On the linearity of the generalized Lorentz transformation

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(Received 23 December 2020; accepted 30 March 2022)

Lorentz transformations between inertial observers, along with Einstein's theory of special relativity, remedied discrepancies between Newtonian physics and Maxwell's electromagnetism caused by the use of the same time in all inertial frames. In view of the fundamental importance of the relativity between inertial observers, there have been several papers deriving generalized Lorentz transformations without using light. Proving that general transformations are linear in space and time can be done in several ways, most commonly relying on a four-dimensional Minkowski spacetime, but other approaches are possible. A method is presented here that establishes the linearity of the transformation by considering velocity transformations in the light of Einstein's first relativity postulate of 1905. Once linearity is obtained, the remainder is fairly straightforward and parallels results and methods found in the literature.

https://doi.org/10.1119/10.0010234

I. INTRODUCTION

In pre-relativistic physics, inertial reference frames move in a homogeneous, isotropic Euclidean space, allowing free choice of origin and orientation. Time is the same for all inertial observers. The associated coordinate transformation is the well-known Galilean transformation, given below.

This worked fine for mechanics, but after electromagnetism became well established, it turned out that its basic equations were not invariant under Galilean transformations between inertial observers, where Newton's first law holds. Once this problem was realized, it became necessary to establish a transformation in which electromagnetism was invariant at the cost of abandoning the concept of absolute time. This led, after several refinements, to what is now called the Lorentz transformation within the context of which it is natural to consider a four-dimensional spacetime.

A 1905 paper[1] by Einstein established the special theory of relativity, postulating that (i) the laws of physics are invariant in all inertial frames of reference and (ii) the speed of light in vacuum is the same for all inertial observers. The concepts of the aether and absolute time were swept away. Galilean transformations were shown to be correct in the limit of relative speeds that are small compared to the universal limit. From the vast literature, we only refer to some of the standard books incorporating this matter,[2–4] and, below, to selected publications.

After the general acceptance of the special relativity theory, there arose efforts to establish a generalized Lorentz transformation not linked to the invariant speed of light but based on the relativity between inertial observers in homogeneous and isotropic three-dimensional Euclidean space with time differing for each observer; see, for example, Refs. 5–9. These efforts are important because, as pointed out by Lévy-Leblond,[7] among others, the emphasis on the relation to electromagnetism, in particular, to the invariant speed of light, tends to obscure the fact that all of physics should obey a basic tenet of the special theory of relativity regarding inertial observers. The question is: How and where to start a derivation of the generalized Lorentz transformation if the invariant speed of light is not explicitly required? How do we incorporate the transformation of time from one inertial observer to the next? Of course, we all know the Lorentz transformation, but how to get there?

A first line of approach is to start from the properties of homogeneity and isotropy in Euclidean space and in time, from which it is often concluded that the transformations between the reference frames of two inertial observers in uniform rectilinear translation are linear in the coordinates and in the observer-dependent time. This is only true with certain nuances:[4] transformations should be affine functions and form upon composition an affine group.[10]

With linearity, application of the first relativity principle suffices to derive the generalized Lorentz form.[11–14] The original concept of a four-dimensional (Minkowski) spacetime[15] is based on the special relativity postulates, inclusive of the constant speed of light. Hence, it is not *a priori* obvious that it holds true for a generalized Lorentz transformation not involving light.

Other authors prove the linearity of the generalized Lorentz transformation based on alternative arguments, involving distance and time intervals rather than coordinates,[5–7,16–18] composition of velocities,[19] conservation of uniform rectilinear motions,[20,21] or an interesting Gedankenexperiment to establish an upper limit velocity.[8] In a recent paper,[9] Mathews discussed and compared several of these methods and included an extended, annotated bibliography. Some of these proofs that the generalized Lorentz transformation is linear require quite lengthy and involved analyses.

The exposition that follows in Sec. II is based on physical and mathematical arguments, not using symmetry or similar properties, but involves time through velocities, studiously ignoring results of the four-dimensional spacetime method. The proof proceeds by comparing the velocities of a point particle P observed in two inertial reference frames. This is done for three special cases: P is at rest in one frame, then in the other, and finally moves in both under the first law of Newton. This method has the advantage that it generates simple analytical relations in one reference frame, in a way that is—to my knowledge—novel. Their combination yields the linearity of the generalized Lorentz transformation.

Once linearity is established, the remainder of the derivation of the generalized Lorentz transformation becomes fairly straightforward and uses arguments found in the literature, involving, e.g., moving rods and/or clocks,[5,7,12,18,21] and

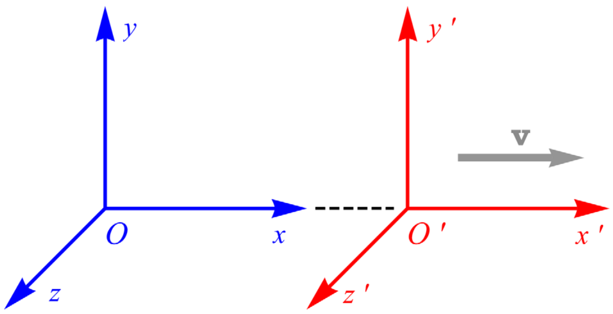

Fig. 1. (Color online) Sketch of the coordinate axes for two inertial frames $\mathcal{O}$ and $\mathcal{O}'$, as observed in $\mathcal{O}$. The x (blue) and x' (red) axes are parallel to $\mathbf{v}$.

the composition of velocities among three inertial observers.[4,5,7–10,12–14,16,18,19,21]

Section III briefly summarizes the conclusions.

II. MATHEMATICAL ANALYSIS

A. Preliminary considerations

Before we get to the mathematical details, we need to discuss the reference frames of a pair of inertial observers. In principle, one can freely choose origin and orientations of the axes of right-handed frames because of the assumed homogeneity and isotropy of three-dimensional space. Given a uniform translation velocity $\mathbf{v}$ of one observer relative to the other, it is simplest to choose the x and x' coordinate axes parallel to $\mathbf{v}$, such that $x = x'$ and $t = t'$ for $\mathbf{v} = 0$. In a plane perpendicular to $\mathbf{v}$, we can take *any* right-handed pair of mutually orthogonal axes for y and z, and y' and z'. We also assume that $y = y'$ and $z = z'$ when $\mathbf{v} = 0$, as illustrated in Fig. 1.

Furthermore, without loss of generality, we can take the spatial origins to coincide when the times in the two frames are both zero, i.e., $x = x' = 0$ when $t = t' = 0$. The unprimed quantities are for one observer and the primed for the other.

In our exposition, as in many others, the perpendicular directions y and z can be omitted because they are irrelevant to the relations between $\{x, x', t, t'\}$. Explicit proofs for the property that $y = y'$ and $z = z'$ remain also when $\mathbf{v} \neq 0$ are available,[5,19] all stressing that the choice of the $\{y, z\}$ and $\{y', z'\}$ axes is arbitrary and that parallel motions are not affected by purely geometric shifts in the directions perpendicular to that of $\mathbf{v}$.

Turning first to Newtonian mechanics, the change of coordinates between two inertial reference frames is given by the Galilean transformation

$$x' = x - vt, \tag{1}$$

and time is an absolute concept. The interpretation of this equation is that a point particle P is observed at the position x at time t by an inertial observer $\mathcal{O}$, and at x' at the same time t by another inertial observer $\mathcal{O}'$, and v along the x and x' axes is the (constant) velocity of $\mathcal{O}'$ measured by $\mathcal{O}$.

The Galilean transformation involves an absolute time but obeys, nevertheless, *a posteriori*, the special relativity principle, stating that all inertial observers are equivalent as to the form of the laws of physics with the exception of Maxwell's theory of classical electromagnetism. In other words, the inverse transformation from $\mathcal{O}'$ to $\mathcal{O}$ is of the same functional form

$$x = x' - v't. \tag{2}$$

This requires the interchange of $x \leftrightarrow x'$ and $v \leftrightarrow v'$, but the inverse transformation can also be obtained by simply solving Eq. (1) for x, yielding

$$x = x' + vt. \tag{3}$$

It also implies, within the context of the Galilean transformation, that $v' = -v$, telling us that when $\mathcal{O}'$ moves in the positive x and x' directions relative to $\mathcal{O}$ with speed $|v|$, $\mathcal{O}$ moves in the negative x and x' directions relative to $\mathcal{O}'$ with speed $|v'| = |v|$.

The Galilean transformation indicates that Newtonian physics obeys the special relativity theory, provided $t = t'$. However, this Newtonian concept of absolute time must be abandoned in order to obtain the generalized Lorentz transformation using relativity arguments.

Thus, to derive a generalized Lorentz transformation, we start from a general form

$$x' = F(x, t; v), \quad t' = G(x, t; v) \tag{4}$$

and require that its inverse is

$$x = F(x', t'; v'), \quad t = G(x', t'; v'). \tag{5}$$

It is important to remark here that the functional form of the generalized Lorentz transformation (4), and its inverse Eq. (5), must be the same,[5] because if that were not the case, a kind of classification could be established that invalidates the relativity and equivalence between inertial observers. Mathematically speaking, functions linear in x and in t (or in x' and in t') obviously would fit this requirement[5,9,20,21] but is it the only possibility, in this formalism?

The interpretation of Eq. (4) at this stage is that $\mathcal{O}$ sees P at x at time t, and $\mathcal{O}'$ at x' at t'. Also, v along x is the velocity of $\mathcal{O}'$ measured by $\mathcal{O}$, and v' along x' that of $\mathcal{O}$ seen by $\mathcal{O}'$.

The derivations of the generalized Lorentz transformation transcend electromagnetic theory and apply to all of classical physics within the limitations underlying those derivations.[4] We will establish from first principles, in a novel way, that transformation (4) is linear in its arguments x and t.

Once linearity is proved, the remainder of the derivation is straightforward and leads to the generalized Lorentz transformation with a constant maximal speed V. This includes, in the mathematical limit $V \to \infty$, the Galilean transformation as a special example.

B. Linearity of the generalized Lorentz transformation

A first step is to assume the reciprocity of the translation velocities v and v' so that $v' = -v$, as in Galilean relativity. Indeed, as $\mathcal{O}'$ moves away from $\mathcal{O}$, with a certain constant velocity, $\mathcal{O}$ is seen by $\mathcal{O}'$ to recede in the opposite direction with the same velocity in absolute value. This is used by Einstein[1] and von Ignatowski,[5] and often presented in the literature as following from the special relativity theory. A more exhaustive discussion of the velocity reciprocity property is given in a very recent paper by Moylan,[22] arguing rightly that this property does not always hold true for all situations and for all kinematics. However, for the Galilean and Lorentz transformations velocity reciprocity is true.[22]

Start by considering a point particle P with position x at time t and velocity p, as observed by $\mathcal{O}$, and position x', time t', and velocity p', as observed by $\mathcal{O}'$. By differentiating Eq. (4) and defining $p = dx/dt$ and $p' = dx'/dt'$, one obtains that

$$p' = \frac{dx'}{dt'} = \frac{\dfrac{\partial F}{\partial x}\,dx + \dfrac{\partial F}{\partial t}\,dt}{\dfrac{\partial G}{\partial x}\,dx + \dfrac{\partial G}{\partial t}\,dt} = \frac{p\,\dfrac{\partial F}{\partial x} + \dfrac{\partial F}{\partial t}}{p\,\dfrac{\partial G}{\partial x} + \dfrac{\partial G}{\partial t}}. \tag{6}$$

For notational simplicity, the arguments x, t, and v of F and G have been suppressed. We will discuss Eq. (6) for three special cases.

In inertial frames, a point particle can, in the absence of a force, only be at rest or move on a uniform rectilinear trajectory according to the first law of Newton. If there is no force in one reference frame, there is no force in the other frame, since there is no source for it.

For $p=0$, P is at rest relative to $\mathcal{O}$, so no force is working on it and consequently in $\mathcal{O}'$ p' is constant. Homogeneity of space allows the origin of $\mathcal{O}$ to be chosen arbitrary. So, one could first take it at P, which should not affect the velocity of $\mathcal{O}$ relative to $\mathcal{O}'$, hence $p=0$ leads to $p' = v' = -v$. Thus, Eq. (6) yields

$$\frac{\partial F}{\partial t} + v\,\frac{\partial G}{\partial t} = 0. \tag{7}$$

The converse case $p' = 0$ means that P is at rest relative to $\mathcal{O}'$ with $p = v$ and, thus,

$$v\,\frac{\partial F}{\partial x} + \frac{\partial F}{\partial t} = 0. \tag{8}$$

This implies that F is a function of the combined argument $x - vt$ as well as of v. Comparing Eqs. (7) and (8) indicates that

$$\frac{\partial G}{\partial t} = \frac{\partial F}{\partial x}. \tag{9}$$

While Eqs. (7)–(9) were derived using particular assumptions concerning p and p', they must hold for all situations, as F and G are functions of $\{x, x', t, t', v\}$ and are independent of p and p'.

For the next step, let P move with constant velocity p along the x axis, seen by $\mathcal{O}$, with the logical consequence that P moves with constant velocity p' too, along the x' axis, seen by $\mathcal{O}'$. Substitution of Eqs. (8) and (9) into Eq. (6) yields

$$p' = \frac{p - v}{p\,C(v) + 1}, \tag{10}$$

after dividing out the common factor $\partial F/\partial x$ and introducing

$$C(v) = \frac{\partial G}{\partial x}\bigg/\frac{\partial F}{\partial x}. \tag{11}$$

Indeed, because in Eq. (10), p, p', and v are constant velocities, the ratio $(\partial G/\partial x)/(\partial F/\partial x)$ cannot depend on x and t, at most on v.

As an aside, Galilean relativity implies $t' = t$, so from Eq. (4) we see that $\partial G/\partial x$ is zero, and thus, $C(v) = 0$. To obtain the generalized Lorentz transformation, one will need to assume that $C(v) \neq 0$.

Taking the derivative of Eq. (7) with respect to x and of Eq. (11) with respect to t yields

$$\frac{\partial^2 F}{\partial t\,\partial x} + v\,\frac{\partial^2 G}{\partial t\,\partial x} = 0, \quad C(v)\,\frac{\partial^2 F}{\partial t\,\partial x} - \frac{\partial^2 G}{\partial t\,\partial x} = 0. \tag{12}$$

This is a set of two homogeneous linear equations in the mixed derivatives of F and G with a nonzero determinant of the coefficients, $[1 + vC(v)]$. This set can only have the trivial solution

$$\frac{\partial^2 F}{\partial t\,\partial x} = 0, \quad \frac{\partial^2 G}{\partial t\,\partial x} = 0. \tag{13}$$

When the mixed derivative of a function of two variables vanishes, the function is additively separable. This proves that F and G are additively separable in x and t and, thus, can be written in the form

$$\begin{aligned} F(x, t; v) &= F_a(x; v) + F_b(t; v), \\ G(x, t; v) &= G_a(x; v) + G_b(t; v). \end{aligned} \tag{14}$$

In Eq. (8), this yields that

$$v\,\frac{dF_a}{dx} + \frac{dF_b}{dt} = 0. \tag{15}$$

This can only hold if the derivatives dF_a/dx and dF_b/dt, depending on either x or t, are constant, indicating that F_a and F_b are linear in their respective arguments x and t. Hence, F is a linear function of x and t through the combination $x - vt$, and we obtain that

$$x' = F(x, t; v) = A(v)(x - vt), \tag{16}$$

having invoked the requirement that $x = t = 0$ yields $x' = 0$. For $v = 0$, the two frames coincide, requiring that $A(0) = 1$.

Similarly, using Eqs. (9) and (11), the structure of G is

$$t' = G(x, t; v) = A(v)[C(v)x + t]. \tag{17}$$

When both frames coincide for $v = 0$, $t' = t$, and thus, $C(0) = 0$. Introducing $B(v)$ through $C(v) = vB(v)$, Eq. (17) becomes

$$t' = G(x, t; v) = A(v)[vB(v)x + t]. \tag{18}$$

For $B(v) \neq 0$, Eq. (18) indicates that x and t are intrinsically linked.

C. Further to the generalized Lorentz transformation

Now we start by inverting Eqs. (16) and (18), which can be done in two different ways. One is by interchanging the primed and unprimed variables and, invoking the velocity reciprocity, replacing v by – v, so that

$$x = A(-v)(x' + vt'), \quad t = A(-v)\left[-vB(-v)x' + t'\right] \tag{19}$$

results. The other method is simply solving Eqs. (16) and (18) for x and t, yielding

$$x = \frac{x' + vt'}{A(v)\left[1 + B(v)v^2\right]}, \quad t = \frac{-vB(v)x' + t'}{A(v)\left[1 + B(v)v^2\right]}. \tag{20}$$

Equating in Eqs. (19) and (20), the coefficients of x' lead to

$$A(-v) = \frac{1}{A(v)\left[1 + B(v)v^2\right]},$$
$$A(-v)B(-v) = \frac{B(v)}{A(v)\left[1 + B(v)v^2\right]}. \tag{21}$$

Satisfying both these equations requires

$$B(-v) = B(v), \tag{22}$$

and $B(v)$ is an even function of v. Equating the coefficients of t' does not give additional results.

To prove that $A(v)$, too, is even in v, we consider the "classic" problem of moving rods. Use Eq. (16) for a rod with length $\Delta x'$ at rest relative to $\mathcal{O}'$, which moves relative to $\mathcal{O}$, so that this observer has to make a snapshot of the endpoints at the same time, $\Delta t = 0$. This gives

$$(\Delta x')_{at\, rest\, in\, \mathcal{O}'} = A(v)(\Delta x)_{moving\, in\, \mathcal{O}}. \tag{23}$$

Considering instead a rod at rest relative to $\mathcal{O}$ yields

$$(\Delta x)_{at\, rest\, in\, \mathcal{O}} = A(-v)(\Delta x')_{moving\, in\, \mathcal{O}'}. \tag{24}$$

If the rods are identical for observers seeing them at rest in their respective frames, the special relativity principle requires that in the frames where they are observed to be moving, the observed lengths must be equal,[5] yielding

$$A(-v) = A(v). \tag{25}$$

Note that this has been proved *after* linearity was established, not the other way around, where length changes have been used to prove that the generalized Lorentz transformation is linear in space and in time.[5,6,12,18,21] This discussion points to the Lorentz–FitzGerald contraction, the shortening of an object along the direction of its motion relative to an observer. Dimensions in other directions are not contracted.

One can give a similar computation for ticking clocks and the dilatation of time intervals between moving observers, leading again to Eq. (25).

From Eqs. (21), (22), and (25), there follows also that

$$A^2(v) = \frac{1}{1 + B(v)v^2}, \tag{26}$$

and the form of $B(v)$ determines the expression for $A(v)$. We note that $A(v)$ is dimensionless, and $B(v)$ has the dimensions of an inverse squared velocity.

To complete the description, three observers are considered, two by two. Use $\{x, t\}$ for $\mathcal{O}$, $\{x', t'\}$ for $\mathcal{O}'$ and $\{x'', t''\}$ for $\mathcal{O}''$. Furthermore, take v as the velocity of $\mathcal{O}'$ seen by $\mathcal{O}$, w as the velocity of $\mathcal{O}''$ seen by $\mathcal{O}'$ and u as the velocity of $\mathcal{O}''$ seen by $\mathcal{O}$. Thus, the transformations from $\mathcal{O}$ to $\mathcal{O}'$ give Eqs. (16) and (18), from $\mathcal{O}'$ to $\mathcal{O}''$,

$$x'' = A(w)(x' - wt'), \quad t'' = A(w)\left[wB(w)x' + t'\right], \tag{27}$$

and finally, from $\mathcal{O}$ to $\mathcal{O}''$,

$$x'' = A(u)(x - ut), \quad t'' = A(u)[uB(u)x + t]. \tag{28}$$

Substitution of Eqs. (16) and (18) into Eq. (27) yields

$$x'' = A(v)A(w)\left\{[1 - B(v)vw]x - (v + w)t\right\},$$
$$t'' = A(v)A(w)\left\{[B(v)v + B(w)w]x + [1 - B(w)vw]t\right\}. \tag{29}$$

Gathering information from Eqs. (28) and (29), the coefficients of x in x'' and of t in t'' indicate that

$$A(u) = A(v)A(w)[1 - B(v)vw]$$
$$= A(v)A(w)[1 - B(w)vw]. \tag{30}$$

Similarly, the coefficients of t in x'' yield

$$A(u)u = A(v)A(w)(v + w). \tag{31}$$

The result (30) proves that $B(v) = B(w)$, for arbitrary v and w, and thus, B is a true constant, independent of v. Dividing Eq. (31) by Eq. (30) gives

$$u = \frac{v + w}{1 - Bvw}. \tag{32}$$

The above reasoning is based on the group property of the generalized Lorentz transformations, but because of the linearity of these transformations, it is a mere question of calculation.

It is clear that positive and finite velocities v and w imply that u has to be positive and finite, too. If B were positive, that could not always be the case, so that B has to be either zero or negative.

For $B = 0$, we obtain from Eq. (26) that $A(v)^2 = 1$, and hence, $A = 1$, as $A(0) = 1$ was chosen from the outset. All this yields the Galilean transformation (1) with the standard addition of different translation velocities as

$$u = v + w. \tag{33}$$

On the other hand, when B is negative, introduce a constant velocity V such that $B = -1/V^2$, given the dimensions of B. Now Eq. (26) yields

$$A(v) = \frac{1}{\sqrt{1 - \frac{v^2}{V^2}}}. \tag{34}$$

Before going to possible interpretations of V, note from Eq. (32) that

$$u = \frac{v + w}{1 + \frac{vw}{V^2}}. \tag{35}$$

Consequently, the "addition" of $v = V$ to another velocity $0 < w \leq V$ still would yield $u = V$, nothing larger than V. In other words, V is a maximal speed. We thus obtain the generalized Lorentz transformation

$$x' = \frac{x - vt}{\sqrt{1 - \frac{v^2}{V^2}}}, \quad t' = \frac{t - \frac{v}{V^2}x}{\sqrt{1 - \frac{v^2}{V^2}}}. \tag{36}$$

The Galilean transformation (1), for $B = 0$, can formally be obtained from Eq. (36) in the limit $V \to +\infty$.

It has been proved[9,12] that V can be measured and shown to be c, the speed of electromagnetic radiation in vacuum, without any use of electromagnetic radiation in the measurement process. With $V = c$, Eq. (36) becomes the Lorentz transformation.

III. CONCLUSIONS

In view of the fundamental importance of the relativity between inertial observers, there have been several papers deriving generalized Lorentz transformations without using the light to show that abandoning the idea of absolute space and absolute time is relevant to classical mechanics, too. All derivations eventually lead to the concept of a four-dimensional spacetime, as formally worked out by Minkowski.[15]

Despite an abundance of ways to prove that the generalized Lorentz transformation is linear in x and t, I am not aware that the proof established here has been presented before. As others, it starts from the relativity part of Einstein's postulates of 1905 but involves velocities and their relative changes in three simple cases. Once linearity is obtained, the remainder is fairly straightforward and parallels results and methods found in the literature.

ACKNOWLEDGMENTS

The author thanks the referees for their critical but constructive assessments.

AUTHOR DECLARATIONS

Conflict of Interest

The author has no conflict of interest to disclose.

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[1]A. Einstein, "Zur elektrodynamik bewegter Körper," Ann. Phys. **17**, 891–921 (1905), see page 895: "Die Gesetze, nach denen sich die Zustände der physikalischen Systeme ändern, sind unabhängig davon, auf welches von zwei relativ zueinander in gleichförmiger Translationsbewegung befindlichen Koordinatensystemen diese Zustands-anderungen bezogen werden." In English: "The laws by which the states of physical systems undergo change are not affected, whether these changes of state be referred to the one or the other of two systems of coordinates in uniform translatory motion."

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[4]O. L. De Lange and J. Pierrus, *Solved Problems in Classical Mechanics: Analytical and Numerical Solutions with Comments* (Oxford U. P., Oxford, 2010), pp. 557–562. This is an excellent and delightful treatise on classical mechanics, through problems. It contains as its final chapter a nice introduction of the special relativity formalism, a rather unusual feature in classical mechanics textbooks.

[5]W. v. Ignatowsky, "Das Relativitätsprinzip," Arch. Math. Phys. **17**, 1–24 (1911); **18**, 17–40 (1911), see page 22 of the second reference: "daß der Übergang von dem ungestrichenen zum gestrichenen System einfach dadurch bewerkstelligt wird, daß man die gestrichenen Größen durch die ungestrichenen ersetzt und umgekehrt, und q durch $-q$. Diese Regel ist im folgenden stets anzuwenden." My translation of the German text: "The transition from the unprimed to the primed [coordinate] system is simply achieved by replacing the primed variables by the unprimed and vice versa, and q by $-q$. This rule is always to be used in what follows."

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