{2nd}, h) = S(E, \sigma, k^{2nd})$ instead.

3.9. Theorem. *Let k be a separably closed field with* $\mathrm{char}(k) \neq 2, 3, 5$. *Tables* 1 *and* 2 *contain a complete, duplicate-free list of isomorphism classes of central simple structurable algebras over k, and their Θ-invariants.*

Proof. Based on the Allison–Smirnov Theorem 3.1, we show that every central simple structurable algebra over $k = k^s$ is in one of the tables. The claim that the tables contain no duplicates will follow from Proposition 3.11 and Theorem 3.14.

(a) The only central simple associative k-algebras are full matrix algebras $M_n(k)$ [94, Theorem 8.20]. Every k-linear involution on $M_n(k)$ is the adjoint involution of a nondegenerate bilinear form that is either symmetric or skew-symmetric [101, Proposition 2.1]. The involution is orthogonal if it corresponds to a symmetric bilinear form, and symplectic if it corresponds to a skew-symmetric one. The degree of $M_n(k)$ is n and the skew-dimension of its involution is determined by the type of the involution, as described in [101, Proposition 2.6].

Since $\operatorname{char} k \neq 2$, we have $k^{\times 2} = k^{\times}$ and every nondegenerate symmetric bilinear form over k is the linearisation of $n\langle 1\rangle$ for some n. Therefore there is exactly one orthogonal involution on $M_n(k)$, up to isomorphism, for all $n \geq 1$. If n is even, there exists exactly one n-dimensional nondegenerate skew-symmetric bilinear form over k, up to isometry, and if n is odd there are none [93, Theorem 6.3]. Therefore there is precisely one isomorphism class of symplectic involutions on $M_n(k)$ if n is even, and there are none if n is odd.

If A is a simple algebra with involution such that $Z(A, -) = k$ but $Z(A) \neq k$, then $Z(A)$ is a 2-dimensional reduced k-algebra, and the only possibility is $Z(A) \simeq \mathbb{B}$. The algebra A is isomorphic to $M_n(\mathbb{B}) \simeq M_n(k) \times M_n(k)$ with the exchange involution $(x, y) \mapsto (y^t, x^t)$ [101, Proposition 2.14]. The generic degree of $M_n(k) \times M_n(k)$ is $2n$. Therefore Table 1 (a) exhausts all possibilities for central simple associative algebras with involution over k.

(b) Every special central simple Jordan algebra of degree $d \geq 3$ is isomorphic to $\mathcal{H}(B, \tau)$ for some associative central simple algebra B with involution τ [91, V. Theorem 11]. Moreover, $(B, \tau) \simeq (C, \sigma)$ as algebras with involution if and only if $\mathcal{H}(B, \tau) \simeq \mathcal{H}(C, \sigma)$. So, we can account for all special central simple Jordan algebras of degree $d \geq 3$ by taking the hermitian subspaces of algebras in Table 1 (a). The generic degree of $\mathcal{H}(B, \tau)$ is $1/m$ times the generic (k-)degree of B, where $m = 1$ if τ is orthogonal, and $m = 2$ if τ is symplectic or unitary [91, §VI.3].

Isomorphism classes of separable Jordan algebras of degree 2 and dimension n are in one-to-one correspondence with isomorphism classes of nondegenerate quadratic spaces of dimension $n - 1$, via $(V, q) \leftrightarrow J(V, q)$ [101, Proposition 37.4]. If $\dim V = 1$ then $J(V, q)$ is not central simple [101, Remark 37.5], so there are no central simple Jordan algebras of degree 2 and dimension 2. Nondegenerate quadratic spaces over $k = k^s$ are isometric if and only if they have the same dimension, so for each $n \geq 3$ there is exactly one simple n-dimensional Jordan algebra of degree 2; these are the algebras $\mathcal{J}Spin_{n-1}(k)$. It follows that Table 1 (a)–(b) exhausts the possibilities for central simple special Jordan algebras over k.

(c) We now claim that structurable algebras $S(E, \sigma, W, h)$ of nonsingular hermitian spaces (W, h) over central simple algebras with involution (E, σ) are all included in Table 1 (a)–(c), with the exception of the quartic Cayley algebra which finds itself in Table 2.

For $E = M_n(k)$, Morita theory implies that every unital E-module is isomorphic to k^{nd} for some d. Hermitian forms over (k, id) are just bilinear forms, and their structurable algebras are just Jordan algebras of quadratic forms, which are already covered in part (b). The case where $W = \{0\}$ is covered in part (a) because these structurable algebras are associative. Therefore we assume $E = M_n(k)$ and $W = k^{nd}$ for some $n \geq 2$ and $d \geq 1$. If σ is orthogonal, then the category of nonsingular hermitian forms on k^{nd} with respect to (E, σ) is equivalent to the category of nondegenerate symmetric k-bilinear forms on k^d [100, I. §9.2–9.3]. But as observed in (a), there is exactly one symmetric bilinear form on k^d, up to isometry. If (E, σ) is symplectic then n is even and the category of nonsingular hermitian forms on k^{nd} with respect to (E, σ) is equivalent to the category of nondegenerate skew-symmetric bilinear forms on k^d. Depending on the parity of d, there are either one or no skew-symmetric forms on k^d, up to isometry. This implies that either $S(E, \sigma, W, h) \simeq S(\mathbb{M}_n(k), k^{nd})$ or n and d are both even and $S(E, \sigma, W, h) \simeq S(\mathbb{M}_{n/2}(\mathbb{H}), k^{nd})$.

On the other hand, if $E = M_n(k) \times M_n(k)$, and σ is the exchange involution, then the category of nonsingular hermitian forms over (E, σ) is isomorphic to the category of nonsingular hermitian forms over $\mathbb{B}$. But all nonsingular hermitian forms over $\mathbb{B}$ are hyperbolic [100, I. §6.7], so they are uniquely determined up to isometry by their k-dimensions. Therefore, every nonsingular hermitian form over (E, σ) is uniquely determined by its k-dimension, and these dimensions can be any multiple of $2n$.

Table 1 (a)–(c) therefore contains all structurable algebras of hermitian forms, except for two deliberate exclusions: $S(\mathbb{B}, k^2)$ is associative by Lemma 3.2, and the split quartic Cayley algebra $S(\mathbb{M}_2(k), k^4)$ is isomorphic to $M(k^3)$ by Lemma 3.3. We deliberately view the quartic Cayley algebra as one of the exceptional algebras, and accordingly it appears in Table 2 (f). The Θ-invariants of $S(E, \sigma, W, h)$ are easy to compute: for instance, the degree of the structurable algebra $S(E, \sigma, W)$ with $W \neq \{0\}$ is twice the degree of the Jordan algebra $\mathcal{H}(E, \sigma)$ [10, Theorem 6.1].

(d) Every exceptional central simple Jordan algebra over $k = k^s$ is 27-dimensional and isomorphic to the split Albert algebra $\mathcal{H}_3(\mathbb{O})$.

(e) A tensor product of two composition algebras is determined up to isomorphism by its dimension alone, because all composition algebras are split over $k = k^s$. These tensor product algebras are associative unless one of the factors is an octonion algebra, so Table 1 (a) and Table 2 (e) contain all of the examples in this class. Smirnov algebras over $k = k^s$ are all isomorphic too, because they are constructed from octonion algebras.

(f) Now we conclude by showing that every structurable algebra $(B, -)$ of skew-dimension one is either already in Table 1, or is in Table 2 (f). Besides $\mathbb{B}$, which is associative, every central simple structurable algebra of skew-dimension 1 over k is a matrix algebra $M(J, N, T, \zeta) = \begin{pmatrix} k & J \\ J & k \end{pmatrix}$, in the language and notation of [7], parameterised by a unital Jordan algebra J of generic degree $d \leq 3$, a cubic form $N : J \to k$ which may or may not be zero, a bilinear form $T : J \times J \to k$ which may or may not be zero, and a scalar $\zeta \in k^\times$ (see [7, Example 1.9, Theorem 1.13]). The scalar ζ is only significant up to its coset in $k^\times / k^{\times 3}$. But $k^{\times 3} = k^\times$ so we may assume that $(B, -)$ is of the form $M(J, N, T, 1)$ for appropriate parameters J, N, and T.

Structurable matrix algebras all have degree 2 or 4 [7, Proposition 4.4]. Those of degree 2 are just structurable algebras of hermitian forms over 2-dimensional composition algebras [7, Theorem 4.11], so we can ignore those because they are already covered in (c). The structurable matrix algebras of degree 4 have constraints on their parameters [7, Lemma 4.2, Example 1.9], namely that $N, T, 1$ are part of a so-called nondegenerate Jordan cubic norm structure and J is the Jordan algebra of generic degree ≤ 3 constructed from it.

According to [129, Theorem 2.1, Proposition 2.6], the Jordan algebras arising from nondegenerate Jordan cubic norm structures are either $J = k^i$ where $1 \leq i \leq 3$, $J = k \times \mathcal{J}Spin_{n-1}$ where $n \geq 3$, or $J = \mathcal{H}_3(C)$ where $C = k$, $\mathbb{B}$, $\mathbb{H}$, or $\mathbb{O}$. However, if $J = k \times k$ or $J = k \times \mathcal{J}Spin_{n-1}$ $(n \geq 3)$ then we know from [4, p. 1872] that $(B, -)$ is isomorphic to one of the hermitian type structurable algebras $S(\mathbb{M}_2(k), k^2)$ or $S(\mathbb{M}_2(k), k^{2n})$, respectively, and so it appears in Table 2 (c). The last remaining possibilities for $(B, -)$ are $M(J)$ for $J = k$, k^3, or $\mathcal{H}_3(C)$, and accordingly these algebras appear in Table 2 (f). □

3.10. *Descent in the infinite families.* Associativity is a property that is preserved by descent, in the sense that an algebra A is associative if and only if A_{k^s} is associative. Similarly, the class of special Jordan algebras is closed under descent.

The class of structurable algebras of hermitian forms is *not* closed under descent, because there exist quartic Cayley algebras that are not the structurable algebra of any hermitian form [31, Proposition 6.4.2], even though the split algebra $M(k^3) \simeq S(\mathbb{M}_2(k), k^4)$ manifestly is.

It turns out that this is the only exception. Allison's theorems [3, Theorem 2.5] and [7, Theorem 4.11] tell us (somewhat indirectly) that if $(A, -)$ becomes isomorphic over a field extension to some $S(E, \sigma, W, h)$ and $\Theta(A, -) \neq (8, 1, 4)$, then $(A, -)$ is necessarily also of hermitian type.

3.11. Proposition. *Suppose k is separably closed with* $\mathrm{char}(k) \neq 2, 3, 5$*, and $(A, -)$ and $(B, -)$ are central simple structurable algebras over k. Then $\Theta(A, -) = \Theta(B, -)$ implies $(A, -) \simeq (B, -)$ except if $\Theta(A, -) = (32, 10, 4)$ or $\Theta(A, -) = (x, 1, 4)$ where $x \in \{14, 20, 32, 56\}$.*

The proof of this proposition is very boring. Before printing it, it is at least interesting to make a remark about why Θ is not injective.

3.12. *Failure of Θ to be injective.* There are five pairs of nonisomorphic structurable algebras with identical Θ-invariants. The first coincidence is that

$$\Theta\big(\mathbb{O} \otimes \mathbb{H}\big) = \Theta\big(S(\mathbb{M}_2(\mathbb{H}), k^{4 \cdot 4})\big).$$

The second co-incidence is that

$$\Theta\big(M(\mathcal{H}_3(C))\big) = \Theta\big(M(k \times \mathcal{J}Spin_{n-1})\big)$$

when the cubic Jordan algebras $\mathcal{H}_3(C)$ and $k \times \mathcal{J}Spin_{n-1}$ have the same dimension, i.e. when $n = 2 + 3 \dim C \in \{5, 8, 14, 26\}$.

These are genuine coincidences: $\mathbb{O} \otimes \mathbb{H} \not\simeq S(\mathbb{M}_2(\mathbb{H}), k^{4 \cdot 4})$ because the one algebra is generated by its skew subspace and the other is not, and it is also true that $M(\mathcal{H}_3(C)) \not\simeq M(k \times \mathcal{J}Spin_{n-1})$, which we will prove later. We even prove the stronger statement that these algebras are not isotopic.

Recall also that $M(k \times \mathcal{J}Spin_{n-1}) \simeq S(\mathbb{M}_2(k), k^{2n})$, and we regard these as members of the infinite family of hermitian type structurable algebras, rather than as exceptional skew-dimension one structurable algebras.

Proof of Proposition 3.11. Every associative central simple algebra with involution maps to a triple in the set:

$$\Theta(\mathsf{Assoc}) = \Big\{(n^2, \tfrac{n(n-1)}{2}, n), (m^2, \tfrac{m(m+1)}{2}, m), (\tfrac{m^2}{2}, \tfrac{m^2}{4}, m) : n \in \mathbb{N} + 1, m \in 2\mathbb{N} + 2\Big\}.$$

There are no positive integer solutions to $\frac{m(m-1)}{2} = \frac{m(m+1)}{2}$ or $\frac{m(m+1)}{2} = \frac{m^2}{4}$, and the only solution to $\frac{m(m-1)}{2} = \frac{m^2}{4}$ is $m = 2$. This demonstrates, after fixing the third coordinate $n = m$ and comparing the second coordinates (and also the first coordinates for $m = 2$), that $\Theta(A,-) = \Theta(B,-)$ implies $(A,-) \simeq (B,-)$ for all central simple associative algebras $(A,-)$ and $(B,-)$ over k.

Substituting $m = 2n$ into $\Theta(S(\mathbb{M}_n(k), k^{nd}) = (n(n+d), \frac{n(n-1)}{2}, 2n)$ gives

$$\Theta(S(\mathbb{M}_{m/2}(k), k^{md/2})) = (m(\tfrac{m}{4} + \tfrac{d}{2}), \tfrac{m(m-2)}{8}, m),$$

which is more convenient to compare with the other formulas. Our list of hermitian type structurable algebras from Table 1 (c) is mapped to the set:

$$\Theta(\mathsf{Herm}) = \Big\{(m(\tfrac{m}{4} + \tfrac{d}{2}), \tfrac{m(m-2)}{8}, m),\ (n(n+2d), \tfrac{n(n+1)}{2}, n),\ (n(\tfrac{n}{2} + d), \tfrac{n^2}{4}, n) \\ : d \in \mathbb{N}+1, n \in 2\mathbb{N}+2, m \in 2\mathbb{N}+4\Big\} \setminus \Big\{(8,1,4), (4,1,2)\Big\}.$$

There are no nonisomorphic hermitian structurable algebras with the same Θ-invariant. This can be demonstrated by fixing the third coordinate $m = n$ and comparing the second coordinates, showing that there are no positive integers m such that any of $\frac{m(m-2)}{8}$, $\frac{m(m+1)}{2}$, or $\frac{m^2}{4}$ are equal.

Every special nonassociative Jordan algebra maps to a triple in the set:

$$\Theta(\mathsf{Special}) = \{(n,0,2), (\tfrac{n(n+1)}{2}, 0, n), (n(2n-1), 0, n), (n^2, 0, n) : n \in \mathbb{N}+3\}.$$

It is clear after fixing the third coordinate and comparing the first coordinates that $\Theta(A, \mathrm{id}) = \Theta(B, \mathrm{id})$ implies $A \simeq B$ for all special central simple Jordan algebras A and B over k.

The exceptional structurable algebras all have distinct Θ-invariants in the set:

$$\Theta(\mathsf{Except}) = \{(27,0,3), (8,7,2), (16,8,4), (32,10,4), (64,14,8), (35,7,7), \\ (4,1,4), (8,1,4), (14,1,4), (20,1,4), (32,1,4), (56,1,4)\}.$$

It is clear that $(\Theta(\mathsf{Except}) \cup \Theta(\mathsf{Herm}) \cup \Theta(\mathsf{Assoc})) \cap \Theta(\mathsf{Special}) = \emptyset$ because only Jordan algebras have skew-dimension zero and there are no special Jordan algebras with degree 3 and dimension 27. We show that $\Theta(\mathsf{Except}) \cap \Theta(\mathsf{Assoc}) = \emptyset$. For all $(x,y,z) \in \Theta(\mathsf{Assoc})$ we have $x = z^2$ or $x = \frac{z^2}{2}$. The only $(x,y,z) \in \Theta(\mathsf{Except})$ with $x = z^2$ or $x = \frac{z^2}{2}$ are $(16,8,4)$, $(64,14,8)$, and $(8,1,4)$. But $(16,y,4) \in \Theta(\mathsf{Assoc})$ implies $y = 6$ or 10, $(64,y,8) \in \Theta(\mathsf{Assoc})$ implies $y = 28$ or 36, and $(8,y,4) \in \Theta(\mathsf{Assoc})$ implies $y = 4$.

One can show that $\Theta(\mathsf{Assoc}) \cap \Theta(\mathsf{Herm}) = \emptyset$, again by first fixing n and observing that $(x,y,n) \in \Theta(\mathsf{Assoc}) \cap \Theta(\mathsf{Herm})$ puts conditions on x and y which are not possible to meet. Namely, previous calculations imply that any $(x,y,z) \in \Theta(\mathsf{Assoc}) \cap \Theta(\mathsf{Herm})$ must be of the form $(m^2, \frac{m(m-1)}{2}, m) = (m(\frac{m}{4} + \frac{d}{2}), \frac{m(m-2)}{8}, m)$ for some even $m \geq 4$, or $(m(\frac{m}{2} + d), \frac{m^2}{4}, m) = (m^2, \frac{m(m-1)}{2}, m)$ for $m = 2$ and some $d \geq 1$. But the first equation has no solutions, and the only solution to the second one is $d = 1$.

Recall that there is an isomorphism $M_2(k) \simeq S(\mathbb{B}, k^2)$, which is why we have excluded this algebra from Table 1 (c), and accordingly we have excluded $(4,1,2)$ from $\Theta(\mathsf{Herm})$. Therefore $\Theta(\mathsf{Assoc}) \cap \Theta(\mathsf{Herm}) = \emptyset$.

The last case to check is the following: suppose $(x, y, z) \in \Theta(\mathsf{Except}) \cap \Theta(\mathsf{Herm})$. Then z is even, $y = \frac{z^2}{4}$, $\frac{z(z+1)}{2}$, or $\frac{z(z-2)}{8}$, and $(x, y, z) \neq (8, 1, 4)$. By inspection, $(x, y, z) = (32, 10, 4)$, $(14, 1, 4)$, $(20, 1, 4)$, $(32, 1, 4)$, or $(56, 1, 4)$. □

3.13. *The Ψ-invariant.* In order to address the problem that Θ is not injective, one can define another numerical invariant Ψ of central simple structurable algebras:

$$\Psi(A, -) = \big(\dim_k A,\ \dim_k \operatorname{Skew}(A, -),\ \deg_k(A, -),\ \dim_k V_{A,A}\big) \in \mathbb{N}^4.$$

It is just a slightly richer invariant than Θ. Like the Θ-invariant, the Ψ-invariant is obviously stable under isotopy and scalar extensions.

The following theorem uses some calculations from Chapter III in its proof.

3.14. Theorem. *Suppose k is separably closed with $\operatorname{char}(k) \neq 2, 3, 5$, and $(A, -)$ and $(B, -)$ are central simple structurable algebras over k. Then $\Psi(A, -) = \Psi(B, -)$ implies $(A, -) \simeq (B, -)$.*

Proof. We only need to prove that if $(A, -)$, $(B, -)$ is one of the pairs from 3.12 with identical Θ-invariants, then $\dim V_{A,A} \neq \dim V_{B,B}$. By the decomposition (2.1.2), it suffices to show that $\dim D_{A,A} \neq \dim D_{B,B}$.

Consider first the case where $(A, -) = \mathbb{O} \otimes \mathbb{H}$ and $(B, -) = S(\mathbb{M}_2(\mathbb{H}), k^{4 \cdot 4})$. Then $\Theta(A, -) = \Theta(B, -)$. But by Proposition 9.10, $\dim D_{A,A} = \dim(\mathfrak{sl}_2) + \dim(\mathfrak{g}_2) = 17$, and by Corollary 8.5, $\dim D_{B,B} = 2\dim(\mathfrak{sp}_4) = 20$. So $\Psi(A, -) \neq \Psi(B, -)$.

Now suppose $(A, -) = M(\mathcal{H}_3(C))$ and $(B, -) = M(k \times \mathcal{J}Spin_{n-1})$ where C is a composition algebra of dimension $m \in \{1, 2, 4, 8\}$, and $n = 2 + 3m \in \{5, 8, 14, 26\}$. Then $\Theta(A, -) = \Theta(B, -)$. But according to 10.11 and Proposition 10.12,

$$\dim D_{A,A} = \begin{cases} 8 & m = 1 \\ 16 & m = 2 \\ 35 & m = 4 \\ 78 & m = 8 \end{cases} \qquad \dim D_{B,B} = 1 + n(n-1) = \begin{cases} 11 & n = 5 \\ 22 & n = 8 \\ 78 & n = 14 \\ 300 & n = 26. \end{cases}$$

So, $\Psi(A, -) \neq \Psi(B, -)$. □

3.15. Corollary. *If $\operatorname{char}(k) \neq 2, 3, 5$ and $(A, -)$ and $(B, -)$ are isotopic central simple structurable algebras over k, then there exists a finite separable extension K/k such that $(A_K, -) \simeq (B_K, -)$.*

Proof. Suppose A and B are isotopic central simple structurable algebras over k. Then $\Psi(A) = \Psi(B)$, which implies $A_{k^s} \simeq B_{k^s}$. Then A_{k^s} and B_{k^s} can be equipped with bases $\{a_1, \ldots, a_n\}$ and $\{b_1, \ldots, b_n\}$ whose structure constants are the same. The separable closure k^s is a direct limit of all finite separable extensions of k, so A_{k^s} can be viewed as the direct limit of A_K as K varies over finite separable extensions of k, and likewise for B_{k^s}. Hence there is some finite separable K for which $\{a_1, \ldots, a_n\} \subset A_K$ and $\{b_1, \ldots, b_n\} \subset B_K$, implying $A_K \simeq B_K$. (This is the type of argument that appears in [72, §2.2].) □

3.16. Corollary. *Central simple structurable algebras over a separably closed field (of characteristic not 2, 3, or 5) are isotopic if and only if they are isomorphic.*

Chapter II

Graded Lie algebras

This chapter introduces some tools for working with graded Lie algebras and their (graded) automorphism groups. The TKK construction is defined, and a proof is given to justify Table 4 from the introduction. Following this, the more general AF construction is defined, together with some structural results on the Lie algebra of Lie related triples of a structurable algebra. Exploiting an S_4-symmetry on the AF construction, we give a formula for the Killing form of these Lie algebras.

4. $\mathbb{Z}$-graded Lie algebras

A combinatorial invariant of $\mathbb{Z}$-graded simple Lie algebras is introduced, called the labelled Dynkin diagram. It will be used later for the purpose of identifying which graded Lie algebras emerge from the TKK construction.

4.1. *Graded algebras.* Let Γ be an abelian group. A k-algebra L is Γ-graded if it is a direct sum of subspaces $L = \bigoplus_{\gamma\in\Gamma} L_\gamma$ and $L_\gamma L_\delta \subset L_{\gamma+\delta}$ for all $\gamma, \delta \in \Gamma$. We write $\mathbf{Aut}_{\mathrm{gr}}(L)$ for the (algebraic) subgroup of $\mathbf{Aut}(L)$ consisting of graded automorphisms:

$$\mathbf{Aut}_{\mathrm{gr}}(L)(R) = \{\alpha \in \mathbf{Aut}(L)(R) \colon \alpha((L_\gamma)_R) \subset (L_\gamma)_R \text{ for all } \gamma \in \Gamma\}.$$

Suppose now that $L = \bigoplus_{i\in\mathbb{Z}} L_i$ is $\mathbb{Z}$-graded. If $L_i = 0$ for all $|i| > n$, then we say that L is $(2n+1)$-graded. We say L is strictly $(2n+1)$-graded if it is not also $(2n-1)$-graded, and that L is *trivially graded* if $L = L_0$.

Define the *grading cocharacter* $\lambda : \mathbf{G}_m \to \mathbf{Aut}(L)$ by $\lambda_R(c)(x_i) = c^i x_i$ for all $c \in R^\times$ and $x_i \in (L_i)_R$, $i \in \mathbb{Z}$. Any cocharacter $\lambda : \mathbf{G}_m \to \mathbf{Aut}(L)$ is the grading cocharacter of a unique $\mathbb{Z}$-grading on L [49, Proposition 1.28].

4.2. Lemma. *Suppose $L = \bigoplus_{i\in\mathbb{Z}} L_i$ is a $\mathbb{Z}$-graded algebra over an arbitrary field K. Let $G = \mathbf{Aut}(L)$, and let $\lambda : \mathbf{G}_m \to \mathbf{Aut}(L)$ be the grading cocharacter. Then $\mathbf{Aut}_{\mathrm{gr}}(L) = C_G(\lambda)$ and $\mathbf{Aut}_{\mathrm{gr}}(L)^\circ = C_G(\lambda)^\circ = C_{G^\circ}(\lambda)$.*

Proof. It is straightforward to show from the definition [122, §1.k] that $\mathbf{Aut}_{\mathrm{gr}}(L)(R) \subset C_G(\lambda)(R)$. On the other hand, if $\beta \in C_G(\lambda)(R)$ and $x_i \in (L_i)_R$, then for all R-algebras S and all $c \in S^\times$,

$$\lambda_S(c) \circ \beta(x_i) = \beta \circ \lambda_S(c)(x_i) = \beta(c^i x_i) = c^i \beta(x_i). \tag{4.2.1}$$

Writing $\beta(x_i) = y$ and decomposing it as $y = \sum_{j\in\mathbb{Z}} y_j$ where $y_j \in (L_j)_R$, we also have $\lambda_S(c)(y) = \sum_{j\in\mathbb{Z}} \lambda_S(c)(y_j) = \sum_{j\in\mathbb{Z}} c^j y_j = \sum_{j\in\mathbb{Z}} c^i y_j$ (comparing with (4.2.1)). Since the homogeneous components of L_R are linearly independent, this implies $(c^j - c^i)y_j = 0$ for all $j \in \mathbb{Z}$ and all $c \in S^\times$. Letting $S = R[t]$ and $c = t$, this implies $y_j = 0$ whenever $i \neq j$. Therefore $y = y_i \in L_i$ and $\beta \in \mathbf{Aut}_{\mathrm{gr}}(L)(R)$.

For the final claim, $\mathbf{Aut}_{\mathrm{gr}}(L)^\circ = C_G(\lambda)^\circ$ is a λ-centralising connected algebraic subgroup of G, which implies $C_G(\lambda)^\circ \leq C_{G^\circ}(\lambda)$. On the other hand, $C_{G^\circ}(\lambda)$ is a connected algebraic subgroup of $C_G(\lambda)$ [122, Theorem 17.38, Remark 17.40 (b)], and therefore $C_{G^\circ}(\lambda) \leq C_G(\lambda)^\circ$. □

4.3. *From adjoint simple groups to central simple Lie algebras and back again.* Let G be an adjoint simple algebraic group over k (recalling that $\mathrm{char}(k) \neq 2, 3$) and let $L = \mathrm{Lie}(G)$. As a rule, L is central simple, with the only exceptions being when $\mathrm{char}(k) = p > 0$ and G is of type A_{mp-1} for some $m \geq 1$ [80, Table 1]. In these unusual cases, the ideal $L' = [L, L]$ is central simple and $\dim L' = \dim L - 1$ [31, Lemma 4.1.6 (i)].

Moreover, the adjoint homomorphism $\mathrm{Ad} : G \to \mathbf{Aut}(L)^\circ$ and the restriction homomorphism $\cdot|_{L'} : \mathbf{Aut}(L)^\circ \to \mathbf{Aut}(L')^\circ$ are isomorphisms [31, Lemma 4.1.6 (ii)]. These facts imply that there are one-to-one correspondences:

$$\boxed{\mathbb{Z}\text{-gradings on } L} \quad \longleftrightarrow \quad \boxed{\mathbb{Z}\text{-gradings on } L'} \quad \longleftrightarrow \quad \boxed{\begin{array}{c}\text{Cocharacters}\\ \lambda : \mathbf{G}_m \to G\end{array}}$$

If $L = \bigoplus_{i\in\mathbb{Z}} L_i$ is a $\mathbb{Z}$-grading on L, then L' is a graded ideal and $L'_i = L_i$ for all $i \neq 0$ [31, Lemma 4.3.2]. Therefore $(2n+1)$-gradings on L are in one-to-one correspondence with $(2n+1)$-gradings on L'.

4.4. *Labelled Dynkin diagrams.* As in 4.3, let G be an adjoint simple algebraic k-group and let $L = \mathrm{Lie}(G)$. We can attach a combinatorial invariant to a $\mathbb{Z}$-grading of L or, equivalently, to a pair (G, λ) where λ is a G-valued cocharacter. This invariant is the *labelled Dynkin diagram*, and it owes its origins to Dynkin himself: he called it the *characteristic* in [48, Ch. III]. I learned about it from [96, 108, 165].

To define the labelled Dynkin diagram of (G, λ), we temporarily extend scalars until G has a split maximal torus. Suppose T is a maximal torus in G containing the image of λ. For each root $\alpha \in \Phi(G, T)$ there is a unique integer $\ell(\alpha)$ such that $\alpha \circ \lambda(c) = c^{\ell(\alpha)}$ for all $c \in \bar{k}^\times$. Equivalently, the root space L_α is contained in the homogeneous component $L_{\ell(\alpha)}$ of the $\mathbb{Z}$-grading associated to λ. We can choose a Weyl chamber for $\Phi(G, T)$ containing all the roots $\alpha \in \Phi(G, T)$ with $\ell(\alpha) > 0$, thus furnishing the root system with a base Π such that $\ell(\beta) \geq 0$ for every $\beta \in \Pi$. The Dynkin diagram of $\Phi(G, T)$ together with the labels $\{\ell(\beta)\}_{\beta\in\Pi}$ is called the labelled Dynkin diagram of (G, λ) with respect to (T, Π).

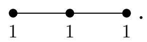
Different choices for T or Π lead to isomorphic labelled diagrams [165, p. 579]. Every possible labelling of the Dynkin diagram of G with non-negative integers is the labelled Dynkin diagram of some cocharacter defined over $\bar{k}$ [108, Lemma 10.2].

4.5. Example. Let $G = \mathbf{PGL}_4$, T the standard (diagonal) torus in G, $\mathfrak{g} = \mathrm{Lie}(G) = \mathfrak{sl}_4$, and $\mathfrak{h}$ the standard Cartan subalgebra of diagonal trace zero matrices. The root system $\Phi(G,T)$ of type A_3 has a base $\Pi = \{\alpha_1, \alpha_2, \alpha_3\}$ where $\alpha_i : \mathfrak{h} \to k$ is the root $\alpha_i(\lambda_1, \ldots, \lambda_4) = \lambda_i - \lambda_{i+1}$ [157, VII. §6]. The root space $\mathfrak{g}_{\alpha_i}$ is spanned by the elementary matrix $E_{i,i+1} \in \mathfrak{sl}_4$.

There are many different $\mathbb{Z}$-gradings on $\mathfrak{g}$. For example, the 7-grading given by

$$\mathfrak{g}_{-3} = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ * & \cdot & \cdot & \cdot \end{pmatrix}, \qquad \mathfrak{g}_{-2} = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ * & \cdot & \cdot & \cdot \\ \cdot & * & \cdot & \cdot \end{pmatrix}, \qquad \ldots, \qquad \mathfrak{g}_{3} = \begin{pmatrix} \cdot & \cdot & \cdot & * \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

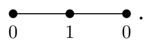
has $\mathfrak{g}_{\alpha_i} \subset \mathfrak{g}_1$ for $1 \leq i \leq 3$, so its labelled Dynkin diagram with respect to (T, Π) is



The 3-grading on $\mathfrak{g}$ given by

$$\mathfrak{g}_{-1} = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ * & * & \cdot & \cdot \\ * & * & \cdot & \cdot \end{pmatrix}, \qquad \mathfrak{g}_{0} = \begin{pmatrix} * & * & \cdot & \cdot \\ * & * & \cdot & \cdot \\ \cdot & \cdot & * & * \\ \cdot & \cdot & * & * \end{pmatrix}, \qquad \mathfrak{g}_{1} = \begin{pmatrix} \cdot & \cdot & * & * \\ \cdot & \cdot & * & * \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

has labelled Dynkin diagram



4.6. *How to read a labelled Dynkin diagram.* We can use the labelled Dynkin diagram of (G, λ) to extract some information about the grading on L, such as its support and the dimensions of its homogeneous components.

Suppose G has a split maximal torus T of rank n. Let $X^*(T) \simeq \mathbb{Z}^n$ be the character group of T in G. The root space decomposition

$$L = \bigoplus_{\omega \in X^*(T)} L_\omega = L_{\underline{0}} \oplus \Big(\bigoplus_{\alpha \in \Phi(G,T)} L_\alpha \Big)$$

is a fine $\mathbb{Z}^n$-grading called the *Cartan grading* [49]. Here, L_ω is the ω-weight space, which is either n-dimensional, 1-dimensional, or 0-dimensional according as ω is zero, a root, or neither. The 0-weight space $L_{\underline{0}} = \mathrm{Lie}(T)$ is an n-dimensional Cartan subalgebra. The grading induced on L by a cocharacter $\lambda : \mathbf{G}_m \to T$ is the following coarsening of the Cartan grading:

$$L = \bigoplus_{i \in \mathbb{Z}} L_i \tag{4.6.1}$$

where

$$L_i = \bigoplus_{\substack{\alpha \in \Phi(G,T) \\ \ell(\alpha) = i}} L_\alpha \quad \text{for } i \neq 0, \qquad L_0 = L_{\underline{0}} \oplus \Big(\bigoplus_{\substack{\alpha \in \Phi(G,T) \\ \ell(\alpha) = 0}} L_\alpha \Big).$$

Calculating $\dim L_i$ boils down to counting the number of roots

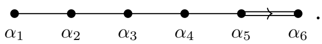
$$\alpha = \sum_{\beta \in \Pi} m_\beta(\alpha) \beta \in \Phi(G,T)$$

such that
$$\ell(\alpha) = \sum_{\beta \in \Pi} m_\beta(\alpha)\ell(\beta) = i.$$
This clearly depends only on the labelled Dynkin diagram of (G, λ), since the labelled diagram determines $\ell(\alpha)$ by linearity for all roots $\alpha \in \Phi(G,T)$.

One can calculate the support of the grading very easily using the coefficients of the highest root. If
$$\tilde{\alpha} = \sum_{\beta \in \Pi} m_\beta(\tilde{\alpha})\beta \in \Phi(G,T)$$
is the highest root with respect to Π then let
$$n = \ell(\tilde{\alpha}) = \sum_{\beta \in \Pi} m_\beta(\tilde{\alpha})\ell(\beta).$$
Since $\tilde{\alpha}$ is the highest root, we have $n \geq \sum_{\beta\in\Pi} m_\beta(\alpha)\ell(\beta) = \ell(\alpha)$ for all $\alpha \in \Phi(G,T)$, so λ induces a strict $(2n+1)$-grading.

If G does not have a split maximal torus T, the $\mathbb{Z}$-grading (4.6.1) is still defined over k provided that λ is defined over k. The dimensions of the L_i are unaffected by scalar extension, so there is no harm in going to an algebraic closure and doing these calculations there if necessary.

4.7. Example. Let $G = \mathbf{O}^+_{13}$. Then $\mathfrak{g} = \mathrm{Lie}(G)$ is of type B_6 [101, p. 368], and has Dynkin diagram



The highest root is $\tilde{\alpha} = \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_6$ [34, Plate II]. If the labelled Dynkin diagram $\{\ell(\alpha_i)\}_{i=1}^6$ has more than one positive label then $\ell(\tilde{\alpha}) \geq 3$, so the corresponding grading supports at least 7 nonzero components. If $\ell(\alpha_1) = 1$ is the only positive label, the labelled diagram corresponds to a strict 3-grading. If $\ell(\alpha_i) = 1$ for some $i > 1$ and $\ell(\alpha_j) = 0$ for $j \neq i$, then the labelled diagram corresponds to a strict 5-grading.

4.8. *The centraliser of the grading cocharacter.* Keeping with the notation, let G be an adjoint simple group and λ a G-valued cocharacter. From the discussion in 4.1 and 4.3, λ is the grading cocharacter of a unique $\mathbb{Z}$-grading on $\mathrm{Lie}(G)$. Let $H = C_G(\lambda)$. Recall from Lemma 4.2 that H is the graded automorphism group of $\mathrm{Lie}(G)$.

Its connected component $H^\circ = C_{G^\circ}(\lambda)$ is reductive [122, Corollary 17.59], and H° is a Levi subgroup of a parabolic subgroup of G° [32, Proposition 20.4].

4.9. Lemma. *Let* $L = \mathrm{Lie}(G)$. *If* $L = \bigoplus_{i\in\mathbb{Z}} L_i$ *is the* $\mathbb{Z}$*-grading determined by* λ *then* $\mathrm{Lie}(H) = L_0$.

Proof. See [122, Proposition 10.34, Corollary 17.59]. □

Despite knowing something about its Lie algebra, it is not always easy to describe H. But since H° is reductive, its derived subgroup $M = (H^\circ)^{\mathrm{der}}$ is semisimple. One can use the labelled Dynkin diagram of (G, λ) to determine the absolute type of M up to isogeny, because Lemma 4.9 and [162, Theorem 8.1.5 (i)] imply that M is the semisimple group generated by the root groups U_α where α ranges over all roots in $\Phi(G,T)$ with $\ell(\alpha)=0$.

To find the Dynkin diagram of M, just delete the positively labelled vertices from the Dynkin diagram of (G,λ). The (unlabelled) Dynkin diagram that remains is the Dynkin diagram of M.

5. The TKK construction

The Tits–Kantor–Koecher (TKK) construction was originally a way of producing simple 3-graded Lie algebras from simple Jordan algebras [91, VIII. §5]. Allison in [4] showed how this generalises to allow the construction of simple 5-graded Lie algebras from simple structurable algebras.

5.1. *Definition of* $K(A,-)$. Let $(A,-)$ be a structurable k-algebra. We shall define a 5-graded Lie algebra on the vector space:

$$K(A,-)=K_{-2}\oplus K_{-1}\oplus K_0\oplus K_1\oplus K_2$$

where K_1 and K_{-1} are copies of A,

$$K_{\pm 1}=\{a_\pm : a\in A\},$$

where K_2 and K_{-2} are copies of $\mathrm{Skew}(A,-)$,

$$K_{\pm 2}=\{s_\pm : s\in \mathrm{Skew}(A,-)\},$$

and where $K_0=V_{A,A}$ is the subspace of $\mathfrak{gl}(A)$ generated by the operators

$$\{V_{x,y} : x,y\in A\}.$$

Recall that $V_{A,A}$ is already a Lie algebra. We define a Lie bracket on $K(A,-)$ which extends the Lie bracket on K_0. Firstly, define the antisymmetric bilinear map

$$\psi : A\times A\to S, \qquad \psi(x,y)=x\bar{y}-y\bar{x}. \tag{5.1.1}$$

Secondly, define a K_0-module structure on K_i for $i\neq 0$ by $[-,-]:K_0\times K_i\to K_i$:

$$\begin{aligned}
K_0\times K_1\to K_1 &\qquad [V_{x,y},z_+]=(V_{x,y}z)_+\\
K_0\times K_{-1}\to K_{-1} &\qquad [V_{x,y},z_-]=(-V_{y,x}z)_-\\
K_0\times K_2\to K_2 &\qquad [V_{x,y},s_+]=-\psi(x,sy)_+\\
K_0\times K_{-2}\to K_{-2} &\qquad [V_{x,y},s_-]=\psi(y,sx)_-
\end{aligned}$$

for all $x,y,z\in A$ and $s\in\mathrm{Skew}(A,-)$. Thirdly, define bilinear maps

$$[-,-]:K_i\times K_j\to K_{i+j}$$

as follows:

$$\begin{aligned}
K_1\times K_1\to K_2 &\qquad [x_+,y_+]=\psi(x,y)_+\\
K_{-1}\times K_{-1}\to K_{-2} &\qquad [x_-,y_-]=\psi(x,y)_-\\
K_1\times K_{-1}\to K_0 &\qquad [x_+,y_-]=V_{x,y}\\
K_2\times K_{-2}\to K_0 &\qquad [s_+,t_-]=L_sL_t\\
K_2\times K_{-1}\to K_1 &\qquad [s_+,x_-]=(sx)_+\\
K_{-2}\times K_1\to K_{-1} &\qquad [s_-,x_+]=(sx)_-
\end{aligned}$$

for all $x, y \in A$ and $s, t \in \mathrm{Skew}(A, -)$. Finally, the bracket is required to satisfy anticommutativity

$$[l, m] = -[m, l] \qquad \text{for all } l, m \in K(A, -)$$

and $[K_i, K_j] = 0$ if $|i + j| > 2$. By linear extension, this defines a bilinear map

$$[-, -] : K(A, -) \times K(A, -) \to K(A, -)$$

which satisfies the Jacobi identity and so gives $K(A, -)$ the structure of a $\mathbb{Z}$-graded Lie algebra [4, §3]. If $(A, -)$ is a central simple structurable algebra then $K(A, -)$ is a central simple Lie algebra, and conversely [4, §5]. The decomposition (2.1.2) gives us a dimension formula:

$$\begin{aligned} \dim K(A, -) = \sum_{i=-2}^{2} \dim K_i &= 2 \dim A + 2 \dim \mathrm{Skew}(A, -) + \dim V_{A,A} \\ &= 3 \dim A + 2 \dim \mathrm{Skew}(A, -) + \dim D_{A,A}. \end{aligned} \tag{5.1.2}$$

5.2. *The output of the TKK construction.* If L is a simple Lie algebra over a field of characteristic 0 or $p > 3$, we have a classification due to Stavrova [165, Theorem 5.11] of all possible gradings on L such that L is graded-isomorphic to $K(A, -)$ for some central simple structurable algebra $(A, -)$. The labelled Dynkin diagrams of these gradings are symmetric and made of 0's and either one or two 1's.

In particular, it is an interesting discovery (long known in characteristic 0, but quite new in characteristic $p > 3$) that every simple Lie algebra L admitting a 5-grading is isomorphic to $K(A, -)$ for some central simple structurable algebra $K(A, -)$ [165, Theorem 1.1].

5.3. Theorem. *If $(A, -)$ is a central simple structurable algebra that is exceptional, a $(4, m)$-product algebra, or an octonion algebra with nonstandard involution, then the labelled Dynkin diagram of $K(A, -)$ is given in Table* 4.

Proof. Stavrova has proven in [165, Theorem 3.4] that the only nontrivially 5-graded simple Lie algebras over algebraically closed fields of characteristic not 2 or 3 are the "classical" ones, i.e. $L = [\mathrm{Lie}(G), \mathrm{Lie}(G)]$ for some adjoint simple group G. (In the literature on modular Lie algebras, it is normal to call Lie algebras of type G_2, F_4, and E classical.) We are therefore in the setting of 4.3 and can assign to $K(A, -)$ a labelled Dynkin diagram.

Step 1: Determine the absolute type of $K(A, -)$. If L is a classical simple Lie algebra with root system Φ, then

$$\dim L = \mathrm{rank}(\Phi) + |\Phi|$$

except when $\mathrm{char}(k) = p > 0$ and Φ is of type A_{mp-1}, in which case $\dim L = \mathrm{rank}(\Phi) + |\Phi| - 1$.

If L is of type B_n and L' of type C_n, then $\dim L = \dim L' = 2n^2 + n$. And if $n = 6$ then $\dim L = \dim L' = \dim \mathfrak{e}_6 = 78$. In all other cases, two simple Lie algebras of the same dimension are isomorphic. (This is easy to check using the data from [34, Plates I–IX].)

The dimensions of $D_{A,A}$ for each of the algebras in the table can be extracted from the results of the subsequent chapter (see Propositions 8.5, 9.10, and 10.12). By the dimension formula (5.1.2), the dimensions of the $K(A,-)$'s in the table are:

$$21, 52, 36, 35, 66, 78, 133, 248, 133, 133, 14, 28, 52, 78, 133, \text{ and } 248.$$

This determines the absolute type of $K(A,-)$, except when the dimension is 21 (which could be B_3 or C_3), 36 (which could be B_4 or C_4), or 78 (which could be B_6, C_6, or E_6). These question marks are resolved in the next step.

Step 2: Determine the 5*-grading.* On a simple Lie algebra L of reasonably low rank, there are usually very few strict 3- or 5-gradings, and one can single them out using the highest root method from 4.6. Example 4.7 is instructive here. So, the procedure is to check all the possible labellings corresponding to 5-gradings on the Lie algebras L of these types and match the structurable algebras in Table 4 to some 5-grading $L = \bigoplus_{i=-2}^{2} L_i$.

The matching is done by comparing dimensions. The purely combinatorial task of calculating the component dimensions of $L = \bigoplus_{i=-2}^{2} L_i$ from its labelled Dynkin diagram was done using the *Root Systems* package in SageMath [147]. In each case there is only one possible grading (up to a diagram automorphism) with components of dimension exactly

$$(\dim L_{-2}, \ldots, \dim L_2) = (\dim S, \dim A, \dim V_{A,A}, \dim A, \dim S).$$

For example, if $(A,-)$ is a quaternion algebra, then $K(A,-)$ has components of dimension $(3, 4, 7, 4, 3)$, so the type of $K(A,-)$ is C_3 and not B_3, because B_3 has no such 5-grading. Similarly, B_4 has no 5-grading with components of dimensions $(3, 8, 11, 8, 3)$, so the type of $K(A,-)$ for an octonion algebra with nonstandard involution is C_4 and not B_4. It turns out that there are no 5-gradings on B_6, C_6, or E_6 with the same series of component dimensions; i.e., if $L = \bigoplus_{i=-2}^{2} L_i$ is one of these types, the data $(\dim L_{-2}, \ldots, \dim L_2)$ is sufficient to say what type L is, and what the grading is, uniquely up to isomorphism. □

5.4. *Isotopies and graded isomorphisms.* Let $(A,-)$ and $(B,-)$ be any pair of structurable algebras. By [14, Proposition 12.3], the set of graded isomorphisms $K(A,-) \xrightarrow{\sim} K(B,-)$ is naturally isomorphic to the set of isotopies $(A,-) \to (B,-)$, where the isomorphism is given by restriction onto the 1-component $K_1 = A$:

$$\begin{aligned} \mathsf{Graded\text{-}Isomorphisms}\big(K(A,-), K(B,-)\big) &\xrightarrow{\sim} \mathsf{Isotopies}\big((A,-),(B,-)\big) \\ f &\longmapsto \pi_1(f) = f|_{K_1}. \end{aligned}$$

Of course, the situation is symmetric and one also obtains an isomorphism between these sets by restricting graded isomorphisms of the Lie algebras to their -1-component instead of their $+1$-component. Consequently, [14, Proposition 12.3] implies:

5.5. Lemma. *Every graded automorphism of $K(A,-)$ is uniquely determined by its restriction to the $+1$-component or the -1-component, and there are isomorphisms*

$$\mathbf{Str}(A,-) \xleftarrow[\pi_{-1}]{\sim} \mathbf{Aut}_{\mathrm{gr}}(K(A,-)) \xrightarrow[\pi_1]{\sim} \mathbf{Str}(A,-).$$

5.6. *An involution on the structure group.* The map

$$\begin{aligned}\pi_{-1} \circ \pi_1^{-1} = \pi_1 \circ \pi_{-1}^{-1} : \mathbf{Str}(A, -) &\longrightarrow \mathbf{Str}(A, -) \\ \alpha &\longmapsto \hat{\alpha}\end{aligned}$$

is an order 2 automorphism of $\mathbf{Str}(A, -)$ that fixes $\mathbf{Aut}(A, -)$ pointwise. It is the same as the map $\alpha \mapsto \hat{\alpha}$ from 2.9.

The structure group contains a one-parameter central subgroup $T \subset \mathbf{Str}(A, -)$:

$$T(R) = \{c\,\mathrm{id} \colon c \in R^\times\}$$

We have $\hat{c} = c^{-1}$ for all $c \in R^\times$. The norm-multiplier homomorphism

$$\mu : \mathbf{Str}(A, -) \to \mathbf{G}_m$$

restricts to $\mu(c\,\mathrm{id}) = c^{\deg A}$ on the subgroup T. It is also clear from definitions that $\mu(\hat{\alpha}) = \mu(\alpha)^{-1}$ for all $\alpha \in \mathbf{Str}(A, -)(R)$.

6. The AF construction

The Allison–Faulkner (AF) construction described in [12, §4] is an extremely general construction which takes as input a structurable algebra $(A, -)$ and three square classes $\gamma_1, \gamma_2, \gamma_3 \in k^\times/k^{\times 2}$. The output is a Lie algebra graded by the Klein 4-group

$$V_4 = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.$$

The construction encompasses both the TKK construction (from a structurable algebra) and the Tits construction (from a reduced Jordan algebra and an alternative algebra).

6.1. *Lie-related triples.* Let $(A, -)$ be a unital algebra with involution over k. A *Lie related triple* (in the sense of [12, §3]) is a triple $T = (T_1, T_2, T_3)$ where $T_i \in \mathrm{End}(A)$ and

$$\overline{T_i(\,\overline{xy}\,)} = T_j(x)y + xT_k(y) \tag{6.1.1}$$

for all $x, y \in A$ and all (i, j, k) that are cyclic permutations of $(1, 2, 3)$. Define $\mathcal{T}$ to be the subspace of $\mathrm{End}(A)^3$ spanned by the set of Lie related triples. It is easily verified that $\mathcal{T}$ is a Lie subalgebra of $\mathfrak{gl}(A) \times \mathfrak{gl}(A) \times \mathfrak{gl}(A)$.

For $a, b \in A$ and $1 \leq i \leq 3$, define

$$T^i_{a,b} = (T_1, T_2, T_3)$$

where (taking indices mod 3):

$$\begin{aligned}T_i &= L_{\bar{b}}L_a - L_{\bar{a}}L_b, \\ T_{i+1} &= R_{\bar{b}}R_a - R_{\bar{a}}R_b, \\ T_{i+2} &= R_{\bar{a}b-\bar{b}a} + L_bL_{\bar{a}} - L_aL_{\bar{b}}.\end{aligned}$$

Let $\mathcal{T}_I$ be the subspace of $\mathrm{End}(A)^3$ spanned by $\{T^i_{a,b} \colon a, b \in A, 1 \leq i \leq 3\}$. If $(A, -)$ is structurable, $\mathcal{T}_I$ is a Lie subalgebra of $\mathcal{T}$ [12, Lemma 5.4]; in other words, $T^i_{a,b}$ satisfies (6.1.1). Sometimes $\mathcal{T}_I$ is called the algebra of *inner Lie related triples.*

6.2. Lemma. *Let* $(A, -)$ *be a structurable algebra and let*

$$U = \{(s_1, s_2, s_3) : s_i \in \mathrm{Skew}(A, -), s_1 + s_2 + s_3 = 0\}.$$

There is an isomorphism of vector spaces

$$\begin{aligned} \mathrm{Der}(A, -) \oplus U &\xrightarrow{\sim} \mathcal{T} \\ (D, s_1, s_2, s_3) &\longmapsto (D, D, D) + (L_{s_2} - R_{s_3}, L_{s_3} - R_{s_1}, L_{s_1} - R_{s_2}) \end{aligned}$$

and the image of U *is contained in* $\mathcal{T}_I$.

Proof. It is clear that (6.1.1) is satisfied by (D, D, D) when $D \in \mathrm{Der}(A, -)$. Skew-alternativity and [12, Lemma 3.4] imply $(L_{s_2} - R_{s_3}, L_{s_3} - R_{s_1}, L_{s_1} - R_{s_2})$ is a Lie-related triple. So the map is well-defined. If $D + L_{s_1} - R_{s_2} = 0$ then $D(1) + L_{s_1}(1) - R_{s_2}(1) = s_1 - s_2 = 0$. If also $D + L_{s_2} - R_{s_3} = 0$ then $3s_2 = 0$, hence $s_1 = s_2 = s_3 = 0$, and $D = 0$ too. So the map is injective. Surjectivity of this map is proved in [12, Corollary 3.5]. For the final claim, observe that for all $(s_1, s_2, s_3) \in U$,

$$(L_{s_2} - R_{s_3}, L_{s_3} - R_{s_1}, L_{s_1} - R_{s_2}) = \frac{1}{2}(T^1_{s_2,1} - T^3_{s_3,1}) \in \mathcal{T}_I. \qquad \square$$

By all accounts, it seems that the map from Lemma 6.2 restricts to an isomorphism $D_{A,A} \oplus S \oplus S \xrightarrow{\sim} \mathcal{T}_I$, although it is hard to give a reference and I do not know how to express $(D_{x,y}, D_{x,y}, D_{x,y})$ as a linear combination of $T^i_{a,b}$'s. In any case, we do not need this fact.

One might also hope at this point to find an elegant expression for the Lie bracket that $\mathrm{Der}(A, -) \oplus S \oplus S$ inherits from $\mathcal{T}$. This bracket can be derived from [8, (1.7)]), but it is way more complicated than one would expect, involving a great many terms as well as scalars $\frac{1}{2}$ and $\frac{1}{3}$. We do, however, have the following situation:

6.3. Lemma. *Let* (A, σ) *be a unital associative algebra with involution, and let* $S = \mathrm{Skew}(A, \sigma)$. *There is an injective homomorphism of Lie algebras:*

$$\begin{aligned} S^- \times S^- \times S^- &\longrightarrow \mathcal{T} \\ (x, y, z) &\longmapsto (L_x - R_y, L_y - R_z, L_z - R_x) \end{aligned}$$

It is an isomorphism if $\mathrm{Der}(A, \sigma) = \mathrm{ad}_S$.

Proof. We first show that $(T_1, T_2, T_3) = (L_x - R_y, L_y - R_z, L_z - R_x) \in \mathcal{T}_I$. Let $u = \frac{1}{3}(x + y + z)$. Then

$$\mathrm{ad}_u = L_u - R_u \in \mathrm{Der}(A, -)$$

and $(x - u) + (y - u) + (z - u) = 0$, so by Lemma 6.2,

$$(T_1, T_2, T_3) = (\mathrm{ad}_u, \mathrm{ad}_u, \mathrm{ad}_u) + (L_{x-u} - R_{y-u}, L_{y-u} - R_{z-u}, L_{z-u} - R_{x-u}) \in \mathcal{T}.$$

Since A is unital, it is clear that the map is injective. Lemma 6.2 implies it is bijective if $\mathrm{Der}(A, \sigma) = \mathrm{ad}_S$. It is also easy to show that the map is a homomorphism; this rests on the associative identity: $[L_x, R_y] = 0$ for all $x, y \in A$. $\square$

6.4. *Definition of $K(A,-,\gamma)$.* Let $(A,-)$ be a central simple structurable algebra and let

$$\gamma = (\gamma_1, \gamma_2, \gamma_3) \in k^\times \times k^\times \times k^\times.$$

For $(i,j) = (1,2)$, $(2,3)$, $(3,1)$ define a vector space $A[ij] = \{a[ij] : a \in A\} \simeq A$, and define

$$K(A,-,\gamma) = \mathcal{T}_I \oplus A[12] \oplus A[23] \oplus A[31].$$

Equip $K(A,-,\gamma)$ with an algebra structure defined by the multiplication:

$$\begin{aligned} [a[ij], b[jk]] &= -[b[jk], a[ij]] = -\gamma_i\gamma_k^{-1}\overline{ab}[ki] \\ [T, a[ij]] &= -[a[ij], T] = T_k(a)[ij] \\ [a[ij], b[ij]] &= \gamma_i\gamma_j^{-1}T^i_{a,b} \end{aligned} \tag{6.4.1}$$

for all $a, b \in A$, $T = (T_1, T_2, T_3) \in \mathcal{T}_I$, and (i,j,k) a cyclic permutation of $(1,2,3)$. Then $K(A,-,\gamma)$ is clearly a V_4-graded algebra, and it is in fact a central simple Lie algebra [12, Theorems 4.1, 4.3, 4.4, & 5.5].

A useful notation is

$$\delta_{ij} = \gamma_i\gamma_j^{-1}.$$

(This should not be confused with the Kronecker delta.) Obviously, $\delta_{ij} = \delta_{ji}^{-1}$ and $\delta_{ij}\delta_{jk} = \delta_{ik}$ for all $i, j \in \{1,2,3\}$.

6.5. *Relation to the TKK construction.* If $\langle\gamma_1, \gamma_2, \gamma_3\rangle$ is isotropic, [8, Corollary 4.7] shows that there is a (noncanonical) isomorphism

$$K(A,-,\gamma) \xrightarrow{\sim} K(A,-).$$

One concrete isomorphism

$$K(A,-,(\gamma_1, -\gamma_1\rho^{-2}, \gamma_3)) \xrightarrow{\sim} K(A,-)$$

appears in [8, Theorem 2.2] and another concrete isomorphism

$$K(A,-,(1,-1,2\alpha)) \xrightarrow{\sim} K(A,-)$$

appears in [50, Proposition 4.4].

6.6. *Dependence on the scalar parameters.* In [8, Proposition 4.1] it is proved that $K(A,-,\gamma) \simeq K(A,-,\gamma')$ if $\langle\gamma_1,\gamma_2,\gamma_3\rangle$ and $\langle\gamma_1',\gamma_2',\gamma_3'\rangle$ are similar quadratic forms. (There is not necessarily a graded isomorphism, however.) A *graded* isomorphism $K(A,-,\gamma) \to K(A,-,(1,1,1))$ can be found if $\gamma_1\gamma_2^{-1}$ and $\gamma_2\gamma_3^{-1}$ have square roots, as the following lemma shows:

6.7. Lemma. *If $\gamma_1\gamma_2^{-1} = \alpha^2$ and $\gamma_2\gamma_3^{-1} = \beta^2$ for some $\alpha, \beta \in k^\times$, then the map*

$$\begin{aligned} u : K(A,-,(\gamma_1,\gamma_2,\gamma_3)) &\longrightarrow K(A,-,(1,1,1)) \\ T + a[12] + b[23] + c[31] &\longmapsto T + \alpha a[12] + \beta b[23] + \alpha^{-1}\beta^{-1}c[31] \end{aligned}$$

is a graded isomorphism of Lie algebras.

Proof. Visibly, u is linear, bijective, graded, and respects the relations (6.4.1). $\square$

6.8. *Preferred generators of S_4.* The next few subsections use the approach and the notation of [50], so we adopt their generators for S_4:

$$\tau = (12) \qquad \tau_1 = (12)(34) \qquad \tau_2 = (23)(14) \qquad \varphi = (123)$$

These generators satisfy the relations

$$\begin{aligned} 1 = \tau_i^2 = \tau^2 = [\tau_1, \tau_2] = [\tau, \tau_1] = \varphi^3 \qquad && \tau_2\tau = \tau\tau_2\tau_1 \qquad && \varphi\tau_2 = \tau_1\tau_2\varphi \\ && \tau\varphi = \varphi^2\tau \qquad && \varphi\tau_1 = \tau_2\varphi. \end{aligned}$$

Some important subgroups of S_4 are

$$\begin{aligned} S_3 &= \langle \tau, \varphi \rangle && \\ V_4 &= \langle \tau_1, \tau_2 \rangle && (\textit{Klein 4-group}) \\ A_4 &= \langle \tau_1, \tau_2, \varphi \rangle && (\textit{Alternating group}) \end{aligned}$$

of which A_4 and V_4 are normal. It helps to remember that

$$S_4 = A_4 \rtimes \langle \tau \rangle \qquad\qquad A_4 = V_4 \rtimes \langle \varphi \rangle.$$

6.9. *Action of S_3 on Lie related triples.* Let $(A, -)$ be a structurable algebra. The symmetric group S_3 acts linearly on the algebra $\mathcal{T}$ of Lie related triples by

$$\begin{aligned} \varphi \cdot (T_1, T_2, T_3) &= (T_3, T_1, T_2) \\ \tau \cdot (T_1, T_2, T_3) &= (T_2^*, T_1^*, T_3^*) \end{aligned}$$

where $T_i^*(a) = \overline{T_i(\bar{a})}$ for all $T_i \in \mathrm{End}(A)$. Note that

$$\varphi \cdot T^i_{a,b} = T^{\varphi(i)}_{a,b} = T^{i+1 \bmod 3}_{a,b} \tag{6.9.1}$$

for $i = 1, 2, 3$ and $a, b \in A$, and one can also calculate that

$$\tau \cdot T^1_{a,b} = T^1_{\bar{a},\bar{b}} \tag{6.9.2}$$

for all $a, b \in A$. (Hint: show that the triples $\tau \cdot T^1_{a,b}$ and $T^1_{\bar{a},\bar{b}}$ agree in first two entries, and use (6.1.1) to get the third entry.) Hence $\mathcal{T}_I$ is stabilised by S_3. Combining (6.9.1), (6.9.2), and the relation $\tau\varphi = \varphi^2\tau$ yields

$$\tau \cdot T^2_{a,b} = T^3_{\bar{a},\bar{b}} \qquad\qquad \tau \cdot T^3_{a,b} = T^2_{\bar{a},\bar{b}}.$$

6.10. *Action of S_4 on $K(A, -, (1, 1, 1))$.* The S_3-action on $\mathcal{T}_I$ extends to an S_4-action on $K(A, -, (1, 1, 1))$:

$$\begin{aligned} \tau_1(T + a[12] + b[23] + c[31]) &= T + a[12] - b[23] - c[31] \\ \tau_2(T + a[12] + b[23] + c[31]) &= T - a[12] + b[23] - c[31] \\ \varphi(T + a[12] + b[23] + c[31]) &= \varphi \cdot T + c[12] + a[23] + b[31] \\ \tau(T + a[12] + b[23] + c[31]) &= \tau \cdot T - \bar{a}[12] - \bar{c}[23] - \bar{b}[31]. \end{aligned}$$

The job of the subgroup V_4 is to describe the grading: each of the four graded components is an intersection of eigenspaces for τ_1 and τ_2. The following result is more or less implicit in [50].

6.11. Proposition. *The S_4-action on $L = K(A, -, (1, 1, 1))$ is an action by Lie algebra automorphisms. The centraliser of S_4 in* $\mathrm{Aut}(L)$ *is isomorphic to* $\mathrm{Aut}(A, -)$.

Proof. The fact that S_4 acts by automorphisms is a computation (for comparison, see [50, Proposition 2.8]). Suppose f is an S_4-equivariant automorphism of L. Then f is a graded automorphism because it commutes with $V = \langle \tau_1, \tau_2 \rangle \leq S_4$. Since f also commutes with φ, there exists a linear bijection $g \in \mathrm{GL}(A)$ such that

$$f(a[12] + b[23] + c[31]) = g(a)[12] + g(b)[23] + g(c)[31].$$

for all $a, b, c \in A$. This dictates that $f(T^i_{a,b}) = T^i_{g(a),g(b)}$ for $a, b \in A$ and $1 \leq i \leq 3$. Since f also commutes with τ, we have $g(\bar{a}) = \overline{g(a)}$. Moreover, $[a[12], b[23]] = -\overline{ab}[31]$ implies

$$-g(\overline{ab})[31] = [g(a)[12], g(b)[23]] = -\overline{g(a)g(b)}[31].$$

This gives $\overline{g(ab)} = g(\overline{ab}) = \overline{g(a)g(b)}$ and hence $g(ab) = g(a)g(b)$ for all $a, b \in A$. Conversely, if g is any automorphism of $(A, -)$ then it extends to an S_4-equivariant automorphism of L given by

$$f(T^i_{a,b} + a[12] + b[23] + c[31]) = T^i_{g(a),g(b)} + g(a)[12] + g(b)[23] + g(c)[31]. \qquad \square$$

6.12. *The Killing form of an AF construction.* Recall that the *Killing form* of a Lie algebra L is the bilinear form

$$\kappa : L \times L \to k, \qquad \kappa(x, y) = \mathrm{tr}(\mathrm{ad}_x \, \mathrm{ad}_y).$$

The *quadratic Killing form* of L is the quadratic form $\kappa(x) = \kappa(x, x) = \mathrm{tr}({\mathrm{ad}_x}^2)$.

Under the assumption that $\mathrm{char}(k) = 0$, the Killing form of $K(A, -, \gamma)$ was determined in [8, Theorem 5.4]. Under the assumption that $\mathrm{char}(k) \neq 2, 3$ and $(A, -)$ is a bioctonion algebra, the Killing form of $K(A, -, \gamma)$ was calculated by the author and Victor Petrov in [133, Proposition 2.2]. In the following theorem, I provide a new proof which partially generalises both these results, with the caveat that the Killing form on the zero component still needs to be calculated separately.

6.13. Theorem. *Let $(A, -)$ be a central simple structurable algebra, $\gamma = (\gamma_1, \gamma_2, \gamma_3) \in (k^\times)^3$, $L = K(A, -, \gamma)$, and let κ be the quadratic Killing form of L. The homogeneous components of L are pairwise orthogonal with respect to κ. If $\kappa \neq 0$ and g is a nondegenerate quadratic form on A such that $g(1) \neq 0$ and the linearisation of g is an invariant bilinear form, then*

$$\kappa \simeq \kappa_0 \perp \langle d \rangle \langle \delta_{12}, \delta_{23}, \delta_{31} \rangle g$$

where $\delta_{ij} = \gamma_i \gamma_j^{-1}$, κ_0 is the restriction of κ to $\mathcal{T}_I$, and

$$d = g(1)^{-1}(-2 \dim A - 8 \dim \mathrm{Skew}(A, -)).$$

Note that if $\mathrm{char}(k)$ does not divide $2 \dim A$, then the quadratic trace form T_A is nondegenerate and invariant, so one may take $g = T_A$. The coefficient in this case is $d = (-\dim A - 4 \dim \mathrm{Skew}(A, -))(\dim A)^{-1}$.

Proof. If $v_1 \neq v_2$ in $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and $\ell_i \in L_{v_i}$, then $\mathrm{ad}_{\ell_1} \mathrm{ad}_{\ell_2}(L_v) \subset L_{v+v_1+v_2}$ for all $v \in V_4$, so $\mathrm{tr}(\mathrm{ad}_{\ell_1} \mathrm{ad}_{\ell_2}) = 0$. This proves the first claim. Let κ_{ij} be the restriction of κ to the $A[ij]$ component, i.e., the quadratic form

$$\kappa_{ij} : A \to k, \qquad \kappa_{ij}(a) = \kappa(a[ij]) = \mathrm{tr}(\mathrm{ad}_{a[ij]} \mathrm{ad}_{a[ij]}).$$

We now have

$$\kappa = \kappa_0 \perp \kappa_{12} \perp \kappa_{23} \perp \kappa_{31}.$$

Step 1: We show that $\kappa_{jk} = \delta_{ij}\delta_{jk}^{-1}\kappa_{ij}$. We may extend scalars to a field extension F/k and assume $\delta_{12} = \gamma_1\gamma_2^{-1} = \alpha^2$ and $\delta_{23} = \gamma_2\gamma_3^{-1} = \beta^2$ for some $\alpha, \beta \in F$. Let $u : L_F \xrightarrow{\sim} K(A_F, -, (1,1,1))$ be the isomorphism from Lemma 6.7, and φ the automorphism of $K(A_F, -, (1,1,1))$ that cyclically permutes the homogeneous components, as in 6.10. The composition $u^{-1} \circ \varphi \circ u : L_F \to L_F$ is

$$T + a[12] + b[23] + c[31] \longmapsto \varphi \cdot T + \alpha^{-2}\beta^{-1}c[12] + \alpha\beta^{-1}a[23] + \alpha\beta^2 b[31].$$

By elementary calculations,

$$(\alpha\beta^{-1})^2 = \delta_{12}\delta_{23}^{-1} \qquad (\alpha\beta^2)^2 = \delta_{23}\delta_{31}^{-1} \qquad (\alpha^{-2}\beta^{-1})^2 = \delta_{31}\delta_{12}^{-1}.$$

Since automorphisms of L_F are isometries of its Killing form, this implies

$$\kappa(x[ij]) = \delta_{ij}\delta_{jk}^{-1}\kappa(x[jk])$$

for all $x \in A$ and all cyclic permutations (i,j,k) of $(1,2,3)$.

Step 2: We show that (the linearisation of) κ_{12} is an invariant form in the sense of Definition 2.15. Like in the previous step, we can extend k to a bigger field F and use the maps $u : L_F \xrightarrow{\sim} K(A_F, -, (1,1,1))$ and $\tau \in \mathrm{Aut}(K(A_F, -, (1,1,1)))$. The composition $u^{-1} \circ \tau \circ u : L_F \to L_F$ is:

$$T + a[12] + b[23] + c[31] \longmapsto \tau \cdot T - \bar{a}[12] - \alpha^{-1}\bar{c}[23] - \alpha\beta^2\bar{b}[31].$$

This implies

$$\kappa(x[12], y[12]) = \kappa(-\bar{x}[12], -\bar{y}[12]) = \kappa(\bar{x}[12], \bar{y}[12]). \tag{6.13.1}$$

Applying (6.13.1) and then (6.4.1) yields

$$\kappa(zx[12], y[12]) = \kappa(\overline{zx}[12], \overline{y}[12]) = \kappa(\delta_{12}[x[31], z[23]],\ \overline{y}[12])$$

The Killing form is associative (also called Lie invariant, meaning $\kappa([m,\ell], n) = \kappa(m, [\ell, n])$ for $m, \ell, n \in L$), so the right hand side equals:

$$\begin{aligned}\delta_{12}\kappa(x[31], [z[23], \bar{y}[12]]) &= \delta_{12}\kappa(x[31], \delta_{31}^{-1}\overline{\bar{y}z}[31]) = \delta_{12}\delta_{31}^{-1}\kappa(x[31], \bar{z}y[31]) \\ &= \delta_{12}\delta_{31}^{-1}\delta_{31}\delta_{12}^{-1}\kappa(x[12], \bar{z}y[12]) = \kappa(x[12], \bar{z}y[12]).\end{aligned}$$

Hence $\kappa(zx[12], y[12]) = \kappa(x[12], \bar{z}y[12])$, so κ_{12} is invariant.

Step 3: We calculate $\kappa_{12}(1)$. By definition, $\kappa_{12}(1) = \kappa(1[12]) = \mathrm{tr}(\mathrm{ad}_{1[12]} \mathrm{ad}_{1[12]})$. The homogeneous components of $K(A, -, \gamma)$ are invariant under ${\mathrm{ad}_{1[12]}}^2$, so we work out the trace separately for each of these components.

For all $y \in A$, we have

$$[1[12],[1[12],y[23]]] = -\delta_{13}[1[12],\bar{y}[31]] = -\delta_{13}\delta_{32}y[23] = -\delta_{12}y[23]$$

so ${\mathrm{ad}_{1[12]}}^2|_{A[23]} = -\delta_{12}\,\mathrm{id}$, and $\mathrm{tr}({\mathrm{ad}_{1[12]}}^2|_{A[23]}) = -(\dim A)\delta_{12}$.

Similarly, for all $y \in A$,

$$[1[12],[1[12],y[31]]] = \delta_{32}[1[12],\bar{y}[23]] = -\delta_{32}\delta_{13}y[31] = -\delta_{12}y[31]$$

so ${\mathrm{ad}_{1[12]}}^2|_{A[31]} = -\delta_{12}\,\mathrm{id}$, and $\mathrm{tr}({\mathrm{ad}_{1[12]}}^2|_{A[31]}) = -(\dim A)\delta_{12}$.

In contrast, for all $y \in A$,

$$\begin{aligned}[1[12],[1[12],y[12]]] &= [1[12],\delta_{12}T^1_{1,y}] = -\delta_{12}(T^1_{1,y})_3(1)\\ &= -\delta_{12}(R_{y-\bar{y}} + L_y - L_{\bar{y}})(1) = -2\delta_{12}(y-\bar{y}).\end{aligned}$$

Therefore the eigenvalues of ${\mathrm{ad}_{1[12]}}^2|_{A[12]}$ are 0 and $-4\delta_{12}$, with corresponding eigenspaces

$$\{h[12] : \bar{h} = h\}, \qquad\qquad \{s[12] : \bar{s} = -s\}.$$

This proves that $\mathrm{tr}({\mathrm{ad}_{1[12]}}^2|_{A[12]}) = -4(\dim \mathrm{Skew}(A,-))\delta_{12}$.

Finally, if $T = (T_1,T_2,T_3) \in \mathcal{T}_I$, then

$$[1[12],[1[12],T]] = [1[12],-T_3(1)[12]] = -\delta_{12}T^1_{1,T_3(1)}.$$

We can use Lemma 6.2 to write

$$T = (D,D,D) + (L_{s_2} - R_{s_3}, L_{s_3} - R_{s_1}, L_{s_1} - R_{s_2})$$

for some unique $D \in \mathrm{Der}(A,-)$ and $s_i \in \mathrm{Skew}(A,-)$ such that $s_1+s_2+s_3 = 0$. Then $T_3(1) = D(1) + L_{s_1}(1) - R_{s_2}(1) = s_1 - s_2$, so

$$\begin{aligned}{\mathrm{ad}_{1[12]}}^2(T) &= -\delta_{12}T^1_{1,s_1-s_2}\\ &= -2\delta_{12}(-L_{s_1-s_2}, -R_{s_1-s_2}, R_{s_1-s_2} + L_{s_1-s_2}).\end{aligned}$$

The eigenvalues of ${\mathrm{ad}_{1[12]}}^2|_{\mathcal{T}_I}$ are 0 and $-4\delta_{12}$, and the latter one has eigenspace

$$\{(L_s, R_s, -L_s - R_s) : s \in \mathrm{Skew}(A,-)\}.$$

This implies that $\mathrm{tr}({\mathrm{ad}_{1[12]}}^2|_{\mathcal{T}_I}) = -4(\dim \mathrm{Skew}(A,-))\delta_{12}$.

Putting this all together,

$$\kappa_{12}(1) = \mathrm{tr}(\mathrm{ad}_{1[12]}\,\mathrm{ad}_{1[12]}) = (-2\dim A - 8\dim \mathrm{Skew}(A,-))\delta_{12}.$$

Step 4: If g is a quadratic form such that $g(1) \neq 0$ and g is invariant, then by Step 2 and Lemma 2.16 (iii), $\kappa_{12} = \langle c\rangle g$ for some $c \in k^\times$. By Step 3,

$$c = g(1)^{-1}\kappa_{12}(1) = g(1)^{-1}(-2\dim A - 8\dim \mathrm{Skew}(A,-))\delta_{12}.$$

The theorem follows from this observation and the result of Step 1. $\square$

Chapter III

Automorphism groups and structure groups of structurable algebras

In this chapter we determine the automorphism groups of various structurable algebras, and the split and quasi-split forms of their structure groups.

7. Smoothness of the groups

We proceed in this section to show that the automorphism group and structure group of a central simple structurable algebra are smooth when the characteristic of the base field is not 2 or 3. These facts rest ultimately on the following Theorem 7.1, which in turn builds on very deep foundations, including the classification of simple Lie algebras in characteristic $p \geq 5$.

An algebraic group G is *absolutely simple* if $G \neq 1$ and $G \times_k k^a$ has no nontrivial connected normal subgroups. An absolutely simple group is *adjoint* if $Z(G) = 1$.

7.1. Theorem (Stavrova [31, 165])**.** *If $(A, -)$ is a central simple structurable algebra and $\gamma_1, \gamma_2, \gamma_3 \in k^\times$, then $\mathbf{Aut}(K(A, -, \gamma))^\circ$ is an adjoint absolutely simple algebraic group.*

Proof. Let $L = K(A, -, \gamma)$. If $\delta = -\gamma_1\gamma_2^{-1}$ is not a square, let $k' = k(\sqrt{\delta})$; otherwise let $k' = k$. Then $L_{k'} \simeq K(A_{k'}, -)$ [8, Theorem 2.2]. We can now apply [31, Lemma 3.1.7 & Theorem 4.1.1] for $\mathrm{char}(k) \neq 2, 3, 5$, or [165, Theorem 4.8] for $\mathrm{char}(k) \neq 2, 3$. These show that the algebraic k'-group $\mathbf{Aut}(L_{k'})^\circ = \mathbf{Aut}(L)^\circ \times_k k'$ is adjoint absolutely simple. This entails that $\mathbf{Aut}(L)^\circ$ is an adjoint absolutely simple k-group. $\square$

The proof of the following Theorem 7.2 is deceptively short: it relies almost entirely on Theorem 7.1. In the examples that we look at in this chapter, we compute the Lie algebras $\mathrm{Lie}(\mathbf{Aut}(A, -)) = \mathrm{Der}(A, -)$ and $D_{A,A}$ as a matter of course, and this can always be used to justify the smoothness of $\mathbf{Aut}(A, -)$ in a more direct and elementary way.

7.2. Theorem. *If $(A, -)$ is a central simple structurable algebra, $\mathbf{Str}(A, -)$ and $\mathbf{Aut}(A, -)$ are smooth.*

Proof. By Lemma 4.2 and the isomorphism $\mathbf{Str}(A,-) \simeq \mathbf{Aut}_{\mathrm{gr}}(K(A,-))$ (see 5.5), $\mathbf{Str}(A,-)^\circ$ is isomorphic to the centraliser of a torus in $\mathbf{Aut}(K(A,-))^\circ$. By 7.1, $\mathbf{Aut}(K(A,-))^\circ$ is smooth, so $\mathbf{Str}(A,-)^\circ$ is too [122, Corollary 13.10].

We shall again use a centraliser argument to prove that $\mathbf{Aut}(A,-)$ is smooth. Finite-dimensional representations of S_4 are all semisimple because we are assuming that $\mathrm{char}(k) \neq 2,3$. The representation theory of a constant algebraic group F is the same as the representation theory of $F(k)$ [178, Exercise 5]. In particular, the constant algebraic group S_4 is linearly reductive in the sense of [122, Definition 12.52].

Let $L = K(A,-,(1,1,1))$. By 7.1, $\mathbf{Aut}(L)$ is smooth. Now let H be the scheme-theoretic centraliser of S_4 in $\mathbf{Aut}(L)$. By [122, Corollary 13.9], H is smooth. Let H' be the image of the canonical (injective) homomorphism $\mathbf{Aut}(A,-) \to H$. Proposition 6.11 implies $H(k^a) = H'(k^a)$ and this implies $H = H'$ [122, Remark 1.95]. □

7.3. *Notation for some algebraic groups.* The automorphism groups and other groups encountered in this chapter are assembled from recognisable finite, semisimple, or reductive parts. Most of the notation here is exactly the same as in [101], and that is where to look for more precise definitions if necessary.

- The multiplicative group is the k-group scheme $\mathbf{G}_m$ where $\mathbf{G}_m(R) = R^\times$ for all commutative unital k-algebras R. We may write $\mathbf{G}_{m,k}$ to be precise about the field of definition. The k-group scheme of n-th roots of unity is $\boldsymbol{\mu}_n$, or $\boldsymbol{\mu}_{n,k}$ if we want to be precise about the field of definition.

- For a finite-dimensional vector space V: $\mathbf{GL}(V)$, $\mathbf{SL}(V)$, and $\mathbf{PGL}(V)$ are the linear, special linear, and projective linear groups, abstractly written as $\mathbf{GL}_n$, $\mathbf{SL}_n$, and $\mathbf{PGL}_n$ where $n = \dim V$.

- For an associative central simple algebra A: the group of units is $\mathbf{GL}_1(A)$, the automorphism group is $\mathbf{PGL}_1(A)$, and $\mathbf{SL}_1(A) = \ker(\mathrm{Nrd}_A : \mathbf{GL}_1(A) \to \mathbf{G}_m)$.

- For an associative central simple algebra with involution (A,σ): $\mathbf{Sim}(A,\sigma) \subset \mathbf{GL}_1(A)$ is the group of similitudes and $\mathbf{Iso}(A,\sigma) \subset \mathbf{Sim}(A,\sigma)$ is the group of isometries. Their k-points are

$$\begin{aligned}\mathrm{Sim}(A,\sigma) &= \{x \in A \colon \sigma(x)x = x\sigma(x) \in k^\times 1\}\\ \mathrm{Iso}(A,\sigma) &= \{x \in A \colon \sigma(x)x = x\sigma(x) = 1\}.\end{aligned}$$

  To keep track of different types of involutions these groups are sometimes written as:

$$\mathbf{Sim}(A,\sigma) = \begin{cases}\mathbf{GO}(A,\sigma) & \text{if } \sigma \text{ is orthogonal}\\ \mathbf{GU}(A,\sigma) & \text{if } \sigma \text{ is unitary}\\ \mathbf{GSp}(A,\sigma) & \text{if } \sigma \text{ is symplectic.}\end{cases}$$

$$\mathbf{Iso}(A,\sigma) = \begin{cases}\mathbf{O}(A,\sigma) & \text{if } \sigma \text{ is orthogonal}\\ \mathbf{U}(A,\sigma) & \text{if } \sigma \text{ is unitary}\\ \mathbf{Sp}(A,\sigma) & \text{if } \sigma \text{ is symplectic.}\end{cases}$$

  We write $\mathbf{SU}(A,\sigma) = \mathbf{U}(A,\sigma)^\circ$ and $\mathbf{O}^+(A,\sigma) = \mathbf{O}(A,\sigma)^\circ$. (The groups $\mathbf{Sp}(A,\sigma)$ are already connected.)

– For a nondegenerate quadratic space (V,q): the orthogonal group is $\mathbf{O}(V,q)$, the group of similitudes is $\mathbf{GO}(V,q)$, and the group of projective similitudes is $\mathbf{PGO}(V,q)$. The special orthogonal group is $\mathbf{O}^+(V,q) = \mathbf{O}(V,q)^\circ$, and the groups of proper similitudes and proper projective similitudes are $\mathbf{GO}^+(V,q) = \mathbf{GO}(V,q)^\circ$ and $\mathbf{PGO}^+(V,q) = \mathbf{PGO}(V,q)^\circ$ respectively. The spin group is $\mathbf{Spin}(V,q)$, and (only if $\dim V = 4m \geq 12$) the half-spin group $\mathbf{HSpin}(V,q)$ is the image of the half-spin representation of $\mathbf{Spin}(V,q)$.

  The split forms are denoted by $\mathbf{O}_n^+ = \mathbf{O}^+(V,q)$, $\mathbf{PGO}_n^+ = \mathbf{PGO}_n^+(V,q)$, $\mathbf{Spin}_n = \mathbf{Spin}(V,q)$, etc., for the n-dimensional quadratic space (V,q) of maximal Witt index and signed discriminant equal to 1. By this convention, $\mathbf{O}_n$ is *not* the same as $\mathbf{O}(n) = \mathbf{O}(\langle 1,\dots,1\rangle)$.

– $\mathbf{SU}_{n,K} = \mathbf{SU}(V,q)$ is the special unitary group of a nonsingular K/k-hermitian form of rank n and maximal Witt index.

– Exceptional groups: G_2, F_4, E_6^{sc}, E_6^{ad}, E_7^{sc}, E_7^{ad}, and E_8 refer to the split forms, superscripts meaning adjoint or simply connected.

– If K/k is a field extension, $G \times_k K$ (or just G_K) is the extension of a k-group G and $R_{K/k}(G)$ is the Weil restriction of an K-group G. It is also quite reasonable to define $R_{E/k}(G)$ where E is an étale extension [101, Remark 20.9].

7.4. *The stabiliser group of a polynomial.* In this subsection, let K be an arbitrary field of any characteristic and let V be an n-dimensional K-vector space. Let $K[V] \simeq K[x_1,\dots,x_n]$ be the ring of polynomial functions on V. Let $P \in K[V]$ be a homogeneous polynomial of degree $m \geq 1$, and define the k-group functors

$$\begin{aligned}\mathbf{Iso}(P)(R) &= \{g \in \mathbf{GL}_n(R) \colon P \circ g = P\},\\ \mathbf{Sim}(P)(R) &= \{g \in \mathbf{GL}_n(R) \colon P \circ g = \lambda_g P \text{ for some } \lambda_g \in R^\times\}.\end{aligned}$$

These are scheme-theoretic stabilisers of the closed point $P \in K[V]$ with respect to actions of $\mathbf{GL}(V)$ and $\mathbf{GL}(V) \times \mathbf{G}_m$, respectively. So they are representable functors corresponding to subgroup schemes $\mathbf{Iso}(P) \subset \mathbf{Sim}(P) \subset \mathbf{GL}(V)$ [122, Corollary 1.81]. We have an exact sequence

$$1 \longrightarrow \mathbf{Iso}(P) \longrightarrow \mathbf{Sim}(P) \xrightarrow{\ \mu\ } \mathbf{G}_m \longrightarrow 1$$

where $\mu : g \mapsto \lambda_g$ is the multiplier homomorphism.

7.5. Lemma. *Let $P \in K[V]$ be a homogeneous polynomial of degree m such that* $\mathrm{char}(K) \nmid m$. *Then* $\mathbf{Iso}(P)$ *and* $\mathbf{Sim}(P)$ *are smooth.*

Proof. For $c \in R^\times$, where R is any commutative unital k-algebra, the scalar transformation $c\,\mathrm{id} \in \mathbf{Sim}(P)(R)$ has multiplier $\mu_R(c\,\mathrm{id}) = c^m$. Hence the differential $d\mu : \mathrm{Lie}(\mathbf{Sim}(P)) \to \mathrm{Lie}(\mathbf{G}_m) = k\varepsilon$ is surjective because its image contains $mk\varepsilon$ and because $\frac{1}{m} \in k$. In turn, this implies $\mathbf{Iso}(P)$ is smooth [101, Proposition 22.13]. Since $\mathbf{G}_m$ is smooth, so is $\mathbf{Sim}(P)$ [101, Corollary 22.12]. ☐

8. Structurable algebras of rank one hermitian forms

The most basic example of a nonsingular hermitian form on a central simple algebra with involution (E, σ) is a rank one form $h : E \times E \to E$ where $h(x, y) = xp\sigma(y)$ for some invertible $p \in \mathrm{Herm}(A, -)$. The situation is even easier if $p \in k$. This section is about the structurable algebras $S(E, \sigma, E, h)$ of such hermitian forms; these are the classical Cayley–Dickson algebras with nonstandard involutions that were mentioned in the first paragraph of 2.5. Included among them are octonion algebras with nonstandard involutions, as well as some (but not all) twisted forms of quartic Cayley algebras.

8.1. *Octonion algebras with nonstandard involution.* An alternative central simple k-algebra is either associative or an octonion algebra over its centre [99, Theorem 3]. Besides the standard involution $x \mapsto \bar{x}$, an octonion k-algebra C always carries a number of other involutions that are called nonstandard.

For any quaternion subalgebra $Q \subset C$ and an element $a \in C$ such that $n(a) \neq 0$ and $n(a, Q) = 0$, we have $C = Q \oplus aQ$ [164, Proposition 1.5.1]. One can define a nonstandard involution τ_Q on C by

$$\tau_Q(q_1 + aq_2) = \overline{q_1} + aq_2 \qquad \text{for all } q_1, q_2 \in Q. \tag{8.1.1}$$

The definition of τ_Q depends only on Q and not on a. Conversely, if τ is any k-linear involution on C that is not the standard involution, then τ composed with the standard involution is an order 2 automorphism of C. This automorphism fixes a unique quaternion subalgebra $Q \subset C$ and its -1-eigenspace is the orthogonal complement of Q with respect to n [87, p. 66]. From this, one can easily derive that $\tau = \tau_Q$. See also [5, p. 376] or [137, Proposition 2.5] for this characterisation of involutions on octonion algebras.

Octonion algebras with nonstandard involutions are characterised among central simple structurable algebras by $(\dim A, \dim \mathrm{Skew}(A, -)) = (8, 3)$. They are hermitian type structurable algebras constructed from a rank one hermitian form on a quaternion algebra with standard symplectic involution, for it is easy to see that

$$(C, \tau_Q) = S(Q, -, W, h)$$

where $W = Q$ and $h : W \times W \to Q$ is the hermitian form

$$h(w_1, w_2) = -n(a)w_1\overline{w_2}.$$

In other words, the absolute type of (C, τ_Q) corresponds to the second-last row of Table 1 with $(n, d) = (2, 1)$.

One can derive from [164, §2.1] that the (abstract) automorphism group of such an algebra is of the form $\mathrm{Aut}(A, -) \simeq \mathrm{SL}_1(Q) \rtimes \mathrm{PGL}_1(Q)$ where $\mathrm{PGL}_1(Q) = \mathrm{Aut}(Q)$ acts on $\mathrm{SL}_1(Q)$ in the obvious way.

We prove a generalisation of this for structurable algebras of rank one hermitian forms $h(x, y) = cx\sigma(y)$ where $c \in k^\times$. The method of proof probably works just as well for more complicated hermitian forms, only it is more technical to write down what the automorphism group actually looks like in that case.

8.2. Theorem. *Let (E,σ) be any central simple algebra with involution such that* $\mathrm{Skew}(E,\sigma)$ *generates E. Let $c \in k^\times$ and let (W,h) be the rank 1 hermitian space:*

$$W = E, \qquad h(x,y) = cx\sigma(y) \qquad \text{for all } x,y \in W.$$

If $(B,-) = S(E,\sigma,W,h) = E \oplus W$ is the structurable algebra described in 2.5, then

$$\mathbf{Aut}(B,-) \simeq \mathbf{Iso}(E,\sigma) \rtimes \mathbf{Aut}(E,\sigma).$$

The condition that $\mathrm{Skew}(E,\sigma)$ generates E is met by all central simple algebras with involution except for quaternion algebras with orthogonal involutions [89, p. 304]. We shall see in §10 that the quartic Cayley algebra $S(\mathbb{M}_2(k),k^4) = M(k^3)$ has an automorphism group with 12 connected components, so it is strictly larger than $\mathbf{Iso}(\mathbb{M}_2(k)) \rtimes \mathbf{Aut}(\mathbb{M}_2(k)) = \mathbf{O}_2 \rtimes \mathbf{PGO}_2$, which has just 4 connected components.

Proof of Theorem 8.2. Any involution-preserving automorphism $f : B \to B$ must map $\mathrm{Skew}(B,-) = \mathrm{Skew}(E,\sigma)$ to itself. But $\mathrm{Skew}(E,\sigma)$ generates E, so f maps E isomorphically onto itself. The facts that $WE = EW = W$ and $WW \subset E$ imply $\mathrm{tr}(L_w) = 0$ for all $w \in W$. The orthogonal complement of E with respect to T_B contains W, since $ew = w\bar{e}$ and so $T_B(e,w) = \mathrm{tr}(L_{ew+w\bar{e}}) = 2\,\mathrm{tr}(L_{ew}) = 0$ for all $e \in E$ and $w \in W$. By dimension count, $E^\perp = W$, and consequently $f(W) \subset W$.

From $(0,1)^2 = (c,0)$ it is easy to derive that $f((0,1)) = (0,u)$ for some $u = \sigma(u)^{-1} \in \mathrm{Iso}(E,\sigma)$. Since $(0,w) = (0,1)(w,0)$ for all $w \in W$, we have $f(0,w) = (0,u)f((w,0))$, so f is fully determined by its restriction to E and the value of $f((0,1))$. Specifically, there is an injective homomorphism

$$\begin{aligned} \mathrm{Aut}(B,-) &\to \mathrm{Iso}(E,\sigma) \rtimes \mathrm{Aut}(E,\sigma) \\ f &\mapsto (u^{-1}, f_0) \end{aligned} \tag{8.2.1}$$

where $f((e,0)) = (f_0(e),0)$ for all $e \in E$ and $f((0,1)) = (0,u)$. The homomorphism (8.2.1) is also surjective, since

$$(e,w) \mapsto (f_0(e), f_0(w)u^{-1}) \tag{8.2.2}$$

is an automorphism of $(B,-)$ for any $f_0 \in \mathrm{Aut}(E,\sigma)$ and any $u \in \mathrm{Iso}(E,\sigma)$.

To make this a statement about algebraic groups, probably the shortest way is to define a homomorphism $F : \mathbf{Iso}(E,\sigma) \rtimes \mathbf{Aut}(E,\sigma) \to \mathbf{Aut}(B,-)$ by mapping $(u,f_0) \in (\mathbf{Aut}(E,\sigma) \rtimes \mathbf{Iso}(E,\sigma))(R)$ to an automorphism $F_R(u,f_0) \in \mathbf{Aut}(B,-)(R)$, as in (8.2.2). This is quite clearly injective (i.e., it is injective on R-points for all k-algebras R). One can conclude using the surjectivity criterion [101, Proposition 22.3] that F is an isomorphism because $\mathbf{Aut}(B,-)$ is smooth and we already showed that F_{k^a} has an inverse, namely (8.2.1). $\square$

The description of $\mathbf{Aut}(B,-)$ as a semidirect product is quite clumsy. It is preferable to convert it to something that is visibly an almost-direct product of simple algebraic groups.

8.3. Lemma. *If (A,σ) is a central simple algebra with involution of the first kind,*

$$\mathbf{Iso}(A,\sigma) \rtimes \mathbf{Aut}(A,\sigma) \simeq \frac{\mathbf{Iso}(A,\sigma) \times \mathbf{Iso}(A,\sigma)}{\boldsymbol{\mu}_2}$$

where $\boldsymbol{\mu}_2$ denotes the subgroup generated by $(-1,-1)$.

Proof. Consider the homomorphism

$$\phi : \mathbf{Iso}(A,\sigma) \times \mathbf{Iso}(A,\sigma) \to \mathbf{Iso}(A,\sigma) \rtimes \mathbf{Aut}(A,\sigma)$$
$$\phi_R(u,w) = (wu^{-1}, \mathrm{Int}(u)).$$

for all $u, w \in \mathrm{Iso}(A_R, \sigma)$. The group $\mathbf{Iso}(A,\sigma)$ is smooth and $\mathrm{Int} : \mathbf{Iso}(A,\sigma) \to \mathbf{Aut}(A,\sigma)$ is surjective with kernel $\boldsymbol{\mu}_2$ (see [101, p. 347, 351]). So it is easy to see that ϕ is surjective and has kernel $\ker(\phi_R) = \{(c,c) : c \in \boldsymbol{\mu}_2(R)\} \simeq \boldsymbol{\mu}_2$. □

The following corollary is merely a restatement of facts.

8.4. Corollary. *If (A,σ) is a central simple algebra with involution of the first kind and $(B,-) = S(A,\sigma,A,h)$ for some rank 1 hermitian form $h(x,y) = cx\sigma(y)$, $c \in k^\times$, then*

$$\mathbf{Aut}(B,-) \simeq \begin{cases} \dfrac{\mathbf{O}(A,\sigma) \times \mathbf{O}(A,\sigma)}{\boldsymbol{\mu}_2} & \text{if } \sigma \text{ is orthogonal} \\[2ex] \dfrac{\mathbf{Sp}(A,\sigma) \times \mathbf{Sp}(A,\sigma)}{\boldsymbol{\mu}_2} & \text{if } \sigma \text{ is symplectic.} \end{cases}$$

In some cases, due to exceptional isomorphisms, there are even more ways of writing this group. For instance, $\mathbf{Sp}(Q,-) \simeq \mathbf{SL}_1(Q)$ for a quaternion algebra Q with symplectic involution.

8.5. Proposition. *With $(B,-)$ as in Theorem* 8.2*, we have*

$$D_{B,B} = \mathrm{Der}(B,-) \simeq \mathrm{Skew}(E,\sigma) \oplus \mathrm{Der}(E,\sigma)$$

where "$\simeq$" is isomorphism of vector spaces (not a direct product of Lie algebras).

Proof. Let us first establish the "$\simeq$" using an argument similar to the proof of 8.2. Any derivation $d \in \mathrm{Der}(B,-)$ must map $\mathrm{Skew}(E,\sigma)$ to itself and satisfy $d(st) = sd(t) + d(s)t$ for all $s,t \in \mathrm{Skew}(E,\sigma)$, hence $d(E) \subset E$. To show that $d(W) \subset W$, notice that $\mathrm{tr}(L_{d(x)}) = \mathrm{tr}([d, L_x]) = 0$ for all $x \in B$, so $\mathrm{tr}(L_{d(x)y}) = -\mathrm{tr}(L_{xd(y)})$ for all $x, y \in B$. If $e \in E$ and $w \in W$ then

$$T_B(e, d(w)) = \mathrm{tr}(L_{ed(w)-d(w)\bar{e}}) = \mathrm{tr}(-L_{d(e)w-wd(\bar{e})}) = 0$$

because $d(e)w - wd(\bar{e}) \in EW + WE = W$, hence $d(W) \subset E^\perp = W$. Now notice that d is fully determined by its restriction to E and the value $d((0,1)) \in W$, since

$$d(0,w) = d((0,1)(w,0)) = d((0,1))(w,0) + (0,1)d((w,0)).$$

The map $d_1 : E \to E$ defined by $d((e,0)) = (d_1(e), 0)$ is a derivation of (E,σ). And if $d((0,1)) = (0,s)$ then $s \in \mathrm{Skew}(A,-)$ because

$$0 = d(1) = d((0,1)^2) = (0,s)(0,1) + (0,1)(0,s) = (\sigma(s) + s, 0).$$

So, we have an injective map

$$\mathrm{Der}(B,-) \to \mathrm{Skew}(E,\sigma) \oplus \mathrm{Der}(E,\sigma) \qquad\qquad d \mapsto (-s, d_1).$$

(This is the differential of (8.2.1).) It is straightforward to write down its inverse and show it is an isomorphism.

Now we show that $D_{B,B} = \mathrm{Der}(B, -)$. We have $D_{E,E} = \mathrm{Der}(E, \sigma)$ by [3, Proposition 15], and clearly $d_{e_1,e_2}(e, 0) = (d_{e_1,e_2}(e), 0)$ for all $e, e_1, e_2 \in E$. By a routine calculation

$$D_{(0,1),(0,w)}(0,1) = \frac{4c}{3}(0, w - \sigma(w))$$

for all $w \in E$, which shows that for any $s \in \mathrm{Skew}(E, \sigma)$ there is a $d \in D_{W,W}$ such that $d(0, 1) = (0, s)$. Now it is clear that $D_{B,B} = D_{E,E} \oplus D_{W,W} = \mathrm{Der}(B, -)$. □

9. Tensor products of two composition algebras

Let $C(m)$ be the split composition algebra of dimension m over k^s. We assume that $m_1, m_2 \in \{1, 2, 4, 8\}$ but $(m_1, m_2) \neq (2, 2)$ throughout the chapter. The algebra $C(m_1) \otimes_{k^s} C(m_2)$ equipped with the canonical involution (obtained by tensoring the standard involutions on the two factors) is a central simple structurable algebra [3, §8].

Our reason for excluding $(m_1, m_2) = (2, 2)$ is that $C(2) \otimes C(2) \simeq (k^s)^4$ is not simple as an algebra with involution. However, these algebras are by no means uninteresting: the automorphism group of $C(2) \otimes C(2)$ as an algebra with involution is the dihedral group of order 8. Many insights on twisted forms of $C(2) \otimes C(2)$ and their cohomological invariants can be found in [102], [74, §5.1], and [58, Exercise 3.9].

9.1. Definition. A k-algebra with involution $(A, -)$ is called an *(m_1, m_2)-product algebra* if $(A_{k^s}, -) \simeq C(m_1) \otimes_{k^s} C(m_2)$ as k^s-algebras with involution.

We say that $(A, -)$ is *decomposable* if there are composition subalgebras C_1, $C_2 \subset A$ such that $\overline{C_i} = C_i$ and $A = C_1 \otimes C_2$.

We say that $(A, -)$ is *split* if there are split composition subalgebras C_1, $C_2 \subset A$ such that $\overline{C_i} = C_i$ and $A = C_1 \otimes C_2$

By a theorem of Albert [101, Theorem 16.1], every associative central simple algebra with involution of degree 4 is a tensor product of two quaternion subalgebras. These are called biquaternion algebras, so a $(4, 4)$-product algebra is the same thing as a *biquaternion algebra with orthogonal involution.* We sometimes call $(8, 8)$-algebras *bioctonion algebras*, and in general we can call any (m_1, m_2)-product algebra a *bicomposition algebra.* An $(m, 1)$-product algebra is just a composition algebra with its standard involution, and an $(m, 2)$-product algebra may be thought of as a composition algebra with an involution of the second kind.

An (m_1, m_2)-product algebra is associative if and only if $m_1, m_2 \leq 4$, and it is alternative if and only if $(m_1, m_2) \neq (4, 8)$ or $(8, 8)$ [86, §1 Lemma 2]. Not only do $(4, 8)$- and $(8, 8)$-product algebras fail to be alternative, they also fail to be power-associative [35, Corollary 1].

The purpose of this section is to determine the groups $\mathbf{Aut}(A, -)$ for all bicomposition algebras.

9.2. *Automorphisms and derivations of composition algebras.* Let $(C, -)$ be composition algebra with its canonical involution, where $m = \dim C = 1$, 2, 4, or 8. Recall that

$$\mathbf{Aut}(C) = \mathbf{Aut}(C, -) = \begin{cases} 1 & \text{if } m = 1 \\ \mathbb{Z}/2\mathbb{Z} & \text{if } m = 2 \\ \mathbf{PGL}_1(C) & \text{if } m = 4 \end{cases}$$

and $\mathbf{Aut}(C) = \mathbf{Aut}(C, -)$ is a simple algebraic group of type G_2 if $m = 8$. We refer to [87, p. 62], [101, §23], and [164, Theorem 2.3.5].

In all cases, $\mathbf{Aut}(C)$ is smooth, and it is connected and absolutely simple except when $m = 2$. Moreover, $\mathrm{Der}(C) = 0$ if $m = 1$ or 2, and $\mathrm{Der}(C)$ is simple and either 3- or 14-dimensional according as $m = 4$ or 8 [164, Lemma 2.4.4].

9.3. *Automorphisms of Smirnov algebras.* For an octonion algebra C, the Smirnov algebra $T(C) \subset C \otimes C$ was mentioned in 2.4. In [16, Theorem 4.3], the authors prove that there is an isomorphism:

$$\mathbf{Aut}(C) \simeq \mathbf{Aut}(T(C), -).$$

Since $\mathrm{Lie}(\mathbf{Aut}(C))$ is simple (a form of $\mathfrak{g}_2$), it follows that the inner derivation algebra $D_{T(C),T(C)}$ is equal to $\mathrm{Der}(T(C), -)$.

By Galois cohomology (or otherwise, as in [11, Theorem 2.5]), one can deduce that every form of a Smirnov algebra is isomorphic to $T(C)$ for some octonion algebra C.

9.4. *Some equivalent categories of algebras.* Consider the following categories, for an arbitrary field k of characteristic not 2 or 3:

– $\mathsf{Prod}_{m_1,m_2}(k)$ is the groupoid of (m_1, m_2)-product algebras over k, where the morphisms are involution-preserving k-algebra isomorphisms;

– $\mathsf{Comp}_m(k)$ is the groupoid of m-dimensional composition algebras over k, where the morphisms are k-algebra isomorphisms;

– $\mathsf{Comp}_m\mathsf{\acute{E}t}_2(k)$ is the groupoid of m-dimensional composition algebras over quadratic étale extensions of k, where the morphisms are k-algebra isomorphisms.

  That is, the objects are k-algebras either of the form C for an m-dimensional composition algebra C over a quadratic field extension E/k, or of the form $C_1 \times C_2$ where C_1, C_2 are m-dimensional composition algebras over k (we view $C_1 \times C_2$ as a composition algebra over $E = k \times k$);

– $\mathsf{Malc}_7(k)$ is the groupoid of exceptional simple Malcev algebras over k, where the morphisms are k-algebra isomorphisms.

Clearly $\mathsf{Prod}_{1,m}(k)$ is equivalent to $\mathsf{Comp}_m(k)$ and $\mathsf{Prod}_{m_1,m_2}(k)$ is equivalent to $\mathsf{Prod}_{m_2,m_1}(k)$.

9.5. Theorem (Kuzmin)**.** *Every exceptional simple Malcev algebra is 7-dimensional and isomorphic to C_0^- for some octonion algebra C, unique up to isomorphism, and $\mathrm{Aut}(C_0^-) \simeq \mathrm{Aut}(C)$.*

Proof. Kuzmin proved that if C is an octonion algebra over k, then C_0^- is an exceptional simple Malcev algebra over k, and if S is an exceptional simple Malcev algebra over k then there is a unique octonion structure on $k \oplus S$ such that $(1, 0)$ is the unit and $(k \oplus S)_0^- = S$ (see [104, Theorems 11–13] or [105, Theorems 3.11 & 3.12]). Moreover, if C and D are octonion algebras then every isomorphism $C \xrightarrow{\sim} D$ restricts to an isomorphism $C_0^- \xrightarrow{\sim} D_0^-$, and every isomorphism $C_0^- \xrightarrow{\sim} D_0^-$ is the restriction of a unique isomorphism $C \xrightarrow{\sim} D$ (see [123, Remark 3.5] or [105, Theorem 3.12]). □

By Kuzmin's Theorem and [101, Proposition 12.37], there is an equivalence

$$\mathsf{Comp}_8(k) \to \mathsf{Malc}_7(k), \qquad\qquad C \mapsto C_0^-.$$

We proceed to describe some more equivalences between these categories.

9.6. Theorem. *If $m_1 > m_2$, the functor*

$$\mathsf{Comp}_{m_1}(k) \times \mathsf{Comp}_{m_2}(k) \to \mathsf{Prod}_{m_1,m_2}(k), \qquad (C_1, C_2) \mapsto (C_1 \otimes C_2, \gamma_1 \otimes \gamma_2),$$

where γ_i is the standard involution on C_i, defines an equivalence of categories. In particular, every (m_1, m_2)-product algebra is decomposable.

Proof. For $m_2 = 1$, the statement is trivial. Let $(m_1, m_2) = (4, 2)$ or $(8, 2)$, and let $(A, -)$ be in $\mathsf{Prod}_{m_1,2}(k)$. It is clear that $Z(A) = Z(A_{k^s})^{\Gamma_k}$ because $Z(A_{k^s})$ is stabilised by the action of Γ_k on A_{k^s}. Since $Z(A_{k^s}) = k^s \times k^s$, Galois descent [25, Lemma III.8.21] implies that $Z(A_{k^s})^{\Gamma_k}$ is a quadratic étale extension of k. Of course, $\overline{Z(A)} = Z(A)$. Furthermore, there is a unique involution-invariant m_1-dimensional composition k-subalgebra C_A of A such that $A = C_A \otimes Z(A)$ (see [101, Proposition 2.22] for $m_1 = 4$ and [137, Theorem 3.2] for $m_1 = 8$). In fact, C_A is the k-subalgebra generated by $[\mathrm{Skew}(A, -), \mathrm{Skew}(A, -)]$. Every isomorphism $(A, -) \xrightarrow{\sim} (A', -)$ in $\mathsf{Prod}_{m_1,2}(k)$ clearly restricts to isomorphisms $Z(A) \xrightarrow{\sim} Z(A')$ and $C_A \xrightarrow{\sim} C_{A'}$ and is uniquely determined by these restrictions, so the functor $(A, -) \mapsto (C_A, Z(A))$ is inverse to the one displayed above.

Let $(m_1, m_2) = (8, 4)$, and let $(A, -)$ be in $\mathsf{Prod}_{8,4}(k)$. By definition, $A_{k^s} \simeq C(8) \otimes C(4)$ as k^s-algebras. It is clear that $\mathrm{Nuc}\big(C(8) \otimes C(4)\big) = C(4)$, and the centraliser of $C(4)$ in $C(4) \otimes C(8)$ is $C(8)$. Since both $\mathrm{Nuc}(A_{k^s})$ and its centraliser are stabilised by Γ_k, $\mathrm{Nuc}(A)$ is a quaternion algebra and its centraliser $C_A(\mathrm{Nuc}(A))$ is an octonion algebra such that $A = C_A(\mathrm{Nuc}(A)) \otimes \mathrm{Nuc}(A)$. The involution on A stabilises both factors because this is true after extending to k^s. Any isomorphism $(A, -) \xrightarrow{\sim} (A', -)$ in $\mathsf{Prod}_{8,4}(k)$ is, in particular, an isomorphism of algebras $A \xrightarrow{\sim} A'$ so it restricts to unique isomorphisms on the nuclei and their centralisers. The functor $(A, -) \mapsto \big(C_A(\mathrm{Nuc}(A)), \mathrm{Nuc}(A)\big)$ is inverse to the one displayed in the statement of the theorem. □

9.7. *Corestriction of composition algebras.* Now let $m = 4$ or 8. Not all (m, m)-product algebras are decomposable, so we need to introduce a construction called the corestriction. This construction can be found in [101, (3.12)] or [6, §2].

Let C be an m-dimensional composition algebra over a quadratic étale extension E/k, with its standard E-linear involution τ. Let ι be the unique nontrivial k-automorphism of E. We define a set of symbols ${}^\iota C = \{{}^\iota x : x \in C\}$ and give it the structure of an E-algebra with involution, as follows:

$${}^\iota x + {}^\iota y = {}^\iota(x + y), \qquad {}^\iota x\, {}^\iota y = {}^\iota(xy), \qquad {}^\iota(ex) = \iota(e)\, {}^\iota(x), \qquad {}^\iota\tau({}^\iota x) = {}^\iota\tau(x)$$

for $x, y \in C$ and $e \in E$. If $n : C \to E$ is the canonical norm of C then $\iota.n : {}^\iota C \to E$, defined by $\iota.n({}^\iota x) = \iota(n(x))$ for all $x \in C$, is the canonical norm of ${}^\iota C$.

Now, $({}^\iota C \otimes_E C, {}^\iota\tau \otimes \tau)$ is an E-algebra with involution. The map $s : {}^\iota C \otimes_E C \to {}^\iota C \otimes_E C$, $s({}^\iota x \otimes y) = {}^\iota y \otimes x$, is an E/k-semilinear automorphism. The set of points in ${}^\iota C \otimes_E C$ fixed by s is a k-algebra, which we denote by

$$\mathrm{cor}_{E/k}(C) = ({}^\iota C \otimes_E C)^s.$$

We give $\mathrm{cor}_{E/k}(C)$ the canonical involution, namely the restriction of ${}^\iota\tau \otimes \tau$.

If $E = k \times k$, the nontrivial k-automorphism is $\iota : (x, y) \mapsto (y, x)$. In this case $C = C_1 \times C_2$ for some composition algebras C_1, C_2 over k. Building $\mathrm{cor}_{E/k}(C)$ from the definition leads to some heavy notation, but nevertheless there are copies of C_1 and C_2 inside $\mathrm{cor}_{E/k}(C)$ such that $\overline{C_i} = C_i$ and $\mathrm{cor}_{E/k}(C) = C_1 \otimes C_2$.

9.8. *Malcev structure on the skew subspace.* If $(A, -) = C_1 \otimes C_2$ is a decomposable (m_1, m_2)-product algebra, since $\overline{z_1 \otimes z_2} = \overline{z_1} \otimes \overline{z_2}$ it is clear that

$$\mathrm{Skew}(A, -) = (C_1)_0 \oplus (C_2)_0. \tag{9.8.1}$$

The subspaces $(C_i)_0^-$ are ideals in the Malcev algebra $\mathrm{Skew}(A, -)^-$.

On the other hand, if $(A, -) = \mathrm{cor}_{E/k}(C)$ where E is a field, then $\mathrm{Skew}(A, -)^- = \{{}^\iota s \otimes 1 + 1 \otimes s : s \in C_0\}$ is a simple (but not central simple) Malcev algebra, because

$$\mathrm{Skew}(A, -)^- \xrightarrow{\sim} C_0^-, \qquad {}^\iota s \otimes 1 + 1 \otimes s \longmapsto s \tag{9.8.2}$$

is an isomorphism.

9.9. Theorem. *If $m = 4$ or 8, the functor*

$$F_k : \mathsf{Comp}_m\acute{\mathsf{E}}\mathsf{t}_2(k) \to \mathsf{Prod}_{m,m}(k), \qquad C \mapsto \mathrm{cor}_{Z(C)/k}(C)$$

defines an equivalence of categories. In particular, if C in $\mathsf{Comp}_m\acute{\mathsf{E}}\mathsf{t}_2(k)$ and $(A, -)$ in $\mathsf{Prod}_{m,m}(k)$ correspond to each other under this equivalence, then:

(i) $\mathrm{Aut}_k(C) \simeq \mathrm{Aut}_k(A, -)$.

(ii) *The centre of C is isomorphic to the centroid of the Malcev algebra* $\mathrm{Skew}(A, -)^-$.

(iii) *$(A, -)$ is decomposable as $(A, -) \simeq C_1 \otimes C_2$ if and only if $C \simeq C_1 \times C_2$.*

Proof. Most of the proof concerns the main statement about the equivalence of categories. It is clear that (i) and (iii) follow from the main statement, and (ii) already follows from (9.8.2).

For $m = 4$, the theorem is proved completely in [101, Theorem 15.7], so we focus on $m = 8$. Owing to the conceptual and notational differences between the decomposable and indecomposable cases, it is convenient to follow the strategy of [101, Proposition 12.37], instead of formally writing down an inverse to F_k. This involves showing that F_k induces a bijection between the isomorphism classes of objects in $\mathsf{Comp}_m\acute{\mathsf{E}}\mathsf{t}_2(k)$ and in $\mathsf{Prod}_{m,m}(k)$, and a bijection between the automorphism groups of C and $F_k(C)$ for all C in $\mathsf{Comp}_m\acute{\mathsf{E}}\mathsf{t}_2(k)$.

Decomposable case: Let $(A, -)$ be an $(8, 8)$-product algebra, and write $S = \mathrm{Skew}(A, -)$. If $(A, -) = C_1 \otimes C_2$ is decomposable, then S^- is a product of two exceptional simple Malcev algebras over k and the centroid of S^- is $k \times k$; see (9.8.1). We claim the converse: if the centroid of S^- is isomorphic to $k \times k$, then $(A, -)$ is decomposable. Indeed, $S^- = S_1 \times S_2$ for a pair of simple Malcev subalgebras S_i. Since $[S_1, S_2] = 0$, the subspaces $k1 \oplus S_1$ and $k1 \oplus S_2$ centralise each other in A, and $A = (k1 \oplus S_1) \otimes (k1 \oplus S_2)$. There is an isomorphism $(A_{k^s}, -) \xrightarrow{\sim} C(8) \otimes C(8)$ that can be arranged to map $k1 \oplus S_1$ into $C(8) \otimes 1$ and $k1 \oplus S_2$ into $1 \otimes C(8)$, so the subalgebras $k1 \oplus S_i \subset A$ are octonion algebras over k. By Theorem 9.5, there is only one octonion algebra structure on $k1 \oplus S_i$ compatible with $(k \oplus S_i)_0^- = S_i$. Therefore $(A, -) = F_k(C_1 \times C_2)$, where $C_1 = k1 \oplus S_1$ and $C_2 = k1 \oplus S_2$ (and these factors are unique up to re-ordering!).

Any k-automorphism α of $(A, -)$ restricts to a k-automorphism α' of $S^- = S_1 \times S_2$. This α' must map S_i isomorphically to $S_{\sigma(i)}$ for some permutation $\sigma \in S_2$, because these are the unique maximal ideals of S. Theorem 9.5 now implies that α' is the restriction of a unique k-automorphism α'' of $C_1 \times C_2$. Then $F_k(\alpha'') = \alpha$. Moreover, if β is any k-automorphism of $C_1 \times C_2$, then $F_k(\beta)'' = \beta$. Therefore F_k puts the automorphism group of $C_1 \times C_2$ in bijection with that of $(A, -)$.

Indecomposable case: If $(A, -)$ is an indecomposable $(8, 8)$-product algebra, the centroid of S^- is a quadratic field extension E/k and S is an exceptional simple Malcev algebra over E. The extension $(S^-)_E$ has centroid $E \otimes_k E \simeq E \times E$ so it is a product $S_1 \times S_2$ of two simple Malcev E-algebras. Following the reasoning from the previous paragraph, there is a unique decomposition $A_E = C_1 \otimes_E C_2$ where $C_i = E1 \oplus S_i$ is an octonion subalgebra of A_E. The generator $\iota \in \mathcal{G}al(E/k)$ acts on A_E by a bijection that is k-linear but not E-linear. Since the centroid of S is E, ι swaps the subspaces $S_1, S_2 \subset A$, and it provides a bijection $C_1 \xrightarrow{\sim} C_2$ that is k-linear and multiplicative, but not E-linear. Composing this with the k-isomorphism $C_2 \to {}^{\iota}C_2$, $x \mapsto {}^{\iota}x$, we can naturally identify C_1 with ${}^{\iota}C_2$. Then $A_E = {}^{\iota}C_2 \otimes_E C_2$, ι acts on A_E by the map ${}^{\iota}x \otimes y \mapsto {}^{\iota}y \otimes x$, and consequently $A = \mathrm{cor}_{E/k}(C_2)$. The pair of E-algebras C_1, C_2 is uniquely determined by $(A, -)$, and of course $C_1 \simeq C_2$ as k-algebras, so the only octonion E-algebras C such that $(A, -) \simeq \mathrm{cor}_{E/k}(C)$ are the ones that are k-isomorphic to C_2.

If α is an automorphism of $(A, -) = F_k(C) = \mathrm{cor}_{E/k}(C)$, then by (9.8.2) it induces a k-automorphism α' of C_0^-. If α' is E-linear then it extends to a unique E-automorphism α'' of C (directly applying Theorem 9.5). If α' is not E-linear, then the map $C_0^- \to ({}^{\iota}C)_0^-$, $x \mapsto {}^{\iota}(\alpha'(x))$, is an E-isomorphism and so it extends to an E-isomorphism $C \to {}^{\iota}C$. Composing this with the k-isomorphism ${}^{\iota}C \to C$, ${}^{\iota}x \mapsto x$ yields a unique extension α'' of α' that is actually a k-isomorphism of C. It is clear that $F_k(\alpha'') = \alpha$ and also that any k-automorphism β of C satisfies $F_k(\beta)'' = \beta$. $\square$

9.10. Proposition. *If $(A, -) = C_1 \otimes C_2$ is a decomposable (m_1, m_2)-product algebra, then*

$$\mathrm{Der}(A, -) = D_{A,A} = D_{S,S} = D_{C_1,C_1} \times D_{C_2,C_2} \simeq \mathrm{Der}(C_1) \times \mathrm{Der}(C_2).$$

Proof. Allison [3, p. 148] shows that $\mathrm{Der}(A, -) \supset D_{A,A} = D_{S,S} = D_{C_1,C_1} \times D_{C_2,C_2}$. But D_{C_i,C_i} embeds as an ideal of $\mathrm{Der}(C_i)$ via the restriction map $D_{x,y} \mapsto D_{x,y}|_{C_i}$, and $\mathrm{Der}(C_i)$ is either 0 or simple, so $D_{C_i,C_i} \simeq \mathrm{Der}(C_i)$. Let $S = \mathrm{Skew}(A, -)$ and $S_i = S \cap C_i$. The rest of the proof is based on [123, Proposition 3.6]. Every $d \in \mathrm{Der}(A, -)$ kills $1 \in A$, satisfies $d([x, y]) = [x, d(y)] + [d(x), y]$, and maps S to itself, so

$$d(C_i) = d(S_i) = d([S_i, S_i]) \subset [S_i, d(S_i)] + [d(S_i), S_i] \subset [S_i, S] + [S, S_i] = S_i \subset C_i.$$

The map $\mathrm{Der}(A,-) \to \mathrm{Der}(C_1) \times \mathrm{Der}(C_2)$, $d \mapsto (d|_{C_1}, d|_{C_2})$, is evidently an isomorphism because $C_1 + C_2$ generates A and $C_1 \cap C_2 = k1$ is annihilated by $\mathrm{Der}(A,-)$. This proves $\mathrm{Der}(A,-) \subset D_{C_1,C_1} \times D_{C_2,C_2}$. □

9.11. Theorem. *Let $(A,-) = C_1 \otimes C_2$ be an (m_1,m_2)-product algebra with $m_1 \neq m_2$. The canonical homomorphism $\varphi : \mathbf{Aut}(C_1) \times \mathbf{Aut}(C_2) \to \mathbf{Aut}(A,-)$ is an isomorphism.*

Proof. It is clear that φ is injective, and Theorem 9.6 proves that φ_{k^a} is surjective. The conclusion follows from [101, Proposition 22.3] because $\mathbf{Aut}(A,-)$ is smooth. □

9.12. Theorem. *Let $m = 4$ or 8, and let $(A,-) = \mathrm{cor}_{E/k}(C)$ be an (m,m)-product algebra, where C is an m-dimensional composition algebra over a quadratic étale extension E/k. Let $C_{/k}$ be the k-algebra with the same underlying set, multiplication, and k-vector space structure as C. Then $\mathbf{Aut}(A,-) \simeq \mathbf{Aut}(C_{/k})$. Consequently,*

(i) $\mathbf{Aut}(A,-)^\circ \simeq R_{E/k}(\mathbf{Aut}(C))$.

(ii) $\mathbf{Aut}(A,-)$ *has two connected components, and the nonidentity component has k-points if and only if $C \simeq {}^\iota C$ as E-algebras.*

(iii) *If $A = C_1 \otimes C_2$ is decomposable then* $\mathbf{Aut}(A,-) \simeq \mathbf{Aut}(C_1 \times C_2)$ *and* $\mathbf{Aut}(A,-)^\circ \simeq \mathbf{Aut}(C_1) \times \mathbf{Aut}(C_2)$.

Proof. We define $\varphi : \mathbf{Aut}(C_{/k}) \to \mathbf{Aut}(A,-)$ by

$$\varphi_R : \mathrm{Aut}_R((C_{/k})_R) \to \mathrm{Aut}_R(A_R,-)$$
$$\varphi_R(\alpha)\big(\textstyle\sum({}^\iota x_i \otimes y_i) \otimes r_i\big) = \sum \big({}^\iota \alpha(x_i) \otimes \alpha(y_i)\big) \otimes r_i$$

for all R-automorphisms α of $(C_{/k})_R = C \otimes_k R$, and all $x_i, y_i \in C$, $r_i \in R$. Then φ_R is injective because, for instance, $\varphi_R(\alpha)(({}^\iota s \otimes 1 + 1 \otimes s) \otimes 1)$ is sent to $\alpha(s \otimes 1)$ by the isomorphism $\mathrm{Skew}(A,-)_R \xrightarrow{\sim} (C_0)_R$. Theorem 9.9 proves that φ_{k^a} is surjective, and the conclusion that φ is an isomorphism follows from the smoothness of $\mathbf{Aut}(A,-)$.

The kernel of the natural homomorphism $\mathbf{Aut}(C_{/k}) \to \mathbf{Aut}(E) = S_2$ is isomorphic to $R_{E/k}(\mathbf{Aut}(C))$, which is connected. Now (i) follows from the uniqueness of the connected-étale sequence [122, Proposition 5.58]. For (ii), it is clear that k-automorphisms of C acting nontrivially on E are the same as E-algebra isomorphisms $C \simeq {}^\iota C$. Finally, (iii) is just a specialisation of the main statement and (i). □

Item (iii) of the above theorem was also proved in [16, Theorem 3.6].

9.13. Theorem. *If $(A,-)$ is an (m_1,m_2)-product algebra with $m_1 \geq m_2$, then $\mathbf{Aut}(A,-) = \mathbf{Aut}(A)$ if and only if $(m_1,m_2) = (1,1)$, $(2,1)$, $(4,1)$, $(8,1)$, $(8,4)$, or $(8,8)$.*

Proof. A theorem of Brešar [36, Theorem 3.1] says that

$$\mathrm{Der}(C_1 \otimes C_2) = L_{Z(C_1)} \otimes \mathrm{Der}(C_2) + \mathrm{Der}(C_1) \otimes L_{Z(C_2)} + \mathrm{ad}(\mathrm{Nuc}(C_1 \otimes C_2)).$$

By considering each case individually and comparing with Proposition 9.10, one can show that $\mathrm{Der}(A,-) = \mathrm{Der}(A)$ if and only if $(m_1,m_2) = (8,4)$, $(8,8)$, or $(m,1)$ for some m. Since $\mathrm{Der}(A) = \mathrm{Lie}(\mathbf{Aut}(A))$ and $\mathrm{Der}(A,-) = \mathrm{Lie}(\mathbf{Aut}(A,-))$, this shows $\mathbf{Aut}(A,-) \neq \mathbf{Aut}(A)$ if (m_1,m_2) is not in this list.

Every automorphism of a composition algebra preserves the standard involution (see 9.2), which settles the case of $m_2 = 1$. In the $(8,4)$ case, we have shown in Theorem 9.11 that $\mathbf{Aut}(C_1 \otimes C_2) \simeq \mathbf{Aut}(C_1) \times \mathbf{Aut}(C_2)$. Since $\overline{C}_i = C_i$ and $\mathbf{Aut}(C_i) = \mathbf{Aut}(C_i, -)$, it follows that $\mathbf{Aut}(C_1 \otimes C_2, -) = \mathbf{Aut}(C_1 \otimes C_2)$.

For decomposable bi-octonion algebras, [16, Lemma 3.5] shows that $\mathbf{Aut}(C_1 \otimes C_2, -) = \mathbf{Aut}(C_1 \otimes C_2)$. The key step is from [123, §3], where is it proven that $C_1 + C_2 = k \oplus \mathrm{Skew}(A, -)$ has the following first-order definition in the language of algebras without involution:

$$C_1 + C_2 = \{a \in A \colon (a, x, y) = -(x, a, y) = (x, y, a) \ \forall x, y \in A\}.$$

If $(A, -)$ is an indecomposable bi-octonion algebra, then one can either use the same proof or argue that $\mathbf{Aut}(A, -) = \mathbf{Aut}(A)$ because $\mathbf{Aut}(A, -)$ is smooth and $\mathbf{Aut}(A_F, -) = \mathbf{Aut}(A_F)$ for some field extension F/k which decomposes $(A, -)$. $\square$

By Galois descent, we obtain the following corollary:

9.14. Corollary. *Let $(A, -)$ and $(A', -)$ be (m_1, m_2)-product algebras where $(m_1, m_2) = (1,1)$, $(2,1)$, $(4,1)$, $(8,1)$, $(8,4)$ or $(8,8)$. Then $(A, -) \simeq (A', -)$ as algebras with involution if and only if $A \simeq A'$ as algebras.*

10. Structurable algebras of skew-dimension one

By Theorem 3.9 and the remarks in 3.10, a central simple structurable algebra of skew-dimension one is exactly one of the following:

(i) A quadratic étale algebra with its standard nontrivial automorphism.

(ii) A quaternion algebra with an orthogonal involution.

(iii) The structurable algebra of a nonsingular hermitian form over (E, σ) where (E, σ) is either (i) or (ii).

(iv) A form of the quartic 2×2 matrix algebra $M(k)$ (see 2.3).

(v) A form of a matrix structurable algebra $M(J)$ co-ordinatised by a separable cubic Jordan algebra J (see 2.6).

Our main focus is the algebras of type (v). We begin with some structure theory of these algebras and then proceed to study their automorphisms and derivations.

10.1. *A large Jordan subalgebra.* Let J be a separable cubic Jordan algebra with generic trace t. Within $\mathrm{Herm}(M(J))$ there is a commutative unital subalgebra

$$V(J) = \left\{ \begin{pmatrix} \alpha & j \\ j & \alpha \end{pmatrix} : \alpha \in k, j \in J \right\}$$

of half the dimension of $M(J)$. Since it is structurable, it must be a Jordan algebra. I found the following isomorphism $k \times J \xrightarrow{\sim} V(J)$ by tracing through the details of [9, §5–6]:

10.2. Proposition. *There is an isomorphism of algebras $F : k \times J \xrightarrow{\sim} V(J)$ with inverse $G : V(J) \xrightarrow{\sim} k \times J$ defined by*

$$F(\alpha,\ j) = \begin{pmatrix} \frac{1}{4}\alpha + \frac{1}{4}t(j) & \frac{1}{2}j + \frac{1}{4}\alpha 1 - \frac{1}{4}t(j)1 \\ \frac{1}{2}j + \frac{1}{4}\alpha 1 - \frac{1}{4}t(j)1 & \frac{1}{4}\alpha + \frac{1}{4}t(j) \end{pmatrix}$$

$$G\begin{pmatrix} \alpha & j \\ j & \alpha \end{pmatrix} = \big(\alpha + t(j),\ 2j + \alpha 1 - t(j)1\big) \qquad \textit{for all } \alpha \in k, j \in J.$$

Perhaps an easier way to think about these maps is that they are specified by the conditions: F and G are linear, $F(1) = 1$, $G(1) = 1$, and

$$F(-t(j), j) = \begin{pmatrix} 0 & \frac{1}{2}j \\ \frac{1}{2}j & 0 \end{pmatrix}, \qquad G\begin{pmatrix} 0 & j \\ j & 0 \end{pmatrix} = (t(j), 2j - t(j)1).$$

Proof. The proposition can be verified directly without much trouble. For the off-diagonal entries, one needs the following identities:

$$j \times 1 = t(j)1 - j, \qquad j\ell = \frac{1}{2}\big(j \times \ell + t(j)\ell + t(\ell)j + t(j\ell)1 - t(j)t(\ell)1\big) \tag{10.2.1}$$

for all $j, \ell \in J$ [115, p. 189–190]. (Of course, the identity $t(1) = 3$ is essential too.) □

10.3. *Classification of cubic Jordan algebras.* By a *cubic Jordan algebra*, we mean a Jordan algebra of generic degree 3. A separable cubic Jordan algebra J over k is either isomorphic to

$$k \times J(V, q)$$

for some nondegenerate quadratic space (V, q) of dimension $n \geq 2$ or it has dimension 3, 6, 9, 15, or 27 and exactly one of the following holds (see [129, Theorem 2.1], [101, Theorem 37.12], and [110, §3]):

(3) J is a cubic étale algebra L.

(6) $J = \mathcal{H}(M_3(k), \mathrm{ad}_b)$ where ad_b is the adjoint involution of some 3-dimensional bilinear form b.

(9') $J = C^+ = \mathcal{H}(C \times C^{\mathrm{op}}, \epsilon)$ where C is a central simple algebra of degree 3 over k and ϵ is the exchange involution.

(9) $J = \mathcal{H}(B, \tau)$ where B is a central simple algebra of degree 3 over a quadratic field extension F/k and τ is an F/k-unitary involution.

(15) $J = \mathcal{H}(M_3(Q), \sigma)$ where Q is a quaternion algebra and σ is a symplectic involution on $M_3(Q) = M_3(k) \otimes Q$ which decomposes as a tensor product $\sigma = \mathrm{ad}_b \otimes -$, for some 3-dimensional bilinear form b and the standard symplectic involution "$-$" on Q.

(27) J is an Albert algebra.

In particular, central simple cubic Jordan algebras exist only in dimensions 6, 9, 15, and 27 (compare with Table 1 (b) and Table 2 (d)). Cubic Jordan division algebras can only exist in dimensions 3, 9, and 27.

Note that item (9') above can be viewed as a special case of (9) if one allows B to be an algebra over $E = k \times k$ with an E/k-semilinear involution τ such that (B, τ) is central simple over k as an algebra with involution.

Recall that we have used the notation $k^s, C(2), C(4)$, and $C(8)$ for the split composition algebras over k^s, and $k, \mathbb{B}, \mathbb{H}$, and $\mathbb{O}$ for the split composition algebras over the base field k. Recall also the definition of $M(J, \eta)$ from 2.6.

10.4. Definition. An algebra with involution $(B, -)$ is called a *green algebra* if $(B_{k^s}, -) \simeq M(\mathcal{H}_3(k^s))$, a *blue algebra* if $(B_{k^s}, -) \simeq M(M_3(k^s)^+)$, a *red algebra* if $(B_{k^s}, -) \simeq M(\mathcal{H}_3(C(4)))$, and a *Brown algebra* if $(B_{k^s}, -) \simeq M(\mathcal{H}_3(C(8)))$.

We say that $(B, -)$ is a *matrix* green, blue, red, or Brown algebra if there is a central simple cubic Jordan algebra J of dimension 6, 9, 15, or 27 respectively and a scalar $\eta \in k^\times$ such that $(B, -) \simeq M(J, \eta)$.

We say that $(B, -)$ is a *split* green, blue, red, or Brown algebra (respectively) if $(B, -)$ is k-isomorphic to $M(J)$ where

$$J = \begin{cases} \mathcal{H}(M_3(k), \mathrm{ad}_{\langle 1,-1,1\rangle}) \\ M_3(k)^+ \\ \mathcal{H}_3(\mathbb{H}) \\ \mathcal{H}_3(\mathbb{O}) & \text{(respectively).} \end{cases}$$

The main goal of this section is to determine the automorphism groups of the colourful matrix algebras $M(J, \eta)$. To a limited extent, we are also interested in the algebras $M(J, \eta)$ where J is nonsimple of the form $k \times J(V, q)$ or is a cubic étale algebra L. The reason for our limited interest in the latter is that we still owe the reader a proof of some claims used in Theorem 3.14, particularly that $M(k \times J(V, q))$ is not isotopic to $M(J)$ for any simple cubic Jordan algebra J.

In the following theorem we determine the norm-preserving group $\mathbf{Iso}(N_J)$ for every separable cubic Jordan algebra J. The groups of points $\mathrm{Sim}(N_J) = \mathbf{Sim}(N_J)(k)$ and $\mathrm{Iso}(N_J) = \mathbf{Iso}(N_J)(k)$ have been known for a very long time, but a translation in terms of modern algebraic group theory is lacking. The summary in [103, Remark 10], while less detailed, is useful for comparison purposes.

10.5. Theorem. *Let J be one of the cubic Jordan algebras described in* 10.3*. Then $\mathbf{Iso}(N_J)$ is smooth. If $J = k \times J(V, q)$, then*

$$\mathbf{Iso}(N_J) \simeq \mathbf{GO}(\langle 1 \rangle \perp \langle -1 \rangle q).$$

In the other cases:

(3)
$$\mathbf{Iso}(N_{L/k}) \simeq \mathbf{G}^1_{m,L} \rtimes \mathbf{Aut}(L)$$

where $\mathbf{G}^1_{m,L} = \ker(\ R_{L/k}(\mathbf{G}_{m,L}) \xrightarrow{N_{L/k}} \mathbf{G}_m\ ).$

(6)
$$\mathbf{Iso}(N_J) \simeq \mathbf{SL}_3.$$

(9')
$$\mathbf{Iso}(N_J)^\circ \simeq \frac{\mathbf{SL}_1(C) \times \mathbf{SL}_1(C)}{\boldsymbol{\mu}_3}$$

where $\boldsymbol{\mu}_3$ *is embedded diagonally in* $\mathbf{SL}_1(C) \times \mathbf{SL}_1(C)$. *Moreover,* $\mathbf{Iso}(N_J)$ *has two connected components and the nonidentity component has k-points if and only if* $C \simeq M_3(k)$.

(9)

$$\mathbf{Iso}(N_J)^\circ \simeq \frac{R_{F/k}(\mathbf{SL}_1(B))}{\boldsymbol{\mu}_{3[F]}}$$

where $\boldsymbol{\mu}_{3[F]} = \ker(\ R_{F/k}(\boldsymbol{\mu}_{3,F}) \xrightarrow{N_{F/k}} \boldsymbol{\mu}_3\ )$ *is embedded centrally in* $R_{F/k}(\mathbf{SL}_1(B))$. *Moreover,* $\mathbf{Iso}(N_J)$ *has two connected components and the nonidentity component has k-points if and only if* $B \simeq M_3(F)$.

(15)

$$\mathbf{Iso}(N_J) \simeq \frac{\mathbf{SL}_3(Q)}{\boldsymbol{\mu}_2}$$

where $\boldsymbol{\mu}_2$ *is embedded centrally in* $\mathbf{SL}_3(Q)$.

(27) $\mathbf{Iso}(N_J)$ *is a simply connected group of inner type* E_6 *with trivial Tits class.*

Proof. By Lemma 7.5, $\mathbf{Iso}(N_J)$ and $\mathbf{Sim}(N_J)$ are smooth. We apply the following strategy in each of the cases: to prove that some group G is isomorphic to $\mathbf{Iso}(N_J)$ it suffices to show that there is an injective homomorphism $G \to \mathbf{Iso}(N_J)$ which is an isomorphism on the k^a-points $G(k^a) \xrightarrow{\sim} \mathbf{Iso}(N_J)(k^a)$.

$\boxed{\text{Case } J = k \times J(V,q)}$ Let $J' = J(V,q) = k \oplus V$. The generic norm of J' is the quadratic form $\tilde{q} = \langle 1 \rangle \perp \langle -1 \rangle q$, i.e. $\tilde{q}(\alpha, v) = \alpha^2 - q(v)$ for all $\alpha \in k$ and $v \in V$ [90, p. 37]. So the generic norm on J is the cubic form

$$N_J(\lambda, j') = \lambda \tilde{q}(j') = \lambda(\alpha^2 - q(v))$$

for all $\lambda \in k$ and $j' = (\alpha, v) \in J'$. There is an injective homomorphism

$$\begin{aligned} &\phi : \mathbf{GO}(J', \tilde{q}) \to \mathbf{Iso}(N_J) \\ &\quad \phi_R(f)(\lambda, j') = (\mu(f)^{-1}\lambda, f(j')) \qquad \text{for all } f \in \mathbf{GL}(J', \tilde{q})(R), \lambda \in R, j' \in J'_R. \end{aligned}$$

To show this is surjective, suppose $f \in \mathrm{Iso}(N_J)$. Then $f^*(p) = p \circ f$ is a k-algebra automorphism $f^* : k[V] \to k[V]$ such that $f^*(N_J) = N_J$. Since $k[V]$ has unique factorisation and N_J has two irreducible factors, of degrees 1 and 2 respectively, f^* must preserve each of these irreducible factors up to an invertible scalar. Hence f preserves the decomposition $J = k \oplus J'$ and has the form $f(\lambda, j') = (\mu\lambda, g(j'))$ for some $\mu \in k^\times$ and $g \in \mathrm{GL}(J')$. Clearly now, g is a similitude of $\tilde{q}$ and $\mu = \mu(g) \in k^\times$ is its multiplier. Hence $\phi_k : \mathrm{GO}(J', \tilde{q}) \to \mathrm{Iso}(N_J)$ is surjective (and also $\phi_{k^a} : \mathbf{GO}(J', \tilde{q})(k^a) \to \mathbf{Iso}(N_J)(k^a)$ is surjective).

$\boxed{\text{Case (3)}}$ It is clear that $\mathbf{G}^1_{m,L} \rtimes \mathbf{Aut}(L)$ acts faithfully on L by norm-presering linear maps $(\ell, f) \cdot x = \ell f(x)$, so we have an injective homomorphism $\mathbf{G}^1_{m,L} \rtimes \mathbf{Aut}(L) \hookrightarrow \mathbf{Iso}(N_{L/k})$. Now suppose $f \in \mathrm{Iso}(N_{L/k})$. If $f(1) = 1$ then $f \in \mathrm{Aut}_k(L)$ [90, p. 45]. If $f(1) = x \neq 1$ then x is invertible, $N_{L/k}(x) = 1$, and $L_{x^{-1}} f$ is an automorphism. This shows $\mathbf{Iso}(N_{L/k})(k) = \mathbf{G}^1_{m,L}(k).\mathbf{Aut}(L)(k)$, and also $\mathbf{Iso}(N_{L/k})(k^a) = \mathbf{G}^1_{m,L}(k^a).\mathbf{Aut}(L)(k^a)$, so it proves the claim.

$\boxed{\text{Case (6)}}$ For $X \in M_3(k)$, let $X^* = \mathrm{ad}_b(X)$. The norm of $J = \mathcal{H}(M_3(k), \mathrm{ad}_b)$ is the restriction of the usual determinant to symmetric matrices $P = P^* \in M_3(k)$. Since $\det(X^*) = \det(X^t) = \det(X)$ [101, Proposition 2.7 (2)], there is a norm-preserving action of $\mathbf{SL}_3$ on J,

$$\begin{aligned} &\phi : \mathbf{SL}_3 \to \mathbf{Iso}(N_J) \\ &\phi_R(X)(P) = XPX^* \qquad\qquad \text{for all } X \in \mathbf{SL}_3(R), P \in J_R. \end{aligned}$$

If $\phi_R(X) = 1$ then $X^* = X^{-1}$ and $XPX^{-1} = P$ for all $P \in J_R$. This implies X is central in $\mathrm{SL}_3(R)$, hence $X = \zeta\,\mathrm{id}$ for some $\zeta \in \boldsymbol{\mu}_3(R)$. But then $XX^* = \zeta^2\,\mathrm{id} = 1$ forces $\zeta = 1$. So ϕ is injective. It is easy to deduce from [88, Theorem 9] that ϕ_{k^a} is surjective, proving our claim.

$\boxed{\text{Case (9')}}$ The norm of $J = C^+$ is just the reduced norm Nrd_C. There is a norm-preserving action of $\mathbf{SL}_1(C) \times \mathbf{SL}_1(C)$ on J given by

$$\begin{aligned} &\phi : \mathbf{SL}_1(C) \times \mathbf{SL}_1(C) \to \mathbf{Iso}(\mathrm{Nrd}_C) \\ &\phi_R(X,Y)(P) = XPY^{-1} \qquad \text{for all } X, Y \in \mathbf{SL}_1(C)(R), P \in C_R^+. \end{aligned} \tag{10.5.1}$$

It is straightforward to show that $\phi_R(X,Y) = 1$ if and only if $X = Y = \zeta\,\mathrm{id}$ for some $\zeta \in \boldsymbol{\mu}_3(R)$. Hence ϕ induces an injective homomorphism $(\mathbf{SL}_1(C) \times \mathbf{SL}_1(C))/\boldsymbol{\mu}_3 \to \mathbf{Iso}(\mathrm{Nrd}_C)$. By [88, Theorem 7], the image of ϕ_{k^a} is a closed subgroup of index two in $\mathbf{Iso}(\mathrm{Nrd}_C)(k^a)$, which implies $(\mathbf{SL}_1(C)\times\mathbf{SL}_1(C))/\boldsymbol{\mu}_3 \simeq \mathbf{Iso}(N_J)^\circ$ and $\pi_0(\mathbf{Iso}(N_J)) = \mathbb{Z}/2\mathbb{Z}$.

If $C \simeq M_3(k)$ is split then it has an anti-automorphism (the transpose) which is not in the image of ϕ_{k^a}, therefore not in $\mathbf{Iso}(\mathrm{Nrd}_C)^\circ(k)$. If $C \not\simeq M_3(k)$ then $C \not\simeq C^{\mathrm{op}}$ by Brauer group considerations [101, Corollary 2.8 (1)], so [88, Theorem 7] (or [92, Theorem 7]) implies $\mathbf{Iso}(\mathrm{Nrd}_C)^\circ(k) = \mathbf{Iso}(\mathrm{Nrd}_C)(k)$.

$\boxed{\text{Case (9)}}$ The norm of $J = \mathcal{H}(B, \tau)$ is the restriction of Nrd_B (actually Nrd_B is an F-valued function, but $\mathrm{Nrd}_B(h) \in k \subset F$ when $h = \tau(h) \in B$ [101, Corollary 2.16]). There is a norm-preserving action of $R_{F/k}(\mathbf{SL}_1(B))$ on $J = \mathcal{H}(B, \tau)$ given by

$$\begin{aligned} &\phi : R_{F/k}(\mathbf{SL}_1(B)) \to \mathbf{Iso}(N_J) \\ &\phi_R(S)(P) = SP\tau(S) \qquad \text{for all } S \in \mathbf{SL}_1(B)(F \otimes_k R) \text{ and } P \in J_R. \end{aligned} \tag{10.5.2}$$

It is straightforward to work out that the kernel of ϕ is $\boldsymbol{\mu}_{3[F]}$. After canonically identifying

$$R_{F/k}(\mathbf{SL}_1(B))(R) = \{b \in B \otimes_k R\colon\ \mathrm{Nrd}_B(b) = 1\},$$

for all k-algebras R and

$$(B \otimes_k F, \tau \otimes \mathrm{id}_F) = (B \times B^{\mathrm{op}}, \epsilon)$$

we have

$$R_{F/k}(\mathbf{SL}_1(B))(S) = \{(x, y^{\mathrm{op}})\colon x, y \in B \otimes_k S, \mathrm{Nrd}(x) = \mathrm{Nrd}(y) = 1\}$$

for all F-algebras S, and $(x, y^{\mathrm{op}}) \mapsto (x, y^{-1})$ is a natural isomorphism

$$R_{F/k}(\mathbf{SL}_1(B)) \times_k F \xrightarrow{\sim} \mathbf{SL}_1(B) \times \mathbf{SL}_1(B).$$

With these identifications, the ϕ in (10.5.2) restricts to the ϕ in (10.5.1), so it is surjective onto $\mathbf{Iso}(N_J)^\circ$, and the group of components of $\mathbf{Iso}(N_J)$ has order two.

By [88, Theorem 8] (or [92, Theorem 8]), $\mathbf{Iso}(N_J)(k)$ has a point not in the image of ϕ_{k^a} if and only if B has an F-linear anti-automorphism, which can only be the case if $B \simeq M_3(F)$.

$\boxed{\text{Case (15)}}$ The norm of $J = \mathcal{H}(M_3(Q), \sigma)$ is the *pfaffian norm*, a square root of $\mathrm{Nrd}_{M_3(Q)}|_J$ [101, p. 19]. Consider the norm-preserving map

$$\begin{aligned} &\phi : \mathbf{SL}_3(Q) \to \mathbf{Iso}(N_J) \\ &\phi_R(X)(A) = XQ\sigma(X) \qquad\qquad \text{for all } X \in \mathbf{SL}_3(Q)(R), A \in J_R. \end{aligned}$$

The kernel of ϕ_R is the set of all $X = \sigma(X)^{-1}$ lying in the centre of $\mathbf{SL}_3(Q)(R)$, and these are the scalar matrices $c\,\mathrm{id}$ such that $c^2 = 1$. Hence $\ker\phi = \boldsymbol{\mu}_2$. By [88, Theorem 9], ϕ_{k^a} is surjective onto $\mathbf{Iso}(N_J)(k^a)$.

$\boxed{\text{Case (27)}}$ Like the previous cases, this is more or less ancient knowledge. The group $\mathbf{Iso}(N_J)$ is simple with root system of type E_6 [161, p. 150–151]. It is simply-connected of inner type because it contains $\boldsymbol{\mu}_3$ in its centre (simply connected E_6's of outer type have centre $\boldsymbol{\mu}_{3[F]} \not\simeq \boldsymbol{\mu}_3$), and it has trivial Tits class because the 27-dimensional irreducible representation J is defined over k [174, 6.4.2]. □

10.6. *Automorphisms of $M(J, \eta)$.* Let J be a separable cubic Jordan algebra, $\eta \in k^\times$, and let $(B, -) = M(J, \eta)$ be the structurable algebra described in 2.6. Any $f \in \mathbf{Aut}(B, -)(R)$ must map the subalgebra

$$R1 \oplus \mathrm{Skew}(B_R, -) = \left\{ \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix} : c \in R \right\} \simeq R \times R$$

isomorphically to itself. So, there is a homomorphism

$$\epsilon : \mathbf{Aut}(B, -) \to \mathbf{Aut}(k \times k) \simeq \mathbb{Z}/2\mathbb{Z}$$

whose kernel is the closed subgroup $\mathbf{Aut}_S(B, -) \subset \mathbf{Aut}(B, -)$ of automorphisms fixing $S = \mathrm{Skew}(B, -)$ pointwise. Note that $\mathbf{Aut}(B, -)^\circ \subset \mathbf{Aut}_S(B)$, but this inclusion may be proper. If $\eta = 1$ then ϵ is split surjective because

$$\omega : \begin{pmatrix} \alpha & j \\ j' & \beta \end{pmatrix} \mapsto \begin{pmatrix} \beta & j' \\ j & \alpha \end{pmatrix} \tag{10.6.1}$$

is an order 2 automorphism of $(B, -)$ which does not fix $\mathrm{Skew}(B, -)$. Therefore if $\eta = 1$ we have $\mathbf{Aut}(B, -) \simeq \mathbf{Aut}_S(B, -) \rtimes \mathbb{Z}/2\mathbb{Z}$.

10.7. Theorem. *Let J be a separable cubic Jordan algebra, $\eta \in k^\times$, and let $(B, -) = M(J, \eta)$. Then $\mathbf{Aut}_S(B, -)$ is a normal subgroup of index 2 in $\mathbf{Aut}(B, -)$, and*

$$\mathbf{Aut}_S(B, -) \simeq \mathbf{Iso}(N_J).$$

The nontrivial coset in $\mathbf{Aut}(B, -)/\mathbf{Aut}_S(B, -)$ has k-points if and only if there is a similitude of N_J with multiplier η^{-1}.

The proof of this theorem goes along the lines of [55, Theorems 2.8 & 2.9].

Proof. We have already established that $\mathbf{Aut}_S(B,-)$ is normal of index two in $\mathbf{Aut}(B,-)$, because it is the kernel of the k-defined homomorphism ϵ.

If $f \in \mathbf{Aut}_S(B,-)(R)$, it fixes the subspace $k \oplus \mathrm{Skew}(B,-) = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$ pointwise. Since

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \cdot B \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & J \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \cdot B \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ J & 0 \end{pmatrix} \tag{10.7.1}$$

it follows that f has the form

$$f\begin{pmatrix} \alpha & j \\ j' & \beta \end{pmatrix} = \begin{pmatrix} \alpha & \varphi(j) \\ \varphi'(j') & \beta \end{pmatrix} \tag{10.7.2}$$

for some $\varphi, \varphi' \in \mathbf{GL}(J)(R)$. Inspecting the multiplication on $M(J,\eta)$, we have

$$T(\varphi(j), \varphi'(j')) = T(j, j') \tag{10.7.3}$$
$$\varphi'(j) \times \varphi'(j) = \varphi(j \times j) \tag{10.7.4}$$
$$\varphi(j) \times \varphi(j) = \varphi'(j \times j) \tag{10.7.5}$$

for all $j \in J$, hence $\varphi'(j^\sharp) = \varphi(j)^\sharp$ and $\varphi(j^\sharp) = \varphi'(j)^\sharp$. Specialising (2.6.1) with $x = y = j$ and $\lambda = 1$ yields

$$N(j) = \frac{1}{6} T(j, j^\sharp) \tag{10.7.6}$$

for all $j \in J$. Therefore

$$N(\varphi(j)) = \frac{1}{6} T(\varphi(j), \varphi(j)^\sharp) = \frac{1}{6} T(\varphi(j), \varphi'(j^\sharp)) = \frac{1}{6} T(j, j^\sharp) = N(j),$$

so $\varphi \in \mathbf{Iso}(N_J)(R)$. Moreover, (10.7.3) implies $\varphi' = \mathrm{ad}_T(\varphi)^{-1}$ is uniquely determined by φ, so we have an injective homomorphism

$$\mathbf{Aut}_S(B,-)(R) \to \mathbf{Iso}(N_J)(R), \qquad f \mapsto \varphi.$$

To prove this is in fact surjective, suppose $\varphi \in \mathbf{Iso}(N_J)(R)$. Letting $\varphi' = \mathrm{ad}_T(\varphi)^{-1}$, the equality (10.7.3) holds by definition. Hence $T(x^\sharp, y) = T(\varphi'(x^\sharp), \varphi(y))$. An application of (2.6.1) yields

$$\begin{aligned} 0 &= N(\varphi(x + \lambda y)) - N(x + \lambda y) \\ &= \lambda(T(x^\sharp, y) - T(\varphi(x)^\sharp, \varphi(y)) + \lambda^2(T(y^\sharp, x) - T(\varphi(y)^\sharp, \varphi(x)), \end{aligned}$$

so

$$T(x^\sharp, y) = T(\varphi(x)^\sharp, \varphi(y)) = T(\varphi'(x^\sharp), \varphi(y)).$$

Now $\varphi(x)^\sharp = \varphi'(x^\sharp)$ because T is nondegenerate, and by linearisation (10.7.4) holds too. Similarly, (10.7.5) holds for φ and φ'. Therefore we can define f as in (10.7.2) and it is an automorphism of $(B_R, -)$. This concludes the argument that $\mathbf{Aut}_S(B,-)(R) \simeq \mathbf{Iso}(N_J)$.

Now suppose there is an automorphism $f \in \mathbf{Aut}(B,-)(k)$ such that f is nontrivial on $k \oplus \mathrm{Skew}(B,-)$. Then by (10.7.1) it must be of the form

$$f\begin{pmatrix} \alpha & j \\ j' & \beta \end{pmatrix} = \begin{pmatrix} \beta & \varphi'(j') \\ \varphi(j) & \alpha \end{pmatrix} \tag{10.7.7}$$

for some $\varphi, \varphi' \in \mathbf{GL}(J)(R)$. Since f is multiplicative, you can see (from the top-left entry) that $\varphi' = \mathrm{ad}_T(\varphi)^{-1}$ and (from the top-right entry) that

$$\varphi'(j_1 \times j_2) = \eta(\varphi(j_1) \times \varphi(j_2)).$$

Then $\varphi'(j^\sharp) = \eta\varphi(j)^\sharp$. Applying (10.7.6) yields that

$$N(\varphi(j)) = \frac{1}{6}T(\varphi(j), \varphi(j)^\sharp) = \frac{1}{6}T(\varphi(j), \eta^{-1}\varphi'(j^\sharp)) = \eta^{-1}N(j).$$

Conversely, if φ is a similitude of N with multiplier η^{-1}, then f defined by (10.7.7) with $\varphi' = \mathrm{ad}_T(\varphi)^{-1}$ is an automorphism not fixing $\mathrm{Skew}(B, -)$. □

10.8. Corollary. *If $(B, -) = M(J)$ for a separable cubic Jordan algebra J, then*

$$\mathbf{Aut}(B, -) \simeq \mathbf{Iso}(N_J) \rtimes \mathbb{Z}/2\mathbb{Z}$$

where $\mathbb{Z}/2\mathbb{Z}$ acts on $\mathbf{Iso}(N_J)$ by sending $\varphi \mapsto \hat{\varphi} = \mathrm{ad}_T(\varphi)^{-1}$.

Here, $\wedge$ is the automorphism of $\mathbf{Str}(J)$ defined in 5.6. This restricts to an automorphism of $\mathbf{Iso}(N_J)$ because $\mathbf{Iso}(N_J) \subset \mathbf{Sim}(N_J) = \mathbf{Str}(J)$ for all separable Jordan algebras J [91, VI. Theorem 7], and $\mathbf{Iso}(N_J)$ is stabilised by $\wedge$.

Proof. It follows from 10.6 and 10.7 that $\mathbf{Aut}(B, -)$ is a semidirect product of its subgroups $\mathbf{Aut}_S(B, -) \simeq \mathbf{Iso}(N_J)$ and $\langle\omega\rangle \simeq \mathbb{Z}/2\mathbb{Z}$. It is clear from the proof of 10.7 that conjugation by ω interchanges the roles of φ and $\varphi' = \mathrm{ad}_T(\varphi)^{-1}$. Further, it is known that $\mathrm{ad}_T(\varphi)^{-1} = \hat{\varphi}$ for $\varphi \in \mathrm{Iso}(N_J)$ [91, p. 246]. □

Quite analogously, one can prove the following:

10.9. Proposition. *Let $(B, -) = M(k)$ be the quartic 2×2 matrix algebra defined in 2.3. Then*

$$\mathbf{Aut}(B, -) \simeq \boldsymbol{\mu}_3 \rtimes \mathbb{Z}/2\mathbb{Z}$$

Proof sketch. Like in the proof of Theorem 10.7, one can show that if $f \in \mathbf{Aut}(B, -)(R)$ then there is a unique $\delta \in \boldsymbol{\mu}_3(R)$ such that either f or $f\omega$ is the map

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & \delta b \\ \delta^{-1}c & d \end{pmatrix}.$$ □

10.10. *The automorphism $\wedge$ for norm-isometry groups of cubic Jordan algebras.* We run through the cases of Theorem 10.5 as quickly as possible to identify the automorphism $\wedge : \varphi \mapsto \hat{\varphi} = \mathrm{ad}_T(\varphi)^{-1}$ of $\mathbf{Iso}(N_J)$. This is the legwork behind many of the footnotes in Table 5.

Let $J = k \times J'$ where $J' = J(V, q) = k \oplus V$. If $\varphi \in \mathrm{Iso}(N_J)$, then both φ and $\varphi^* = \mathrm{ad}_{T_J}(\varphi)$ are determined by their restrictions $g = \varphi|_{J'}$ and $g^* = \varphi^*|_{J'}$, which live in the structure group $\mathrm{Str}(J') = \mathrm{GO}(\langle -1\rangle \perp q)$. Since $T_J|_{J'} = T_{J'}$, we have also that $g^* = \mathrm{ad}_{T_{J'}}(g) = \hat{g}^{-1}$. One can further show that for $g \in \mathrm{Str}(J')$,

$$\hat{g} = \mu(g)^{-1}s_1 g s_1$$

where $s_1 : J' \to J'$ is the reflection $s_1(\alpha, v) = (\alpha, -v)$ for all $(\alpha, v) \in J'$ [95, p. 16].

If L is a cubic étale algebra, then $\hat{f} = f$ for all $f \in \mathbf{Aut}(L)$ and $\hat{L}_\ell = L_\ell^{-1} = L_{\ell^{-1}}$ for all $\ell \in L^\times$ (see Lemma 2.11), so with respect to the isomorphism $\mathbf{Iso}(N_{L/k}) \simeq \mathbf{G}_{m,L}^1 \rtimes \mathbf{Aut}(L)$, $\wedge$ fixes $\mathbf{Aut}(L)$ pointwise and inverts elements of the subgroup $\mathbf{G}_{m,L}^1$.

If $J = \mathcal{H}(M_3(k), \mathrm{ad}_b)$ for some bilinear form b, every norm-isometry of J is of the form $\phi(X) : P \mapsto XPX^*$ for some $X \in \mathbf{SL}_3(k^a)$ and $X^* = \mathrm{ad}_b(X)$. Then for $P, Q \in J$ we have

$$T_J(\phi(X)(P), Q) = \mathrm{Tr}(XPX^*Q) = \mathrm{Tr}(PX^*QX) = T_J(P, \phi(X^*)(Q)),$$

so $\phi(X)^* = \phi(X^*)$. In other words, $\wedge$ acts by $X \mapsto (X^*)^{-1}$ with respect to the isomorphism $\mathbf{SL}_3 \simeq \mathbf{Iso}(N_J)$.

If $J = C^+$ for a degree 3 central simple algebra C, every norm-isometry of J is of the form $\phi(X, Y) : P \mapsto XPY^{-1}$ for some $X, Y \in \mathbf{SL}_1(C)(k^a)$. We have for $P, Q \in C$,

$$T_J(\phi(X,Y)(P), Q) = \mathrm{Trd}_C(XPY^{-1}Q) = \mathrm{Trd}(PY^{-1}QX) = T_J(P, \phi(Y^{-1}, X^{-1})(Q)),$$

so $\wedge$ acts by $(X, Y) \mapsto (Y, X)$ with respect to the isomorphism $\mathbf{Iso}(\mathrm{Nrd}_C)^\circ \simeq (\mathbf{SL}_1(C) \times \mathbf{SL}_1(C))/\boldsymbol{\mu}_3$. If $C \simeq M_3(k)$ is split, then

$$\mathrm{Tr}(P^tQ) = \mathrm{Tr}(Q^tP) = \mathrm{Tr}(PQ^t),$$

which tells that $\wedge$ commutes with the transpose, and determines how $\wedge$ acts on the nonidentity component of $\mathbf{Iso}(\mathrm{Nrd}_C)$.

If $J = \mathcal{H}(B, \tau)$ for a degree 3 central simple algebra over a quadratic field extension F/k, this is quite similar to the previous case: with respect to the isomorphism $\mathbf{Iso}(N_J)^\circ \simeq R_{F/k}(\mathbf{SL}_1(B))/\boldsymbol{\mu}_{3[F]}$, $\wedge$ sends $S \mapsto \tau(S)^{-1}$ for all $S \in \mathrm{SL}_1(B)$, and commutes with the transpose if $(B, \tau) \simeq M_3(F)$.

If $J = \mathcal{H}(M_3(Q), \sigma)$ for a quaternion algebra Q and symplectic involution σ on $M_3(Q)$, then with respect to the isomorphism $\mathbf{Iso}(N_J) \simeq \mathbf{SL}_3(Q)$ the automorphism $\wedge$ is $X \mapsto \sigma(X)^{-1}$.

Finally, if J is an Albert algebra then $G = \mathbf{Iso}(N_J)$ is a simple group of type E_6. In this instance, $\mathbf{Aut}(J)$ is equal to the fixed point group $G^\wedge$ (not just contained in it) and it is a subgroup of type F_4. If k is separably closed, $T \subset \mathbf{Iso}(N_J)$ is a maximal torus stabilised by $\wedge$, and Δ is a system of simple roots for the root system $\Phi = \Phi(G, T)$, then (up to conjugation by some element of the Weyl group) $\wedge$ is precisely the automorphism of G defined by permuting the root groups U_α, $\alpha \in \Delta$, according to the nontrivial automorphism of the Dynkin diagram of Φ [56, Example 2.4].

10.11. *Derivations of $M(J, \eta)$.* Let $(B, -) = M(J, \eta)$ for a separable cubic Jordan algebra J. We have

$$\mathrm{Der}(B, -) = \mathrm{Lie}(\mathbf{Aut}(B, -)) = \mathrm{Lie}(\mathbf{Aut}(B, -)^\circ) = \mathrm{Lie}(\mathbf{Iso}(N_J)^\circ).$$

If k is separably closed, then in each respective case

$$\operatorname{Der}(B,-) \simeq \begin{cases} k\varepsilon \times \mathfrak{so}_n & J = k \times \mathcal{J}Spin_{n-1}\ (n \geq 3) \\ k\varepsilon \times k\varepsilon & J = k^3 \\ \mathfrak{sl}_3 & J = \mathcal{H}_3(k) \\ \mathfrak{sl}_3 \times \mathfrak{sl}_3 & J = M_3(k)^+ \\ \mathfrak{sl}_6 & J = \mathcal{H}_3(\mathbb{H}) \\ \mathfrak{e}_6 & J = \mathcal{H}_3(\mathbb{O}). \end{cases}$$

where $k\varepsilon$ is the one-dimensional abelian Lie algebra. This follows from Proposition 8.5 and Corollary 10.8, and Theorem 10.5. For the case $J = k \times \mathcal{J}Spin_{n-1}$, one needs to use $\operatorname{char}(k) \neq 3$ to justify the decomposition $\operatorname{Lie}(\mathbf{GO}_n) \simeq k\varepsilon \times \mathfrak{so}_n$.

10.12. Proposition. *If J is a separable cubic Jordan algebra and $(B,-) = M(J,\eta)$, then $D_{B,B} = \operatorname{Der}(B,-)$.*

Proof. We may assume that k is algebraically closed. First let $(B,-) = M(k^3)$ be a quartic Cayley algebra. We would like to show that $D_{B,B}$ is at least 2-dimensional, which would obviously imply $D_{B,B} = \operatorname{Der}(B,-)$. Lemma 3.3 lets us use the model

$$(B,-) = S(E,-,W,h) = E \oplus W$$

where $(E,-) = (M_2(k),{}^t)$ and (W,h) is the hermitian space $W = M_2(k)$, $h(x,y) = xy^t$. Let

$$e = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} f = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

One can check directly that $D_{(e,0),(f,0)}(E) \neq 0$ and $D_{(e,0),(f,0)}(W) = 0$. On the other hand, as in the proof of Proposition 8.5 8.5, if $w \neq \tau(w)$ then $D_{(0,1),(0,w)}(W) \neq 0$. Hence $D_{(e,0),(f,0)}$ and $D_{(0,1),(0,w)}$ are linearly independent.

Now let J be one of the cubic Jordan algebras of dimension ≥ 3, and $(B,-) = M(J)$. The ideal $D_{B,B} \subset \operatorname{Der}(B,-)$ is nonzero because $(B,-)$ contains a subalgebra $(B',-) \simeq M(k^3)$ [7, Corollary 9.3] and $D_{B',B'} \neq 0$ by the previous paragraph. If $J = \mathcal{H}_3(k)$, $\mathcal{H}_3(\mathbb{H})$, or $\mathcal{H}_3(\mathbb{O})$ we can conclude that $D_{B,B} = \operatorname{Der}(B,-)$ because $\mathfrak{sl}_3$, $\mathfrak{sl}_6$, and $\mathfrak{e}_6$ are all simple when $\operatorname{char}(k) \neq 2,3$.

If $J = M_3(k)^+$, then either $D_{B,B} = \operatorname{Der}(B,-)$ or $D_{B,B}$ is one of the two simple subalgebras isomorphic to $\mathfrak{sl}_3$. For any automorphism $f \in \operatorname{Aut}(B,-)$ we have

$$f^{-1} D_{f(x),f(y)} f = D_{x,y}$$

for all $x, y \in B$, so $f^{-1} D_{B,B} f = D_{B,B}$. There are automorphisms of $(B,-)$ that switch the copies of $\mathfrak{sl}_3$ in $\mathfrak{sl}_3 \times \mathfrak{sl}_3$ (see 10.10), so it must be that $D_{B,B} = \operatorname{Der}(B,-)$.

Now suppose $J = k \times \mathcal{J}Spin_{n-1}$ where $n \geq 3$, and let $(B,-) = M(J)$. Consider the ideal $D' = [\operatorname{Der}(B,-), \operatorname{Der}(B,-)]$; this is the simple subalgebra isomorphic to $\mathfrak{so}_n$. Looking at the isomorphism $\mathbf{Aut}(B,-)^\circ \xrightarrow{\sim} \mathbf{GO}_n^+$ from 10.5 and 10.7, it is clear that D' is the set of derivations of $(B,-)$ that annihilate the subspace

$$N = \left\{ \begin{pmatrix} a & (b,0) \\ (c,0) & d \end{pmatrix} : a,b,c,d \in k \right\}.$$

Consider the fixed-point subalgebra $M = V(J) = B^\omega$ where ω is the outer automorphism (10.6.1). Then $\overline{M} = M$ and there is an isomorphsim $F : (J, \mathrm{id}) \xrightarrow{\sim} (M, -)$ by Proposition 10.2. Clearly $D_{M,M} \subset \mathrm{Der}(B, -)^\omega$, where

$$\mathrm{Der}(B, -)^\omega = \{d \in \mathrm{Der}(B, -) \colon \omega d \omega = d\}.$$

Moreover, $(\mathbf{Aut}(A, -)^\circ)^\omega$ fixes N pointwise (see 10.10), so $\mathrm{Der}(B, -)^\omega$ is contained in D'. We have $D_{M,M}(M) \subset M$, so $D_{M,M}$ maps surjectively onto $D_{J,J}$ by $D_{m_1,m_2} \mapsto F^{-1} \circ D_{m_1,m_2} \circ F = D_{F^{-1}(m_1),F^{-1}(m_2)}$. But $D_{J,J} = \mathrm{Der}(J) \simeq \mathfrak{so}_{n-1} \neq 0$ [95, p. 36]. This implies $0 \neq D_{M,M} \subset D' \cap D_{B,B}$ and so $D' \subset D_{B,B}$.

Since $\dim(D') = \dim(\mathrm{Der}(B, -)) - 1$, the proof is over once we show that $D' \neq D_{B,B}$. Conveniently, N is just an associative algebra:

$$N \xrightarrow{\sim} M_2(k), \qquad \begin{pmatrix} a & (b,0) \\ (c,0) & d \end{pmatrix} \longmapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is an isomorphism. (This has to do with the fact that $(c, 0)^\sharp = 0$ for $c \in k$; see [129, Example 2.2].) Also $\overline{N} = N$, and the involution is orthogonal (adjoint to the hyperbolic form). Like in the first paragraph of the proof, it is easy to produce some elements $n_1, n_2 \in N$ such that $D_{n_1,n_2}(N) \neq 0$, implying $D_{N,N} \not\subset D'$. □

11. Automorphism and structure groups: the split forms

The data collected so far in this chapter is much easier to process and remember if we focus on the split cases. The following theorem summarises the situation.

11.1. Theorem. *Let $(A, -)$ be a central simple structurable algebra that is exceptional, a $(4, m)$-product algebra, or an octonion algebra with nonstandard involution. Then $\mathbf{Aut}(A, -)^\circ$ is a reductive group and, if this is split, $\mathbf{Aut}(A, -)$ is isomorphic to the group in column 1 of Table 5.*

Proof. In all the cases mentioned, we have shown that $\mathbf{Aut}(A, -)^\circ$ is reductive (in most cases it is even semisimple). We have also demonstrated in each case that $\mathbf{Aut}(A, -) \to \pi_0(\mathbf{Aut}(A, -))$ has a canonical section whenever $\mathbf{Aut}(A, -)^\circ$ is split.

For quaternion algebras, octonion algebras, Smirnov algebras, and other tensor products of composition algebras, the facts can be found in 8.4, 9.2, 9.3, 9.11, and 9.12. For the automorphism group of the Albert algebra, see [161, §14.24]. The last six entries are proved in 10.5–10.10. □

11.2. *Automorphism groups of TKK Lie algebras.* Let $(A, -)$ be a central simple structurable algebra, and let $G = \mathbf{Aut}(K(A, -))$. Then G° is an adjoint absolutely simple algebraic group (see Theorem 7.1) and it has k-rank ≥ 1 because it receives a grading cocharacter $\lambda : \mathbf{G}_m \to G$ (see 4.1).

The quotient of $G(k^s)$ by $G^\circ(k^s)$ is isomorphic to the automorphism group of the Dynkin diagram of $K(A, -)$ [166, 4.7]. This implies that:

$\pi_0(G) \simeq \mathbb{Z}/2\mathbb{Z}$ if G has type A_n $(n \geq 2)$, D_n $(n \neq 4)$, or E_6.

$\pi_0(G)$ is an étale group scheme of order 6 (and $\pi_0(G) \times_k k^s \simeq S_3$) if G has type D_4.

$\pi_0(G) = 1$ otherwise.

Moreover, if G is split then the connected-étale sequence $G^\circ \to G \to \pi_0(G)$ splits because the Chevalley Lie algebra $[\mathrm{Lie}(G), \mathrm{Lie}(G)] \simeq K(A, -)$ admits "diagram automorphisms" [166, p. 1122].

11.3. Theorem. *Let $(A, -)$ be a central simple structurable algebra that is exceptional, a $(4, m)$-product algebra, or an octonion algebra with nonstandard involution. If $G = \mathbf{Aut}(K(A, -))^\circ$ is split, it is isomorphic to the group in column* 3 *of Table* 5.

Proof. In light of the discussion above, the work of determining these groups is essentially done way back in Theorem 5.3, because it is easy to look up, say in [101, §25], what the split adjoint groups are. □

11.4. *Structure groups revisited.* Now let $H = \mathbf{Aut}_{\mathrm{gr}}(K(A, -))$. By Lemmas 4.2 and 5.5, there is an isomorphism

$$H = C_G(\lambda) \simeq \mathbf{Str}(A, -)$$

where λ is the grading cocharacter. The connected group $C_G(\lambda)^\circ = C_{G^\circ}(\lambda) \simeq \mathbf{Str}(A, -)^\circ$ is reductive. Its derived subgroup

$$M = (\mathbf{Str}(A, -)^\circ)^{\mathrm{der}}$$

is semisimple. We name this subgroup the *semisimple structure group.*

11.5. Proposition. *Suppose $(A, -)$ is a central simple structurable algebra, let $G = \mathbf{Aut}(K(A, -))$, and let $H = \mathbf{Aut}_{\mathrm{gr}}(K(A, -))$. Then $\pi_0(H) \simeq \pi_0(G)$.*

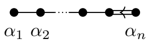
Proof. The labelled Dynkin diagram of $K(A, -)$ is preserved by all automorphisms of the underlying unlabelled Dynkin diagram [165, Theorem 5.11]. This implies that the diagram automorphisms (which are defined over k^s) centralise λ and that each of the connected components of G has nonempty intersection with $H = C_G(\lambda)$. By Lemma 4.2, the group $H \cap G^\circ = C_{G^\circ}(\lambda) = H^\circ$ is connected, so the inclusion map $H \to G$ induces an isomorphism $\pi_0(H) \simeq \pi_0(G)$. □

11.6. Theorem. *Let $(A, -)$ be a central simple structurable algebra that is exceptional, a $(4, m)$-product algebra, or an octonion algebra with nonstandard involution. If its semisimple structure group is split, it is isomorphic to the group in column* 2 *of Table* 5.

Proof. Let us change the notation from the previous proposition, and now write $G = \mathbf{Aut}(K(A, -))^\circ$ and $M = C_G(\lambda)^{\mathrm{der}}$. Recall from 4.8 that the Dynkin diagram of M is determined by deleting the positively labelled vertices from the labelled Dynkin diagram of (G, λ). These diagrams are available in Table 4. If G is simply connected, so is M [162, Exercise 8.4.6 (6)]. Adjoint groups of types G_2, F_4, and E_8 are simply connected, so this determines M for octonion algebras, $(8, 8)$-product algebras, quartic 2×2 matrix algebras, green algebras, and Brown algebras. (For information on split simply connected simple groups and their centres, see [101, (25.9)–(25.14)] or [122, §25].)

If G is not simply connected, let $\tilde{G}$ be its simply connected cover and $\pi: \tilde{G} \to G$ the universal covering isogeny. The kernel of π is $Z = Z(\tilde{G})$. There is a unique simply connected semisimple subgroup $\tilde{M} \subset \tilde{G}$ such that $\pi(\tilde{M}) = M$. We know what $\tilde{M}$ is, so to determine M we just need to work out how $Z \cap \tilde{M}$ sits inside $Z(\tilde{M})$. If $(A, -)$ is a Smirnov algebra then $\tilde{G} \simeq E_7^{\mathrm{sc}}$, $Z \simeq \boldsymbol{\mu}_2$, $\tilde{M} \simeq \mathbf{SL}_7$, and $Z(\tilde{M}) \simeq \boldsymbol{\mu}_7$. Then $Z \cap \tilde{M} = 1$ because 7 is prime to 2, so $M = \tilde{M}$. For $(8,2)$-product algebras too, $M = \tilde{M} \simeq \mathbf{Spin}_8$; the proof rests on the fact that 3, the order of $Z(E_6^{\mathrm{sc}})$, is prime to 4, the order of $Z(\mathbf{Spin}_8)$. The case of the Albert algebra is extremely well-known, and already featured in the proof of Theorem 10.5: we have $M = \tilde{M} \simeq E_6^{\mathrm{sc}}$.

Suppose $(A, -)$ is a quaternion algebra. Then $G \simeq \mathbf{PGSp}_6$ is of type C_3, $\tilde{G} \simeq \mathbf{Sp}_6$, $Z \simeq \boldsymbol{\mu}_2$, $\tilde{M} \simeq \mathbf{SL}_2 \times \mathbf{SL}_2$, and $Z(\tilde{M}) = \boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2$. The Dynkin diagram of type C_n is



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There is a nice expression for the centre of $\mathbf{Sp}_{2n}$ in terms of Chevalley generators (see [66, Example 8.4], where the notation is also explained):

$$Z(\mathbf{Sp}_{2n})(k) = \Big\{1, \prod_{\substack{i=1 \\ i \text{ odd}}}^{n} h_{\alpha_i}(-1)\Big\}.$$

Now the two simple $\mathbf{SL}_2$ subgroups of $\tilde{M}$ are generated by $\{x_{\alpha_1}(c)\colon c \in k\}$ and $\{x_{\alpha_3}(c)\colon c \in k\}$ respectively, and $Z(\tilde{M})(k)$ is generated by $h_{\alpha_1}(-1)$ and $h_{\alpha_3}(-1)$. Hence $(Z \cap \tilde{M})(k) = \{1, h_{\alpha_1}(-1)h_{\alpha_3}(-1)\}$. This makes it clear that $Z \cap \tilde{M} \simeq \boldsymbol{\mu}_2$ is embedded diagonally into $\tilde{M} \simeq \mathbf{SL}_2 \times \mathbf{SL}_2$, i.e., by $\pm 1 \mapsto (\pm 1, \pm 1)$.

If $(A, -)$ is an octonion algebra with nonstandard involution, then $G \simeq \mathbf{PGSp}_8$ is of type C_4. Just like in the previous paragraph, we find that $M \simeq (\mathbf{SL}_2 \times \mathbf{Sp}_4)/\boldsymbol{\mu}_2$ where $\boldsymbol{\mu}_2$ is embedded diagonally in $Z(\mathbf{SL}_2 \times \mathbf{Sp}_4) \simeq \boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2$.

For the last few rows of Table 5, direct references are available to tell us what M is. If $(A, -)$ is the quartic Cayley algebra, M is isomorphic to $\mathbf{SL}_2 \times \mathbf{SL}_2 \times \mathbf{SL}_2$ modulo the subgroup $\{(c_1, c_2, c_3)\colon c_i = \pm 1, c_1c_2c_3 = 1\} \simeq \boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2$ [162, 17.9.3 (a)]. This can also be checked using the Chevalley generators of the centre of $\mathbf{Spin}_8$ [66, Example 8.6]. If $(A, -)$ is a blue algebra, then M is isomorphic to $\mathbf{SL}_6/\boldsymbol{\mu}_3$ [162, 17.7.2]. If $(A, -)$ is a red algebra, then [162, 17.8.3 (a)] implies $M \simeq \mathbf{HSpin}_{12}$ (the image of $\mathbf{Spin}_{12}$ in one of its half-spin representations).

In §18, the structure groups of arbitrary bicomposition algebras are computed from first principles; the results are in Table 6. It is a little subtle to compute them using the elementary methods above. And unfortunately, [162, Chapter 17] is not free of mistakes: for instance, it seems to imply that M for $(8,4)$-product algebras is $\mathbf{SO}_{10} \times \mathbf{PGL}_2$. (This is not right because $\ker(E_7^{\mathrm{sc}} \to E_7^{\mathrm{ad}})$ has order 2 while the isogeny $\tilde{M} = \mathbf{Spin}_{10} \times \mathbf{SL}_2 \to \mathbf{SO}_{10} \times \mathbf{PGL}_2$ has kernel of order 4.) On the contrary, M is the quotient of $\tilde{M}$ by the subgroup $\boldsymbol{\mu}_2$ embedded diagonally in $Z(\tilde{M}) \simeq \boldsymbol{\mu}_4 \times \boldsymbol{\mu}_2$. So for these reasons I postpone the remaining part of proof until Theorem 18.14, where it is done thoroughly and in a more general setting. □

11.7. *The colourful series.* It is interesting to give some context to the "colourful series" which I have called the green, blue, red, and Brown algebras. The connected automorphism groups, semisimple structure groups, and connected automorphism

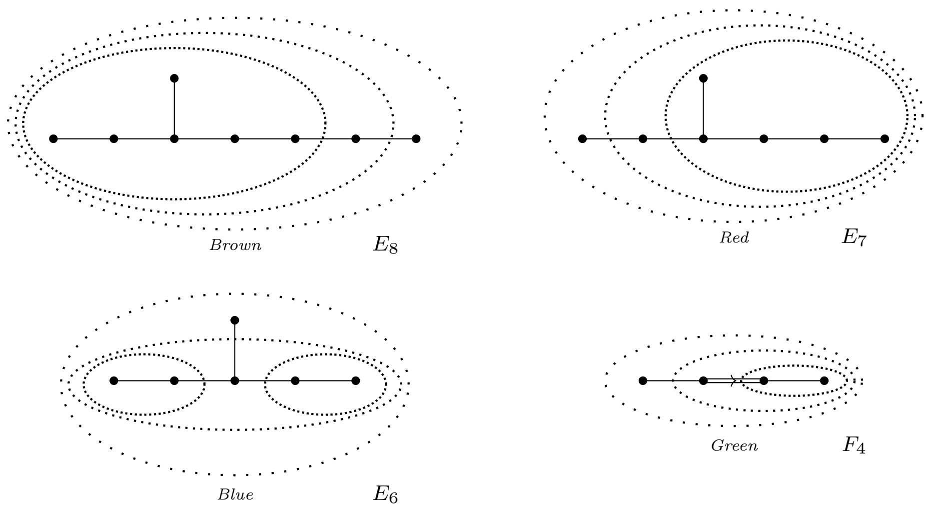

FIGURE 1: Nested Dynkin diagrams of algebraic groups $\mathbf{Aut}(B,-) \subset (\mathbf{Str}(B,-)^\circ)^{\mathrm{der}} \subset \mathbf{Aut}(K(B,-))^\circ$ for certain exceptional structurable algebras $(B,-)$ of skew-dimension one.

groups of their TKK Lie algebras can be visualised as a series of nested Dynkin diagrams – see Figure 1. One paper that began to draw attention to this phenomenon is [55].

In [58, §12], the representations corresponding to the colourful series are constructed internally from certain overgroups (the groups in the third column of Table 5) and these representations are called internal Chevalley modules. The references in [58, §12] give some further historical context from various perspectives in algebraic group theory, nonassociative algebra, and representation theory.

These representations of the structure groups are examples of *prehomogeneous vector spaces*, and this is another source from which they have independently attracted interest. (In fact, any central simple structurable algebra is a prehomogeneous vector space for its connected structure group; see Lemma 13.6 (i).) The papers [43] and [103] classify the orbits of these prehomogeneous spaces and explore interesting connections to geometry and number theory.

11.8. *Quasi-split semisimple structure groups.* Simple algebraic groups with symmetric Dynkin diagrams have outer automorphisms. The presence of these outer automorphisms means that the split form of the full structure group does much more twisting than the semisimple structure group can. In other words, the semisimple structure group does not control all of the twisted isotopes of a structurable algebra.

Recall that a semisimple algebraic k-group G is called *quasi-split* if it has a maximal torus T and a system of simple roots $\Pi \subset \Phi(G,T)$ which is stabilised by the natural action of the absolute Galois group Γ_k [101, 27.C]. Then there is a homomorphism $\Gamma_k \to \mathrm{Sym}(\mathrm{Dyn}(G,T))$ which defines a quadratic étale extension F/k if G is of type A_n ($n \geq 2$), D_n ($n \neq 4$), or E_6, or a cubic étale extension L/k if G is of type D_4. The group is split if and only if the action of Γ_k on $\mathrm{Dyn}(G,T)$ is trivial.

We again use the notation from [101, p. 418] for the twisted group of n-th roots of unity (this group often turns up in the centres of quasi-split simply connected semisimple algebraic groups):

$$\boldsymbol{\mu}_{n[E]} = \ker\left(R_{E/k}(\boldsymbol{\mu}_{n,E}) \xrightarrow{N_{E/k}} \boldsymbol{\mu}_n \right).$$

11.9. Proposition. *Let $(A,-)$ be any form of one of the algebras from Table 5. If its semisimple structure group M is quasi-split but not split, then either:*

(i) *$(A,-)$ is a $(4,2)$-product algebra and*

$$M \simeq \frac{R_{F/k}(\mathbf{SL}_2 \times_k F) \times \mathbf{SL}_2}{\boldsymbol{\mu}_2}.$$

(ii) *$(A,-)$ is a $(4,4)$-product algebra and*

$$M \simeq \frac{\mathbf{SL}_4 \times \mathbf{SO}(N_{F/k} \perp \mathbb{H})}{\boldsymbol{\mu}_2}$$

(iii) *$(A,-)$ is an $(8,2)$-product algebra and*

$$M \simeq \mathbf{Spin}(N_{F/k} \perp 3\mathbb{H})$$

(iv) *$(A,-)$ is a quartic Cayley algebra and*

$$M \simeq \frac{R_{L/k}(\mathbf{SL}_2 \times_k L)}{\boldsymbol{\mu}_2}$$

(v) *$(A,-)$ is a blue algebra and*

$$M \simeq \frac{\mathbf{SU}_{6,F}}{\boldsymbol{\mu}_{3[F]}}.$$

where the F's are quadratic field extensions and the L is a nonsplit cubic étale extension (in both cases unique for M).

Proof. The quasi-split forms of simply connected absolutely simple algebraic groups of type A_n and D_n ($n \neq 4$) are described in [101, 27.C]. If the action of Γ_k permutes irreducible components of the Dynkin diagram, then the quasi-split form is a Weil restriction of a split group of the same type [173, 3.1.2].

For (i) it is visible that a symmetry of the labelled Dynkin diagram of type A_5 fixes one of the $\mathbf{SL}_2$ subgroups (corresponding to vertex 3) and permutes the other two $\mathbf{SL}_2$ subgroups. Also note that $\mathbf{SL}_2$ has no nontrivial symmetries in its one-vertex Dynkin diagram, so it is not affected by this Galois action (see [101, VI. Exercise 11]). For (ii), the $\mathbf{SL}_4$ subgroup is not affected by the diagram automorphism, only the $\mathbf{SO}_4$ subgroup is. In (iv), the three $\mathbf{SL}_2$ subgroups are permuted by symmetries of the Dynkin diagram, but the central subgroup $\boldsymbol{\mu}_2$ is stabilised and hence fixed because it has no automorphisms. For (v), the symmetry of the Dynkin diagram acts nontrivially on the $\boldsymbol{\mu}_3$ lying in the centre of $\mathbf{SL}_6$ and so the corresponding central subgroup of $\mathbf{SU}_{6,F}$ is $\boldsymbol{\mu}_{3[F]}$. □

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Chapter IV

Galois cohomology

The contents of this chapter are mostly well-known to experts, or at least intuitive. The purpose is mainly to lay down consistent definitions and necessary foundations. Section 13 is the most original of the work done in this chapter.

12. Introduction to Galois cohomology with examples

This section contains a rudimentary introduction to Galois cohomology, with some examples pertaining to structurable algebras.

12.1. *Nonabelian Galois cohomology.* Recall that $\Gamma_k = \mathcal{G}al(k^s/k)$ is the absolute Galois group of k. It is a profinite group, so it carries a locally compact totally disconnected topology. A Γ_k*-group* is a discrete group M with a continuous action

$$\Gamma_k \times M \to M \qquad\qquad (\sigma, m) \to \sigma \cdot m$$

by group automorphisms. The nonabelian cohomology sets

$$\begin{aligned} H^0(k, M) &= M^{\Gamma_k} \\ H^1(k, M) &= H^1(\Gamma_k, M) \end{aligned}$$

are defined as in [156, I.§5.1]. Briefly, a *cocycle* is a continuous map

$$\begin{aligned} a : \Gamma_k &\to M \\ \sigma &\mapsto a_\sigma, \end{aligned}$$

satisfying

$$a_{\sigma\tau} = a_\sigma \sigma{\cdot}a_\tau \qquad\qquad \text{for all } \sigma, \tau \in \Gamma_k.$$

The set of cocycles $\Gamma_k \to M$ is denoted by $Z^1(k, M)$. We say $a, a' \in Z^1(k, M)$ are *cohomologous* and write $a \sim a'$ if

$$a'_\sigma = b^{-1} a_\sigma \sigma{\cdot}b \qquad\qquad \text{for some } b \in M.$$

By definition, $H^1(k, M)$ is the set of cohomology classes

$$H^1(k, M) = Z^1(k, M)/\sim .$$

If G is a smooth algebraic group over k, then $G(k^s)$ is naturally a Γ_k-group. We write

$$H^d(k, G) = H^d(k, G(k^s)) \qquad\qquad d = 0, 1.$$

In this notation,

$$\begin{aligned} &H^0(k, G) = G(k) \\ &H^1(k, G) = \{\text{equivalence classes of cocycles } \Gamma_k \to G(k^s)\}. \end{aligned}$$

12.2. *Abelian Galois cohomology.* An abelian Γ_k-group M is called a Γ_k-module, or a discrete Galois module. For example, if M is a commutative algebraic k-group, $M(k^s)$ is a Γ_k-module.

If M is a Γ_k-module, the abelian cohomology groups

$$H^d(k, M) = H^d(\Gamma_k, M)$$

are defined for all $d \geq 0$. By definition,

$$H^d(k, M) = \varinjlim H^d(\mathcal{G}al(L/k), M^{\Gamma_L})$$

where $H^d(\mathcal{G}al(L/k), M^{\Gamma_L})$ is the d-th cohomology group (in the sense of ordinary finite group cohomology) and the limit ranges over all finite Galois extensions L/k.

If M and N are Γ_k-modules, their tensor product $M \otimes N = M \otimes_{\mathbb{Z}} N$ is a Γ_k-module too. For all i, j there is an associative product operation called the *cup product* [72, §3.4],

$$\begin{aligned} \cup : H^i(k, M) \times H^j(k, N) &\to H^{i+j}(k, M \otimes N) \\ (\mu, \nu) &\mapsto \mu \cup \nu. \end{aligned}$$

The cup product is graded commutative:

$$\mu \cup \nu = (-1)^{ij}(\nu \cup \mu).$$

12.3. *Functoriality.* Galois cohomology is functorial in both arguments. Any morphism of Γ_k-groups $g : M \to N$ induces a morphism

$$g_* : H^d(k, M) \to H^d(k, N).$$

If G is a smooth algebraic k-group and L/k is any field extension, one can define $H^d(L, G) = H^d(L, G \times_k L)$ for $d = 0, 1$. If $L_1/k \to L_2/k$ is any morphism of field extensions, there is a restriction map

$$\operatorname{res}_{L_2/L_1} : H^d(L_1, G) \to H^d(L_2, G)$$

that makes $H^0(*, G)$ and $H^1(*, G)$ into (covariant) functors

$$\begin{aligned} H^0(*, G) &: \mathsf{Fields}_{/k} \to \mathsf{Groups} \\ H^1(*, G) &: \mathsf{Fields}_{/k} \to \mathsf{Sets}_* \end{aligned}$$

where $\mathsf{Fields}_{/k}$ is the category of field extensions of k and Sets_* is the category of pointed sets [156, II.§1.1]. The distinguished point in $H^1(k, G)$ is the class of the trivial cocycle: $a_\sigma = 1$ for all $\sigma \in \Gamma_k$.

If G is commutative, then $H^d(*, G) : \mathsf{Fields}_{/k} \to \mathsf{Abelian\ Groups}$ is a functor for all $d \geq 0$, not just $d = 0, 1$.

12.4. *Principal homogeneous spaces.* Let G be a smooth algebraic k-group. A principal homogeneous space for G (or *G-torsor*) is a scheme $P \neq \emptyset$ over k with a right action $P \times G \to P$ such that the map

$$P \times G \to P \times P \qquad\qquad (p, g) \mapsto (p, pg)$$

is an isomorphism of schemes [122, Definition 2.66]. A G-torsor P is isomorphic to G if and only if $P(k) \neq \emptyset$; we call these torsors trivial.

The set $H^1(k, G)$ is naturally isomorphic to the set of k-isomorphism classes of G-torsors; see [101, Proposition 28.14], [122, Proposition 3.50], or [156, I.§5.2]. The trivial class in $H^1(k, G)$ corresponds to the trivial G-torsor.

12.5. *Twisting an algebra by a cocycle.* Let W be a k-algebra, $G = \mathbf{Aut}(W)$, and let K/k be a separable field extension. A *K/k-form* of W is a k-algebra W' such that $W'_K \simeq W_K$ as algebras. Denote by $E(K/k)$ the set of k-isomorphism classes of K/k-forms of W.

The Galois group $\mathcal{G}al(K/k)$ acts on W_K by k-algebra automorphisms. Given a cocycle $a \in Z^1(\mathcal{G}al(K/k), G(K))$, one can define a twisted action of G on W_K by

$$\sigma * v = a_\sigma(\sigma \cdot v).$$

Denote by ${}_aW_K$ the $\mathcal{G}al(K/k)$-module of this twisted action. The fixed-point subspace

$$({}_aW_K)^{\mathcal{G}al(K/k)}$$

is a k-subalgebra of W_K and it is a K/k-form of W. The process of going from W to $({}_aW_K)^{\mathcal{G}al(K/k)}$ is called *twisting by a*.

If $b \in Z^1(\mathcal{G}al(K/k), G(K))$ is cohomologous to a then ${}_aW_K$ and ${}_bW_K$ are isomorphic $\mathcal{G}al(K/k)$-modules. In other words, twisting by cohomologous cocycles results in isomorphic twisted forms. This twisting construction determines an isomorphism

$$\begin{aligned} H^1(\mathcal{G}al(K/k), G(K)) &\xrightarrow{\sim} E(K/k) \\ [a] &\mapsto [({}_aW_K)^{\mathcal{G}al(K/k)}] \end{aligned}$$

For more details and complete proofs, see for instance [72, §2.3]. In particular, there is an isomorphism

$$H^1(k, G) \xrightarrow{\sim} E(k^s/k).$$

The G-torsor corresponding to $[W'] \in E(k^s/k) \simeq H^1(k, G)$ is the affine variety $\mathrm{Isom}(W, W')$ whose points are given by the representable functor

$$R \mapsto \mathrm{Isom}_R(W_R, W'_R)$$

where $\mathrm{Isom}_R(W_R, W'_R)$ is the set of R-isomorphisms $W_R \to W'_R$.

The situation described here works not only for algebras, but also for many other kinds of algebraic structures. Say, if $(W, -)$ is an algebra with involution, then

$H^1(k, \mathbf{Aut}(A, -))$ classifies up to k-isomorphism the k-algebras with involution that become isomorphic to $(W, -)$ over k^s. Or, if $P \in k[V]$ is a polynomial function on a vector space V, then $H^1(k, \mathbf{Iso}(P))$ classifies the $\mathbf{GL}(V)(k)$-orbits of polynomials $P' \in k[V]$ that are in the same $\mathbf{GL}(V)(k^s)$-orbit as P.

12.6. Example. Let $(A, -)$ be a structurable algebra, and let $L = K(A, -, (1, 1, 1))$. Then L has a V_4-grading, so there is a homomorphism $V_4 \to \mathbf{Aut}(L)$ (see 6.10). Precomposing it with

$$\begin{aligned}(\mathbb{Z}/2\mathbb{Z})^3 &\longrightarrow (\mathbb{Z}/2\mathbb{Z})^2 = V_4\\ (x, y, z) &\longmapsto (x - y, y - z).\end{aligned}$$

gives a homomorphism

$$t : (\mathbb{Z}/2\mathbb{Z})^3 \to \mathbf{Aut}(L).$$

Since $(\mathbb{Z}/2\mathbb{Z})^3$ is a constant algebraic group, the Galois action is trivial and a cocycle

$$b \in Z^1(k, (\mathbb{Z}/2\mathbb{Z})^3)$$

is just a group homomorphism $b : \Gamma_k \to (\mathbb{Z}/2\mathbb{Z})^3$. Composing it with t, we can view b as a cocycle in $Z^1(k, \mathbf{Aut}(L))$ and consider the Lie algebra L' obtained when you twist L by b.

Since $\mathrm{char}(k) \neq 2$ we have $\mathbb{Z}/2\mathbb{Z} \simeq \boldsymbol{\mu}_2$, and can use Kummer Theory [72, Proposition 4.3.6] to canonically identify the abelian groups

$$H^1(k, \mathbb{Z}/2\mathbb{Z}) \simeq H^1(k, \boldsymbol{\mu}_2) \simeq k^\times/k^{\times 2}.$$

The following proposition describes the map

$$t_* : H^1(k, \mathbb{Z}/2\mathbb{Z}^3) \to H^1(k, \mathbf{Aut}(L)).$$

12.7. Proposition. *Suppose the cocycle $b \in Z^1(k, (\mathbb{Z}/2\mathbb{Z})^3)$ corresponds to the triple $(\gamma_1, \gamma_2, \gamma_3) \in (k^\times/k^{\times 2})^3$. Then the Lie algebra L' (i.e., L twisted by b) is isomorphic to $K(A, -, (\gamma_1, \gamma_2, \gamma_3))$.*

Proof. It can be checked directly by following the definition of twisting, choosing a suitable basis for the twisted algebra L', and writing down an isomorphism. ☐

We can make a connection here to Allison's result (see 6.6) that $K(A, -, \gamma) \simeq K(A, -, \gamma')$ if $\langle \gamma_1, \gamma_2, \gamma_3\rangle$ and $\langle \gamma_1', \gamma_2', \gamma_3'\rangle$ are similar quadratic forms. The proof of this result in [8, Proposition 4.1] rests implicitly on the fact that there is an intermediate subgroup $H \simeq \mathbf{O}^+(3)$ such that $V_4 \subset H \subset \mathbf{Aut}(L)$. The result follows from the observation that $H^1(k, V_4) \to H^1(k, \mathbf{Aut}(L))$ factors through $H^1(k, \mathbf{O}^+(3))$.

12.8. *Cohomology and short exact sequences.* While there are plenty of references on the effects of applying Galois cohomology to exact sequences, it is worth recapping some details. An exact sequence of smooth algebraic groups

$$1 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 1 \qquad (12.8.1)$$

gives rise to a map δ and an exact sequence of pointed sets [156, I.§5 Proposition 38]:

$$1 \longrightarrow A(k) \longrightarrow B(k) \longrightarrow C(k) \xrightarrow{\delta} H^1(k,A) \xrightarrow{f_*} H^1(k,B) \xrightarrow{g_*} H^1(k,C). \quad (12.8.2)$$

If A is abelian and $f(A) \subset Z(B)$, there is a map $\Delta : H^1(k,C) \to H^2(k,A)$ and appending this to (12.8.2) gives a slightly longer exact sequence [156, I.§5.6 Proposition 42]. We call δ and Δ the first and second connecting maps, respectively.

The group $C(k)$ acts on $H^1(k,A)$ from the right. Given a point $z \in C(k)$, lift it to a point $y \in B(k^s)$ such that $g(y) = z$. There is a cocycle $a' \in Z^1(k,A)$ such that $\sigma{\cdot}y = ya'_\sigma$ for all $\sigma \in \Gamma_k$. Now to define the action of $C(k)$, for each cocycle $a \in Z^1(k,A)$, set

$$[a]{\cdot}z = [b] \in H^1(k,A)$$

where $b \in Z^1(k,A)$ is the cocycle

$$b_\sigma = y^{-1}a_\sigma \sigma{\cdot}y = y^{-1}a_\sigma y a'_\sigma.$$

The main reason to care about this action is to understand the kernels of f^* and g^*. The point is that $\ker(f_*)$ is the orbit of the basepoint in $H^1(k,A)$, and $\ker(g_*)$ can be identified with the set of orbits of $H^1(k,A)$ by the group $C(k)$ [156, I.§5.5 Proposition 39].

If g has a section $s : C \to B$, then g_* also a section $s_* : H^1(k,C) \to H^1(k,B)$. Also, $C(k)$ has a right action on $A(k^s)$ by Γ_k-automorphisms:

$$a \cdot z = s(z)^{-1}as(z)$$

for $a \in A(k^s)$ and $z \in C(k)$. Since $H^1(k,A)$ is functorial in A, this induces a right action of $C(k)$ on $H^1(k,A)$ by basepoint-preserving bijections (see [25, Example II.3.20]). This action is the same as the one from the previous paragraph.

12.9. *Twisting a short exact sequence.* If $b \in Z^1(k,B)$ is a cocycle, then the sequence (12.8.1) can be twisted to obtain a new exact sequence of algebraic groups

$$1 \longrightarrow {}_bA \xrightarrow{f_b} {}_bB \xrightarrow{g_b} {}_bC \longrightarrow 1$$

where for $G = A$, B, or C we define ${}_bG$ to be the smooth algebraic group such that ${}_bG(k^s) = G(k^s)$ with a different action of Γ_k, namely:

$$\sigma * x = b_\sigma \sigma{\cdot}x b_\sigma^{-1}$$

for all $x \in G(k^s)$. We have canonical bijections

$$\tau_b : Z^1(k, {}_bG) \to Z^1(k,G)$$

for $G = B$ or C, *but not* A, defined by $\tau_b(g)_\sigma = g_\sigma b_\sigma$ for all $g \in Z^1(k, {}_bG)$. Since they are compatible with cohomology, these maps define bijections $\tau_b : H^1(k, {}_bG) \to H^1(k,G)$. These maps τ_b are not basepoint-preserving. The following diagram of sets commutes [156, I.§5.3–5.4]:

$$\begin{tikzcd}
H^1(k, {}_bA) \arrow[r, "(f_b)_*"] & H^1(k, {}_bB) \arrow[r, "(g_b)_*"] \arrow[d, "\tau_b"] & H^1(k, {}_bC) \arrow[d, "\tau_b"] \\
H^1(k,A) \arrow[r, "f_*"] & H^1(k,B) \arrow[r, "g_*"] & H^1(k,C).
\end{tikzcd}$$

Since $\ker((g_b)_*) = \tau_b^{-1}(g_*^{-1}([b]))$, one of the main insights gained by twisting is that the fibre of g_* over $g_*([b])$ is isomorphic to the orbit of the basepoint of $H^1(k, {}_bA)$ under the action of $C_b(k)$, as per 12.8.

12.10. *The Brauer group.* This subsection is about unital associative central simple algebras.Two central simple algebras A, B are called Brauer equivalent (or Morita equivalent) if there is a pair of positive integers m, n such that $M_m(A) \simeq M_n(B)$. The *Brauer group* $\mathrm{Br}(k)$ is the abelian group of Brauer equivalence classes of central simple algebras over k, with the operation $[A] + [B] = [A \otimes B]$. There is a canonical isomorphism [72, Theorem 4.4.7]

$$H^2(k, \mathbf{G}_m) \simeq \mathrm{Br}(k).$$

By definition, the *exponent* of A is the order of $[A]$ in $\mathrm{Br}(k)$. The exponent is a divisor of the index, and these two numbers have the same prime factors [94, p. 497].

12.11. Example (Groups of type A_ℓ)**.** Since $\mathbf{PGL}_n = \mathbf{Aut}(M_n(k))$, we can identify $H^1(k, \mathbf{PGL}_n)$ with the set of central simple algebras of degree n, up to isomorphism. If n is prime to $\mathrm{char}(k)$, there are canonical isomorphisms

$$H^1(k, \mathbf{PGL}_n) \xrightarrow[\Delta]{\sim} H^2(k, \boldsymbol{\mu}_n) \xrightarrow{\sim} {}_n\,\mathrm{Br}(k).$$

where ${}_n\,\mathrm{Br}(k) \subset \mathrm{Br}(k)$ is the n-torsion part of the Brauer group, Δ is the second connecting map associated to the short exact sequence $\boldsymbol{\mu}_n \to \mathbf{SL}_n \to \mathbf{PGL}_n$, and the final map is induced by the inclusion $\boldsymbol{\mu}_n \subset \mathbf{G}_m$ [72, Corollary 4.4.9].

Let A be an associative central simple algebra of degree $n = \ell + 1$. Since A can be realised as $M_n(k)$ twisted by some cocycle in $Z^1(k, \mathbf{PGL}_n)$, there is a canonical isomorphism of sets $H^1(k, \mathbf{PGL}_n) \to H^1(k, \mathbf{PGL}_1(A))$ that shifts the basepoint. The same does not apply when $\mathbf{PGL}_n$ is replaced by $\mathbf{SL}_n$.

The exact sequence

$$1 \longrightarrow \mathbf{SL}_1(A) \longrightarrow \mathbf{GL}_1(A) \xrightarrow{\mathrm{Nrd}} \mathbf{G}_m \longrightarrow 1$$

induces an exact sequence in cohomology:

$$\mathrm{GL}_1(A) \xrightarrow{\mathrm{Nrd}} k^\times \xrightarrow{\delta} H^1(k, \mathbf{SL}_1(A)) \longrightarrow H^1(k, \mathbf{GL}_1(A)).$$

Since $H^1(k, \mathbf{GL}_1(A)) = 1$ by Hilbert's Theorem 90 [25, Proposition III.8.24], we can canonically identify

$$H^1(k, \mathbf{SL}_1(A)) = k^\times / \mathrm{Nrd}(A^\times).$$

In particular, $H^1(k, \mathbf{SL}_n) = 1$ because $\det : \mathrm{GL}_n(k) \to k^\times$ is surjective.

Besides the adjoint group $\mathbf{PGL}_n$ and the simply connected group $\mathbf{SL}_n$, there may exist other split absolutely simple algebraic groups of type A_ℓ. This is because the centre of $\mathbf{SL}_n$ is isomorphic to $\boldsymbol{\mu}_n$ and, unless n is a prime, $\boldsymbol{\mu}_n$ has nontrivial subgroups.

If $n = md$, the best description of $H^1(k, \mathbf{SL}_n/\boldsymbol{\mu}_d)$ is probably the one stated in [40, Lemma 3.1] and [29, Proposition 4.2]. Assume that $\mathrm{char}(k)$ does not divide n. Using the exact sequence

$$1 \longrightarrow \boldsymbol{\mu}_m \longrightarrow \mathbf{SL}_n/\boldsymbol{\mu}_d \longrightarrow \mathbf{PGL}_n \longrightarrow 1,$$

one can conclude by some standard cohomological reasoning that:

12.12. Lemma. *The image of the map*

$$H^1(k, \mathbf{SL}_n/\boldsymbol{\mu}_d) \to H^1(k, \mathbf{PGL}_n)$$

is the set of isomorphism classes of central simple algebras of degree n and exponent dividing d, and the fibre over some $[A] \in H^1(k, \mathbf{PGL}_n)$ carries a simply transitive action of the group $k^\times / \operatorname{Nrd}(A^\times) k^{\times m}$.

12.13. Example (A group of type $A_\ell + A_\ell$)**.** Assume $\operatorname{char}(k)$ does not divide $n = \ell+1$. Consider the diagonal embedding $\boldsymbol{\mu}_n \to \mathbf{SL}_n \times \mathbf{SL}_n$, i.e. the homomorphism given on R-points by $\zeta \mapsto (\zeta\,\mathrm{id}, \zeta\,\mathrm{id})$ for all $\zeta \in \boldsymbol{\mu}_n(R)$, and the semisimple group of type $A_\ell + A_\ell$:

$$H_n = \frac{\mathbf{SL}_n \times \mathbf{SL}_n}{\boldsymbol{\mu}_n}$$

Groups of this kind appear three times in Table 5.

How do we interpret the set $H^1(k, H_n)$? We can fit H_n into a commutative diagram with exact rows

$$\begin{tikzcd}
1 \arrow[r] & \boldsymbol{\mu}_n \arrow[r] \arrow[d] & H_n \arrow[r] \arrow[d, equal] & \mathbf{PGL}_n \times \mathbf{PGL}_n \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \mathbf{SL}_n \times 1 \arrow[r] & H_n \arrow[r] & 1 \times \mathbf{PGL}_n \arrow[r] & 1.
\end{tikzcd}$$

Note that the bottom sequence splits, as there is a diagonal embedding $\mathbf{PGL}_n \to H_n$. Recalling that $H^1(k, \mathbf{SL}_n) = 1$, we extract a commutative diagram with exact rows:

$$\begin{tikzcd}
 & H^1(k, \boldsymbol{\mu}_n) \arrow[r] \arrow[d] & H^1(k, H_n) \arrow[r] \arrow[d, equal] & H^1(k, \mathbf{PGL}_n \times \mathbf{PGL}_n) \arrow[r, "\Delta"] \arrow[d] & H^2(k, \boldsymbol{\mu}_n) \\
\mathrm{PGL}_n(k) \arrow[r, "\delta"] & 1 \arrow[r] & H^1(k, H_n) \arrow[r, "p"] & H^1(k, \mathbf{PGL}_n). &
\end{tikzcd} \tag{12.13.1}$$

The connecting map Δ takes values in $H^2(k, \boldsymbol{\mu}_n)$, or the n-torsion part of the Brauer group. Concretely, the connecting map is $\Delta([A],[B]) = [A] - [B]$ for all degree n central simple algebras A, B. So

$$\ker(\Delta) = \{([A],[A]) : [A] \in H^1(k, \mathbf{PGL}_n)\},$$

and the projection $\ker(\Delta) \to H^1(k, \mathbf{PGL}_n)$ is bijective. It follows that p is surjective with trivial kernel.

Now let $b \in Z^1(k, H_n)$ be a cocycle whose image in $H^1(k, \mathbf{PGL}_n \times \mathbf{PGL}_n)$ is $([A],[A])$. We may assume that b is "symmetric", i.e., takes values in the diagonally embedded copy of $\mathbf{PGL}_n$ in H_n. Then

$${}_bH_n \simeq \frac{\mathbf{SL}_1(A) \times \mathbf{SL}_1(A)}{\boldsymbol{\mu}_n}$$

and this twisted group fits into the following exact sequence, which is the second row of (12.13.1) twisted by b:

$$\mathrm{PGL}_1(A) \longrightarrow k^\times / \operatorname{Nrd}(A^\times) \longrightarrow H^1(k, {}_bH_n) \longrightarrow H^1(k, \mathbf{PGL}_1(A)).$$

The kernel of $H^1(k, {}_bH_n) \to H^1(k, \mathbf{PGL}_1(A))$ is canonically isomorphic to the fibre of $H^1(k, H_n) \to H^1(k, \mathbf{PGL}_n)$ over $[A]$. Moreover, this kernel can be identified with the set of orbits of $k^\times / \operatorname{Nrd}(A^\times)$ by the group $\mathrm{PGL}_1(A)$ [156, I.§5.5 Corollary 2]. Tracing through the details of how $\mathrm{PGL}_1(A)$ acts on $k^\times / \operatorname{Nrd}(A^\times)$, one can show that this action is trivial because automorphisms of A preserve the reduced norm. Hence the fibre of $H^1(k, H_n)$ over $[A]$ is isomorphic to $H^1(k, \mathbf{SL}_1(A)) = k^\times / \operatorname{Nrd}(A^\times)$.

We have shown that there is a one-to-one correspondence:

$$\boxed{H^1\left(k, \frac{\mathbf{SL}_n \times \mathbf{SL}_n}{\boldsymbol{\mu}_n}\right)} \quad \longleftrightarrow \quad \boxed{\begin{array}{c}\text{Isomorphism classes of pairs } (A, c\operatorname{Nrd}(A^\times)) \\ \text{where } A \text{ is a central simple algebra of degree } n, \\ \text{and } c\operatorname{Nrd}(A^\times) \in k^\times / \operatorname{Nrd}(A^\times).\end{array}}$$

This example differs from Lemma 12.12 in one important aspect. The fibre of $H^1(k, H_n) \to H^1(k, \mathbf{PGL}_n)$ over $[A]$ has a canonical basepoint, mainly because the surjective map $H_n \to 1 \times \mathbf{PGL}_n$ has a section. In contrast, the fibre of $H^1(k, \mathbf{SL}_n/\boldsymbol{\mu}_d) \to H^1(k, \mathbf{PGL}_n)$ over $[A]$ does not have a canonical basepoint, so it is impossible to identify it with $k^\times / \operatorname{Nrd}(A^\times) k^{\times m}$, as one would like to.

13. Galois cohomology of structurable algebras

To interpret the Galois cohomology set $H^1(k, \mathbf{Str}(A, -))$, we follow standard practice and look for an algebraic object whose automorphism group is $\mathbf{Str}(A, -)$.

13.1. Definition. A *(nonassociative) pair* over a k-algebra R is a pair of R-modules $P = (P_+, P_-)$ equipped with a pair of R-bilinear maps

$$\begin{aligned} \mathcal{V} &: P_+ \times P_- \to \operatorname{End}(P_+) \\ \mathcal{V} &: P_- \times P_+ \to \operatorname{End}(P_-). \end{aligned}$$

(We deliberately use the same notation $\mathcal{V} : (x, y) \mapsto \mathcal{V}_{x,y}$ for both these maps, and it tends not to cause confusion.)

An isomorphism of pairs $P \to Q$ is a pair of R-module isomorphisms $f_\sigma : P_\sigma \to Q_\sigma$ such that $f_\sigma \mathcal{V}_{x,y} = \mathcal{V}_{f_\sigma(x), f_{-\sigma}(y)} f_\sigma$ for all $x, y \in P_\sigma \times P_{-\sigma}$. For an R-module S, we define the scalar extension of P in the obvious way and denote it by $P_S = (P_{+,S}, P_{-,S})$.

13.2. *The automorphism group scheme of a pair.* Let P be a nonassociative pair over k, and let $W_\sigma = \operatorname{Hom}(P_\sigma \otimes P_{-\sigma} \otimes P_\sigma, P_\sigma)$. Consider the following representations ρ_+ and ρ_-:

$$\begin{gathered} \rho_\sigma : \mathbf{GL}(P_+) \times \mathbf{GL}(P_-) \to \mathbf{GL}(W_\sigma) \\ \rho_{\sigma,R}(f_+, f_-)(E)(x \otimes y \otimes z) = f_\sigma(E(f_\sigma^{-1}(x) \otimes f_{-\sigma}^{-1}(y) \otimes f_\sigma^{-1}(z))) \end{gathered}$$

for all $f_\sigma \in \mathrm{GL}(P_{\sigma,R})$, $E \in W_{\sigma,R}$, and $(x, y, z) \in P_{\sigma,R} \times P_{-\sigma,R} \times P_{\sigma,R}$. The direct sum $\rho = \rho_+ \oplus \rho_-$ is a representation of $\mathbf{GL}(P_+) \times \mathbf{GL}(P_-)$ in the vector space $W = W_+ \oplus W_-$.

Note that W can be identified with the set of all nonassociative pairs with underlying vector spaces P_+ and P_-. Let $w \in W$ be the representative of $\mathcal{V}$; that is,

$w = (w_+, w_-)$ where $w_\sigma(x,y,z) = \mathcal{V}_{x,y}(z)$. Define the group functor $\mathbf{S}_w \subset \mathbf{GL}(W)$ as the stabiliser of w in $\mathbf{GL}(W)$; that is,

$$\mathbf{S}_w(R) = \{\alpha \in \mathrm{GL}(W_R) \colon \alpha(w) = w \text{ for all } w \in W_R\}.$$

Finally, define the group functor $\mathbf{Aut}(P) = \rho^{-1}(\mathbf{S}_w) \subset \mathbf{GL}(P_+) \times \mathbf{GL}(P_-)$, meaning:

$$\mathbf{Aut}(P)(R) = \{(f_+, f_-) \in \mathrm{GL}(P_{+,R}) \times \mathrm{GL}(P_{-,R}) \colon \rho_R(f_+, f_-) \in \mathbf{S}_w(R)\}.$$

The functor $\mathbf{Aut}(P)$ is representable [101, Examples 20.3 (2) & 20.4 (2)], and we call it the *automorphism group scheme of P*. The group of R-points $\mathbf{Aut}(P)(R)$ is precisely the automorphism group of the pair P_R.

13.3. *Galois cohomology of Kantor pairs.* Since $H^1(k, \mathbf{GL}(W)) = 1$ by Hilbert 90, [101, Proposition 29.1] implies that there is a one-to-one correspondence

$$\boxed{H^1\big(k, \mathbf{Aut}(P)\big)} \quad \longleftrightarrow \quad \boxed{\begin{array}{c} k\text{-isomorphism classes of pairs } Q \text{ such that the} \\ k^s\text{-pairs } Q_{k^s} \text{ and } P_{k^s} \text{ are isomorphic.} \end{array}}$$

13.4. Definition. [13, §3] A pair $P = (P_+, P_-)$ is called a *Kantor pair* if it satisfies the identities:

(KP1) $[\mathcal{V}_{x,y}, \mathcal{V}_{z,w}] = \mathcal{V}_{\mathcal{V}_{x,y}(z),w} - \mathcal{V}_{z,\mathcal{V}_{y,x}(w)}$

(KP2) $\mathcal{K}_{a,b}\mathcal{V}_{x,y} + \mathcal{V}_{y,x}\mathcal{K}_{a,b} = \mathcal{K}_{\mathcal{K}_{a,b}(x),y}$

for all $(x,y), (z,w) \in P_\sigma \times P_{-\sigma}$ and $(a,b) \in P_\sigma \times P_\sigma$, where

$$\mathcal{K}_{z,w}(x) = \mathcal{V}_{z,x}(w) - \mathcal{V}_{w,x}(z).$$

13.5. *Kantor pairs from structurable algebras.* If $(A, -)$ is a structurable algebra, then the pair

$$\mathrm{KP}(A, -) = (A, A) \qquad\qquad \mathcal{V}_{x,y} = 2V_{x,y}$$

is a Kantor pair [13, p. 533]. Let $(A, -)$ and $(B, -)$ be structurable R-algebras. An R-module isomorphism $f : A \to B$ is an isotopy if and only if there exists a linear map $\hat{f} : A \to B$ such that $(f, \hat{f})$ is an isomorphism of the Kantor pairs $\mathrm{KP}(A, -) \to \mathrm{KP}(B, -)$. The map $\hat{f}$, if it exists, is uniquely determined by f. Consequently, $(A, -)$ is isotopic to $(B, -)$ if and only if $\mathrm{KP}(A, -)$ is isomorphic to $\mathrm{KP}(B, -)$, and there is an isomorphism of algebraic groups

$$\mathbf{Str}(A, -) \xrightarrow{\sim} \mathbf{Aut}(\mathrm{KP}(A, -))$$

defined by $\alpha \mapsto (\alpha, \hat{\alpha})$ for all $\alpha \in \mathrm{Str}(A_R, -)$.

Not every Kantor pair is of the form $\mathrm{KP}(A, -)$ for a structurable algebra $(A, -)$. Item (ii) of the upcoming lemma implies, in particular, that the class of Kantor pairs associated to structurable algebras is closed under Galois descent. In other words, if P is a Kantor pair such that $P_K \simeq \mathrm{KP}(A_K, -)$ for some separable field extension K/k and some structurable algebra $(A, -)$ over K, then $P \simeq \mathrm{KP}(A', -)$ for some structurable algebra $(A', -)$ over k.

The following lemma is the main result of this section (and probably the most important result in the whole chapter).

13.6. Lemma. *Let $(A,-)$ be a central simple structurable algebra over k.*

(i) *If k is algebraically closed, the action of $\mathbf{Str}(A,-)^\circ$ on A has a dense open orbit.*

(ii) *The inclusion $i : \mathbf{Aut}(A,-) \subset \mathbf{Str}(A,-)$ induces a surjective map*

$$i_* : H^1(k, \mathbf{Aut}(A,-)) \longrightarrow H^1(k, \mathbf{Str}(A,-)).$$

(iii) *Let $M = (\mathbf{Str}(A,-)^\circ)^{\mathrm{der}}$ be the semisimple structure group of $(A,-)$. The inclusion $M \subset \mathbf{Str}(A,-)^\circ$ induces a surjective map*

$$H^1(k, M) \longrightarrow H^1(k, \mathbf{Str}(A,-)^\circ).$$

(iv) *The embedding $\mathbf{Str}(A,-) \simeq \mathbf{Aut}_{\mathrm{gr}}(K(A,-)) \subset \mathbf{Aut}(K(A,-))$ from Lemma 5.5 induces injective maps*

$$H^1(k, \mathbf{Str}(A,-)) \longrightarrow H^1(k, \mathbf{Aut}(K(A,-))$$
$$H^1(k, \mathbf{Str}(A,-)^\circ) \longrightarrow H^1(k, \mathbf{Aut}(K(A,-)^\circ).$$

Proof. To fix some notation, let $L = K(A,-) = \bigoplus_{i=-2}^{2} K_i$, let $G = \mathbf{Aut}(L)^\circ$, and let $\nu : \mathbf{G}_m \to G$ be the grading cocharacter. By Lemmas 4.2 and 5.5, the diagram $C_G(\nu) \to \mathbf{GL}(K_1)$ is isomorphic to the diagram $\mathbf{Str}(A,-)^\circ \to \mathbf{GL}(A)$.

(i) Set $\lambda = 2\nu$. Then by [165, Theorem 5.8], the pair (G, λ) satisfies the conditions of [165, Theorem 5.6 (1)] and it follows that $C_G(\lambda) = C_G(\nu)$ has a unique dense open orbit in K_1 (since the 2-weight space of λ is the same as the 1-weight space of ν). Hence the same is true about $\mathbf{Str}(A,-)^\circ$ and its representation in A.

(ii) Firstly assume k is infinite. Rost's Theorem on prehomogeneous vector spaces [58, Theorem 9.3 & Context, p. 29] implies that i_* is surjective. Secondly, if k is a finite field, then i_* is surjective provided that $\pi_0(i) : \pi_0(\mathbf{Aut}(A,-)) \to \pi_0(\mathbf{Str}(A,-))$ is surjective [156, III. §2.4 Corollaries 2 & 3], and $\pi_0(i)$ is indeed surjective by Proposition 11.5.

(iii) Note that $C_G(\nu)$ has a maximal torus T containing the image of ν, and T is also a maximal torus in G. In particular, G and $C_G(\nu) \simeq \mathbf{Str}(A,-)^\circ$ have the same rank. The quotient $S = \mathbf{Str}(A,-)^\circ/M$ is a torus (see [122, §19 d.]) of dimension $r = \operatorname{rank}(G) - \operatorname{rank}(M) = 1$ or 2, depending on the number of positively labelled vertices on the labelled Dynkin diagram of (G, ν). There is a nonzero character $\mu : \mathbf{Str}(A,-)^\circ \to \mathbf{G}_m$ (the multiplier homomorphism) which necessarily factors through a character $\bar{\mu} \in X^*(S) = \operatorname{Hom}(S, \mathbf{G}_m)$. If $r = 1$ this implies S is split. If $r = 2$ and S is not split, then the action of Γ_k on $X^*(S) \simeq \mathbb{Z}^2$ fixes $\mathbb{Z}\bar{\mu}$ and therefore consists of a reflection. This action uniquely determines S (see [122, Theorem 12.23]). The stabiliser is an index 2 subgroup $\Gamma_F \subset \Gamma_k$ corresponding to some quadratic field extension F, and F is the unique quadratic extension that splits S. Hence $S = R_{F/k}(\mathbf{G}_{m,F})$. Whether $S = \mathbf{G}_m$, $S = \mathbf{G}_m^2$, or $S = R_{F/k}(\mathbf{G}_{m,F})$, Hilbert 90 implies $H^1(k, S) = 1$. By the exact sequence (12.8.2) in Galois cohomology, the map $H^1(k, M) \to H^1(k, \mathbf{Str}(A,-)^\circ)$ is surjective.

(iv) We first deal with the connected scenario. Consider the parabolic subgroup $P = P_G(\nu) \subset G$, whose Levi subgroup is $C_G(\nu)$. By [156, §III.2 Exercise 1], the natural map $H^1(k, P) \to H^1(k, G)$ is injective. (This exercise only relies on the fact that the map $G(k) \to G/P(k)$ is surjective [32, Proposition 20.5], and does not require

k to be perfect.) Since $P = U \rtimes C_G(\nu)$, where U is the unipotent radical of P, the map $C_G(\nu) \to P \to P/U$ is an isomorphism. This implies $H^1(k, C_G(\nu)) \to H^1(k, P) \to H^1(k, P/U)$ is an isomorphism, and in particular that $H^1(k, C_G(\nu)) \to H^1(k, P)$ is injective (this argument is used in [44, Corollary 5.4.8] for Čech cohomology instead of Galois cohomology). Hence $H^1(k, C_G(\nu)) \to H^1(k, G)$ is injective because it is a composition of two injective maps.

Now suppose $\mathbf{Aut}(L)$ is not connected, and $\alpha, \beta \in H^1(k, \mathbf{Str}(A, -))$ have the same image in $H^1(k, \mathbf{Aut}(L))$. Then their images in $H^1(k, \pi_0(\mathbf{Aut}(L)))$ are equal too. By twisting (say, replacing $(A, -)$ with a twisted form $(A', -)$ whose image in $H^1(k, \mathbf{Str}(A, -))$ is α) we can assume that α and β come from cocycle classes $\alpha', \beta' \in H^1(k, \mathbf{Str}(A, -)^\circ)$. By the previous paragraph, α' and β' have the same image in $H^1(k, G)$, so they are in the same orbit by $\pi_0(\mathbf{Aut}(L))(k)$. Since $\pi_0(\mathbf{Str}(A, -)) \simeq \pi_0(\mathbf{Aut}(L))$, this implies α' and β' are in the same orbit of $\pi_0(\mathbf{Str}(A, -))(k)$, and hence $\alpha = \beta \in H^1(k, \mathbf{Str}(A, -))$. □

Lemma 13.6 (i) is a well-known situation: see [20, 58, 165] and the references therein. I have cited [165] in the proof because it is convenient and requires the least translation work.

Here is an alternative proof of Lemma 13.6 (i) that gives an explicit description of the dense open orbit. By Corollary 3.15, two central simple structurable algebras over an algebraically (even separably) closed field are isotopic if and only if they are isomorphic. Then for all $u \in A^*$, $(A, -) \simeq (A^{\langle u \rangle}, -^{\langle u \rangle})$, so $\mathrm{Str}(A, -)$ acts transitively on A^*, and A^* is known to be dense and open in A [10, Theorem 10.5].

Lemma 13.6 (iii) is also a well-known situation, and its proof is similar to the argument in [175, p. 657] (which was, unfortunately, not quite general enough to serve as a direct reference for the claim). Probably (iv) is well-known too, or at least very intuitive.

13.7. *Galois cohomology of structurable algebras.* If $(A, -)$ is any algebra with involution, there is a one-to-one correspondence:

$H^1\big(k, \mathbf{Aut}(A, -)\big)$	$\longleftrightarrow$	k-isomorphism classes of algebras with involution $(A', -)$ such that $(A'_{k^s}, -)$ and $(A_{k^s}, -)$ are isomorphic.

Assume $(A, -)$ is a structurable algebra. By 13.3 and 13.5, we can identify the cohomology set $H^1(k, \mathbf{Str}(A, -))$ with the set of isomorphism classes of Kantor pairs that become isomorphic to $\mathrm{KP}(A, -)$ after extending scalars to k^s. The map

$$i_* : H^1\big(k, \mathbf{Aut}(A, -)\big) \to H^1\big(k, \mathbf{Str}(A, -)\big)$$

sends the isomorphism class of $(A', -)$ to the isomorphism class of $\mathrm{KP}(A', -)$. Two k-forms of $(A, -)$ have the same image under i_* if and only if they are isotopic over k. We can therefore identify the image of i_* with the set of k-isotopy classes of structurable algebras that become isotopic to $(A, -)$ over k^s. If $(A, -)$ is central simple, then i_* is surjective by Lemma 13.6 (ii), so there is a one-to-one correspondence:

$H^1\big(k, \mathbf{Str}(A, -)\big)$	$\longleftrightarrow$	k-isotopy classes of structurable algebras $(A', -)$ such that $(A'_{k^s}, -)$ and $(A_{k^s}, -)$ are isotopic.

Recall also from Corollary 3.15 that $(A'_{k^s}, -)$ is isotopic to $(A_{k^s}, -)$ if and only if it is isomorphic to $(A_{k^s}, -)$.

13.8. *The isotopy problem.* The isotopy problem for a structurable algebra $(A, -)$ is the problem of deciding whether another structurable algebra $(A', -)$ is isotopic to $(A, -)$. In other words, it is about understanding the kernel of $H^1(k, \mathbf{Aut}(A, -)) \to H^1(k, \mathbf{Str}(A, -))$.

It was proved in [11, Theorem 4.4] that the isotopy problem has a trivial solution for Smirnov algebras over a field of characteristic 0. That is, all Smirnov algebras over a field of characteristic 0 are isotopic to each other. The solution to the isotopy problem is also trivial for exceptional 14-dimensional skew-dimension one structurable algebras (i.e., green algebras), and quartic 2×2 matrix algebras.

13.9. Theorem. *If $(A_1, -)$ and $(A_2, -)$ are both Smirnov algebras, both green algebras, or both forms of the quartic 2×2 matrix algebra, then $(A_1, -)$ is isotopic to $(A_2, -)$.*

Proof. Let $(A, -) = T(\mathbb{O})$ be the split Smirnov algebra. Two Smirnov algebras $(A_1, -)$ and $(A_2, -)$ are isotopic if and only if their classes in $H^1(k, \mathbf{Aut}(A, -))$ have the same image in $H^1(k, \mathbf{Str}(A, -))$. But the semisimple structure group of $(A, -)$ is $\mathbf{SL}_7$ and $H^1(k, \mathbf{SL}_7) = 1$. We have $\mathbf{Str}(A, -) = \mathbf{Str}(A, -)^\circ$ by Proposition 11.5, so $H^1(k, \mathbf{Str}(A, -)) = H^1(k, \mathbf{Str}(A, -)^\circ) = H^1(k, \mathbf{SL}_7) = 1$ by Lemma 13.6 (iii).

Similarly, if $(A, -) = M(\mathcal{H}_3(k))$ is the split green algebra then its structure group is connected, its semisimple structure group is $\mathbf{Sp}_6$, and $H^1(k, \mathbf{Str}(A, -)) = H^1(k, \mathbf{Sp}_6) = 1$ (for the last equality, see [101, Example 29.25]). If $(A, -)$ is the quartic 2×2 matrix algebra, then $H^1(k, \mathbf{Str}(A, -)) = H^1(k, \mathbf{SL}_2) = 1$. $\square$

14. Galois cohomology of quadratic forms

In this section, we take a closer look at some reductive groups associated to a quadratic space, and assemble some information about their Galois cohomology. The Clifford algebra of a quadratic space [94, §4.8] is of central importance here.

14.1. *Groups of units in the even Clifford algebra.* Let (V, q) be an even-dimensional quadratic space over k. (We shall assume quadratic spaces are nondegenerate and might not mention this every time.) Every isometry $\nu \in \mathrm{O}(V, q)$ induces an automorphism $\tilde{C}(\nu)$ of the full Clifford algebra $C(V, q)$, and every similitude $\beta \in \mathrm{GO}(V, q)$ induces an automorphism $C(\beta)$ of the even Clifford algebra $C^+(V, q)$ [101, Proposition 13.1]. In concrete terms,

$$\begin{aligned} \tilde{C}(\nu)(v_1 \dots v_r) &= \nu(v_1) \dots \nu(v_r), \\ C(\beta)(v_1 \dots v_{2r}) &= \mu(\beta)^{-r} \beta(v_1) \dots \beta(v_{2r}) \end{aligned}$$

for all $v_1, \dots, v_{2r} \in V \subset C(V, q)$, where $\mu(\beta) \in k^\times$ is the multiplier of β.

Of course $\tilde{C}(\nu)$ preserves the $\mathbb{Z}/2\mathbb{Z}$-grading on $C(V, q)$, and the restriction of $\tilde{C}(\nu)$ to $C^+(V, q)$ is $C(\nu)$. Moreover, $C(\beta)$ fixes the centre $Z = Z(C^+(V, q))$ if and only if $\beta \in \mathrm{GO}^+(V, q)$; i.e., β is a proper similitude [101, Proposition 13.2].

This situation can be summarised by saying there are canonical homomorphisms

$$\begin{aligned}\tilde{C} &: \mathrm{O}(V,q) \to \mathrm{Aut}(C(V,q))\\ C &: \mathrm{GO}(V,q) \to \mathrm{Aut}(C^+(V,q))\end{aligned}$$

and C restricts to a homomorphism $C : \mathrm{GO}^+(V,q) \to \mathrm{Aut}_Z(C^+(V,q))$ whose kernel is $k^\times$ id.

This all works on the level of algebraic groups, so there are canonical homomorphisms

$$\begin{aligned}\tilde{C} &: \mathbf{O}(V,q) \to \mathbf{Aut}(C(V,q))\\ C &: \mathbf{GO}^+(V,q) \to \mathbf{Aut}_Z(C^+(V,q)).\end{aligned}$$

Since all k-automorphisms of $C(V,q)$ and all Z-automorphisms of $C^+(V,q)$ are inner, the homomorphisms

$$\begin{aligned}\mathrm{Int} &: \mathbf{GL}_1(C(V,q)) \to \mathbf{Aut}(C(V,q))\\ \mathrm{Int} &: \mathbf{GL}_1(C^+(V,q)) \to \mathbf{Aut}_Z(C^+(V,q))\end{aligned}$$

are both surjective.

Let $\tau : s_1 \dots s_{2r} \mapsto s_{2r} \dots s_1$ be the standard involution on $C^+(V,q)$. The following subgroups of $\mathbf{GL}_1(C^+(V,q))$ are well-known and the equalities below can serve as their definitions (see [101, 151]):

$$\mathbf{\Omega}(V,q) = \mathrm{Int}^{-1}(C(\mathbf{GO}^+(V,q)) \qquad \textit{(the extended Clifford group)}$$

$$\begin{aligned}\mathbf{\Gamma}^+(V,q) &= \mathrm{Int}^{-1}(\tilde{C}(\mathbf{O}^+(V,q))) \qquad \textit{(the even Clifford group)}\\ &= \mathrm{Int}^{-1}(\tilde{C}(\mathbf{O}(V,q))) \cap \mathbf{GL}_1(C^+(V,q))\end{aligned}$$

$$\mathbf{Spin}(V,q) = \mathbf{Iso}(C^+(V,q),\tau) \cap \mathbf{\Gamma}^+(V,q) \qquad \textit{(the Spin group).}$$

A consequence of the Cartan–Dieudonné Theorem is that the k-points of $\mathbf{\Gamma}^+(V,q)$ are products $v_1 \dots v_{2r}$, where $v_1, \dots, v_{2r} \in V$ are vectors such that $q(v_i) \neq 0$ [94, Theorem 4.15]. This is in contrast with $\mathbf{\Omega}(V,q)$ whose k-points may include units of $C^+(V,q)$ that are nontrivial linear combinations of such terms.

All three of these groups act on $(C^+(V,q),\tau)$ by involution-preserving inner automorphisms. In summary, we have a series of subgroups:

$$\mathbf{Spin}(V,q) \subset \mathbf{\Gamma}^+(V,q) \subset \mathbf{\Omega}(V,q) \subset \mathbf{Sim}(C^+(V,q),\tau) \subset \mathbf{GL}_1(C^+(V,q)).$$

The *vector representation* of $\mathbf{\Gamma}^+(V,q)$ is the homomorphism

$$\chi : \mathbf{\Gamma}^+(V,q) \to \mathbf{O}^+(V,q)$$

where $\chi(x)(v) = xvx^{-1}$ for all $x \in \Gamma^+(V,q)$ and all $v \in V_R$, i.e. $\chi = \mathrm{Int}\,|_V$.

There is a unique homomorphism

$$\chi' : \mathbf{\Omega}(V,q) \to \mathbf{PGO}^+(V,q)$$

such that $\chi'(x) = \bar{\beta}$ where $\beta \in \mathrm{GO}^+(V,q)$ is a similitude such that $\mathrm{Int}(x) = C(\beta)$, and $\bar{\beta}$ is its image in $\mathrm{PGO}^+(V,q)$ [101, (13.19)].

We have the following commutative diagram of algebraic groups:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbf{G}_m & \longrightarrow & \mathbf{\Gamma}^+(V,q) & \xrightarrow{\chi} & \mathbf{O}^+(V,q) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathbf{GL}_1(Z) & \longrightarrow & \mathbf{\Omega}(V,q) & \xrightarrow{\chi'} & \mathbf{PGO}^+(V,q) & \longrightarrow & 1. \end{array} \tag{14.1.1}$$

The rows are exact, the first two columns are injective, and the third column is surjective [101, (13.24), p. 352]. Since the groups in the first and third columns are smooth and connected, so are the groups in the middle column [122, Propositions 1.62 & 5.59].

If (V,q) is an odd-dimensional quadratic form, a modified version of this diagram is still valid but it is somewhat degenerate because the two rows are isomorphic: $\mathbf{GL}_1(Z) \simeq \mathbf{G}_m$, $\mathbf{PGO}(V,q) \simeq \mathbf{O}^+(V,q)$ [101, Proposition 12.4]. The groups $\mathbf{PGO}^+(V,q)$ and $\mathbf{\Omega}(V,q)$ are not usually defined if $\dim V$ is odd.

If (V,q) is $2n$-dimensional with a split Clifford algebra, then

$$C^+(V,q) \simeq M_{2^{n-1}}(k) \times M_{2^{n-1}}(k).$$

The two *half-spin representations* $\rho_i : \mathbf{Spin}(V,q) \to \mathbf{SL}_{2^{n-1}}(k)$ are the projections onto each component of $C^+(V,q)$; in general these are inequivalent representations.

14.2. *Galois cohomology of projective orthogonal groups.* Let $(A,-) = (\operatorname{End} V, \mathrm{ad}_q)$ be a matrix algebra with orthogonal involution adjoint to a quadratic form q on a vector space V of even dimension n. Then $\mathbf{Aut}(A,-) = \mathbf{PGO}(V,q)$ and so $H^1(k, \mathbf{PGO}(V,q))$ is in one-to-one correspondence with the isomorphism classes of associative central simple algebras with orthogonal involution of degree n [101, 29.F].

By Hilbert 90, the natural map

$$H^1(k, \mathbf{O}(V,q)) \to H^1(k, \mathbf{GO}(V,q))$$

is surjective. By Hilbert 90 and twisting, the natural map

$$H^1(k, \mathbf{GO}(V,q)) \to H^1(k, \mathbf{PGO}(V,q))$$

is injective. As such, $H^1(k, \mathbf{GO}(V,q))$ is in natural one-to-one correspondence with the similitude classes of n-dimensional quadratic spaces. (See also [25, III, Exercise 2 & Lemma VIII.21.21] for another perspective.)

The set $H^1(k, \mathbf{PGO}^+(V,q))$ is in one-to-one correspondence with isomorphism classes of triples $[(A,\sigma),\varphi]$ where (A,σ) is an orthogonal involution of degree n and $\varphi : Z(C(A,\sigma)) \to Z(C^+(V,q))$ is an isomorphism [101, 29.F], $C(A,\sigma)$ being the Clifford algebra of (A,σ) as defined in [101, 8.B]. In the event that $Z(C^+(V,q)) = k \times k$, making a choice of an isomorphism $\varphi : Z(C(A,\sigma)) \to k \times k$ is the same as ordering the two simple subalgebras of $C(A,\sigma)$ and labelling them as C_+ and C_- [101, Remark 29.31].

While the map $H^1(k, \mathbf{PGO}^+(V,q)) \to H^1(k, \mathbf{PGO}(V,q))$ is not generally injective, it does have trivial fibres over the subset corresponding to quadratic forms. In other words, the composition

$$H^1(k, \mathbf{GO}^+(V,q)) \to H^1(k, \mathbf{PGO}^+(V,q)) \to H^1(k, \mathbf{PGO}(V,q))$$

is injective. One can show this by an argument similar to the one on [100, p. 407].

14.3. *The Witt and Grothendieck–Witt rings.* We refer to [106, II. §1] for full definitions of the Witt and Grothendieck–Witt rings. Briefly, the Grothendieck–Witt group $\widehat{W}(k)$ is the universal enveloping group of the cancellative abelian monoid of isometry classes of nondegenerate quadratic forms, with the additive operation $\perp$. The commutative operation $\otimes$ distributes over $\perp$ and so gives $\widehat{W}(k)$ the structure of a commutative unital ring with unit $1 = \langle 1 \rangle$.

The Witt ring is the quotient of $\widehat{W}(k)$ by the ideal generated by $\mathbb{H}$. Hence, there is a natural homomorphism

$$\pi : \widehat{W}(k) \to W(k)$$

whose kernel is generated by $\mathbb{H}$. Two nondegenerate quadratic forms q and q' are called *Witt equivalent* if and only if $q \perp n\mathbb{H} \simeq q' \perp m\mathbb{H}$ for some $n, m \in \mathbb{N}$. So $W(k)$ can be identified with the set of Witt equivalence classes of nondegenerate finite-dimensional quadratic forms over k.

The fundamental ideal in the Witt ring $W(k)$ is the ideal $I(k)$ consisting of the Witt classes of all even-dimensional quadratic forms. We write $I^n(k) = I(k)^n$; this ideal is generated as a subgroup of $W(k)$ by the classes of n-Pfister forms. The ideal $I^2(k)$ is the set of Witt classes of forms with trivial discriminant. According to Merkurjev's Theorem of 1981 (see [106, §V.3]), $I^3(k)$ is the set of Witt classes of forms whose Clifford algebra is split.

The Arason–Pfister Hauptsatz [19] is a theorem that says 2^n is the minimum dimension of an anisotropic quadratic form whose Witt class is in $I^n(k)$.

For an even number $j \geq 2$, we define $I^n_j(k)$ to be the set of isometry classes of j-dimensional forms whose Witt class is $I^n(k)$, and we define $PI^n_j(k)$ to be the set $I^n_j(k)$ modulo the equivalence relation of similitude.

14.4. *I^2_n, I^3_n, and PI^3_n as Galois cohomology sets..* Let (V, q) be a quadratic space. The Galois cohomology of the exact sequence

$$1 \longrightarrow \boldsymbol{\mu}_2 \longrightarrow \mathbf{Spin}(V, q) \longrightarrow \mathbf{O}^+(V, q) \longrightarrow 1 \tag{14.4.1}$$

is well-understood: see [25, IV.11.2], [101, p. 437], or [58, 16.2]. The set $H^1(k, \mathbf{O}^+(V, q))$ injects into $H^1(k, \mathbf{O}(V, q))$ and so it corresponds bijectively with the set of isometry classes of quadratic forms with the same dimension and discriminant as (V, q). The image of $H^1(k, \mathbf{Spin}(V, q)) \to H^1(k, \mathbf{O}^+(V, q))$ is the set of isometry classes of quadratic forms with the same dimension, discriminant, and Clifford algebra as (V, q). By appending (14.4.1) above the top row of (14.1.1), one demonstrates as in [41, p. 461–462] that there is a one-to-one correspondence:

$H^1\big(k, \mathbf{\Gamma}^+(V, q)\big)$	$\longleftrightarrow$	Isometry classes of quadratic forms with the same dimension, discriminant, and Clifford algebra as (V, q).

Assume now that V is even-dimensional. The extended Clifford group $\mathbf{\Omega}(V, q)$ is connected and reductive. Let $Z = Z(C^+(V, q))$. We can show by chasing the diagram (14.1.1), or referring to [101, Lemma 13.20] and using the smoothness of $\mathbf{\Omega}(V, q)$, that

$\mathbf{\Omega}(V,q)$ is generated by its centre $\mathbf{GL}_1(Z)$ and the subgroup $\mathbf{\Gamma}^+(V,q)$. Hence $\mathbf{\Gamma}^+(V,q)$ is a normal subgroup of $\mathbf{\Omega}(V,q)$. We extend (14.1.1) to make the first two columns into exact sequences:

$$\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d] & & \\
1 \arrow[r] & \mathbf{G}_m \arrow[r] \arrow[d] & \mathbf{\Gamma}^+(V,q) \arrow[r, "\chi"] \arrow[d] & \mathbf{O}^+(V,q) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \mathbf{GL}_1(Z) \arrow[r] \arrow[d, "z\mapsto z^{-2}N_{Z/k}(z)"'] & \mathbf{\Omega}(V,q) \arrow[r, "\chi'"] \arrow[d] & \mathbf{PGO}^+(V,q) \arrow[r] & 1 \\
 & \mathbf{G}^1_{m,Z} \arrow[r] \arrow[d] & T \arrow[d] & & \\
 & 1 & 1 & &
\end{tikzcd}$$

Here, $\mathbf{G}^1_{m,Z} = \ker(N_{Z/k} : \mathbf{GL}_1(Z) \to \mathbf{G}_m) \simeq \mathbf{GL}_1(Z)/\mathbf{G}_m$, and $T = \mathbf{\Omega}(V,q)/\mathbf{\Gamma}^+(V,q)$. We have from [101, Corollary 13.16] the fact that $Z^\times \cap \Gamma^+(V,q) = \{z \in Z^\times : z^2 \in k^\times\}$. Hence $\mathbf{GL}_1(Z)\cap\mathbf{\Gamma}^+(V,q)$ is the kernel of the map $\mathbf{GL}_1(Z) \to \mathbf{G}^1_{m,Z}$, $z \mapsto z^{-2}N_{Z/k}(z)$. By the isomorphism theorem [122, Theorem 5.52],

$$T = \frac{\mathbf{\Gamma}^+(V,q).\mathbf{GL}_1(Z)}{\mathbf{\Gamma}^+(V,q)} \simeq \frac{\mathbf{GL}_1(Z)}{\mathbf{GL}_1(Z)\cap\mathbf{\Gamma}^+(V,q)} \simeq \mathbf{G}^1_{m,Z}.$$

Standard arguments, as in [101, p. 416], show that $H^1(k, \mathbf{G}^1_{m,Z}) \simeq k^\times/N_{Z/k}(Z^\times)$. If $N_{Z/k}$ is surjective then $H^1(k, \mathbf{G}^1_{m,Z}) = 1$ and

$$H^1(k, \mathbf{\Gamma}^+(V,q)) \to H^1(k, \mathbf{\Omega}(V,q)) \tag{14.4.2}$$

is surjective. The fibre in $H^1(k, \mathbf{\Gamma}^+(V,q))$ over some $\omega \in H^1(k, \mathbf{\Omega}(V,q))$ is the set of all $\gamma \in H^1(k, \mathbf{\Gamma}^+(V,q))$ such that $\chi_*(\gamma)$ maps to $\chi'_*(\omega)$ in $H^1(k, \mathbf{PGO}^+(V,q))$. This means that the fibres of the map (14.4.2) are similitude classes of quadratic forms.

In summary, for any even-dimensional quadratic space (V,q) such that $N_{Z/k}$ is surjective, we have a natural one-to-one correspondence:

$H^1\big(k, \mathbf{\Omega}(V,q)\big)$	$\longleftrightarrow$	Similitude classes of quadratic forms with the same dimension, discriminant, and Clifford algebra as (V,q).

If q is the hyperbolic n-dimensional form, we use the notations $\mathbf{O}^+_n = \mathbf{O}^+(V,q)$, $\mathbf{\Gamma}^+_n = \mathbf{\Gamma}^+(V,q)$, $\mathbf{\Omega}_n = \mathbf{\Omega}(V,q)$, etc. From the discussion above and in 1.4 and 14.2, we summarise that for all even $n > 0$ there are isomorphisms of functors:

$$\begin{array}{lll}
H^1(*, \mathbf{O}_n) \simeq I_n(*) & H^1(*, \mathbf{O}^+_n) \simeq I^2_n(*) & H^1(*, \mathbf{\Gamma}^+_n) \simeq I^3_n(*) \\
H^1(*, \mathbf{GO}_n) \simeq PI_n(*) & H^1(*, \mathbf{GO}^+_n) \simeq PI^2_n(*) & H^1(*, \mathbf{\Omega}_n) \simeq PI^3_n(*).
\end{array}$$

In Serre's notation from [158], $I_n = \mathrm{Quad}_n$ and $I^2_n = \mathrm{Quad}_{n,(-1)^{n/2}}$.

15. Introduction to cohomological invariants

A cohomological invariant is a function from a set to a Galois cohomology group that is compatible with base change. In other words, it is a natural transformation $F \to T$ between two functors

$$F, T : \mathsf{Fields}_{/k} \rightrightarrows \mathsf{Sets}$$

where T is the functor corresponding to some Galois cohomology group. Typically, $F(*) = H^1(*, G)$ for some smooth algebraic k-group G. Before getting to a more formal definition, we introduce some cohomology groups that frequently appear as the targets for cohomological invariants.

The notation in this section is designed to be as compatible as possible with the main references [58, 101, 116, 158].

15.1. *Mod* 2 *Galois cohomology.* The constant Galois module $\mathbb{Z}/2\mathbb{Z}$ (alternatively written as S_2 or as $\boldsymbol{\mu}_2$ since $\operatorname{char}(k) \neq 2$) is important because the functor

$$H(*) = \bigoplus_{d \geq 0} H^d(*, \mathbb{Z}/2\mathbb{Z}) \tag{15.1.1}$$

is the target of many cohomological invariants. There is a canonical isomorphism of abelian groups

$$k^\times / k^{\times 2} \simeq H^1(k, \mathbb{Z}/2\mathbb{Z})$$

where a square class $ck^{\times 2} \in k^\times/k^{\times 2}$ corresponds to the symbol $(c) \in H^1(k, \mathbb{Z}/2\mathbb{Z})$. Note the change from multiplicative to additive notation: we have $(ab) = (a) + (b)$ for all $a, b \in k^\times$.

Since $\mathbb{Z}/2\mathbb{Z} \otimes \mathbb{Z}/2\mathbb{Z} \simeq \mathbb{Z}/2\mathbb{Z}$ and $H(*)$ is 2-torsion, the cup product (which we denote by $\cdot$) gives a commutative unital $\mathbb{Z}$-graded ring structure to $H(*)$ with some nice properties [25, Proposition III.9.15]:

$$(a)\cdot(a) = (-1)\cdot(a) \tag{15.1.2}$$

$$(a)\cdot(-a) = (a)\cdot(1-a) = 0$$

$$(a)\cdot(b) = 0 \text{ if and only if } a \in k^{\times 2} \text{ or } b \in N_{E/k}(E^\times) \text{ where } E = k(\sqrt{a}). \tag{15.1.3}$$

Consequent to the Milnor Conjecture, the entire cohomology ring $H(k)$ is additively generated by symbols; i.e., elements of the form $(a_1)\cdot \cdots \cdot(a_m)$. For an element $\alpha \in H^d(k, \mathbb{Z}/2\mathbb{Z})$, the *symbol length* of α is the least $\ell \geq 0$ such that α is a sum of ℓ symbols.

15.2. Definition. Let $F : \mathsf{Fields}_{/k} \to \mathsf{Sets}$ be a functor. A *mod* 2 *cohomological invariant* of F is a natural transformation $F \to H$.

The set of all mod 2 cohomological invariants of F is denoted by $\operatorname{Inv}(F, 2)$. Via the cup product, it is a commutative unital $\mathbb{Z}$-graded $H(k)$-algebra:

$$\operatorname{Inv}(F, 2) = \bigoplus_{d \geq 0} \operatorname{Inv}^d(F, 2)$$

where $\operatorname{Inv}^d(F, 2)$ is the subgroup of invariants taking values in $H^d(*, \mathbb{Z}/2\mathbb{Z})$. The multiplicative unit of $\operatorname{Inv}(F, 2)$ is the constant invariant 1 that maps everything in $F(L)$ to the point $1 \in H^0(L, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$.

15.3. *Mod n Galois cohomology.* Let $n > 1$ be prime to $\operatorname{char}(k)$. Define Γ_k-modules:

$$\mathbb{Z}/n\mathbb{Z}(d) = \begin{cases} \operatorname{Hom}(\boldsymbol{\mu}_n, \mathbb{Z}/n\mathbb{Z}) & d = -1 \\ \mathbb{Z}/n\mathbb{Z} & d = 0 \\ \boldsymbol{\mu}_n^{\otimes d} = \boldsymbol{\mu}_n \otimes \cdots \otimes \boldsymbol{\mu}_n & d \geq 1. \end{cases}$$

The functor

$$\bigoplus_{d \geq 0} H^d(*, \mathbb{Z}/n\mathbb{Z}(d-1)).$$

will be the target of what we call "mod n" cohomological invariants. Note that $\mathbb{Z}/2\mathbb{Z}(d) \simeq \mathbb{Z}/2\mathbb{Z}$ for all d, so for $n = 2$ this is the same as (15.1.1).

More generally, for any Galois module C whose every element has finite order prime to $\operatorname{char}(k)$, and any $d \in \mathbb{Z}$, one can define the d-th Tate twist $C(d)$ as in [158, §7.8]. We can identify

$$\boldsymbol{\mu}_n(d) = \mathbb{Z}/n\mathbb{Z}(d+1)$$

Another particular case is the Galois module $\mathbb{Q}/\mathbb{Z}(d)$ defined in [116, Appendix A]:

$$\mathbb{Q}/\mathbb{Z}(d) = \coprod_{\substack{p \text{ prime} \\ p \neq \operatorname{char}(k)}} \mathbb{Q}_p/\mathbb{Z}_p(d) \qquad \text{where } \mathbb{Q}_p/\mathbb{Z}_p(d) = \varinjlim_m \mathbb{Z}/p^m\mathbb{Z}(d).$$

The functor

$$H^d(*, \mathbb{Q}/\mathbb{Z}(d-1))$$

is another important target for cohomological invariants, especially of degree $d \leq 3$. If $\operatorname{char}(k) = p > 0$, this functor can be modified by adding a p-primary component "$H^d(*, \mathbb{Q}_p/\mathbb{Z}_p(d-1))$" to facilitate characteristic-free theorems.

For example, there is an isomorphism

$$H^2(k, \mathbb{Q}_p/\mathbb{Z}_p(1)) = \varinjlim_m H^2(k, \boldsymbol{\mu}_{p^m}) \xrightarrow{\sim} \operatorname{Br}(k)\{p\}$$

where $\operatorname{Br}(k)\{p\}$ is the p-primary component of the Brauer group [116, Examples A.2, A.3].

15.4. Definition (Cohomological invariant)**.** Let $F : \mathsf{Fields}_{/k} \to \mathsf{Sets}$ be a functor, and let C be a Galois module whose every element has finite order prime to $\operatorname{char}(k)$. A *cohomological invariant* of F with coefficients in C is a natural transformation

$$a : F(*) \longrightarrow \bigoplus_{d \geq 0} H^d(*, C(d-1)).$$

If the image of a_L is contained in $H^d(L, C(d-1))$ for all field extensions L/k, we say a is a cohomological invariant of *degree* d.

The set of all cohomological invariants of F with coefficients in C is denoted by $\operatorname{Inv}(F, C)$. This is an abelian group, with a decomposition

$$\operatorname{Inv}(F, C) = \bigoplus_{d \geq 0} \operatorname{Inv}^d(F, C(d-1))$$

where $\operatorname{Inv}^d(F, C(d-1))$ is the set of invariants of degree d.

15.5. *Cohomological invariants of algebraic groups.* We take some liberties with the notation. If G is an algebraic group we write

$$\mathrm{Inv}(G, C) = \mathrm{Inv}(F, C) \qquad\qquad \text{where } F = H^1(*, G),$$

and say that $a \in \mathrm{Inv}(G, C)$ is a cohomological invariant of G.

An invariant $a \in \mathrm{Inv}(G, C)$ is called *normalised* if it takes the value 0 at the trivial torsor in $H^1(k, G)$, and it is called *constant* if $a_L : H^1(L, G) \to \bigoplus_{d \geq 0} H^d(L, C(d-1))$ is a constant function for all field extensions L/k. We have

$$\mathrm{Inv}(G, C) = \mathrm{Inv}(G, C)_{\mathrm{const}} \oplus \mathrm{Inv}(G, C)_{\mathrm{norm}}$$

where

$$\mathrm{Inv}(G, C)_{\mathrm{const}} \simeq \mathrm{Inv}(1, C) \simeq \bigoplus_{d \geq 0} H^d(L, C(d-1))$$

is the group of constant invariants, and $\mathrm{Inv}(G, C)_{\mathrm{norm}}$ is the group of normalised invariants.

An invariant $a \in \mathrm{Inv}(G, C)$ is called *nontrivial* if for all field extensions L/k there exists a field extension L'/L and some $\zeta \in H^1(L', G)$ such that $a(\zeta) \neq 0$. For example, the cup product $\lambda{\cdot}a$ of a constant $\lambda \in H^{d+1}(k, \mathbb{Z}/2\mathbb{Z})$ and any invariant $a \in \mathrm{Inv}(G, 2)$ is a trivial invariant because $\mathrm{res}_{k^s/k}(\lambda) = 0$.

15.6. *Coefficients in $\mathbb{Z}/n\mathbb{Z}$ and $\mathbb{Q}/\mathbb{Z}$.* A cohomological invariant with coefficients in $\mathbb{Z}/n\mathbb{Z}$ is called a *mod n cohomological invariant.* We write

$$\mathrm{Inv}(F, n) = \bigoplus_{d \geq 0} \mathrm{Inv}^d(F, n) = \bigoplus_{d \geq 0} \mathrm{Inv}^d(F, \mathbb{Z}/n\mathbb{Z}(d-1)).$$

for the set of mod n cohomological invariants of F. We write

$$\mathrm{Inv}(F) = \bigoplus_{d \geq 0} \mathrm{Inv}^d(F) = \bigoplus_{d \geq 0} \mathrm{Inv}^d(F, \mathbb{Q}/\mathbb{Z}(d-1))$$

for the set of cohomological invariants of F with coefficients in $\mathbb{Q}/\mathbb{Z}$.

If n is prime to $\mathrm{char}(k)$ and has prime factorisation $n = p_1^{m_1} \ldots p_r^{m_r}$, then

$$\mathbb{Z}/n\mathbb{Z}(d-1) \simeq \bigoplus_i \mathbb{Z}/p_i^{m_i}\mathbb{Z}(d-1) \subset \mathbb{Q}/\mathbb{Z}(d-1),$$

so there are natural maps $\mathrm{Inv}^d(F, n) \to \mathrm{Inv}^d(F)$ and $\mathrm{Inv}(F, n) \to \mathrm{Inv}(F)$.

15.7. *Maps between groups of invariants.* For any C,

$$\mathrm{Inv}(*, C) : \mathsf{Sets}^{\mathsf{Fields}_{/k}} \longrightarrow \mathsf{Abelian\ Groups}$$

is a contravariant functor. This means that any morphism of set-valued functors $F_1, F_2 : \mathsf{Fields}_{/k} \rightrightarrows \mathsf{Sets}$

$$F_1 \xrightarrow{\ \phi\ } F_2$$

comes with a group homomorphism

$$\mathrm{Inv}(F_1, C) \xleftarrow{\ \phi^*\ } \mathrm{Inv}(F_2, C)$$

where $\phi^*(a) = a \circ \phi$. In particular, a homomorphism of algebraic groups $G_1 \to G_2$ comes with a homomorphism $\mathrm{Inv}(G_2, C) \to \mathrm{Inv}(G_1, C)$.

If $a \in \mathrm{Inv}(F_1, C)$ and $a = \phi^*(b)$ for some $b \in \mathrm{Inv}(F_2, C)$, we say that a is the restriction of b to F_1, and that b is an extension of a to F_2.

15.8. Proposition (Uniqueness and existence of extensions). *Let* $F_1, F_2 : \mathsf{Fields}_{/k} \rightrightarrows \mathsf{Sets}$ *be two functors with a morphism* $\phi : F_1 \to F_2$.

(i) *Suppose that for all field extensions* L/k, *the map* $\phi_L : F_1(L) \to F_2(L)$ *is surjective. The homomorphism*

$$\phi^* : \mathrm{Inv}(F_2, C) \to \mathrm{Inv}(F_1, C)$$

is injective. Its image is the set of invariants $a \in \mathrm{Inv}(F_1, C)$ *with the property that* a_L *factors through* ϕ_L *for all field extensions* L/k.

(ii) *Let* p *be a prime, and suppose that for all field extensions* L/k *and all* $y \in F_2(L)$ *there exists a finite extension* L'/L *of degree prime to* p *such that* $y_{L'}$ *is in the image of* $\phi_{L'} : F_1(L') \to F_2(L')$. *Then*

$$\phi^* : \mathrm{Inv}(F_2, p) \to \mathrm{Inv}(F_1, p)$$

is injective. Its image is the set of invariants $a \in \mathrm{Inv}(F_1, p)$ *with the property that* a_L *factors through* ϕ_L *for all field extensions* L/k. *In particular, if* ϕ_L *is injective for all* L/k, *then* ϕ^* *is an isomorphism.*

Proof. (i) If $\phi^*(b) = 0$, then $b_L \circ \phi_L(x) = 0$ for all $x \in F_1(L)$. The surjectivity assumption implies $b(y) = 0$ for all $y \in F_2(L)$, so $b = 0$. This shows a is injective. Clearly if a is in the image of ϕ^* then a_L must factor through ϕ_L. On the other hand, if a_L factors through ϕ_L then we can define an invariant $b \in \mathrm{Inv}(F_2, C)$ by $b_L(y) = a_L(x)$ where $x \in \phi_L^{-1}(y)$.

(ii) A complete proof is given in [58, §7]. It mostly rests on the fact that restriction $\mathrm{res}_{L'/L} : H^d(L, \mathbb{Z}/p\mathbb{Z}(d-1)) \to H^d(L', \mathbb{Z}/p\mathbb{Z}(d-1))$ has a left inverse $[L' : L]^{-1} \mathrm{cor}_{L'/L} : H^d(L', \mathbb{Z}/p\mathbb{Z}(d-1)) \to H^d(L, \mathbb{Z}/p\mathbb{Z}(d-1))$ whenever L'/L is separable with $[L' : L]$ not divisible by p [72, Proposition 4.2.10]. □

When the assumption of Proposition 15.8 (ii) is satisfied, we say that ϕ is *surjective up to prime-to-p extensions.*

Without some surjectivity condition on $\phi : F_1 \to F_2$, there is no guarantee that invariants of F_1 extend to F_2, even if they do factor through ϕ. And if an invariant of F_1 does extend to F_2, there is no guarantee that the extension is unique.

15.9. *Invariants of degree* 1. Let G be any algebraic group. The homomorphism $G \to G/G^\circ = \pi_0(G)$ is the unique homomorphism from G to an étale group with connected kernel [122, Proposition 5.58]. Suppose $\pi_0(G)$ has order n, and let $g : \pi_0(G)(k^s) \to \mathbb{Z}/n\mathbb{Z}$ be a Γ_k-equivariant group homomorphism. The composition

$$a^g : H^1(k, G) \longrightarrow H^1(k, \pi_0(G)) \xrightarrow{g_*} H^1(k, \mathbb{Z}/n\mathbb{Z})$$

induces an isomorphism [101, Proposition 31.15]

$$\mathrm{Hom}_{\Gamma_k}(\pi_0(G)(k^s), \mathbb{Z}/n\mathbb{Z}) \xrightarrow{\sim} \mathrm{Inv}^1(G, n)_{\mathrm{norm}} = \mathrm{Inv}^1(G)_{\mathrm{norm}} \tag{15.9.1}$$

$$g \longmapsto a^g. \tag{15.9.2}$$

In particular, if G is connected then $\mathrm{Inv}^1(G)_{\mathrm{norm}} = 0$.

15.10. *Invariants of degree* 2. Suppose now that G is semisimple. It has a universal covering $\tilde{G} \to G$ with $\tilde{G}$ a simply connected semisimple group [122, §18 d.]. Let $Z = \pi_1(G) = \ker(\tilde{G} \to G)$, and let $Z^* = \mathrm{Hom}(Z, \mathbf{G}_m)$ be its character group. The group Z is finite, so any $\chi \in Z^*$ factors through $\boldsymbol{\mu}_m$, where m is the order of Z. (Assume for simplicity that $\mathrm{char}(k) \nmid m$.) This gives a map $\chi' : H^2(k, Z) \to H^2(k, \boldsymbol{\mu}_m) \subset H^2(k, \mathbf{G}_m)$. The short exact sequence $Z \to \tilde{G} \to G$ induces a second connecting map, and the composition

$$b_\chi : H^1(k, G) \xrightarrow{\Delta} H^2(k, Z) \xrightarrow{\chi'} H^2(k, \boldsymbol{\mu}_m)$$

is a mod m cohomological invariant. This gives an isomorphism [101, Proposition 31.19]

$$\begin{aligned} Z^* &\xrightarrow{\sim} \mathrm{Inv}^2(G, m)_{\mathrm{norm}} = \mathrm{Inv}^2(G)_{\mathrm{norm}} \\ \chi &\longmapsto b_\chi. \end{aligned}$$

In particular, if G is simply connected then $\mathrm{Inv}^2(G)_{\mathrm{norm}} = 0$.

15.11. *The Rost invariant.* If G is an absolutely simple simply connected algebraic group, $\mathrm{Inv}^3_{\mathrm{norm}}(G)$ is a finite cyclic group with a canonical generator r_G, called the *Rost invariant* [116, Theorem 9.11]. The order of r_G is a positive integer n_G called the *Dynkin index* of G. So we have

$$\mathrm{Inv}^3_{\mathrm{norm}}(G) = \mathrm{Inv}^3_{\mathrm{norm}}(G, n_G) \simeq \mathbb{Z}/n_G\mathbb{Z}.$$

Exact values of n_G have been calculated by Merkurjev for all simply connected groups [116, Appendix B].

For any homomorphism $\alpha : G \to G'$ between absolutely simple simply connected groups, there is an integer $n_\alpha \geq 0$ defined in [116, p. 122] and called the *Rost multiplier*, such that the composition

$$H^1(k, G) \xrightarrow{\alpha_*} H^1(k, G') \xrightarrow{r_{G'}} H^3(k, \mathbb{Q}/\mathbb{Z}(2))$$

is equal to $n_\alpha r_G$. Rost multipliers have some convenient properties, like respecting composition of homomorphisms: $n_{\beta\circ\alpha} = n_\beta n_\alpha$ [116, Proposition 7.9].

15.12. Example (Invariants of $\mathbf{SL}_1(A)$ and $(\mathbf{SL}_n \times \mathbf{SL}_n)/\boldsymbol{\mu}_n$)**.** Let A be a central simple (associative) algebra of degree n not divisible by $\mathrm{char}(k)$. Because $\mathbf{SL}_1(A)$ is simply connected, we have

$$\mathrm{Inv}^1(\mathbf{SL}_1(A)) = \mathrm{Inv}^2(\mathbf{SL}_1(A)) = 0.$$

In Example 12.11 we identified

$$H^1(k, \mathbf{SL}_1(A)) = k^\times / \mathrm{Nrd}(A^\times).$$

There is a cohomological invariant $r \in \mathrm{Inv}^3(\mathbf{SL}_1(A), n)$ [116, Example 2.2] which assigns an element $c\,\mathrm{Nrd}(A^\times) \in k^\times / \mathrm{Nrd}(A^\times)$ to

$$r(c\,\mathrm{Nrd}(A^\times)) = \{c\} \cup [A] \in H^3(k, \boldsymbol{\mu}_n \otimes \boldsymbol{\mu}_n) \subset H^3(k, \mathbb{Q}/\mathbb{Z}(2)).$$

Here, $\{c\}$ is the representative of $ck^{\times n}$ under the isomorphism $H^1(k, \boldsymbol{\mu}_n) \simeq k^\times/k^{\times n}$ (the Galois symbol) and $[A]$ is the representative of A in $H^2(k, \boldsymbol{\mu}_n) \simeq {}_n\operatorname{Br}(k)$. In fact, r is equal to the Rost invariant of $\mathbf{SL}_1(A)$ [71, Théorème 1.1].

Example 12.13 is about the group

$$H_n = \frac{\mathbf{SL}_n \times \mathbf{SL}_n}{\boldsymbol{\mu}_n}.$$

There are two cohomological invariants

$$p \in \operatorname{Inv}^2(H_n, n), \qquad q \in \operatorname{Inv}^3(H_n, n)$$

which assign to a pair $(A, c\operatorname{Nrd}(A^\times))$ the values

$$\begin{aligned} p(A, c\operatorname{Nrd}(A^\times)) &= [A] \in H^2(k, \boldsymbol{\mu}_n) \\ q(A, c\operatorname{Nrd}(A^\times)) &= \{c\} \cup [A] \in H^3(k, \boldsymbol{\mu}_n \otimes \boldsymbol{\mu}_n). \end{aligned}$$

16. Cohomological invariants of quadratic forms

Quadratic form theory is fertile ground for cohomological invariants, and almost all the known mod 2 cohomological invariants have something to do with a quadratic form.

16.1. *Stiefel–Whitney invariants.* Given a quadratic form $q = \langle a_1, \dots, a_n\rangle$ over k, the Stiefel–Whitney classes $w_d(q) \in H^d(k, \mathbb{Z}/2\mathbb{Z})$ are defined as follows and are independent of the choice of diagonalisation of q:

$$\begin{aligned} w_0(q) &= 1, \\ w_1(q) &= \sum_i (a_i) = (a_1 a_2 \dots a_n), \\ w_2(q) &= \sum_{i<j} (a_i)\cdot(a_j), \\ &\vdots \\ w_n(q) &= (a_1)\cdot(a_2)\cdot\dots\cdot(a_n), \\ w_d(q) &= 0 \text{ for all } d > n. \end{aligned}$$

These are mod 2 cohomological invariants of $\mathbf{O}_n$. They have the following properties (as in [158, §17] or [53, (5.7)]):

$$w_d(q \perp q') = \sum_{j=0}^{d} w_j(q) w_{d-j}(q') \tag{16.1.1}$$

$$w_d(q) = \prod_{i \in R} w_{2^i}(q) \qquad \text{if } d = \sum_{i\in R} 2^i, R \text{ any finite subset of } \mathbb{N}. \tag{16.1.2}$$

From (16.1.2) it is clear that $w_1(q) = 0$ implies $w_n(q) = 0$ for all odd n.

16.2. *The ideals $J_m(k)$.* Let us define the chain of ideals

$$J_1(k) \subset J_2(k) \subset \dots \subset H(k)$$

$$\begin{aligned}
J_1(k) &= \{h \in H(k) \colon h\cdot(-1) = 0\}, \\
J_2(k) &= \{h \in H(k) \colon h\cdot(-1)\cdot(-1) = 0\}, \\
J_3(k) &= \dots
\end{aligned}$$

It is clear that $(-1)\cdot J_{m+1}(k) \subset J_m(k)$ for all $m \geq 1$. Also, $J_m(k) = H(k)$ if and only if k has length $\leq 2^{m-1}$ (in the sense of [106, §XI.2]). On the other extreme, if k is a real-closed field then $J_m(k) = \{0\}$ for all $m \geq 1$.

16.3. *Cohomological invariants of I_n^2.* Given $h \in J_1(k)$ and $q = \langle a_1, \dots, a_n \rangle \in I_n^2(L)$, define

$$b^h(q) = h\cdot(a_1)\cdot(a_2)\cdot \dots \cdot(a_{n-1}).$$

Serre showed in [158, Proposition 20.1] that if $n > 2$ and $h \in J_1(k)$, the element $b^h(q)$ does not depend on the choice of diagonalisation of q. Moreover, $h \mapsto b^h$ is an injective map $J_1(k) \to \mathrm{Inv}(I_n^2, 2)$.

Serre classified not only the cohomological invariants of $\mathbf{O}_n$, but also the cohomological invariants of $\mathbf{O}_n^+$. The latter group for $n = 2 \bmod 4$ and $J_1(k) \neq H(k)$ was probably the first known instance of an algebraic group whose ring of mod 2 cohomological invariants is not a free $H(k)$-module.

16.4. Theorem (Serre [158, Theorems 17.3, 19.1, & 20.6])**.**

(i) $\mathrm{Inv}(\mathbf{O}_n, 2)$ *is a free $H(k)$-module with basis* $\{1, w_1, \dots, w_n\}$.

(ii) *If n is odd,* $\mathrm{Inv}(\mathbf{O}_n^+, 2)$ *is a free $H(k)$-module with basis* $\{1, w_2, \dots, w_{n-1}\}$.

(iii) *If $n = 0$ mod 4,* $\mathrm{Inv}(\mathbf{O}_n^+, 2) = \mathrm{Inv}(I_n^2, 2)$ *is a free $H(k)$-module with basis* $\{1, w_2, \dots, w_{n-2}, b^1\}$.

(iv) *If $n = 2$ mod 4 and $n \geq 4$,* $\mathrm{Inv}(\mathbf{O}_n^+, 2) = \mathrm{Inv}(I_n^2, 2)$ *is a direct sum of the free $H(k)$-module with basis* $\{1, w_2, \dots, w_{n-2}\}$ *and the $H(k)$-module*

$$\{b^\lambda \colon \lambda \in J_1(k)\} \simeq J_1(k).$$

16.5. Example. Let $q = \langle a_1, a_2, a_3 \rangle$ be a 3-dimensional quadratic form and $c \in k^\times$. Since $(a_i a_j) = (a_i) + (a_j)$ and $(c)\cdot(c) = (-1)\cdot(c)$ in $H(k)$, it is easy to derive that

$$\begin{aligned}
&w_1(\langle c \rangle q) - w_1(q) = (c) \\
&w_2(\langle c \rangle q) - w_2(q) = (c)\cdot(-1) \\
&w_3(\langle c \rangle q) - w_3(q) = (c)\cdot(-1)\cdot(-1) + (c)\cdot(-1)\cdot w_1(q) + (c)\cdot w_2(q).
\end{aligned}$$

The invariant $v = (-1)\cdot(-1) + (-1)\cdot w_1 + w_2 \in \mathrm{Inv}(\mathbf{O}_3, 2)$ is normalised and constant on similitude classes. On q, it takes the value

$$v(q) = (-a_1 a_2^{-1})\cdot(-a_2 a_3^{-1}).$$

(By our definition of $\mathbf{O}_3$, the basepoint of $H^1(k, \mathbf{O}_3)$ is the class of $\langle 1, -1, 1 \rangle$.) Moreover, any mod 2 invariant of $\mathbf{O}_3$ that is constant on similitude classes and remains so over arbitrary field extensions is an $H(k)$-linear combination of 1 and v.

16.6. *Degree n invariants of I^n.* Since the Milnor Conjectures are true [125, 177], there exists an infinite sequence of mod 2 cohomological invariants

$$e_n : I^n \to H^n(*, \mathbb{Z}/2\mathbb{Z}), \qquad n \geq 0,$$

such that for all fields L/k, $e_n : I^n(L) \to H^n(L, \mathbb{Z}/2\mathbb{Z})$ is the unique additive group homomorphism with

$$e_n(\langle\langle a_1, \dots, a_n \rangle\rangle) = (a_1) \cdot \dots \cdot (a_n) \qquad \text{for all } a_1, \dots, a_n \in L^\times.$$

Moreover, e_n is surjective with kernel I^{n+1}, so

$$I^n(k)/I^{n+1}(k) \simeq H^n(k, \mathbb{Z}/2\mathbb{Z}).$$

The existence of these invariants is a very deep result. However, the invariants e_0 (dimension modulo 2), e_1 (signed discriminant), and e_2 (Clifford invariant), are more classical; see [171, §1] for a nice exposition. The existence of e_3 (Arason invariant [18]) and e_4 [84] was also established prior the resolution of the Milnor Conjectures. A concise exposition on the higher-degree invariants e_n can be found in [24, §1.1].

A basic property of these invariants worth mentioning is that they are constant on similitude classes: since $e_n(I^{n+1}(k)) = 0$, clearly $e_n(\langle c\rangle q) = e_n(q)$ for all $c \in k^\times$ and all $q \in I^n(k)$.

16.7. *The exterior square.* Let (V, q) be an n-dimensional quadratic space over k, with $n \geq 2$. The exterior square of (V, q) is the $\binom{n}{2}$-dimensional k-quadratic space $(\Lambda^2 V, \lambda^2 q)$ defined by

$$\lambda^2 q(v_1 \wedge v_2) = \det \begin{pmatrix} q(v_1, v_1) & q(v_1, v_2) \\ q(v_2, v_1) & q(v_2, v_2) \end{pmatrix}$$

for all $v_1, v_2 \in V$. Clearly, if $q = \langle a_1, \dots, a_n \rangle$ is a diagonalisation of q, then

$$\lambda^2(q) = \mathop{\perp}_{1 \leq i < j \leq n} \langle a_i a_j \rangle.$$

It is clear that for all $c \in k^\times$,

$$\lambda^2(\langle c\rangle q) = \lambda^2(q). \tag{16.7.1}$$

The λ^2-operation extends to a unique map $\lambda^2 : \widehat{W}(k) \to \widehat{W}(k)$ such that $\lambda^2([q]) = [\lambda^2(q)]$ and the following equation holds for all $x, y \in \widehat{W}(k)$ [58, (19.5)]:

$$\lambda^2(x - y) = \lambda^2(x) - xy + \dim y + \lambda^2(y).$$

16.8. Examples. (i) For a 3-dimensional quadratic form $q = \langle a_1, a_2, a_3 \rangle$, it is easy to check that

$$\lambda^2(q) \simeq \langle d \rangle q$$

where $d = a_1 a_2 a_3$ is the unsigned discriminant of q.

(ii) [58, Example 19.1] If $q = n\mathbb{H}$, then

$$\lambda^2(q) \simeq (n^2 - n)\mathbb{H} \perp n\langle -1 \rangle.$$

(iii) [58, Lemma 19.8] If q is an n-Pfister form with $n \geq 1$, then

$$\lambda^2(q) \simeq 2^{n-1}q'.$$

16.9. *Degree $2n$ invariants of I^n.* There is a canonical homomorphism

$$I(k) \longrightarrow \widehat{W}(k), \qquad\qquad q \mapsto \hat{q} = q - \tfrac{\dim q}{2}\mathbb{H}$$

whose image we denote by

$$\widehat{I}(k) = \{\hat{q} \colon q \in I(k)\} \subset \widehat{W}(k).$$

The projection

$$\pi : \widehat{W}(k) \longrightarrow W(k)$$

restricts to an isomorphism

$$\pi|_{\widehat{I}(k)} : \widehat{I}(k) \xrightarrow{\sim} I(k).$$

For $n \geq 1$, there is a map $P_n : I(k) \to W(k)$ defined as follows:

$$P_n(q) = \pi(\lambda^2(\hat{q})) - 2^{n-1}q.$$

By [58, Example 19.7], the following equation holds in $W(k)$, for all even-dimensional quadratic forms q:

$$P_n(q) = \frac{\dim q}{2} + \lambda^2(q) - 2^{n-1}q. \tag{16.9.1}$$

The maps P_n are neither additive nor multiplicative, but they have the following important properties [58, p. 57]:

$$P_n(\langle\langle a_1, \dots, a_n\rangle\rangle) = 0 \qquad \text{for all } a_1, \dots, a_n \in k^\times, \tag{16.9.2}$$
$$P_n(\langle c\rangle q) = P_n(q) + 2^{n-1}\langle\langle c\rangle\rangle q \qquad \text{for all } c \in k^\times, q \in I(k), \tag{16.9.3}$$
$$P_n(x+y) = P_n(x) + xy + P_n(y) \qquad \text{for all } x, y \in I(k), \tag{16.9.4}$$

Using (16.9.2)–(16.9.4), it is easy to show that if ϕ_i are n-Pfister forms and $c_i \in k^\times$,

$$P_n\Big(\sum_i \langle c_i\rangle\phi_i\Big) = \sum_{i<j}\langle c_i c_j\rangle\phi_i\phi_j + 2^{n-1}\sum_i \langle\langle c_i\rangle\rangle\phi_i. \tag{16.9.5}$$

This gives a concrete way of expressing $P_n(q)$ for any $q \in I^n(k)$, and it also implies that

$$P_n(I^n(k)) \subset I^{2n}(k). \tag{16.9.6}$$

For all n, the composition

$$I^n \xrightarrow{P_n} I^{2n} \xrightarrow{e_{2n}} H^{2n}(*, \mathbb{Z}/2\mathbb{Z}) \subset H(*)$$

is a mod 2 cohomological invariant. These invariants appear in Garibaldi's work [58], based on unpublished work by Rost.

17. Cohomological invariants of exceptional groups: a very short survey

We close the chapter with a short survey of known cohomological invariants of exceptional algebraic groups.

17.1. *Invariants of G_2.* The set $H^1(k, G_2)$ classifies octonion algebras over k up to isomorphism. The standard norm n_C of an octonion algebra C is a 3-Pfister quadratic form, so there is an invariant $e \in \mathrm{Inv}^3(G_2, 2)$ defined by

$$e(C) = e_3(n_C) \in H^3(k, \mathbb{Z}/2\mathbb{Z}).$$

This is the Rost invariant of G_2. Serre [158, §18.4] proved that $\mathrm{Inv}(G_2, 2)$ is a free $H(k)$-module

$$\mathrm{Inv}(G_2, 2) = H(k)\cdot 1 \oplus H(k)\cdot e.$$

So, e and 1 are the only nontrivial mod 2 cohomological invariants of G_2.

17.2. *Invariants of F_4.* The set $H^1(k, F_4)$ classifies Albert algebras over k up to isomorphism. To the quadratic trace form $T_J(x) = t(x^2)$ of an Albert algebra J can be associated unique 3- and 5-Pfister forms q_3 and q_5 [158, Theorem 22.4] such that

$$T_J \perp \langle 2\rangle q_3 \simeq \langle 1, 1, 1\rangle \perp \langle 2\rangle q_5.$$

Consequently, there are cohomological invariants $f_3 \in \mathrm{Inv}^3(F_4, 2)$ and $f_5 \in \mathrm{Inv}^5(F_4, 2)$ defined by

$$f_3(J) = e_3(q_3) \in H^3(k, \mathbb{Z}/2\mathbb{Z}), \qquad f_5(J) = e_5(q_5) \in H^5(k, \mathbb{Z}/2\mathbb{Z}).$$

On a reduced Albert algebra $J = \mathcal{H}_3(C, \gamma)$, where C is an octonion algebra and $\gamma \in (k^\times)^3$, these invariants take the values [164, p. 118]

$$f_3(J) = e(C) \qquad f_5(J) = (-\gamma_1\gamma_2^{-1})\cdot(-\gamma_2\gamma_3^{-1})\cdot e(C). \tag{17.2.1}$$

As for mod 3 invariants, there is a subgroup

$$H_3 = \frac{\mathbf{SL}_3 \times \mathbf{SL}_3}{\boldsymbol{\mu}_3} \subset F_4$$

which induces a map, the *first Tits construction* [101, §39], which is surjective up to quadratic extensions:

$$H^1(k, H_3) \longrightarrow H^1(k, F_4) \qquad [(A, \lambda\,\mathrm{Nrd}(A^\times))] \mapsto J(A, \lambda).$$

The invariant $-q \in \mathrm{Inv}(H_3, 3)$ from Example 15.12 factors through the first Tits construction (a difficult theorem!) so it extends uniquely to a cohomological invariant $g_3 \in \mathrm{Inv}^3(F_4, 3)$ called the Serre–Rost invariant [131]. It takes the value

$$g_3(J(A, \lambda)) = [A] \cup \{\lambda\}$$

on first Tits constructions. This invariant detects division algebras: $g_3(J) \neq 0$ if and only if J is a division algebra. The Rost invariant of F_4 is equal to $f_3 + g_3$ [71, Corollaire 6.7].

Serre [158, Theorem 22.5] proved that

$$\mathrm{Inv}(F_4, 2) = H(k)\cdot 1 \oplus H(k)\cdot f_3 \oplus H(k)\cdot f_5$$

and Garibaldi [62, Proposition 8.6] proved that

$$\mathrm{Inv}_{\mathrm{norm}}(F_4, 3) = \bigoplus_{d\geq 0} H^d(k, \boldsymbol{\mu}_3^{\otimes d}) \cdot g_3.$$

In other words, f_3, f_5 and $\pm g_3$ are the only nontrivial normalised invariants of F_4. (Taking cup products by constants from $H(k)$ or $H^d(k, \boldsymbol{\mu}_3^{\otimes d})$ just creates trivial invariants that eventually vanish.)

17.3. *Invariants of E_6^{sc}.* There is a subgroup $F_4 \times \boldsymbol{\mu}_3 \subset E_6^{\mathrm{sc}}$ such that the map

$$H^1(k, F_4 \times \boldsymbol{\mu}_3) \to H^1(k, E_6^{\mathrm{sc}})$$

is surjective [62, Example 9.12] and induces an injective homomorphism

$$\mathrm{Inv}(E_6^{\mathrm{sc}}) \to \mathrm{Inv}(F_4 \times \boldsymbol{\mu}_3).$$

The invariants $f_3, g_3 \in \mathrm{Inv}(F_4)$ extend to E_6^{sc}, but f_5 does not [169]. The following invariant of $F_4 \times \boldsymbol{\mu}_3$

$$([J], ck^{\times 3}) \mapsto g_3(J) \cup \{c\}$$

extends uniquely to an invariant $g_4 \in \mathrm{Inv}^4(E_6^{\mathrm{sc}}, 3)$ [55, Remark 2.12].

By [158, Exercise 22.8],

$$\mathrm{Inv}_{\mathrm{norm}}(E_6^{\mathrm{sc}}, 2) = H(k)\cdot 1 \oplus H(k)\cdot f_3.$$

On the other hand, [62, Proposition 11.9] says that

$$\mathrm{Inv}_{\mathrm{norm}}(E_6^{\mathrm{sc}}, 3) = \bigoplus_{d\geq 0} H^d(k, \boldsymbol{\mu}_3^{\otimes d}) \cdot g_3 \oplus \bigoplus_{d\geq 0} H^d(k, \boldsymbol{\mu}_3^{\otimes d}) \cdot g_4.$$

17.4. *The remaining split and quasi-split exceptional groups.* Besides $\mathrm{Inv}(G_2, p)$, $\mathrm{Inv}(F_4, p)$, and $\mathrm{Inv}(E_6^{\mathrm{sc}}, p)$ for $p = 2, 3$, the following groups are known:

$\mathrm{Inv}(E_6^{\mathrm{ad}}, 2)$	$\mathrm{Inv}(E_7^{\mathrm{sc}}, 3)$	$\mathrm{Inv}(E_8, 5)$
$\mathrm{Inv}({}^2E_6^{\mathrm{sc}}, 3)$	$\mathrm{Inv}(E_7^{\mathrm{ad}}, 3)$	

where ${}^2E_6^{\mathrm{sc}}$ denotes a quasi-split "outer" form of E_6^{sc} that splits over some quadratic extension K/k. Some of these groups are easy enough to calculate, once 17.1–17.3 is known, that it is often given as exercises: see the references in [62, Table 5]. The groups

$\mathrm{Inv}({}^2E_6^{\mathrm{sc}}, 2)$	$\mathrm{Inv}(E_7^{\mathrm{sc}}, 2)$	$\mathrm{Inv}(E_8, 2)$
$\mathrm{Inv}({}^2E_6^{\mathrm{ad}}, 2)$	$\mathrm{Inv}(E_7^{\mathrm{ad}}, 3)$	$\mathrm{Inv}(E_8, 3)$
$\mathrm{Inv}(E_6^{\mathrm{ad}}, 3)$		
$\mathrm{Inv}({}^2E_6^{\mathrm{ad}}, 3)$		

potentially have interesting invariants in them. However, the calculation of any these groups of invariants is so difficult that I am not aware of any progress, either proving or refuting the existence of any new invariants, in the last decade or more.

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Chapter V

Results on bicomposition algebras

This chapter seeks to expand the theory of bicomposition algebras and the algebraic groups that act on them. It is a rich subject, with analogues of most of the classical results on biquaternion algebras, like Albert's and Jacobson's Theorems [2, 112]. Allison's original approach to the subject [6] makes heavy use of Lie algebras, and a characteristic 0 assumption remained a major limitation in most of the important results until now. The approach here uses algebraic groups instead of Lie algebras, and removes that limitation (although we still need $\mathrm{char}(k) \neq 2, 3$). It also makes Galois cohomology available, which is put to use for proving several theorems in new ways.

The bioctonion algebras are particularly interesting, of course, because of the connection to E_8. Sometimes, these algebras surface in different contexts too because of the connection to $\mathbf{Spin}_{14}$. When they are used for questions in representation theory [61, 135], geometry [1], or quadratic form theory [142], the algebraic structure gets forgotten to some extent because there are other tools available.

To explain why it is valuable to study the algebraic structure on these tensor products, it is fitting to draw a comparison to Albert algebras. Thanks to an intensive record of research, we understand a lot about Albert algebras, including information about constructing and classifying them, the division algebras among them, their cubic norms, quadratic traces, special subalgebras, isotopes, automorphism groups, structure groups, and cohomological invariants [81, 114, 126, 128, 130, 131, 167, 169].

If we had no substantial theory of Jordan algebras, we would still know about the 27-dimensional representation for F_4 and E_6^{sc} and a certain cubic invariant polynomial. But we would understand much less about nonsplit forms of F_4 and ${}^1E_6$, and many mysteries would have persisted for much longer, like the answer to the Kneser–Tits problem for groups of type $E_{7,1}^{78}$ and $E_{8,2}^{78}$ [15, 168], and even Serre's Conjecture II for F_4 [22, §6.2]. With this kind of motivation in mind, we aim to advance the theory of bioctonion algebras as far as possible.

Our approach also leads to a new proof of Rost's Theorem on 14-dimensional quadratic forms in I^3, which is valuable because the original proof has only ever been published as a sketch (in [142] and [58, Theorem 21.3]). In addition, we make significant contributions to understanding the octic $\mathbf{Spin}_{14}$-invariant polynomial. For instance, we give an explicit description of its matrix factorisation, discovered in [1], and prove that this factorisation exists also for twisted forms of the octic over non-algebraically closed fields.

18. Structure groups

The goal of this section is to determine precisely the connected structure groups of all bicomposition algebras. So, in some sense it is an extension of Section 11, but here the approach is rational and deals uniformly with all forms of these structurable algebras, rather than just the split forms.

We deal with the associative algebras first, followed by the nonassociative algebras. A quadratic form called the Albert form plays a very important role. Some of the results in this section appear implicitly or explicitly in [6]. However, we make fewer restrictions on the characteristic of k (only assuming $\mathrm{char}(k) \neq 2, 3$) and we work exclusively with algebraic groups rather than their Lie algebras.

18.1. *Structure groups of associative central simple algebras with involution.* Let $(A, -)$ be an associative central simple algebra with involution over k and let $F = Z(A)$, which is either k or a quadratic étale extension of k. There are two obvious subgroups of $\mathbf{Str}(A, -)$, namely $\mathbf{Aut}(A, -)$ and the group of invertible left-multiplication operators. The latter is a copy of $\mathbf{GL}_1(A)$ (see Lemma 2.11). In [14, §10 (1)] it is shown that the abstract group $\mathrm{Str}(A, -)$ is indeed generated by $L_{A^\times}$ and $\mathrm{Aut}(A, -)$, which gives a semidirect product decomposition:

$$\mathrm{Str}(A, -) = L_{A^\times} \rtimes \mathrm{Aut}(A, -). \tag{18.1.1}$$

Using the fact that $\mathbf{Str}(A, -)$ and $\mathbf{Aut}(A, -)$ are smooth, one can make this into a statement about algebraic groups, namely that there is a split short exact sequence with $e : \alpha \mapsto L_{\alpha(1)}^{-1}\alpha$,

$$1 \longrightarrow \mathbf{GL}_1(A) \longrightarrow \mathbf{Str}(A, -) \underset{i}{\overset{e}{\rightleftarrows}} \mathbf{Aut}(A, -) \longrightarrow 1. \tag{18.1.2}$$

However, we prefer a different description so that we can express the semisimple structure group as an almost-direct product of simple groups (this rewriting is comparable to Lemma 8.3). Consider the homomorphism

$$\phi : \mathbf{GL}_1(A) \times \mathbf{Sim}(A, -) \to \mathbf{Str}(A, -)^\circ, \qquad \phi_R(x, y) = L_x R_{y^{-1}}$$

for all $x \in A_R^\times$ and $y \in \mathrm{Sim}(A_R, -)$. It is easy to show that

$$\ker(\phi_R) = \{(c, c^{-1})\colon c \in F_R^\times\} \simeq F_R^\times.$$

We have $\mathbf{Aut}(A, -)^\circ \subset \mathbf{Aut}_F(A, -) = \mathrm{Int}(\mathbf{Sim}(A, -))$ [101, (23.3)], so by (18.1.1) every isotopy in $\mathbf{Str}(A, -)^\circ(k^a)$ is of the form $L_w \,\mathrm{Int}(z) = L_w L_z R_{z^{-1}} = L_{wz} R_{z^{-1}} = \phi(wz, z^{-1})$ for some $w \in \mathbf{GL}_1(A)(k^a)$ and $z \in \mathbf{Sim}(A, -)(k^a)$. Since all the groups involved are smooth, ϕ is a surjection. This yields the exact sequence:

$$1 \longrightarrow \mathbf{GL}_1(F) \longrightarrow \mathbf{GL}_1(A) \times \mathbf{Sim}(A, -) \xrightarrow{\phi} \mathbf{Str}(A, -)^\circ \longrightarrow 1.$$

We have proved:

18.2. Theorem. *Let $(A, -)$ be an associative central simple algebra with involution, and let $F = Z(A)$. Then*

$$\mathbf{Str}(A, -)^\circ \simeq \frac{\mathbf{GL}_1(A) \times \mathbf{Sim}(A, -)}{\mathbf{GL}_1(F)}.$$

To situate this in the context of this chapter, suppose $(A,-)$ is an $(4,m)$-product algebra where $m \leq 4$. Using the notation set up in 7.3,

$$\mathbf{Sim}(A,-) = \begin{cases} \mathbf{GO}(A,-) & \text{if } (m_1,m_2) = (4,4) \\ \mathbf{GU}(A,-) & \text{if } (m_1,m_2) = (4,2) \\ \mathbf{GSp}(A,-) & \text{if } (m_1,m_2) = (4,1). \end{cases}$$

18.3. *The Albert form Q and the $\natural$-map.* Allison in [6, §3] defined some maps which turn out to be important for the classification of (m_1,m_2)-product algebras up to isotopy.

Let $(A,-)$ be an (m_1,m_2)-product algebra where $m_1 \geq m_2$ and $S = \mathrm{Skew}(A,-)$. To begin with, suppose that $(A,-) = C_1 \otimes C_2$ is decomposable and write $S_i = \mathrm{Skew}(C_i) = (C_i)_0$ for the skew subspace of C_i, and n_i for its norm. The *Albert form* is the following nondegenerate quadratic form defined on $S = S_1 \oplus S_2$:

$$Q(s_1+s_2) = n_1(s_1) - n_2(s_2) \qquad \text{for all } s_i \in S_i. \tag{18.3.1}$$

In other words $Q = n_1' \perp \langle -1\rangle n_2'$. The identity $Q = n_1 - n_2 = n_1' - n_2'$ holds in $W(k)$.

Another important map is the isometry $\natural \in \mathrm{O}(S,Q)$, defined as:

$$(s_1+s_2)^\natural = s_1 - s_2 \qquad \text{for all } s_i \in S_i. \tag{18.3.2}$$

Note that if $m_2 = 1$ then $(A,-)$ is just a composition algebra, Q is the pure norm, and $\natural$ is the identity map. If $m_1 = m_2$, the definitions (18.3.1) and (18.3.2) depend on a choice of decomposition for $(A,-)$. By Theorem 9.9, the only possible decompositions (into involution-stable subalgebras) are $C_1 \otimes C_2$ and $C_2 \otimes C_1$. Changing the order flips the sign of Q, but this shall have no material effect on anything that follows.

18.4. *Q and $\natural$ for indecomposable algebras.* If $(A,-)$ is indecomposable then it is of the form $\mathrm{cor}_{E/k}(C)$ for a quadratic field extension E/k and an m-dimensional composition algebra C over E, where $m = 4$ or 8. Write n for the norm of C. Since $(A_E,-) = {}^\iota C \otimes_E C$ is decomposable, we have an Albert form $\iota.n' \perp \langle -1\rangle n'$ defined on S_E, in the sense of 18.3. Unfortunately, this quadratic form is never in the image of $\mathrm{res}_{E/k} : W(k) \to W(E)$ and so there is no quadratic form on S whose extension to E is isometric to $\iota.n' \perp \langle -1\rangle n'$. Nevertheless, it is possible to nominate a pair of maps Q and $\natural$, which play more or less the same role as the ones from 18.3.

Suppose that $E = k(\sqrt{d})$ for some non-square $d \in k$. Define the following nondegenerate quadratic form on S:

$$Q({}^\iota s \otimes 1 + 1 \otimes s) = \sqrt{d}\big(\iota(n(s)) - n(s)\big) \qquad \text{for all } s \in C_0.$$

For context, Q is the Scharlau transfer (which is defined only later in 22.1) of n' along the linear functional $s : E \to k$,

$$s(a + b\sqrt{d}) = -2bd \qquad \text{for all } a, b \in k.$$

In other words, $Q = T_{E/k}(\langle -\sqrt{d}\rangle n')$.

Likewise, we define a map $\natural : S \to S$ as follows:

$$({}^\iota s \otimes 1 + 1 \otimes s)^\natural = \sqrt{d}({}^\iota s \otimes 1 - 1 \otimes s) = -{}^\iota(\sqrt{d}s) \otimes 1 - 1 \otimes (\sqrt{d}s) \quad \text{for all } s \in C_0.$$

Clearly Q_E is similar to the Albert form of $(A_E, -)$; the two forms merely differ by a factor $\sqrt{d}$. And similarly, $\natural_E$ is $\sqrt{d}$ times the $\natural$-map on S_E. The definitions of Q and $\natural$ depend on a choice of d, but this choice does not make a material difference in any subsequent statements. Also note that $\natural$ is no longer an isometry of Q but rather a similitude of Q with multiplier d.

18.5. *Generalities on invertible skew elements and isotopies.* Up to and including 18.8, we continue to assume that $(A, -)$ is any (m_1, m_2)-product algebra over k, $S = \mathrm{Skew}(A, -)$, and Q and $\natural$ are as defined in 18.3 or 18.4.

Recall from 2.10 that if R is a commutative unital k-algebra and $s \in S_R$, then s is conjugate-invertible if and only if $L_s \in \mathrm{GL}(A_R)$. In that case $L_s^{-1} = -L_{\hat{s}}$. Therefore, if $c \in R$ and $s \in S_R$, then $cs \in {S_R}^*$ if and only if $c \in R^\times$ and $s \in {S_R}^*$. Given $s \in S_R$, one can calculate directly (as in [6, Proposition 3.3]) that

$$L_s L_{s^\natural} = -Q_R(s)\,\mathrm{id}\,. \tag{18.5.1}$$

This implies that $s \in {S_R}^*$ if and only if $Q_R(s) \in R^\times$. We can calculate the conjugate inverse of s if it has one:

$$\hat{s} = -L_s^{-1}(1) = Q_R(s)^{-1} L_{s^\natural}(1) = Q_R(s)^{-1} s^\natural. \tag{18.5.2}$$

Recall the map $\psi : A_R \times A_R \to S_R$ defined in (5.1.1). Given $\alpha \in \mathrm{Str}(A_R, -)$ we define $\alpha_S \in \mathrm{End}_R(S_R)$,

$$\alpha_S(s) = \tfrac{1}{2}\psi\big(\alpha(s), \alpha(1)\big) \qquad \text{for all } s \in S. \tag{18.5.3}$$

It turns out that $\alpha_S \in \mathrm{GL}(S_R)$ [14, Lemma 12.1], and for all $s \in S_R$,

$$L_{\alpha_S(s)}\hat{\alpha} = \alpha L_s. \tag{18.5.4}$$

Exchanging the roles of α and $\hat{\alpha}$, it also holds that $L_{\hat{\alpha}_S(s)}\alpha = \hat{\alpha}L_s$. If $s, t \in S_R$, then

$$\alpha L_s L_t \alpha^{-1} = (\alpha L_s \hat{\alpha}^{-1})(\hat{\alpha} L_t \alpha^{-1}) = L_{\alpha_S(s)} L_{\hat{\alpha}_S(t)}. \tag{18.5.5}$$

Furthermore, if s is conjugate-invertible then setting $t = \hat{s}$ yields

$$L_{\alpha_S(s)} L_{\hat{\alpha}_S(\hat{s})} = \alpha L_s L_{\hat{s}} \alpha^{-1} = \alpha(-\mathrm{id})\alpha^{-1} = -\mathrm{id}\,.$$

It follows that $\alpha_S(s)$ is conjugate-invertible if and only if s is so. We can then derive:

$$\widehat{\alpha_S(s)} = L_{\widehat{\alpha_S(s)}}(1) = -L^{-1}_{\alpha_S(s)}(1) = L_{\hat{\alpha}_S(\hat{s})}(1) = \hat{\alpha}_S(\hat{s}). \tag{18.5.6}$$

18.6. Proposition. *There is a homomorphism of algebraic groups*

$$\gamma : \mathbf{Str}(A, -) \to \mathbf{GO}(S, Q)$$

with $\gamma(\alpha) = \alpha_S$ for all $\alpha \in \mathrm{Str}(A, -)$, where α_S is the linear map defined in (18.5.3)*.*

Proof. It is shown in [14, Lemma 12.1] that $\alpha_S(\psi(x, y)) = \psi(\alpha(x), \alpha(y))$ for all $\alpha \in \mathrm{Str}(A_R, -)$ and all $s \in S_R$. It follows that

$$(\alpha\beta)_S(s) = \frac{1}{2}\psi(\alpha\beta(s), \alpha\beta(1)) = \alpha_S(\frac{1}{2}\psi(\beta(s), \beta(1))) = \alpha_S\beta_S(s)$$

for all $\alpha, \beta \in \mathrm{Str}(A_R, -)$. Clearly $(\mathrm{id}_A)_S = \mathrm{id}_S$, so $\alpha \mapsto \alpha_S$ is a homomorphism $\mathrm{Str}(A_R, -) \to \mathrm{GL}(S_R)$. It is also clear that this homomorphism is functorial in R, so it defines a morphism of algebraic groups $\mathbf{Str}(A, -) \to \mathbf{GL}(S)$.

By Theorem 7.2, $\mathbf{Str}(A, -)$ is smooth. To show that the morphism of the previous paragraph factors through $\mathbf{GO}(S, Q)$, it suffices to show $\gamma_{k^a}(\mathrm{Str}(A_{k^a}, -)) \subset \mathrm{GO}(S_{k^a}, Q_{k^a})$. This is covered by the remaining part of the proof, where we assume that $k = k^a$.

We follow the same approach as [6, Proposition 5.2], but with slightly more details included, to show that for all $\alpha \in \mathrm{Str}(A, -)$, α_S is a similitude of Q. Applying (18.5.2) to both sides of (18.5.6) yields that

$$Q(\alpha_S(s))^{-1}\alpha_S(s)^\natural = \hat{\alpha}_S(Q(s)^{-1}s^\natural) = Q(s)^{-1}\hat{\alpha}_S(s^\natural) \qquad \text{for all } s \in S^*.$$

Define the rational function $\rho(s) = Q(\alpha_S(s))Q(s)^{-1}$ on S, which gives

$$\alpha_S(s)^\natural = \rho(s)\hat{\alpha}_S(s^\natural) \qquad \text{for all } s \in S^*. \tag{18.6.1}$$

We shall show that $\rho : S^* \to k$ is a constant function. Fix a particular $s_0 \in S^*$, and let $t \in S^*$ be arbitrary with the intention of showing that $\rho(t) = \rho(s_0)$. Clearly $\rho(s_0) = \rho(\lambda s_0)$ for all $\lambda \in k^\times$, so we may assume that s_0 and t are linearly independent. Working with (18.6.1), we obtain:

$$\begin{aligned}\rho(s_0)\hat{\alpha}_S(s_0^\natural) + \rho(t)\hat{\alpha}_S(t^\natural) = \alpha_S(s_0)^\natural + \alpha_S(t)^\natural = \alpha_S(s_0 + t)^\natural = \rho(s_0 + t)\hat{\alpha}_S((s_0 + t)^\natural)\\ = \rho(s_0 + t)\hat{\alpha}_S(s_0^\natural) + \rho(s_0 + t)\hat{\alpha}_S(t^\natural).\end{aligned}$$

Since s_0 and t are linearly independent, so are $\hat{\alpha}_S(s_0^\natural)$ and $\hat{\alpha}_S(t^\natural)$, and it follows that $\rho(s_0) = \rho(s_0 + t) = \rho(t)$. Let $\mu(\alpha_S) = \rho(s_0) \in k^\times$. We have proved

$$Q(\alpha_S(s)) = \mu(\alpha_S)Q(s) \qquad \text{for all } s \in S^*. \tag{18.6.2}$$

Since we showed in 18.5 that $Q(s) = Q(\alpha_S(s)) = 0$ for all $s \in S \setminus S^*$, equation (18.6.2) holds for all $s \in S$. The conclusion is that $\alpha_S \in \mathrm{GO}(S, Q)$ and its multiplier is $\mu(\alpha_S)$. □

Note that we can use (18.6.1) to determine the effect of composing $\wedge$ with γ:

$$\gamma_R(\hat{\alpha})(s) = \hat{\alpha}_S(s) = \mu(\alpha_S)^{-1}\alpha_S(s^\natural)^\natural \quad \text{for all } \alpha \in \mathrm{Str}(A, -) \text{ and } s \in S. \tag{18.6.3}$$

18.7. *A composition property of the Albert form.* The multiplication operators L_s, $s \in S^*$, are important examples of isotopies (see 2.10). Applying the map γ to L_s leads us to an interesting observation. It is easy to show from the definition that $\gamma(L_s)(t) = (L_s)_S(t) = -sts$ for all $s \in S^*$, $t \in S$. (Note that we can write sts without brackets because of the skew-alternativity property of structurable algebras.) By (18.5.1), this yields $(L_s)_S(s^\natural) = -ss^\natural s = Q(s)s$ and so $\mu((L_s)_S) = Q(s)^2$ is the relevant multiplier. Since $Q((L_s)_S(t) = Q(-sts) = Q(sts)$, we have derived the interesting identity

$$Q(sts) = Q(s)^2Q(t) \qquad \text{for all } s, t \in S^*. \tag{18.7.1}$$

Additionally, we have $L_{sts} = L_sL_tL_s$ by (2.7.2), so $Q(sts) = 0$ if and only if $Q(s)^2Q(t) = 0$, hence the identity (18.7.1) holds for all $s, t \in S$.

18.8. Proposition (Allison [6, Proposition 4.2]). *There exists a unique algebra homomorphism* $\theta : C^+(S, Q) \to \mathrm{End}_k(A)$ *such that*

$$\theta(st) = -L_s L_{t^\natural} \qquad \text{for all } s, t \in S.$$

Proof. By the universal property of the even Clifford algebra [101, Lemma 8.1], it is good enough to show that $-L_s L_{s^\natural} = Q(s)\,\mathrm{id}$ and $(-L_r L_{s^\natural})(-L_s L_{t^\natural}) = -Q(s) L_r L_{t^\natural}$ for all $r, s, t \in S$. These identities are evident from (18.5.1). ☐

18.9. *Structure groups of octonion algebras.* Let $(C, -)$ be an octonion algebra over k with its standard involution and norm n. Since $C^+(C_0, n') \simeq M_8(k) \simeq \mathrm{End}_k(C)$, the representation θ is an isomorphism. The generators

$$\{st : s, t \in C_0, Q(s), Q(t) \neq 0\}$$

of $\Gamma^+(C_0, n')$ are mapped to isotopies $-L_s L_t \in \mathrm{Str}(C, -)$. Since $\mathbf{\Gamma}^+(C_0, n')$ is connected, θ induces an injective homomorphism $\theta' : \mathbf{\Gamma}^+(C_0, n') \to \mathbf{Str}(C, -)^\circ$. It is easy to calculate from the definition that $\gamma_R(L_s) = \gamma_R(-L_s) = -L_s R_s|_{S_R}$ for all $s \in (C_0)^*_R$. We calculate that for all $s, t \in (C_0)^*_R$,

$$\gamma_R \circ \theta'_R(st) = \gamma_R(-L_s L_t) = \gamma_R(-L_s)\gamma_R(L_t) = L_s R_s L_t R_t|_{(C_0)_R}.$$

Let $\rho_s \in \mathrm{O}(C_0, n')$ be the reflection about an anisotropic vector $s \in C_0$. It turns out that $L_s R_s|_{C_0} = n(s)\rho_s$ [164, p. 44], so $\gamma_R \circ \theta'_R(st) = n(st)\rho_s\rho_t$. Meanwhile, in the vector representation $\chi : \mathbf{\Gamma}^+(V, q)$ it is well-known that $\chi_R(s) = \rho_s$ [94, p. 239], so $\chi_R(st) = \rho_s\rho_t$. This proves that $\gamma_R \circ \theta'_R$ agrees with χ_R modulo scalars.

Let $\gamma' : \mathbf{Str}(C, -)^\circ \to \mathbf{O}^+(C_0, n')$ be the composition of γ and the natural surjection $\mathbf{GO}^+(C_0, n') = \mathbf{G}_m \cdot \mathbf{O}^+(C_0, n') \to \mathbf{O}^+(C_0, n')$. We have proved that the following diagram commutes:

$$\begin{tikzcd}
1 \arrow[r] & \mathbf{G}_m \arrow[r] \arrow[d, "\simeq"] & \mathbf{\Gamma}^+(C_0, n') \arrow[r, "\chi"] \arrow[d, "\theta'"] & \mathbf{O}^+(C_0, n') \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & \mathbf{G}_m \arrow[r] & \mathbf{Str}(C, -)^\circ \arrow[r, "\gamma'"] & \mathbf{O}^+(C_0, n') \arrow[r] & 1.
\end{tikzcd} \tag{18.9.1}$$

Since $\chi = \gamma' \circ \theta'$ is surjective, γ' must be surjective too. Now suppose $\alpha \in \mathrm{Str}(C_R, -)$ and $\gamma_R(\alpha) = r\,\mathrm{id}$ for some $r \in R^\times$. Then $\gamma_R(\hat{\alpha}) = r^{-1}\,\mathrm{id}$, according to (18.6.3). By (18.5.5), we have $\alpha L_s L_t \alpha^{-1} = L_{rs} L_{r^{-1}t} = L_s L_t$ for all $s, t \in (C_0)_R$. But $\{L_s L_t : s, t \in C_0\}$ generates $\mathrm{End}_R(C_R)$ as an R-algebra because θ is surjective. So $\alpha \in Z(\mathrm{GL}(C_R)) = R^\times\,\mathrm{id} \subset \mathrm{Str}(C_R, -)$. Therefore $\ker(\gamma')$ is the central torus $\mathbf{G}_m$. This shows that the bottom row of (18.9.1) is exact. All the groups involved are smooth and a standard diagram chase (the five lemma) proves:

18.10. Theorem. *Let* $(C, -)$ *be an octonion algebra over* k *with standard involution and norm* n. *The map*

$$\theta' : \mathbf{\Gamma}^+(C_0, n') \to \mathbf{Str}(C, -)^\circ$$

is an isomorphism.

For context, the bottom row of (18.9.1) is exactly the same as the second exact sequence in [127, §4.13], although this is not obvious at first sight. If C is an octonion division algebra, $\mathrm{Str}(C,-)$ is the same as the group X_1 studied in [176, (37.23)].

18.11. *Structure groups of* $(8,m)$*-product algebras.* The next few subsections are all about adapting the arguments from 18.9 to $(8,m)$-product algebras where $m \geq 2$, i.e. fitting their structure groups into commutative diagrams that are easy to understand.

First let us fix some notation. The right-multiplication operators give $\mathrm{End}_k(A)$ the structure of a right N-module, where $N = \mathrm{Nuc}(A)$. Define $\mathrm{End}_N(A)$ to be the ring of N-endomorphisms. In light of Lemma 2.11, there is an embedding

$$R : \mathbf{GL}_1(N) \to \mathbf{Str}(A,-).$$

Define $\mathbf{Str}_N(A,-)$ to be the centraliser of the image of $\mathbf{GL}_1(N)$ in $\mathbf{Str}(A,-)$. We also fix the notation $Z = Z(C^+(S,Q))$.

18.12. Lemma. *Let* $(A,-)$ *be an* $(8,m)$*-product algebra with* $m \geq 2$*.*

(i) [6, Theorem 4.5] *The image of* $\theta : C^+(S,Q) \to \mathrm{End}_k(A)$ *is* $\mathrm{End}_N(A)$*.*

(ii) *The algebra homomorphism* θ *induces a homomorphism of algebraic groups*

$$\theta' : \mathbf{\Omega}(S,Q) \to \mathbf{Str}_N(A,-)^\circ.$$

(iii) *If* $m = 2$ *then* θ' *is injective. If* $m = 4$ *or* 8 *then*

$$\ker(\theta'_R) = \{ce_1 + 1e_2 : c \in R^\times\} \simeq R^\times$$

where $e_1, e_2 \in Z$ *are orthogonal idempotents such that* $1 = e_1 + e_2$*.*

(iv) *If* k *is algebraically closed, then the* k*-subalgebra of* $\mathrm{End}_k(A)$ *generated by* $\theta'_k(\mathrm{Spin}(S,Q))$ *is* $\mathrm{End}_N(A)$*.*

Proof. (i) Clearly $0 \neq \theta(C^+(S,Q)) \subset \mathrm{End}_N(A)$ because left-multiplications commute with right-multiplications by nuclear elements. If $m = 8$ then Q is a 14-dimensional form with trivial discriminant and Clifford invariant, so the even Clifford algebra

$$C^+(S,Q) \simeq M_{64}(k) \times M_{64}(k)$$

is split. Since $N = k$ we have $\mathrm{End}_N(A) = \mathrm{End}_k(A) \simeq M_{64}(k)$, so θ must kill one of the full matrix subalgebras of $C^+(S,Q)$ and map the other one isomorphically onto $\mathrm{End}_k(A)$.

If $m = 4$ then Q is a 10-dimensional form whose discriminant is trivial and whose Clifford invariant is the Brauer class of $N = C_2$, so

$$C^+(S,Q) \simeq M_8(C_2) \times M_8(C_2).$$

Then $\theta(C^+(S,Q)) \subset \mathrm{End}_N(A,-) = \mathrm{End}_{C_2}(C_1 \otimes C_2) \simeq M_8(C_2)$ and the conclusion follows as it did before.

If $m = 2$ then Q is an 8-dimensional form whose discriminant is the class of $N = Z(A) = C_2$, and by [94, Theorem 4.14],

$$C^+(S,Q) \simeq M_8(C_2).$$

Now $\operatorname{End}_N(A) = \operatorname{End}_{C_2}(C_1 \otimes C_2) \simeq M_8(C_2)$. If C_2 is a field, the conclusion is clear. More generally, if $s_1, \ldots, s_7 \in S_1$ and $t \in S_2$ constitute an orthogonal basis for S, then the centre of $C^+(S,Q)$ is $Z = k[s_1 \ldots s_7 t] \simeq C_2$ [94, p. 237] and it is easy to see that

$$\theta(s_1 \ldots s_7 t) = (-L_{s_1} L_{s_2^\natural})(-L_{s_3} L_{s_4^\natural})(-L_{s_5} L_{s_6^\natural})(-L_{s_7} L_{t^\natural}) = -L_{s_1} \ldots L_{s_7} L_t \notin k\,\mathrm{id}\,.$$

because $L_{s_1} \ldots L_{s_7} L_t(C_1) \subset tC_1$ and $C_1 \cap tC_1 = \{0\}$. Therefore θ is injective on Z, so it is an isomorphism onto its image $\operatorname{End}_N(A)$.

(ii) Since $\mathbf{\Omega}(S,Q)$ is smooth and connected, it suffices to show that

$$\theta'_{k^a}\big(\mathbf{\Omega}(S,Q)(k^a)\big) \subset \mathbf{Str}_N(A,-)(k^a).$$

We assume $k = k^a$ for ease of notation; this implies $A = C_1 \otimes C_2$ is decomposable (and the C_i's are split composition subalgebras of A). The group $\Omega(S,Q)$ is generated by its subgroups $\Gamma^+(S,Q)$ and $Z^\times$ [101, Lemma 13.20]. Since

$$\theta'(s_1 s_2) = -L_{s_1} L_{s_2^\natural} \in \operatorname{Str}_N(A,-)$$

for all $s_1, s_2 \in S^*$, and $\Gamma^+(S,Q)$ is generated by elements of the form $s_1 s_2$ where the s_i are anisotropic, we have $\theta'(\Gamma^+(S,Q)) \subset \operatorname{Str}_N(A,-)$. Now $Z^\times = k^\times e_1 \times k^\times e_2$ where $e_1, e_2 \neq 1$ are a pair of orthogonal idempotents such that $e_1 + e_2 = 1$. From the proof of (i), it is clear that if $m = 4$ or 8 then $\theta(e_i) \in \{0, \mathrm{id}\}$, which implies $\theta'(Z^\times) = k^\times \,\mathrm{id} \subset \operatorname{Str}_N(A,-)$. If $m = 2$ then θ is an isomorphism so $\theta(Z) = Z(\operatorname{End}_{Z(A)}(A)) = R_{Z(A)}$ and $\theta'(Z^\times) = R^\times_{Z(A)} \subset \operatorname{Str}_N(A,)$ by Lemma 2.11.

(iii) For $m = 2$, this is clear because θ itself is injective. For $m = 4$ or 8, it follows from the proof of (i) that

$$\ker(\theta'_R) = \{xe_1 + 1e_2 \colon x \in C^+(S_R, Q_R)\} \cap \Omega(S_R, Q_R).$$

Recall that $\Omega(S_R, Q_R) \subset \operatorname{Sim}(C^+(S_R, Q_R), \tau)$ where τ is the main involution on $C^+(S,Q)$. In this situation, τ is an involution of the second kind [101, Proposition 8.4]. So if $y = xe_1 + 1e_2 \in \Omega(S_R, Q_R)$ then $y\tau(y) = xe_1 + \tau(x)e_2 \in R^\times(e_1 + e_2)$ implies $x = \tau(x) \in R^\times$. This shows that

$$\ker(\theta'_R) = \{ce_1 + 1e_2 \colon c \in R^\times\} \simeq R^\times.$$

(iv) This is simply because $C^+(S,Q)$ is linearly spanned by $\Gamma^+(S,Q)$, and it is easy to show that k being algebraically closed implies $\Gamma^+(S,Q) = Z^\times.\operatorname{Spin}(S,Q)$. Therefore $Z^\times \cup \operatorname{Spin}(S,Q)$ generates $C^+(S,Q)$ as a k-algebra, so the set

$$\theta'_k(Z^\times \cup \operatorname{Spin}(S,Q)) = k^\times \,\mathrm{id} \cup \theta'_k(\operatorname{Spin}(S,Q))$$

generates $\theta'_k(C^+(S,Q)) = \operatorname{End}_N(A)$ as a k-algebra. □

Recall the definitions of the following homomorphisms from 14.1, 18.6, and 18.12 (ii).

$$\chi' : \mathbf{\Omega}(V,q) \to \mathbf{PGO}^+(V,q),$$
$$\gamma : \mathbf{Str}(A,-) \to \mathbf{GO}(S,Q),$$
$$\theta' : \mathbf{\Omega}(S,Q) \to \mathbf{Str}(A,-)^\circ.$$

Let $\gamma' : \mathbf{Str}(A,-) \to \mathbf{PGO}(S,Q)$ be the composition of γ with the natural homomorphism $\mathbf{GO}(S,Q) \to \mathbf{PGO}(S,Q)$, and let $\gamma'' : \mathbf{Str}(A,-)^\circ \to \mathbf{PGO}^+(S,Q)$ be the restriction of γ' to the identity component.

18.13. Lemma. *Let $(A,-)$ be an $(8,m)$-product algebra with $m \geq 2$, and let $N = \mathrm{Nuc}(A)$.*

(i) $\ker(\gamma')$ *is the group of right multiplications by nuclear elements, i.e., the image of* $\mathbf{GL}_1(N) \to \mathbf{Str}(A,-)$.

(ii) $\gamma'' \circ \theta' = \chi'$.

Proof. Towards (i), suppose $\alpha \in \ker(\gamma'_R) = \gamma_R^{-1}(R^\times \mathrm{id})$. This means there exists $r \in R^\times$ with $\alpha_S = r\,\mathrm{id}$. According to (18.6.3) and (18.5.5), we have $\alpha L_s L_t \alpha^{-1} = L_{rs} L_{r^{-1}t} = L_s L_t$ for all $s, t \in S_R$. Now $\{L_s L_t \colon s, t \in S\}$ generates $\mathrm{End}_{N_R}(A_R)$ as an R-algebra because $\theta(C^+(S,Q)) = \mathrm{End}_N(A)$, as we showed in Lemma 18.12 (i). So α centralises $\mathrm{End}_{N_R}(A_R)$ in $\mathrm{End}_R(A_R)$. The double centraliser theorem [94, Theorem 4.10] implies $\alpha \in N_R$. So $\ker(\gamma'_R)$ is contained in the image of $\mathbf{GL}_1(N) \to \mathbf{Str}(A,-)$. It is easy to check that this containment is an equality.

For (ii), let $x \in \Omega(S_R, Q_R)$ and let $\alpha = \theta'_R(x)$. By definition, $\chi'_R(x) = \bar{\beta} \in \mathrm{PGO}^+(S_R, Q_R)$ for some $\beta \in \mathrm{GO}^+(S_R, Q_R)$ such that

$$\mathrm{Int}(x)(st) = C_R(\beta)(st) = \mu(\beta)^{-1}\beta(s)\beta(t)$$

for all $s, t \in S_R$. Applying θ'_R to the preceding equation yields

$$\alpha(-L_s L_{t^\natural})\alpha^{-1} = \theta'_R \circ C_R(\beta)(st) = -\mu(\beta)^{-1} L_{\beta(s)} L_{\beta(t)^\natural}.$$

Meanwhile, $\gamma''_R(\alpha) = \overline{\beta'}$ for some $\beta' \in \mathrm{GO}^+(S_R, Q_R)$ such that $\beta' = \gamma_R(\alpha) = \alpha_S$. This implies that

$$\alpha(-L_s L_{t^\natural})\alpha^{-1} = -\mu(\beta')^{-1} L_{\beta'(s)} L_{\beta'(t)^\natural} = \theta'_R \circ C_R(\beta')(st) \text{ for all } s, t \in S_R,$$

the first equality being a consequence of (18.6.3) and (18.5.5). We have shown that $\theta_R \circ C_R(\beta) = \theta_R \circ C_R(\beta')$. We would like to show that $C_R(\beta) = C_R(\beta')$. If $m_1 = 2$, then θ_R is injective so this goal is achieved. If $m_1 = 4$ or 8, the main involution τ on $C^+(S,Q)$ is of the second kind [101, (8.4)], so it swaps the two full matrix subalgebras of $C^+(S,Q)$. Equality in $C^+(S_R, Q_R)$ can be tested by applying θ_R and $\theta_R \circ \tau$ successively. Any automorphism in the image of C_R commutes with τ, so

$$\theta_R \circ \tau \circ C_R(\beta) = \theta_R \circ C_R(\beta) \circ \tau = \theta \circ C_R(\beta') \circ \tau = \theta_R \circ \tau \circ C_R(\beta').$$

This proves $C_R(\beta) = C_R(\beta')$. Given that $\ker(C_R) = R^\times \mathrm{id}$, this proves that $\overline{\beta} = \overline{\beta'} \in \mathrm{PGO}^+(S_R, Q_R)$. $\square$

18.14. Proposition. *Let $(A,-)$ be an $(8,8)$-product algebra. The following diagram commutes, the rows are exact, and the vertical arrows are surjective.*

$$\begin{tikzcd}
1 \arrow[r] & \mathbf{G}_m \times \mathbf{G}_m \arrow[r] \arrow[d] & \mathbf{\Omega}(S,Q) \arrow[r, "\chi'"] \arrow[d, "\theta'"] & \mathbf{PGO}^+(S,Q) \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & 1 \times \mathbf{G}_m \arrow[r] & \mathbf{Str}(A,-)^\circ \arrow[r, "\gamma''"] & \mathbf{PGO}^+(S,Q) \arrow[r] & 1.
\end{tikzcd} \tag{18.14.1}$$

The middle column is part of a short exact sequence:

$$1 \longrightarrow \mathbf{G}_m \times 1 \longrightarrow \mathbf{\Omega}(S,Q) \xrightarrow{\theta'} \mathbf{Str}(A,-)^\circ \longrightarrow 1. \tag{18.14.2}$$

Proof. The top row of the diagram comes from (14.1.1), using the fact that $Z = Z(C^+(S,Q))$ is split. Lemma 18.13 proves that $\ker(\gamma'')$ is the group of scalar matrices and that the right square is commutative (so the left square is too, by design). It follows that γ'' is surjective onto $\mathbf{PGO}^+(S,Q)$, hence the bottom row is exact. A diagram chase (the five lemma) applied to the R-points of (18.14.1) proves that θ'_R is surjective onto $\mathrm{Str}(A_R,-)$ for all R. The kernel of θ' is one of the copies of $\mathbf{G}_m$ in the centre of $\mathbf{\Omega}(S,Q)$ by Lemma 18.12 (iii). Hence (18.14.2) is exact. ☐

18.15. Proposition. *Let $(A,-)$ be an $(8,4)$-product algebra and let $N = \mathrm{Nuc}(A)$ be its quaternion factor. The following diagram commutes and the rows are exact.*

$$\begin{tikzcd}
1 \arrow[r] & \mathbf{G}_m \times \mathbf{G}_m \arrow[r] \arrow[d, "(c_1,c_2)\mapsto c_2"'] & \mathbf{\Omega}(S,Q) \arrow[r, "\chi'"] \arrow[d, "\theta'"] & \mathbf{PGO}^+(S,Q) \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & \mathbf{GL}_1(N) \arrow[r, "R"] & \mathbf{Str}(A,-)^\circ \arrow[r, "\gamma''"] & \mathbf{PGO}^+(S,Q) \arrow[r] & 1.
\end{tikzcd} \tag{18.15.1}$$

Proof. The proof is again very straightforward using Lemma 18.13. ☐

The major difference between and (18.14.1) and (18.15.1) is that in the latter diagram the left two vertical arrows are neither injective nor surjective. Rather, it is clear from Lemma 18.12 (i) that $\theta'(\mathbf{\Omega}(S,Q)) \subset \mathbf{Str}_N(A,-)^\circ$ and by diagram chasing it is easy to deduce that $\theta'(\mathbf{\Omega}(S,Q)) = \mathbf{Str}_N(A,-)^\circ$, which by Lemma 18.12 (iii) is isomorphic to $\mathbf{\Omega}(S,Q)/(\mathbf{G}_m \times 1)$. We can apply another diagram chase to (18.15.1) and find that $\mathbf{Str}(A,-)^\circ$ is generated by the two commuting subgroups $\mathbf{GL}_1(N)$ and $\mathbf{Str}_N(A,-)^\circ \simeq \mathbf{\Omega}(S,Q)/(\mathbf{G}_m \times 1)$. These two subgroups intersect in a central torus $\mathbf{G}_m$, so we can summarise by:

18.16. Corollary. *Let $(A,-) = (C \otimes N,-)$ be an $(8,4)$-product algebra. Then*

$$\mathbf{Str}(A,-)^\circ \simeq \frac{\mathbf{Str}_N(A,-)^\circ \times \mathbf{GL}_1(N)}{\mathbf{G}_m} \simeq \frac{\mathbf{\Omega}(S,Q) \times \mathbf{GL}_1(N)}{T}$$

where T is a split torus of rank 2 in $\mathbf{\Omega}(S,Q) \times \mathbf{GL}_1(N)$.

18.17. Proposition. *Let $(A,-)$ be an $(8,2)$-product algebra and let $Z(A) = F$. The following diagram commutes, the rows are exact, and the vertical arrows are isomorphisms:*

$$\begin{tikzcd}
1 \arrow[r] & \mathbf{GL}_1(Z) \arrow[r] \arrow[d, "\simeq"] & \mathbf{\Omega}(S,Q) \arrow[r, "\chi'"] \arrow[d, "\theta'"] & \mathbf{PGO}^+(S,Q) \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & \mathbf{GL}_1(F) \arrow[r] & \mathbf{Str}(A,-)^\circ \arrow[r, "\gamma''"] & \mathbf{PGO}^+(S,Q) \arrow[r] & 1
\end{tikzcd}$$

Proof. The exactness of the bottom row again follows from Lemma 18.13. Clearly the leftmost vertical arrow is an isomorphism (using [101, Theorem 8.2] or just the fact from Lemma 18.12 that θ' is injective). By diagram chase, θ' is an isomorphism. ☐

18.18. Proposition. *Suppose $(A,-)$ is an (m_1,m_2)-product algebra, and let $H = \mathbf{Str}(A,-)$. Then H is connected if $(m_1,m_2) = (8,8)$, $(8,4)$, $(8,1)$, $(4,1)$, and $(1,1)$, and otherwise it has two connected components.*

Proof. This is a direct consequence of Proposition 11.5 and Theorem 11.6. ☐

In the following theorem, we summarise our results on the connected structure group H°. The approach taken means that this theorem is independent of 18.18 and in fact it does not rely on any results from Chapter III.

18.19. Theorem. *Let $(A,-)$ be an (m_1,m_2)-product algebra with an Albert form Q on $S = \mathrm{Skew}(A,-)$, and let $N = \mathrm{Nuc}(A)$ and $F = Z(A)$. The connected structure group $H^\circ = \mathbf{Str}(A,-)^\circ$ is the reductive group determined up to isomorphism by the data in Table* 6.

		(**A**)	(**B**)	(**C**)	(**D**)	(**E**)
m_1	m_2	H°	M	$Z(H^\circ)$	$Z(M)$	$\Phi(M)$
8	8	$\dfrac{\mathbf{\Omega}(S,Q)}{\mathbf{G}_m\times 1}$	$\mathbf{Spin}(S,Q)$	$\mathbf{G}_m$	$\boldsymbol{\mu}_4$	D_7
	4	$\dfrac{\mathbf{\Omega}(S,Q)\times\mathbf{GL}_1(N)}{\mathbf{G}_m^2}$	$\dfrac{\mathbf{Spin}(S,Q)\times\mathbf{SL}_1(N)}{\boldsymbol{\mu}_2}$	$\mathbf{G}_m$	$\boldsymbol{\mu}_4$	$D_5{+}A_1$
	2	$\mathbf{\Omega}(S,Q)$	$\mathbf{Spin}(S,Q)$	$\mathbf{GL}_1(F)$	$\boldsymbol{\mu}_2\times\boldsymbol{\mu}_2$	D_4
	1	$\mathbf{\Gamma}^+(S,Q)$	$\mathbf{Spin}(S,Q)$	$\mathbf{G}_m$	$\boldsymbol{\mu}_2$	B_3
4	4	$\dfrac{\mathbf{GL}_1(A)\times\mathbf{GO}(A,-)}{\mathbf{G}_m}$	$\dfrac{\mathbf{SL}_1(A)\times\mathbf{O}^+(A,-)}{\boldsymbol{\mu}_2}$	$\mathbf{G}_m$	$\boldsymbol{\mu}_4$	$A_3{+}A_1{+}A_1$
	2	$\dfrac{\mathbf{GL}_1(A)\times\mathbf{GU}(A,-)}{\mathbf{GL}_1(F)}$	$\dfrac{\mathbf{SL}_1(A)\times\mathbf{SU}(A,-)}{\boldsymbol{\mu}_2}$	$\mathbf{GL}_1(F)$	$\boldsymbol{\mu}_2$	$A_1{+}A_1{+}A_1$
	1	$\dfrac{\mathbf{GL}_1(A)\times\mathbf{GL}_1(A)}{\mathbf{G}_m}$	$\dfrac{\mathbf{SL}_1(A)\times\mathbf{SL}_1(A)}{\boldsymbol{\mu}_2}$	$\mathbf{G}_m$	$\boldsymbol{\mu}_2$	$A_1{+}A_1$
2	1	$\mathbf{GL}_1(F)$	1	$\mathbf{GL}_1(F)$	1	
1	1	$\mathbf{G}_m$	1	$\mathbf{G}_m$	1	

Table 6: If $(A,-)$ is an (m_1,m_2)-product algebra, the table displays: (**A**) The group $H^\circ = \mathbf{Str}(A,-)^\circ$. (**B**)–(**D**) The isomorphism classes of $M = (H^\circ)^{\mathrm{der}}$, $Z(H^\circ)$, and $Z(M)$. (**E**) The type of the root system of M.

Notation: $F = Z(A)$, $N = \mathrm{Nuc}(A)$, $S = \mathrm{Skew}(A,-)$, and Q is the Albert quadratic form. All quotients in column (**B**) are by diagonally-embedded central subgroups.

Proof. The entries in column (A) come directly from 18.2, 18.10, 18.14, 18.16, and 18.17. So we begin with Column (B). Since $\mathbf{Spin}(V,q)^{\mathrm{der}} = \mathbf{Spin}(V,q)$, it is not difficult to show using the definitions that $\mathbf{\Omega}(V,q)^{\mathrm{der}} = \mathbf{\Gamma}^+(V,q)^{\mathrm{der}} = \mathbf{Spin}(V,q)$ for any quadratic space (V,q). Similar facts like $\mathbf{GL}_1(A)^{\mathrm{der}} = \mathbf{SL}_1(A)$, $\mathbf{GO}(A,-)^{\mathrm{der}} = \mathbf{SO}(A,-)$, and $\mathbf{GU}(A,-)^{\mathrm{der}} = \mathbf{SU}(A,-)$ are standard and easy to prove. To complete Column (B) we also used the fact that $[B/A, B/A] \simeq [B,B]/(A\cap[B,B])$ for abstract groups $A \lhd B$, in combination with the characterisation of G^{der} from [122, Proposition 6.18]. The remaining columns are based on direct and easily replicable calculations, using facts from [101, §23–26] or [122, §24] about the root systems and centres of semisimple groups. ☐

19. Norm-similitude groups

Algebraists have been studying the norm-similitude and norm-preserving groups of algebras for a very long time. The following Theorem 19.1 is as classical as it gets, proved in an early form by Frobenius [54] and generalised by Dieudonné [46], Jacobson [92], Waterhouse [179], and many others. It is stated below in terms of an exact sequence, to be consistent with the style elsewhere in this chapter.

Let (A, σ) be an associative central simple algebra with involution. Consider the group $\mathbf{GL}_1(A)^2 \rtimes \mathbb{Z}/2\mathbb{Z}$, where $\mathbb{Z}/2\mathbb{Z}$ acts by $(x, y) \mapsto (\sigma(y)^{-1}, \sigma(x)^{-1})$. There is a homomorphism

$$\phi_\sigma : \mathbf{GL}_1(A)^2 \rtimes \mathbb{Z}/2\mathbb{Z} \to \mathbf{Sim}(\mathrm{Nrd}_A), \qquad \phi_\sigma(x, y, \epsilon) = L_x R_{y^{-1}} \sigma^\epsilon.$$

Clearly, the image of the subgroup $\mathbf{SL}_1(A)^2 \rtimes \mathbb{Z}/2\mathbb{Z}$ is contained in $\mathbf{Iso}(\mathrm{Nrd}_A)$.

19.1. Theorem (after Frobenius)**.** *Let (A, σ) be a central simple associative algebra of degree n with involution of the first kind over an arbitrary field K with* $\mathrm{char}(K) \nmid n$. *The following diagram commutes, the columns are injective, and the rows are exact:*

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \boldsymbol{\mu}_n & \longrightarrow & \mathbf{SL}_1(A)^2 \rtimes \mathbb{Z}/2\mathbb{Z} & \longrightarrow & \mathbf{Iso}(\mathrm{Nrd}_A) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathbf{G}_m & \longrightarrow & \mathbf{GL}_1(A)^2 \rtimes \mathbb{Z}/2\mathbb{Z} & \xrightarrow{\phi_\sigma} & \mathbf{Sim}(\mathrm{Nrd}_A) & \longrightarrow & 1.
\end{array}$$

Proof. It is clear that the kernel of ϕ_σ is the one-parameter subgroup $T \simeq \mathbf{G}_m$ where $T(R) = \{(c\,\mathrm{id}, c\,\mathrm{id}, 0) : c \in R^\times\}$. Also clear is that

$$\ker(\phi_\sigma) \cap (\mathbf{SL}_1(A)^2 \rtimes \mathbb{Z}/2\mathbb{Z}) \simeq \boldsymbol{\mu}_n.$$

Both $\mathbf{Iso}(\mathrm{Nrd}_A)$ and $\mathbf{Sim}(\mathrm{Nrd}_A)$ are smooth, by Lemma 7.5. According to [88, Theorem 7], $\mathbf{Iso}(\mathrm{Nrd}_A)(k)$ is the set of maps $L_x R_{y^{-1}} \sigma^\epsilon$ where $x, y \in A^\times$, $\mathrm{Nrd}_A(x) = \mathrm{Nrd}(y)$, and $\epsilon = 0$ or 1. In particular, $\mathbf{Iso}(\mathrm{Nrd}_A)(k^a)$ is the set of maps $L_x R_{y^{-1}} \sigma^\epsilon$ where $x, y \in A_{k^a}^\times$ and $\mathrm{Nrd}_A(x) = \mathrm{Nrd}_A(y) = 1$. The surjectivity criterion [101, Proposition 22.3] implies the top row is exact. Similarly, $\mathbf{Sim}(\mathrm{Nrd}_A)(k^a)$ is the set of maps $L_x R_{y^{-1}} \sigma^\epsilon$ where $x, y \in A_{k^a}^\times$ are elements of any norm and $\epsilon = 0$ or 1, so the bottom row is exact too. □

19.2. *The norm-similitude group for involutions of the second kind.* Suppose K is an arbitrary field and (B, τ) is a central simple associative algebra over K with involution of the second kind. Let $F = Z(B)$, a quadratic étale extension of K. The generic norm of B, as a K-algebra, is $N_B = N_{F/K} \circ \mathrm{Nrd}_B$.

It is clear that we have an exact sequence

$$1 \longrightarrow \mathbf{G}_{m,F} \longrightarrow \mathbf{GL}_1(A)^2 \rtimes \mathbb{Z}/2\mathbb{Z} \xrightarrow{\phi_\tau} \mathbf{Sim}(N_A).$$

In the trivial case where $A = F$, it is easy to work out that ϕ_τ is surjective. However, if $A \neq F$ then ϕ_τ is not surjective. Indeed, if $A = M_n(F)$, $n > 1$, then there is a conjugate-transpose involution $* = t \circ \iota = \iota \circ t$ on A (of the second kind), where $t \in \mathrm{End}(M_n(F))$ is the transpose map and ι is the nontrivial automorphism of F/K applied entrywise. There is a self-adjoint unit $u = u^* \in \mathbf{GL}_n(F)$ such that $\tau = \mathrm{Int}(u) \circ *$ [101, Proposition 2.20 (2)]. By [46, Théorème 3],

$$\mathrm{Sim}(N_A) \simeq \frac{(\mathrm{GL}_1(A) \times \mathrm{GL}_1(A)) \rtimes \{1, t, \iota, *\}}{F^\times}.$$

and this group has no fewer than four Zariski-connected components with K-defined points, unlike the image of ϕ_τ which has only two connected components.

19.3. *The norm-similitude group for octonion algebras.* If C is an octonion algebra over k with norm n, the similitude group is $\mathbf{GO}(C, n) = \mathbf{G}_m.\mathbf{O}(C, n)$. The k-points of $\mathbf{GO}(C, n)$ are all of the form $L_z t$ where z is an invertible element of C and $t \in \mathrm{O}(C, n)$ is a product of some reflections [164, p. 38]. If $y \in C$ is invertible then the reflection about y is

$$\rho_y = -N(y)^{-1} L_y R_y \tau = -N(y)^{-1} L_y \tau L_{\overline{y}}$$

where τ is the standard involution [164, p. 44], so the group of norm-similitudes is generated as an abstract group by left-multiplications and the standard involution.

19.4. Theorem. *If* $(A, -)$ *is an* $(8,8)$*-product algebra, then* $\mathbf{Str}(A, -) = \mathbf{Sim}(N_A)$.

Proof. We have $\mathbf{Str}(A, -) \subset \mathbf{Sim}(N_A) \subset \mathbf{GL}(A)$, all of these groups are smooth (by Proposition 7.2 and Lemma 7.5, respectively), and $\mathbf{Str}(A, -)$ is connected (by Proposition 11.5). Since $\mathbf{Str}(A, -)$ is reductive, it is generated by its centre $Z = Z(\mathbf{Str}(A, -))$ and its derived subgroup $X = \mathbf{Str}(A, -)^{\mathrm{der}}$. Proposition 18.14 implies that $Z = Z(\mathbf{GL}(A))$, which is a one-dimensional torus, and that X is the image of the (faithful) half-spin representation $\theta'|_{\mathbf{Spin}(S,Q)} : \mathbf{Spin}(S, Q) \to \mathbf{GL}(A)$. In summary,

$$\mathbf{Str}(A, -) = X.Z \qquad\qquad \text{where } X \simeq \mathbf{Spin}(S, Q) \text{ and } Z \simeq \mathbf{G}_m.$$

Clearly $X \subset \mathbf{Iso}(N_A) \subset \mathbf{SL}(A)$ because X is contained in the kernel of any homomorphism from $\mathbf{Str}(A, -)$ to an abelian group. The half-spin representations of X satisfy the conditions of [60, Lemma 5.1]; details on this are postponed to 19.5. Moreover, N_A is of degree 8 and generates $k[A]^X$ (for details, look ahead to 21.8), so we can apply [60, Lemma 5.1] to conclude that $\mathbf{Iso}(N_A)^\circ \times_k k^a = X \times_k k^a$. Therefore $\mathbf{Iso}(N_A)^\circ = X$, using [122, Corollary 1.18].

We claim that $\mathbf{Str}(A, -)$ contains the normaliser of X in $\mathbf{GL}(A)$; since $\mathbf{Iso}(N_A)^\circ$ is normal in $\mathbf{Sim}(N_A)$, this clearly implies that $\mathbf{Sim}(N_A) \subset \mathbf{Str}(A, -)$. If $g \in N_{\mathbf{GL}(A)}(X)(k^a)$, then $\mathrm{Int}(g)|_X$ is an automorphism of X. There is only one non-trivial outer automorphism class of X and it acts nontrivially on $Z(X) \subset Z(\mathbf{GL}(A))$, so there is no element of $\mathbf{GL}(A)(k^a)$ whose conjugation action is an outer automorphism of X. Therefore, $\mathrm{Int}(g)|_X$ is an inner automorphism of X and this implies $g \in X.C_{\mathbf{GL}(A)}(X)(k^a)$. But $C_{\mathbf{GL}(A)}(X) = Z(\mathbf{GL}(A)) = Z$ because, according to Lemma 18.12 (iv), anything in $\mathbf{GL}(A)(k^a)$ that commutes with $X(k^a)$ must commute with all of $\mathbf{GL}(A)(k^a)$. Therefore $g \in X.Z(k^a) = \mathbf{Str}(A, -)(k^a)$. ☐

19.5. *Some pertinent details on half-spin representations of* D_7. For completeness, we record some facts about the half-spin representations of D_7, which were used in the proof of 19.4. It is tidier to keep this separate, also because these facts are true in any characteristic $p \geq 0$ (and can be generalised to some other D_ℓ's too, by checking the details in the sources cited).

Let (V,q) be a 14-dimensional quadratic space. The half-spin representations of $\mathbf{Spin}(V,q)$ are defined if (V,q) has trivial discriminant. These inequivalent representations have dimension $2^6 = 64$, and are irreducible [42, II.4.3]. They are p-restricted (in the sense of [60, p. 3]), since the highest weight is a fundamental dominant weight of the form $\lambda = \frac{1}{2}(x_1 + x_2 + \cdots \pm x_7)$ [136, 11§7.2]. And they are tensor-indecomposable, because the smallest irreducible representation of D_7 is the vector representation of dimension 14 [109, Theorem 4.4 & Appendix A.44], and $14^2 > 64$ makes it impossible to tensor-decompose the half-spin representations.

By combining Proposition 11.5, Table 6, and the results of 19.1–19.4 we arrive at the following conclusion.

19.6. Theorem. *Let $(A,-)$ be an (m_1,m_2)-product algebra. If $(m_1,m_2) = (8,8)$, $(4,1)$, $(2,1)$, or $(1,1)$ then $\mathbf{Str}(A,-) = \mathbf{Sim}(N_A)$. If $(m_1,m_2) = (8,1)$, $(4,4)$, or $(4,2)$ then $\mathbf{Str}(A,-)$ is a proper subgroup of positive codimension in $\mathbf{Sim}(N_A)$.*

I have not been able to answer this question for $(8,4)$- or $(8,2)$-product algebras.

20. Albert forms, isotopy, and division algebras

The goal of this section is to establish some theorems about isotopy of (m_1,m_2)-product algebras, and some criteria for being a division algebra. For instance, we prove that isotopic algebras have similar Albert forms, and that the Albert forms classify $(8,m)$-product algebras up to isotopy – these were previously known only in characteristic 0. Galois cohomology affords us some elegant proofs of these results, which are also new proofs of the known results in characteristic 0.

20.1. Proposition. *If $(A,-)$ and $(A',-)$ are isotopic (m_1,m_2)-product algebras, then their Albert forms are similar.*

Proof. The map $\gamma : \mathbf{Str}(A,-) \to \mathbf{GO}(S,Q)$ from Proposition 18.6, when restricted to $\mathbf{Aut}(A,-)$, becomes $\gamma(f) = f|_{S_R}$ for all $f \in \mathbf{Aut}(A,-)(R)$. This induces a commutative triangle:

$$\begin{tikzcd}
H^1(k,\mathbf{Aut}(A,-)) \arrow[d, "i_*"] \arrow[dr, "(\gamma|_{\mathbf{Aut(A,-)}})_*"] & \\
H^1(k,\mathbf{Str}(A,-)) \arrow[r, "\gamma_*"] & H^1(k,\mathbf{GO}(S,Q))
\end{tikzcd}$$

It is clear that the arrow $(\gamma|_{\mathbf{Aut}(A,-)})_*$ sends the isomorphism class of an (m_1,m_2)-product algebra $(A',-)$ to the similitude class of its Albert form Q'. The fact that this factors through $H^1(k,\mathbf{Str}(A,-))$ gives us the lemma. □

The following theorem was proved by Allison in [5, Corollary 7.6]. We give a new proof using cohomology.

20.2. Theorem (Allison)**.** *Let $(A,-)$ and $(A',-)$ be central simple algebras with involution over k such that A is either associative or an $(8,m_2)$-product algebra for $m_2 \leq 2$. Then $(A,-)$ and $(A',-)$ are isotopic if and only if they are isomorphic.*

Proof. The claim to be proven is clearly equivalent to the statement that

$$i_* : H^1(k, \mathbf{Aut}(A, -)) \to H^1(k, \mathbf{Str}(A, -))$$

is injective (and therefore an isomorphism, according to Lemma 13.6 (ii)).

If $(A, -)$ is associative, applying $H^1(k, *)$ to the split short exact sequence (18.1.2) makes it clear that i_* is injective. If $(A, -)$ is not associative, then it is either an octonion algebra over k with standard or nonstandard involution of the first kind, or it is an $(8, 2)$-product algebra over k [5, Theorem 5.1].

If $(A, -)$ is an octonion algebra with standard involution, then so are its isotopes, and one can deduce from (18.9.1) using Hilbert 90 and twisting that $(A, -)$ and $(A', -)$ are isotopic if and only if they have the same image in $H^1(k, \mathbf{O}^+(n'))$; that is, their pure norms are isometric. This of course implies $(A, -) \simeq (A', -)$.

If $(A, -)$ is an $(8, 2)$-product algebra then so are its isotopes. By Proposition 20.1, the isotopes $(A, -)$ and $(A', -)$ have similar Albert forms, say $Q \simeq \langle c\rangle Q'$. In the Witt ring we have $Q = n - \langle\langle d\rangle\rangle$ and $Q' = m - \langle\langle e\rangle\rangle$ for some $d, e \in k^\times$ and 3-Pfister forms $n, m \in W(k)$. Then $n - \langle c\rangle m = \langle\langle d\rangle\rangle - \langle c\rangle\langle\langle e\rangle\rangle = 0$ by the Arason–Pfister Hauptsatz, so $n = \langle c\rangle m$ and $\langle\langle d\rangle\rangle = \langle c\rangle\langle\langle e\rangle\rangle$. It follows that $n = m$ and $\langle\langle d\rangle\rangle = \langle\langle e\rangle\rangle$ in $W(k)$ [106, X. Corollary 5.4], so $(A, -)$ and $(A', -)$ have isomorphic octonion and quadratic étale factors, and consequently $(A, -) \simeq (A', -)$. □

In fact, Allison proved the above theorem also holds for octonion algebras with nonstandard involution (hence for all alternative central simple structurable algebras). One can give a cohomological proof in that case too, but we omit the details.

To proceed further towards an isotopy criterion for $(8, m)$-product algebras, we summarise some information from §18, now taking into account the whole structure group and not just its connected component.

20.3. Lemma. *If $(A, -)$ is an $(8, m)$-product algebra, $m \geq 2$, and $N = \mathrm{Nuc}(A)$, then the following sequence is exact:*

$$1 \longrightarrow \mathbf{GL}_1(N) \xrightarrow{R} \mathbf{Str}(A, -) \xrightarrow{\gamma'} \mathbf{PGO}(S, Q).$$

If $m = 2$, then γ' is surjective; otherwise its image is $\mathbf{PGO}^+(S, Q)$. The map

$$\gamma'_* : H^1(k, \mathbf{Str}(A, -)) \to H^1(k, \mathbf{PGO}(S, Q))$$

is injective, and sends the isotopy class of $(A', -)$ to the isomorphism class of $(\mathrm{End}\, S', \tau_{Q'})$ where $S' = \mathrm{Skew}(A', -)$ and $\tau_{Q'}$ is the adjoint involution to an Albert form Q' of $(A', -)$.

Proof. The exactness of the sequence is proved in Lemma 18.13 (i). By Proposition 11.5, $\mathbf{Str}(A, -)$ is connected if $m = 8$ or 4, and has two connected components if $m = 2$. Therefore, by Propositions 18.14, 18.15, and 18.17, the image of γ' is $\mathbf{PGO}^+(S, Q)$ if $m = 8$ or 4 and all of $\mathbf{PGO}(S, Q)$ if $m = 2$. If $m = 2$ then Hilbert 90 and a twisting argument can be used to show that $\gamma'_* : H^1(k, \mathbf{Str}(A, -)) \to H^1(k, \mathbf{PGO}(S, Q))$ is injective. Similarly, if $m = 8$ or 4 then $\gamma''_* : H^1(k, \mathbf{Str}(A, -)) \to H^1(k, \mathbf{PGO}^+(S, Q))$ is injective. As mentioned in 14.2, the image of γ''_* maps injectively into $H^1(k, \mathbf{PGO}(S, Q))$, hence γ'_* is injective too. The interpretation of γ'_* is clear from the definition of γ'. □

20.4. Corollary. *Let $(A,-)$ and $(A',-)$ be $(8,m)$-product algebras over k, where $m = 1, 2, 4$, or 8. Then $(A,-)$ and $(A',-)$ are isotopic if and only if they have similar Albert forms.*

Proof. For the case for $m = 1$, this is implied by Theorem 20.2 and for the case $m \geq 2$ it is implied by Lemma 20.3. $\square$

20.5. Example (Isotopic but not isomorphic)**.** Theorem 20.2 does not hold for $(8,4)$ or $(8,8)$-product algebras. Take for example the field $K = k(t_1, t_2, t_3)$, the octonion algebras C_1 and C_2 over K with the following norms

$$n_{C_1} = \langle\!\langle t_1, t_2, t_3 \rangle\!\rangle, \qquad n_{C_2} = \langle\!\langle 1, 1, 1 \rangle\!\rangle \text{ (i.e., hyperbolic)},$$

and the quaternion algebra Q over K with norm

$$n_Q = \langle\!\langle t_1, t_2 \rangle\!\rangle.$$

The $(8,4)$-product algebras $C_1 \otimes Q$ and $C_2 \otimes Q$ are isotopic because they have similar Albert forms:

$$\langle\!\langle t_1, t_2, t_3 \rangle\!\rangle - \langle\!\langle t_1, t_2 \rangle\!\rangle = \langle -t_3 \rangle \langle\!\langle t_1, t_2 \rangle\!\rangle.$$
$$\langle\!\langle 1, 1, 1 \rangle\!\rangle - \langle\!\langle t_1, t_2 \rangle\!\rangle = \langle -1 \rangle \langle\!\langle t_1, t_2 \rangle\!\rangle$$

However, $C_1 \otimes Q$ and $C_2 \otimes Q$ are not isomorphic because $C_1 \not\simeq C_2$ (see Theorem 9.6).

We now move towards various characterisations of division structurable algebras among the bicomposition algebras.

20.6. Lemma. *Let $q = \langle 1 \rangle \perp q'$ be an m-Pfister form ($m \geq 1$) over k, and let $F = k(\sqrt{a})$ be a quadratic field extension. Then q_F is isotropic if and only if q' represents $-a$.*

Proof. This is an easy exercise using properties of Pfister forms (say, [106, Theorem VII.3.1 & Theorem X.1.8]). $\square$

The next theorem generalises [6, Theorem 3.14], where it was proved in the characteristic 0 case.

20.7. Theorem. *Let $(A,-)$ be an (m_1, m_2)-product algebra. Then $(A,-)$ is a structurable division algebra if and only if its Albert form Q is anisotropic and $Z(A)$ is a field.*

Proof. ($\Rightarrow$) If the Albert form Q is isotropic, there exists a nonzero element $s \in S = \mathrm{Skew}(A,-)$ such that $Q(s) = 0$, and by (18.5.1) this implies that s is not invertible. If $Z(A)$ is not a field, then clearly $(A,-)$ fails to be a structurable division algebra.

($\Leftarrow$) If $A = Z(A)$ is a field, then A is obviously a structurable division algebra; see Lemma 2.11. If $m_2 = 1$ and $m_1 \geq 4$, then A is a composition algebra and Q is the pure norm of A. If Q is anisotropic then the standard norm $\langle 1 \rangle \perp Q$ is anisotropic too because $\dim Q > \frac{1}{2}(\dim Q + 1)$, so A is an alternative division algebra, and therefore also a structurable division algebra [5, Corollary 3.6].

Suppose $m_2 = 2$, $m_1 \geq 4$, $Z(A) = k(\sqrt{a})$ is a field, and Q is anisotropic. Let C be the m_1-dimensional composition algebra in the (unique) decomposition $A = C \otimes Z(A)$, and let n be the norm of C. Then $Q \simeq n' \perp \langle a \rangle$, so n' does not represent $-a$, and $n_{k(\sqrt{a})}$ is anisotropic by Lemma 20.6. This implies that A is an alternative division algebra, because $n_{k(\sqrt{a})}$ is its generic norm, hence $(A, -)$ is a structurable division algebra [5, Corollary 3.6].

If $(m_1, m_2) = (4, 4)$, then the statement is essentially Albert's Theorem – see [101, Theorem 16.5 & Corollary 16.28].

This leaves two remaining cases, which we approach using algebraic group theory and some results of [31]. Suppose $(m_1, m_2) = (8, 4)$, let $N = \mathrm{Nuc}(A)$ be the quaternion factor of A, and assume Q is anisotropic. Then $\mathbf{Spin}(S, Q)$ is anisotropic. Say the norm of N is $n = \langle 1 \rangle \perp n'$; then n' is anisotropic because it is a subform of Q, and therefore n is anisotropic too. This implies N is a quaternion division algebra, so $\mathbf{SL}_1(N)$ is anisotropic. By Theorem 18.19, the semisimple anisotropic kernel of $\mathbf{Aut}(K(A, -))^\circ$ is strictly k-isogenous to $\mathbf{Spin}(S, Q) \times \mathbf{SL}_1(N)$, which is anisotropic of absolute rank 6. Therefore $\mathbf{Aut}(K(A, -))^\circ$, being of absolute rank 7, has k-rank equal to 1. Now the proof of [31, Theorem 4.3.1] implies $(A, -)$ is a structurable division algebra. If $(m_1, m_2) = (8, 8)$ and Q is anisotropic, then the semisimple anisotropic kernel of $\mathbf{Aut}(K(A, -))^\circ$ is $\mathbf{Spin}(S, Q)$, which is anisotropic of absolute rank 7. Therefore $\mathbf{Aut}(K(A, -))^\circ$, being of absolute rank 8, has k-rank equal to 1, which implies $(A, -)$ is a division algebra. $\square$

The statement of Theorem 20.7 for biquaternion algebras (Albert's Theorem) has been proved many times, and a discussion of its history can be found in the notes in [101, p. 275] and [106, p. 71]. The statement is also provable in elementary ways for $(8, 2)$- and $(8, 4)$-product algebras; see [30, Lemma 3.16] for a sketch. However, an elementary proof of Theorem 20.7 for $(8, 8)$-product algebras seems far out of reach because of the difficulty of working with the elements of such algebras.

The following is somewhat useful reformulation of the above theorem for $(8, 8)$-product algebras. (For a similar theorem on biquaternion algebras in all characteristics, see [23, Theorem 1.1].)

20.8. Corollary. *Let E/k be a quadratic étale extension and let C be an octonion algebra over E. The following are equivalent:*

(1) $\mathrm{cor}_{E/k}(C)$ *is not a structurable division algebra.*

(2) *The pure norm n' of C represents an element of $k \subset E$.*

(3) *C contains a quadratic étale extension K/k which is linearly disjoint from E/k (in the sense that EK is a 4-dimensional k-vector space).*

Proof. An Albert form of $\mathrm{cor}_{E/k}(C)$ is $Q = T_{E/k}(\langle \delta \rangle n')$ where n' is the pure norm of C and $\delta \in E^\times$ is an element of trace zero. By definition Q is isotropic if and only if there is a nonzero element $z \in C_0$ such that $\mathrm{tr}_{E/k}(\delta n(z)) = 0$, which is equivalent to $n(z) \in k$. Together with Theorem 20.7, this observation yields (1) $\Leftrightarrow$ (2). If (2) holds and $z \in C_0$ has $n(z) \in k$, let $K = k(z)$. Then $\{z, \delta z, 1, \delta\}$ is a k-basis for EK because $Ez \cap E = \{0\}$, implying (3). Conversely, suppose (3) holds. There is a generator y of K/k such that $y^2 \in k \subset E$. It is a basic property of composition algebras over fields of characteristic not 2 that an element whose square is a central scalar is either in the centre or has trace zero: so either $y \in E$ or $y \in C_0$. The former is impossible since y generates K and $EK \neq E$. Hence $y^2 = -n(y) \in k$, which gives us (2). $\square$

20.9. Theorem. *Let $(A,-)$ be an $(8,8)$- or $(8,4)$-product algebra over k. The following are equivalent:*

(1) *$(A,-)$ is not a structurable division algebra.*

(2) *$(A,-)$ has an isotropic Albert form.*

(3) *$(A,-)$ has a noninvertible skew element.*

(4) *$(A,-)$ has a nondivision biquaternion subalgebra stabilised by the involution.*

(5) *$(A,-)$ has a nondivision associative subalgebra stabilised by the involution.*

Proof. (1) $\Leftrightarrow$ (2) is given by Theorem 20.7, and (2) $\Leftrightarrow$ (3) is a direct consequence of (18.5.1).

To show that (2) $\Rightarrow$ (4), assume first that $(A,-)$ is an $(8,8)$-product algebra with an isotropic Albert form. Then $(A,-)=\mathrm{cor}_{E/k}(C)$ for some quadratic étale extension E/k and an octonion algebra C over E. By Corollary 20.8 there is a $z\in C_0$ such that $n(z)\in k$. There is a quaternion subalgebra $Q\subset C$ containing $z\subset Q_0$ [164, Proposition 1.6.4]. Then $\mathrm{cor}_{E/k}(Q)$ is a biquaternion subalgebra of $(A,-)$ stabilised by the involution whose Albert form is isotropic, and which is therefore not a division algebra.

Secondly, if $(A,-)$ is an $(8,4)$-product algebra with an isotropic Albert form, then $(A,-)=C\otimes Q$ where C is an octonion algebra and Q is a quaternion algebra, such that either C or Q is split, or the pure norms n'_C and n'_Q represent a common element $a\in k^\times$. If C or Q is split, then it is easy to find a nondivision biquaternion subalgebra stabilised by the involution. Otherwise, $n_C=\langle\!\langle -a,b,c\rangle\!\rangle$ and $n_Q=\langle\!\langle -a,d\rangle\!\rangle$ for some $b,c,d\in k^\times$ [106, Pure Subform Theorem 1.5]. Let $Q'\subset C$ be a subspace on which n_C restricts to $\langle\!\langle -a,b\rangle\!\rangle$; then Q' is a quaternion algebra [164, Propositions 1.2.3 & 1.5.1] and $Q'\otimes Q$ is a biquaternion subalgebra stabilised by the involution. Since $Q'\otimes Q$ has an isotropic Albert form $\langle\!\langle -a,b\rangle\!\rangle'-\langle\!\langle -a,d\rangle\!\rangle'$, it is not a division algebra. Therefore we have shown that (2) implies (4).

Clearly (4) $\Rightarrow$ (5). If $(A,-)$ has an associative subalgebra $(B,-)$ with involution that is not a division algebra, then $(B,-)$ is not a structurable division algebra by Lemma 2.11, so neither is $(A,-)$. This settles (5) $\Rightarrow$ (1). □

It is interesting to ask if there exists a characterisation of division $(8,4)$- and $(8,8)$-product algebras, where the characterisation does not make reference to the involution. For example, if A has a nontrivial idempotent $e=e^2$, does this imply $(A,-)$ is not a structurable division algebra? The answer is yes if A is alternative [5, Corollary 3.6] but for $(8,4)$- and $(8,8)$-product algebras it is unknown to me.

20.10. *Split isotopes.* After studying division structurable algebras, it is also interesting to look at the other extreme: algebras that have a split isotope. This amounts to taking a structurable algebra $(A,-)$ such that $L=K(A,-)$ is a split central simple Lie algebra, and asking what is the kernel of the map

$$H^1(k,\mathbf{Aut}(A,-))\longrightarrow H^1(k,\mathbf{Str}(A,-))\subset H^1(k,\mathbf{Aut}(L))? \tag{20.10.1}$$

The "$\subset$" here is used informally, but makes sense because of Lemma 13.6 (iv).

This question has already been answered for some other types of structurable algebras [56, 0.4]. For instance, if $(A,-)$ is an Albert algebra then this kernel is trivial. If $(A, -)$ is a Brown algebra, the kernel is isomorphic to $k^\times/k^{\times 2}$, which is as small as you could hope for.

We show now that the kernel of (20.10.1) is trivial if $(A, -)$ is a split (m_1, m_2)-product algebra where $(m_1, m_2) \neq (8, 8)$. Emphatically, this does not imply that (20.10.1) is injective – there was a counterexample in 20.5. We also determine the kernel when $(m_1, m_2) = (8, 8)$.

20.11. Proposition. *Assume $(A, -)$ is an (m_1, m_2)-product algebra that is isotopic to the split (m_1, m_2)-product algebra. Then either $(A, -)$ itself is split, or $(m_1, m_2) = (8, 8)$ and*

$$(A, -) = \mathrm{cor}_{E/k}(M_E)$$

for some quadratic étale extension E/k and some octonion algebra M over k.

Proof. Since Theorem 20.2 renders most cases trivial, we may assume $(m_1, m_2) = (8, 8)$ or $(8, 4)$. Suppose $(A, -) = C_1 \otimes C_2$ is decomposable and let n_i be the norm of C_i. Proposition 20.1 implies the Albert form of $(A, -)$ is hyperbolic, so $n_1 - n_2 = 0$ in $W(k)$. If $(m_1, m_2) = (8, 4)$ this implies $n_1 = n_2 = 0$ because a 2-Pfister form cannot be Witt equivalent to a 3-Pfister form, unless they are both hyperbolic. If $(m_1, m_2) = (8, 8)$ then $n_1 = n_2$ implies $C_1 = C_2$, so $(A, -) \simeq C_1 \otimes C_1$. (Note that we can write $C_1 \otimes C_1 = \mathrm{cor}_{k \times k/k}(C_1 \times C_1)$ to match the statement of the theorem.)

Finally, suppose $(A, -) = \mathrm{cor}_{E/k}(C)$ for some quadratic field extension $E = k(\sqrt{d})$ and octonion algebra C over E with norm n. Then the Albert form is hyperbolic, i.e., $T_{E/k}(\langle \sqrt{d} \rangle n') = T_{E/k}(\langle \sqrt{d} \rangle n) = 0$ in $W(k)$. The Transfer Principle [106, XI.4.13] implies $n \simeq q_E$ for some 3-Pfister form q over k. In turn, this implies $C \simeq M_E$ where M is the octonion k-algebra with norm q. $\square$

By the proposition, there is a natural-in-k map of pointed sets

$$\begin{aligned} H^1(k, \mathbb{Z}/2\mathbb{Z}) \times H^1(k, G_2) &\longrightarrow \ker\Big(H^1(k, (G_2 \times G_2) \rtimes \mathbb{Z}/2\mathbb{Z})) \longrightarrow H^1(k, E_8)\Big) \\ ([E], [M]) &\longmapsto [\mathrm{cor}_{E/k}(M_E)] \end{aligned}$$

whose kernel is the set of pairs $([E], [M])$ such that

$$[M] \in \ker\Big(\ H^1(k, G_2) \xrightarrow{\mathrm{res}_{E/k}} H^1(E, G_2)\ \Big).$$

21. 14-dimensional forms in I^3 and octic polynomials

Recall from 14.2 that if (V, q) is a quadratic space with trivial discriminant, the cohomology set $H^1(k, \mathbf{PGO}^+(V, q))$ can be seen as the set of isomorphism classes of quadruples (B, σ, C_+, C_-), where (B, σ) is an orthogonal involution of degree $n = \dim V$ and C_+, C_- are k-algebras such that the Clifford algebra $C(B, \sigma)$ is k-isomorphic to $C_+ \times C_-$. If $n = 2 \mod 4$, the fundamental relation [101, (9.16)] says that $[C_+] + [C_-] = 0$ in $\mathrm{Br}(k)$; that is, $C_+ \simeq (C_-)^{\mathrm{op}}$.

21.1. Proposition. *Let $(A, -)$ be a bioctonion algebra. The connecting map*

$$\Delta : H^1(k, \mathbf{PGO}^+(S, Q)) \longrightarrow H^2(k, \mathbf{G}_m) = \mathrm{Br}(k)$$

induced by the second row of (18.14.1) *is* $[(B, \sigma, C_+, C_-)] \mapsto [C_-] \in \mathrm{Br}(k)$.

Proof. The diagram (18.14.1) induces a pair of exact sequences with a vertical arrow between them:

$$\begin{array}{ccccc}
\ldots \longrightarrow H^1(k,\mathbf{\Omega}(S,Q)) & \xrightarrow{\chi'_*} & H^1(k,\mathbf{PGO}^+(S,Q)) & \xrightarrow{\Delta'} & H^2(k,\mathbf{G}_m\times\mathbf{G}_m) = \mathrm{Br}(k\times k) \\
\downarrow & & \| & & \downarrow \\
\ldots \longrightarrow H^1(k,\mathbf{Str}(A,-)) & \xrightarrow{\gamma''_*} & H^1(k,\mathbf{PGO}^+(S,Q)) & \xrightarrow{\Delta} & H^2(k,\mathbf{G}_m) = \mathrm{Br}(k).
\end{array}$$

The connecting map Δ' is described in [101, VII. Exercise 15]: since $C^+(S,Q) \simeq M_{64}(k) \times M_{64}(k)$, the class $[(B,\sigma,C_+,C_-)]$ is mapped to the class $[C_+ \times C_-]$ in $\mathrm{Br}(k \times k)$. The vertical arrow $\mathrm{Br}(k \times k) \to \mathrm{Br}(k)$ is induced by the projection $\mathbf{G}_m \times \mathbf{G}_m \to 1 \times \mathbf{G}_m \simeq \mathbf{G}_m$, so it sends $[C_+ \times C_-] \mapsto [C_-]$. Therefore the connecting map Δ sends $[(B,\sigma,C_+,C_-)] \mapsto [C_-]$. □

Note that there is no canonical way of doing this, and had we made different conventions back in §18 or elsewhere, then Proposition 21.1 might have said that Δ maps $[(B,\sigma,C_+,C_-)] \mapsto C_+$. In more ambidextrous terms, the proposition would say: "the connecting map Δ associated to γ'' sends an element of $H^1(k,\mathbf{PGO}^+(S,Q))$ to one of the components of its Clifford algebra, and if ∇ is the connecting map associated to $\gamma'' \circ f$, where f is an outer automorphism of $\mathbf{Str}(A,-)$, then ∇ sends an element of $H^1(k,\mathbf{PGO}^+(S,Q))$ to the other component of its Clifford algebra."

21.2. Corollary. *Let $(A,-) = C(8) \otimes C(8)$ be the split bioctonion algebra. The map $H^1(k,\mathbf{Str}(A,-)) \to PI^3_{14}(k)$, sending the isotopy class of a bioctonion algebra to the similitude class of its Albert form, is an isomorphism of pointed sets.*

Proof. Suppose (V,q) is a 14-dimensional quadratic space whose Clifford algebra is isomorphic to $M_{128}(k)$; for short, $q \in I^3_{14}(k)$. Let $(B,\sigma) = (M_{14}(k),\tau_q)$ be the orthogonal involution adjoint to q. Then $C(B,\sigma) = C_+ \times C_-$ where $C_+ \simeq C_- \simeq M_{64}(k)$ [106, Theorem V.2.5 (3)], and hence $[(B,\sigma,C_+,C_-)] \in \ker(\Delta) = \mathrm{im}(\gamma''_*)$. Therefore $(B,\sigma) \simeq (M_{14}(k),\tau_Q)$ for some Q which is an Albert form of a bioctonion algebra $(A,-)$, and hence q and Q are similar. This shows $H^1(k,\mathbf{Str}(A,-)) \to PI^3_{14}(k)$ is surjective. The injectivity was established in Corollary 20.4. □

Note that since $\mathrm{im}(\gamma''_*) = \ker(\Delta)$, Proposition 21.1 also implies that for an orthogonal involution (A,σ) of degree 14, if one of the components of $C(A,\sigma)$ is split, then A is split and σ is adjoint to a quadratic form in I^3. But this fact is actually easy to prove for any degree $n \equiv 2 \mod 4$ [57, Lemma 1.5].

21.3. Corollary (Rost)**.** *Let $Q \in I^3_{14}(k)$. Then either:*

(1) *There exist 3-Pfister forms ϕ_1 and ϕ_2 over k and a scalar $c \in k^\times$ such that*

$$Q \simeq \langle c\rangle(\phi'_1 \perp \langle -1\rangle\phi'_2).$$

(2) *There exists a quadratic field extension E/k, a 3-Pfister form ϕ over E, and an element $\delta \in E^\times$ of trace zero such that*

$$Q \simeq T_{E/k}(\langle\delta\rangle\phi').$$

Proof. By Corollary 21.2, Q is similar to the Albert form of a bioctonion algebra, and these are precisely the two possibilities for a form that is similar to the Albert form of a bioctonion algebra. □

The above proof of Rost's Theorem is different from those that have previously been sketched in [142] and [58, Theorem 21.3].

21.4. *Rationally parameterised classes of quadratic forms.* Corollary 21.3 is an example of a "rational parameterisation" of 14-dimensional forms in I^3 (for a formal definition of what this means, see [119]). For all even $n \leq 14$, there is a rational parameterisation theorem for the functor $I^3_n(*)$ [79, Theorem 2.1]. (The case of $n = 12$ will be discussed further in 21.6.) Notably, there is no known rational parameterisation of $I^3_n(*)$ for $n > 14$, and it is conjectured by Merkurjev that no rational parameterisation exists (see [118, Conjecture 4.5] and [119, Corollary 11]).

Another class of quadratic forms which can be rationally parameterised is 10-dimensional quadratic forms with trivial discriminant and Clifford invariant of index ≤ 2. Every such quadratic form is similar to the difference of the pure parts of a 3-Pfister and a 2-Pfister form [77, Theorems 4.1 & 5.1], i.e., similar to the Albert form of an $(8, 4)$-product algebra. This can probably also be proved in a similar manner to Corollary 21.3.

21.5. *Existence results on quadratic forms in I^3_{14}.* The two possibilities in Rost's Theorem are not mutually exclusive. In fact, any isotropic form q in $I^3_{14}(k)$ satisfies (1) – more on this in 21.6. Quadratic forms satisfying (2) but not (1) must therefore be anisotropic. They exist, and examples have been found by Izhboldin and Karpenko [83, Corollary 17.4] and Hoffmann and Tignol [79, 6.2–6.3]; probably the most accessible example is over the field $\mathbb{Q}(t_1, t_2, t_3, t_4)$. Quadratic forms satisfying (1) but not (2) also exist: for example over the field $k_n = \mathbb{R}((t_1, \ldots, t_n))$ where $n \geq 3$, every anisotropic form in $I^3_{14}(k_n)$ satisfies (1) but not (2) [6, Theorems 7.3 & 7.13].

21.6. *Pfister's parameterisation of quadratic forms in I^3_{12}.* By a theorem of Pfister (see [134, p. 123–124], [97, Théorème 8.1.1], or [58, Theorem 17.13]), for every quadratic form $q \in I^3_{12}(k)$ there exists a 6-dimensional quadratic form $r \in I^2_6(k)$ and a scalar $c \in k^\times$ such that

$$q \simeq \langle\langle c \rangle\rangle r.$$

Moreover, for any form $r \in I^2_6(k)$, there exist scalars $d, x_1, x_2, y_1, y_2 \in k^\times$ such that

$$r = \langle d \rangle (\psi'_1 \perp \langle -1 \rangle \psi'_2) \qquad\qquad \psi_i = \langle\langle x_i, y_i \rangle\rangle.$$

That is,

$$q \simeq \langle d \rangle \langle\langle c \rangle\rangle (\psi'_1 \perp \langle -1 \rangle \psi'_2). \tag{21.6.1}$$

This also means that every isotropic quadratic form $Q \in I^3_{14}(k)$ is of the form

$$Q = q + \mathbb{H} = \langle d \rangle (\langle\langle c, x_1, y_1 \rangle\rangle' \perp \langle -1 \rangle \langle\langle c, x_2, y_2 \rangle\rangle').$$

Combining this result with Corollary 20.4 and Theorem 20.7 yields the following theorem:

21.7. Theorem. *Let $(A, -)$ be a bioctonion algebra. The following are equivalent:*

(1) *$(A, -)$ is not a structurable division algebra.*

(2) *$(A, -)$ is isotopic to a decomposable bioctonion algebra $C_1 \otimes C_2$ where C_1 and C_2 are octonion algebras whose norms have a common slot.*

(3) *The Albert form of $(A, -)$ is similar to $\phi_1' \perp \langle -1 \rangle \phi_2'$ where the ϕ_i are 3-Pfister forms with a common slot.*

21.8. *The octic norm of a bioctonion algebra.* The norm N_A of a structurable algebra $(A, -)$ is preserved up to similitude under the action of $\mathbf{Str}(A, -)$, hence N_A is preserved up to isomorphism under the action of $\mathbf{Str}(A, -)^{\mathrm{der}}$. If $(A, -)$ is a bioctonion algebra, this means that its norm is an invariant polynomial for $\mathbf{Spin}(S, Q)$, where Q is an Albert form for $(A, -)$. When $(A, -)$ is split, the polynomial N_A is traditionally denoted by J. It follows from Lemma 13.6 (i) and then [28, Proposition 6.1] applied to $\mathbf{Spin}_{14}$ that J in fact generates the ring of $\mathbf{Spin}_{14}$-invariant polynomials; that is,

$$k[A]^{\mathbf{Spin}_{14}} = k[J].$$

It is well-known in invariant theory that $\deg J = 8$: in characteristic zero this was proved in [148, Proposition 40] and [135, Proposition 13]. Also including characteristic $p > 3$, Allison and Faulkner proved that $\deg J = 8$ and even gave a formula for N_A for arbitrary bioctonion algebras [10, Theorem 9.6]. Their formula is reproduced in (21.12.1). Another concrete but unwieldy expression for J (over the complex numbers) was calculated by Gyoja in [73, Theorem 5].

In this subsection and the next one, we assemble some curious and remarkable facts about this octic form J and build a picture of how bioctonion algebras, quadratic forms, octic forms, and algebraic groups fit in with one another. From Theorem 19.4 and Corollary 21.2, we have $PI^3_{14}(k) \simeq H^1(k, \mathbf{Str}(C(8) \otimes C(8))) = H^1(k, \mathbf{Sim}(J))$. Since $H^1(k, \mathbf{Str}(C(8) \otimes C(8)))$ classifies bioctonion algebras up to isotopy (see 13.7), and $H^1(k, \mathbf{Sim}(J))$ classifies k-forms of J up to similitude [25, III, Exercise 2], there are one-to-one correspondences:

Similitude classes of 14-dimensional quadratic forms in $I^3(k)$	$\longleftrightarrow$	Isotopy classes of bioctonion k-algebras	$\longleftrightarrow$	Similitude classes of 64-dimensional octic forms over k that become isomorphic to J over k^s.

The leftward arrow sends a bioctonion algebra to the class of its Albert quadratic form, and the rightward arrow sends the algebra to the class of its norm. Moreover, Theorem 20.7 implies that a quadratic form in $I^3_{14}(k)$ is anisotropic if and only if its corresponding octic form is anisotropic.

By Theorem 5.3, Theorem 18.19, and classification results on simple algebraic groups [173], the three-way correspondence above can also be extended to a five-way correspondence including:

Isomorphism classes of strongly inner simply connected simple algebraic k-groups of type D_7	$\longleftrightarrow$	Isomorphism classes of simple algebraic k-groups with Tits index $E_{8,1}^{91}$, $E_{8,2}^{66}$, $E_{8,4}^{28}$, or $E_{8,8}^{0}$.

Concretely, these last two correspondences are defined by sending a bioctonion algebra to the derived subgroup of its structure group (a simply connected group of type D_7) or to the automorphism group of its TKK Lie algebra (an isotropic group of type E_8). Equivalently, the simply connected group of type D_7 contains the semisimple anisotropic kernel of the corresponding group of type E_8.

21.9. *Matrix factorisations.* Let $P \in k[x_1, \dots, x_n]$ be a nonzero polynomial in n variables. A *matrix factorisation* of P is a pair (N, M) where N, M are square $t \times t$ matrices ($t \geq 1$) with entries in $k[x_1, \dots, x_n]$, such that $NM = P.\,\mathrm{id}_{t\times t}$. What is interesting about this concept is that many irreducible polynomials (i.e., having no factorisation by a pair of nonconstant polynomials) do have matrix factorisations. If P is homogeneous, say of degree d, and the entries of N and M are all of degree $j \geq 1$ and $d - j$ respectively, then the number of coefficients appearing in N and M is potentially much smaller than the number of coefficients appearing in P. This is especially true if j is close to $\frac{1}{2}d$, and even more so if $N = M$. In other words, a matrix factorisation is often an incredibly efficient way to represent P.

Recently it was discovered that over the complex numbers the $\mathbf{Spin}_{14}$-invariant octic polynomial J admits a factorisation as the square of a certain 14×14 matrix [1, Theorem 2.3.2]. We are able to describe this matrix factorisation explicitly in terms of bioctonion algebras. Surprisingly, nonsplit forms of J are also matrix-factorisable over the base field – a fact that cannot easily be deduced from the complex case. The matrices in the factorisation can even be calculated in reasonable time using computer algebra software.

21.10. *P-operators.* In any structurable algebra $(A, -)$, there is a family of operators $\{P_x : x \in A^*\} \subset \operatorname{End} A$, defined as

$$P_x(a) = \tfrac{1}{3} U_x(5a - 2V_{a,x}\hat{x})$$

for all $a \in A$. Despite the exotic definition, these operators have a number of nice properties: for example $P_x \in \operatorname{Str}(A, -)$ and

$$\hat{P}_x = P_{\hat{x}} = P_x^{-1} \tag{21.10.1}$$

for all $x \in A^*$ [14, Theorem 8.3]. Since $P_x \in \operatorname{Str}(A, -)$, we can define the map $(P_x)_S \in \operatorname{End}_k S$ as in (18.5.3).

21.11. Lemma. *Let $(A, -)$ be a structurable algebra. For all $x \in A^*$ and $s \in \operatorname{Skew}(A, -)$,*

$$(P_x)_S(s) = \tfrac{1}{6}\psi(x, U_x(sx)). \tag{21.11.1}$$

Proof. The proof is loosely based on [31, Lemma 3.3.4]. Using (18.5.4), we have

$$L_{(P_x)_S(s)}\hat{P}_x = P_x L_s. \tag{21.11.2}$$

An expression for $\hat{x}$ derived by Allison (see [5, Proposition 2.6], or the more accessible sources [10, eq. (1.7)] and [31, eq. (2.14)]), combined with the fact that $L_s^{-1} = -L_{\hat{s}}$, yields:

$$U_x(sx) = -\tfrac{1}{2}\psi(x, U_x(sx))\hat{x}. \tag{21.11.3}$$

The following identities appear in [31, p. 33]; these are (reasonably direct) consequences of the definition of P_x:

$$U_x(sx) = -3P_x(sx), \tag{21.11.4}$$

$$P_x(\hat{x}) = x. \tag{21.11.5}$$

Applying (21.11.3), (21.11.4), (21.11.2), and (21.11.5) in that order, it follows that

$$\begin{aligned}\tfrac{1}{6}\psi(x, U_x(sx))\hat{x} &= -\tfrac{1}{3}U_x(sx) = P_x(sx) = P_x L_s(x)\\ &= L_{(P_x)_S(s)}\hat{P}_x(x) = L_{(P_x)_S(s)}\hat{x} = (P_x)_S(s)\hat{x}.\end{aligned}$$

The linear map $R_{\hat{x}}|_S : S \to S\hat{x}$, $s \mapsto s\hat{x}$, is bijective because $S\hat{x} = \mathrm{Skew}(A^{\langle\hat{x}\rangle}, -^{\langle\hat{x}\rangle})$ and $\dim(\mathrm{Skew}(A, -)) = \dim \mathrm{Skew}(A^{\langle\hat{x}\rangle}, -^{\langle\hat{x}\rangle})$ [14, Corollary 12.2], hence (21.11.1). □

An important role of Lemma 21.11 is to show that $x \mapsto (P_x)_S$ extends to a globally defined rational mapping $A \to \mathrm{End}(S)$ such that A^* is mapped into $\mathrm{GL}(S)$. That is, even though P_x might not be defined for all $x \in A$, it is clear that $\frac{1}{6}\psi(x, U_x(sx))$ is defined for all $x \in A$ and all $s \in \mathrm{Skew}(A, -)$.

21.12. *A matrix factorisation of the octic norm.* Now assume that $(A, -)$ is a bioctonion algebra. The composition

$$A^* \xrightarrow{P} \mathbf{Str}(A, -) \xrightarrow{\gamma} \mathbf{GO}(S, Q) \xrightarrow{\mu} \mathbf{G}_m$$

is a well-defined map of varieties. By Lemma 21.11, this composition is the restriction of a genuine polynomial function $A \to k$, which we can define by picking an arbitrary conjugate-invertible basepoint $s_0 \in \mathrm{Skew}(A, -)$ and sending

$$x \mapsto \frac{1}{36Q(s_0)}Q(\psi(x, U_x(s_0x))). \tag{21.12.1}$$

This polynomial function is none other than the norm of $(A, -)$ [10, Theorem 9.6], a claim which can be justified by the uniqueness property of the norm (namely that it is the unique normalised invertibility-detecting polynomial function of minimal degree on A). We repeat for emphasis that $N_A(x) = \mu((P_x)_S)$ if $x \in A^*$.

21.13. Theorem. *Let $(A, -)$ be a bioctonion algebra with* $\mathrm{Skew}(A, -) = S$. *Define for all $x \in A$ the linear map $M_x \in \mathrm{End}\, S$,*

$$M_x(s) = \frac{1}{6}\psi(x, U_x(s^\sharp x)).$$

Then

$$M_x^2 = N_A(x).\, \mathrm{id}_S\,.$$

In other words, (M, M) is a matrix factorisation of N_A.

Proof. Suppose $x \in A^*$. Note that $M_x(s) = (P_x)_S(s^\sharp) = \gamma(P_x)(s^\sharp)$ by Lemma 21.11. We have by (18.6.3) and (21.10.1) that

$$\begin{aligned}\mathrm{id}_S = \gamma(P_x P_x^{-1}) = \gamma(P_x \hat{P}_x) &= \gamma(P_x)\gamma(\hat{P}_x) \\ &= (P_x)_S(\hat{P}_x)_S = \frac{1}{\mu((P_x)_S)}(P_x)_S \circ \sharp \circ (P_x)_S \circ \sharp\end{aligned}$$

and hence

$$N_A(x).\,\mathrm{id}_S = \mu((P_x)_S).\,\mathrm{id}_S = M_x^2 \qquad \text{for all } x \in A^*.$$

Extending to an infinite field if necessary, M_x and $N_A(x)$ remain polynomial and A^* is Zariski-dense in A. Therefore $N_A(x).\,\mathrm{id}_S = M_x^2$ for all $x \in A$. $\square$

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Chapter VI

Cohomological invariants of bicomposition algebras

We introduce a number of mod 2 cohomological invariants of bicomposition algebras. For $(8,8)$-product algebras, the three invariants b_1, b_3, and b_6 originate from the Malcev algebra on the skew elements, whose centroid is a quadratic étale algebra, the Albert form, which is in $I^3(k)$, and the quadratic trace, which is a 6-Pfister neighbour. We provide some applications of these invariants: the first invariant detects decomposability, the second invariant has a small kernel, and the second and third invariants give sufficient (but not necessary) conditions for being a division algebra. We sketch the analogous results for $(4,4)$-product algebras, and briefly investigate the invariants of other (m_1,m_2)-product algebras.

We then study the AF constructions of E_8 and E_7, investigating the extent to which our invariants survive the passage from structurable algebras to Lie algebras. In the case of E_8, all three of the bioctonion invariants play a role either in the Killing form or the Rost invariant. Some of this work, §25–26, is based on collaboration with Victor Petrov.

22. Transfer of quadratic forms

This section introduces the additive transfer of quadratic forms. This is also known as the Scharlau transfer, and is quite classical. An analogous notion of multiplicative transfer (or norm transfer) first appeared in work by Rost [143] and Tignol [170]. For simplicity's sake we limit discussion to transfers along quadratic extensions.

Let E/k be a quadratic étale extension: that is, either a quadratic field extension or the split quadratic étale extension $E = k \times k$. To make sense of the split case, it is helpful to view a nondegenerate n-dimensional quadratic space (V,q) over $E = k\times k$ as an ordered pair of nondegenerate n-dimensional quadratic spaces (V_1,q_1) and (V_2,q_2) over k, where $V_1 = (1,0)V$, $V_2 = (0,1)V$, and $q(v_1+v_2) = (q_1(v_1), q_2(v_2))$ for all $v_i \in V_i$.

22.1. *Additive transfer.* Let (V,q) be an n-dimensional quadratic space over E, and let $s : E \to k$ be a linear functional. The *additive transfer* of (V,q) along s is the $2n$-dimensional k-quadratic space $(V, s_*(q))$ where

$$s_*(q)(v) = s(q(v))$$

for all $v \in V$. We write $T_{E/k}$ for the operation $(\mathrm{tr}_{E/k})_*$. This operation is compatible with Witt equivalence of quadratic forms, so it defines an additive homomorphism

$$T_{E/k} : W(E) \to W(k).$$

If $E = k(\sqrt{d})$ is a field, for some non-square $d \in k^\times$, then by [151, Lemma 5.8] or direct calculation, we have:

$$T_{E/k}(1) = \langle 2 \rangle \langle\langle -d \rangle\rangle. \tag{22.1.1}$$

In the split case where $E = k \times k$, if (V, q) corresponds to the pair (V_1, q_1), (V_2, q_2), then the definition entails $T_{E/k}(q) \simeq q_1 \perp q_2$.

22.2. *Multiplicative transfer.* Let (V, q) be a quadratic space over E. Let ι be the nonidentity automorphism of E/k. Define ${}^\iota V$ to be the E-module ${}^\iota V = V$ with $e \cdot v = \iota(e)v$ for all $e \in E$, $v \in V$, and define ${}^\iota q(v) = \iota(q(v))$ for all $v \in {}^\iota V$. This defines a quadratic space $({}^\iota V, {}^\iota q)$ over E. The *multiplicative transfer* of (V, q) is the n^2-dimensional k-quadratic space

$$N_{E/k}(V, q) = (N_{E/k}(V), N_{E/k}(q))$$

defined as follows:

- $N_{E/k}(V)$ is the subspace of ${}^\iota V \otimes_E V$ fixed by the switch map $x \otimes y \mapsto y \otimes x$.
- $N_{E/k}(q)$ is the restriction of ${}^\iota q \otimes_E q$ to $N_{E/k}(V)$.

For one-dimensional quadratic forms, the multiplicative transfer behaves straightforwardly [180, Lemma 2.6 (i)]:

$$N_{E/k}(\langle e \rangle) = \langle N_{E/k}(e) \rangle \qquad \text{for all } e \in E. \tag{22.2.1}$$

For quadratic forms of dimension > 1, it is less straightforward. For example, if $E = k(\sqrt{d})$ is a field then by [180, Lemma 2.13],

$$N_{E/k}(\mathbb{H}) = \langle 2 \rangle \langle\langle d \rangle\rangle + \mathbb{H} = \langle 2, -2d, 1, -1 \rangle.$$

Unlike the additive transfer $T_{E/k}$, the multiplicative transfer $N_{E/k}$ is not generally compatible with Witt equivalence, so it does not define an operation on $W(k)$. However, $N_{E/k}$ does extend to the Grothendieck–Witt ring; specifically, [180, Lemma 2.6 (iii) & Satz 2.9] show that there is a unique multiplicative map

$$N_{E/k} : \widehat{W}(E) \to \widehat{W}(k)$$

such that $N_{E/k}([q]) = [N_{E/k}(q)]$ for all quadratic forms q over E, and $N_{E/k}(-1) = 3 - \langle 2 \rangle \langle\langle d \rangle\rangle$.

In the split case where $E = k \times k$, if (V, q) corresponds to the pair (V_1, q_1), (V_2, q_2), then the definition entails $N_{E/k}(q) \simeq q_1 \otimes q_2$.

The following theorem says what $N_{E/k}$ does to Pfister forms; it is a direct consequence of [180, Satz 2.16 (ii)], whose statement also appears without proof in [144].

22.3. Theorem (Rost–Wittkop). *Let $E = k(\sqrt{d})$ be a quadratic field extension, and let $q = \langle\langle c_1, \ldots, c_n \rangle\rangle$ where $c_i \in E$, $n \geq 1$. If $\mathrm{tr}_{E/k}(c_i) = 0$ for some i, then*

$$N_{E/k}(q) \simeq 2^{n-1}(2^n - 1)\mathbb{H} \perp 2^{n-1}\langle\langle d \rangle\rangle.$$

Otherwise,

$$N_{E/k}(q) \perp 2^{n-1}\mathbb{H} \simeq 2^{n-1}\langle\langle d \rangle\rangle \perp \bigotimes_{i=1}^{n} \langle\langle \mathrm{tr}_{E/k}(c_i), -dN_{E/k}(c_i) \rangle\rangle.$$

Consequently, in $W(k)$ we have $N_{E/k}(q) - 2^{n-1}\langle\langle d \rangle\rangle \in I^{2n}(k)$.

The following formula makes a seriously useful connection between the operations of additive transfer, multiplicative transfer, and exterior square (see 16.7):

22.4. Theorem (Rost, Wittkop [180, Satz 2.12]). *Let $E = k(\sqrt{d})$ be a quadratic field extension, and let $x \in \widehat{W}(E)$. Then*

$$\lambda^2(T_{E/k}(x)) = T_{E/k}(\lambda^2(x)) + \langle d \rangle N_{E/k}(x).$$

It is worth emphasising that this identity holds in $\widehat{W}(E)$ but not in $W(E)$, because neither λ^2 nor $N_{E/k}$ are well-defined operations on $W(E)$.

23. Invariants of bioctonion algebras

In this section we define three mod 2 cohomological invariants of bioctonion algebras.

23.1. *Galois cohomology of bioctonion algebras.* To lighten the notation in this section, write

$$G = G_2.$$

That is, G is the split absolutely simple algebraic group of type G_2, or the automorphism group of the split octonion algebra over k. Let S_2 be the constant finite algebraic group of order 2. The group $H^1(k, S_2) \simeq k^\times/k^{\times 2}$ classifies quadratic étale k-algebras up to k-isomorphism.

By Theorem 9.12, the automorphism group of the split bioctonion algebra is

$$\mathbf{Aut}(\mathbb{O} \otimes \mathbb{O}, -) = G^2 \rtimes S_2.$$

The split short exact sequence of algebraic groups

$$1 \longrightarrow G^2 \longrightarrow G^2 \rtimes S_2 \longrightarrow S_2 \longrightarrow 1$$

induces an exact sequence of pointed sets, in which we denote the third arrow by b_1:

$$S_2 \longrightarrow H^1(k, G^2) \longrightarrow H^1(k, G^2 \rtimes S_2) \xrightarrow{b_1} k^\times/k^{\times 2} \longrightarrow 1. \tag{23.1.1}$$

23.2. Lemma. *If $\beta \in H^1(k, G^2 \rtimes S_2)$ and $(A, -)$ is a bioctonion algebra corresponding to β, then $b_1(\beta)$ is the isomorphism class of the centroid of* $\mathrm{Skew}(A, -)^-$.

Proof. Clearly b_1 and the map $[(A, -)] \mapsto [\mathrm{Centr}(\mathrm{Skew}(A, -)^-)]$ are both nonzero normalised cohomological invariants $H^1(*, G^2 \rtimes S_2) \to H^1(*, \mathbb{Z}/2\mathbb{Z})$. By (15.9.1), there is exactly one such invariant. $\square$

Now suppose E/k is a quadratic étale extension corresponding to some $\varepsilon \in H^1(k, S_2)$. The group S_2 acts on E/k by k-automorphisms. By functoriality it acts on $H^1(k, R_{E/k}(G_E))$. At this point, the background theory from 12.9 is very relevant.

23.3. Lemma. *The fibre $b_1^{-1}(\varepsilon)$ is in natural bijective correspondence with the orbit space*

$$\frac{H^1(k, R_{E/k}(G \times_k E))}{S_2}.$$

Proof. If $\mathbb{O}_E$ is the split octonion algebra over E, then the automorphism group of $\mathrm{cor}_{E/k}(\mathbb{O}_E)$ is $R_{E/k}(G \times_k E) \rtimes S_2$ by Theorem 9.12. We have an exact sequence:

$$S_2 \longrightarrow H^1(k, R_{E/k}(G \times_k E)) \longrightarrow H^1(k, R_{E/k}(G \times_k E) \rtimes S_2) \longrightarrow k^\times / k^{\times 2} \longrightarrow 1. \tag{23.3.1}$$

This is the sequence (23.1.1) twisted by a cocycle $b \in Z^1(k, G^2 \rtimes S_2)$ representing the isomorphism class of $\mathrm{cor}_{E/k}(\mathbb{O}_E)$. The claim follows from [156, I.§5.5 Corollary 2]. $\square$

23.4. *Decomposable bioctonion algebras.* The exact sequence (23.1.1) carries some meaning. We can identify

$$H^1(k, G^2) = H^1(k, G) \times H^1(k, G)$$

as S_2-sets, where S_2 acts on $H^1(k, G) \times H^1(k, G)$ by swapping. One should think of $H^1(k, G^2)$ as the set of ordered pairs of octonion k-algebras, and $H^1(k, G^2 \rtimes S_2)$ as the set of bioctonion algebras up to isomorphism.

From this point of view the map $H^1(k, G^2) \to H^1(k, G^2 \rtimes S_2)$ sends the class of (C_1, C_2) to the class of $C_1 \otimes C_2$. The exactness of (23.1.1) implies $\ker(b_1)$ is the set of isomorphism classes of decomposable bioctonion algebras. Lemma 23.3 for $\varepsilon = 0$ essentially says that $C_1 \otimes C_2 \simeq C_1' \otimes C_2'$ if and only if $C_1 \simeq C'_{\sigma(1)}$ and $C_2 \simeq C'_{\sigma(2)}$ for some $\sigma \in S_2$, which is something we proved directly in Theorem 9.9.

23.5. *Partitioning the cohomology set.* By Lemma 23.3 and Shapiro's Lemma [101, Lemma 29.6], there are isomorphisms:

$$H^1(k, G^2 \rtimes S_2) \simeq \coprod_{[E] \in H^1(k, S_2)} \frac{H^1\big(k, R_{E/k}(G \times_k E)\big)}{S_2} \simeq \coprod_{[E] \in H^1(k, S_2)} \frac{H^1(E, G \times_k E)}{S_2}$$

For $E = k(\sqrt{a})$, the pointed set $H^1(E, G)$ is identified with the set of E-isomorphism classes of octonion algebras over E. The quotient of $H^1(E, G)$ by S_2 is identified with the set of k-isomorphism classes of octonion algebras over E. In other words, the partition displayed above is a cohomological version of Theorem 9.9 for $(8, 8)$-product algebras.

23.6. *Classification of bioctonion algebras by successive invariants.* We can define two successive invariants of bioctonion algebras which classify them up to isomorphism. The first invariant is b_1 from Lemma 23.2. If

$$b_1(A,-) = [E] \in H^1(k, S_2)$$

then $(A,-) \simeq \mathrm{cor}_{E/k}(C)$ for a certain octonion algebra C over E, and C is unique up to k-isomorphism.

The group $S_2 \simeq \mathrm{Aut}_k(E) = \{1, \iota\}$ acts on $H^3(E, \mathbb{Z}/2\mathbb{Z})$ by functoriality; on symbols this action is $(a)\cdot(b)\cdot(c) \mapsto ({}^\iota a)\cdot({}^\iota b)\cdot({}^\iota c)$. Let n_C be the norm of C and define

$$b_{[E]}(A,-) = \{e_3(n_C), e_3({}^\iota n_C)\} \in \frac{H^3(E, \mathbb{Z}/2\mathbb{Z})}{S_2}.$$

Recall that octonion algebras over E are classified up to E-isomorphism by the invariant $C \mapsto e_3(n_C)$ [131, Theorem 5.4], and therefore they are classified up to k-isomorphism by the invariant $b_{[E]}$. In other words, the map

$$b_{[E]} : \frac{H^1(k, R_{E/k}(G \times_k E))}{S_2} \longrightarrow \frac{H^3(E, \mathbb{Z}/2\mathbb{Z})}{S_2}$$

is both well-defined and injective. In summary, we have two invariants b_1 and $b_{[*]}$ that, when applied successively, classify bioctonion algebras up to k-isomorphism.

Note however, that $b_{[*]}$ is not a mod 2 cohomological invariant in the sense of Definition 15.2, i.e., it is not an element of $\mathrm{Inv}(G^2 \rtimes S_2, 2)$. This situation is comparable to the classification of cubic étale algebras by cohomological invariants [101, Proposition 30.18]. In both situations, the invariants need to be applied successively in order to obtain classifying data, and the second invariant takes values in an orbit space of a cohomology group.

We discover later in Corollary 38.18 that bioctonion algebras are *not* classified by invariants in $\mathrm{Inv}(G^2 \rtimes S_2, 2)$.

23.7. *Cohomological invariants of $G^2 \rtimes S_2$ in degree ≤ 3.* We have a degree 1 invariant $b_1 : H^1(*, G^2 \rtimes S_2) \to H^1(*, \mathbb{Z}/2\mathbb{Z})$ from Lemma 23.2, which in concrete terms is

$$b_1(A,-) = [\mathrm{Centr}(\mathrm{Skew}(A,-)^-)].$$

We can define a degree 3 invariant as follows. Let $e_3 : I^3(*) \to H^3(*, \mathbb{Z}/2\mathbb{Z})$ be the Arason invariant (see 16.6), and define $b_3 : H^1(*, G^2 \rtimes S_2) \to H^3(*, \mathbb{Z}/2\mathbb{Z})$:

$$b_3(A,-) = e_3(Q)$$

where Q is an Albert form for $(A,-)$. Since e_3 is constant on similitude classes, no ambiguity arises if we choose a different Albert form $\langle c\rangle Q$ instead of Q.

To no-one's surprise, b_3 is the restriction of the Rost invariant of E_8.

23.8. Proposition. *For all field extensions L/k, the following diagram commutes:*

$$\begin{tikzcd}
H^1(L, G^2 \rtimes S_2) \arrow[r, "b_3"] \arrow[d] & H^3(L, \mathbb{Z}/2\mathbb{Z}) \arrow[d] \\
H^1(L, E_8) \arrow[r, "r_{E_8}"] & H^3(L, \mathbb{Q}/\mathbb{Z}(2))
\end{tikzcd}$$

where the left vertical arrow is induced by the inclusion

$$G^2 \rtimes S_2 = \mathbf{Aut}(\mathbb{O} \otimes \mathbb{O}, -) \subset \mathbf{Aut}(K(\mathbb{O} \otimes \mathbb{O}, -)) = E_8,$$

and the right vertical arrow comes from the inclusion $\mathbb{Z}/2\mathbb{Z} = \boldsymbol{\mu}_2^{\otimes 2} \subset \mathbb{Q}/\mathbb{Z}$.

Proof. Let $\zeta \in H^1(k, E_8)$ be the class corresponding to $K(A, -)$. Lemmas 13.6 (iii)–(iv) and 20.3 imply that there is a class $\xi \in H^1(k, \mathbf{Spin}_{14})$ whose image in $H^1(k, E_8)$ is equal to ζ, and whose corresponding quadratic form $q_\xi \in I^3_{14}(k)$ has the same similitude class as the Albert form of $(A, -)$. The inclusion $\mathbf{Spin}_{14} \subset E_8$ has Rost multiplier 1 because E_8's root system is simply laced [66, §5.7], so

$$r_{E_8}(\zeta) = r_{\mathbf{Spin}_{14}}(\xi).$$

It is well-known that $r_{\mathbf{Spin}_{14}}(\xi) = e_3(q_\xi)$ [101, p. 437]. By definition of b_3, we have $b_3(A, -) = e_3(q_\xi)$. □

We now move towards higher-degree invariants. Recall from 2.14 how the trace T_A of a structurable algebra $(A, -)$ is defined.

23.9. Lemma. *Let C be an octonion algebra over a quadratic étale extension E/k with norm n, and let $(A, -) = \mathrm{cor}_{E/k}(C)$. Then $T_A = \langle 128 \rangle N_{E/k}(n)$.*

Proof. The symmetric bilinear form associated to $N_{E/k}(n)$ is nondegenerate and invariant (see [6, Proposition 2.2 (i)]), so Lemma 2.16 implies it is a scalar multiple of T_A. The scalar is 128, since $T(1) = \mathrm{tr}(L_2) = 128$ and $N_{E/k}(n)(1) = n(1)^\iota n(1) = 1$. □

23.10. *A cohomological invariant of $G^2 \rtimes S_2$ in degree* 6. Suppose L/k is a field extension and $(A, -) = \mathrm{cor}_{E/L}(C)$ for some quadratic étale extension E/L and an octonion algebra C over E. Define the invariant $b_6 : H^1(*, G^2 \rtimes S_2) \to H^6(*, \mathbb{Z}/2\mathbb{Z})$ by

$$b_6(A, -) = e_6(N_{E/L}(n_C) - 4n_E)$$

where n_C and n_E are the standard norms of C and E respectively. (Note that $n_E = \langle\langle d \rangle\rangle$ if $E = L(\sqrt{d})$ is a field and $n_E = \mathbb{H}$ if $E = L \times L$ is split.) This invariant is well-defined because by Theorem 22.3, $N_{E/L}(n_C) - 4n_E$ is Witt equivalent to a 6-Pfister form.

In light of Lemmas 23.2 and 23.9, this invariant has an even more direct description:

$$b_6(A, -) = e_6(\langle 128 \rangle T_A - 4S)$$

where $T_A \in W(L)$ is the quadratic trace form of $(A, -)$ and $S \in W(L)$ is the norm of the centroid of $\mathrm{Skew}(A, -)^-$.

23.11. Example (Real bioctonion algebras)**.** There are exactly two isomorphism classes of octonion algebras over $\mathbb{R}$, the split octonion algebra $\mathbb{O}_{\mathrm{split}}$ and the division octonion algebra $\mathbb{O}_{\mathrm{div}}$. Let $\mathbb{O}_{\mathbb{C}}$ denote the unique (split) complex octonion algebra. There are four isomorphism classes of bioctonion algebras over $\mathbb{R}$, listed in the table below. The algebra $\mathbb{O}_{\mathrm{split}} \otimes \mathbb{O}_{\mathrm{div}}$ has an Albert form of signature -7, and the other three algebras have hyperbolic Albert forms. Therefore there are exactly two isotopy classes of real bioctonion algebras. The real bioctonion algebras are classified by the invariants b_1, b_3, and b_6, which take the following values:

$(A,-)$	$b_1(A,-)$	$b_3(A,-)$	$b_6(A,-)$
$\mathbb{O}_{\text{split}} \otimes \mathbb{O}_{\text{split}}$	0	0	0
$\mathbb{O}_{\text{split}} \otimes \mathbb{O}_{\text{div}}$	0	$(-1)^3$	0
$\mathbb{O}_{\text{div}} \otimes \mathbb{O}_{\text{div}}$	0	0	$(-1)^6$
$\mathrm{cor}_{\mathbb{C}/\mathbb{R}}(\mathbb{O}_{\mathbb{C}})$	(-1)	0	0

Table 7: Values of the cohomological invariants on real bioctonion algebras.

We return to assuming k is an arbitrary field with $\mathrm{char}(k) \neq 2, 3$. The following theorem provides some applications of the cohomological invariants of $G^2 \rtimes S_2$.

23.12. Theorem. *Let $(A,-)$ be a bioctonion algebra.*

(i) *$b_1(A,-) = 0$ if and only if $(A,-)$ is decomposable.*

(ii) *$b_3(A,-)$ has symbol length ≤ 3.*

(iii) *If $b_3(A,-)$ has symbol length 3 then $(A,-)$ is an indecomposable division algebra.*

(iv) *$b_3(A,-) = 0$ if and only if $(A,-)$ is isotopic to the split bioctonion algebra.*

(v) *$b_6(A,-)$ is a symbol.*

(vi) *If $(A,-)$ is not a division algebra then $b_6(A,-) \in (-1)\cdot H(k)$.*

(vii) *If $\sqrt{-1} \in k$ then $b_6(A,-) \neq 0$ implies $(A,-)$ is a division algebra.*

Proof. (i) This follows from the fact that (23.1.1) is exact. (A more expanded discussion was given in 23.4.)

(ii) By definition $b_3(A,-) = e_3(Q)$ where $Q \in I^3_{14}(k)$ is an Albert form for $(A,-)$. A result proved independently by Izhboldin [83, Remark 17.7] and Hoffmann [79, Proposition 2.3] says that any $Q \in I^3_{14}(k)$ is a sum of at most three forms that are similar to 3-Pfister forms, hence the symbol length of $e_3(Q)$ is at most 3.

(iii) If $(A,-)$ is not a division algebra, Theorem 21.7 implies Q is similar to a difference of two 3-Pfister forms, hence the symbol length of $e_3(Q)$ is at most 2. If $(A,-)$ is decomposable, Q is a difference of two 3-Pfister forms.

(iv) We have $e_3(Q) = 0$ if and only if $Q \in I^4_{14}(k)$, if and only if Q is hyperbolic (by the Arason–Pfister Hauptsatz). By Corollary 20.4, Q is hyperbolic if and only if $(A,-)$ is isotopic to the split bioctonion algebra.

(v) If $(A,-) = C_1 \otimes C_2$ is decomposable, then $b_6(A,-)$ is the symbol:

$$b_6(A,-) = e_6(N_{k\times k/k}((n_1,n_2))) = e_6(n_1 n_2) = e_3(n_1)e_3(n_2)$$

where n_i is the norm of C_i. If $(A,-) = \mathrm{cor}_{E/k}(C)$ where $E = k(\sqrt{d})$ is a field then $b_6(A,-) = e_6(N_{E/k}(n_C) - 4\langle\!\langle d\rangle\!\rangle)$. Say $n_C = \langle\!\langle c_1, c_2, c_3\rangle\!\rangle$. By Theorem 22.3, $b_6(A,-)$ is either zero or equal to the symbol

$$e_6\Big(\bigotimes_{i=1}^{3}\langle\!\langle \mathrm{tr}_{E/k}(c_i), -dN_{E/k}(c_i)\rangle\!\rangle\Big) = \prod_{i=1}^{3}(\mathrm{tr}_{E/k}(c_i))\cdot(-dN_{E/k}(c_i)).$$

(vi) Suppose $(A,-) = \mathrm{cor}_{E/k}(C)$ where E is a quadratic étale k-algebra and n is the norm of C. If $(A,-)$ is not a division algebra then n' represents an element $c \in k$ (Corollary 20.8). If n' represents 0 then n is hyperbolic and $b_6(A,-) = 0$. Otherwise, n' represents some $c \in k^\times$. In this case, $n = \langle\langle c, c_1, c_2\rangle\rangle$ for some $c_1, c_2 \in E^\times$ and

$$\begin{aligned} b_6(A,-) &= (2c)\cdot(-dc^2)\cdot(\mathrm{tr}_{E/k}(c_1))\cdot(-dN_{E/k}(c_1))\cdot(\mathrm{tr}_{E/k}(c_2))\cdot(-dN_{E/k}(c_2)) \\ &= (-1)\cdot(2c)\cdot(\mathrm{tr}_{E/k}(c_1))\cdot(-dN_{E/k}(c_1))\cdot(\mathrm{tr}_{E/k}(c_2))\cdot(-dN_{E/k}(c_2)) \end{aligned}$$

because $(-dc^2) = (-1) + (d)$ and $(d)\cdot(-dN_{E/k}(c_1)) = (d)\cdot(N_{E/k}(c_1\sqrt{d})) = 0$ by (15.1.3).

(vii) This is just a special case of (vi). □

The bound in 23.12 (ii) is sharp: there are examples of quadratic forms $Q \in I^3_{14}(k)$ for which $e_3(Q)$ has symbol length 3 [83, 17.1–17.3], and by Corollary 21.3 every such form is an Albert form of a bioctonion algebra. The assumption $\sqrt{-1} \in k$ really is necessary in 23.12 (vii): recall from Example 23.11 that there is a nondivision real bioctonion algebra with nonzero b_6. In view of 23.12 (iii), the kernel of b_3 is described in Proposition 20.11.

One wonders if the invariant b_6 "detects" the property of being a division algebra. In other words, does the reverse implication hold in (vii)? The answer is no, as the following example shows.

23.13. Example (Division algebras on which b_6 vanishes)**.** Let $(A,-)$ be a bioctonion division algebra over any field K/k, such that $b_6(A,-) \neq 0$. Let ψ be the 6-Pfister form associated to $(A,-)$, as in 23.10. By definition, $b_6(A,-) = e_6(\psi)$. If Q is an Albert form for $(A,-)$ then Q is anisotropic by Theorem 20.7. Hoffmann's Separation Theorem [75, Theorem 1] implies that Q remains anisotropic over the function field of ψ, which implies $(A_{K(\psi)},-)$ is a division algebra. But $\psi_{K(\psi)}$ is hyperbolic, so $b_6(A_{K(\psi)},-) = 0$.

24. Invariants of other (m_1, m_2)-product algebras

Most of the results of §23 concerning $(8,8)$-product algebras also have analogues for $(4,4)$-product algebras. Recall that $(4,4)$-product algebras are better known as central simple algebras with orthogonal involution of degree 4.

24.1. *Orthogonal involutions of degree* 4. It should be no trouble to reproduce the proofs, so we shall only summarise. Let hyp be the hyperbolic involution on $M_4(k)$. The automorphism group of $(M_4(k), \mathrm{hyp})$ is $\mathbf{PGO}_4 \simeq (\mathbf{PGL}_2 \times \mathbf{PGL}_2) \rtimes S_2$. There are natural-in-$k$ bijections

$$H^1(k, \mathbf{PGO}_4) \simeq \mathsf{Prod}_{4,4}(k) \simeq \mathsf{Comp}_4\acute{\mathsf{E}}\mathsf{t}_2(k).$$

One can define three nontrivial mod 2 cohomological invariants of $\mathbf{PGO}_4$. There are various ways to think about the unique nontrivial degree 1 invariant y_1: it is the map $\delta : H^1(*, \mathbf{PGO}_4) \to H^1(*, \mathbb{Z}/2\mathbb{Z})$ induced by the projection $\mathbf{PGO}_4 \to S_2$; it is the discriminant in the sense of [101, Definition 7.2]; it is the map sending (A,σ) to the centroid of the 6-dimensional semisimple Lie algebra $\mathrm{Skew}(A,\sigma)^-$ of type $A_1 \times A_1$; and it is the map sending (A,σ) to the centre $Z(Q)$ of the unique (up to k-isomorphism) quaternion algebra Q such that $(A,\sigma) \simeq \mathrm{cor}_{Z(Q)/k}(Q)$.

The next nontrivial invariant y_2 is of degree 2, and again there are several ways to think about it: it is the map $(A, \sigma) \mapsto [A] \in {}_2\operatorname{Br}(k) = H^2(k, \mathbb{Z}/2\mathbb{Z})$, and it is the map sending (A, σ) to $e_2(Q)$ where Q is an Albert form of (A, σ).

The next nontrivial invariant is of degree 4, and it is entirely analogous to the invariant described in 23.10. We shall describe it here for the record. Suppose (A, σ) is a central simple associative algebra with orthogonal involution of degree 4, and let Trd be its reduced trace. The bilinear form

$$b(x, y) = \operatorname{Trd}(x\sigma(y) + y\sigma(x))$$

is an invariant symmetric bilinear form on A. So is the form $N_{E/k}(n_Q)$, where n_Q is the norm on the unique quaternion algebra Q over a quadratic étale extension E such that $(A, \sigma) = \operatorname{cor}_{E/k}(Q)$. Since invariant forms are unique up to scaling, we have $b = \langle 8 \rangle N_{E/k}(n_Q)$. By Theorem 22.3,

$$N_{E/k}(n_Q) - 2n_E = \varphi$$

for some 4-Pfister form $\varphi \in W(k)$. We can define a degree 4 invariant y_4 by taking

$$y_4(A, \sigma) = e_4(\varphi).$$

24.2. Proposition. *If (A, σ) is a $(4, 4)$-product algebra which is not a division algebra, then $y_4(A, \sigma) \in (-1) \cdot H^3(k, \mathbb{Z}/2\mathbb{Z})$.*

Proof. The proof is entirely similar to 23.12 (vi), using the fact that if Q is a quaternion algebra over a quadratic étale extension E/k, and $\operatorname{cor}_{E/k}(Q)$ is not a division algebra, then $n_Q \simeq \langle\langle a, b \rangle\rangle$ for some $a \in k^\times$ and $b \in E^\times$ [101, Corollary 16.28]. ☐

24.3. *Comparison with invariants of* $\mathbf{PGL}_4$. For any n, there is a cohomological invariant $\delta \in \operatorname{Inv}^2(\mathbf{PGL}_n, n)$ that sends a central simple (associative) algebra A of degree n to its Brauer class in $H^2(k, \boldsymbol{\mu}_n)$.

Under the assumption $\sqrt{-1} \in k$, Rost, Serre, and Tignol showed that a central simple algebra A of degree 4 has the property that the quadratic form $x \mapsto \operatorname{Trd}(x^2)$ on A is Witt equivalent to the sum $q_2 \perp q_4$ of a 2-Pfister and a 4-Pfister form [145]. These summands are unique, so they lead to cohomological invariants $f_2 \in \operatorname{Inv}^2(\mathbf{PGL}_4, 2)$ and $f_4 \in \operatorname{Inv}^4(\mathbf{PGL}_4, 2)$, taking the following values on a central simple algebra A of degree 4:

$$f_2(A) = e_2(q_2) = 2[A], \qquad\qquad f_4(A) = e_4(q_4).$$

The form q_4 is hyperbolic if and only if A is cyclic, and q_2 is hyperbolic if and only if $[A]$ has order 2 in the Brauer group, which is the case if A supports an involution of the first kind. If $\sqrt{-1} \in k$, the invariants $1, f_2, f_4$ are $H(k)$-linearly independent generators for $\operatorname{Inv}(\mathbf{PGL}_4, 2)$; this was asked as [58, Question 6.14] and answered in [21, Corollary 3.3].

Not surprisingly, the composition

$$H^1(*, \mathbf{PGO}_4) \longrightarrow H^1(*, \mathbf{PGL}_4) \xrightarrow{f_4} H^4(*, \mathbb{Z}/2\mathbb{Z})$$

is the same as the invariant y_4; see [145, Exemple].

24.4. *The case* $m_1 > m_2$. For completeness we shall describe the invariants of (m_1, m_2)-product algebras where $m_1 > m_2$. The mod 2 invariants of G_2 [158, §18.4], $\mathbf{PGL}_2$ [58, Proposition 6.1], and $\mathbb{Z}/2\mathbb{Z}$ [58, Proposition 2.1] are well-understood and classified. This leaves the question of invariants of $(8,4)$-, $(8,2)$-, and $(4,2)$-product algebras.

Let G be the automorphism group of the split (m_1, m_2)-product algebra. Recall from Theorem 9.11 9.11 that in the three respective cases

$$G = \begin{cases} G_2 \times \mathbf{PGL}_2 & (m_1, m_2) = (8,4) \\ G_2 \times \mathbb{Z}/2\mathbb{Z} & (m_1, m_2) = (8,2) \\ \mathbf{PGL}_2 \times \mathbb{Z}/2\mathbb{Z} & (m_1, m_2) = (4,2). \end{cases}$$

Using the product formula from [58, §6.7], it is a simple exercise to prove:

24.5. Proposition. *With G as above,* $\mathrm{Inv}(G, 2)$ *is a free $H(k)$-module with a basis of nontrivial invariants of degrees*

$$\begin{cases} 0,2,3,5 & \textit{if } G = G_2 \times \mathbf{PGL}_2 \\ 0,1,3,4 & \textit{if } G = G_2 \times \mathbb{Z}/2\mathbb{Z} \\ 0,1,2,3 & \textit{if } G = \mathbf{PGL}_2 \times \mathbb{Z}/2\mathbb{Z}. \end{cases}$$

The degree 0 invariant is the constant invariant 1. The next two nontrivial invariants classify the smaller and the larger of the factor algebras, respectively. Taken together, this pair of invariants classifies (m_1, m_2)-product algebras up to isomorphism. The third invariant is just the cup product of the first two invariants.

24.6. *Invariants of* $(8,4)$*-product algebras.* For example, the three nontrivial normalised invariants of $(8,4)$-product algebras are

$$c_2 \in \mathrm{Inv}^2(G_2 \times \mathbf{PGL}_2, 2), \quad c_3 \in \mathrm{Inv}^3(G_2 \times \mathbf{PGL}_2, 2), \quad c_2 \cdot c_3 \in \mathrm{Inv}^5(G_2 \times \mathbf{PGL}_2, 2),$$

where for any $(A, -) = C \otimes N$,

$$\begin{aligned} c_2(A,-) &= [N] \in H^2(k, \mathbb{Z}/2\mathbb{Z}) = {}_2\operatorname{Br}(k) \\ c_3(A,-) &= e_3(n_C) \in H^3(k, \mathbb{Z}/2\mathbb{Z}) \\ c_2 \cdot c_3(A,-) &= c_2(A,-) \cdot c_3(A,-) \in H^5(k, \mathbb{Z}/2\mathbb{Z}). \end{aligned}$$

One can also take $H(k)$-linear combinations of these invariants, for example the invariant

$$d_5 \in \mathrm{Inv}^5(G_2 \times \mathbf{PGL}_2, 2) \qquad d_5 = (-1)\cdot(-1)\cdot c_3 - c_2 \cdot c_3.$$

24.7. Proposition. *If* $(A_1, -)$ *and* $(A_2, -)$ *are isotopic* $(8,4)$*-product algebras, then* $c_2(A_1, -) = c_2(A_2, -)$ *and* $d_5(A_1, -) = d_5(A_2, -)$.

Proof. Suppose $(A_1, -) = C_1 \otimes N_1$ and $(A_2, -) = C_2 \otimes N_2$ for octonion algebras C_i with norms n_i and quaternion algebras N_i with norms q_i. By Proposition 20.1, their Albert forms are similar, so in $W(k)$ we have

$$n_1 - q_1 = \langle u \rangle (n_2 - q_2) \tag{24.7.1}$$

for some $u \in k^\times$. The Albert forms are in $I^2(k)$, and

$$\begin{aligned} c_2(A_1, -) = [N_1] = e_2(q_1) = e_2(n_1 - q_1) &= e_2(\langle u \rangle (n_2 - q_2)) \\ &= e_2(n_2 - q_2) = e_2(q_2) = [N_2] = c_2(A_2, -). \end{aligned}$$

Moreover, $e_2(q_1) = e_2(q_2)$ implies $q_1 \simeq q_2$ [52, Theorem 1.8], and now (24.7.1) yields

$$\langle\langle u \rangle\rangle q_1 = n_1 - \langle u \rangle n_2. \tag{24.7.2}$$

The fact that $\langle\langle a, a \rangle\rangle \simeq \langle\langle -1, a \rangle\rangle$ for all $a \in k^\times$ [52, Corollary 1.9] implies $q_1 q_1 = \langle\langle -1, -1 \rangle\rangle q_1$ in $W(k)$. Now, multiplying both sides of (24.7.2) by $\langle\langle -1, -1 \rangle\rangle - q_1$ yields

$$0 = (\langle\langle -1, -1 \rangle\rangle - q_1)(n_1 - \langle u \rangle n_2) = (\langle\langle -1, -1 \rangle\rangle - q_1) n_1 - \langle u \rangle (\langle\langle -1, -1 \rangle\rangle - q_1) n_2$$

and hence

$$\begin{aligned} d_5(A_1, -) = e_5(\langle\langle -1, -1 \rangle\rangle n_1 - q_1 n_1) &= e_5(\langle u \rangle (\langle\langle -1, -1 \rangle\rangle n_2 - q_1 n_2) \\ &= e_5(\langle\langle -1, -1 \rangle\rangle n_2 - q_2 n_2) = d_5(A_2, -). \quad \square \end{aligned}$$

Example 20.5 provides a pair of $(8, 4)$-product algebras $(A_1, -)$ and $(A_2, -)$ over $k(t_1, t_2, t_3)$ which are isotopic but have $e_3(n_1) = (t_1)\cdot(t_2)\cdot(t_3) \neq 0$ and $e_3(n_2) = 0$. Hence the invariant c_3 is not constant on isotopy classes. One can show using that same example that the invariant $c_2 \cdot c_3$ is not constant on isotopy classes of $(8, 4)$-product algebras over the field $\mathbb{R}(t_1, t_2, t_3)$.

24.8. *The Tits algebra.* If $(A, -)$ is an $(8, 4)$-product algebra, the invariant

$$c_2(A, -) = [\mathrm{Nuc}(A, -)] \in H^2(k, \mathbb{Z}/2\mathbb{Z}) = {}_2\operatorname{Br}(k)$$

is the Tits class of $G = \mathbf{Aut}(K(A, -))$. Equivalently, $c_2(A, -)$ is the Brauer class of the Tits algebra of G (as defined in [101, §27]). Note that since G is of type E_7, the cocentre of its simply connected cover is $\mathbb{Z}/2\mathbb{Z}$, so it has only one (minimal) Tits algebra; it is defined over the base field and has exponent ≤ 2.

We also have the following analogue of Theorem 23.12 (vi).

24.9. Proposition. *If $(A, -)$ is an $(8, 4)$-product algebra that is not a division algebra, then $d_5(A, -) \in (-1)\cdot H(k)$.*

Proof. Since the Albert form of $(A, -)$ is isotropic, the norms of the octonion and quaternion factors represent a common element $a \in k^\times$. We can write these norms as $n = \langle\langle a, x, y \rangle\rangle$ and $q = \langle\langle a, z \rangle\rangle$, respectively, and then calculate using (15.1.2) that

$$\begin{aligned} d_5(A, -) &= (-1)\cdot(-1)\cdot(a)\cdot(x)\cdot(y) - (a)\cdot(z)\cdot(a)\cdot(x)\cdot(y) \\ &= (-1)\cdot(-1)\cdot(a)\cdot(x)\cdot(y) - (-1)\cdot(a)\cdot(z)\cdot(x)\cdot(y). \end{aligned}$$ $\square$

25. The AF construction of E_8

The subject of the next three sections is Lie algebras of type E_8 that are of the form $L = K(A, -, \gamma)$ for some bioctonion algebra $(A, -)$. Two of these sections, §25 and §26, are based on joint work with Victor Petrov that has been published in [133], but I have not included all the results of that paper (or even the main result).

25.1. *Local triality for octonion algebras.* Let $(C,-)$ be an octonion algebra with norm n. Each of the projections on the algebra of inner Lie related triples

$$\begin{aligned}\pi_i : \mathcal{T}_I &\longrightarrow \mathfrak{gl}(C)\\ (T_1,T_2,T_3) &\longmapsto T_i, \qquad\qquad (i=1,2,3)\end{aligned}$$

is injective [164, Theorem 3.5.5]. This is known as the *principle of local triality.* Moreover, [164, Lemma 3.5.2] shows that there is an isomorphism of Lie algebras

$$\mathcal{T}_I \simeq \mathfrak{so}(n).$$

25.2. Lemma. *Let (V,q) be an n-dimensional quadratic space, and let τ be the quadratic Killing form of $\mathfrak{so}(V,q)$. Then*

$$\tau \simeq \langle 4-2n\rangle \lambda^2(q).$$

Proof. This is [58, Exercise 19.2]. One way to do this exercise is to use the explicit representation of $\mathfrak{so}(q)$ as the subalgebra of $C(V,q)^-$ spanned by elements of the form

$$[u,v] \qquad\qquad u,v\in V.$$

These elements satisfy the following relations from [89, p. 232 (30)]:

$$[[u,v],[u',v']] = -2q(u,u')[v,v'] + 2q(u,v')[v,u'] + 2q(v,u')[u,v'] - 2q(v,v')[u,u'].$$

Given a diagonal quadratic form $q=\langle c_1,\dots,c_n\rangle$, it is not very difficult to find an orthogonal basis for this model of $\mathfrak{so}(V,q)$ such that the Killing form diagonalises to $\langle 4-2n\rangle \perp_{i<j}\langle c_ic_j\rangle$. □

25.3. Proposition. *Suppose $(A,-)$ is a bioctonion algebra of the form*

$$(A,-)=\mathrm{cor}_{E/k}(C)$$

for some quadratic étale extension E/k and some octonion algebra C over E. All its Lie related triples are inner, i.e. $\mathcal{T}_I=\mathcal{T}$, and there is an isomorphism

$$\mathcal{T}_I \simeq \mathrm{Lie}(R_{E/k}(\mathbf{Spin}(n)))$$

where n is the norm of C.

Proof. We have $\mathcal{T}_I\subset\mathcal{T}$ by definition. By Lemma 6.2 and Proposition 9.10,

$$\dim\mathcal{T} = \dim\mathrm{Der}(A,-) + 2\dim\mathrm{Skew}(A,-) = 28+28=56.$$

On the other hand, $\mathcal{T}_I$ (as an E-module) is isomorphic to $\mathrm{Lie}(\mathbf{Spin}(n))$ [164, Theorem 3.5.5] and so $\mathcal{T}_I$ (as a k-vector space) is 56-dimensional and isomorphic to $\mathrm{Lie}(R_{E/k}(\mathbf{Spin}(n)))$. □

25.4. *Subgroups of E_8 implied by the V_4-grading on $K(A,-,\gamma)$.* Applying the AF construction to a split bioctonion algebra and parameters $\gamma=(1,-1,1)$ produces the split Lie algebra of type E_8. There is a short discussion in [59, Example 4.6] that sheds some light on the V_4-grading

$$\mathfrak{e}_8 = \mathcal{T}_I \oplus A[12] \oplus A[23] \oplus A[31]$$

that comes out of this construction. The semisimple subgroup of E_8 corresponding to the subalgebra $\mathcal{T}_I$ is of type $D_4 + D_4$ (a highly symmetric Dynkin diagram!). The split group in question is

$$C_{E_8}(V_4)^\circ \simeq \frac{\mathbf{Spin}_8 \times \mathbf{Spin}_8}{\boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2},$$

with $\boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2$ diagonally embedded in the centre.

The semisimple subgroup of E_8 corresponding to the subalgebra $\mathcal{T}_I \oplus A[ij]$ is the connected centraliser of one of the transpositions in V_4. This subgroup has type D_8, and is isomorphic to $\mathbf{HSpin}_{16}$ [59, Example 4.3].

If $(A, -) = C_1 \otimes C_2$ is decomposable but not necessarily split, there exist simple subgroups

$$H_{ij} \subset \mathbf{Aut}(K(A, -, \gamma))$$

of half-spin type D_8 with $\mathrm{Lie}(H_{ij}) = \mathcal{T}_I \oplus A[ij]$. The author and Victor Petrov proved in [133, Proposition 1.5] that

$$\mathcal{T}_I \oplus A[ij] \simeq \mathfrak{so}(\langle\gamma_i\rangle n_1 \perp \langle -\gamma_j^{-1}\rangle n_2).$$

If $(A, -)$ is indecomposable, I do not know how to describe the subalgebra $\mathcal{T}_I \oplus A[ij] \subset K(A, -, \gamma)$ or the D_8 subgroup of $\mathbf{Aut}(K(A, -, \gamma))$ to which it corresponds; this subgroup is not necessarily of the form $\mathbf{HSpin}(q)$ for a quadratic form q (it may have a nontrivial Tits algebra).

25.5. Theorem. *Let C be an octonion algebra with norm n over a quadratic étale extension E/k. Let $(A, -) = \mathrm{cor}_{E/k}(C)$, $\gamma \in (k^\times)^3$, and let κ be the quadratic Killing form of $K(A, -, \gamma)$. Then*

$$\kappa \simeq \langle -15\rangle\big(T_{E/k}(\lambda^2(n)) \perp \langle \gamma_1\gamma_2^{-1}, \gamma_2\gamma_3^{-1}, \gamma_3\gamma_1^{-1}\rangle N_{E/k}(n)\big). \tag{25.5.1}$$

Proof. By Theorem 6.13,

$$\kappa \simeq \kappa_0 \perp \langle d\rangle\langle \gamma_1\gamma_2^{-1}, \gamma_2\gamma_3^{-1}, \gamma_3\gamma_1^{-1}\rangle N_{E/k}(n) \tag{25.5.2}$$

where κ_0 is the restriction of κ to $\mathcal{T}_I$, and

$$d = -2\dim A - 8\dim \mathrm{Skew}(A, -) = -240.$$

Let τ be the Killing form of $\mathcal{T}_I$. The Killing form of $\mathfrak{so}(n)$ is $\langle -12\rangle\lambda^2(n)$, by Lemma 25.2. Since $\mathcal{T}_I \simeq \mathrm{Lie}(R_{E/k}(\mathbf{Spin}(n))$ by (i), we have

$$\tau \simeq T_{E/k}(\langle -12\rangle\lambda^2(n)) = \langle -12\rangle T_{E/k}(\lambda^2(n)).$$

There is an automorphism of $K(A, -, \gamma) \otimes_k k^a$ that swaps the two simple subalgebras of $\mathcal{T}_I \otimes_k k^a$, and this implies κ_0 is a scalar multiple of τ; say

$$\kappa_0 \simeq \langle\phi_0\rangle\langle -12\rangle T_{E/k}(\lambda^2(n))$$

for some $\phi_0 \in k^\times$. Let us determine ϕ_0. The grading on $K(A, -, \gamma)$ makes it a sum of four $\mathcal{T}_I$-modules. For $P, Q \in \mathcal{T}_I$, $a \in A$, and any cyclic permutation (i, j, k) of $(1, 2, 3)$, we have

$$[P, [Q, a[ij]]] = P_k(Q_k(a))[ij].$$

Therefore

$$\kappa(P, Q) = \operatorname{tr}(\operatorname{ad}_P \operatorname{ad}_Q) = \tau(P, Q) + \operatorname{tr}(P_1 Q_1) + \operatorname{tr}(P_2 Q_2) + \operatorname{tr}(P_3 Q_3).$$

The trace forms of the three irreducible representations

$$\begin{aligned} \mathcal{T}_I &\to \mathfrak{gl}(A) \\ P &\mapsto P_\ell \qquad\qquad\qquad\qquad (\ell = 1, 2, 3) \end{aligned}$$

are all equal (despite them being inequivalent representations) and so $\operatorname{tr}(P_1 Q_1) = \operatorname{tr}(P_2 Q_2) = \operatorname{tr}(P_3 Q_3)$ for all $P, Q \in \mathcal{T}_I$. Moreover, $\operatorname{tr}(P_1 Q_1)$ is a scalar multiple of $\tau(P, Q)$. To determine the ratio between $\operatorname{tr}(P_1 Q_1)$ and $\tau(P, Q)$, we can assume $A = C_1 \otimes C_2$ is decomposable, and consider the subalgebra

$$\mathfrak{so}(n_1) \subset \mathfrak{so}(n_1) \times \mathfrak{so}(n_2) \simeq \operatorname{Lie}(R_{E/k}(\mathbf{Spin}(n)),$$

where n_ℓ is the norm on C_ℓ. It is well-known that the Killing form κ_1 on $\mathfrak{so}(n_1)$ is 6 $(= 8 - 2)$ times the trace form of its vector representation $\mathfrak{so}(n_1) \to \mathfrak{gl}(C_1)$, while the trace form of the representation $\mathfrak{so}(n_1) \to \mathfrak{gl}(C_1 \otimes C_2)$ is clearly 8 times the trace form of the vector representation. But κ_1 is equal to the restriction of the Killing form τ on $\mathfrak{so}(n_1) \times \mathfrak{so}(n_2)$, so this means that if $P \in \mathcal{T}_I$ belongs to the $\mathfrak{so}(n_1)$ subalgebra we have

$$\operatorname{tr}(P_1^2) = 8 \operatorname{tr}({P_1|_{C_1}}^2) = \frac{8}{6}\kappa_1(P) = \frac{8}{6}\tau(P).$$

In conclusion, $\phi_0 = 5$, so $\kappa_0 \simeq \langle -60 \rangle T_{E/k}(\lambda^2(n))$. We can plug this into (25.5.2) and simplify to get (25.5.1) because -60 and -240 are both in the same square class as -15. $\square$

25.6. Example (Real forms of E_8)**.** There are three Lie algebras of type E_8 over $\mathbb{R}$, which are called the *split* form, the *compact* form, and the *intermediate* form [59, §5]. There are various numerical invariants that distinguish these algebras (or groups) from each other: the rank of a maximal split torus; the signature of the Killing form; or the Rost invariant (which is like a numerical invariant because $H^3(\mathbb{R}, \mathbb{Q}/\mathbb{Z}(2)) \simeq \mathbb{Z}/2\mathbb{Z}$). The data is displayed in Table 8.

	$\mathbb{R}$-rank	signature	Rost invariant
compact	0	-248	0
intermediate	4	-24	1
split	8	8	0

Table 8: Real forms of E_8 and their numerical invariants [59].

There exist four real bioctonion algebras (see Example 23.11) and two 3-dimensional real quadratic forms up to similitude, $\langle 1, 1, 1 \rangle$ and $\langle 1, 1, -1 \rangle$. So there are eight possible combinations of inputs for this version of the AF construction. We can use Theorem 25.5 or one of many other arguments to decide what the output $K(A, -, \gamma)$ is for each of these combinations of inputs. The results are in Table 9.

	$(1,1,-1)$	$(1,1,1)$
$\mathbb{O}_{\text{split}} \otimes \mathbb{O}_{\text{split}}$	split	split
$\mathbb{O}_{\text{split}} \otimes \mathbb{O}_{\text{div}}$	intermediate	intermediate
$\mathbb{O}_{\text{div}} \otimes \mathbb{O}_{\text{div}}$	split	compact
$\text{cor}_{\mathbb{C}/\mathbb{R}}(\mathbb{O}_{\mathbb{C}})$	split	intermediate

Table 9: The real Lie algebra $K(A,-,\gamma)$ based on a real bioctonion algebra (rows) and a triple of real numbers (columns).

25.7. *A reduced Killing form in characteristic* 5. If $\text{char}(k) = 5$, there is the issue that the Killing form on E_8 is zero. However, if $(A,-) = \text{cor}_{E/k}(C)$ then the symmetric bilinear form on $K(A,-,\gamma)$ associated to

$$\kappa' = T_{E/k}(\lambda^2(n)) \perp \langle \gamma_1\gamma_2^{-1}, \gamma_2\gamma_3^{-1}, \gamma_3\gamma_1^{-1}\rangle N_{E/k}(n). \tag{25.7.1}$$

is nondegenerate and Lie invariant. This can be proved in at least two ways: one can factor out $\langle -30\rangle$ in the Killing form of the Chevalley Lie algebra of type E_8 defined over $\mathbb{Z}$, extend the new bilinear form to the split E_8 over k, and then twist it to get the form (25.8.1) on $K(A,-,\gamma)$. This form is clearly invariant and nondegenerate (its radical is a nonzero ideal and E_8 is a simple Lie algebra in all characteristics). Alternatively, one use the hint from [158, Exercise 27.21 (2)]: lift the Killing form of $K(A,-,\gamma)$ to the ring of Witt vectors, divide by -30 up there, and reduce modulo 5 to get (25.8.1).

25.8. Lemma. *Let n be a 3-Pfister form over a quadratic étale extension E/k, and let*

$$\kappa' = T_{E/k}(\lambda^2(n)) + \langle \gamma_1\gamma_2^{-1}, \gamma_2\gamma_3^{-1}, \gamma_3\gamma_1^{-1}\rangle N_{E/k}(n) \in W(k). \tag{25.8.1}$$

Then $\kappa' + 8 \in I^5(k)$ and

$$e_5(\kappa' + 8) = (-1)\cdot(-1)\cdot \text{cor}_{E/k}(e_3(n)) + (-1)\cdot(-1)\cdot(-\gamma_1\gamma_2^{-1})\cdot(-\gamma_2\gamma_3^{-1})\cdot(d).$$

Here, $\text{cor}_{E/k}$ refers to a map $H^d(E,\mathbb{Z}/2\mathbb{Z}) \to H^d(k,\mathbb{Z}/2\mathbb{Z})$ called the *corestriction*; for its definition see for instance [72, Construction 3.3.6].

Proof. By Example 16.8 (iii), $\lambda^2(n) = 4n'$. By Theorem 22.3, there is unique 6-Pfister form q_6 and a $d \in k^\times$ such that $N_{E/k}(n) = q_6 + 4\langle\langle d\rangle\rangle$. So we can write

$$\begin{aligned}\kappa' &= T_{E/k}(4n') + \langle \gamma_1\gamma_2^{-1}, \gamma_2\gamma_3^{-1}, \gamma_3\gamma_1^{-1}\rangle(q_6 + 4\langle\langle d\rangle\rangle)\\ &= T_{E/k}(4n) - T_{E/k}(4) - 4\langle\langle d\rangle\rangle + 4\langle\langle -\gamma_1\gamma_2^{-1}, -\gamma_2\gamma_3^{-1}, d\rangle\rangle + \langle\langle -\gamma_1\gamma_2^{-1}, -\gamma_2\gamma_3^{-1}\rangle\rangle' q_6.\end{aligned}$$

Now, $T_{E/k}(4) = 4T_{E/k}(1) = 4\langle 2\rangle\langle\langle -d\rangle\rangle$ by (22.1.1). The identities $4\langle 2\rangle = 4$ and $\langle\langle -d\rangle\rangle + \langle\langle d\rangle\rangle = 2$ hold in the Witt ring of any field (of characteristic not 2). Hence

$$\kappa' = T_{E/k}(4n) - 8 + 4\langle\langle -\gamma_1\gamma_2^{-1}, -\gamma_2\gamma_3^{-1}, d\rangle\rangle + \langle\langle -\gamma_1\gamma_2^{-1}, -\gamma_2\gamma_3^{-1}\rangle\rangle' q_6.$$

The term $T_{E/k}(4n)$ belongs to $I^5(k)$ and has $e_5(T_{E/k}(4n)) = (-1)\cdot(-1)\cdot \text{cor}_{E/k}(e_3(n))$ [53, Corollary 21.5, Lemma 40.1]. The term $\langle\langle -\gamma_1\gamma_2^{-1}, -\gamma_2\gamma_3^{-1}\rangle\rangle' q_6$ belongs to $I^6(k)$ so it is killed by e_5. The term $4\langle\langle -\gamma_1\gamma_2^{-1}, -\gamma_2\gamma_3^{-1}, d\rangle\rangle$ belongs to $I^5(k)$ and its image under e_5 is $(-1)\cdot(-1)\cdot(-\gamma_1\gamma_2^{-1})\cdot(-\gamma_2\gamma_3^{-1})\cdot(d)$. ☐

25.9. Lemma. *Let κ' be any nondegenerate Lie invariant bilinear form on $K(A, -, \gamma)$. If -1 is a sum of two squares in k, then $\kappa' \in I^6(k)$ and there is a unique 64-dimensional form $q \in I^6(k)$ such that $q + \kappa' \in I^8(k)$.*

Proof. The Lie algebra $K(A, -, \gamma)$ is central simple, so κ' is unique up to a scalar multiple [34, p. 105, Exercise 18 (a)]. Hence we can assume that κ' is the form displayed in (25.8.1). The assumption that -1 is a sum of two squares is equivalent to the identity $4 = 0$ in $W(k)$, or $(-1)\cdot(-1) = 0$ in $H(k)$. So the lemma implies $\kappa' \in I^6(k)$. Setting $q = N_{E/k}(n)$ yields

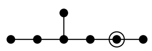
$$q + \kappa' = \langle 1, \gamma_1\gamma_2^{-1}, \gamma_2\gamma_3^{-1}, \gamma_3\gamma_1^{-1}\rangle N_{E/k}(n) = \langle\!\langle -\gamma_1\gamma_2^{-1}, -\gamma_2\gamma_3^{-1}\rangle\!\rangle N_{E/k}(n) \in I^8(K).$$

The uniqueness of q follows from the Arason–Pfister Hauptsatz. $\square$

26. Some partial invariants of E_8

Let us say that a *partial cohomological invariant* of an algebraic group G is a cohomological invariant of a subfunctor $\mathsf{S}(*) \subset H^1(*, G)$. (The terminology is my own, and is motivated by the concept of a partial function.)

26.1. Example. By Proposition 24.7, d_5 is a partial cohomological invariant of E_7^{ad} defined on the set of all $[\zeta] \in H^1(k, E_7^{\mathrm{ad}})$ such that ${}_\zeta(E_7^{\mathrm{ad}}/P_6)$ has a rational point, i.e., on torsors corresponding to groups of type E_7 whose Tits index is



(or has more circles).

26.2. *Subfunctors of $H^1(*, E_8)$.* Since E_8 is the automorphism group of its own Lie algebra, the cohomology set $H^1(k, E_8)$ is in natural bijection with the set of isomorphism classes of Lie algebras of type E_8. Let

$$\mathsf{Q}(*) \subset \mathsf{R}(*) \subset H^1(*, E_8)$$

be the functors $\mathsf{Fields}_{/k} \to \mathsf{Sets}$ such that for all fields F/k:

- $\mathsf{Q}(F)$ is the set of isomorphism classes of Lie algebras of type E_8 that are isomorphic to $K(A, -, \gamma)$ for some bioctonion algebra $(A, -)$ over F and some $\gamma = (\gamma_1, \gamma_2, \gamma_3) \in (K^\times)^3$. That is, $\mathsf{Q}(F)$ is the image of the AF construction (see Example 12.6):

$$H^1(F, (G_2 \times G_2 \rtimes \mathbb{Z}/2\mathbb{Z}) \times V_4) \to H^1(F, E_8).$$

- $\mathsf{R}(F)$ is the set of isomorphism classes of Lie algebras L of type E_8 such that the class of L is contained in $\mathsf{Q}(F')$ for some odd-degree extension F'/F.

By Lemma 15.8 (ii), a cohomological invariant of Q has a unique extension to R. By applying the quadratic form invariants $e_n : I^n(*) \to H^n(*, \mathbb{Z}/2\mathbb{Z})$ for $n = 6$ and 8 to the forms q and $q + \kappa'$ from Lemma 25.9, we obtain cohomological invariants of Q, hence also of R.

26.3. Theorem. *Suppose -1 is a sum of two squares in k. There exist nontrivial cohomological invariants $h_6 \in \mathrm{Inv}^6(\mathsf{R}, 2)$ and $h_8 \in \mathrm{Inv}^8(\mathsf{R}, 2)$ such that if $(A, -) = \mathrm{cor}_{E/F}(C)$ for some field extension F/k, quadratic étale F-algebra E, and octonion E-algebra C, then*

$$\begin{aligned} h_6(K(A, -, \gamma)) &= e_6(N_{E/F}(n)), \\ h_8(K(A, -, \gamma)) &= (-\gamma_1\gamma_2^{-1}) \cdot (-\gamma_2\gamma_3^{-1}) \cdot e_6(N_{E/F}(n)). \end{aligned}$$

Note that $e_6(N_{E/F}(n)) = b_6(A, -)$ if $\sqrt{-1} \in k$, which explains the claim from the chapter introduction that b_6 survives the TKK and AF constructions.

26.4. *The Tits construction of exceptional Lie algebras.* Tits in [172] introduced a groundbreaking construction of exceptional Lie algebras. It remains probably the best-known construction for the E series of Lie algebras.

Let C be a unital alternative algebra and J a Jordan algebra. Denote by C_0 and J_0 the subspaces of generic trace zero, and define binary operations $\circ$ and bilinear forms $(-, -)$ on C_0 and J_0 by the formula

$$ab = a \circ b + (a, b)1.$$

Two elements a, b in J and C define an inner derivation $\langle a, b\rangle$ of the respective algebra, namely:

$$\langle a, b\rangle(x) = \tfrac{1}{4}[[a, b], x] - \tfrac{3}{4}[a, b, x].$$

Then there is a Lie algebra structure on the vector space

$$T(C, J) = \mathrm{Der}(J) \oplus J_0 \otimes C_0 \oplus \mathrm{Der}(C)$$

defined by the formulas

$$\begin{aligned} &[\mathrm{Der}(J), \mathrm{Der}(C)] = 0; \\ &[B + D, a \otimes c] = B(a) \otimes c + a \otimes D(c); \\ &[a \otimes c, a' \otimes c'] = (c, c')\langle a, a'\rangle + (a \circ a') \otimes (c \circ c') + (a, a')\langle c, c'\rangle \end{aligned}$$

for all $B \in \mathrm{Der}(J)$, $D \in \mathrm{Der}(C)$, $a, a' \in J_0$, and $c, c' \in C_0$.

For any pair of composition algebras C_1, C_2, and any $\gamma \in (k^\times)^3$, there is an isomorphism

$$K(C_1 \otimes C_2, -, \gamma) \simeq T(C_1, \mathcal{H}_3(C_2, \gamma)). \tag{26.4.1}$$

This fact is stated in [8, Remark 1.9 (c)] and a detailed proof is given in [51, Theorem 6.4]. If one of the C_i's is an octonion algebra and the other is a composition algebra of dimension 1, 2, 4, or 8 then $T(C_1, \mathcal{H}_3(C_2, \gamma))$ is of type F_4, E_6, E_7, or E_8 accordingly.

Let us define functors

$$\mathsf{S}(*) \subset \mathsf{T}(*) \subset H^1(k, E_8)$$

such that for all fields F/k,

– $\mathsf{S}(F)$ is the set of isomorphism classes of Lie algebras of type E_8 that are isomorphic to $T(C, J)$ for some reduced Albert algebra J and some octonion algebra C over F.

– $\mathsf{T}(F)$ is the set of isomorphism classes of Lie algebras of type E_8 that are isomorphic to $T(C, J)$ for some Albert algebra J (which may be a division algebra) and some octonion algebra C over F. That is, $\mathsf{T}(F)$ is the image of the Tits construction:

$$H^1(F, G_2 \times F_4) \to H^1(F, E_8).$$

It follows from (26.4.1) that

$$\mathsf{S}(*) \subset \mathsf{Q}(*).$$

Moreover, since every Albert algebra becomes reduced over a degree field extension dividing 3 [164, Proposition 6.1.1], we also have

$$\mathsf{T}(*) \subset \mathsf{R}(*).$$

We have shown in [133, Corollary 3.6] that if $L \in \mathsf{T}(k)$ and $h_8(L) \neq 0$ then L is anisotropic.

Incidentally, another overlap between the AF and the Tits constructions occurs for algebras of the form $(A, -) = M(J)$ where J is a cubic Jordan algebra and $\mathbb{O}$ is the split octonion algebra. In this case, [51, Theorem 4.4] shows that

$$K(A, -, (1, 1, 1)) \simeq T(\mathbb{O}, J).$$

26.5. *Comparison with invariants of* $G_2 \times F_4$. Since

$$\mathsf{R}(*) \subset \mathsf{T}(*) = \mathrm{im}\left(H^1(*, G_2 \times F_4) \to H^1(*, E_8)\right),$$

there is a restriction map $\mathrm{Inv}(\mathsf{R}, 2) \to \mathrm{Inv}(G_2 \times F_4, 2)$. The images of h_6 and h_8 are the unique cohomological invariants

$$h_6^* \in \mathrm{Inv}^6(G_2 \times F_4, 2), \qquad h_8^* \in \mathrm{Inv}^8(G_2 \times F_4, 2),$$

such that $h_d^*(C, J) = h_d(T(C, J))$ for all octonion algebras C and Albert algebras J.

Comparing (26.4.1), (17.2.1), and Corollary 26.3 yields that, for $d = 6$ or 8,

$$h_d^*(C_1, \mathcal{H}_3(C_2, \gamma)) = h_d(K(C_1 \otimes C_2, -, \gamma)) = e(C_1) \cdot f_{d-3}(\mathcal{H}_3(C_2, \gamma))$$

for all pairs of octonion algebras C_1, C_2 and scalars $\gamma_1, \gamma_2, \gamma_3$. If two mod 2 cohomological invariants agree up to odd-degree extensions, then they are equal by Lemma 15.8 (ii), so it follows that

$$h_6^*(C, J) = e(C) \cdot f_3(J), \qquad h_8^*(C, J) = e(C) \cdot f_5(J)$$

for all octonion algebras C and Albert algebras J.

27. The value of the Rost invariant

We calculate the value of the Rost invariant on the Lie algebra $K(A, -, \gamma)$ where $(A, -)$ is a bioctonion algebra.

The Rost invariant of E_8 has order 60 [116, Theorem 16.8], so it takes values in

$$H^3(*, \mathbb{Z}/4\mathbb{Z}) \times H^3(*, \mathbb{Z}/3\mathbb{Z}) \times H^3(*, \boldsymbol{\mu}_5 \otimes \boldsymbol{\mu}_5) \subset H^3(*, \mathbb{Q}/\mathbb{Z}(2)),$$

after identifying $\boldsymbol{\mu}_4 \otimes \boldsymbol{\mu}_4 \simeq \mathbb{Z}/4\mathbb{Z}$ and $\boldsymbol{\mu}_3 \otimes \boldsymbol{\mu}_3 \simeq \mathbb{Z}/3\mathbb{Z}$ canonically. See [101, VIII. Exercise 11] and [154, Example 8.4].

If L is a central simple Lie algebra of type E_8, there is a unique class $[z] \in H^1(k, E_8)$ such that L is isomorphic to the split Lie algebra $\mathfrak{e}_8$ twisted by z. We define the value of the Rost invariant on L to be $r_{E_8}(L) = r_{E_8}([z])$.

If L splits over a field extension whose degree is prime to 15 then $4r_{E_8}(L) = 0$ [58, §5.4] and so $r_{E_8}(L)$ lies in $H^3(k, \mathbb{Z}/4\mathbb{Z})$, the 4-torsion part of $H^3(k, \mathbb{Q}/\mathbb{Z}(2))$.

27.1. Lemma. *If $L = K(A, -, \gamma)$ for a bioctonion algebra $(A, -)$, then $2r_{E_8}(L) = 0$.*

Proof. We have $4r_{E_8}(L) = 0$ because there is a field extension of some degree 2^i that splits L. Since $\kappa' + 8 \in I^5(k)$ by Lemma 25.8, it follows from [57, Lemma 13.7] that $30r_{E_8}(L) = 0$, hence $2r_{E_8}(L) = 0$. □

27.2. Theorem. *Assume* $\operatorname{char}(k) = 0$. *Let $(A, -)$ be a bioctonion algebra and let $L = K(A, -, \gamma)$. Then*

$$r_{E_8}(L) = b_3(A, -) + b_1(A, -)\cdot(-\gamma_1\gamma_2^{-1})\cdot(-\gamma_2\gamma_3^{-1}) \in H^3(k, \mathbb{Z}/2\mathbb{Z}).$$

Proof. Proposition 23.8 implies the theorem if γ is isotropic, because then $L \simeq K(A, -)$. For Tits constructions we know $r_{E_8}(T(C, J)) = e(C) + f_3(J)$ [68, p. 3750]. If $(A, -) = C_1 \otimes C_2$ is decomposable then L is a Tits construction and

$$r_{E_8}(L) = e(C_1) + f_3(\mathcal{H}_3(C_2, \gamma)) = e(C_1) + e(C_2) = b_3(A, -),$$

so the formula is correct in this case because $b_1(A, -) = 0$.

Now fix a quadratic field extension E/k. There is a map

$$\begin{aligned} H^1(k, R_{E/k}(G_2 \times_k E)) &\longrightarrow \operatorname{Inv}^3_{\mathrm{norm}}(\mathbf{O}_3, 2) \\ \nu &\longmapsto b_\nu \end{aligned}$$

defined as follows. For $\nu \in H^1(k, R_{E/k}(G_2 \times_k E))$ corresponding to an octonion algebra C over E, let $(A, -) = \operatorname{cor}_{E/k}(C)$ and define

$$b_\nu(\langle\gamma_1, \gamma_2, \gamma_3\rangle) = r_{E_8}(K(A, -, (\gamma_1, \gamma_2, \gamma_3))) - r_{E_8}(K(A, -)).$$

Since b_ν is normalised and constant on similitude classes, Example 16.5 implies it must be of the form

$$b_\nu(\langle\gamma_1, \gamma_2, \gamma_3\rangle) = p(\nu)\cdot(-\gamma_1\gamma_2^{-1})\cdot(-\gamma_2\gamma_3^{-1})$$

for some unique element $p(\nu) \in H^1(k, \mathbb{Z}/2\mathbb{Z})$. Upon reflection, p is a cohomological invariant of $R_{E/k}(G_2 \times_k E)$. But $R_{E/k}(G_2 \times_k E)$ is connected, so p is a constant invariant. Also, p must be a multiple of $b_1(A, -) = [E] \in H^1(k, \mathbb{Z}/2\mathbb{Z})$ because it vanishes when scalars extend from k to E. All this implies that

$$r_{E_8}(K(A, -, \gamma)) = r_{E_8}(K(A, -)) + x\cdot b_1(A, -)\cdot(-\gamma_1\gamma_2^{-1})\cdot(-\gamma_2\gamma_3^{-1})$$

where x is either 0 or 1. (Until now, all of this is valid in any characteristic $\neq 2, 3$.) The solution for x when $k = \mathbb{Q}$ is the same as when k is any other field of characteristic 0, because the invariants r_{E_8}, b_1, b_3 are all defined over $\mathbb{Q}$. We learn from Example 25.6 that $x = 1$, because if $(A, -) = \mathrm{cor}_{\mathbb{C}/\mathbb{R}}(\mathbb{O}_{\mathbb{C}})$ and $\gamma = (1, 1, 1)$, then

$$r_{E_8}(K(A, -, \gamma)) = r_{E_8}(K(A, -)) + (-1)\cdot(-1)\cdot(-1) \in H^3(\mathbb{R}, \mathbb{Z}/2\mathbb{Z}). \qquad \square$$

If t_1, t_2 are indeterminates and $(A, -)$ is an indecomposable bioctonion division algebra such that $b_3(A, -)$ has symbol length 3, then $K(A, -, (t_1, 1, t_2))$ has a Rost invariant of symbol length 4 in $H^3(k(t_1, t_2), \mathbb{Z}/2\mathbb{Z})$. There are other known examples of Lie algebras of type E_8 with such a long Rost invariant [57, Example 2.4].

28. The AF construction of E_7

The following is a version of Proposition 25.3 and Theorem 25.5 for $(8, 4)$-product algebras, and the proof goes along the same lines.

28.1. Theorem. *Suppose $(A, -) = C \otimes Q$ is an $(8, 4)$-product algebra, where C is an octonion algebra with norm n_C and Q is a quaternion algebra with norm n_Q.*

(i) *All Lie related triples are inner, and there is an isomorphism*

$$\mathcal{T}_I \simeq \mathfrak{so}(n_C) \times Q_0^- \times Q_0^- \times Q_0^-.$$

(ii) *For $\gamma \in (k^\times)^3$, let κ be the quadratic Killing form on $K(A, -, \gamma)$. Then*

$$\kappa \simeq \langle -1 \rangle \big(4n'_C \perp 3\langle 2 \rangle n'_Q \perp \langle \gamma_1\gamma_2^{-1}, \gamma_2\gamma_3^{-1}, \gamma_3\gamma_1^{-1} \rangle n_C n_Q \big).$$

Proof. (i) Let $\mathcal{T}_{I,C} \subset \mathcal{T}_I$ be the subalgebra generated by the inner Lie related triples $\{T^i_{a,b} \colon a, b \in C, i = 1, 2, 3\}$. Similarly, let $\mathcal{T}_{I,Q} \subset \mathcal{T}_I$ be the subalgebra generated by $\{T^i_{a,b} \colon a, b, \in Q, i = 1, 2, 3\}$. Clearly $[T_{I,C}, T_{I,Q}] = \{0\}$. It is also easy to show that $\mathcal{T}_{I,C} \cap \mathcal{T}_{I,Q} = \{0\}$. (Hint: show that $\mathcal{T}_{I,C} \cap \mathcal{T}_{I,Q} \subset (k\,\mathrm{id}_A)^3$ and that $(c_1\,\mathrm{id}_A, c_2\,\mathrm{id}_A, c_3\,\mathrm{id}_A)$ is a Lie related triple if and only if $c_1 = c_2 = c_3 = 0$.) Hence

$$\mathcal{T}_{I,C} \times \mathcal{T}_{I,Q} \subset \mathcal{T}_I.$$

The structure of $\mathcal{T}_{I,C}$ is well-known: it is isomorphic to $\mathfrak{so}(n_C)$ [164, Theorem 3.5.5]. By Lemma 6.3, $\mathcal{T}_{I,Q} \simeq Q_0^- \times Q_0^- \times Q_0^-$. To conclude, compare dimensions with the help of Lemma 6.2:

$$37 = \dim(\mathcal{T}_{I,C} \oplus \mathcal{T}_{I,Q}) \leq \dim \mathcal{T}_I \leq \dim \mathcal{T} = \dim \mathrm{Der}(A, -) + 2 \dim \mathrm{Skew}(A, -) = 37.$$

(ii) Note that $n_C \otimes n_Q$ is an nondegenerate invariant form on $(A, -)$. By Theorem 6.13,

$$\kappa \simeq \kappa_0 \perp \langle d \rangle \langle \gamma_1\gamma_2^{-1}, \gamma_2\gamma_3^{-1}, \gamma_3\gamma_1^{-1} \rangle n_C n_Q. \tag{28.1.1}$$

where κ_0 is the restriction of κ to $\mathcal{T}_I$, and $d = -2 \dim A - 8 \dim \mathrm{Skew}(A, -) = -144$. Also, $\mathcal{T}_{I,C}$ and $\mathcal{T}_{I,Q}$ are orthogonal with respect to κ, so $\kappa_0 = \kappa_{0,C} \perp \kappa_{0,Q}$ where $\kappa_{0,C}$ and $\kappa_{0,Q}$ are the restrictions of κ to $\mathcal{T}_{I,C}$ and $\mathcal{T}_{I,Q}$ respectively.

We proceed to calculate $\kappa_{0,C}$. Applying Lemma 25.2, the Killing form of $\mathfrak{so}(n_C)$ is $\langle -12 \rangle \lambda^2(n_C)$. The representation $\rho_k : \mathcal{T}_{I,C} \simeq \mathfrak{so}(n_C) \to \mathrm{End}\, A[ij]$, i.e. $\rho_k : P \mapsto P_k$, is 4 times the vector representation. The trace form of ρ_k is therefore 4 times the trace of the vector representation, which in turn is 1/6 times the Killing form of $\mathfrak{so}(n_C)$. Hence

$$\kappa_{0,C} \simeq \langle 1 + 2/3 + 2/3 + 2/3\rangle\langle -12\rangle\lambda^2(n_C) = \langle -36\rangle\lambda^2(n_C).$$

Example 16.8 (iii) yields the simplification

$$\kappa_{0,C} = 4\langle -36\rangle n'_C. \tag{28.1.2}$$

Next, we calculate $\kappa_{0,Q}$. Since $\mathfrak{sl}(Q) \simeq \mathfrak{so}(n'_Q)$, Lemma 25.2 and Example 16.8 (i) combined imply that the Killing form of $\mathfrak{sl}(Q)$ is

$$\langle -2\rangle\lambda^2(n'_Q) = \langle -2\rangle n'_Q.$$

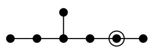
For each of the three copies of $\mathfrak{sl}(Q)$ in $\mathcal{T}_{I,Q}$, the representation in $A[ij]$ is 8 times the adjoint representation, so its trace form 8 times the Killing form. It follows that

$$\kappa_{0,Q} \simeq 3\langle 1 + 8 + 8 + 8\rangle\langle -2\rangle n'_Q = 3\langle -50\rangle n'_Q. \tag{28.1.3}$$

Combining (28.1.1)–(28.1.3) and simplifying yields the formula in (ii) above. ☐

We can use this to write the Killing form of some isotropic E_7's in terms of invariants. This is a continuation of the line of work done in [158, Example 27.20] and [111] to find formulas for the Killing forms of exceptional Lie algebras in terms of invariants.

28.2. Corollary. *Suppose $L = \mathrm{Lie}(G)$ for an adjoint absolutely simple algebraic group G whose Tits index is*



or has more circles. Then the quadratic Killing form of L is isometric to

$$\langle -1\rangle(4q'_3 \perp 3\langle 2\rangle q'_2 \perp q_2q_3) \perp 32\mathbb{H}$$

where q_2 is a 2-Pfister form such that $e_2(q_2)$ is the Tits class of L, and q_3 is a 3-Pfister form such that $d_5(L) = e_5(\langle\langle -1, -1\rangle\rangle q_3 + q_2q_3)$.

Proof. A Lie algebra with such a Tits index is isomorphic to $K(A, -)$ for some $(8, 4)$-product algebra $(A, -)$. So it follows from 28.1 with $\gamma = (1, -1, 1)$. ☐

The Lie algebras $K(A, -)$ for $(8, 2)$-product algebras $(A, -)$ are of type ${}^1E_6$ or ${}^2E_6$. Their Tits indices are displayed in [111, (6), Table 2], and formulas for the Killing forms in terms of cohomological invariants are available there. Very strong results are also available in [64] concerning the Tits construction of ${}^2E_6$.

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Chapter VII

Algebras of skew-dimension one

We look at various constructions of the exceptional skew-dimension one structurable algebras, which I have also been calling the "colourful series" (Definition 10.4). We calculate the trace forms and Rost invariants of Brown algebras whenever this is possible (i.e., on any of the known rational constructions). Finally, we give some cohomological criteria for properties of Brown algebras based mainly on what is known about the Rost invariant for quasi-split E_6 and E_7. At the end of the chapter we give some examples, and prove a theorem about the intersection of the two 5-graded constructions of E_8 – building a bridge between bioctonion algebras and Brown algebras.

As a reminder, we are now interested in twisted forms of the algebra

$$(B,-) = M(J) = \left\{ \begin{pmatrix} \alpha & j \\ j' & \beta \end{pmatrix} : \alpha, \beta \in k^\times, j, j' \in J \right\}$$

where J is a separable Jordan algebra of generic degree 3. The multiplication and involution on $M(J)$ are as in 2.6 (when you substitute $\eta = 1$):

$$\begin{pmatrix} \alpha_1 & j_1 \\ j_1' & \beta_1 \end{pmatrix} \begin{pmatrix} \alpha_2 & j_2 \\ j_2' & \beta_2 \end{pmatrix} = \begin{pmatrix} \alpha_1\alpha_2 + T(j_1, j_2') & \alpha_1 j_2 + \beta_2 j_1 + j_1' \times j_2' \\ \alpha_2 j_1' + \beta_1 j_2' + j_1 \times j_2 & \beta_1\beta_2 + T(j_2, j_1') \end{pmatrix}$$

$$\overline{\begin{pmatrix} \alpha & j \\ j' & \beta \end{pmatrix}} = \begin{pmatrix} \beta & j \\ j' & \alpha \end{pmatrix}.$$

29. Types of constructions

This section takes the form of a survey (mainly for specialists). It outlines various approaches to understanding the colourful series of algebras and their related algebraic groups (visualised in Figure 1).

29.1. *Constructions from subgroups.* Recall from Corollary 10.8 that

$$\mathbf{Aut}(B,-) \simeq \mathbf{Iso}(N_J) \rtimes \mathbb{Z}/2\mathbb{Z}$$

where $\mathbb{Z}/2\mathbb{Z}$ acts on norm-isometries $\varphi \in \mathbf{Iso}(N_J)(R)$ by $\varphi \mapsto \hat{\varphi} = \mathrm{ad}_T(\varphi)^{-1}$.

Most of the known constructions of the colourful algebras can be viewed as maps in Galois cohomology $H^1(k, G) \to H^1(k, \mathbf{Aut}(B,-))$ induced by subgroups $G \subset$

$\mathbf{Aut}(B,-)$. The objective is that G should be large enough to produce interesting twists but still small enough, or classical enough, to understand its G-torsors.

Constructing inner forms, i.e., the algebras twisted by cocycles from $\mathbf{Iso}(N_J)$, is reasonably straightforward: they are all matrix structurable algebras. The main difficulty is the outer forms.

In the next section we will discuss various constructions of forms of $M(J)$ that come from interesting subgroups $G \subset \mathbf{Aut}(B,-)$. This is the only type of construction we will explore any further, so 29.2–29.3 can safely be skipped if one so wishes.

29.2. *Hermitian cubic norm structures.* A totally different approach to parameterising the twisted forms of $(B,-)$, or simply connected groups of type ${}^2E_6$ with trivial Tits algebras, can be found in various places in the literature [4, 45, 64, 163]. The idea is usually to involve a Jordan cubic N over a quadratic étale extension E/k, enriched with some additional data such as a hermitian form T and an E/k-semilinear quadratic mapping $\sharp$ that collectively satisfy some identities.

Candidates for such a structure were introduced in the papers mentioned above, and called *hermitian cubic norm structures*, *cubic norm structures with a semilinear self-adjoint autotopy* [45], and *hermitian E_6-structures* [163]. A common feature is that not every Jordan cubic N over E/k can be enriched with the necessary extra hermitian structure, and if it can then there may be more than one way of doing it. Unfortunately, we do not have enough examples to really understand the situation.

These are not constructions involving any subgroup of $\mathbf{Aut}(B,-)$. Perhaps a way to understand where they come from, say, in the case of Brown algebras, is that there are two distinct maps

$$H^1(k, E_6^K) \to H^1(K, E_6)$$

that are worth caring about. Here we are referring to simply connected split and quasi-split groups of type E_6. One of these maps is the restriction $\mathrm{res}_{K/k}$. The other is a twisted version of the embedding of E_6 into $E_6 \times E_6$ by $\varphi \mapsto (\varphi, \hat{\varphi})$. This embedding is $\mathbb{Z}/2\mathbb{Z}$-equivariant with respect to an action on E_6 (by diagram automorphisms) and on $E_6 \times E_6$ (by swapping components). Twisting by a quadratic field extension K/k gives a map $E_6^K \to R_{K/k}(E_6)$ and also a map $H^1(k, E_6^K) \to H^1(k, R_{K/k}(E_6)) = H^1(K, E_6)$.

These maps are, in general, neither injective nor surjective. Any information on their images and fibres, even in some special cases, would be very revealing. Of course, this is just a paraphrasing in my own terms of what has already been said by others, and it remains a difficult problem.

29.3. *Classical objects.* In some cases, there is another possible approach that circumvents structurable algebras altogether. If $(B,-)$ is a blue or red algebra, the automorphism group $X = \mathbf{Aut}(B,-)$, full structure group $S = \mathbf{Str}(A,-)$, and semisimple structure group $M \subset S^\circ$ are of classical type. So they are related to classical objects, like hermitian forms and associative central simple algebras with involution.

For example, looking at the second last row of Table 5 together with Lemma 13.6, there are maps

$$\begin{tikzcd} H^1\big(k, \frac{\mathbf{SL_6}}{\boldsymbol{\mu}_2} \rtimes \mathbb{Z}/2\mathbb{Z}\big) \arrow[r, two heads] & H^1(k,S) \arrow[r, hook] & H^1(k, E_7^{\mathrm{ad}}) \\ & H^1(k, \mathbf{HSpin}_{12}) \arrow[u, two heads] & \end{tikzcd}$$

The image of $H^1\big(k, \frac{\mathbf{SL_6}}{\boldsymbol{\mu}_2} \rtimes \mathbb{Z}/2\mathbb{Z}\big) \to H^1(k, \mathbf{PGL_6} \rtimes \mathbb{Z}/2\mathbb{Z})$ classifies unitary involutions on central simple algebras of degree 6 with exponent ≤ 2 (equivalently, rank 3 hermitian forms over quaternion algebras with involution of the second kind). Unitary involutions of this kind have a nice descent theorem [139, Theorem 1.3].

The image of $H^1(k, \mathbf{HSpin}_{12}) \to H^1(k, \mathbf{PGO}_{12})$ classifies orthogonal involutions of degree 12 with trivial discriminant and one split Clifford algebra [57, Lemma 4.1]. (We are looking at the cohomology of the adjoint groups as a kind of proxy for $H^1(k,S)$.)

This suggests (but falls short of proving) that there ought to be a way of constructing *all* orthogonal involutions of degree 12 with trivial discriminant and Clifford invariant from unitary involutions of degree 6 and exponent ≤ 2. One also expects a relationship between the invariants of these involutions.

While the picture above is suggestive, it does nothing to explain what the mechanism is. Fortunately, the relationship between orthogonal involutions of degree 12 and unitary involutions of degree 6 has been investigated and explained beautifully in [67, Theorem 3.1]. This theory was further expanded and strengthened in [138,140].

Using quite different methods, the mod 2 cohomological invariants of $\mathbf{HSpin}_{12}$ have recently been classified in degree ≤ 3 by Ruether [146]. Higher-degree invariants of $\mathbf{HSpin}_{12}$ are elusive, but may well exist.

30. The two basic subgroup constructions

In this section, we describe the two most basic constructions of the colourful algebras, and the subgroups they come from. One of these, the matrix construction $M(J,\eta)$, was already introduced in 2.6 of the first chapter.

30.1. *A subgroup of order* 6. Let $(B,-) = M(J)$ for a separable cubic Jordan algebra J. There is a subgroup $\boldsymbol{\mu}_3 \rtimes \mathbb{Z}/2\mathbb{Z} \subset \mathbf{Aut}(B,-)$ whose group of R-points is generated by the automorphisms ω and $\{g_\zeta : \zeta \in \boldsymbol{\mu}_3(R)\}$ as defined below:

$$\omega \begin{pmatrix} \alpha & j \\ j' & \beta \end{pmatrix} = \begin{pmatrix} \beta & j' \\ j & \alpha \end{pmatrix}, \qquad g_\zeta \begin{pmatrix} \alpha & j \\ j' & \beta \end{pmatrix} = \begin{pmatrix} \alpha & \zeta^{-1} j \\ \zeta j' & \beta \end{pmatrix}.$$

A posteriori, based on results collected in §10, this is the subgroup generated by the centre $Z(\mathbf{Iso}(N_J)^\circ) \simeq \boldsymbol{\mu}_3$ and one of the diagram automorphisms of $\mathbf{Iso}(N_J)$. This finite subgroup centralises the image of $\mathbf{Aut}(J)$ in $\mathbf{Aut}(B,-)$, so we have a reasonably large disconnected subgroup

$$\mathbf{Aut}(J) \times (\boldsymbol{\mu}_3 \rtimes \mathbb{Z}/2\mathbb{Z}) \subset \mathbf{Aut}(B,-).$$

and there is a corresponding map in Galois cohomology

$$H^1(k, \mathbf{Aut}(J)) \times H^1(k, \boldsymbol{\mu}_3 \rtimes \mathbb{Z}/2\mathbb{Z}) \to H^1(k, \mathbf{Aut}(B,-)). \tag{30.1.1}$$

If k contains all the cube roots of unity, $\boldsymbol{\mu}_3 \rtimes \mathbb{Z}/2\mathbb{Z}$ is the constant group S_3, and $H^1(k, S_3)$ classifies cubic étale algebras. In that case, one can think of the construction (30.1.1) as twisting $M(J)$ by a cubic étale algebra. It is hard to write down the twisted algebras explicitly, and also not necessary. We do, however, take a closer look at two special cases in the next subsections.

Table 10 displays the groups $\mathbf{Aut}(J)$ and $\mathbf{Iso}(N_J)$, where J is a simple cubic Jordan algebra of dimension 6, 9, 15, or 27. A procedure is outlined in [56, §2.4–2.5] to obtain the root system of $\mathbf{Aut}(J)^\circ$ by folding the root system of $\mathbf{Iso}(N_J)^\circ$. The mod 2 cohomological invariants of $\mathbf{Aut}(J)$ have been classified by MacDonald [110].

J	$\dim J$	$\mathbf{Aut}(J)$	$\mathbf{Iso}(N_J)$
$\mathcal{H}(M_3(k), \mathrm{ad}_{\langle 1,-1,\rangle})$	6	$\mathbf{O}_3^+$	$\mathbf{SL}_3$
$M_3(k)^+$	9	$\mathbf{PGL}_3 \rtimes \mathbb{Z}/2\mathbb{Z}$	$\dfrac{\mathbf{SL}_3 \times \mathbf{SL}_3}{\boldsymbol{\mu}_3} \rtimes \mathbb{Z}/2\mathbb{Z}$
$\mathcal{H}_3(\mathbb{H})$	15	$\mathbf{PGSp}_6$	$\dfrac{\mathbf{SL}_6}{\boldsymbol{\mu}_2}$
$\mathcal{H}_3(\mathbb{O})$	27	F_4	E_6^{sc}

Table 10: Split forms of the subgroups involved in the basic constructions of exceptional skew-dimension one structurable algebras.

30.2. *Twisting by a cubic scalar.* If we identify $H^1(k, \boldsymbol{\mu}_3) = k^\times/k^{\times 3}$ via the Kummer exact sequence [25, Remark III.8.28], the restriction of (30.1.1) to $H^1(k, \boldsymbol{\mu}_3)$ is a familiar construction, namely:

$$\begin{aligned} k^\times/k^{\times 3} &\longrightarrow H^1(k, \mathbf{Aut}(B, -)) \\ \eta k^{\times 3} &\longmapsto [M(J, \eta)]. \end{aligned}$$

This can be verified using the recipe in 12.5: pick an element $x \in k^s$ with $x^3 = \eta \in k^\times$ and show that the twist of $M(J)$ by the cocycle $z \in Z^1(k, \boldsymbol{\mu}_3)$, $z_\sigma = \sigma(x)x^{-1}$, is isomorphic to $M(J, \eta)$. As such, the image of

$$H^1(\mathbf{Aut}(J)) \times H^1(k, \boldsymbol{\mu}_3) \to H^1(k, \mathbf{Aut}(B, -))$$

comprises all the twisted forms of $(B, -)$ that are matrix algebras in the sense of Definition 10.4.

30.3. *Twisting by a quadratic extension.* The restriction of (30.1.1) to $H^1(k, \mathbb{Z}/2\mathbb{Z})$ is also a rather straightforward construction, described in [55, Example 2.4] for the case where J is an Albert algebra. The twisting works as follows (for any separable cubic Jordan algebra J).

Let E/k be a quadratic étale extension with $\mathrm{Aut}_k(E) = \{1, \iota\}$, and consider the k-automorphism $\varpi = \omega \otimes \iota$ of $M(J) \otimes E$. It has the effect that

$$\varpi \begin{pmatrix} e_1 & e_2 j \\ e_3 j' & e_4 \end{pmatrix} = \begin{pmatrix} \iota(e_4) & \iota(e_3) j' \\ \iota(e_2) j & \iota(e_1) \end{pmatrix}$$

for all $e_i \in E$ and $j, j' \in J$. The fixed point set is a k-subalgebra of $M(J) \otimes E$ that can be written as

$$M(J,E) = (M(J) \otimes E)^{\varpi} = \left\{ \begin{pmatrix} c_1 + \delta c_2 & j_1 + \delta j_2 \\ j_1 - \delta j_2 & c_1 - \delta c_2 \end{pmatrix} : c_i \in k, j_i \in J \right\} \tag{30.3.1}$$

where $\delta \in E$ is a fixed nonzero element with $\mathrm{tr}_{E/k}(\delta) = 0$. This is a twisted form of $M(J)$ because $M(J,E)_E \simeq M(J_E)$. In particular, $M(J, k \times k) \simeq M(J) = (B, -)$. The map that one gets by restricting (30.1.1) is the construction

$$\begin{aligned} H^1(k, \mathbb{Z}/2\mathbb{Z}) &\to H^1(k, \mathbf{Aut}(B,-)) \\ [E] &\mapsto [M(J,E)]. \end{aligned}$$

30.4. *A $\mathbb{Z}/2\mathbb{Z}$-grading on $M(J,E)$.* Fix a nonzero $\delta \in E$ with $\mathrm{tr}_{E/k}(\delta) = 0$, and let $\mu = \delta^2 \in k^\times$. The one-dimensional space $\mathrm{Skew}(M(J,E))$ is spanned by the element

$$s_0 = \begin{pmatrix} \delta & 0 \\ 0 & -\delta \end{pmatrix}.$$

We have

$$s_0^2 = \mu 1, \qquad s_0 \begin{pmatrix} c_1 + \delta c_2 & j_1 + \delta j_2 \\ j_1 - \delta j_2 & c_1 - \delta c_2 \end{pmatrix} = \begin{pmatrix} \mu c_2 + \delta c_1 & \mu j_2 + \delta j_1 \\ -\mu j_2 - \delta j_1 & -\mu c_2 - \delta c_1 \end{pmatrix},$$

and can use (30.3.1) to make a $\mathbb{Z}/2\mathbb{Z}$-grading

$$M(J,E) = V \oplus s_0 V, \qquad \text{where } V = \left\{ \begin{pmatrix} \alpha & j \\ j & \alpha \end{pmatrix} : \alpha \in k, j \in J \right\}.$$

Let us denote by $\Omega \in \mathrm{Aut}(M(J,E))$ the grading automorphism of this $\mathbb{Z}/2\mathbb{Z}$-grading; it is the unique automorphism that fixes V pointwise and sends $s_0 \mapsto -s_0$. The automorphism Ω has order 2 and is not in the identity component of $\mathrm{Aut}(M(J,E))$.

To lighten the notation, we shall write elements of V as

$$\begin{pmatrix} \alpha & j \\ j & \alpha \end{pmatrix} = \begin{bmatrix} \alpha \\ j \end{bmatrix}.$$

In summary, V is a Jordan algebra with identity $1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and multiplication

$$\begin{bmatrix} \alpha \\ j \end{bmatrix} \begin{bmatrix} \beta \\ \ell \end{bmatrix} = \begin{bmatrix} \alpha\beta + T(j,\ell) \\ \alpha\ell + \beta j + j \times \ell \end{bmatrix} \qquad \text{for all } \alpha, \beta \in k \text{ and } j, \ell \in J. \tag{30.4.1}$$

By Proposition 10.2, there is an isomorphism

$$\begin{aligned} F \colon k \times J &\xrightarrow{\sim} V \\ (\alpha, j) &\longmapsto \begin{bmatrix} \frac{1}{4}\alpha + \frac{1}{4}t(j) \\ \frac{1}{2}j + \frac{1}{4}\alpha 1 - \frac{1}{4}t(j)1 \end{bmatrix}, \end{aligned}$$

where t is the generic trace of J.

Define a map $\theta : V \to V$ by

$$v \longmapsto v^\theta = -v + \tfrac{1}{2} t_V(v)1, \tag{30.4.2}$$

where t_V is the generic trace of V. In matrix notation, we have for all $\alpha \in k$ and $j \in J$,

$$t_V\Big(\begin{bmatrix} \alpha \\ j \end{bmatrix} \Big) = 4\alpha, \qquad \begin{bmatrix} \alpha \\ j \end{bmatrix}^\theta = \begin{bmatrix} \alpha \\ -j \end{bmatrix}$$

For all $v_1, v_2 \in V$, the following relations are easily checked (cf. [9, (6.3)]):

$$v_1(s_0 v_2) = s_0(v_1^\theta v_2) \tag{30.4.3}$$

$$(s_0 v_1)v_2 = s_0(v_1^\theta v_2^\theta)^\theta \qquad (s_0 v_1)(s_0 v_2) = \mu(v_1 v_2^\theta)^\theta. \tag{30.4.4}$$

Since $\overline{v} = v$ and $\overline{s_0} = -s_0$, relation (30.4.3) implies

$$\overline{s_0 v} = -v s_0 = -s_0 v^\theta. \tag{30.4.5}$$

The algebra $M(J, E)$ can be reconstructed from V and the above relations, which gives another way of defining $M(J, E)$.

31. Doubling constructions

From this point on, we shall assume $(B, -)$ is blue, red, or Brown. So we study $(B, -) = M(J, E)$ where J is a cubic Jordan algebra of dimension $d \in \{9, 15, 27\}$ and E is a quadratic étale algebra.

Green algebras, corresponding to 6-dimensional simple cubic Jordan algebras, are less interesting (see Theorem 13.9) and would needlessly complicate the exposition. Quartic Cayley algebras, in contrast, are highly interesting but they demand a different approach, like in [31, §6.4]. One needs to work with cubic and quartic étale algebras, whereas we will be working with cubic and quartic simple Jordan algebras. Even though there are some commonalities in the structure theory, it is hard to combine them into the same exposition.

We are going to find another large semisimple subgroup in $\mathbf{Aut}(B, -)$ that produces a whole different set of twisted forms than the subgroup $\mathbf{Aut}(J)$ does. Readers familiar with structurable algebras might recognise this as Allison and Faulkner's *Cayley–Dickson* construction [9]. However, we are going to do this construction internally and try to make it less opaque. This approach is therefore quite different from other sources.

31.1. *Another large Jordan subalgebra.* We continue to study $M(J, E)$ in a particular case where J is reduced and its coordinate algebra has a common slot with E. I learned about much of what follows from [17].

Begin by assuming $(C, -)$ is a composition algebra of dimension $m \in \{2, 4, 8\}$ with standard involution and norm n. Let $Q \subset C$ be a composition subalgebra of dimension $m/2$, and $u \in Q^\perp$ an element with $u^2 = -n(u) = \mu \neq 0$. Note that

$$C = Q \oplus uQ,$$

and there is a unique automorphism $\lambda_u \in \mathrm{Aut}(C)$ that fixes Q and sends $u \mapsto -u$. We have the following Cayley–Dickson type relations:

$$\begin{array}{lll} x(uy) = u(\bar{x}y) & (ux)y = u(yx) & (ux)(uy) = \mu y\bar{x} \\ x\bar{y} + y\bar{x} = n(x,y) = n(\bar{x},\bar{y}) & n(ux,uy) = -\mu n(x,y) & \overline{ux} = -ux = -\bar{x}u \end{array} \tag{31.1.1}$$

for all $x, y \in Q$ (see [164, Lemma 1.3.1, (1.26)–(1.27)] and [93, p. 441]).

Consider the reduced Jordan algebra

$$J = \mathcal{H}_3(C) = \{x \in M_3(C) \colon x_{ij} = \overline{x_{ji}}\}.$$

This is an Albert algebra or a special simple cubic Jordan algebra of dimension 9 or 15. Let

$$(B, -) = M(J, E) \qquad \text{where } E = k[t]/(t^2 - \mu).$$

Choose a $\delta \in E$ with $\delta^2 = \mu$, and for the generator of $\mathrm{Skew}(B, -)$ choose

$$s_0 = \begin{pmatrix} \delta & 0 \\ 0 & -\delta \end{pmatrix}.$$

The automorphism $\lambda_u \in \mathrm{Aut}(C)$ extends to J by applying it entrywise, and in turn it extends uniquely to an automorphism $\Lambda_u \in \mathrm{Aut}(B, -)^\circ$ that fixes s_0. This automorphism respects the $\mathbb{Z}/2\mathbb{Z}$-grading from 30.4, and it looks like:

$$\Lambda_u\left(\begin{bmatrix}\alpha \\ j\end{bmatrix} + s_0\begin{bmatrix}\beta \\ \ell\end{bmatrix}\right) = \left(\begin{bmatrix}\alpha \\ \lambda_u(j)\end{bmatrix} + s_0\begin{bmatrix}\beta \\ \lambda_u(\ell)\end{bmatrix}\right) \quad \text{for all } \alpha, \beta \in k \text{ and } j, \ell \in J.$$

Actually, we are more interested in the automorphism $\Lambda_u \circ \Omega \in \mathrm{Aut}(B, -)$ from the nonidentity component, which negates s_0:

$$\Lambda_u \circ \Omega\left(\begin{bmatrix}\alpha \\ j\end{bmatrix} + s_0\begin{bmatrix}\beta \\ \ell\end{bmatrix}\right) = \left(\begin{bmatrix}\alpha \\ \lambda_u(j)\end{bmatrix} - s_0\begin{bmatrix}\beta \\ \lambda_u(\ell)\end{bmatrix}\right).$$

Its fixed point subspace is

$$H = (B, -)^{\Lambda_u \circ \Omega} = \left\{ \begin{bmatrix}\alpha \\ j\end{bmatrix} + s_0\begin{bmatrix}0 \\ \ell\end{bmatrix} \;:\; \begin{array}{c} \alpha \in k, j \in \mathcal{H}_3(Q) \\ \ell \in \mathcal{H}_3(C), \ell = -\lambda_u(\ell) \end{array} \right\}.$$

By (30.4.5) we have $H \subset \mathrm{Herm}(B, -)$, so H is a Jordan algebra.

Since $L_{s_0}^2 = L_{s_0{}^2} = \mu\,\mathrm{id}$ (2.7), we have

$$s_0\left(\begin{bmatrix}\alpha \\ j\end{bmatrix} + s_0\begin{bmatrix}0 \\ \ell\end{bmatrix}\right) = \begin{bmatrix}0 \\ \mu\ell\end{bmatrix} + s_0\begin{bmatrix}\alpha \\ j\end{bmatrix}$$

and there is a $\mathbb{Z}/2\mathbb{Z}$-grading

$$B = H \oplus s_0 H$$

of which $\Lambda_u \circ \Omega$ is the grading automorphism.

Define a map $\theta : H \to H$, $h \mapsto h^\theta$ by

$$\left(\begin{bmatrix}\alpha \\ j\end{bmatrix} + s_0\begin{bmatrix}0 \\ \ell\end{bmatrix}\right)^\theta = \begin{bmatrix}\alpha \\ -j\end{bmatrix} - s_0\begin{bmatrix}0 \\ \ell\end{bmatrix}. \tag{31.1.2}$$

Notice that if $h = v + s_0 w \in H$ for some $v, w \in V$, then

$$h^\theta = v^\theta + s_0 w^\theta,$$

where $\theta : V \to V$ is the map defined in (30.4.2). (This is why it does not cause confusion to use the same notation for these two maps: they agree on $H \cap V$.)

31.2. Lemma. *For all $h_1, h_2 \in H$,*

$$\begin{aligned} h_1(s_0h_2) &= s_0(h_1^\theta h_2) \\ (s_0h_1)h_2 &= s_0(h_1^\theta h_2^\theta)^\theta \qquad\qquad (s_0h_1)(s_0h_2) = \mu(h_1h_2^\theta)^\theta. \end{aligned}$$

Proof. Let $h_i = v_i + s_0 w_i$. By the relations (30.4.3)–(30.4.4), we have

$$h_1(s_0h_2) = \mu(w_1v_2^\theta)^\theta + \mu v_1 w_2 + s_0(v_1^\theta v_2 + \mu(w_1w_2^\theta)^\theta)$$

On the other hand, $h_1^\theta = v_1^\theta + s_0 w_1^\theta$. Using the same relations again, we obtain

$$s_0(h_1^\theta h_2) = \mu(w_1^{\theta\theta}v_2^\theta)^\theta + \mu v_1^{\theta\theta} w_2 + s_0(v_1^\theta v_2 + \mu(w_1^{\theta\theta}v_2^\theta)^\theta).$$

Since $\theta^2 = \mathrm{id}$, this yields $h_1(s_0h_2) = s_0(h_1^\theta h_2)$.

The other relations in the lemma are proved similarly. $\square$

31.3. *Identifying the Jordan algebra H.* We shall determine that H is isomorphic to the simple Jordan algebra

$$\mathcal{H}_4(Q) = \{a \in M_4(Q) \colon a_{ij} = \overline{a_{ji}}\}.$$

There is a $\mathbb{Z}/2\mathbb{Z}$-grading $\mathcal{H}_4(Q) = \mathcal{H}_4(Q)_0 \oplus \mathcal{H}_4(Q)_1$, where

$$\mathcal{H}_4(Q)_0 = \left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} & 0 \\ \overline{a_{12}} & a_{22} & a_{23} & 0 \\ \overline{a_{13}} & \overline{a_{23}} & a_{33} & 0 \\ 0 & 0 & 0 & a_{44} \end{pmatrix} \middle| \begin{matrix} a_{ij} \in Q \\ a_{ii} = \overline{a_{ii}} \in k1 \end{matrix} \right\}$$

$$\mathcal{H}_4(Q)_1 = \left\{ \begin{pmatrix} 0 & 0 & 0 & b_1 \\ 0 & 0 & 0 & b_2 \\ 0 & 0 & 0 & b_3 \\ \overline{b_1} & \overline{b_2} & \overline{b_3} & 0 \end{pmatrix} \middle| b_{ij} \in Q \right\}. \tag{31.3.1}$$

Clearly $\mathcal{H}_4(Q)_0 \simeq k \times \mathcal{H}_3(Q)$, so it embeds canonically as a unital subalgebra of $k \times J$.

The composition $\Phi : \mathcal{H}_4(Q)_0 \longrightarrow k \times J \xrightarrow{F} V$ is an injective homomorphism whose image is $H \cap V$. For $a \in \mathcal{H}_4(Q)_0$, we have

$$\Phi(a) = \Phi \begin{pmatrix} a_{11} & a_{12} & a_{13} & 0 \\ \overline{a_{12}} & a_{22} & a_{23} & 0 \\ \overline{a_{13}} & \overline{a_{23}} & a_{33} & 0 \\ 0 & 0 & 0 & a_{44} \end{pmatrix} = \begin{bmatrix} \frac{1}{4}\sum a_{ii} \\ \frac{1}{2}\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ \overline{a_{12}} & a_{22} & a_{23} \\ \overline{a_{13}} & \overline{a_{23}} & a_{33} \end{pmatrix} + \frac{1}{4}(a_{44} - \sum_{i=1}^3 a_{ii}) \end{bmatrix}.$$

Extend this to a map $\Phi : \mathcal{H}_4(Q) \longrightarrow H \subset M(J,E)$ by setting

$$\Phi(b) = \Phi \begin{pmatrix} 0 & 0 & 0 & b_1 \\ 0 & 0 & 0 & b_2 \\ 0 & 0 & 0 & b_3 \\ \overline{b_1} & \overline{b_2} & \overline{b_3} & 0 \end{pmatrix} = s_0 \begin{bmatrix} & 0 \\ \frac{1}{2\mu}\begin{pmatrix} 0 & -ub_3 & ub_2 \\ ub_3 & 0 & -ub_1 \\ -ub_2 & ub_1 & 0 \end{pmatrix} & \end{bmatrix}$$

for all $b \in \mathcal{H}_4(Q)_1$.

31.4. Lemma. *The map $\Phi\colon \mathcal{H}_4(Q) \to H$ is an isomorphism.*

This isomorphism is the restriction to H of the map $\varphi_{\mathfrak{C}\mathfrak{D}}$ mentioned on [17, p. 292], where it says that "it is straightforward to verify that the mapping... is an isomorphism of algebras". I have indeed verified this using just (30.4.1), (30.4.4), and (31.1.1).

A direct, matrix-based proof is large and difficult to typeset. Probably it would be neater to use Jacobson-style generators and relations for $\mathcal{H}_4(Q)$ and $\mathcal{H}_3(C)$, like in [115, p. 193–194]. Either way, the set-up and the proof would run to several pages and cause a long diversion from the task at hand, so it is omitted.

31.5. Lemma. *The map $\theta : H \to H$ defined in* (31.1.2) *is characterised by*

$$h^\theta = -h + \tfrac{1}{2}t_H(h)1$$

where t_H is the generic trace of H.

Proof. If $a \in \mathcal{H}_4(Q)_0$, then $t_H(\Phi(a)) = \sum a_{ii}$. It is clear from the definition of θ that $\Phi(a)^\theta = -\Phi(a) + \frac{1}{2}(\sum a_{ii})1$. If $b \in \mathcal{H}_4(Q)_1$ then $t_H(\Phi(b)) = 0$ and by definition $\Phi(b)^\theta = -\Phi(b)$. By linearity, $h^\theta = -h + \frac{1}{2}t_H(h)1$ for all $h \in H$. □

31.6. Lemma. *An automorphism of H extends in exactly two ways to an involution-preserving automorphism of $(B,-)$, one fixing s_0 and the other negating s_0.*

Proof. Let $f \in \mathrm{Aut}(H)$. An automorphism of $(B,-)$ restricts to an automorphism of $k[s_0] \simeq E$, which means it sends $s_0 \mapsto \pm s_0$. Since $B = H \oplus s_0H$, there are at most two extensions of f.

So extend f to B by declaring $f(s_0) = s_0$ or $-s_0$ and requiring f to be linear. We have $t_H(f(h)) = t_H(h)$ for all $h \in H$. Lemma 31.5 implies $f(h^\theta) = f(h)^\theta$. Lemma 31.2 now implies

$$f(h_1(s_0h_2)) = f(h_1)f(s_0h_2) \qquad f((s_0h_1)(s_0h_2)) = f(s_0h_1)f(s_0h_2)$$
$$f((s_0h_1)h_2) = f(s_0h_1)f(h_2)$$

for all $h_1, h_2 \in H$, so the extension of f is an automorphism of $(B,-)$. □

31.7. Corollary. *The group $\mathbf{Aut}(B,-)$ contains a subgroup isomorphic to*

$$\mathbf{Aut}(\mathcal{H}_4(Q)) \times \mathbb{Z}/2\mathbb{Z}.$$

A canonical generator of the factor $\mathbb{Z}/2\mathbb{Z}$ is the automorphism $\Lambda_u \circ \Omega \in \mathbf{Aut}(B,-)$ that determines the $\mathbb{Z}/2\mathbb{Z}$-grading $B = H \oplus s_0H$. Also, $\mathcal{H}_4(Q)$ generates $M_4(Q)$ as an associative algebra, so $\mathbf{Aut}(\mathcal{H}_4(Q)) \simeq \mathbf{Aut}(M_4(Q),-)$ where "$-$" is the conjugate-transpose involution.

Corollary 31.7 gives a construction

$$H^1(k, \mathbf{Aut}(\mathcal{H}_4(Q)) \times k^\times/k^{\times 2} \to H^1(k, \mathbf{Aut}(B, -)). \tag{31.7.1}$$

One can describe this construction by revisiting 31.1. We now state the "external" version of the doubling construction.

31.8. *Definition of* $\mathrm{CD}(H, \mu)$. Let H be any simple Jordan algebra of (generic) degree 4, and let $\mu \in k^\times$. Define $h^\theta = -h + \frac{1}{2}t_H(h)$ for all $h \in H$. Define a set of formal symbols $s_0H = \{s_0h \colon h \in H\}$ with the vector space structure: $s_0h_1 + s_0h_2 = s_0(h_1 + h_2)$ and $\alpha(s_0h) = s_0(\alpha h)$ for all $h, h_1, h_2 \in H$ and all $\alpha \in k$. Define on

$$B = H \oplus s_0H$$

the structure of an algebra by setting

$$(h_1 + s_0h_2)(h_3 + s_0h_4) = h_1h_3 + \mu(h_2h_4^\theta)^\theta + s_0(h_1^\theta h_4 + (h_2^\theta h_3^\theta)^\theta)$$

for all $h_1, \dots, h_4 \in H$. Further define the involution

$$\overline{h_1 + s_0h_2} = h_1 - s_0h_2^\theta.$$

Denote this algebra with involution by $\mathrm{CD}(H, \mu)$. If $H = \mathcal{H}(C, \sigma)$ for a central simple algebra with involution (C, σ), we also write $\mathrm{CD}((C, \sigma), \mu) = \mathrm{CD}(\mathcal{H}(C, \sigma), \mu)$.

31.9. Proposition (Allison–Faulkner)**.** *If H is a central simple Jordan algebra of degree* 4*, then* $\mathrm{CD}(H, \mu)$ *is a central simple structurable algebra of degree* 4 *and skew-dimension* 1*.*

Proof. Over some field extension L/k, there is a composition algebra Q such that $H_L \simeq \mathcal{H}_4(Q)$. By Lemma 31.4, $\mathrm{CD}(H, \mu)_L \simeq M(\mathcal{H}_3(C), k(\sqrt{\mu}))$ where $C = Q \oplus uQ$ is the Cayley–Dickson double of Q with parameter μ. □

This was originally proved in a different way in [9, Proposition 6.5]. In fact, the original definition of the construction attempts to be much more general: possible inputs are not only simple Jordan algebras but any separable Jordan algebras of degree ≤ 4. If one takes $H = k \times J$, it is easy to see from 30.4 that $\mathrm{CD}(k \times J, \mu) \simeq M(J, k(\sqrt{\mu}))$. If one takes other nonsimple separable Jordan algebras of degree 4, for example $H = \mathcal{J}Spin_n(k) \times \mathcal{J}Spin_m(k)$, then $\mathrm{CD}(H, \mu)$ is also a central simple structurable algebra of skew-dimension 1, but it is not one of the exceptional types. So for our purposes, there is nothing to lose by only putting simple Jordan algebras into the doubling construction.

31.10. *The subgroups.* In case Q is a split composition algebra, the group $\mathbf{Aut}(\mathcal{H}_4(Q))$ is displayed in Table 11, alongside (the main index 2 subgroup of) the automorphism group $\mathbf{Aut}(\mathrm{CD}(\mathcal{H}_4(Q), 1))$ of the corresponding split structurable algebra.

The subgroups $\mathbf{PGSp}_8, F_4 \subset E_6^{\mathrm{sc}}$ are each centralised by a different outer automorphism of E_6^{sc}. These are the only types of subgroups that are fixed by order 2 outer automorphisms of E_6^{sc} [152, Theorem 3.4]. An explicit outer automorphism of E_6^{sc} fixing $\mathbf{PGSp}_8$ is described in [63, §5.1] in terms of Chevalley generators. (We have given a complementary, algebraic description of this automorphism $\Lambda_u \circ \Omega$ in 31.1.)

$(B,-)$	$\dim B$	type of $(M_4(Q),-)$	$\mathbf{Aut}(\mathcal{H}_4(Q))$	$\mathbf{Aut}_S(B,-)$
Blue	20	orthogonal	$\mathbf{PGO}_4$	$\dfrac{\mathbf{SL}_3\times\mathbf{SL}_3}{\boldsymbol{\mu}_3}\rtimes\mathbb{Z}/2\mathbb{Z}$
Red	32	unitary	$\mathbf{PGL}_4\rtimes\mathbb{Z}/2\mathbb{Z}$	$\dfrac{\mathbf{SL}_6}{\boldsymbol{\mu}_2}$
Brown	56	symplectic	$\mathbf{PGSp}_8$	E_6^{sc}

Table 11: Split forms of the subgroups involved in the doubling construction of exceptional skew-dimension one structurable algebras. Q is a split composition algebra of dimension ≤ 4, and $(B,-)=\mathrm{CD}(\mathcal{H}_4(Q),1)$. $\mathbf{Aut}_S(B,-)$ is the subgroup of $\mathbf{Aut}(B,-)$ fixing $\mathrm{Skew}(B,-)$.

32. Trace forms of Brown algebras

Trace forms are important invariants in virtually all kinds of separable algebras: associative, alternative, étale, Jordan, Lie, bicomposition, and so on. So it is no surprise that traces are interesting for the colourful algebras too. We limit the scope here to Brown algebras; the arguments are easily adaptable to the other colours, only the calculations are different.

The trace on a Brown algebra $(B,-)$ is nondegenerate if $\mathrm{char}(k)\neq 2,7$ because $(B,-)$ is simple and $\dim B = 56 = 2^3\cdot 7$.

If (C,σ) is a central simple algebra with symplectic involution, the quadratic Jordan trace of $\mathcal{H}(C,\sigma)$ is

$$T_\sigma^+(x)=\tfrac{1}{2}\,\mathrm{Trd}_C(x^2).$$

(The scalar $\frac{1}{2}$ is accounted for by $\deg\mathcal{H}(C,\sigma)=\frac{1}{2}\deg C$.)

32.1. Theorem. *Assume* $\mathrm{char}(k)\neq 2,3,7$ *and let E/k be a quadratic étale extension with norm $N_{E/k}=\langle\langle\mu\rangle\rangle$.*

(i) *If J is an Albert algebra with quadratic trace T_J and $(B,-)=M(J,E)$, then*

$$T_B=\langle 7\rangle\langle\langle\mu\rangle\rangle(\langle 1\rangle\perp T_J).$$

(ii) *If (C,σ) is a central simple algebra of degree 8 with symplectic involution and $(B,-)=\mathrm{CD}((C,\sigma),\mu)$, then*

$$T_B=\langle 7\rangle\langle\langle\mu\rangle\rangle T_\sigma^+.$$

(iii) *If $(B,-)=M(J,\eta)$ is a matrix Brown algebra then T_B is hyperbolic.*

Proof. We can treat cases (i) and (ii) simultaneously, to a point. Let $X=V\simeq k\times J$ or $X=\mathcal{H}(C,\sigma)$. There is a $\mathbb{Z}/2\mathbb{Z}$-grading $B=X\oplus s_0X$, where $s_0\in\mathrm{Skew}(B,-)$ is an element with $s_0^2=\mu 1$. If $x,y\in X$ then

$$T_B(x,s_0y)=\mathrm{tr}(L_{x(\overline{s_0y})+(s_0x)\overline{y}})=0$$

because $L_{x(\overline{s_0 y})+(s_0 y)\overline{x}}$ is homogeneous of degree 1 with respect to the grading. So X is orthogonal to s_0X, and T_B is nondegenerate on X and s_0X separately.

Since T_B is an invariant form (Definition 2.15) and $L_{s_0}^2 = L_{s_0^2} = \mu\,\mathrm{id}$, we have

$$T_B(s_0x, s_0y) = T_B(x, -s_0(s_0y)) = -\mu T_B(x, y).$$

So $T_B = \langle\!\langle \mu \rangle\!\rangle (T_B|_X)$. In case (ii), Lemma 2.16 (iii) implies $T_B|_X$ is a scalar multiple of T_C^+. We have $T_B(1) = 2\dim B = 112$, while $T_\sigma^+(1) = 4$. So $T_B|_X = \langle 28\rangle T_\sigma^+ \simeq \langle 7\rangle T_\sigma^+$.

In case (i), write $1 = e_1 + e_2$, where

$$e_1 = \begin{bmatrix} 1/4 \\ 1/4 \end{bmatrix} \qquad\qquad e_2 = \begin{bmatrix} 3/4 \\ -1/4 \end{bmatrix}.$$

These are the orthogonal idempotents such that $e_1V = ke_1$ and $e_2V \simeq J$; see Proposition 10.2. For $v_1, v_2 \in V$, we have by (30.4.3)

$$L_{e_i}(v_1 + s_0v_2) = e_iv_1 + e_i(s_0v_2) = e_iv_1 + s_0(e_i^\theta v_2).$$

and

$$e_1^\theta = \begin{bmatrix} 1/4 \\ -1/4 \end{bmatrix} = -\tfrac{1}{2}e_1 + \tfrac{1}{2}e_2, \qquad\qquad e_2^\theta = \begin{bmatrix} 3/4 \\ 1/4 \end{bmatrix} = \tfrac{3}{2}e_1 + \tfrac{1}{2}e_2.$$

By the above calculations,

$$\begin{aligned} \mathrm{tr}(L_{e_1}) &= \mathrm{tr}(L_{e_1}|_V) + \mathrm{tr}(L_{e_1}|_{s_0V}) = 1 - \tfrac{1}{2} + \tfrac{27}{2} = 14 \\ \mathrm{tr}(L_{e_2}) &= \mathrm{tr}(L_{e_2}|_V) + \mathrm{tr}(L_{e_2}|_{s_0V}) = 27 + \tfrac{3}{2} + \tfrac{27}{2} = 42 \end{aligned}$$

so $T_B(e_1) = 2\,\mathrm{tr}(L_{e_1}) = 28$ and $T_B(e_2) = 2\,\mathrm{tr}(L_{e_2}) = 84$. We have $T_J(e_2) = 3$, and the uniqueness of invariant forms implies $T_B|_{e_2V} = \langle 28\rangle T_J$. Therefore $T_B|_V = \langle 28\rangle(\langle 1\rangle \perp T_J) \simeq \langle 7\rangle(\langle 1\rangle \perp T_J)$.

(iii) Let J be an Albert algebra, $\eta \in k^\times$, and $(B, -) = M(J, \eta)$. There is a field extension L/k of degree dividing 3 such that J is reduced [164, Proposition 6.1.1]. Then $(B_L, -) = M(J_L, \eta) \simeq M(J_L, 1) \simeq M(J_L, k \times k)$ [55, Lemma 2.8 (3)]. By (i), $(T_B)_L = T_{(B_L)}$ is a multiple of $\mathbb{H}$ so it is hyperbolic. Now Springer's Theorem on odd-degree extensions [106, VII. Theorem 2.7] implies T_B is hyperbolic. □

32.2. Example. Let $(C, \sigma) = (Q \otimes M_4(k), - \otimes \mathrm{ad}_b)$ where $(Q, -)$ is a quaternion algebra with its standard symplectic involution and b is a 4-dimensional bilinear form. The adjoint involution ad_b depends only on the similitude class of b. We can rescale b if necessary so that it represents its own discriminant d, and write $b = \langle d, x, y, xy\rangle$ for some $x, y \in k^\times$. In [27, Lemma 11], it is calculated that

$$T_\sigma^+ \simeq 4\langle 1\rangle \perp \langle 2\rangle\langle\!\langle \alpha, \beta\rangle\!\rangle \lambda^2(b)$$

where $\langle\!\langle \alpha, \beta\rangle\!\rangle$ is the norm of Q. Since $\lambda^2(b) = \langle dx, dy, dxy, xy, y, x\rangle = \langle\!\langle -d\rangle\!\rangle\langle x, y, xy\rangle$, we can write this as

$$T_\sigma^+ \simeq 4\langle 1\rangle \perp \langle 2\rangle\langle\!\langle \alpha, \beta, -d\rangle\!\rangle\langle x, y, xy\rangle.$$

In particular, if Q is split or $d = -1$, then $T_\sigma^+ \simeq 4\langle 1\rangle \perp 12\mathbb{H}$.

32.3. Example. [164, (5.3), p. 118] If $J = \mathcal{H}_3(C, \gamma)$ is a reduced Albert algebra,

$$T_J = 3\langle 1\rangle \perp \langle 2\rangle\langle\langle -\gamma_1\gamma_2^{-1}, -\gamma_2\gamma_3^{-1}\rangle\rangle' n_C.$$

This together with Theorem 32.1 (i) shows that the trace of the "quasi-split" Brown algebra $M(\mathbb{A}, K)$ is Witt equivalent to $4\langle 7\rangle\langle\langle \mu\rangle\rangle$.

32.4. Example. In terms of the hermitian cubic norm structures devised in [45, §4], the bilinear trace of a Brown algebra $\mathcal{A}(J, N, \sharp, T)$ is similar to the bilinear form $q(x, y) = \mathrm{tr}_{K/k}(T(x, y)) = T(x, y) + T(y, x)$.

33. Cohomological invariants of Brown algebras

To simplify the notation in this section, let us write $E_6 = E_6^{\mathrm{sc}}$ for the split simply connected group of type E_6. If K/k is a quadratic field extension, we write E_6^K for the unique quasi-split simply connected group of type E_6 such that $E_6^K \times_k K$ is split. Let $\mathbb{A} = \mathcal{H}_3(\mathbb{O})$ be the split Albert algebra. The automorphism group of the split Brown algebra is

$$\mathbf{Aut}(M(\mathbb{A})) = E_6 \rtimes S_2.$$

By [55, Theorem 2.9 (2)],

$$\mathbf{Aut}(M(\mathbb{A}, K)) = E_6^K \rtimes S_2.$$

33.1. *The first invariant.* For any Brown algebra $(B, -)$, let $s_0 \in \mathrm{Skew}(B, -)$ and define

$$f_1(B, -) = \mu k^{\times 2} \in k^\times/k^{\times 2}$$

where $s_0^2 = \mu 1$. This is a mod 2 cohomological invariant $f_1 \in \mathrm{Inv}^1(E_6 \rtimes S_2, 2)$.

The split short exact sequence

$$1 \longrightarrow E_6 \longrightarrow E_6 \rtimes S_2 \longrightarrow S_2 \longrightarrow 1$$

induces an exact sequence of pointed sets, in which the third arrow is f_1:

$$S_2 \longrightarrow H^1(k, E_6) \longrightarrow H^1(k, E_6 \rtimes S_2) \xrightarrow{f_1} k^\times/k^{\times 2} \longrightarrow 1. \tag{33.1.1}$$

33.2. *Matrix Brown algebras.* Since $E_6 \subset E_6 \rtimes S_2$ is the group of automorphisms of $M(\mathbb{A})$ fixing s_0, there is a natural bijection

$$\boxed{H^1(k, E_6)} \longleftrightarrow \boxed{\begin{array}{c} k\text{-isomorphism classes of pairs } ((B, -), \phi) \text{ with} \\ (B, -) \text{ a Brown algebra and} \\ \phi : \mathrm{Skew}(B, -) \to k \times k \text{ a } k\text{-algebra isomorphism.} \end{array}}$$

We have

$$\ker\Big(H^1(k, E_6 \rtimes S_2) \xrightarrow{f_1} H^1(k, S_2) \Big) \simeq \frac{H^1(k, E_6)}{S_2}.$$

The subgroup $F_4 \times \boldsymbol{\mu}_3$ from 30.1 induces a map

$$H^1(k, F_4 \times \boldsymbol{\mu}_3) \longrightarrow H^1(k, E_6) \tag{33.2.1}$$

For an Albert algebra J and an $\eta \in k^\times$, the class $([J], \eta k^{\times 3}) \in H^1(k, F_4 \times \boldsymbol{\mu}_3)$ gets sent to $[M(J, \eta)] \in H^1(k, E_6 \rtimes S_2)$. The action of S_2 on $H^1(k, F_4 \times \boldsymbol{\mu}_3)$ sends $([J], \eta) \mapsto ([J], \eta^{-1})$, which is consistent with the fact that $M(J, \eta) \simeq M(J, \eta^{-1})$ [55, Lemma 2.8].

The map (33.2.1) is surjective because $F_4 \times \boldsymbol{\mu}_3$ is the stabiliser of a point in an open orbit of the projective 27-dimensional representation of E_6 [58, Example 9.12]. (This is the orbit of the line through 1 when E_6 is viewed as the semisimple structure group of $\mathbb{A}$.) This surjection explains, or is equivalent to:

33.3. Proposition (Allison–Faulkner [9, Proposition 4.5]). $f_1(B, -) = 0$ *if and only if* $(B, -)$ *is a matrix Brown algebra.*

33.4. *Nonmatricial Brown algebras.* Suppose $b \in Z^1(k, S_2)$ is a cocycle corresponding to a quadratic field extension $K = k(\sqrt{\mu})$. There is a section $s : S_2 \to E_6 \rtimes S_2$, and twisting (33.1.1) by $s_*(b)$ yields the diagram

$$\begin{array}{ccccccc}
H^1(k, E_6^K) & \longrightarrow & H^1(k, E_6^K \rtimes S_2) & \longrightarrow & k^\times/k^{\times 2} & \longrightarrow & 1 \\
 & & \simeq \big\downarrow \tau & & \big\downarrow \cdot\mu & & \\
H^1(k, E_6) & \longrightarrow & H^1(k, E_6 \rtimes S_2) & \xrightarrow{f_1} & k^\times/k^{\times 2} & \longrightarrow & 1
\end{array}$$

There are natural bijections

$H^1(k, E_6^K)$	$\longleftrightarrow$	k-isomorphism classes of pairs $((B, -), \phi)$ with $(B, -)$ a Brown algebra and $\phi : \mathrm{Skew}(B, -) \to K$ a k-algebra isomorphism.

and

$$f_1^{-1}([\mu]) \simeq \frac{H^1(k, E_6^K)}{S_2}.$$

Unlike the situation with $\mu = 1$, it is very difficult to parameterise E_6^K-torsors in terms of more elementary objects.

33.5. *A degree* 3 *(Rost) invariant.* We define an invariant

$$r \in \mathrm{Inv}^3(E_6 \rtimes S_2, \mathbb{Q}/\mathbb{Z}(2)).$$

Suppose $\beta \in H^1(k, E_6 \rtimes S_2)$ and $f_1(\beta) = [K]$. Lift β to an element $\tilde{\beta} \in H^1(k, E_6^K)$ and define

$$r(\beta) = r_{E_6^K}(\tilde{\beta}).$$

This makes sense because there are at most two ways of lifting β to $H^1(k, E_6^K)$ and they are interchanged by an outer automorphism ω of E_6^K whose Rost multiplier is obviously 1 (it is a positive integer with $n_\omega^2 = 1$). The value of r also does not depend on which cocycle does the twisting in 33.4.

In more concrete terms, suppose $(B,-)$ is a Brown algebra with $f_1(B,-)=[K]$. Since $\mathbf{Aut}(M(\mathbb{A},K))^\circ = E_6^K$, there is a cocycle $c \in Z^1(k, E_6^K)$ such that $(B,-)$ is isomorphic to the twist of $M(\mathbb{A},K)$ by c. We then have $r(B,-) = r_{E_6^K}([c])$.

There is a relationship between r and the invariant a defined in [64, §2.1]. If $(B,-)$ is any Brown algebra, $\mathbf{Aut}(B,-)^\circ$ is a simply connected group of type E_6 with trivial Tits algebras [55, Theorem 2.9 (2)]. The relationship between these invariants is

$$r(B,-) = -a(\mathbf{Aut}(B,-)^\circ).$$

33.6. *The order of* r. The Rost invariant of E_6 has order 6, while the Rost invariant of E_6^K has order 12 [116]. This means r has order 12 and takes values in

$$H^3(*, \mathbb{Z}/12\mathbb{Z}(2)) = H^3(*, \mathbb{Z}/4\mathbb{Z}) \oplus H^3(*, \mathbb{Z}/3\mathbb{Z}).$$

We can split r into even and odd components:

$$\begin{aligned} f_3 &= -3r \in \mathrm{Inv}^3(E_6 \rtimes S_2, 2) \\ g_3 &= 4r \in \mathrm{Inv}^3(E_6 \rtimes S_2, 3). \end{aligned}$$

33.7. Proposition. *For any field extension L/k, the following diagram commutes:*

$$\begin{tikzcd} H^1(L, E_6 \rtimes S_2) \arrow[r, "r"] \arrow[d] & H^3(L, \mathbb{Q}/\mathbb{Z}(2)) \arrow[d, equal] \\ H^1(L, E_8) \arrow[r, "r_{E_8}"] & H^3(L, \mathbb{Q}/\mathbb{Z}(2)) \end{tikzcd}$$

where the left vertical arrow is induced by the inclusion

$$E_6 \rtimes S_2 = \mathbf{Aut}(M(\mathbb{A}),-) \subset \mathbf{Aut}(K(M(\mathbb{A}),-)) = E_8.$$

Proof. If $\zeta \in H^1(L, E_6 \rtimes S_2)$ is in the kernel of f_1 then it is in the image of $H^1(L, E_6)$. The inclusion $E_6 \subset E_8$ (the one that is visible on the Dynkin diagram) has Rost multiplier 1, which proves the claim.

Otherwise, suppose $[z] \in H^1(L, E_6 \rtimes S_2)$ and $f_1([z]) = [K]$. Let $b \in Z^1(L, E_6 \rtimes S_2)$ be the image of z under the map induced by the composition $E_6 \rtimes S_2 \to S_2 \to E_6 \rtimes S_2$. Twisting by b obtains a diagram

$$\begin{tikzcd} H^1(L, {}_b(E_6 \rtimes S_2)) \arrow[r, "\sim", "\tau_b"'] \arrow[d] & H^1(L, E_6 \rtimes S_2) \arrow[r, "r"] \arrow[d] & H^3(L, \mathbb{Q}/\mathbb{Z}(2)) \\ H^1(L, {}_bE_8) \arrow[r, "\sim", "\tau_b"'] & H^1(L, E_8) \arrow[r, "r_{E_8}"] & H^3(L, \mathbb{Q}/\mathbb{Z}(2)). \end{tikzcd}$$

It is clear that $\tau_b^{-1}([z])$ is in the image of $H^1(L, {}_b(E_6 \rtimes S_2)^\circ)$ and ${}_b(E_6 \rtimes S_2)^\circ \simeq E_6^K$.

The algebras $M(\mathbb{A},K)$ and $M(\mathbb{A})$ are isotopic [55, Proposition 5.12], so b is equivalent in $Z^1(L, E_8)$ to the trivial cocycle, and τ_b (in the bottom row) is induced by a Γ_k-group isomorphism $f : {}_bE_8 \xrightarrow{\sim} E_8$. The Rost multiplier of f is 1, so $r_{E_8}([z]) = r_{{}_bE_8}(\tau_b^{-1}([z]))$. The Rost multiplier of $E_6^K \to {}_bE_8$ is also 1 (because it is after extending scalars from L to K). So we have

$$r([z]) = r_{E_6^K}(\tau_b^{-1}([z])) = r_{{}_bE_8}(\tau_b^{-1}([z])) = r_{E_8}([z]). \qquad \square$$

The statement and the proof of Proposition 33.7 are very similar to [56, Proposition 3.6], which shows that E_6^K embeds in E_7 with Rost multiplier 1, and has an intermediate subgroup $E_6^K \subset E_6 \rtimes \boldsymbol{\mu}_4 \subset E_7$. Actually this implies our result because it is about the same subgroup E_6^K. The main difference is that $E_6 \rtimes S_2$ is not a subgroup of E_7, which is why we have compared it to E_8 instead. Also, $E_6 \rtimes S_2$ is a bit easier to work with than $E_6 \rtimes \boldsymbol{\mu}_4$.

33.8. Theorem. *Let $(B,-)$ be a Brown algebra with $f_1(B,-) = (\mu) \in H^1(k, \mathbb{Z}/2\mathbb{Z})$. It has a unique nondegenerate quadratic form q_B that is invariant and has $q_B(1) = 1$. Moreover,*

$$q_B - 4\langle\!\langle \mu \rangle\!\rangle \in I^3(k) \qquad \textit{and} \qquad e_3(q_B - 4\langle\!\langle \mu \rangle\!\rangle) = 2f_3(B,-) \in H^3(k, \mathbb{Z}/2\mathbb{Z}).$$

Proof. If $\mathrm{char}(k) \neq 7$, let $q_B = \frac{1}{112} T_B$. If $\mathrm{char}(k) = 7$, a reduced trace q_B can be defined like in 25.7, because there is a model of the split Brown algebra over $\mathbb{Z}$ whose trace is nondegenerate and divisible by 112. The uniqueness of q_B is Lemma 2.16 (iii).

If $f_1(B,-) = 0$ then q_B is hyperbolic by Theorem 32.1 (iii), so clearly it is in $I^3(k)$. Also $2f_3(B,-) = 0$ because the Dynkin index of E_6 equals 6.

So let $K = k(\sqrt{\mu})$ be a field. The rest of the proof uses a technique from [57, Lemmas 13.5 & 13.7]. If $B = M(\mathbb{A}, K)$, then $q_B - 4\langle\!\langle \mu \rangle\!\rangle = 0$ by Example 32.3. The homomorphism

$$\mathbf{Aut}(M(\mathbb{A}, K))^\circ = E_6^K \longrightarrow \mathbf{O}_{64}^+ = \mathbf{O}^+(M(\mathbb{A}, K), q_{M(\mathbb{A},K)} - 4\langle\!\langle \mu \rangle\!\rangle)$$

lifts to a homomorphism

$$E_6^K \longrightarrow \mathbf{Spin}_{64}$$

because E_6^K is simply connected [33, Proposition 2.24 (i)]. Any $(B,-)$ with $f_1(B,-) = [K]$ is twisted from $M(\mathbb{A}, K)$ by some cocycle in $Z^1(k, E_6^K)$; the form q_B is twisted by the image of this cocycle in $Z^1(k, \mathbf{Spin}_{64})$ and that is why $q_B \in I^3(k)$.

There is a commutative diagram where the arrows are labelled by their Rost multipliers:

$$\begin{tikzcd} E_6^K \arrow[r, "?"] \arrow[dr, "12"'] & \mathbf{Spin}_{64} \arrow[d, "2"] \\ & \mathbf{SL}_{64} \end{tikzcd}$$

By the composition property of Rost multipliers, the multiplier of the top arrow is 6, and we have

$$e_3(q_B - 4\langle\!\langle \mu \rangle\!\rangle) = r_{\mathbf{Spin}_{64}}(q_B - 4\langle\!\langle \mu \rangle\!\rangle) = 6r(B,-) = 2f_3(B,-). \qquad \square$$

33.9. Proposition. *If $(B,-) = M(J, \eta)$ is a matrix Brown algebra then $f_3(B,-) = f_3(J)$ and $g_3(B,-) = g_3(J)$.*

Proof. First let $\eta = 1$. Then $(B,-)$ corresponds to a class in the image of

$$H^1(k, F_4) \to H^1(k, E_6) \to H^1(k, E_6 \rtimes S_2).$$

Since the Rost multiplier of $F_4 \subset E_6$ equals 1 [56, Example 2.4], we have

$$r(B,-) = r_{F_4}(J) = f_3(J) + g_3(J) \in H^3(k, \mathbb{Q}/\mathbb{Z}(2)).$$

For arbitrary $\eta \in k^\times$, the algebras $(B, -) = M(J, \eta)$ and $(B', -) = M(J)$ are isotopic [55, Lemma 4.13] so they have the same image in $H^1(k, E_8)$, which is the class of $K(B, -) \simeq K(B', -)$. It follows from Proposition 33.7 and the first part of the proof that $r(B, -) = r(B', -) = f_3(J) + g_3(J)$. □

33.10. Proposition. *Let K/k be a quadratic étale extension, and J an Albert algebra. If $(B, -) = M(J, K)$, then $f_3(B, -) = f_3(J)$ and $g_3(B, -) = g_3(J)$.*

Proof. The algebras $M(J, K)$ and $M(J)$ are isotopic [55, Proposition 5.12], so they have the same image in $H^1(k, E_8)$ and $r(M(J, K)) = r(M(J)) = f_3(J) + g_3(J)$. □

33.11. *The discriminant of a symplectic involution.* For any $n \geq 1$ there is a normalised cohomological invariant

$$\Delta \in \mathrm{Inv}^3(\mathbf{PGSp}_{8n}, 2)$$

called the *discriminant*. A relative version of this invariant was first introduced in [27], for comparing different symplectic involutions on the same central simple algebra of any degree divisible by 4. It was proved later in [63] that Δ exists as an absolute invariant when the degree is divisible by 8.

The invariant $\Delta \in \mathrm{Inv}^3(\mathbf{PGSp}_8, 2)$ vanishes on hyperbolic involutions and on involutions that decompose completely as a tensor product of three quaternion algebras with standard involutions $(C, \sigma) = \bigotimes_{i=1}^3 (Q_i, -)$.

There is also the main degree 2 invariant

$$\delta \in \mathrm{Inv}^2(\mathbf{PGSp}_8, 2), \qquad \delta(C, \sigma) = [C] \in {}_2\mathrm{Br}(k) = H^2(k, \boldsymbol{\mu}_2).$$

We now calculate the value of r on the image of the map

$$H^1(k, \mathbf{PGSp}_8) \times H^1(k, \mathbb{Z}/2\mathbb{Z}) \to H^1(k, E_6 \rtimes S_2).$$

33.12. Theorem. *Let $(B, -) = \mathrm{CD}((C, \sigma), \mu)$ for a symplectic involution (C, σ) of degree 8 and some $\mu \in k^\times$. Then $g_3(B, -) = 0$ and*

$$f_3(B, -) = \Delta(C, \sigma) + (\mu)\cdot[C] \in H^3(k, \mathbb{Z}/2\mathbb{Z}).$$

Proof. It has been shown in [63, §5] that the diagram

$$\begin{array}{ccc} H^1(k, \mathbf{PGSp}_8) & \xrightarrow{\Delta} & H^3(k, \mathbb{Z}/2\mathbb{Z}) \\ \downarrow & & \downarrow \\ H^1(k, E_6) & \xrightarrow{r_{E_6}} & H^3(k, \mathbb{Z}/6\mathbb{Z}(2)) \end{array}$$

commutes. This proves the theorem in the case where $(\mu) = 0$.

For the general case, define a temporary invariant

$$f' \in \mathrm{Inv}^3(\mathbf{PGSp}_8 \times \mathbb{Z}/2\mathbb{Z},\ \mathbb{Z}/12\mathbb{Z}(2)), \quad f'((C, \sigma), \mu) = f_3(\mathrm{CD}((C, \sigma), \mu)) - \Delta(C, \sigma).$$

The first paragraph implies that for any $\mu \in k^\times \setminus k^{\times 2}$,

$$\mathrm{res}_{k(\sqrt{\mu})/k}(f'((C, \sigma), \mu))) = 0.$$

This implies $2f' = 0$ [72, Proposition 4.2.10], so f' takes its values in the 2-torsion part of $H^3(k, \mathbb{Z}/12\mathbb{Z}(2))$, which is $H^3(*, \mathbb{Z}/2\mathbb{Z})$ [116, p. 154].

Fixing μ, we can define another temporary invariant $f_\mu \in \mathrm{Inv}^3(\mathbf{PGSp}_8, \mathbb{Z}/2\mathbb{Z})$ by

$$f_\mu(C, \sigma) = f_3(\mathrm{CD}(C, \sigma), \mu).$$

By [63, Proposition 4.1], there are unique constants $\lambda_0, \lambda_0' \in \{0, 1\}$ and $\lambda_1 \in H^1(k, \mathbb{Z}/2\mathbb{Z})$ so that

$$f_\mu = \lambda_0 \cdot 1 + \lambda_0' \cdot \Delta + \lambda_1 \cdot \delta.$$

Take $\mathbb{M}_4(\mathbb{H}) = (M_8(k), -)$ with a symplectic involution (there is only one up to conjugacy) and $(B_0, -) = \mathrm{CD}(\mathbb{M}_4(\mathbb{H}), \mu)$. Then $\Delta(\mathbb{M}_4(\mathbb{H})) = \delta(\mathbb{M}_4(\mathbb{H})) = 0$. But Lemma 31.4 and Proposition 33.10 imply $f_3(B_0, -) = 0$ too, because

$$(B_0, -) \simeq M(\mathcal{H}_3(\mathbb{O}), k(\sqrt{\mu})),$$

which means $\lambda_0 = 0$. We have $\lambda_0' = 1$, because $\mathrm{res}_{k(\sqrt{\mu})/k}(f_\mu) = \Delta$ by the first paragraph of the proof. To calculate λ_1, take a quaternion division algebra $(Q, -)$ over $k(t_1, t_2)$ with norm $\langle\langle t_1, t_2 \rangle\rangle$. Let $(C, \sigma) = (M_4(Q), - \otimes \mathrm{ad}_{\langle 1,1,1,1 \rangle})$, so $\mathcal{H}(C, \sigma) = \mathcal{H}_4(Q)$. Let $(B_1, -) = \mathrm{CD}((C, \sigma), \mu)$. Then Lemma 31.4 implies $(B_1, -) \simeq M(\mathcal{H}_3(O), k(\sqrt{\mu}))$ where $O = Q \oplus uQ$ is the octonion algebra with norm $\langle\langle \mu, t_1, t_2 \rangle\rangle$. By Proposition 33.10,

$$f_3(B_1, -) = f_3(\mathcal{H}_3(O)) = e(O) = (t_1) \cdot (t_1) \cdot (\mu) = (\mu) \cdot [Q] \cdot.$$

We also have $\Delta(C, \sigma) = 0$ (see the main theorem of [155]). In conclusion, $\lambda_1 = (\mu)$. $\square$

It is possible that Theorem 33.12 is a special case of Garibaldi's [57, Theorem 9.1]. Looking at his Table 7B, it is not clear to me that we are talking about the same subgroup $\mathbf{PGSp}_8 \times \boldsymbol{\mu}_2$ in E_8. In any case, it is not too bad to have a different proof.

33.13. *Limitations in the constructions.* As Victor Petrov pointed out to me, the formulas in 33.9, 33.10, and particularly 33.12 settle an open question raised in [31, Remark 2.3.10]:

> "It is a major open problem whether every structurable division algebra of skew-dimension one is either a hermitian structurable algebra or is obtained from a Cayley–Dickson process on a Jordan division algebra with a Jordan norm of degree 4."

The answer is negative for Brown algebras: since $r_{E_6^K}$ has order 12 for any quadratic field K/k, there exists a field extension F/k and a $[y] \in H^1(F, E_6^K)$ such that $6r_{E_6^K}([y]) \neq 0$ [116, Theorem 3.3, Proposition A.9]. Twisting the quasi-split Brown algebra $M(\mathbb{A} \otimes F, K \otimes F)$ by such a cocycle y produces a Brown algebra $(Y, -)$ with $6r(Y, -) \neq 0$. By our calculations, $(Y, -)$ cannot be isomorphic or even isotopic to $M(J, \eta)$, $M(J, K)$, or $\mathrm{CD}((C, \sigma), \mu)$ for any inputs J, η or (C, σ), because all of these have $6r(B, -) = 0$. One can even take the 2-special closure F'/F (so that F' has no more odd-degree extensions), and $2r(Y \otimes F', -)$ will still be nonzero, and not reachable by the doubling construction.

By Theorem 33.8, a necessary and sufficient condition for $2r(B, -) = 0$, say if $\mathrm{char}(k) \neq 7$, is that $T_B - 4\langle 7 \rangle \langle\langle \mu \rangle\rangle \in I^4(k)$ where $(\mu) = f_1(B, -)$. So in order to find concrete counterexamples over fields that we understand, one needs to look for a $(B, -)$ with $T_B - 4\langle 7 \rangle \langle\langle \mu \rangle\rangle \in I^3(k) \setminus I^4(k)$.

34. Overview and applications

If $(B, -)$ is a Brown algebra, the invariant r has a lot to say about the groups:

> $\mathbf{Aut}(B, -)^\circ$, which is simply connected of type E_6 with trivial Tits algebras;
>
> $\mathbf{Str}(B, -)^{\mathrm{der}}$, which is simply connected of type E_7 with trivial Tits algebras;
>
> $\mathbf{Aut}(K(B, -))$, which is of type E_8 with its semisimple anisotropic kernel contained in E_7.

If $f_1(B, -) \neq 0$ and $G = \mathbf{Aut}(B, -)^\circ$ is isotropic, the Tits index and sometimes even the exact isomorphism class of G can be predicted from the value of r using [64, Propositions 2.3 & 2.9]. If $f_1(B, -) = 0$, it is just an exercise to determine the Tits index of G from the value of r; there are four possible indices for ${}^1E_6$ and only three of them can have trivial Tits algebras; see [153, V. §3.1].

34.1. *Reduced and division algebras.* A Brown algebra $(B, -)$ is called *reduced* if there is an element $0 \neq e \in B$ such that $U_e(B) \subset ke$. A result by Allison and Faulkner [9, Theorem 4.6] says that

> $(B, -)$ is reduced if and only if it is isotopic to a matrix Brown algebra.

So $(B, -)$ is reduced if and only if it comes from $H^1(k, E_6)$. A major result by Garibaldi, Petrov, and Semenov [65, Theorem 10.10] implies directly that

> $(B, -)$ is reduced if and only if $6r(B, -) = 0$ and $3r(B, -) \in H^3(k, \mathbb{Z}/2\mathbb{Z})$ is a symbol.

A Brown algebra $(B, -)$ is called *quasi-split* if it is isomorphic to $M(\mathbb{A}, K)$ for some quadratic étale extension K/k and the split Albert algebra $\mathbb{A}$. A result by Garibaldi [56, Theorem 0.1, §0.4] (and independently by Chernousov [39]) with quite lengthy proofs is that:

> $(B, -)$ is isotopic to $M(\mathbb{A})$ if and only if it is quasi-split,
> if and only if $r(B, -) = 0$.

In summary, we have a dictionary in Table 12 for matching properties of $(B, -)$ against the Tits index of its semisimple structure group, and against the value of $r(B, -)$. The possibilities for the Tits index of $\mathbf{Str}(B, -)^\circ$ are easily deduced from [173, p. 60] and the labelled Dynkin diagram of $K(M(\mathbb{A}))$; see Table 4. The conditions on $r(B, -)$ are derived from the theorems mentioned above.

A *necessary* condition for $(B, -)$ to belong in the second row of Table 12 is deduced from [58, Proposition A.1]:

> If $(B, -)$ is neither reduced nor division, then $r(B, -) \in H^3(k, \mathbb{Z}/2\mathbb{Z})$, it has symbol length 2, and is a sum of two symbols with a common slot.

If k has no odd-degree extensions, this is also a sufficient condition by [65, Theorem 10.18].

The two Tits indices in the third row of Table 12 are distinguished by whether their Levi subgroup of type E_6 is anisotropic or not, i.e., whether it is the norm-similitude group of an Albert division algebra or a reduced Albert algebra. From

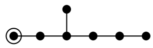
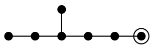
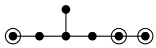
$(B, -)$	$\mathbf{Str}(B, -)^{\text{der}}$	$r(B, -)$
division	anisotropic	?
neither division nor reduced		?
reduced but not quasi-split	 or 	$3r(B, -) \in H^3(k, \mathbb{Z}/2\mathbb{Z})$ and is a nonzero symbol.
quasi-split	split	$r(B, -) = 0$.

Table 12: Dictionary between Brown algebras and their semisimple structure groups of type E_7. A Brown algebra $(B, -)$ has the property in the left column if and only if its semisimple structure group has the Tits index in the middle column, if and only if $r(B, -)$ has the condition in the right column.

Proposition 33.9 and the sensitivity of $g_3 \in \mathrm{Inv}(F_4, 3)$ to division algebras [131, §3.3], it is easy to see that the two cases are distinguished by whether $g_3(B, -) = 4r(B, -)$ is nonzero or zero.

34.2. Example (Existence of certain Tits indices)**.** As a sample application, we can sketch a Brown algebraic proof of the existence of algebraic groups with Tits index $E_{8,1}^{133}$. The other examples (e.g., [9, Theorem 7.1], [175], and [132, Theorem 2]) all have some common features, like a division algebra of degree 4 whose reduced norm is not surjective.

Suppose (Q_1, σ_1), (Q_2, σ_2), (Q_3, σ_3) are quaternion algebras with standard involutions, and let $(D, \sigma) = (Q_1 \otimes Q_2 \otimes Q_3, \sigma_1 \otimes \sigma_2 \otimes \sigma_3)$. Then $\Delta(D, \sigma) = 0$ because (D, σ) is completely decomposable [63, Theorem B]. For $(B, -) = \mathrm{CD}((D, \sigma), \mu)$, Theorem 33.12 shows that

$$r(B, -) = (\mu) \cdot [D] = (\mu) \cdot [Q_1] + (\mu) \cdot [Q_2] + (\mu) \cdot [Q_3].$$

If this has symbol length 3 then $(B, -)$ is a division algebra, by [58, Proposition A.1]. (As an aside, one can show that D is a division algebra if and only if $[D]$ has symbol length 3.) If $r(B, -)$ has symbol length 2 then $(B, -)$ is neither division nor reduced. In these two respective cases, $\mathbf{Aut}(B, -)$ has Tits index $E_{8,1}^{133}$ or $E_{8,2}^{66}$.

For interest's sake, we give another example where the Δ part of r is nonzero.

34.3. Example. Let $K = k(\sqrt{d})$, and suppose $(Q, -)$ is a quaternion algebra over K. Consider $\mathrm{cor}_{K/k}(Q)$ with its usual (orthogonal) involution γ as in 9.7. Suppose further that (Q_1, σ) is a quaternion algebra over k with standard involution.

Let $(A, \rho) = (Q_1 \otimes \mathrm{cor}_{K/k}(Q), \sigma \otimes \gamma)$, and $(B, -) = \mathrm{CD}((A, \rho), \mu)$. We have $\Delta(A, \rho) = (d) \cdot [Q_1]$ by [63, Example 3.1]. Theorem 33.12 yields that

$$\begin{aligned} r(B, -) &= (d) \cdot [Q_1] + (\mu) \cdot [A] \\ &= (d\mu) \cdot [Q_1] + (\mu) \cdot [\mathrm{cor}_{K/k}(Q)]. \end{aligned}$$

The last results of this chapter will build a bridge with Chapter VI. We need the following result, which derives from a theorem of Hoffmann.

34.4. Theorem. *Let* $\mathbf{Spin}_{12} \subset E_8$ *be the subgroup coming from the obvious inclusion of root systems. The Rost invariant* $r_{E_8} : H^1(k, E_8) \to H^3(k, \mathbb{Q}/\mathbb{Z}(2))$ *is injective on the image of* $H^1(k, \mathbf{Spin}_{12}) \to H^1(k, E_8)$.

Proof. Let $[\zeta], [\gamma] \in H^1(k, \mathbf{Spin}_{12})$ and suppose $r_{E_8}([\zeta]) = r_{E_8}([\gamma])$. Let $q_\zeta, q_\gamma \in I^3_{12}(k)$ be the quadratic forms identified with the images of $[\zeta], [\gamma]$ in $H^1(k, \mathbf{O}_{12})$. Then ${}_\zeta\mathbf{Spin}_{12} = \mathbf{Spin}(q_\zeta)$ is a subgroup of $G = {}_\zeta E_8$. Let $[\rho] = \tau_\zeta([\gamma]) \in H^1(k, \mathbf{Spin}(q_\zeta))$.

The Rost multiplier of $\mathbf{Spin}_{12} \to E_8$ is 1 because all the roots of E_8 have the same length (see [116, Proposition 7.9 (2)]), and so the multiplier of $\mathbf{Spin}(q_\zeta) \to G$ is also 1. We have $r_G([\rho]) = r_{\mathbf{Spin}(q_\zeta)}([\rho]) = e_3(q_\gamma - q_\zeta)$; see [101, (31.42)]. Also, by [70, Lemme 7], $r_G([\rho]) = r_{E_8}([\gamma]) - r_{E_8}([\zeta])$, which is zero by assumption.

So we have $q_\gamma - q_\zeta \in I^4(k)$. By Pfister's Theorem, q_γ and q_η are each a product of a binary form and an Albert form in $I^2_6(k)$. The theorem of Hoffmann [78, Corollary] that we apply says that q_γ is similar to q_η. There is an intermediate subgroup $\mathbf{Spin}_{12} \subset \mathbf{\Omega}_{14}/\mathbf{G}_m \subset E_8$; see (18.15.1). So $H^1(k, \mathbf{Spin}_{12}) \to H^1(k, E_8)$ factors through $H^1(k, \mathbf{\Omega}_{14})$. The classes $[\zeta], [\gamma] \in H^1(k, \mathbf{Spin}_{12})$ have the same image in $H^1(k, \mathbf{\Omega}_{14})$ because $q_\zeta \perp \mathbb{H}$ is similar to $q_\gamma \perp \mathbb{H}$; see (14.4.2). Consequently, ζ and γ are cohomologically equivalent as cocycles in $Z^1(k, E_8)$. ☐

34.5. Theorem. *Let* $(A, -)$ *be a bioctonion algebra and* $(B, -)$ *a Brown algebra. Then*

$$K(A, -) \simeq K(B, -)$$

if and only if neither $(A, -)$ *nor* $(B, -)$ *are division algebras and*

$$b_3(A, -) = r(B, -).$$

Proof. ($\Rightarrow$) Suppose $L = K(A, -) \simeq K(B, -)$. Since L has a $\mathbb{Z}^2$-grading, $\mathbf{Aut}(L)$ has a split torus of rank 2, which implies neither $(A, -)$ nor $(B, -)$ is a division algebra [31, p. 66, Step 1.]. We have $b_3(A, -) = r(B, -) = r_{E_8}(L)$ by Propositions 23.8 and 33.7.

($\Leftarrow$) If neither $(A, -)$ nor $(B, -)$ are division algebras then $G = \mathbf{Aut}(K(A, -))$ and $H = \mathbf{Aut}(K(B, -))$ have k-rank ≥ 2. The labelled Dynkin diagrams of the gradings on $K(A, -)$ and $K(B, -)$ respectively are

By Tits's classification [173, p. 60], we deduce that G has Tits index $E^{66}_{8,2}$, $E^{28}_{8,4}$, or $E^{0}_{8,8}$ (according to the Witt index of the Albert form of $(A, -)$, but we do not need this information). Likewise, H has one of those indices too, or possibly $E^{78}_{8,2}$. But if it is $E^{78}_{8,2}$ then H would contain a semisimple anisotropic kernel that is simply connected of type ${}^1E_6$. Such a subgroup is isomorphic to $\mathbf{Iso}(N_J)$ for an Albert division algebra J, and $r_{E_8}(K(B, -)) = r_{F_4}(J) = g_3(J) + f_4(J)$ would have order 3 or 6, contradicting that $2r(B, -) = 0$.

Consequently G and H are both twists of E_8 by some cocycle classes $[\zeta], [\gamma]$ in the image of $H^1(k, \mathbf{Spin}_{12}) \to H^1(k, E_8)$. We have $r_{E_8}([\zeta]) = b_3(A, -) = r(B, -) = r_{E_8}([\gamma])$, so Theorem 34.4 yields that ζ and γ are cohomologically equivalent as cocycles in E_8. As such, $G = {}_\zeta E_8 \simeq {}_\gamma E_8 = H$. Since $K(A, -) \simeq \mathrm{Lie}(G)$ and $K(B, -) \simeq \mathrm{Lie}(H)$, the proof is complete. ☐

34.6. Example. One of the applications of the above theorem is to answer a question from Jeroen Meulewaeter's PhD thesis. In [121, Example 4.2.20], he proves that for a given bicomposition algebra $(A, -)$ which is not a division algebra, there is a structurable algebra $(B, -)$ of skew-dimension one (unique up to isotopy) such that $K(A, -) \simeq K(B, -)$. A nondivision bioctonion algebra $(A, -)$ has an invariant of the form

$$b_3(A, -) = (a)\cdot(x_1)\cdot(y_1) + (a)\cdot(x_2)\cdot(y_2) \in H^3(k, \mathbb{Z}/2\mathbb{Z})$$

by Theorem 21.7. Our Example 34.2 clearly gives a way to find a nondivision Brown algebra $(B, -)$ with $r(B, -) = b_3(A, -)$, so also $K(A, -) \simeq K(B, -)$. If we did not have cohomological invariants, it would be a hard and thankless task to prove that these 248-dimensional Lie algebras are isomorphic!

Chapter VIII

Classifying cohomological invariants

The simple algebraic group $\mathbf{Spin}_n$ is a simply connected double cover of $\mathbf{O}_n^+$. Roughly speaking, if we understand the Galois cohomology of spin groups, we understand the nature of quadratic forms in $I_n^3(k)$ (for odd n, this set is defined in [41, §3]). There are various ways to measure the complexity of I_n^3, such as the essential dimension and the Pfister number [37].

Following some remarkable work by many mathematicians, we know the essential dimensions of $\mathbf{Spin}_n$ and I_n^3 for all n [37,41,61]. Unlike $\mathbf{O}_n^+$ whose essential dimension grows linearly with n, the essential dimension of $\mathbf{Spin}_n$ grows exponentially with n. It begins to explode dramatically after $n = 14$, going from $\text{e.d.}(\mathbf{Spin}_{14}) = 7$ to

$$\text{e.d.}(\mathbf{Spin}_{15}) = 23, \qquad \text{e.d.}(\mathbf{Spin}_{16}) = 24, \qquad \text{e.d.}(\mathbf{Spin}_{17}) = 120, \qquad \dots$$

For even n, the Pfister number $\mathrm{Pf}(3, n)$ is the least number of general Pfister forms one needs to sum to get any form in I_n^3. These numbers also grow at least exponentially with n [37] but exact values are not known. Looking at the situation around 14, we have $\mathrm{Pf}(3, 12) = 2$, $\mathrm{Pf}(3, 14) = 3$, and $\mathrm{Pf}(3, 16) \geq 4$ with no known upper bound [98]. In short, $\mathbf{Spin}_{14}$ is a fascinating boundary case among the spin groups.

The essential dimension of $\mathbf{Spin}_n$ has implications for rational parameterisations and cohomological invariants of quadratic forms in I_n^3. There is a rational parameterisation for quadratic forms in I_{14}^3 that takes 7 parameters (this is the content of Rost's Theorem [142], which we reproved in Corollary 21.3), and there exists a degree 7 cohomological invariant of I_{14}^3 (defined by Garibaldi [58], and appearing here in 38.1); both of these results are consistent with the value $\text{e.d.}(\mathbf{Spin}_{14}) = \text{e.d.}(I_{14}^3) = 7$. In contrast, if there even exists a rational parameterisation for quadratic forms in I_{16}^3, which is thought to be unlikely, then it would need to have at least 24 parameters. And for all we know, there could exist cohomological invariants of $\mathbf{Spin}_{16}$ in degree as high as 24.

So we have basically no hope (at least with current methods) of classifying or even finding the high-degree cohomological invariants of $\mathbf{Spin}_n$ beyond the frontier of $n = 14$. Cohomological invariants of $\mathbf{Spin}_n$ for $n \leq 12$ were classified by Garibaldi in [58, Table 23B], with the help of methods developed years ago mainly by Rost and Serre. The complete classification of invariants of $\mathbf{Spin}_{14}$ seemed within reach and yet was still unknown until now – this is what made it an alluring problem.

Outline of the main theorem

Our approach to classifying the mod 2 invariants of $\mathbf{Spin}_{14}$ goes as follows. We approach $\mathbf{Spin}_{14}$ from the following system of subgroups of the extended Clifford group $\mathbf{\Omega}_{14}$ of the hyperbolic 14-dimensional quadratic form:

$$\begin{tikzcd}
G_2 \times G_2 \arrow[r, hook] & (G_2 \times G_2) \rtimes S_2 \arrow[r, hook] & \mathbf{\Omega}_{14} \\
 & & \mathbf{Spin}_{14} \arrow[u, hook]
\end{tikzcd}$$

Since $H^1(k, \mathbf{\Omega}_n) \simeq PI^3_n(k)$, this leads to the following diagram of rings:

$$\begin{tikzcd}
\mathrm{Inv}(G_2 \times G_2, 2)^{S_2} & \mathrm{Inv}((G_2 \times G_2) \rtimes S_2, 2) \arrow[l] & \mathrm{Inv}(PI^3_{14}, 2) \arrow[l, hook'] \arrow[d, hook] \\
 & & \mathrm{Inv}(\mathbf{Spin}_{14}, 2)
\end{tikzcd} \tag{34.6.1}$$

The two monomorphisms in this diagram are important: they are injective because the maps

$$H^1(*, (G_2 \times G_2) \rtimes S_2) \longrightarrow PI^3_{14}(*), \qquad H^1(*, \mathbf{Spin}_{14}) \longrightarrow PI^3_{14}(*)$$

are surjections. Of all these sets of invariants, $\mathrm{Inv}(G_2 \times G_2, 2)^{S_2}$ is the easiest to determine and it becomes the cornerstone of our analysis. All the S_2-invariant invariants of $G_2 \times G_2$ extend to $(G_2 \times G_2) \rtimes S_2$, so the leftmost arrow in (34.6.1) is split surjective. The unique degree 1 invariant of $(G_2 \times G_2) \rtimes S_2$ is clearly in the kernel of this arrow, and we show that it does in fact generate the kernel. This step makes use of a lemma (35.7) by Philippe Gille on unramified elements in the cycle module of a quasitrivial torus. In this way we classify the contents of $\mathrm{Inv}((G_2 \times G_2) \rtimes S_2, 2)$, and from there it is straightforward to determine which invariants of $(G_2 \times G_2) \rtimes S_2$ lie in the image of $\mathrm{Inv}(PI^3_{14}, 2)$. Thereby we have determined $\mathrm{Inv}(PI^3_{14}, 2)$, a crucial subset of $\mathrm{Inv}(\mathbf{Spin}_{14}, 2)$. Interestingly, $\mathrm{Inv}(PI^3_{14}, 2)$ is not generally a free module if $-1 \notin k^{\times 2}$.

For the next step, we use the system of subgroups

$$\begin{tikzcd}
\mathbf{Spin}_{14} \arrow[rr, hook] & & \mathbf{\Omega}_{14} \\
\mathbf{Spin}_{12} \arrow[u, hook] \arrow[r, hook] & \mathbf{\Gamma}^+_{12} \arrow[r, hook] & \mathbf{\Omega}_{12} \arrow[u, hook]
\end{tikzcd}$$

from which the solid arrows in the following diagram are obtained:

$$\begin{tikzcd}
\mathrm{Inv}(PI^3_{14}, 2) \arrow[rr, hook] \arrow[d] & & \mathrm{Inv}(\mathbf{Spin}_{14}, 2) \arrow[r, dashed] \arrow[d] & \mathrm{Inv}(PI^3_{14}, 2) \arrow[d] \\
\mathrm{Inv}(PI^3_{12}, 2) \arrow[r, hook] & \mathrm{Inv}(I^3_{12}, 2) \arrow[r, hook] \arrow[rr, dashed, bend right] & \mathrm{Inv}(\mathbf{Spin}_{12}, 2) \arrow[r, dotted] & \mathrm{Inv}(PI^3_{12}, 2)
\end{tikzcd}$$

The dashed arrows are "residue maps" obtained using the concept of a *fibration* of functors (Definition 36.1). It was previously known that $\mathrm{Inv}(I^3_{12}, 2) \simeq \mathrm{Inv}(\mathbf{Spin}_{12}, 2)$ if $\sqrt{-1} \in k$, and actually this is true even without that assumption – we prove it in Theorem 37.8. This allows us to complete the rightmost square with the lightly dotted arrow, and it turns out that this square is commutative. A nice feature of our fibrations is that the sequences

$$\mathrm{Inv}(PI^3_n, 2) \longrightarrow \mathrm{Inv}(\mathbf{Spin}_n, 2) \longrightarrow \mathrm{Inv}(PI^3_n, 2)$$

derived from them are exact in the middle for $n = 12, 14$. The contents of $\mathrm{Inv}(PI^3_{14}, 2)$, $\mathrm{Inv}(PI^3_{12}, 2)$, and $\mathrm{Inv}(\mathbf{Spin}_{12}, 2)$ are known at this late stage of the process, and it turns out that all of this information is sufficient to determine $\mathrm{Inv}(\mathbf{Spin}_{14}, 2)$ using an elementary argument, but only if -1 is a square in k. Attempts to prove this without assuming $\sqrt{-1} \in k$ failed at the very last step, because in this configuration of residues and restriction maps, there appear many terms involving the Galois symbol (-1) and the presence of these terms gives us insufficient control over the right hand side of the diagram above.

35. Invariants of $\mathbf{PGO}_4$ and $(G_2 \times G_2) \rtimes \mathbb{Z}/2\mathbb{Z}$

In this section, we classify the cohomological invariants of bioctonion algebras. The same method works to classify the invariants of $\mathbf{PGO}_4$, something which does not seem to have been done before, so we state and prove both results in Theorem 35.9.

35.1. *Cohomological invariants of* $G_2 \times G_2$. Let $e \in \mathrm{Inv}^3(G_2, 2)$ be the unique nontrivial normalised cohomological invariant of G_2, as in 17.1. Define invariants $e', e'', r_3, r_6 \in \mathrm{Inv}(G_2 \times G_2, 2)$ by:

$$e'(\alpha, \beta) = e(\alpha), \qquad r_3(\alpha, \beta) = e(\alpha) + e(\beta),$$
$$e''(\alpha, \beta) = e(\beta), \qquad r_6(\alpha, \beta) = e(\alpha) \cdot e(\beta).$$

Recall from Example 23.4 how S_2 acts on $H^1(*, G_2 \times G_2)$. In the obvious way, S_2 acts on $\mathrm{Inv}(G_2 \times G_2, 2)$ too.

35.2. Lemma. $\mathrm{Inv}(G_2 \times G_2, 2)$ *is the free* $H(k)$*-module with basis* $\{1, e', e'', r_6\}$*, and* $\mathrm{Inv}(G_2 \times G_2, 2)^{S_2}$ *is the free* $H(k)$*-module with basis* $\{1, r_3, r_6\}$*.*

Proof. We use the fact that $\mathrm{Inv}(G_2, 2)$ is a free $H(k)$-module with basis $\{1, e\}$ [158, Theorem 18.1]. Applying [158, Exercise 16.5], $\mathrm{Inv}(G_2 \times G_2, 2)$ is a free $H(k)$-module with basis $\{1, e', e'', r_6\}$. Since S_2 fixes 1 and r_6 and it swaps e' with e'', it follows that $\{1, r_3, r_6\}$ is an $H(k)$-basis for $\mathrm{Inv}(G_2 \times G_2, 2)^{S_2}$. □

The fibres of the map $H^1(k, G_2 \times G_2) \to H^1(k, (G_2 \times G_2) \rtimes S_2)$ are orbits of $H^1(k, G_2 \times G_2)$ by S_2; see 23.3–23.4. This means there is an $H(k)$-linear map

$$\mathrm{Inv}((G_2 \times G_2) \rtimes S_2, 2) \to \mathrm{Inv}(G_2 \times G_2, 2)^{S_2} \tag{35.2.1}$$

defined by restricting invariants to the image of $H^1(*, G_2 \times G_2)$ in $H^1(*, (G_2 \times G_2) \rtimes S_2)$.

35.3. *Cohomological invariants of* $\mathbf{PGL}_2 \times \mathbf{PGL}_2$. Consider the unique nontrivial normalised cohomological invariant $\delta \in \mathrm{Inv}^2(\mathbf{PGL}_2, 2)$, sending a quaternion algebra to its Brauer class. We can define invariants $r_2, r_4 \in \mathrm{Inv}(\mathbf{PGL}_2 \times \mathbf{PGL}_2, 2)$,

$$r_2(\alpha, \beta) = \delta(\alpha) + \delta(\beta), \qquad\qquad r_4(\alpha, \beta) = \delta(\alpha) \cdot \delta(\beta).$$

Like the case with G_2, it is clear that $\mathrm{Inv}(\mathbf{PGL}_2 \times \mathbf{PGL}_2, 2)$ is the free $H(k)$-module with basis $\{1, r_2, r_4\}$. There is an exceptional isomorphism

$$\mathbf{PGO}_4 \simeq (\mathbf{PGL}_2 \times \mathbf{PGL}_2) \rtimes S_2,$$

and a map

$$\mathrm{Inv}(\mathbf{PGO}_4, 2) \to \mathrm{Inv}(\mathbf{PGL}_2 \times \mathbf{PGL}_2, 2)^{S_2} \tag{35.3.1}$$

which is entirely analogous to (35.2.1).

35.4. Lemma. *The maps* (35.2.1) *and* (35.3.1) *are split surjective.*

Proof. Recall the definitions of b_1, b_3, b_6 from 23.7 and 23.10, and y_1, y_2, y_4 from 24.1. The first map (35.2.1) sends:

$$1 \mapsto 1, \qquad b_1 \mapsto 0, \qquad b_3 \mapsto r_3, \qquad b_6 \mapsto r_6.$$

It is surjective by Proposition 35.2, and clearly splits. The second map (35.3.1) sends

$$1 \mapsto 1, \qquad y_1 \mapsto 0, \qquad y_2 \mapsto r_2, \qquad y_4 \mapsto r_4$$

and it is just as obviously split surjective. □

The next step is to show that in fact the kernels of (35.2.1) and (35.3.1) are the ideals $H(k){\cdot}b_1$ and $H(k){\cdot}s_1$ respectively.

35.5. *Unramified elements in the cycle module of a quasitrivial torus.* An element of $H^1(k, (G_2 \times G_2) \rtimes S_2)$ in the fibre over a nonzero $[E] \in H^1(k, S_2)$ is an octonion algebra over E. It is specified by three parameters from the quasitrivial torus $R_{E/k}(\mathbf{G}_{m,E})$. We are compelled to consider natural maps $R_{E/k}(\mathbf{G}_{m,E})^r(*) \to H(*)$ of set-valued functors that are killed by $\mathrm{res}_{E/k}$. The next lemma will show that such a map is constant. The lemma is written for cycle modules (much more general than mod 2 Galois cohomology) because that is the right setting for it.

Incidentally, there is a classification in [120] of the natural maps

$$R_{E/k}(\mathbf{G}_m)^r(*) \to \bigoplus_{d \geq 0} H^d(*, \mathbb{Q}/\mathbb{Z}(d-1))$$

that are group homomorphisms, but for the problem at hand we cannot make that reduction and have to consider more than just the homomorphisms.

35.6. *Cycle modules.* Cycle modules were introduced by Rost [141] as a generalisation of, say, the Galois cohomology functor $\bigoplus_{d \geq 0} H^d(*, \mathbb{Q}/\mathbb{Z}(d-1))$, as well as the Milnor K-theory $K_*(*)$. A cycle module over a field F is a functor $M = \bigoplus_{d \in \mathbb{Z}} M_d$ from the category of finitely generated field extensions of F to the category of $\mathbb{Z}$-graded abelian groups, equipped with:

(i) norm homomorphisms $N_{L_1/L_2} : M(L_2) \to M(L_1)$, graded of degree 0, for every finite extension L_2/L_1 of finitely generated fields over F;

(ii) residue homomorphisms $\partial_v : M(L) \to M(E)$, graded of degree -1, for every extension L/F having a discrete valuation v with residue field E, such that v corresponds to the valuation at a codimension 1 point in a normal proper F-variety X with $L/F \simeq F(X)/F$;

(iii) a graded left $K_*(L)$-module structure on $M(L)$, for every finitely generated L/F.

In addition, there are a number of axioms which should be satisfied; see [141, §1–2] or [117, §2].

For a cycle module M over F, an equidimensional F-variety X, and an integer $d \geq 0$, we define the group

$$A^0(X, M_d) = \ker\Big(\bigoplus_{x \in X^{(0)}} M_d(F(x)) \xrightarrow{\partial} \bigoplus_{y \in X^{(1)}} M_{d-1}(F(y)) \Big).$$

If X is of dimension d_X, this group is $A_{d_X}(X, M, d - d_X)$ with the homological notation of [141, p. 356]. Here, $X^{(i)}$ stands for the set of codimension i points of X, and $F(x)$ is the residue field associated to x.

The map ∂ is defined as in [141, p. 337] or [117, p. 54–55]: for $x \in X^{(0)}$, the x, y-component ∂_x^y is trivial if $y \notin \overline{\{x\}}$ and otherwise it equals $\sum N_{F(v)/F(x)} \circ \partial_v$ where v ranges over the discrete valuations on $F(x)$ corresponding to points lying over y in the normal closure of $\overline{\{x\}}$.

If X is normal and irreducible then $X^{(0)}$ has only the generic point ξ and

$$A^0(X, M_d) = \bigcap_{x \in X^{(1)}} \ker \partial_x^\xi$$

is the subset of $M(F(X))$ which is "unramified at all irreducible divisors of X".

35.7. Lemma (Philippe Gille)**.** *Let E/F be a Galois field extension and consider the torus $T = R_{E/F}(\mathbf{G}_m)^r$ for $r \geq 1$. For each $d \geq 0$, we have an isomorphism*

$$\ker\Big(M_d(F) \to M_d(E)\Big) \xrightarrow{\sim} \ker\Big(A^0(T, M_d) \to A^0(T_E, M_d)\Big).$$

Proof. The idea is to construct a partial compactification $T \hookrightarrow U$ where U is open in some affine space X such that $U^{(1)} = X^{(1)}$. We put $n = [E : F]$ and consider the embedding of $T_0 = (\mathbf{G}_m)^{nr}$ in $U_0 = \mathbf{A}^{nr} \setminus \bigcup_{i<j} \{x_i = x_j\}$. We observe that the closed F-subvariety

$$Z_0 = \bigcup_i \{x_i = 0\} \setminus \bigcup_{i<j} \{x_i = x_j\}$$

of U_0 is isomorphic to $\bigsqcup_{i=1,\dots,nr} \mathbf{G}_m$. In particular, Z_0 is smooth and $U_0 \setminus Z_0 = T_0$.

This embedding is S_{nr}-equivariant and S_n-equivariant for the diagonal embedding $S_n \to S_{nr}$. Twisting by the S_n-torsor $P = \mathrm{Isom}(F^n, E)$ provides open embeddings $T \hookrightarrow U \hookrightarrow \mathbf{A}^r(E)$. Furthermore T is the complement of $Z = {}^P Z_0$. We have

$Z_0 = \coprod_{i=1,\ldots,nr} \mathbf{G}_m$ so that $Z = \coprod_{l=1,\ldots,r} \mathbf{G}_{m,E}$. In particular, Z is an E-variety. We consider the commutative diagram of long exact sequences (defined in [141, §5]):

$$\begin{array}{ccccccc} 0 \longrightarrow & A^0(U, M_d) & \longrightarrow & A^0(T, M_d) & \longrightarrow & A^0(Z, M_{d-1}) \\ & \downarrow & & \downarrow & & \downarrow \\ 0 \longrightarrow & A^0(U_E, M_d) & \longrightarrow & A^0(T_E, M_d) & \longrightarrow & A^0(Z_E, M_{d-1}). \end{array}$$

where the vertical maps are pull-backs for $U_E \to U$, $T_E \to T$, $Z_E \to Z$ respectively. The point is that the map $Z_E \to Z$ admits a splitting so that the right vertical map is injective. By diagram chase, we get an isomorphism

$$\ker\Big(A^0(U, M_d) \to A^0(U_E, M_d)\Big) \xrightarrow{\sim} \ker\Big(A^0(T, M_d) \to A^0(T_E, M_d)\Big).$$

Since U is an open subset of $\mathbf{A}^r(E)$ containing all its points of codimension 1, we have $A^0(\mathbf{A}^r(E), M_d) = A^0(U, M_d)$ and $A^0(\mathbf{A}^r(E)_E, M_d) = A^0(U_E, M_d)$. But $\mathbf{A}^r(E)$ is an affine space so $M_d(F) = A^0(\mathbf{A}^r(E), M_d)$ and $M_d(E) = A^0(\mathbf{A}^r(E)_E, M_d)$. Combining those identities, we get the desired isomorphism $\ker(M_d(F) \to M_d(E)) \simeq \ker\big(A^0(T, M_d) \to A^0(T_E, M_d)\big)$. $\square$

35.8. Lemma. *The kernels of the maps* (35.2.1) *and* (35.3.1) *are the ideals* $H(k){\cdot}b_1$ *and* $H(k){\cdot}s_1$ *respectively.*

Proof. Let $(G, r) = (\mathbf{PGL}_2, 2)$ or $(G_2, 3)$. Fix a quadratic field extension E/k. For any field L/k there is a surjection

$$H^1(L \otimes E, \boldsymbol{\mu}_2^r) \simeq H^1(L, R_{E/k}(\boldsymbol{\mu}_2)^r) \longrightarrow H^1(L, R_{E/k}(G)) \simeq H^1(L \otimes E, G)$$

taking parameters $(c_1, \ldots, c_r)$ to the composition algebra over $L \otimes E$ with norm $\langle\!\langle c_1, \ldots, c_r \rangle\!\rangle$. From this, we get an injective map (see Proposition 15.8 (i))

$$\mathrm{Inv}^d(R_{E/k}(G), 2) \hookrightarrow \mathrm{Inv}^d(R_{E/k}(\boldsymbol{\mu}_2)^r, 2)$$

for all $d \geq 0$. The torus $T = R_{E/k}(\mathbf{G}_m)^r$ is a classifying variety for $R_{E/k}(\boldsymbol{\mu}_2)^r$ in the exact sense of [116, §3], so there is an injective map

$$\mathrm{Inv}^d(R_{E/k}(\boldsymbol{\mu}_2)^r, 2) \hookrightarrow H^d(k(T), \mathbb{Z}/2\mathbb{Z}),$$

namely evaluation at the generic torsor

$$Y_\xi \in H^1(k(T), R_{E/k}(\boldsymbol{\mu}_2)^r).$$

Cohomological invariants of $R_{E/k}(\boldsymbol{\mu}_2)^r$ are unramified at every codimension 1 point of T [158, Theorem 11.7], which means that their values at the generic torsor are contained in $A^0(T, H^d(*, \mathbb{Z}/2\mathbb{Z}))$. In summary, we have an injective map

$$\begin{array}{rcl} \mathrm{Inv}^d(R_{E/k}(G), 2) & \hookrightarrow & A^0(T, H^d(*, \mathbb{Z}/2\mathbb{Z})) \subset H^d(k(T), \mathbb{Z}/2\mathbb{Z}). \\ b & \mapsto & b(Y_\xi) \end{array}$$

Note that $R_{E/k}(G)_E = G^2$ (as an algebraic group over E) and consider the commutative diagram, where the left vertical arrow is just precomposition with the forgetful functor $\mathsf{Fields}_{/E} \to \mathsf{Fields}_{/k}$:

$$\begin{array}{ccc} \mathrm{Inv}^d\left(R_{E/k}(G),2\right)^{S_2} & \hookrightarrow & A^0(T, H^d(*,\mathbb{Z}/2\mathbb{Z})) \\ \downarrow & & \downarrow \\ \mathrm{Inv}^d_E\left(G^2,2\right)^{S_2} & \hookrightarrow & A^0(T_E, H^d(*,\mathbb{Z}/2\mathbb{Z})). \end{array}$$

If $b \in \ker(\mathrm{Inv}^d(G^2 \rtimes S_2, 2) \to \mathrm{Inv}^d(G^2,2)^{S_2})$, then $b(Y_\xi)$ is in the kernel of

$$A^0(T, H^d(*,\mathbb{Z}/2\mathbb{Z})) \to A^0(T_E, H^d(*,\mathbb{Z}/2\mathbb{Z})).$$

By Lemma 35.7, $b(Y_\xi)$ is a constant from $H^d(k,\mathbb{Z}/2\mathbb{Z})$. This in turn implies that b is locally constant, by which we mean: constant on the fibres (see 23.5)

$$\frac{H^1(*, R_{E/k}(G))}{S_2} \subset H^1(*, G^2 \rtimes S_2)$$

for all E/k. And indeed b is constantly zero on $H^1(*,G^2)/S_2$. So, b factors through the map $H^1(*,G^2 \rtimes S_2) \to H^1(*,S_2)$. The only normalised invariants in $\mathrm{Inv}(S_2,2)$ are cup products with the identity [158, Proposition 16.2], so this shows there is a unique $\lambda \in H^{d-1}(k,\mathbb{Z}/2\mathbb{Z})$ such that $b = \lambda{\cdot}b_1$ (if $G = G_2$), or $b = \lambda{\cdot}s_1$ (if $G = \mathbf{PGL}_2$). □

35.9. Theorem.

(i) $\mathrm{Inv}(\mathbf{PGO}_4, 2)$ *is a free* $H(k)$*-module with basis* $\{1, y_1, y_2, y_4\}$.

(ii) $\mathrm{Inv}((G_2 \times G_2) \rtimes S_2, 2)$ *is a free* $H(k)$*-module with basis* $\{1, b_1, b_3, b_6\}$.

Proof. We have shown in Lemmas 35.4 and 35.8 that the sequence

$$0 \longrightarrow \mathrm{Inv}(S_2,2) \longrightarrow \mathrm{Inv}((G_2 \times G_2) \rtimes S_2, 2) \longrightarrow \mathrm{Inv}(G_2 \times G_2, 2)^{S_2} \longrightarrow 0$$

is split exact, so $\mathrm{Inv}((G_2 \times G_2) \rtimes S_2, 2) \simeq \mathrm{Inv}(S_2,2) \oplus \mathrm{Inv}(G_2 \times G_2, 2)^{S_2}$. Keeping track of the isomorphism shows that $\{1, b_1, b_3, b_6\}$ is the basis for $\mathrm{Inv}((G_2 \times G_2) \rtimes S_2, 2)$ that we get from this splitting. The proof of (i) is identical with G_2 replaced by $\mathbf{PGL}_2$, and it yields the basis $\{1, y_1, y_2, y_4\}$. □

36. Fibrations

Fibrations have often been used for calculating the essential dimensions of algebraic groups, including groups like $\mathbf{Spin}_n$ and $\mathbf{\Gamma}_n^+$ [26, 37, 41]. The concept turns out to extremely useful for cohomological invariants.

36.1. Definition. Let $F : \mathsf{Fields}_{/k} \to \mathsf{Groups}$ be a functor. Let A, B be functors $\mathsf{Fields}_{/k} \to \mathsf{Sets}$ and $\pi : A \to B$ a surjective morphism of functors, meaning that $\pi_L : A(L) \to B(L)$ is surjective for all L/k. A *fibration* of π by F is an action of F on A such that for all fields L/k,

(i) π_L is $F(L)$-equivariant with respect to the trivial action of $F(L)$ on $B(L)$;

(ii) $F(L)$ acts transitively on each fibre of π_L.

The meaning of (i) and (ii) is that the orbits of $F(L)$ are precisely the fibres of π_L; this is why it is called a fibration. It is standard to denote a fibration by

$$F \rightsquigarrow A \xrightarrow{\pi} B.$$

36.2. Example. In quadratic form theory, for all $j, n \geq 0$ with j even there is an obvious fibration

$$H^1(*, \boldsymbol{\mu}_2) \rightsquigarrow I_j^n \longrightarrow\!\!\!\!\!\!\rightarrow PI_j^n \tag{36.2.1}$$

where $H^1(L, \boldsymbol{\mu}_2) = L^\times/L^{\times 2}$ acts on $I_j^n(L)$ by $cL^{\times 2} \cdot q = \langle c \rangle q$.

36.3. Lemma. *An exact sequence of algebraic groups*

$$1 \longrightarrow A \longrightarrow X \longrightarrow Y \longrightarrow 1,$$

with A a central subgroup of X, yields a fibration

$$H^1(*, A) \rightsquigarrow H^1(*, X) \xrightarrow{\pi} B(*)$$

where $B(L)$ is the image of $H^1(L, X)$ in $H^1(L, Y)$.

Proof. The action of the group $H^1(*, A)$ on $H^1(*, X)$ is just pointwise multiplication on the level of cocycles, and [156, I.§5 Proposition 42] says that this is a fibration. □

The following proposition resembles [69, Propositions 7.1 & 7.4], but we are working more generally here.

36.4. Proposition. *Let* $H^1(*, \boldsymbol{\mu}_{2m}) \rightsquigarrow A \xrightarrow{\pi} B$ *be a fibration.*

(i) *For each $a \in \mathrm{Inv}^d(A, 2)$, there is a unique invariant $\bar{\partial} a$ in the image of π^* : $\mathrm{Inv}^{d-1}(B, 2) \to \mathrm{Inv}^{d-1}(A, 2)$ such that*

$$a(tL^{\times 2m} \cdot x) - a(x) = (t) \cdot \bar{\partial} a(x)$$

for all field extensions L/k, $t \in L^\times$, and $x \in A(L)$.

(ii) *Let $\partial a \in \mathrm{Inv}^{d-1}(B, 2)$ be the unique invariant such that $\pi^*(\partial a) = \bar{\partial} a$. Then $\partial : \mathrm{Inv}^d(A, 2) \to \mathrm{Inv}^{d-1}(B, 2)$ is a homomorphism and the following sequence is exact*

$$0 \longrightarrow \mathrm{Inv}^d(B, 2) \xrightarrow{\pi^*} \mathrm{Inv}^d(A, 2) \xrightarrow{\partial} \mathrm{Inv}^{d-1}(B, 2).$$

(iii) *Suppose* $H^1(*, \boldsymbol{\mu}_{2m}) \rightsquigarrow A' \xrightarrow{\pi'} B'$ *is another fibration and there are morphisms $f : A \to A'$ and $g : B \to B'$ such that $\pi' \circ f = g \circ \pi$ and $f : A(L) \to A'(L)$ is $L^\times/L^{\times 2m}$-equivariant for all fields L/k. Then the following diagram commutes:*

$$\begin{array}{ccc} \mathrm{Inv}^d(A', 2) & \xrightarrow{\partial} & \mathrm{Inv}^{d-1}(B', 2) \\ \Big\downarrow{\scriptstyle f^*} & & \Big\downarrow{\scriptstyle g^*} \\ \mathrm{Inv}^d(A, 2) & \xrightarrow{\partial} & \mathrm{Inv}^{d-1}(B, 2) \end{array}$$

We shall call ∂ the *residue map* with respect to the fibration, and we say that $\partial a \in \mathrm{Inv}^{d-1}(B,2)$ is the residue of $a \in \mathrm{Inv}^d(A,2)$.

Proof. (i) Let $a \in \mathrm{Inv}^d(A,2)$, $x \in A(k)$. We have a normalised invariant in $\mathrm{Inv}^d(\boldsymbol{\mu}_{2m},2)$:

$$tL^{\times 2m} \mapsto a(tL^{\times 2m}\cdot x) - a(x) \qquad \text{for all } t \in L^\times, L/k.$$

By [58, Proposition 2.5], there is a unique element $\bar\partial a(x) \in H^{d-1}(k,\mathbb{Z}/2\mathbb{Z})$ such that

$$a(tL^{\times 2}\cdot x) - a(x) = (t)\cdot \bar\partial a(x)$$

for all field extensions L/k and $t \in L^\times$. Clearly $\bar\partial a$ is a cohomological invariant in $\mathrm{Inv}^{d-1}(A,2)$. Now we claim that $\bar\partial a$ comes from $\mathrm{Inv}^{d-1}(B,2)$; that is, if $\pi(x) = \pi(x')$ then $\bar\partial a(x) = \bar\partial a(x')$. Since $L^\times/L^{\times 2m}$ acts transitively on the fibres of π, we have $x' = rL^{\times 2m}\cdot x$ for some $r \in L^\times$. Then

$$\begin{aligned}(t)\cdot\bar\partial a(x') &= a(tL^{\times 2m}\cdot x') - a(x')\\ &= a(trL^{\times 2m}\cdot x) - a(rL^{\times 2m}\cdot x) = (tr)\cdot\bar\partial a(x) - (r)\cdot\bar\partial a(x) = (t)\cdot\bar\partial a(x).\end{aligned}$$

Therefore $\bar\partial a(x) = \bar\partial a(x')$ by uniqueness.

(ii) The sequence is exact at $\mathrm{Inv}^d(B,2)$ simply because π is surjective. For exactness at $\mathrm{Inv}^d(A,2)$, it is clear that $\partial a = 0$ if and only if a is constant on the fibres of $\pi : A(L) \to B(L)$ for all fields L/k, which means a is the image of an invariant in $\mathrm{Inv}^d(B,2)$.

(iii) It suffices to show that $\bar\partial\circ f^* = f^*\circ\bar\partial$. Suppose $a = f^*(a') = a'\circ f \in \mathrm{Inv}^d(A,2)$. Then for all fields L/k, $x \in A(L,2)$, and $t \in L^\times$:

$$\begin{aligned}(t)\cdot\bar\partial a'(f(x)) &= a'(tL^{\times 2m}\cdot f(x)) - a'(f(x))\\ &= a'(f(tL^{\times 2m}\cdot x)) - a'(f(x))\\ &= a(tL^{\times 2m}\cdot x) - a(x) = (t)\cdot\bar\partial a(x).\end{aligned}$$

By uniqueness, $f^*(\bar\partial a') = \bar\partial a'\circ f = \bar\partial a = \bar\partial f^*(a')$. □

37. Invariants of I^3_{12} and $\mathbf{Spin}_{12}$

Garibaldi in [58, §20] showed that $\mathrm{Inv}(I^3_{12},2) \simeq \mathrm{Inv}(\mathbf{Spin}_{12},2)$ is a free $H(k)$-module with generators in degrees 0, 3, 5, 6, under the rather strong assumption that $\sqrt{-1} \in k$. We shall revisit the classification to remove that assumption.

37.1. *Known invariants of I^3_{12}.* There are several nontrivial invariants in $\mathrm{Inv}(I^3_{12},2)$ whose existence is established in [58, §22.3]. The first nontrivial invariant is

$$z_3(q) = e_3(q) \in H^3(k,\mathbb{Z}/2\mathbb{Z}).$$

This is just the restriction of the Arason invariant $e_3 : I^3 \to H^3(*,\mathbb{Z}/2\mathbb{Z})$ to 12-dimensional forms. The next nontrivial invariant is defined as

$$z_5(q) = e_5(\langle\langle c\rangle\rangle P_2(r)) = (c)\cdot e_4(P_2(r)) \in H^5(k,\mathbb{Z}/2\mathbb{Z}),$$

where $q = \langle\langle c\rangle\rangle r$ is a factorisation of q such that $c \in k^\times$ and $r \in I^2_6(k)$. This makes sense because Pfister's Theorem (see 21.6) shows that such a factorisation always exists, because (16.9.6) shows that $P_2(r) \in I^4(k)$, and because [58, Corollary 20.7] shows that the class of $\langle\langle c\rangle\rangle P_2(r) \in I^5(k)$ does not depend on the way of factorising q. Using the parameterisation (21.6.1) of quadratic forms in $I^3_{12}(k)$, we can write down the values taken by these invariants:

37.2. Lemma. *If $q = \langle d\rangle\langle\langle c\rangle\rangle(\psi_1' \perp \langle -1\rangle\psi_2') \in I^3_{12}(k)$ where $\psi_i = \langle\langle x_i, y_i\rangle\rangle$ and $c, d, x_i, y_i \in k^\times$, then:*

$$\begin{aligned} z_3(q) &= (c)\cdot e_2(\psi_1) + (c)\cdot e_2(\psi_2) \\ &= (c)\cdot(x_1)\cdot(y_1) + (c)\cdot(x_2)\cdot(y_2). \\ z_5(q) &= (c)\cdot e_2(\psi_1)\cdot e_2(\psi_2) + (-1)\cdot(c)\cdot(d)\cdot e_2(\psi_1) + (-1)\cdot(c)\cdot(-d)\cdot e_2(\psi_2) \\ &= (c)\cdot(x_1)\cdot(y_1)\cdot(x_2)\cdot(y_2) + (-1)\cdot(c)\cdot(d)\cdot(x_1)\cdot(y_1) + (-1)\cdot(c)\cdot(-d)\cdot(x_2)\cdot(y_2). \end{aligned}$$

Proof. The form q is Witt equivalent to $q \perp \mathbb{H} = \langle d\rangle(\langle\langle c\rangle\rangle\psi_1 \perp \langle -1\rangle\langle\langle c\rangle\rangle\psi_2)$. The invariant z_3 is the restriction of the Arason invariant, so $z_3(q) = e_3(q) = e_3(\langle\langle c\rangle\rangle\psi_1) + e_3(\langle\langle c\rangle\rangle\psi_2)$ and the first formula is clear. The invariant z_5 is defined so that $z_5(q) = (c)\cdot e_4(P_2(\langle d\rangle\psi_1 + \langle -d\rangle\psi_2))$ and one can derive the second formula using either (16.9.5) or [58, Example 20.9]. □

37.3. Lemma. *If $q \in I^3_{12}(k)$ is isotropic, $z_5(q)$ is a symbol in $(-1)\cdot H^4(k, \mathbb{Z}/2\mathbb{Z})$.*

Proof. If $q \in I^3_{12}(k)$ is isotropic, then it is isometric to some $p \perp \mathbb{H}$ where $p \in I^3_{10}(k)$. But, as it is well-known [79, Theorem 2.1], every form in $I^3_{10}(k)$ is isotropic and similar to a 3-Pfister form, so we can write $q = \langle d\rangle\langle\langle c\rangle\rangle(\langle\langle x, y\rangle\rangle' \perp \langle 1, -1, 1\rangle)$ for some $d, x, y, z \in k^\times$. Hence by Lemma 37.2, $z_5(q) = (-1)\cdot(c)\cdot(d)\cdot(x)\cdot(y)$. □

Given the lemma above, $h\cdot z_5$ vanishes on isotropic forms for all $h \in J_1(k)$ (the ideal defined in 16.2). We can use Rost's technique [58, Proposition 10.2] to define a set of invariants $\{z^h : h \in J_1(k)\}$ by

$$z^h(q) = h\cdot(q(v))\cdot z_5(q)$$

where $v \in k^{12}$ is any anisotropic vector for q. If q is as in Lemma 37.2, then q represents $-dx_1$, but $(-dx_1)\cdot(x_1) = (d)\cdot(x_1)$, hence for all $h \in J_1(k)$

$$z^h(q) = h\cdot(-dx_1)\cdot z_5(q) = h\cdot(d)\cdot(c)\cdot(x_1)\cdot(y_1)\cdot(x_2)\cdot(y_2). \tag{37.3.1}$$

If $J_1(k) = H(k)$ then $z^h = h\cdot z^1$ for all $h \in H(k)$. We write $z_6 = z^1$; this is the invariant from [58, §20.13].

37.4. Lemma. *Let $\partial' : \mathrm{Inv}(I^3_{12}, 2) \to \mathrm{Inv}(PI^3_{12}, 2)$ be the residue map associated to the fibration $H^1(*, \boldsymbol{\mu}_2) \rightsquigarrow I^3_{12} \twoheadrightarrow PI^3_{12}$ described in Example (36.2). We have*

$$\begin{aligned} \partial' z_3 &= 0 \\ \partial' z_5 &= (-1)\cdot z_3 \\ \partial' z^h &= h\cdot z_5 && \textit{for all } h \in J_1(k). \end{aligned}$$

Proof. Since z_3 is in the image of $\mathrm{Inv}(PI^3_{12},2) \hookrightarrow \mathrm{Inv}(I^3_{12},2)$, Proposition 36.4 (ii) implies $\partial' z_3 = 0$. If $q = \langle d\rangle\langle\!\langle c\rangle\!\rangle(\psi'_1 \perp \langle -1\rangle\psi'_2) \in I^3_{12}(L)$ then we can reconcile using Lemma 37.2 that for all $b \in L^\times$,

$$\begin{aligned} z_5(\langle b\rangle q) - z_5(q) &= (-1)\cdot(c)\cdot\big((bd)-(d)\big)\cdot e_2(\psi_1) + (-1)\cdot(c)\cdot\big((-bd)-(-d)\big)\cdot e_2(\psi_2) \\ &= (-1)\cdot(c)\cdot(b)\cdot e_2(\psi_1) + (-1)\cdot(c)\cdot(b)\cdot e_2(\psi_2) \\ &= (b)\cdot(-1)\cdot z_3(q) \end{aligned}$$

If $h \in J_1(k)$ then $h\cdot(-1) = 0$ so $h\cdot z_5(\langle b\rangle q) = h\cdot z_5(q)$. Taking an anisotropic vector v,

$$z^h(\langle b\rangle q) - z^h(q) = h\cdot(\langle b\rangle q(v))\cdot z_5(\langle b\rangle q) - h\cdot(q(v))\cdot z_5(q) = (b)\cdot h\cdot z_5(q),$$

hence $\partial' z^h = h\cdot z_5$. □

There are general methods for determining the invariants of a direct product, for instance [58, Lemma 6.7], but they require that the factors have free modules of invariants. In the absence of freeness, for instance when the group is $\mathbf{O}^+_{4\ell+2}$ and $-1 \notin k^{\times 2}$, we can use the following lemma.

37.5. Lemma. *Let G be an algebraic group and identify* $\mathrm{Inv}(G,2)$ *naturally with its image in* $\mathrm{Inv}(\boldsymbol{\mu}_{2m} \times G, 2)$. *Then* $\mathrm{Inv}(\boldsymbol{\mu}_{2m} \times G,2) = \mathrm{Inv}(G,2) \oplus s\cdot \mathrm{Inv}(G,2)$ *where* $s \in \mathrm{Inv}(\boldsymbol{\mu}_{2m} \times G, 2)$ *is the invariant:*

$$(cL^{\times 2m}, \zeta) \mapsto (c) \in H^1(L, \mathbb{Z}/2\mathbb{Z})$$

for all field extensions L/k, $\zeta \in H^1(L,G)$, and $c \in L^\times$.

Proof. The fibration $H^1(*,\boldsymbol{\mu}_{2m}) \rightsquigarrow H^1(*,\boldsymbol{\mu}_{2m}\times G) \twoheadrightarrow H^1(*,G)$ associated to the short exact sequence $\boldsymbol{\mu}_{2m} \to \boldsymbol{\mu}_{2m} \times G \to G$ induces the exact sequence of $H(k)$-modules

$$0 \longrightarrow \mathrm{Inv}(G,2) \xrightarrow{\pi^*} \mathrm{Inv}(\boldsymbol{\mu}_{2m} \times G,2) \longrightarrow \mathrm{Inv}(G,2).$$

The residue map $\mathrm{Inv}(\boldsymbol{\mu}_{2m} \times G, 2) \to \mathrm{Inv}(G,2)$ admits a section $a \mapsto s\cdot j^*(a)$ where $j : G \to \boldsymbol{\mu}_{2m} \times G$ is the natural inclusion, so we are done. □

37.6. *A surjection onto I^3_{12}.* Because of Pfister's Theorem on 12-dimensional quadratic forms in I^3 (see 21.6), we have for all fields L/k a surjective map

$$\begin{aligned} H^1(L, \boldsymbol{\mu}_2 \times \mathbf{O}^+_6) = (L^\times/L^{\times 2}) \times I^2_6(L) &\longrightarrow I^3_{12}(L) = H^1(L, \boldsymbol{\Gamma}^+_{12}) \\ (cL^{\times 2}, r) &\longmapsto \langle\!\langle c\rangle\!\rangle r. \end{aligned}$$

Consequently, there is an injective homomorphism

$$\mathrm{Inv}(I^3_{12},2) = \mathrm{Inv}(\boldsymbol{\Gamma}^+_{12},2) \hookrightarrow \mathrm{Inv}(\boldsymbol{\mu}_2 \times \mathbf{O}^+_6, 2) = \mathrm{Inv}(\mathbf{O}^+_6,2) \oplus s\cdot \mathrm{Inv}(\mathbf{O}^+_6,2)$$

where $s(cL^{\times 2}, r) = (c)$ for all $c \in L^\times$, $r \in H^1(L,\mathbf{O}^+_6)$. By Serre's Theorem 16.4, the right-hand side is a direct sum of the free module generated by $\{1, w_2, w_4, s, s\cdot w_2, s\cdot w_4\}$ and the module $\{b^h : h \in J_1(k)\} \oplus \{s\cdot b^h : h \in J_1(k)\}$. To determine $\mathrm{Inv}(I^3_{12},2)$, it remains to check which of these invariants is in the image of $\mathrm{Inv}(I^3_{12}, 2)$. Both compositions $H^1(k, \mathbf{O}_6^+) \to H^1(k, \boldsymbol{\mu}_2 \times \mathbf{O}_6^+) \to I^3_{12}(L)$ and $H^1(k, \boldsymbol{\mu}_2) \to H^1(k, \boldsymbol{\mu}_2 \times \mathbf{O}_6^+) \to I^3_{12}(L)$ are the zero maps, so the image of $\mathrm{Inv}(I^3_{12}, 2)$ is contained in the proper submodule

$$H(k){\cdot}s{\cdot}w_2 \oplus H(k){\cdot}s{\cdot}w_4 \oplus \{s{\cdot}b^h \colon h \in J_1(k)\} \subset \mathrm{Inv}(\boldsymbol{\mu}_2 \times \mathbf{O}_6^+, 2). \tag{37.6.1}$$

The following technical lemma settles the matter: this submodule is the image of $\mathrm{Inv}(I^3_{12}, 2)$.

37.7. Lemma. *If* $q = \langle\!\langle c\rangle\!\rangle r$ *where* $r = \langle d\rangle(\psi_1' \perp \langle -1\rangle\psi_2')$, $\psi_i = \langle\!\langle x_i, y_i\rangle\!\rangle$, *and* $c, d, x_i, y_i \in k^\times$, *then*

$$\begin{aligned}
(c){\cdot}w_2(r) &= z_3(q), \\
(c){\cdot}w_4(r) &= z_5(q) + (-1){\cdot}(-1){\cdot}z_3(q), \\
(c){\cdot}b^h(r) &= z^h(q) \qquad\qquad \textit{for all } h \in J_1(k).
\end{aligned}$$

Proof. We have $w_2(r) = e_2(r)$ if the Witt class of r is in $I^2(k)$ [53, p. 31] so $(c){\cdot}w_2(r) = (c){\cdot}e_2(r) = e_3(\langle\!\langle c\rangle\!\rangle d) = z_3(q)$, hence the first identity. Towards the second identity, (16.1.1) implies

$$\begin{aligned}
w_4(r) = w_1(\langle d\rangle\psi_1'){\cdot}w_3(\langle -d\rangle\psi_2') \\
+ w_2(\langle d\rangle\psi_1'){\cdot}w_2(\langle -d\rangle\psi_2') + w_3(\langle d\rangle\psi_1'){\cdot}w_1(\langle -d\rangle\psi_2').
\end{aligned} \tag{37.7.1}$$

Now, $w_1(\langle d\rangle\psi_1') = w_1(\langle -dx_1, -dy_1, dx_1y_1\rangle) = (d)$ and $w_1(\langle -d\rangle\psi_2') = (-d)$. Further,

$$e_2(\langle d\rangle\psi_1) = w_2(\langle d\rangle \perp \langle d\rangle\psi_1') = w_2(\langle d\rangle\psi_1') + w_1(\langle d\rangle){\cdot}w_1(\langle d\rangle\psi_1') = w_2(\langle d\rangle\psi_1') + (d){\cdot}(d)$$

hence $w_2(\langle d\rangle\psi_1') = e_2(\langle d\rangle\psi_1) + (d){\cdot}(d) = e_2(\psi_1) + (-1){\cdot}(d)$. Clearly then $w_2(\langle -d\rangle\psi_2') = e_2(\psi_2) + (-1){\cdot}(-d)$. After some basic manipulations with identities $(ab) = (a) + (b)$ and $(a)(-a) = 0$,

$$w_3(\langle d\rangle\psi_1') = (-dx_1){\cdot}(-dy_1){\cdot}(dx_1y_1) = (x_1){\cdot}(y_1){\cdot}(d), \quad w_3(\langle -d\rangle\psi_2') = (x_2){\cdot}(y_2){\cdot}(-d).$$

Plugging these calculations into (37.7.1), the first and third terms vanish and we are left with

$$\begin{aligned}
w_4(r) &= [e_2(\psi_1) + (-1){\cdot}(d)]{\cdot}[e_2(\psi_2) + (-1){\cdot}(-d)] \\
&= e_2(\psi_1){\cdot}e_2(\psi_2) + (-1){\cdot}(-d){\cdot}e_2(\psi_1) + (-1){\cdot}(d){\cdot}e_2(\psi_2).
\end{aligned}$$

Comparing with Lemma 37.2, we prove the second identity:

$$\begin{aligned}
(c){\cdot}w_4(r) - z_5(q) &= (-1){\cdot}(c){\cdot}e_2(\psi_1){\cdot}[(-d) + (d)] + (-1){\cdot}(c){\cdot}e_2(\psi_2){\cdot}[(d) + (-d)] \\
&= (-1){\cdot}(-1){\cdot}(c){\cdot}[e_2(\psi_1) + e_2(\psi_2)] = (-1){\cdot}(-1){\cdot}z_3(q).
\end{aligned}$$

We prove the third identity by comparing the following calculation with (37.3.1):

$$\begin{aligned}
b^h(r) &= h{\cdot}(-dx_1){\cdot}(-dy_1){\cdot}(dx_1y_1){\cdot}(dx_2){\cdot}(dy_2) \\
&= w_3(\langle d\rangle\psi_1'){\cdot}(dx_2){\cdot}(dy_2) = h{\cdot}(x_1){\cdot}(y_1){\cdot}(d){\cdot}[(d){\cdot}(-x_2y_2) + (x_2){\cdot}(y_2)] \\
&= h{\cdot}(-1){\cdot}(x_1){\cdot}(y_1){\cdot}(d){\cdot}(-x_2y_2) + h{\cdot}(d){\cdot}(x_1){\cdot}(y_1){\cdot}(x_2){\cdot}(y_2) \\
&= h{\cdot}(d){\cdot}(x_1){\cdot}(y_1){\cdot}(x_2){\cdot}(y_2).
\end{aligned}$$

□

37.8. Theorem. *The natural inclusion* $\mathrm{Inv}(I^3_{12},2) \to \mathrm{Inv}(\mathbf{Spin}_{12},2)$ *is an isomorphism, and* $\mathrm{Inv}(I^3_{12},2)$ *is a direct sum of the free* $H(k)$*-module with basis* $\{1,z_3,z_5\}$ *and the* $H(k)$*-module* $\{z^h : h \in J_1(k)\} \simeq J_1(k)$.

Proof. The classification of invariants of I^3_{12} is already completed by 37.6 and 37.7. So it remains to show that $\mathrm{Inv}(I^3_{12},2) \simeq \mathrm{Inv}(\mathbf{Spin}_{12},2)$.

There is an inclusion $\boldsymbol{\mu}_4 \times \mathbf{O}^+_6 \subset \mathbf{Spin}_{12}$ such that $H^1(L,\boldsymbol{\mu}_4 \times \mathbf{O}^+_6) \to H^1(L,\mathbf{Spin}_{12})$ is surjective for all fields L/k [58, Example 17.12]. The composition of this map with $H^1(L,\mathbf{Spin}_{12}) \to H^1(L,\mathbf{O}^+_{12})$ is $(cL^{\times 4}, r) \mapsto \langle\langle c\rangle\rangle r$. In other words, it is just an extension of the surjection from 37.6. There are two fibrations (the horizontal lines) connected by the canonical surjective maps, denoted here by S and T,

$$\begin{tikzcd}
H^1(*,\boldsymbol{\mu}_2) \arrow[r, rightsquigarrow] & H^1(*,\boldsymbol{\mu}_4 \times \mathbf{O}^+_6) \arrow[r, twoheadrightarrow] \arrow[d, "S"] & H^1(*,\boldsymbol{\mu}_2 \times \mathbf{O}^+_6) \arrow[d, "T"] \\
H^1(*,\boldsymbol{\mu}_2) \arrow[r, rightsquigarrow] & H^1(*,\mathbf{Spin}_{12}) \arrow[r, twoheadrightarrow] & I^3_{12}(*)
\end{tikzcd}$$

such that S_L is $L^\times/L^{\times 2}$-equivariant for all fields L/k, and the square formed by S,T, and the horizontal arrows is commutative.

By Lemma 37.5, $\mathrm{Inv}(\boldsymbol{\mu}_2 \times \mathbf{O}^+_6,2) \to \mathrm{Inv}(\boldsymbol{\mu}_4 \times \mathbf{O}^+_6,2)$ is an isomorphism. By Proposition 36.4 (ii) the residue $\partial : \mathrm{Inv}(\boldsymbol{\mu}_4 \times \mathbf{O}^+_6,2) \to \mathrm{Inv}(\boldsymbol{\mu}_2 \times \mathbf{O}^+_6,2)$ associated to the first fibration is zero. Since S^* is injective, Proposition 36.4 (iii) implies the residue $\partial : \mathrm{Inv}(\mathbf{Spin}_{12},2) \to \mathrm{Inv}(I^3_{12},2)$ associated to the second fibration is also zero. In turn, Proposition 36.4 (ii) implies the theorem. □

For an alternative proof that $\mathrm{Inv}(I^3_{12},2) \simeq \mathrm{Inv}(\mathbf{Spin}_{12},2)$, one can work out the image of the embedding $\mathrm{Inv}(\mathbf{Spin}_{12},2) \to \mathrm{Inv}(\boldsymbol{\mu}_4 \times \mathbf{O}^+_6,2) \simeq \mathrm{Inv}(\boldsymbol{\mu}_2 \times \mathbf{O}^+_6,2)$. The submodule (37.6.1) is a lower bound because we proved that this is the image of $\mathrm{Inv}(I^3_{12},2) \to \mathrm{Inv}(\boldsymbol{\mu}_2 \times \mathbf{O}^+_6,2)$. It is also an upper bound, because the compositions $H^1(k,\mathbf{O}^+_6) \to H^1(k,\boldsymbol{\mu}_4 \times \mathbf{O}^+_6) \to H^1(k,\mathbf{Spin}_{12})$ and $H^1(k,\boldsymbol{\mu}_4) \to H^1(k,\boldsymbol{\mu}_4 \times \mathbf{O}^+_6) \to H^1(k,\mathbf{Spin}_{12})$ are the trivial maps.

37.9. Corollary. *The image of the homomorphism* $\mathrm{Inv}(PI^3_{12},2) \hookrightarrow \mathrm{Inv}(I^3_{12},2)$ *is*

$$H(k)\cdot 1 \oplus H(k)\cdot z_3 \oplus \{h\cdot z_5 \colon h \in J_1(k)\}.$$

Proof. By Proposition 36.4 (ii) the image is $\ker(\partial')$, which is already known from Lemma 37.4. □

38. Invariants of I^3_{14} and $\mathbf{Spin}_{14}$

There are three nontrivial invariants of I^3_{14} that are known from [58, §22.3]. These occur in degrees 3, 6, and 7, but the degree 7 invariant is only known to exist for fields k in which $\sqrt{-1} \in k$.

38.1. *Known invariants of* I^3_{14}. The degree 3 invariant, which we denote by $a_3 \in \mathrm{Inv}(I^3_{14},2)$, is the restriction of the Arason invariant $e_3 : I^3(*) \to H^3(*,\mathbb{Z}/2\mathbb{Z})$ to 14-dimensional forms.

The degree 6 invariant is denoted by $a_6 \in \mathrm{Inv}(I^3_{14},2)$ and it is the restriction of $e_6 \circ P_3 : I^3(*) \to H^6(*,\mathbb{Z}/2\mathbb{Z})$, where $P_3 : I^3(*) \to I^6(*)$ is the functor defined in 16.9.

The third invariant (defined as long as $\sqrt{-1} \in k$) is denoted by $a_7 \in \mathrm{Inv}(I^3_{14}, 2)$ and for a quadratic form $Q \in I^3_{14}(k)$ it takes the value

$$a_7(Q) = (Q(v))\cdot a_6(Q)$$

where $v \in k^{14}$ is any anisotropic vector for Q. When $Q \in I^3_{14}$ is similar to the difference of two 3-Pfister forms, as in Corollary 21.3 (1), it is easy to give an expression for $a_6(Q)$:

38.2. Lemma. *If $Q = \langle c\rangle(\phi_1' \perp \langle -1\rangle\phi_2')$ where ϕ_i are 3-Pfister forms over k, then*

$$a_6(Q) = e_3(\phi_1)\cdot e_3(\phi_2) + (-1)\cdot(-1)\cdot(c)\cdot e_3(\phi_1) + (-1)\cdot(-1)\cdot(-c)\cdot e_3(\phi_2).$$

Proof. By (16.9.5), $P_3(Q) = \langle -1\rangle\phi_1\phi_2 + 4\langle\langle c\rangle\rangle\phi_1 + 4\langle\langle -c\rangle\rangle\phi_2$ and the lemma follows by applying e_6 to this expression. □

Provided -1 is a sum of two squares in k, we can also express $a_6(Q)$ quite easily for any $Q \in I^3_{14}(k)$, and this is done in (38.6.1) and (38.6.2), but in full generality it is hard. The following lemma generalises [58, Proposition 22.2 (2)]. Unfortunately [58, Proposition 22.2 (1)] has a flaw in it: Lemma 38.2 implies that the isotropic form $Q = \langle\langle -1, t_1, t_2\rangle\rangle' \perp \langle -1\rangle\langle\langle -1, t_3, t_4\rangle\rangle'$ over $\mathbb{R}(t_1, t_2, t_3, t_4)$ evaluates to a nonsymbol:

$$a_6(Q) = (-1)\cdot(-1)\cdot(t_1)\cdot(t_2)\cdot(t_3)\cdot(t_4) + (-1)\cdot(-1)\cdot(-1)\cdot(-1)\cdot(t_3)\cdot(t_4).$$

38.3. Lemma. *If $Q \in I^3_{14}(k)$ is isotropic then $a_6(Q) \in (-1)\cdot H^5(k, \mathbb{Z}/2\mathbb{Z})$.*

Proof. An isotropic $Q \in I^3_{14}(k)$ is of the form $Q = \langle c\rangle(\phi_1' \perp \langle -1\rangle\phi_2')$ where the $\phi_i = \langle\langle x, y_i, z_i\rangle\rangle$ have a common slot (21.6). By Lemma 38.2, $a_6(Q) = e_3(\phi_1)\cdot e_3(\phi_2)$ modulo $(-1)\cdot H^5(k, \mathbb{Z}/2\mathbb{Z})$, and

$$e_3(\phi_1)\cdot e_3(\phi_2) = (x)\cdot(y_1)\cdot(z_1)\cdot(x)\cdot(y_2)\cdot(z_2) = (-1)\cdot(x)\cdot(y_1)\cdot(z_1)\cdot(y_2)\cdot(z_2). \qquad \square$$

With this lemma at hand, one can generalise the invariant a_7. For $h \in J_1(k)$, let

$$a^h(Q) = h\cdot Q(v)\cdot a_6(Q)$$

where v is an anisotropic vector for Q. This defines an invariant because $h\cdot a_6$ vanishes on isotropic forms in $I^3_{14}(k)$ and [58, Proposition 10.2] implies that the quantity $h\cdot Q(v)\cdot a_6(Q)$ does not depend on the choice of v. If $-1 \in k^{\times 2}$, then $J_1(k) = H(k)$ and $a^h = h\cdot a^1$ for all $h \in H(k)$.

38.4. Lemma. *For all $Q \in I^3_{14}(k)$ and $c \in k^\times$,*

$$a_6(\langle c\rangle Q) - a_6(Q) = (-1)\cdot(-1)\cdot(c)\cdot a_3(Q).$$

Proof. This follows directly from (16.9.3). □

We would like to compare the degree 6 invariant of $H^1(*, (G_2 \times G_2) \rtimes S_2)$ with the degree 6 invariant of $I^3_{14}(*)$, but at first glance it is not clear how to do this because there is no morphism between these two functors. (Recall that the Albert form of a bioctonion algebra is only defined up to similitude, and a_6 is not always compatible with similitudes.) Theorem 38.5 is the best we can do: it bounds the difference between the quadratic trace invariant b_6 of a bioctonion algebra and the invariant a_6 of a quadratic form similar to one of its Albert forms.

38.5. Theorem. *Suppose $Q \in I^3_{14}(k)$ and $\beta \in H^1(k, (G_2 \times G_2) \rtimes S_2)$ have the same image in $PI^3_{14}(k)$. Then*

$$a_6(Q) - b_6(\beta) \in (-1)\cdot(-1)\cdot H^4(k, \mathbb{Z}/2\mathbb{Z}).$$

Proof. Case 1: Suppose $b_1(\beta) = 0$. Then β is the isomorphism class of a decomposable bioctonion algebra $C_1 \otimes C_2$ where C_i are some octonion algebras with norms n_i. And Q is similar to the Albert form of $C_1 \otimes C_2$, say $Q = \langle c\rangle(n_1' \perp \langle -1\rangle n_2')$ for some $c \in k^\times$. We have $b_6(\beta) = e_6(n_1 \cdot n_2) = e_3(n_1)\cdot e_3(n_2)$. By Lemma 38.2, $a_6(Q) - b_6(Q) \in (-1)\cdot(-1)\cdot H^4(k, \mathbb{Z}/2\mathbb{Z})$.

Case 2: Suppose $b_1(\beta) \neq 0$. The $\beta_1(\beta)$ is the class of a quadratic field extension E/k, and β is the isomorphism class of an indecomposable bioctonion algebra $\mathrm{cor}_{E/k}(C)$ where C is some octonion algebra over E with norm n. Now Q is similar to the Albert form of $\mathrm{cor}_{E/k}(C)$, say $Q = T_{E/k}(\langle\sqrt{d}\rangle n')$ where $d \in k^\times \setminus k^{\times 2}$ is an element whose square root generates E. Note that $Q \perp \mathbb{H} \simeq T_{E/k}(\langle\sqrt{d}\rangle n)$, so $Q = T_{E/k}(\langle\sqrt{d}\rangle n)$ in $W(k)$. Now, by (16.9.1):

$$P_3(Q) = P_3(T_{E/k}(\langle\sqrt{d}\rangle n)) = 8 + \lambda^2(T_{E/k}(\langle\sqrt{d}\rangle n)) - 4T_{E/k}(\langle\sqrt{d}\rangle n).$$

Applying Theorem 22.4 yields

$$P_3(Q) = 8 + T_{E/k}(\lambda^2(\langle\sqrt{d}\rangle n)) + \langle d\rangle N_{E/k}(\langle\sqrt{d}\rangle n) - 4T_{E/k}(\langle\sqrt{d}\rangle n).$$

We have $\lambda^2(\langle\sqrt{d}\rangle n) = \lambda^2(n) = 4n'$ by (16.7.1) and Example 16.8 (iii). And we have $N_{E/k}(\langle\sqrt{d}\rangle n) = N_{E/k}(\langle\sqrt{d}\rangle)N_{E/k}(n) = \langle -d\rangle N_{E/k}(n)$ by multiplicativity of $N_{E/k}$ and (22.2.1). It follows that

$$P_3(Q) = 8 + T_{E/k}(4n') - N_{E/k}(n) - 4T_{E/k}(\langle\sqrt{d}\rangle n)$$

Since $T_{E/k}$ is $H(k)$-linear, we may write

$$T_{E/k}(4n') = 4T_{E/k}(n') = -4T_{E/k}(\langle 1\rangle) + 4T_{E/k}(\langle 1\rangle + n') = -4\langle 2, 2d\rangle + 4T_{E/k}(n).$$

Therefore

$$\begin{aligned} P_3(Q) &= 8 - 4\langle 2, 2d\rangle + 4T_{E/k}(n) - 4T_{E/k}(\langle\sqrt{d}\rangle n) - N_{E/k}(n) \\ &= 8 - 4\langle 2, 2d\rangle + 4T_{E/k}(\langle 1, -\sqrt{d}\rangle n) - N_{E/k}(n) \\ &= 8 - 4\langle 2, 2d\rangle + 4T_{E/k}(\langle\langle\sqrt{d}\rangle\rangle n) - N_{E/k}(n). \end{aligned}$$

Now,

$$\begin{aligned} P_3(Q) + N_{E/k}(n) - 4\langle\langle d\rangle\rangle &= 8 - 4\langle 2, 2d\rangle - 4\langle 1, -d\rangle + 4T_{E/k}(\langle\langle\sqrt{d}\rangle\rangle n) \\ &= 4\langle\langle 2, -d\rangle\rangle + 4T_{E/k}(\langle\langle\sqrt{d}\rangle\rangle n) = 4T_{E/k}(\langle\langle\sqrt{d}\rangle\rangle n) \end{aligned}$$

since $4\langle\langle 2, -d\rangle\rangle \simeq 8\mathbb{H}$ by a straightforward calculation using the usual isometry criterion for binary forms [106, Proposition 5.1]. Finally, since $T_{E/k}(\langle\langle\sqrt{d}\rangle\rangle n) \in I^4(k)$ [53, Corollary 34.17], this implies

$$\begin{aligned} a_6(Q) - b_6(\beta) = a_6(Q) + b_6(\beta) &= e_6(P_3(Q) + N_{E/k}(n) - 4\langle\langle d\rangle\rangle) \\ &= e_6(4T_{E/k}(\langle\langle\sqrt{d}\rangle\rangle n)) = (-1)\cdot(-1)\cdot e_4(T_{E/k}(\langle\langle\sqrt{d}\rangle\rangle n)) \end{aligned}$$

and therefore $a_6(Q) - b_6(\beta) \in (-1)\cdot(-1)\cdot H^4(k, \mathbb{Z}/2\mathbb{Z})$. $\square$

As a consequence of Theorems 23.12 (ii) and 38.5, we also have a generalisation of [58, Proposition 22.2 (3)]:

38.6. Corollary. *If -1 is a sum of two squares in k, then $a_6(Q)$ and $a_7(Q)$ are symbols for all $Q \in I^3_{14}(k)$.*

In fact, if -1 is a sum of two squares then the symbols $a_6(Q)$ and $a_7(Q)$ can be determined explicitly from the proof of Theorem 23.12 (ii). Namely, if Q is of the form $Q = \langle c\rangle(\langle\!\langle x_1, x_2, x_3\rangle\!\rangle' \perp \langle -1\rangle\langle\!\langle y_1, y_2, y_3\rangle\!\rangle')$ then

$$a_6(Q) = (x_1)\cdot(x_2)\cdot(x_3)\cdot(y_1)\cdot(y_2)\cdot(y_3). \tag{38.6.1}$$

If $Q = T_{E/k}(\langle\delta\rangle\langle\!\langle z_1, z_2, z_3\rangle\!\rangle')$ for some quadratic field extension E/k, $z_i \in E$, and $0 \neq \delta \in \ker(\mathrm{tr}_{E/k})$ then either $a_6(Q) = 0$ (if one of the z_i has zero trace) or

$$a_6(Q) = \prod_{i=1}^{3}(\mathrm{tr}_{E/k}(z_i))\cdot(-\delta^2 N_{E/k}(z_i)). \tag{38.6.2}$$

38.7. *Classifying the invariants of PI^3_{14}.* The functors $H^1(*, (G_2 \times G_2) \rtimes S_2) \to PI^3_{14}$ and $I^3_{14} \to PI^3_{14}$ are surjective, so they induce injective homomorphisms $\mathrm{Inv}(PI^3_{14}, 2) \hookrightarrow \mathrm{Inv}(I^3_{14}, 2)$ and $\mathrm{Inv}(PI^3_{14}, 2) \hookrightarrow \mathrm{Inv}((G_2 \times G_2) \rtimes S_2, 2)$.

38.8. Proposition. *Suppose $\beta, \beta' \in H^1(k, (G_2 \times G_2) \rtimes S_2)$ have the same image in $PI^3_{14}(k)$. Then $b_3(\beta) = b_3(\beta')$ and $b_6(\beta) - b_6(\beta') \in (-1)\cdot(-1)\cdot H^4(k, \mathbb{Z}/2Z Z)$.*

(Equivalently, if $(A, -)$ and $(A', -)$ are isotopic bioctonion algebras, then we have $b_3(A, -) = b_3(A', -)$ and $b_6(A, -) - b_6(A', -) \in (-1)\cdot(-1)\cdot H^4(k, \mathbb{Z}/2\mathbb{Z})$.)

Proof. The equivalence of the two statements follows from Proposition 20.1. Assuming that $(A, -)$ and $(A', -)$ are isotopic bioctonion algebras with similar Albert forms Q and Q' respectively, we have by definition, $b_3(A, -) - b_3(A', -) = e_3(Q) - e_3(Q') = 0$. By Lemma 38.4 and Theorem 38.5,

$$(b_6(A, -) - a_6(Q)) + (a_6(Q) - a_6(Q')) + (a_6(Q') - b_6(A', -)) \in (-1)\cdot(-1)\cdot H^4(k, \mathbb{Z}/2\mathbb{Z}).$$

$\square$

38.9. Theorem. *The image of* $\mathrm{Inv}(PI^3_{14}, 2) \hookrightarrow \mathrm{Inv}((G_2 \times G_2) \rtimes S_2, 2)$ *is*

$$H(k)\cdot 1 \oplus H(k)\cdot b_3 \oplus J_2(k)\cdot b_6.$$

Proof. Proposition 38.8 implies

$$H(k)\cdot 1 \oplus H(k)\cdot b_3 \oplus J_2(k)\cdot b_6 \subset \mathrm{im}\left(\mathrm{Inv}(PI^3_{14}, 2) \to \mathrm{Inv}((G_2 \times G_2) \rtimes S_2, 2)\right).$$

For the reverse inclusion, recall from Theorem 35.9 that $\mathrm{Inv}((G_2 \times G_2) \rtimes S_2, 2)$ is the free $H(k)$-module with basis $\{1, b_1, b_3, b_6\}$. Suppose $\lambda, \mu, \nu \in H(k)$ are such that

$$b = \lambda\cdot b_1 + \mu\cdot b_3 + \nu\cdot b_6$$

is in the image of $\mathrm{Inv}(PI^3_{14}, 2)$. This assumption means that $b(A, -) = b(A', -)$ for all pairs of bioctonion algebras $(A, -)$ and $(A', -)$ over any field extension L/k, as long as they have similar Albert forms (equivalently, are isotopic).

Consider the field $K = k(t)$ and the algebras:

$$(B, -) = \text{the split bioctonion algebra over } K,$$
$$(B', -) = \operatorname{cor}_{K(\sqrt{t})/K}(C) \text{ where } C \text{ is the split octonion algebra over } K(\sqrt{t}).$$

The Albert form of $(B', -)$ is hyperbolic because it is the additive transfer of the hyperbolic form $\langle\!\langle 1, 1, 1\rangle\!\rangle$ over $K(\sqrt{t})$. Hence $0 = b(B, -) = b(B', -)$. Clearly $b_3(B', -) = 0$. By [180, Lemma 2.13] together with basic fact that $4\langle 2\rangle = 4$ in $W(k)$ for any field k,

$$b_6(B', -) = e_6(N_{K(\sqrt{t})/K}(4\mathbb{H}) - 4\langle\!\langle t\rangle\!\rangle) = e_6(4\langle 2\rangle\langle\!\langle t\rangle\!\rangle - 4\langle\!\langle t\rangle\!\rangle) = e_6(0) = 0.$$

This all implies

$$0 = b(B', -) = b_1(B', -) = \lambda\cdot(t),$$

so $\lambda = 0$. Now let $K' = k(t_1, t_2, t_3, t_4)$ and consider the decomposable bioctonion algebras

$$(D, -) = C_1 \otimes C_2 \qquad\qquad (D', -) = C_1' \otimes C_2',$$

where the C's are octonion algebras over K' with the following norms:

$$n_{C_1} = \langle\!\langle t_1, t_2, t_3\rangle\!\rangle, \qquad n_{C_2} = \langle\!\langle t_1, t_2, t_4\rangle\!\rangle,$$
$$n_{C_1'} = \langle\!\langle t_1, t_2, t_3^{-1}t_4\rangle\!\rangle, \qquad n_{C_2'} = \langle\!\langle 1, 1, 1\rangle\!\rangle \text{ (i.e., hyperbolic).}$$

The Albert form of $(D, -)$ is similar to the Albert form of $(D', -)$, because in $W(K')$:

$$\begin{aligned}\langle\!\langle t_1, t_2, t_3\rangle\!\rangle - \langle\!\langle t_1, t_2, t_4\rangle\!\rangle &= \langle\!\langle t_1, t_2\rangle\!\rangle(\langle\!\langle t_3\rangle\!\rangle - \langle\!\langle t_4\rangle\!\rangle) = \langle\!\langle t_1, t_2\rangle\!\rangle\langle -t_3, t_4\rangle \\ &= \langle -t_3\rangle\langle\!\langle t_1, t_2, t_3^{-1}t_4\rangle\!\rangle = \langle -t_3\rangle(\langle\!\langle t_1, t_2, t_3^{-1}t_4\rangle\!\rangle - \langle\!\langle 1, 1, 1\rangle\!\rangle).\end{aligned}$$

In particular, $b_3(D, -) = b_3(D', -) = (t_1)\cdot(t_2)\cdot(t_3^{-1}t_4)$. It is also clear that $b_1(D, -) = b_1(D', -) = 0$ because both algebras are decomposable. Since we assumed that b is an isotopy invariant, we have $0 = b(D, -) - b(D', -) = \nu\cdot b_6(D, -) - \nu\cdot b_6(D', -)$. The b_6's are straightforward to calculate:

$$\begin{aligned} b_6(D, -) &= (t_1)\cdot(t_2)\cdot(t_3)\cdot(t_1)\cdot(t_2)\cdot(t_4) = (-1)\cdot(-1)\cdot(t_1)\cdot(t_2)\cdot(t_3)\cdot(t_4), \\ b_6(D', -) &= (1)\cdot(1)\cdot(1)\cdot(t_1)\cdot(t_2)\cdot(t_3^{-1}t_4) = 0.\end{aligned}$$

This implies

$$0 = b(D, -) - b(D', -) = \nu\cdot(-1)\cdot(-1)\cdot(t_1)\cdot(t_2)\cdot(t_3)\cdot(t_4).$$

Symbols of the form $(t_{i_1})\cdot\dots\cdot(t_{i_j})$ in $H(K')$ are $H(k)$-linearly independent (see [21, Lemma 1.1]), so we conclude that $\nu\cdot(-1)\cdot(-1) = 0$, hence $\nu \in J_2(k)$. □

38.10. Corollary. *The image of* $\operatorname{Inv}(PI^3_{14}, 2) \hookrightarrow \operatorname{Inv}(I^3_{14}, 2)$ *is*

$$H(k)\cdot 1 \oplus H(k)\cdot a_3 \oplus J_2(k)\cdot a_6.$$

Proof. Let $\overline{a_3}$ and $\overline{\nu\cdot a_6}$, for $\nu \in J_2(k)$, be the unique elements of $\operatorname{Inv}(PI^3_{14}, 2)$ whose images in $\operatorname{Inv}(G^2 \rtimes S_2, 2)$ are b_3 and $\nu\cdot b_6$ respectively. Then $\operatorname{Inv}(PI^3_{14}, 2)$ is generated by $\overline{a_3}$ and $\{\overline{\nu\cdot a_6}\colon \nu \in J\}$. The image of $\overline{a_3}$ in $\operatorname{Inv}(I^3_{14}, 2)$ is a_3 (simply by comparing the definitions), while the image of $\overline{\nu\cdot a_6}$ in $\operatorname{Inv}(I^3_{14}, 2)$ is $\nu\cdot a_6$ (because of Theorem 38.5). □

38.11. *An important fibration.* Let n be even and let C be the centre of $\mathbf{Spin}_n$, so $C \simeq \boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2$ or $C \simeq \boldsymbol{\mu}_4$ according as $n = 0 \bmod 4$ or $n = 2 \bmod 4$ [122, p. 517]. We have a short exact sequence $C \longrightarrow \mathbf{Spin}_n \longrightarrow \mathbf{PGO}_n^+$, and we know that

$$\operatorname{im}(H^1(*, \mathbf{Spin}_n) \to H^1(*, \mathbf{PGO}_n^+)) \simeq H^1(*, \boldsymbol{\Omega}_n) \simeq PI_n^3(*),$$

so by Lemma 36.3 there is a fibration

$$H^1(*, C) \rightsquigarrow H^1(*, \mathbf{Spin}_n) \twoheadrightarrow PI_n^3.$$

We shall describe in concrete detail the action of $H^1(*, C)$ on $H^1(*, \mathbf{Spin}_n)$ in terms of quadratic forms. If e_1, e_2 are the two nontrivial orthogonal central idempotents in the even Clifford algebra of $\frac{n}{2}\mathbb{H}$, then the points of C are:

$$\begin{aligned} C(R) &= \{\zeta e_1 + \zeta^{-1} e_2 \colon \zeta \in \boldsymbol{\mu}_4(R)\} && \text{if } n = 2 \mod 4, \\ C(R) &= \{\xi_1 e_1 + \xi_2 e_2 \colon \xi_i \in \boldsymbol{\mu}_2(R)\} && \text{if } n = 0 \mod 4. \end{aligned}$$

This fixes an isomorphism $C \simeq \boldsymbol{\mu}_4$ or $C \simeq \boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2$. The kernel $J \simeq \boldsymbol{\mu}_2$ of the map $\mathbf{Spin}_n \to \mathbf{O}_n^+$ has points $J(R) = \{\xi 1 \colon \xi \in \boldsymbol{\mu}_2(R)\} \subset C(R)$. If $n = 2 \bmod 4$, then C/J can be identified with $\boldsymbol{\mu}_2$ by $(\zeta e_1 + \zeta^{-1} e_2)J \mapsto \zeta^2$. Now we have identifications $H^1(k, C) = k^\times/k^{\times 4}$ and $H^1(k, C/J) = k^\times/k^{\times 2}$. The map $H^1(k, C) \to H^1(k, C/J)$ sends $xk^{\times 4} \mapsto xk^{\times 2}$. If $n = 0 \bmod 4$ then C/J can be identified with $\boldsymbol{\mu}_2$ by $(\xi_1 e_1 + \xi_2 e_2)J \mapsto \xi_1 \xi_2$. In this case, the set $H^1(k, C)$ is identified with $k^\times/k^{\times 2} \times k^\times/k^{\times 2}$, and the map $H^1(k, C) \mapsto H^1(k, C/J)$ sends $(xk^{\times 2}, yk^{\times 2}) \mapsto xyk^{\times 2}$.

For $\eta \in H^1(L, \mathbf{Spin}_n)$, let $q_\eta \in I_n^3(L)$ be the corresponding quadratic form. Galois cohomology produces the following commutative diagram with exact rows:

$$\begin{array}{ccccccc} \mathbf{PGO}_n^+(*) & \xrightarrow{\delta} & H^1(*, C) & \longrightarrow & H^1(*, \mathbf{Spin}_n) & \longrightarrow & PI_n^3 \\ & & \downarrow & & \downarrow{\scriptstyle \eta \mapsto q_\eta} & & \downarrow \\ \mathbf{PGO}_n^+(*) & \xrightarrow{\delta} & H^1(*, C/J) & \longrightarrow & H^1(*, \mathbf{O}_n^+) & \longrightarrow & PI_n^2 \end{array}$$

The group $H^1(k, C/J) = k^\times/k^{\times 2}$ acts on $H^1(k, \mathbf{O}_n^+) = I_n^2(k)$ by $ck^{\times 2} \cdot q = \langle c \rangle q$. (To prove this, rather than dealing with cocycles, one can work out that stabiliser of q is $G(q)/k^{\times 2}$, using [156, I.§5 Proposition 39 (iii)] and the calculation of the first connecting map δ from [101, Proposition 13.33].) Consequently, the commutativity of the diagram implies that the action of $H^1(k, C)$ on $H^1(k, \mathbf{Spin}_n)$ has the following effect, for all $\eta \in H^1(k, \mathbf{Spin}_n)$:

$$\begin{aligned} q_{xk^{\times 4} \cdot \eta} &= \langle x \rangle q_\eta && \text{if } n = 2 \mod 4, \\ q_{(xk^{\times 2}, yk^{\times 2}) \cdot \eta} &= \langle xy \rangle q_\eta && \text{if } n = 0 \mod 4. \end{aligned} \tag{38.11.1}$$

For the next lemma, let a_3, a_6, and a^h, $h \in J_2(k)$, be the images of the identically named invariants of I_{14}^3 under the map $\operatorname{Inv}(I_{14}^3, 2) \hookrightarrow \operatorname{Inv}(\mathbf{Spin}_{14}, 2)$; that is, $a_i(\eta) = a_i(q_\eta)$ and $a^h(\eta) = a^h(q_\eta)$ for all $\eta \in H^1(L, \mathbf{Spin}_{14})$. (Doing this avoids having to introduce yet more notation.) The mod 2 part of $\mathbf{Spin}_{14}$'s Rost invariant is therefore called a_3.

38.12. Lemma. *Let* $\partial : \operatorname{Inv}(\mathbf{Spin}_{14}, 2) \to \operatorname{Inv}(PI^3_{14}, 2)$ *be the residue map associated to the fibration* $H^1(*, \boldsymbol{\mu}_4) \rightsquigarrow H^1(*, \mathbf{Spin}_{14}) \twoheadrightarrow PI^3_{14}$ *. We have*

$$\begin{aligned} \partial a_3 &= 0 \\ \partial a_6 &= (-1)\cdot(-1)\cdot a_3 \\ \partial a^h &= h\cdot a_6 && \text{for all } h \in J_1(k). \end{aligned}$$

Proof. Since a_3 is in the image of $\operatorname{Inv}(PI^3_{14}, 2) \hookrightarrow \operatorname{Inv}(\mathbf{Spin}_{14}, 2)$, Proposition 36.4 (ii) implies $\partial a_3 = 0$. Now suppose L/k is any field extension, $\eta \in H^1(L, \mathbf{Spin}_{14})$, and $[c] \in H^1(L, \boldsymbol{\mu}_4) = L^\times / L^{\times 4}$. If $q_\eta \in I^3_{14}(L)$ is the quadratic form corresponding to η, then $q_{[c]\cdot\eta} = \langle c \rangle q_\eta$ by (38.11.1). By Lemma 38.4, we have for all $\eta \in H^1(L, \mathbf{Spin}_n)$,

$$a_6([c] \cdot \eta) - a_6(\eta) = (-1)\cdot(-1)\cdot(c)\cdot a_3(\eta).$$

Hence $\partial a_6 = (-1)\cdot(-1)\cdot(c)\cdot a_3$. Now $h \in J_1(k)$ means $h\cdot(-1) = 0$ and so $h\cdot a_6([c]\cdot\eta) = h\cdot a_6(\eta)$ for all $\eta \in H^1(L, \mathbf{Spin}_{14})$. If $v \in k^{14}$ is an anisotropic vector for q_η, then

$$\begin{aligned} a^h([c]\cdot\eta) - a^h(\eta) &= h\cdot(q_{[c]\cdot\eta}(v))\cdot a_6([c]\cdot\eta) - h\cdot(q_\eta(v))\cdot a_6(q_\eta) \\ &= h\cdot(\langle c \rangle q_\eta(v))\cdot a_6(\eta) - h\cdot(q_\eta(v))\cdot a_6(q_\eta) = (c)\cdot h\cdot a_6(\eta). \end{aligned}$$ □

The subform $6\mathbb{H} \subset 7\mathbb{H}$ induces natural inclusions

$$\begin{aligned} f &: \mathbf{Spin}_{12} \to \mathbf{Spin}_{14} \\ g &: \boldsymbol{\Gamma}^+_{12} \to \boldsymbol{\Gamma}^+_{14} \\ h &: \boldsymbol{\Omega}_{12} \to \boldsymbol{\Omega}_{14}, \end{aligned}$$

each of which is an extension of the former. Recalling from 14.4 that $H^1(*, \boldsymbol{\Omega}_n) = PI^3_n$ and $H^1(*, \boldsymbol{\Gamma}^+_n) = I^3_n$, these maps induce (injective) morphisms

$$\begin{aligned} g^* &: I^3_{12} \to I^3_{14}, && q \mapsto q \perp \mathbb{H}, \\ h^* &: PI^3_{12} \to PI^3_{14}, && [q] \mapsto [q \perp \mathbb{H}]. \end{aligned}$$

and there are corresponding ring homomorphisms $H : \operatorname{Inv}(PI^3_{14}, 2) \to \operatorname{Inv}(PI^3_{12}, 2)$, $G : \operatorname{Inv}(I^3_{14}, 2) \to \operatorname{Inv}(I^3_{12}, 2)$, and $F : \operatorname{Inv}(\mathbf{Spin}_{14}, 2) \to \operatorname{Inv}(\mathbf{Spin}_{12}, 2)$.

38.13. Lemma. *We have*

$$\begin{aligned} G(a_3) &= z_3 \\ G(a_6) &= (-1)\cdot z_5 \\ G(a^h) &= 0 && \text{for all } h \in J_1(k). \end{aligned}$$

Proof. Suppose $q = \langle d \rangle \langle\!\langle c \rangle\!\rangle (\psi_1' \perp \langle -1 \rangle \psi_2') \in I^3_{12}(k)$ where $\psi_i = \langle\!\langle x_i, y_i \rangle\!\rangle$, using the parameterisation (21.6.1). By definition $a_3(q \perp \mathbb{H}) = z_3(q) = e_3(q)$. By Lemmas 38.2 and 37.2,

$$\begin{aligned} &a_6(q \perp \mathbb{H}) \\ &= e_3(\langle\!\langle c \rangle\!\rangle \psi_1)\cdot e_3(\langle\!\langle c \rangle\!\rangle \psi_2) + (-1)\cdot(-1)\cdot(d)\cdot e_3(\langle\!\langle c \rangle\!\rangle \psi_1) + (-1)\cdot(-1)\cdot(-d)\cdot e_3(\langle\!\langle c \rangle\!\rangle \psi_2) \\ &= (c)\cdot e_2(\psi_1)\cdot(c)\cdot e_2(\psi_2) + (-1)\cdot(-1)\cdot(d)\cdot(c)\cdot e_2(\psi_1) + (-1)\cdot(-1)\cdot(-d)\cdot(c)\cdot e_2(\psi_2) \\ &= (-1)\cdot z_5(q). \end{aligned}$$

If $h \in J_1(k)$ then $a^h(q \perp \mathbb{H}) = 0$ because $h\cdot a_6$ vanishes on isotropic forms. □

Recall from Theorem 37.8 that the natural inclusion $\mathrm{Inv}(I^3_{12},2) \hookrightarrow \mathrm{Inv}(\mathbf{Spin}_{12},2)$ is an isomorphism. And recall that we denote by $q_\eta \in I^3_n(L)$ the (isometry class of the) quadratic form corresponding to a cohomology class $\eta \in H^1(L,\mathbf{Spin}_n)$

38.14. Lemma. *The following diagram of $H(k)$-modules commutes:*

$$\begin{tikzcd} \mathrm{Inv}(I^3_{14},2) \arrow[r, hook] \arrow[d, "G"] & \mathrm{Inv}(\mathbf{Spin}_{14},2) \arrow[r, "\partial"] \arrow[d, "F"] & \mathrm{Inv}(PI^3_{14},2) \arrow[d, "H"] \\ \mathrm{Inv}(I^3_{12},2) \arrow[r, hook] \arrow[rr, bend right, "\partial'"'] & \mathrm{Inv}(\mathbf{Spin}_{12},2) & \mathrm{Inv}(PI^3_{12},2) \end{tikzcd}$$

Proof. The small square on the left obviously commutes because the maps in it arise from the (contravariant) functoriality of Inv. The rest of the proof is really about unpacking the definitions. Let $a \in \mathrm{Inv}(I^3_{14},2)$ and denote by $\tilde{a}$ its image in $\mathrm{Inv}(\mathbf{Spin}_{14},2)$. By definition, $\partial\tilde{a} \in \mathrm{Inv}(PI^3_{14},2)$ is the unique invariant satisfying

$$\tilde{a}(cL^{\times 4}\cdot\eta) - \tilde{a}(\eta) = (c)\cdot\partial\tilde{a}([q_\eta])$$

for all field extensions L/k, $c \in L^\times$, and $\eta \in H^1(L,\mathbf{Spin}_{14})$. Similarly, for all $z \in \mathrm{Inv}(I^3_{12},2)$, $\partial' z \in \mathrm{Inv}_{\mathrm{norm}}(PI^3_{12},2)$ is the unique invariant satisfying

$$z(\langle c\rangle q) - z(q) = (c)\cdot\partial' z([q])$$

for all field extensions L/k, $c \in L^\times$, and $q \in I^3_{12}(L)$. Suppose $G(a) = z$. Then for all L/k and $q \in I^3_{12}(L)$, we have

$$z(\langle c\rangle q) - z(q) = a(\langle c\rangle q \perp \mathbb{H}) - a(q \perp \mathbb{H}) = a(\langle c\rangle(q \perp \mathbb{H})) - a(q \perp \mathbb{H}). \qquad (38.14.1)$$

If $\eta \in H^1(L,\mathbf{Spin}_{14})$ is a preimage of $q \perp \mathbb{H}$ (i.e. $q_\eta = q \perp \mathbb{H}$) then (38.11.1) implies that the right-hand side of (38.14.1) is equal to

$$\tilde{a}(cL^{\times 4}\cdot\eta) - \tilde{a}(\eta) = (c)\cdot\partial\tilde{a}([q_\eta]).$$

By definition, $\partial\tilde{a}([q_\eta]) = \partial\tilde{a}([q \perp \mathbb{H}]) = H(\partial\tilde{a})([q])$. Therefore we have shown that $\partial' z = \partial' G(a) = H(\partial\tilde{a})$, which was the goal. □

38.15. Theorem. *Assume $\sqrt{-1} \in k$. The map $\mathrm{Inv}(I^3_{14},2) \to \mathrm{Inv}(\mathbf{Spin}_{14},2)$ is an isomorphism and $\mathrm{Inv}(I^3_{14},2)$ is a free $H(k)$-module generated by the invariants $\{1, a_3, a_6, a_7\}$.*

Proof. By Theorem 37.8, the map $\mathrm{Inv}(I^3_{12},2) \hookrightarrow \mathrm{Inv}(\mathbf{Spin}_{12},2)$ is an isomorphism and by Lemma 38.14 the following square commutes:

$$\begin{tikzcd} \mathrm{Inv}(\mathbf{Spin}_{14},2) \arrow[r, "\partial"] \arrow[d, "F"] & \mathrm{Inv}(PI^3_{14},2) \arrow[d, "H"] \\ \mathrm{Inv}(\mathbf{Spin}_{12},2) \simeq \mathrm{Inv}(I^3_{12},2) \arrow[r, "\partial'"] & \mathrm{Inv}(PI^3_{12},2) \end{tikzcd}$$

Let $a \in \mathrm{Inv}(\mathbf{Spin}_{14}, 2)$. By Theorem 37.8, there exist unique $\alpha, \beta, \gamma, \varepsilon \in H(k)$ such that

$$F(a) = \alpha{\cdot}1 + \beta{\cdot}z_3 + \gamma{\cdot}z_5 + \varepsilon{\cdot}z_6.$$

By Corollary 38.10, there exist unique $\lambda, \kappa, \omega \in H(k)$ such that

$$\partial a = \lambda{\cdot}1 + \kappa{\cdot}a_3 + \omega{\cdot}a_6.$$

Then according to Lemmas 37.4 and 38.13,

$$\varepsilon{\cdot}z_5 = \partial' F(a) = H(\partial a) = \lambda{\cdot}1 + \kappa{\cdot}z_3$$

But $1, z_3, z_5$ are $H(k)$-linearly independent, so this implies $\varepsilon = \lambda = \kappa = 0$. Therefore

$$\begin{aligned} F(a) &= \alpha{\cdot}1 + \beta{\cdot}z_3 + \gamma{\cdot}z_5 \\ \partial a &= \omega{\cdot}a_6 \end{aligned}$$

Now let $a' = \alpha{\cdot}1 + \beta{\cdot}a_3 + \omega{\cdot}a_7 \in \mathrm{Inv}(\mathbf{Spin}_{14}, 2)$. By Lemma 38.12, $\partial a - \partial a' = 0$ so Proposition 36.4 (ii) yields that $a = a' + y$ for some $y = \nu{\cdot}a_3 + \mu{\cdot}a_6$ in the image of $\mathrm{Inv}(PI^3_{14}, 2) \hookrightarrow \mathrm{Inv}(\mathbf{Spin}_{14}, 2)$. By Lemma 38.13,

$$\gamma{\cdot}z_5 = F(a) - F(a') = F(y) = \nu{\cdot}z_3,$$

which implies that $\gamma = \nu = 0$. Therefore

$$a = a' + y = \alpha{\cdot}1 + \beta{\cdot}a_3 + \omega{\cdot}a_7 + \mu{\cdot}a_6.$$

To prove that $\{1, a_3, a_7, a_6\}$ is $H(k)$-linearly independent, consider the quadratic form

$$Q = \langle t_7 \rangle (\langle\langle t_1, t_2, t_3 \rangle\rangle' \perp \langle -1 \rangle \langle\langle t_4, t_5, t_6 \rangle\rangle)$$

over the field $k(t_1, \dots, t_7)$. Then $a_3(Q) = (t_1){\cdot}(t_2){\cdot}(t_3) + (t_4){\cdot}(t_5){\cdot}(t_6)$, while $a_6(Q) = (t_1){\cdot}(t_2){\cdot}(t_3){\cdot}(t_4){\cdot}(t_5){\cdot}(t_6)$ (see (38.6.1)) and $a_7(Q) = (t_1){\cdot}(t_2){\cdot}(t_3){\cdot}(t_4){\cdot}(t_5){\cdot}(t_6){\cdot}(t_7)$. These values are $H(k)$-linearly independent in $H(k(t_1, \dots, t_7))$. □

38.16. *The mod 2 invariants fail to classify anisotropic quadratic forms in I^3_{14}.* In answer to a question of Lam, it was shown by Izhboldin [82, Theorem 4.4] that there exists a field extension F/k with a pair of quadratic forms $q_1, q_2 \in I^3_{14}(F)$ such that $q_1 - q_2 \in I^4(F)$ but q_1 and q_2 are not similar. (Although the statement of [82, Theorem 4.4] only asserts the existence of a field F with this property, the proof rests on Hoffmann's example [76, Theorem 4.3] of a pair of dissimilar anisotropic 8-dimensional forms that are Witt-equivalent modulo $I^4(F)$ and whose Clifford algebras have index 4. It is clear from [76, Lemma 4.2] that one can arrange for F to be an extension of the original k.) Moreover, Izhboldin's q_1 and q_2 are constructed in such a way that they have a common 6-dimensional anisotropic subform [82, p. 348]. In particular, q_1 and q_2 represent a common element $c \in F^\times$.

Clearly, $q_1 - q_2 \in I^4(F)$ implies $a_3(q_1) = e_3(q_1) = e_3(q_2) = a_3(q_2)$. We may assume that $\sqrt{-1} \in F$, and then $q_1 - q_2 \in I^4(F)$ also implies that $P_3(q_1) = P_3(q_2)$ [58, Proposition 19.12 (3)], hence $a_6(q_1) = a_6(q_2)$. The fact that q_1 and q_2 represent a common element c further implies $a_7(q_1) = (c){\cdot}a_6(q_1) = (c){\cdot}a_6(q_2) = a_7(q_2)$. As a consequence:

38.17. Corollary. *Assume $\sqrt{-1} \in k$. There exists a field extension F/k and a pair of quadratic forms $q_1, q_2 \in I^3_{14}(F)$ such that $a(q_1) = a(q_2)$ for all cohomological invariants $a \in \operatorname{Inv}(I^3_{14}, 2)$ but q_1 is not similar to q_2.*

The quadratic forms $q_i \in I^3_{14}(F)$ constructed in [82, Theorem 4.4] are each similar to a difference of two 3-Pfister forms (i.e., of the form (1) in Corollary 21.3). This means (by Corollary 21.2) that there exist decomposable bioctonion algebras $(A_1, -)$ and $(A_2, -)$ over F such that the Albert form of A_i is similar to q_i. The invariants of these algebras agree, despite the algebras not being isomorphic (or even isotopic, by Corollary 20.4):

$$\begin{aligned} &b_1(A_1, -) = b_1(A_2, -) = 0 \\ &b_3(A_1, -) = a_3(q_1) = a_3(q_2) = b_3(A_2, -) \\ &b_6(A_1, -) = a_6(q_1) = a_6(q_2) = b_6(A_2, -) \end{aligned}$$

38.18. Corollary. *There exists a field extension F/k and bioctonion algebras $(A_1, -)$ and $(A_2, -)$ such that $b(A_1, -) = b(A_2, -)$ for all cohomological invariants $b \in \operatorname{Inv}((G_2 \times G_2) \rtimes S_2, 2)$ but $(A_1, -)$ and $(A_2, -)$ are not isotopic.*

In contrast to these two corollaries, the invariant $a_3 \in \operatorname{Inv}(I^3_{14}, 2)$ does separate *isotropic* quadratic forms in I^3_{14} up to similarity (see [78, Corollary] and its application in Theorem 34.4). Consequently the invariant $b_3 \in \operatorname{Inv}((G_2 \times G_2) \rtimes S_2, 2)$ does separate *nondivision* bioctonion algebras up to isotopy.

Cohomologische invarianten van structureerbare algebra's

Nederlandstalige samenvatting

Structureerbare algebra's werden voor het eerst geïntroduceerd met de bedoeling om exceptionele Lie algebra's over willekeurige velden te construëeren. Men neemt de som van een paar kopieën van de algebra zelf, met enkele van haar deelruimten en een bijhorende ruimte van lineaire operatoren. Met behulp van een slimme definitie wordt dit dan een Lie algebra. (Deze aanpak werkt alleen als de karakteristiek van het veld k verschillend is van 2 en 3.) Klassieke voorbeelden van structureerbare algebra's zijn alternatieve en Jordan algebra's. Een exotischer voorbeeld wordt gegeven door het tensorproduct van twee octonionenalgebra's, dat een bioctonionenalgebra genoemd wordt.

Het onderwerp van dit project werd geïnspireerd door het succes van cohomologische invarianten in andere algebraïsche deelgebieden, zoals Jordan algebra's, centrale enkelvoudige algebra's (met involuties), en kwadratische vormentheorie. Een cohomologische invariant is een functie die aan een algebraïsch object (denk bijvoorbeeld aan een algebra) een uniek element van een Galois cohomologiegroep hecht. Een cohomologische invariant moet bovendien compatibel zijn met velduitbreidingen.

Er zijn vele toepassingen van cohomologische invarianten. De eerste en meest fundamentele toepassing is om te weten of twee objecten al dan niet isomorf (identiek hetzelfde) zijn. De mate waarin dit lukt, hangt af van de invariant. Bijvoorbeeld, de invariant δ, die een quaternionenalgebra $(a, b)_k$ afbeeldt op het symbool $(a)\cdot(b) \in H^2(k, \boldsymbol{\mu}_2)$, onderscheidt alle quaternionenalgebra's op isomorfisme na.

Als bijkomende toepassing kan een invariant, in sommige gevallen, bepaalde eigenschappen van het object aan het licht brengen. Een beroemd voorbeeld daarvan is de Serre–Rost invariant van exceptionele Jordan algebra's (ook bekend als Albert algebra's). Deze invariant beeldt een Albert algebra J af op een element $g_3(J) \in H^3(k, \mathbb{Z}/3\mathbb{Z})$ dat weet of J een delingsalgebra is, in de zin dat $g_3(J) \neq 0$ als en slechts als J een delingsalgebra is.

Een derde toepassing, die we in dit project één keer tegenkomen (33.13), is dat invarianten het bestaan kunnen bewijzen van iets dat we niet noodzakelijk direct kunnen observeren. Als we weten dat een invariant niet identiek nul is, dan moet er een bepaald object bestaan waarop ze niet nul is, zelfs als de invariant nulwaardig is op alle gekende voorbeelden van dat soort object.

Op dit moment is het de moeite waard om de juiste terminologie te vermelden. We zijn voornamelijk geïnteresseerd in cohomologische invarianten van algebraïsche groepen. Dat zijn natuurlijke transformaties $H^1(k,G) \to H^d(k,C)$ waarbij G een algebraïsche groep is en C een Galoismoduul, zoals $\boldsymbol{\mu}_p$ of $\mathbb{Z}/p\mathbb{Z}$. De verzameling $H^1(k,G) = H^1(\mathcal{G}al(k^{\mathrm{sep}}/k), G(k^{\mathrm{sep}}))$ wordt opgevat volgens de niet-abelse cohomologie van Serre.

Het begrip invarianten van een algebraïsche groep omvat onder andere de voorgaande voorbeelden. Omdat F_4 de automorfismegroep is van de (gespleten) Albert algebra, zeggen we dat g_3 een cohomologische invariant is van F_4. Op dezelfde manier is δ een cohomologische invariant van $\mathbf{PGL}_2$, want $\mathbf{PGL}_2$ is de automorfismegroep van een quaternionenalgebra, namelijk de 2×2 matrixalgebra. Omgekeerd kan een cohomologische invariant van de orthogonale groep $\mathbf{O}_n$ opgevat worden als een functie op de verzameling van alle kwadratische vormen op k^n, want $\mathbf{O}_n$ is de automorfismegroep van een zekere n-dimensionale kwadratische vorm.

Initieel hoopten we dat interessante nieuwe invarianten ontdekt zouden kunnen worden door enkele van de exceptionele structureerbare algebra's te onderzoeken en, nogal optimistisch, dat deze inspanning eventueel zou leiden tot nieuwe invarianten van exceptionele algebraïsche groepen. Dat laatste blijkt echter moeilijk. Groepen van type E_6, E_7, en E_8 bevatten weinig grote deelgroepen, waardoor het bereik van onze constructies vrij beperkt blijft. Met andere woorden, men kan nooit alle Lie algebra's (of groepen) van type E_r construeren uit basisbouwstenen, of de bouwstenen nu structureerbare algebra's zijn of iets totaal anders.

Als nieuwe invarianten toch op die manier ontdekt worden, dan zijn ze hooguit gedefinieerd op een deelverzameling van $H^1(k,E_r)$, zoals de verzameling Lie algebra's van type E_r over k met een voorgeschreven gradering. Het kan niettemin interessant zijn om een glimp van een invariant te zien, zelfs als er onzekerheid is over of zij gedefinieerd is op heel $H^1(k,E_r)$.

Op een zeker punt tijdens dit project raakten we ook geïnteresseerd in het classificeren van alle invarianten van bepaalde groepen. Zodra ik me realiseerde dat cohomologische invarianten erg moeilijk te vinden zijn, begon ik te denken dat het in sommige gevallen toch makkelijker zou zijn om te bewijzen dat er géén nieuwe invarianten meer bestaan. De verzameling cohomologische invarianten van G met coëfficienten in $\mathbb{Z}/2\mathbb{Z}$ is zowel een abelse groep als een moduul voor de cohomologiering $H^\bullet(k,\mathbb{Z}/2\mathbb{Z})$. Soms is het mogelijk om deze groep $\mathrm{Inv}(G,2)$ te berekenen en er een basis voor te geven, of een korte lijst van voortbrengers. Dit is het onderwerp van het laatste hoofdstuk van de thesis.

Er waren nog enkele kleinere doelstellingen van het onderzoek, zoals kennis opdoen over de exceptionele structureeerbare algebra's. We hadden bijvoorbeeld geen enkel criterium om te bepalen of een bioctonionenalgebra een delingsalgebra is, laat staan een cohomologisch criterium. (We wisten van tevoren dat een zekere 14-dimensionale Albert kwadratische vorm anisotroop is als en slechts als de algebra een delingsalgebra is, maar we hadden alleen een bewijs in karakteristiek 0.) Nog een doel van het project was om de bestaande invarianten, zoals Rost invarianten, te bestuderen om te begrijpen hoe ze zich uitdrukken in de structureerbare algebra's en of ze nuttig kunnen zijn voor eventuele toepassingen in de theorie van niet-associatieve algebra's en algebraïsche groepen.

Overzicht van resultaten

Hoofdstuk I gaat over (centrale enkelvoudige) structureerbare algebra's. We verduidelijken enkele aspecten van de bestaande classificatiestelling door te werken over een separabel gesloten veld.

De classificatie van centrale enkelvoudige structureerbare algebra's over een separabel gesloten veld staat in Tabellen 1 en 2.

Aan de hand van een numerieke invariant gedefinieerd in 3.13, krijgen we een nieuw resultaat:

Gevolg 3.15. *Zij $(A,-)$ en $(B,-)$ isotopische centrale enkelvoudige structureerbare algebra's over k, en $\mathrm{char}(k) \neq 2,3,5$, dan bestaat er een eindige separabele velduitbreiding K/k die zorgt dat $(A_K,-) \simeq (B_K,-)$.*

In Hoofdstuk II onderzoeken we de Tits–Kantor–Koecher (TKK) constructie van $\mathbb{Z}$-gegradeerde Lie algebra's over willekeurige velden k. (In alle hoofdstukken gaan we ervan uit dat $\mathrm{char}(k) \neq 2,3$.)

Gelabelde Dynkin-diagrammen zijn de voorkeursinvariant om een $\mathbb{Z}$-gradering van een enkelvoudige Lie algebra te beschrijven. De resultaten op basis van diverse structureerbare algebra's worden in Tabel 4 opgenomen.

De Allison–Faulkner (AF) constructie wordt ook in dit hoofdstuk onderzocht; het is een veralgemening van de TKK constructie die gebruik maakt van een andere gradering. In verband met deze constructie bewijzen we de volgende stelling.

Stelling 6.13. *Stel dat $(A,-)$ een centrale enkelvoudige structureerbare algebra is, dat $\gamma = (\gamma_1,\gamma_2,\gamma_3) \in (k^\times)^3$ een drievoud van scalairen is, dat $L = K(A,-,\gamma)$ de door de AF-constructie geconstrueerde Lie algebra is, en dat κ de kwadratische Killing vorm is van L. De homogene componenten van L zijn paarsgewijs orthogonaal met betrekking tot κ. Stel dat $\kappa \neq 0$ en g een niet-ontaarde kwadratische vorm is op A waarbij $g(1) \neq 0$ en de bijhorende bilinaire vorm* invariant *is in de zin van Definitie* 2.15*. Dan is*

$$\kappa \simeq \kappa_0 \perp \langle d \rangle \langle \delta_{12}, \delta_{23}, \delta_{31} \rangle g$$

waarbij κ_0 de restrictie van κ tot de nul-component is, $\delta_{ij} = \gamma_i \gamma_j^{-1}$, en

$$d = g(1)^{-1}(-2\dim A - 8\dim \mathrm{Skew}(A,-)).$$

In Hoofdstuk III berekenen we de automorfismegroepen van alle exceptionele structureerbare algebra's en ook die van enkele klassieke structureerbare algebra's. We berekenen ook hun derivatie-algebra's, en de gespleten en quasi-gespleten vormen van hun halfenkelvoudige structuurgroepen. De gespleten vormen van deze automorfisme- en structuurgroepen staan in Tabel 5.

De automorfismegroepen en structuurgroepen van bicompositie-algebra's zijn zeer belangrijk in deze thesis. Hier doen we de moeite om rationeel te werken, dus we moeten rekening houden met twee soorten bicompositie-algebra's.

Algebra's van de eerste soort hebben een over k gedefinieerde decompositie als tensorproduct van twee compositie deelalgebra's die gestabiliseerd worden door de involutie. Die van de tweede soort hebben geen dergelijke decompositie over k, maar wel over een kwadratische velduitbreiding van k. In de volgende stelling verwijzen we naar een *corestrictie* constructie, gedefinieerd in 9.7, die beide soorten omvat.

Stelling 9.12. *Zij* $(A,-) = \mathrm{cor}_{E/k}(C)$ *een bioctonionenalgebra (respectievelijk biquaternionenalgebra), waarbij* C *een octonionenalgebra (respectievelijk quaternionenalgebra) is over een kwadratische étale uitbreiding* E/k. *Zij* $C_{/k}$ *de* k*-algebra met dezelfde onderliggende verzameling, vermenigvuldiging, en* k*-vectorruimte als* C. *Dan is* $\mathbf{Aut}(A,-) \simeq \mathbf{Aut}(C/_k)$. *Bijgevolg:*

(i) $\mathbf{Aut}(A,-)^\circ \simeq R_{E/k}(\mathbf{Aut}(C))$.

(ii) $\mathbf{Aut}(A,-)$ *heeft twee samenhangende componenten, en de niet-identiteit component heeft* k*-punten als en slechts als de* E*-algebra's* C *en* ${}^\iota C$ *isomorf zijn.*

(iii) *Als* $A = C_1 \otimes C_2$ *een decompositie is van* $(A,-)$, *dan hebben we* $\mathbf{Aut}(A,-) \simeq \mathbf{Aut}(C_1 \times C_2)$ *en* $\mathbf{Aut}(A,-)^\circ \simeq \mathbf{Aut}(C_1) \times \mathbf{Aut}(C_2)$.

Hoofdstuk IV gaat over Galois cohomologie en cohomologische invarianten, en bestaat voornamelijk uit fundering voor latere resultaten.

Het volgende technische resultaat is waarschijnlijk het belangrijkste daarvan.

Lemma 13.6. *Zij* $(A,-)$ *een centrale enkelvoudige structureerbare algebra over* k.

(i) *In het geval dat* k *algebraïsch gesloten is, heeft de groepswerking van* $\mathbf{Str}(A,-)^\circ$ *op* A *een dichte open baan.*

(ii) *De inclusie* $i : \mathbf{Aut}(A,-) \subset \mathbf{Str}(A,-)$ *induceert een surjectieve afbeelding*

$$i_* : H^1(k, \mathbf{Aut}(A,-)) \longrightarrow H^1(k, \mathbf{Str}(A,-)).$$

(iii) *Zij* $M = (\mathbf{Str}(A,-)^\circ)^{\mathrm{der}}$ *de halfenkelvoudige structuurgroep van* $(A,-)$. *De inclusie* $M \subset \mathbf{Str}(A,-)^\circ$ *induceert een surjectieve afbeelding*

$$H^1(k, M) \longrightarrow H^1(k, \mathbf{Str}(A,-)^\circ).$$

(iv) *De inbedding* $\mathbf{Str}(A,-) \simeq \mathbf{Aut}_{\mathrm{gr}}(K(A,-)) \subset \mathbf{Aut}(K(A,-))$ *van Lemma* 5.5 *induceert injectieve afbeeldingen*

$$H^1(k, \mathbf{Str}(A,-)) \longrightarrow H^1(k, \mathbf{Aut}(K(A,-))$$
$$H^1(k, \mathbf{Str}(A,-)^\circ) \longrightarrow H^1(k, \mathbf{Aut}(K(A,-)^\circ).$$

Hoofdstuk V gaat over de theorie van bicompositie-algebra's, in het bijzonder bioctonionenalgebra's. Op de scheefdeelruimte $\mathrm{Skew}(A,-)$ van elke bicompositie-algebra is er een kwadratische vorm Q, de Albert vorm genoemd, met de eigenschap dat $Q(s) \neq 0$ als en slechts als de linkse vermenigvuldigingsoperator $L_s \in \mathrm{End}\, A$ inverteerbaar is.

Een berekening van de volledige (reductieve) structuurgroepen blijkt informatief te zijn en leidt tot tal van toepassingen.

Stelling 18.19. *Zij* $(A,-)$ *een bicompositie-algebra met Albert vorm* Q, *en stel* $S = \mathrm{Skew}(A,-)$, $N = \mathrm{Nuc}(A)$, *en* $F = Z(A)$. *De samenhangende structuurgroep* $H^\circ = \mathbf{Str}(A,-)^\circ$ *is de reductieve groep die, op isomorfisme na, beschreven is door de data in Tabel* 6.

Hieronder staat het verwachte criterium voor bicompositie delingsalgebra's.

Stelling 20.7. *Zij $(A, -)$ een bicompositie-algebra. Dan is $(A, -)$ een structureerbare delingsalgebra als en slechts als haar Albert vorm Q anisotroop is en haar centrum $Z(A)$ een veld is.*

Door de structuurgroep en haar cohomologische afbeeldingen verder te bestuderen vinden we een nieuw bewijs van de Stelling van Rost terug.

Gevolg 21.3 (Rost)**.** *Zij Q een 14-dimensionale kwadratische vorm met triviale discriminant en Clifford invariant. Dan geldt er minstens één van de volgende:*

(1) *Er bestaan 3-Pfister vormen ϕ_1 en ϕ_2 over k en een scalair $c \in k^\times$ zodat*

$$Q \simeq \langle c \rangle (\phi_1' \perp \langle -1 \rangle \phi_2').$$

(2) *Er bestaat een kwadratische velduitbreiding E/k, een 3-Pfister vorm ϕ over E, en een spoorloos element $\delta \in E^\times$ zodat*

$$Q \simeq T_{E/k}(\langle \delta \rangle \phi').$$

In Hoofdstuk VI wordt voor het eerst serieus ingegaan op cohomologische invarianten van structureerbare algebra's. De nadruk ligt hier weer op bioctonionalgebra's. De invarianten daarvan kunnen zeer concreet beschreven worden, hebben rijke toepassingen, en drukken zich ook uit in de Lie algebra's.

Als $(A, -)$ een bioctonionen algebra is, dan heeft $\mathrm{Skew}(A, -)$ de structuur van een Malcev algebra wiens centroid E een kwadratische étale algebra is. De eerste invariant is

$$b_1(A, -) = [E] = \text{ de klasse van het centroid van } \mathrm{Skew}(A, -).$$

De tweede invariant is

$$b_3(A, -) = e_3(Q) = \text{ de Arason invariant van de Albert vorm van } (A, -).$$

De spoorvorm $T_A(x, y) = \mathrm{tr}(L_{x\bar{y}+y\bar{x}})$ is een 64-dimensionale symmetrische bilineaire vorm op A. Deze spoorvorm is een Pfister buurvorm omdat $\langle 128 \rangle T_A \perp 4\langle -1 \rangle N_{E/k}$ Witt-equivalent is aan een 6-Pfister vorm. Zo is de derde invariant gedefinieerd:

$$b_6(A, -) = e_6(\langle 128 \rangle T_A \perp 4\langle -1 \rangle N_{E/k}).$$

Stelling 23.12. *Zij $(A, -)$ een bioctonionenalgebra.*

(i) $b_1(A, -) = 0$ *als en slechts als $(A, -)$ een decompositie heeft.*

(ii) $b_3(A, -)$ *heeft symboollengte ≤ 3.*

(iii) *Als de symboollengte van $b_3(A, -)$ gelijk is aan 3, dan is $(A, -)$ een delingsalgebra zonder decompositie.*

(iv) $b_3(A, -) = 0$ *als en slechts als $(A, -)$ isotoop is aan de gespleten bioctonionenalgebra.*

(v) $b_6(A, -)$ *is een symbool.*

(vi) *Als $(A, -)$ geen delingsalgebra is, dan is $b_6(A, -) \in (-1) \cdot H(k)$.*

(vii) *Als* $\sqrt{-1} \in k$ *en* $b_6(A,-) \neq 0$, *dan is* $(A,-)$ *een delingsalgebra.*

Als we een bioctonionenalgebra in de AF-constructie stoppen dan wordt er een Lie algebra van type E_8 geproduceerd. De bovenstaande invarianten hebben een zeer mooie relatie met de bekende Rost invariant r_{E_8}.

Stelling 27.2. *Veronderstel dat* $\mathrm{char}(k) = 0$. *Als* $(A,-)$ *een bioctonionenalgebra is en* $\gamma = (\gamma_1, \gamma_2, \gamma_3) \in (k^\times)^3$, *dan is*

$$r_{E_8}(K(A,-,\gamma)) = b_3(A,-) + b_1(A,-)\cdot(-\gamma_1\gamma_2^{-1})\cdot(-\gamma_2\gamma_3^{-1}) \in H^3(k, \mathbb{Z}/2\mathbb{Z}).$$

Hoofdstuk VII gaat over de exceptionele structureerbare algebra's van scheefdimensie één, in het bijzonder Brown algebra's. We bestuderen alle gekende constructies met het oog op de onderliggende deelgroepen van $E_6^{\mathrm{sc}} \rtimes \mathbb{Z}/2\mathbb{Z}$.

Net als we in Hoofdstuk VI deden, berekenen we ook de spoorvormen van deze Brown algebra's.

Stelling 32.1. *Stel dat* $\mathrm{char}(k) \neq 2, 3, 7$ *en dat* E/k *een kwadratische étale uitbreiding is met norm* $N_{E/k} = \langle\langle \mu \rangle\rangle$.

(i) *Zij* J *een Albert algebra met kwadratische spoor* T_J. *Als* $(B,-) = M(J,E)$, *dan is*

$$T_B = \langle 7 \rangle \langle\langle \mu \rangle\rangle (\langle 1 \rangle \perp T_J).$$

(ii) *Zij* C *een associatieve centrale enkelvoudige algebra van graad* 8 *met symplectische involutie* σ. *Als* $(B,-) = \mathrm{CD}((C,\sigma),\mu)$, *dan is*

$$T_B = \langle 7 \rangle \langle\langle \mu \rangle\rangle T_\sigma^+.$$

(iii) *Als* $(B,-) = M(J,\eta)$ *een Brown matrixalgebra is, dan is* T_B *hyperbolisch.*

In 33.5 definiëren we zorgvuldig een Rost invariant $r \in \mathrm{Inv}(E_6^{\mathrm{sc}} \rtimes \mathbb{Z}/2\mathbb{Z}, 2)$ van Brown algebra's en berekenen de waarde van deze invariant op de gekende constructies. Bijvoorbeeld:

Stelling 33.12. *Zij* $(B,-) = \mathrm{CD}((C,\sigma),\mu)$ *de Brown algebra geconstrueerd door de verdubbeling van een symplectische involutie* (C,σ) *van graad* 8 *met parameter* $\mu \in k^\times$. *Dan is*

$$r(B,-) = \Delta(C,\sigma) + (\mu)\cdot[C] \in H^3(k, \mathbb{Z}/2\mathbb{Z}),$$

waarbij $\Delta(C,\sigma)$ *de discriminant is van de symplectische involutie.*

Volgens Tabel 12 kan de Rost invariant informatie geven over eigenschappen van de Brown algebra's die geen delingsalgebra's zijn. De Rost invariant legt ook de beperkingen bloot in de constructies die ons ter beschikking staan – zie 33.13.

Hoofdstuk VIII bestaat volledig uit het bewijs van één belangrijke stelling: de classificatie van cohomologische invarianten van $\mathbf{Spin}_{14}$, de enkelvoudig samenhangende overdekkingsgroep van de speciale orthogonale groep $\mathbf{O}_{14}^+$.

Stelling 38.15. *Veronderstel dat* $\sqrt{-1} \in k$. *De canonieke afbeelding* $\mathrm{Inv}(I_{14}^3, 2) \to \mathrm{Inv}(\mathbf{Spin}_{14}, 2)$ *is een isomorfisme en* $\mathrm{Inv}(I_{14}^3, 2)$ *is de vrije* $H(k)$*-moduul voortgebracht door de invarianten* $\{1, a_3, a_6, a_7\}$ *gedefinieerd in* 38.1.

Gaandeweg classificeren we de cohomologische invarianten van veel andere groepen en klassen van kwadratische vormen. De resultaten zijn samengevat in Tabel 3.

De invarianten van $\mathbf{Spin}_{14}$ slagen er niet in om kwadratische vormen van elkaar te onderscheiden.

Gevolg 38.17. *Als $\sqrt{-1} \in k$, dan bestaan er een velduitbreiding F/k en een paar kwadratische vormen $q_1, q_2 \in I^3_{14}(F)$ zodat $a(q_1) = a(q_2)$ voor alle cohomologische invarianten $a \in \mathrm{Inv}(I^3_{14}, 2)$ maar toch is $q_1 \not\simeq \langle c \rangle q_2$ voor alle $c \in k^\times$.*

De invarianten van $(G_2 \times G_2) \rtimes \mathbb{Z}/2\mathbb{Z}$, de automorfismegroep van de gespleten bioctonionenalgebra, slagen er ook niet in om alle bioctonionenalgebra's van elkaar te onderscheiden.

Gevolg 38.18. *Er bestaan een velduitbreiding F/k en bioctonionenalgebra's $(A_1, -)$ en $(A_2, -)$ zodat $b(A_1, -) = b(A_2, -)$ voor alle cohomologische invarianten $b \in \mathrm{Inv}((G_2 \times G_2) \rtimes S_2, 2)$ maar $(A_1, -)$ en $(A_2, -)$ toch geen isotopen zijn van elkaar.*

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Bibliography

[1] R. Abuaf and L. Manivel, *Gradings of Lie algebras, magical spin geometries and matrix factorizations*, Representation Theory **25** (2019), 527–542.

[2] A. A. Albert, *Tensor products of quaternion algebras*, Proceedings of the American Mathematical Society **35** (1972), no. 1, 65–66.

[3] B. N. Allison, *A class of nonassociative algebras with involution containing the class of Jordan algebras*, Mathematische Annalen **237** (1978), no. 2, 133–156.

[4] ———, *Models of isotropic simple Lie algebras*, Communications in Algebra **7** (1979), no. 17, 1835–1875.

[5] ———, *Conjugate inversion and conjugate isotopes of alternative algebras with involution*, Algebras, Groups and Geometries **3** (1986), 361–385.

[6] ———, *Tensor products of composition algebras, Albert forms and some exceptional simple Lie algebras*, Transactions of the American Mathematical Society **306** (1988), no. 2, 667–695.

[7] ———, *Simple structurable algebras of skew-dimension one*, Communications in Algebra **18** (1990), no. 4, 1245–1279.

[8] ———, *Construction of* 3×3*-matrix Lie algebras and some Lie algebras of type* D_4, Journal of Algebra **143** (1991), no. 1, 63–92.

[9] B. N. Allison and J. R. Faulkner, *A Cayley-Dickson process for a class of structurable algebras*, Transactions of the American Mathematical Society **283** (1984), no. 1, 185–210.

[10] ———, *Norms on structurable algebras*, Communications in Algebra **20** (1992), no. 1, 155–188.

[11] ———, *The algebra of symmetric octonion tensors*, Nova Journal of Algebra and Geometry **2** (1993), no. 1, 47–58.

[12] ———, *Nonassociative coefficient algebras for Steinberg unitary Lie algebras*, Journal of Algebra **161** (1993), no. 1, 1–19.

[13] ———, *Elementary groups and invertibility for Kantor pairs*, Communications in Algebra **27** (1999), no. 2, 519–556.

[14] B. N. Allison and W. Hein, *Isotopes of some nonassociative algebras with involution*, Journal of Algebra **69** (1981), no. 1, 120–142.

[15] S. Alsaody, V. Chernousov, and A. Pianzola, *On the Tits–Weiss conjecture and the Kneser–Tits conjecture for* $\mathrm{E}_{7,1}^{78}$ *and* $\mathrm{E}_{8,2}^{78}$ (with an Appendix by R. M. Weiss), Forum of Mathematics, Sigma **9** (2021), e75.

[16] D. Aranda-Orna and A. S. Córdova-Martínez, *Gradings on tensor products of composition algebras and on the Smirnov algebra*, Linear Algebra and Its Applications **584** (2020), 1–36.

[17] D. Aranda-Orna, A. Elduque, and M. Kochetov, *A* $\mathbb{Z}_4^3$*-grading on a* 56*-dimensional simple structurable algebra and related fine gradings on the simple Lie algebras of type E*, Commentationes Mathematicae Universitatis Carolinae **55** (2014), no. 3, 285–313.

[18] J. K. Arason, *Cohomologische Invarianten quadratischer Formen*, Journal of Algebra **36** (1975), no. 3, 448–491.

[19] J. K. Arason and A. Pfister, *Beweis des Krullschen Durchschnittsatzes für den Wittring*, Inventiones mathematicae **12** (1971), no. 2, 173–176.

[20] H. Azad, M. Barry, and G. Seitz, *On the structure of parabolic subgroups*, Communications in Algebra **18** (1990), no. 2, 551–562.

[21] S. Baek and G. Berhuy, *Cohomological invariants of central simple algebras of degree* 4, Archiv der Mathematik **98** (2012), 415–425.

[22] E. Bayer-Fluckiger and R. Parimala, *Galois cohomology of the classical groups over fields of cohomological dimension* $\leqq 2$, Inventiones mathematicae **122** (1995), no. 1, 195–229.

[23] K. J. Becher, N. Grenier-Boley, and J.-P. Tignol, *Transfer of quadratic forms and of quaternion algebras over quadratic field extensions*, Archiv der Mathematik **111** (2018), no. 2, 135–143.

[24] G. Berhuy, *Cohomological invariants of quaternionic skew-hermitian forms*, Archiv der Mathematik **88** (2007), no. 5, 434–447.

[25] ______, *An Introduction to Galois Cohomology and Its Applications*, London Mathematical Society Lecture Note Series, vol. 377, London Mathematical Society, 2010.

[26] G. Berhuy and G. Favi, *Essential dimension: A functorial point of view (after A. Merkurjev)*, Documenta Mathematica **8** (2003), 279–330.

[27] G. Berhuy, M. Monsurro, and J.-P. Tignol, *The discriminant of a symplectic involution*, Pacific Journal of Mathematics **209** (2003), no. 2, 201–218.

[28] H. Bermudez, S. Garibaldi, and V. Larsen, *Linear preservers and representations with a* 1*-dimensional ring of invariants*, Transactions of the American Mathematical Society **366** (2014), no. 9, 4755–4780.

[29] H. Bermudez and A. Ruozzi, *Degree* 3 *cohomological invariants of split simple groups that are neither simply connected nor adjoint*, Journal of the Ramanujan Mathematical Society **29** (2014), no. 4, 465–481.

[30] L. Boelaert, *From the Moufang world to the structurable world and back again*, Ph.D. thesis, Ghent University, 2013.

[31] L. Boelaert, T. De Medts, and A. Stavrova, *Moufang sets and structurable division algebras*, Memoirs of the American Mathematical Society **259** (2019), no. 1245, 1–90.

[32] A. Borel, *Linear Algebraic Groups*, second enlarged ed., Graduate Texts in Mathematics, vol. 126, Springer, 1991.

[33] A. Borel and J. Tits, *Compléments à l'article: «Groupes réductifs»*, Publications Mathématiques de l'IHÉS **41** (1972), 253–276.

[34] N. Bourbaki, *Lie Groups and Lie Algebras: Chapters 4–6*, Elements of Mathematics, Springer, 2002.

[35] M. Bremner and I. Hentzel, *Identities for algebras of matrices over the octonions*, Journal of Algebra **277** (2004), no. 1, 73–95.

[36] M. Brešar, *Derivations of tensor products of nonassociative algebras*, Linear Algebra and Its Applications **530** (2017), 244–252.

[37] P. Brosnan, Z. Reichstein, and A. Vistoli, *Essential dimension, spinor groups, and quadratic forms*, Annals of Mathematics (2010), 533–544.

[38] R. B. Brown, *A new type of nonassociative algebras*, Proceedings of the National Academy of Sciences of the United States of America **50** (1963), no. 5, 947–949.

[39] V Chernousov, *The kernel of the Rost invariant, Serre's Conjecture II and the Hasse principle for quasi-split groups* ${}^{3,6}D_4$*,* E_6*,* E_7, Mathematische Annalen **326** (2003), no. 2, 297–330.

[40] V. Chernousov and A. Merkurjev, *Essential p-dimension of split simple groups of type* A_n, Mathematische Annalen **357** (2013), no. 1, 1–10.

[41] ______, *Essential dimension of spinor and Clifford groups*, Algebra & Number Theory **8** (2014), no. 2, 457–472.

[42] C. Chevalley, *The Algebraic Theory of Spinors* (1954), The Algebraic Theory of Spinors and Clifford Algebras (Cartier P. Chevalley, C., ed.), Collected Works, vol. 2, Springer, 1996, pp. 67–192.

[43] J.-L. Clerc, *Special prehomogeneous vector spaces associated to* F_4*,* E_6*,* E_7*,* E_8 *and simple Jordan algebras of rank* 3, Journal of Algebra **264** (2003), no. 1, 98–128.

[44] B. Conrad, *Reductive group schemes*, Autour des Schémas en groupes: Volume I (S. Brochard, B. Conrad, and J. Oesterlé, eds.), Panoramas et Synthèses, vol. 42–43, Société Mathématique de France, 2014, pp. 93–444.

[45] T. De Medts, *Structurable algebras of skew-dimension one and hermitian cubic norm structures*, Communications in Algebra **47** (2019), no. 1, 154–172.

[46] J. Dieudonné, *Sur une généralisation du groupe orthogonal à quatre variables*, Archiv der Mathematik **1** (1948), no. 4, 282–287.

[47] C. Draper and A. Elduque, *Fine gradings on the simple Lie algebras of type E*, Note di Matematica **34** (2014), no. 1, 53–87.

[48] E. Dynkin, *Semisimple subalgebras of simple lie algebras.* Doklady Akademii Nauk SSSR (N.S.) 81, 6 (1951), 987–990, Selected Papers of E. B. Dynkin with Commentary (A. A. Yushkevich, G. M. Seitz, and Onishchik A. L., eds.), American Mathematical Society, 2000, pp. 175–308.

[49] A. Elduque and M. Kochetov, *Gradings on Simple Lie Algebras*, Mathematical Surveys and Monographs, vol. 189, American Mathematical Society, 2013.

[50] A. Elduque and S. Okubo, *Lie algebras with S_4-action and structurable algebras*, Journal of Algebra **307** (2007), no. 2, 864–890.

[51] ———, *S_4-symmetry on the Tits construction of exceptional Lie algebras and superalgebras*, Publicacions Matemàtiques **52** (2008), no. 2, 315–346.

[52] R. Elman and T. Y. Lam, *Pfister forms and K-theory of fields*, Journal of Algebra **23** (1972), no. 1, 181–213.

[53] R. S. Elman, N. Karpenko, and A. Merkurjev, *The Algebraic and Geometric Theory of Quadratic Forms*, Colloquium Publications, vol. 56, American Mathematical Society, 2008.

[54] F. G. Frobenius, *Über die Darstellung der endlichen Gruppen durch lineare Substitutionen.* Sitzungsberichte der Königlich Preußischen Akademie der Wissenschaften zu Berlin (1897) 994–1015, Gesammelte Abhandlungen III (J. P. Serre, ed.), Springer, 1968, pp. 82–103.

[55] R. S. Garibaldi, *Structurable algebras and groups of type E_6 and E_7*, Journal of Algebra **236** (2001), no. 2, 651–691.

[56] ———, *The Rost invariant has trivial kernel for quasi-split groups of low rank*, Commentarii Mathematici Helvetici **76** (2001), no. 4, 684–711.

[57] S. Garibaldi, *Orthogonal involutions on algebras of degree* 16 *and the Killing form of E_8*, Quadratic forms–algebra, arithmetic, and geometry (R. Baeza, W. K. Chan, D. W. Hoffmann, and R. Schulze-Pillot, eds.), Contemporary Mathematics, vol. 493, American Mathematical Society, 2007, pp. 131–162.

[58] ———, *Cohomological Invariants: Exceptional Groups and Spin Groups*, Memoirs of the American Mathematical Society **200** (2009), no. 937, 1–81.

[59] ———, *E_8, the most exceptional group*, Bulletin of the American Mathematical Society **53** (2016), no. 4, 643–671.

[60] S. Garibaldi and R. M. Guralnick, *Simple groups stabilizing polynomials*, Forum of Mathematics, Pi **3** (2015), e3.

[61] ———, *Spinors and essential dimension*, Compositio Mathematica **153** (2017), no. 3, 535–556.

[62] S. Garibaldi, A. Merkurjev, and J.-P. Serre, *Cohomological Invariants in Galois Cohomology*, University Lecture Series, no. 28, American Mathematical Society, 2003.

[63] S. Garibaldi, R. Parimala, and J.-P. Tignol, *Discriminant of symplectic involutions*, Pure and Applied Mathematics Quarterly **5** (2009), no. 1, 349–374.

[64] S. Garibaldi and H. P. Petersson, *Groups of outer type* E_6 *with trivial Tits algebras*, Transformation Groups **12** (2007), no. 3, 443–474.

[65] S. Garibaldi, V. Petrov, and N. Semenov, *Shells of twisted flag varieties and the Rost invariant*, Duke Mathematical Journal **165** (2016), no. 2, 285–339.

[66] S. Garibaldi and A. Quéguiner-Mathieu, *Restricting the Rost invariant to the center*, St. Petersburg Mathematical Journal **19** (2008), no. 2, 197–213.

[67] ———, *Pfister's theorem for orthogonal involutions of degree* 12, Proceedings of the American Mathematical Society **137** (2009), no. 4, 1215–1222.

[68] S. Garibaldi and N. Semenov, *Degree* 5 *invariant of* E_8, International Mathematics Research Notices **2010** (2010), no. 19, 3746–3762.

[69] N. Garrel, *Witt and cohomological invariants of Witt classes*, Annals of K-Theory **5** (2020), no. 2, 213–248.

[70] P. Gille, *Invariants cohomologiques de Rost en caractéristique positive*, K-theory **21** (2000), no. 1, 57–100.

[71] P. Gille and A. Quéguiner-Mathieu, *Formules pour l'invariant de Rost*, Algebra and Number Theory **5** (2011), 1–35.

[72] P. Gille and T. Szamuely, *Central Simple Algebras and Galois Cohomology*, Cambridge Studies in Advanced Mathematics, vol. 165, Cambridge University Press, 2017.

[73] A. Gyoja, *Construction of invariants*, Tsukuba Journal of Mathematics **14** (1990), no. 2, 437–457.

[74] C. Hirsch, *On the decomposability of mod* 2 *cohomological invariants of Weyl groups*, Commentarii Mathematici Helvetici **95** (2020), no. 4, 765–809.

[75] D. W. Hoffmann, *Isotropy of quadratic forms over the function field of a quadric*, Mathematische Zeitschrift **220** (1995), no. 1, 461–476.

[76] ———, *Similarity of quadratic forms and half-neighbors*, Journal of Algebra **204** (1998), no. 1, 255–280.

[77] ———, *Splitting patterns and invariants of quadratic forms*, Mathematische Nachrichten **190** (1998), no. 1, 149–168.

[78] ———, *On a conjecture of Izhboldin on similarity of quadratic forms*, Documenta Mathematica **4** (1999), 61–64.

[79] D. W. Hoffmann and J.-P. Tignol, *On* 14*-dimensional quadratic forms in* I^3*,* 8*-dimensional forms in* I^2*, and the common value property*, Documenta Mathematica **3** (1998), 189–214.

[80] G. M. D. Hogeweij, *Almost-classical Lie algebras. I*, Indagationes Mathematicae (Proceedings) **85** (1982), no. 4, 441–452.

[81] N. Hooda, *A survey on Albert algebras and groups of type F_4*, Elementary Theory of Groups and Group Rings, and Related Topics (P. Baginski, B. Fine, A. Moldenhauer, G. Rosenberger, and V. Shpilrain, eds.), De Gruyter Proceedings in Mathematics, De Gruyter, 2020, pp. 89–102.

[82] O. T. Izhboldin, *Motivic equivalence of quadratic forms*, Documenta Mathematica **3** (1998), 341–351.

[83] O. T. Izhboldin and N. A. Karpenko, *Some new examples in the theory of quadratic forms*, Mathematische Zeitschrift **234** (2000), no. 4, 647–695.

[84] B. Jacob and M. Rost, *Degree four cohomological invariants for quadratic forms*, Inventiones mathematicae **96** (1989), no. 3, 551–570.

[85] N. Jacobson, *Simple Lie algebras over a field of characteristic zero*, Duke Mathematical Journal **4** (1938), no. 3, 534–551.

[86] ______, *A Kronecker factorization theorem for Cayley algebras and the exceptional simple Jordan algebra*, American Journal of Mathematics **76** (1954), no. 2, 447–452.

[87] ______, *Composition algebras and their automorphisms*, Rendiconti del Circolo Matematico di Palermo **7** (1958), no. 1, 55–80.

[88] ______, *Some groups of transformations defined by Jordan algebras. I.*, Journal für die reine und angewandte Mathematik **201** (1959), 178–195.

[89] ______, *Lie Algebras*, Dover Books on Advanced Mathematics, Dover, 1962.

[90] ______, *Generic norm of an algebra*, Osaka Mathematical Journal **15** (1963), no. 1, 25–50.

[91] ______, *Structure and Representations of Jordan Algebras*, Colloquium Publications, vol. 39, American Mathematical Society, 1968.

[92] ______, *Structure groups and Lie algebras of Jordan algebras of symmetric elements of associative algebras with involution*, Advances in Mathematics **20** (1976), no. 2, 106–150.

[93] ______, *Basic Algebra I*, Dover, 2012.

[94] ______, *Basic Algebra II*, Dover, 2012.

[95] N. Jacobson and K. McCrimmon, *Quadratic Jordan algebras of quadratic forms with base points*, The Journal of the Indian Mathematical Society **35** (1971), 1–45.

[96] V. G. Kac, *Some remarks on nilpotent orbits*, Journal of Algebra **64** (1980), no. 1, 190–213.

[97] B. Kahn, *Formes Quadratiques sur un Corps*, Cours Spécialisés, vol. 15, Société Mathématique de France, 2008.

[98] N. A. Karpenko, *Around 16-dimensional quadratic forms in I_q^3*, Mathematische Zeitschrift **285** (2017), 433–444.

[99] E. Kleinfeld, *Alternative nil rings*, Annals of Mathematics (1957), 395–399.

[100] M.-A. Knus, *Quadratic and Hermitian Forms over Rings*, Grundlehren der mathematischen Wissenschaften, vol. 294, Springer-Verlag, 1991.

[101] M.-A. Knus, A. Merkurjev, M. Rost, and J.-P. Tignol, *The Book of Involutions*, Colloquium Publications, vol. 44, American Mathematical Society, 1998.

[102] M.-A. Knus and J.-P. Tignol, *Quartic exercises*, International Journal of Mathematics and Mathematical Sciences **2003** (2003).

[103] S. Krutelevich, *Jordan algebras, exceptional groups, and Bhargava composition*, Journal of Algebra **314** (2007), no. 2, 924–977.

[104] E. Kuzmin, *Mal'tsev algebras and their representations*, Algebra i Logika **7** (1968), no. 4, 48–69.

[105] ______, *Structure and representations of finite dimensional Malcev algebras*, Quasigroups and Related Systems **22** (2014), no. 1, 97–132.

[106] T. Y. Lam, *Introduction to Quadratic Forms over Fields*, Graduate Studies in Mathematics, vol. 67, American Mathematical Society, 2005.

[107] D. W. Lewis and T. Unger, *Hermitian Morita theory: a matrix approach*, Bulletin of the Irish Mathematical Society **62** (2008), 37–41.

[108] M. W. Liebeck and G. M. Seitz, *Unipotent and Nilpotent Classes in Simple Algebraic Groups and Lie Algebras*, Mathematical Surveys and Monographs, vol. 180, American Mathematical Society, 2012.

[109] F. Lübeck, *Small degree representations of finite Chevalley groups in defining characteristic*, LMS Journal of Computation and Mathematics **4** (2001), 135–169.

[110] M. L. MacDonald, *Cohomological invariants of odd degree Jordan algebras*, Mathematical Proceedings of the Cambridge Philosophical Society **145** (2008), no. 2, 295–303.

[111] A. Malagon, *Killing forms of isotropic Lie algebras*, Journal of Pure and Applied Algebra **216** (2012), no. 10, 2213–2224.

[112] P. Mammone and D. B. Shapiro, *The Albert quadratic form for an algebra of degree four*, Proceedings of the American Mathematical Society **105** (1989), no. 3, 525–530.

[113] K. McCrimmon, *The Freudenthal–Springer–Tits constructions of exceptional Jordan algebras*, Transactions of the American Mathematical Society **139** (1969), 495–510.

[114] ______, *The Freudenthal–Springer–Tits constructions revisited*, Transactions of the American Mathematical Society **148** (1970), no. 1, 293–314.

[115] ______, *A Taste of Jordan Algebras*, Universitext, Springer, 2006.

[116] A. Merkurjev, *Rost invariants of simply connected algebraic groups* [with a section by S. Garibaldi], Cohomological Invariants in Galois Cohomology (S. Garibaldi, A. Merkurjev, and J.-P. Serre, eds.), University Lecture Series, no. 28, American Mathematical Society, 2003, pp. 101–158.

[117] ______, *Unramified elements in cycle modules*, Journal of the London Mathematical Society **78** (2008), no. 1, 51–64.

[118] ______, *Invariants of algebraic groups and retract rationality of classifying spaces*, Algebraic Groups: Structure and Actions (M. B. Can, ed.), Proceedings of Symposia in Pure Mathematics, vol. 94, American Mathematical Society, 2017, p. 277.

[119] ______, *Versal torsors and retracts*, Transformation Groups **25** (2020), no. 3, 843–858.

[120] A. S. Merkurjev, R. Parimala, and J.-P. Tignol, *Invariants of quasitrivial tori and the Rost invariant*, Algebra i Analiz **14** (2002), no. 5, 110–151.

[121] J. Meulewaeter, *Bridging exceptional geometries and algebras using inner ideals of Lie algebras*, Ph.D. thesis, Ghent University, 2021.

[122] J. S. Milne, *Algebraic Groups: The Theory of Group Schemes of Finite Type over a Field*, Cambridge Studies in Advanced Mathematics, vol. 170, Cambridge University Press, 2017.

[123] P. J. Morandi, J. M. Pérez-Izquierdo, and S. Pumplün, *On the tensor product of composition algebras*, Journal of Algebra **243** (2001), no. 1, 41–68.

[124] S. Okubo, *Algebras satisfying symmetric triality relations*, Non-Associative Algebra and Its Applications (L. Sabinin, L. Sbitneva, and I. Shestakov, eds.), Pure and Applied Mathematics, CRC Press, 2006, pp. 313–321.

[125] D. Orlov, A. Vishik, and V. Voevodsky, *An exact sequence for $K_*^M/2$ with applications to quadratic forms*, Annals of Mathematics **165** (2007), 1–13.

[126] R. Parimala, R. Sridharan, and M. L. Thakur, *A classification theorem for Albert algebras*, Transactions of the American Mathematical Society **350** (1998), no. 3, 1277–1284.

[127] H. P. Petersson, *The structure group of an alternative algebra*, Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg **72** (2002), 165–186.

[128] ______, *A survey on Albert algebras*, Transformation Groups **24** (2019), no. 1, 219–278.

[129] H. P. Petersson and M. L. Racine, *Jordan algebras of degree* 3 *and the Tits process*, Journal of Algebra **98** (1986), no. 1, 211–243.

[130] ______, *On the invariants mod* 2 *of Albert algebras*, Journal of Algebra **174** (1995), no. 3, 1049–1072.

[131] ______, *An elementary approach to the Serre–Rost invariant of Albert algebras*, Indagationes Mathematicae **7** (1996), no. 3, 343–366.

[132] V. Petrov, *A rational construction of Lie algebras of type* E_7, Journal of Algebra **481** (2017), 348–361.

[133] V. Petrov and S. W. Rigby, *The Allison–Faulkner construction of* E_8, Canadian Mathematical Bulletin **65** (2022), no. 3, 686–701.

[134] A. Pfister, *Quadratische Formen in beliebigen Körpern*, Inventiones mathematicae **1** (1966), no. 2, 116–132.

[135] V. L. Popov, *Classification of spinors of dimension fourteen*, Transactions of the Moscow Mathematical Society **37** (1980), no. 1, 181–232.

[136] C. Procesi, *Lie groups: An approach through invariants and representations*, Universitext, Springer, 2007.

[137] S. Pumplün, *Involutions on composition algebras*, Indagationes Mathematicae **14** (2003), no. 2, 241–248.

[138] A. Quéguiner-Mathieu and J.-P. Tignol, *The Arason invariant of orthogonal involutions of degree* 12 *and* 8*, and quaternionic subgroups of the Brauer group*, Documenta Mathematica, Extra volume, Alexander S. Merkurjev's Sixtieth Birthday (2015), 529–576.

[139] ______, *Outer automorphisms of classical algebraic groups*, Annales scientifiques de l'École normale supérieure **51** (2018), no. 1, 113–141.

[140] ______, *Decomposability of orthogonal involutions in degree* 12, Pacific Journal of Mathematics **304** (2020), no. 1, 169–180.

[141] M. Rost, *Chow groups with coefficients*, Documenta Mathematica **1** (1996), no. 16, 319–393.

[142] ______, *On* 14*-dimensional quadratic forms, their spinors, and the difference of two octonion algebras*, Preprint (1999).

[143] ______, *The multiplicative transfer for the Grothendieck–Witt ring*, Preprint (2003).

[144] ______, *On the Galois cohomology of* Spin(14), Preprint (2006).

[145] M. Rost, J.-P. Serre, and J.-P. Tignol, *La forme trace d'une algèbre simple centrale de degré* 4, Comptes Rendus Mathematique **342** (2006), no. 2, 83–87.

[146] C. Ruether, *Rost multipliers of lifter Kronecker tensor products*, Documenta Mathematica **25** (2020), 1997–2022.

[147] The Sage Developers, *Sagemath, the Sage Mathematics Software System (Version 8.8)*, 2019, `https://www.sagemath.org`.

[148] M. Sato and T. Kimura, *A classification of irreducible prehomogeneous vector spaces and their relative invariants*, Nagoya Mathematical Journal **65** (1977), 1–155.

[149] R. D. Schafer, *On structurable algebras*, Journal of Algebra **92** (1985), no. 2, 400–412.

[150] ———, *Invariant forms on central simple structurable algebras*, Journal of Algebra **122** (1989), no. 1, 112–117.

[151] W. Scharlau, *Quadratic and Hermitian Forms*, Grundlehren der mathematischen Wissenschaften, vol. 270, Springer, 1985.

[152] G. M. Seitz, *Topics in the theory of algebraic groups*, Group Representation Theory (G. Meinolf, D. Testerman, and J. Thévenaz, eds.), Fundamental Sciences, EPFL Press, 2007, pp. 355–404.

[153] M. Selbach, *Klassifikationstheorie halbeinfacher algebraischer Gruppen*, Bonner Mathematische Schriften, vol. 83, 1976.

[154] N. Semenov, *Motivic construction of cohomological invariants*, Commentarii Mathematici Helvetici **91** (2016), no. 1, 163–202.

[155] A. Serhir and J.-P. Tignol, *The discriminant of a decomposable symplectic involution*, Journal of Algebra **273** (2004), no. 2, 601–607.

[156] J.-P. Serre, *Galois Cohomology*, Springer Monographs in Mathematics, Springer, 1997.

[157] ———, *Complex semisimple Lie algebras*, Springer Monographs in Mathematics, Springer, 2000.

[158] ———, *Cohomological invariants, Witt invariants, and trace forms* [Notes by S. Garibaldi], Cohomological Invariants in Galois Cohomology (S. Garibaldi, A. Merkurjev, and J.-P. Serre, eds.), University Lecture Series, no. 28, American Mathematical Society, 2003, pp. 1–100.

[159] O. N. Smirnov, *An example of a simple structurable algebra*, Algebra and Logic **29** (1990), no. 4, 331–336.

[160] ———, *Simple and semisimple structurable algebras*, Algebra and Logic **29** (1990), no. 5, 377–394.

[161] T. A. Springer, *Jordan Algebras and Algebraic Groups*, Ergebnisse der Mathematik und ihrer Grenzgebiete, 1973.

[162] ———, *Linear Algebraic Groups*, Progress in Mathematics, vol. 9, Birkhäuser, 1998.

[163] ———, *Some groups of type E_7*, Nagoya Mathematical Journal **182** (2006), 259–284.

[164] T. A. Springer and F. D. Veldkamp, *Octonions, Jordan Algebras and Exceptional Groups*, Springer Monographs in Mathematics, Springer, 2000.

[165] A. Stavrova, 5-*graded simple Lie algebras, structurable algebras, and Kantor pairs*, Journal of Algebra **611** (2022), 561–588.

[166] R. Steinberg, *Automorphisms of classical Lie algebras*, Pacific Journal of Mathematics **11** (1961), no. 3, 1119–1129.

[167] M. Thakur, *The cyclicity problem for Albert algebras*, Israel Journal of Mathematics **241** (2021), no. 1, 139–145.

[168] ———, *Albert algebras and the Tits–Weiss conjecture*, Transactions of the American Mathematical Society **375** (2022), no. 9, 6075–6091.

[169] M. L. Thakur, *Isotopy and invariants of Albert algebras*, Commentarii Mathematici Helvetici **74** (1999), no. 2, 297–305.

[170] J.-P. Tignol, *La norme des espaces quadratiques et la forme trace des algèbres simples centrales*, Théorie des Nombres. Années 1992/1993–1993/1994, Publications Mathématiques de la Faculté des Sciences Besançon, Université de Franche-Comté, 1994, pp. 1–18.

[171] ———, *Cohomological invariants of central simple algebras with involution*, Quadratic Forms, Linear Algebraic Groups, and Cohomology (J. L. Colliot-Thélène, S. Garibaldi, R. Sujatha, and V. Suresh, eds.), Developments in Mathematics, vol. 18, Springer, 2010, pp. 137–171.

[172] J. Tits, *Algèbres alternatives, algèbres de Jordan et algèbres de Lie exceptionnelles. I. Construction*, Indagationes Mathematicae **28** (1966), no. 223-237, 204.

[173] ———, *Classification of algebraic semisimple groups*, Algebraic Groups and Discontinuous Subgroups (A. Borel and G. D. Mostow, eds.), Proceedings of Symposia in Pure Mathematics, vol. 9, American Mathematical Society, 1966, pp. 33–62.

[174] ———, *Représentations linéaires irréductibles d'un groupe réductif sur un corps quelconque*, Journal für die reine und angewandte Mathematik **247** (1971), 196–220.

[175] ———, *Strongly inner anisotropic forms of simple algebraic groups*, Journal of Algebra **131** (1990), no. 2, 648–677.

[176] J. Tits and R. M. Weiss, *Moufang Polygons*, Springer Monographs in Mathematics, Springer, 2002.

[177] V. Voevodsky, *Motivic cohomology with* $\mathbf{Z}/2$*-coefficients*, Publications Mathématiques de l'IHÉS **98** (2003), 59–104.

[178] W. C. Waterhouse, *Introduction to Affine Group Schemes*, Graduate Texts in Mathematics, vol. 66, Springer, 1979.

[179] ———, *Automorphisms of* $\det(X_{ij})$*: the group scheme approach*, Advances in Mathematics **65** (1987), no. 2, 171–203.

[180] T. Wittkop, *Die Multiplikative Quadratische Norm Für Den Grothendieck–Witt–Ring*, Diplomarbeit, Universität Bielefeld, 2006.