Critical Lattice Model for a Haagerup Conformal Field Theory

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We use the formalism of strange correlators to construct a critical classical lattice model in two dimensions with the *Haagerup fusion category* $\mathcal{H}_3$ as input data. We present compelling numerical evidence in the form of finite entanglement scaling to support a Haagerup conformal field theory (CFT) with central charge $c = 2$. Generalized twisted CFT spectra are numerically obtained through exact diagonalization of the transfer matrix, and the conformal towers are separated in the spectra through their identification with the topological sectors. It is further argued that our model can be obtained through an orbifold procedure from a larger lattice model with input $Z(\mathcal{H}_3)$, which is the simplest modular tensor category that does not admit an algebraic construction. This provides a counterexample for the conjecture that all rational CFT can be constructed from standard methods.

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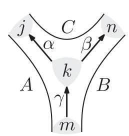
Introduction.—Conformal field theory (CFT) plays a central role throughout the natural sciences, from string theory, to the standard model of fundamental physics, through to the effective description of many body systems at criticality [1,2]. As a consequence, the study of CFTs has been an extremely active and vibrant research area since their introduction [3]. Fortunately, the rich symmetries exhibited by CFTs, particularly in $1 + 1$ dimensions, have enabled dramatic progress in the study of their properties and their classification. A prominent role here is played by rational CFTs (RCFT), which supply a rich family of atomic building blocks for general CFTs. These models are highly constrained and, since the inception of CFT, there has been optimism for their classification [4,5]. Considerable progress toward this goal has been achieved. More specifically, it was shown that the underlying mathematical framework of 2D RCFTs is a modular tensor category (MTC) [6] by establishing a holographic map between 3D topological field theory and 2D CFT. Conversely, it is unknown whether one can construct a CFT based on any modular tensor category and moreover, if standard CFT constructions, such as orbifolds, cosets, and simple-current extensions, can produce all RCFTs when applied to the catalogue of basic rational theories [4,5,7,8].

The conjecture that all RCFTs can be produced via standard constructions has always been perhaps too bold as there are a variety of potential counterexamples. One particularly exotic candidate which has risen to recent prominence is a putative (R)CFT whose chiral modular data would be realized by the quantum double $\mathcal{DH}_3$ [or the Drinfeld center $Z(\mathcal{H}_3)$] of the *Haagerup fusion category* $\mathcal{H}_3$ [7,9]. This fusion category arose in the mathematical theory of subfactors [10] and has, so far, only been constructed via baroque combinatorial methods [11–13].

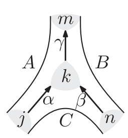
The first serious efforts to produce a Haagerup (R)CFT commenced with the work of Jones, who exploited ideas from tensor networks to directly build CFT-like continuum theories from fusion category data [14–16]. While this initial idea was ultimately unsuccessful [16,17], it did open the door to the application of the Haagerup case to the well-known anyon chains [18–20] and to a host of recently developed methods [21–24], in particular a research program targeting the systematic construction of full CFTs from topological modular data. This program is built on the premise that an arbitrary CFT may be microscopically realized via the strange correlator [22,25,26] applied to different tensor network representations [24] of Levin-Wen string-net models [27–29]. This idea is attractive for a variety of reasons: (1) it provides a systematic way to build the geometric correlation data of a CFT from the purely topological modular data, (2) it supplies a clear and direct realization of the symmetries of the CFT via matrix product operators (MPO), and (3) it enables the direct application of tensor-network methods to study the resulting critical lattice model and enables the selection of topological sectors in a systematic way. The program is essentially a lattice implementation of the description of the topological aspects of 2D RCFTs in terms of MTCs and their representations, as described in a series of detailed papers by Fuchs, Fröhlich, Runkel, and Schweigert [30–32].

In this Letter we report on a critical classical lattice model, obtained via the strange-correlator construction applied to a string-net tensor network with the $\mathcal{H}_3$ fusion category data as input. The argumentation can be summarized into three different parts: (i) we give numerical evidence that this statistical mechanics model is associated to a CFT with central charge $c = 2$; (ii) we argue that this lattice model can equally be obtained from a string-net model based on the monoidal center $Z(\mathcal{H}_3)$ by an orbifold procedure; and (iii) we further speculate that this CFT, in turn, can be obtained from the conjectured Haagerup (R) CFT (which is understood to have central charge divisible by 8) by a coset construction. We apply a complementary portfolio of numerical methods, including anyonic infinite variational tensor network methods and a general numerical method for selecting topological sectors in critical lattice models with input data of potentially nonbraided fusion categories. This method generalizes the special case of modular input categories discussed in Ref. [33].

A strange correlator for the $\mathcal{H}_3$ fusion category.—We construct a two-dimensional lattice model starting from generalized projected entangled pair state (PEPS) representations of string-net ground states as described in Ref. [24]. The construction requires two fusion categories $\mathcal{C}$ and $\mathcal{D}$ and a $(\mathcal{C},\mathcal{D})$-bimodule category $\mathcal{M}$. In the remainder of this Letter, we will use the convention that $\mathcal{C}$ labels the symmetries, $\mathcal{D}$ is the input of the string-net construction, and $\mathcal{M}$ labels the virtual degrees of freedom of the tensor network. We can build the generalized string-net ground state on a honeycomb lattice using the following trivalent PEPS tensors:



$$:= \frac{\left({}^{3}F^{A\alpha\beta}_{B}\right)^{\gamma,km}_{C,jn}}{\sqrt{d_C}},$$

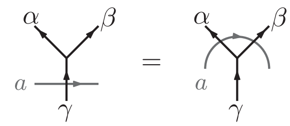


$$:= \frac{\left({}_{3}F^{A\alpha\beta}_{B}\right)^{\gamma,km}_{C,jn}}{\sqrt{d_C}}, \tag{1}$$

where ${}^{3}F$ (${}_{3}F$) is the (inverse) module associator of $\mathcal{M}$ as a right $\mathcal{D}$ module category with $\{A,B,C\} \in \mathcal{M}$ and $\{\alpha,\beta,\gamma\} \in \mathcal{D}$. For our model we will make the choice $\mathcal{C} = \mathcal{M} = \mathcal{D} = \mathcal{H}_3$ with $\mathcal{H}_3$ the $G = \mathbb{Z}_3$ Haagerup-Izumi category [7,34]. This choice coincides with the original PEPS representation for string-net ground states [35,36], in which case ${}^{3}F = F$ is a unitary solution of the pentagon equation, and the fusion multiplicities are trivial [34,37,38]. $\mathcal{H}_3$ has six simple objects $\{\mathbf{1},\alpha,\alpha^*\}$ (we will call type-1) and $\{\rho, {}_{\alpha}\rho, {}_{\alpha^*}\rho\}$ (we will call type-ρ), with nontrivial fusion rules $\alpha^3 = \mathbf{1}$,

$$\begin{aligned} {}_{\alpha}\rho &= \alpha \otimes \rho, & {}_{\alpha^*}\rho &= \alpha^* \otimes \rho \\ \rho \otimes \alpha &= {}_{\alpha^*}\rho, & \rho \otimes \alpha^* &= {}_{\alpha}\rho \end{aligned} \tag{2}$$

and $\rho \otimes \rho = \mathbf{1} \oplus \rho \oplus {}_{\alpha}\rho \oplus {}_{\alpha^*}\rho$. The other fusion rules of the form type-$\rho \otimes$ type-ρ can be obtained from Eq. (2). The fusion rules admit a Fibonacci grading between the type-1 and the type-ρ objects: type-$\rho \otimes$ type-$\rho = \{\text{type-}\mathbf{1}\} \oplus \{\text{type-}\rho\}$. The corresponding quantum dimensions are $d_{\text{type-}\mathbf{1}} = 1$ and $d_{\text{type-}\rho} = (3 + \sqrt{13}/2)$. The PEPS tensors have virtual MPO symmetries [36], i.e., stringlike operators that can be freely pulled through the lattice without any action on the physical indices ($a \in \mathcal{C} = \mathcal{H}_3$):



$$. \tag{3}$$

For diagrammatic convenience, we have omitted the triple line notation and will keep doing so going forward. In the next step, we choose a strange correlator [22] by fixing the physical indices (greek letters) of the ground state to ρ, obtaining a lattice partition function where the degrees of freedom are the loops of the original PEPS. The resulting partition function will inherit the virtual MPO symmetries of the PEPS, which become the lattice manifestation of the continuum CFT topological defects. The result of the strange correlator choice is a hard constraint between neighbouring plaquettes, not allowing any two adjacent plaquettes both labeled by type-one objects. The situation is very similar as in the case of the hard hexagon model [22,39]. The adjacency rules for neighboring plaquettes can be diagramatically shown as a Dynkin diagram (Fig. 1). Note that some of the Boltzmann weights of the model [originating from Eq. (1)] are negative, but that every configuration on a torus has a positive contribution to the partition function.

Besides the internal $\mathbb{Z}_3$ symmetry generated by α, the model has an extra $\mathbb{Z}_3$ sublattice symmetry. This can be seen in the maximally occupied configuration (see Fig. 1) [39]. As a consequence of the sublattice symmetry, we have to define the transfer matrix of the model on a ring of length $L = 3n, n \in \mathbb{Z}$. Choosing $L \neq 3n$ amounts to introducing a nontrivial twist. The $\mathbb{Z}_3$ sublattice symmetry is generated by shifting the lattice by one site and becomes an invertible topological defect line of the CFT in the continuum limit.

The topological sectors of the model are given by the Drinfeld center $Z(\mathcal{H}_3)$, which has twelve simple objects, labeled by (Z,Ω), where Z is an object in $\mathcal{H}_3$ and Ω the half

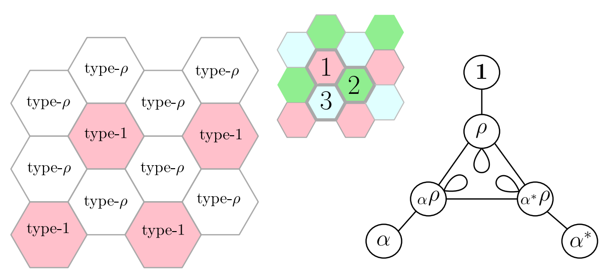


FIG. 1. The partition function of the model on the hexagonal lattice in a maximally occupied configuration. There are three maximally occupied configurations as indicated by the three different sublattices. The adjacency rules of neighboring particles can be shown by the corresponding Dynkin diagram.

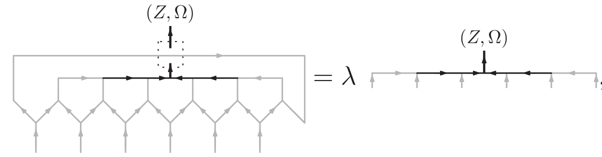
TABLE I. The twelve simple objects of $Z(\mathcal{H}_3)$ labeled by an object $Z \in \mathcal{H}_3$ and a half braiding Ω. The last column is the corresponding quantum dimension.

$Z(\mathcal{H}_3)$	$Z \in \mathcal{H}_3$	Ω	Dim
id	$\mathbf{1}$	id^1	1
$\mu^{1,2,3,4,5,6}$	$\rho \oplus {}_\alpha\rho \oplus {}_{\alpha^*}\rho$	$\mu^{1,2,3,4,5,6}$	$3d_\rho$
π_1	$\mathbf{1} \oplus \rho \oplus {}_\alpha\rho \oplus {}_{\alpha^*}\rho$	π_1^1	$3d_\rho + 1$
π_2	$\mathbf{1} \oplus \mathbf{1} \oplus \rho \oplus {}_\alpha\rho \oplus {}_{\alpha^*}\rho$	π_2^1	$3d_\rho + 2$
$\sigma^{1,2,3}$	$\alpha \oplus \alpha^* \oplus \rho \oplus {}_\alpha\rho \oplus {}_{\alpha^*}\rho$	$\sigma^{1,2,3}$	$3d_\rho + 2$

braiding. If we write Z in the basis of simple objects, we can label the twelve objects as in Table I, using a similar labeling as in Refs. [7,40].

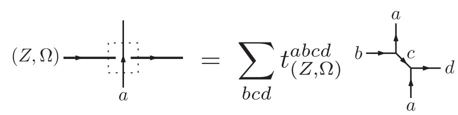
Numerical results.—We have performed variational uniform matrix product state (VUMPS) simulations [41] for the MPS fixed point of the transfer matrix in the thermodynamic limit for increasing bond dimension. The algorithm explicitly preserves the anyonic $\mathcal{H}_3$ symmetry, allowing for higher bond dimensions than standard methods [33,42–44]. As the entanglement entropy in an infinite chain diverges with increasing MPS bond dimension, approximating the critical point, it scales as $S = (c/6)log(\xi)$ [45–50], with ξ the MPS correlation length. The result is shown in Fig. 2 up to $\xi \approx 150$ and strongly indicates a critical theory with central charge $c = 2$.

Secondly, we have performed exact diagonalization with anyonic symmetry on the transfer matrix with periodic boundary conditions [43]. The method, for modular tensor categories, is explained in great detail in Refs. [51–53]. It consists of writing states as an anyonic fusion tree, such that the action of the transfer matrix on these states can be computed using F moves. We then solve the following eigenvalue problem (illustrated here for a ring of six sites):



(4)

where the gray lines are fixed to ρ. The eigenvector is chosen in a specific topological sector (Z, Ω) by fixing the total charge of the fusion tree to Z and using the half braiding Ω whenever a crossing is required. These half braidings can be obtained from the tube algebra idempotents that project on a topological sector in $Z(\mathcal{H}_3)$:



(5)

We refer to Ref. [54] and the Supplemental Material of [22] for a detailed discussion around the tube algebra idempotent decomposition for the toplogical sectors in terms of the coefficients t. To obtain a full spectrum, the diagonalization scheme is repeated for every sector (Z, Ω). The simple objects in the decomposition of $Z = \bigoplus_a a$ for a given sector (see Table I) indicate the presence of that sector in the spectrum of the transfer matrix twisted by the corresponding topological twists a. The spectra of the transfer matrix with a trivial ($a = \mathbf{1}$), α- and ρ-twist on $L = 15$ sites are shown in Fig. 3, together with the trivial sector on $L = 18$ sites. The numerically obtained ground state in the trivial sector has a finite-size correction $E_0 \sim fL + (\pi c v/6L)$ [55], where $\exp(-f)$ is the free energy per site in the thermodynamic limit and v the characteristic velocity, both of which can be determined by fitting the ground state energy for several sizes $L = 6$, 9, 12, 15, 18. We label the spectra with the topological sectors [elements in $Z(\mathcal{H}_3)$] (Table I). The conformal spins in each sector acquire a topological correction shown in Table II.

The torus partition function (twisted in one direction by a) is of the form $Z_a \simeq \sum_{\alpha,\bar{\beta}} \chi_\alpha(q) \tilde{M}^a_{\alpha\bar{\beta}} \bar{\chi}_{\bar{\beta}}(\bar{q})$ [56] and is in particular modular invariant for $a = \mathbf{1}$. Projecting the spectrum onto a topological sector amounts to breaking down the partition function into single (or possibly sums) of sesquilinear character terms [22,26]. The conformal spin $s = h_\alpha - h_\beta$ (h is the conformal weight) of the lowest lying eigenvalue in the tower $\chi_\alpha \bar{\chi}_{\bar{\beta}}$ corresponds to the topological spin of that sector. Note that the multiplicity of the simple object one in the sector π_2 (see Table I) signals an exact degeneracy in the 1-twisted spectrum and a corresponding multiplicity in a term of the partition function ($2|\chi_\alpha|^2$). The partition function is expected to be non-diagonal ($M_{\alpha\beta} \neq \delta_{\alpha,\beta}$).

Discussion—Following Ref. [26], a critical lattice model built from a strange correlator requires two pieces

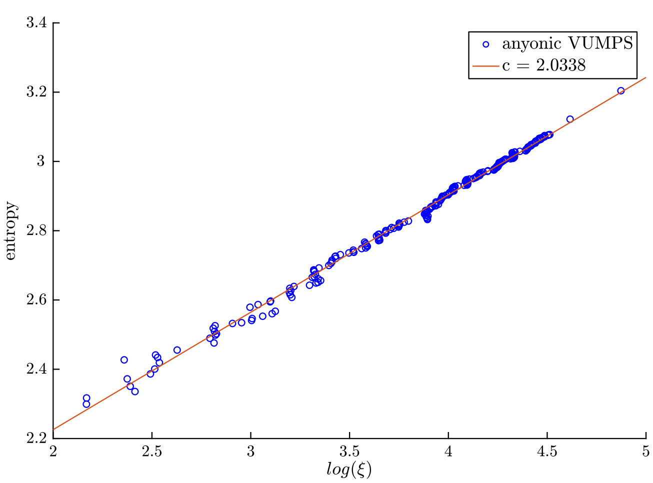


FIG. 2. Finite entanglement scaling for the fixed point MPS of the transfer matrix calculated using VUMPS with explicit $\mathcal{H}_3$ anyonic symmetry.

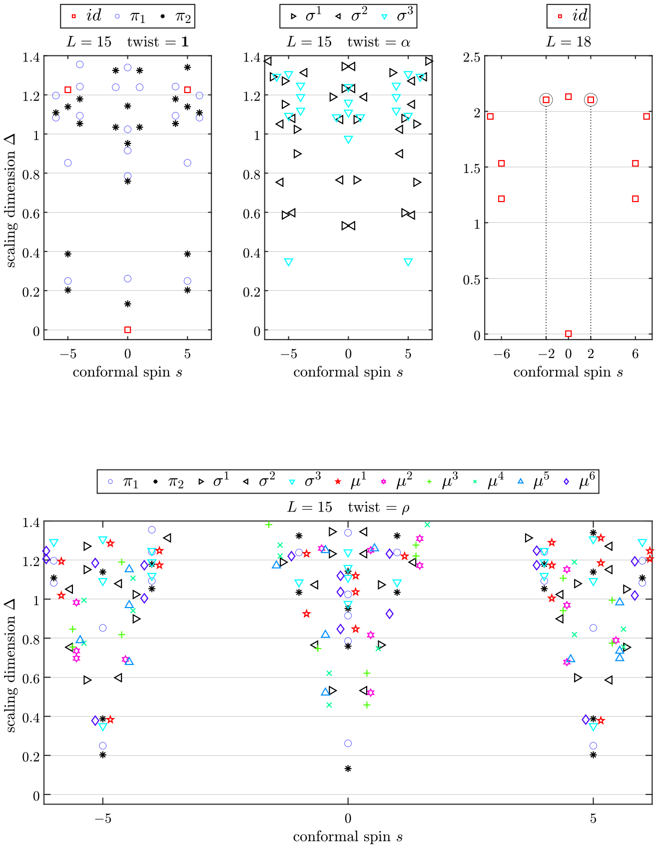


FIG. 3. Spectra for the transfer matrix, twisted with topological defects one (upper left), α (upper middle), and ρ (bottom), numerically obtained with anyonic symmetry-preserving exact diagonalization on $L = 15$ sites. The eigenvalues are labeled by their corresponding topological sectors $Z(\mathcal{H}_3)$ according to Table I. Upper right: the identity sector on $L = 18$ sites. The first excited states of the vacuum ($\Delta = 2$, $s = -2$, 2) are circled in black.

of categorical data: a choice of fusion category $\mathcal{D}$, describing the string-net model, and a right $\mathcal{D}$-module category $\mathcal{M}$, dictating the tensor network representation of the string-net ground state. In CFT terms, the fusion category $\mathcal{D}$ describes the representations of the chiral algebra, whereas the module category $\mathcal{M}$ roughly corresponds to the choice of modular invariant. Nondiagonal partition functions ($M_{\alpha\beta} \neq \delta_{\alpha,\beta}$) are constructed by choosing module categories $\mathcal{M}$ that differ from $\mathcal{D}$ itself. The topological defects are in turn given by objects in a category $\mathcal{C}$, which depends on $\mathcal{D}$ and $\mathcal{M}$ by requiring that $\mathcal{M}$ is an invertible ($\mathcal{C}$,$\mathcal{D}$)-bimodule category. We refer to Ref. [24] for the details regarding the tensor network representations of the corresponding string-net ground states.

In the model described above the choice $\mathcal{D} = \mathcal{M} = \mathcal{H}_3$ was made, but since $\mathcal{H}_3$ is not modular, it does not directly capture the chiral algebra of the underlying CFT. Exactly the same model can be obtained by choosing $\mathcal{D} = Z(\mathcal{H}_3)$ (which is modular) and $\mathcal{M} = \mathcal{H}_3$ by choosing a suitable strange correlator. It was recently shown in Ref. [57] that any strange correlator on a string-net model $\mathcal{D}_1$ can be rewritten as a strange correlator on a string-net model $\mathcal{D}_2$, provided there exists an invertible $(\mathcal{D}_1, \mathcal{D}_2)$-bimodule category. Taking $\mathcal{D}_1 = \mathcal{H}_3 \boxtimes \mathcal{H}_3^{\mathrm{op}}$ and $\mathcal{D}_2 = Z(\mathcal{H}_3)$, we can convert any strange correlator on $\mathcal{H}_3 \boxtimes \mathcal{H}_3^{\mathrm{op}}$ (which includes strange correlators on $\mathcal{H}_3$ as a special case) to strange correlators on $Z(\mathcal{H}_3)$.

Assuming the model we consider should indeed be thought of as a strange correlator on a PEPS with $\mathcal{D} = Z(\mathcal{H}_3)$ and $\mathcal{M} = \mathcal{H}_3$, the corresponding CFT will not have a diagonal partition function, as this is only the case when $\mathcal{M} = \mathcal{D}$. This implies that our model can be obtained from a diagonal one through a (generalized) orbifold construction [58]. The enlarged model based on $\mathcal{D} = Z(\mathcal{H}_3)$ shows that the symmetries of the model we study are actually given by $\mathcal{C} = \mathcal{H}_3 \boxtimes \mathcal{H}_3^{\mathrm{op}}$, implying that we did not consider the full set of topological defects.

We end by noting that the observed central charge of $c = 2$ appears to be in contradiction with an underlying MTC corresponding to $Z(\mathcal{H}_3)$, which should have $c = 0 \bmod 8$ [7]. However, this situation is not unfamiliar; for example, the critical RSOS models as constructed from the $\mathrm{SU}(2)_k$ MTCs (in a certain regime) are not described by a WZW CFT with central charge $3k/(k+2)$, but rather by the minimal models with $c < 1$, described by cosets of $\mathrm{SU}(2)_k$ WZW models [59]. The fact that the coset MTC describing the CFT is not required to construct the critical lattice model can be understood by the fact that the lattice model does not necessarily have all the topological defect symmetries of the continuum CFT [60]. This scenario appears to be quite common, and we speculate that the model we study here is no different and that the MTC $\mathcal{D}_{\mathrm{coset}}$ of the CFT describing our critical lattice model is a coset involving the MTC $Z(\mathcal{H}_3)$ and another MTC with $c = 6$, e.g., $\mathcal{D}_{\mathrm{coset}} = Z(\mathcal{H}_3)/\mathcal{D}_{c=6}$. The precise nature of this coset requires a detailed analysis of the spectrum, as well as a characterization of the possible cosets involving $Z(\mathcal{H}_3)$.

Conclusion.—We have shown strong numerical evidence for a Haagerup CFT with central charge $c = 2$, using the strange correlator prescription for the Haagerup fusion category $\mathcal{H}_3$. The model admits an interpretation as a

TABLE II. Topological spins of the sectors in $Z(\mathcal{H}_3)$.

$Z(\mathcal{H}_3)$	id	π_1	π_2	σ^1	σ^2	σ^3	μ^1	μ^2	μ^3	μ^4	μ^5	μ^6
Topological spin	0	0	0	$-(1/3)$	$(1/3)$	0	$(2/13)$	$(6/13)$	$(5/13)$	$-(5/13)$	$-(6/13)$	$-(2/13)$

nondiagonal modular invariant of a CFT with an MTC corresponding to $Z(\mathcal{H}_3)$. We argue that the observed central charge of the critical lattice model can be obtained as a coset involving $Z(\mathcal{H}_3)$, although an explicit construction requires further analysis. Preliminary checks also indicate that our model is not integrable. Furthermore, it is worth investigating if similar critical lattice models can be constructed (and their corresponding CFTs identified), for the general series of Haagerup-Izumi fusion categories.

Near the completion of this work, we learned that a critical anyonic chain Hamiltonian for the Haagerup fusion category $\mathcal{H}_3$ was obtained independently [61]. Their numerical evidence also indicates a central charge $c = 2$ CFT for this Hamiltonian. A preliminary check indicates that this Hamiltonian is not merely the $(1+1)d$ quantum analog of the 2D classical model discussed in this work. We thank Tzu-Cheng Huang, Ying-Hsuan Lin, Kantaro Ohmori, Yuji Tachikawa, and Masaki Tezuka for insightful discussions and for a coordinated submission of our manuscripts.

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