

DUNKL INTERTWINING OPERATOR FOR SYMMETRIC GROUPS

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ABSTRACT. In this note, we express explicitly the Dunkl kernel and generalized Bessel functions of type A_{n-1} by the Humbert's function $\Phi_2^{(n)}$, with one variable specified. The obtained formulas lead to a new proof of Xu's integral expression for the intertwining operator associated to symmetric groups, which was recently reported in [28].

1. INTRODUCTION

Dunkl operators are a family of commuting differential-difference operators associated with a finite reflection group. They were introduced by Dunkl in the late eighties in [13]. During the last years, Dunkl operators have played an important role in generalizing classical Fourier analysis and have made a deep and lasting impact on special functions theory. Furthermore, in the symmetric group case, Dunkl theory is naturally connected with the Schrödinger operators for Calogero-Sutherland type quantum many body systems, see e.g. [3].

One of the important problems still open in this theory is to construct the explicit formulas for the intertwining operator (see Section 2.1) and the integral kernel of the Dunkl transform [8, 14] for concrete finite groups. Concrete formulas for the Dunkl kernel and intertwining operator are the premise to do a lot of hard analysis, see e.g. [28]. This problem has received considerable attention in very recent times, especially in the case of dihedral groups and symmetric groups.

In the dihedral group setting, some explicit expressions for the Dunkl kernel were obtained in the last years, see [12] for an explicit series expansion and [6] for a complicated integral expression. Without integral expression, the action of the intertwining operator on polynomials has been studied in [16] and [26]. In 2020, for a restricted class of functions, an elegant integral expression was obtained for the intertwining operator in [27], using an integral over the simplex. This result was further explored in the general case in [7] and [11]. In particular, a new integral expression for the intertwining operator was given in [7].

For the symmetric groups, besides the explicit formula for the symmetric group S_3 determined by Dunkl long ago in [15], there are mainly two approaches to study this problem. The first one starts from constructing the explicit formulas of the generalized Bessel functions, then studies the Abel transform and its dual, which coincides with the intertwining operator for functions invariant under the finite reflection group. For example, complicated iterative formulas for the generalized Bessel functions were given in [1, 21] on a hyperplane of the Euclidean space $\mathbb{R}^n$. The other approach starts from determining the intertwining operator directly for

a class of functions. For example, Dunkl himself determined the action of the intertwining operator on polynomials in [17]. Recently, an explicit integral expression of the intertwining operator for functions of single component was obtained by Xu in [28]. Note that his results are proven by direct verification of the intertwining relations (2.1). It was pointed out that the sets of functions considered in the first approach and in [28] do not overlap.

The main contribution of the present paper is that we give an alternative proof of Xu's formula for the symmetric group starting from the generalized Bessel function. To do that, we first express the generalized Bessel function and the Dunkl kernel for a fixed variable by the Humbert function $\Phi_2^{(n)}$. This is obtained using the limiting relations between the generalized Bessel function and the Heckman-Opdam hypergeometric function and the recent result in [23]. This moreover provides the link between both approaches mentioned above.

This note is organized as follows. In Section 2, we give the basic notions of Dunkl theory and the Humbert function. Section 3 is devoted to the explicit formulas of the generalized Bessel function. In Section 4, we study the Dunkl kernel and the intertwining operator. We give a conclusion at the end of this note.

2. PRELIMINARIES

2.1. Dunkl operator and Dunkl's intertwiner. The Dunkl operators associated to the symmetric group S_n (or root system A_{n-1}) over $\mathbb{R}^n$ are defined by

$$D_i f(x) = \frac{\partial}{\partial x_i} f(x) + \kappa \sum_{j=1, j \neq i}^n \frac{f(x) - f(x(i, j))}{x_i - x_j}, \quad 1 \leq i \leq n,$$

where κ is a nonnegative real number and (i, j) is the transposition exchanging the i th and j th coordinates of $x \in \mathbb{R}^n$, see [13].

Denote $\mathcal{P}_m^n$ the space of homogeneous polynomial of degree m in n variables. There exists a unique linear operator $V_\kappa : \mathcal{P}_m^n \rightarrow \mathcal{P}_m^n$, called intertwining operator [15], satisfying the relations

$$(2.1) \quad D_i V_\kappa = V_\kappa \partial_i, \quad 1 \leq i \leq n$$

and $V_\kappa 1 = 1$. It was abstractly proved that the intertwining operator can be expressed as an integral operator in [20]. More precisely, there exists a nonnegative probability measure $d\mu_x$ such that

$$V_\kappa f(x) = \int_{\mathbb{R}^n} f(y) d\mu_x(y).$$

However, the explicit expression of the intertwining operator is only known for a few groups, e.g. $G = \mathbb{Z}_2^n$ and S_3 , see [15]. Recently, partial result for the intertwining operator associated to the dihedral groups was obtained in [27] and a full expression was obtained in [7].

The Dunkl kernel is defined by

$$E_\kappa(x, y) := V_\kappa \left[e^{\langle \cdot, y \rangle} \right] (x), \quad x, y \in \mathbb{R}^n$$

and is the integral kernel of the Dunkl transform [14, 8]. The symmetric analogue of the Dunkl kernel is called the generalized Bessel function. It is denoted by $J_\kappa(x, y)$

and given by

$$(2.2) \quad J_\kappa(x, y) := \frac{1}{n!} \sum_{\sigma \in S_n} E_\kappa(x, y\sigma),$$

in the case of the symmetric group. Some complicated integral expressions for the generalized Bessel function of type A_{n-1} were given in [1] and [21].

2.2. Heckman-Opdam hypergeometric function and the asymptotic relationship. Basics of the trigonometric Dunkl theory can be found in the review [3]. The Cherednik operator T_ξ , $\xi \in \mathbb{R}^n$ associated with the root system R and the non-negative multiplicity function κ is defined by

$$T_\xi f(x) = \partial_\xi f(x) + \sum_{\alpha \in R_+} \kappa_\alpha \langle \alpha, \xi \rangle \frac{f(x) - f(r_\alpha(x))}{1 - e^{-\langle \alpha, x \rangle}} - \langle \rho(\kappa), \xi \rangle f(x),$$

where $\rho(\kappa) = \frac{1}{2} \sum_{\alpha \in R_+} \kappa_\alpha \alpha$ and r_α the reflection in the hyperplane orthogonal to α . The Weyl group for the root system R is denoted by W .

The Heckman-Opdam hypergeometric function F_κ is defined as the unique holomorphic W -invariant function on $\mathbb{C}^n \times (\mathbb{R}^n + iU)$ (U is a W -invariant neighborhood of 0) that satisfies the system of differential equations:

$$p(T_{e_1}, T_{e_2}, \dots, T_{e_n}) F_\kappa(\lambda, \cdot) = p(\lambda) F_\kappa(\lambda, \cdot), \quad F_\kappa(\lambda, 0) = 1$$

for all $\lambda \in \mathbb{C}^n$ and all W -invariant polynomials p on $\mathbb{R}^n$. For a fixed root system R , the hypergeometric function F_κ and the generalized Bessel function J_κ satisfy the following rational limits

$$(2.3) \quad J_\kappa(\lambda, x) = \lim_{m \rightarrow \infty} F_\kappa\left(m\lambda + \rho(\kappa), \frac{x}{m}\right).$$

Such limit transition was first obtained in [4] for integer multiplicity function κ and then later by de Jeu in [9]. It has been used to give an alternative proof for the positivity of the intertwining operator in [20] and to obtain an integral expression for the generalized Bessel functions of type A_{n-1} in [1].

Recently, for the root system of type A_{n-1} , the hypergeometric function F_κ with a certain degenerate parameter was expressed explicitly by the Lauricella hypergeometric function F_D , see Theorem 2.2 and Theorem 3.1 in [23]. Recall that the Lauricella hypergeometric function F_D is the analytic continuation of the series

$$\begin{aligned} & F_D(a, b_1, \dots, b_n, c; x_1, \dots, x_n) \\ &= \sum_{m_1, \dots, m_n \geq 0} \frac{(a)_{m_1 + \dots + m_n} (b_1)_{m_1} \dots (b_n)_{m_n}}{(c)_{m_1 + \dots + m_n}} \frac{x_1^{m_1} \dots x_n^{m_n}}{m_1! \dots m_n!} \end{aligned}$$

where $a, b_1, \dots, b_n, c$ are complex constants with $c \neq -1, -2, \dots$. In the sequel, we denote the hyperplane $\mathbb{V}$ of $\mathbb{R}^n$ given by

$$\mathbb{V} = \{x \in \mathbb{R}^n : x_1 + x_2 + \dots + x_n = 0\}.$$

Theorem 2.1. [23] *Assume $\kappa \geq 0$, $\nu \in \mathbb{C}$ and $x \in \mathbb{V}$. Then the Heckman-Opdam hypergeometric function for the root system of type A_{n-1} can be written as*

$$F_\kappa(\lambda(\nu) + \rho(\kappa), x) = (y_1 \dots y_{n-1})^{-\frac{\kappa}{n}} F_D(-\nu, \kappa, \dots, \kappa, n\kappa; 1 - y_1, \dots, 1 - y_{n-1}),$$

where $\lambda(\nu) = \left(-\frac{\nu}{n}, \dots, -\frac{\nu}{n}, \frac{(n-1)\nu}{n}\right)$, $y_j = e^{x_j - x_n}$, $(1 \leq j \leq n-1)$, $y_n = e^{x_n}$ and $\rho(\kappa) = \frac{\kappa}{2} \sum_{\alpha \in R_+} \alpha$.

This result can be regarded as a generalization of the explicit expressions for the zonal spherical function on $SL(3, \mathbb{R})$ given in [22] and the Jack polynomials obtained in [24]. However, for general λ , the classical hypergeometric representation for the Heckman-Opdam function $F_\kappa(\lambda, x)$ is much more complicated and is still open.

2.3. Humbert functions $\Phi_2^{(n)}$. The Humbert function $\Phi_2^{(n)}$ of n variables is defined by

$$\Phi_2^{(n)}[b_1, \dots, b_n; c; x_1, \dots, x_n] := \sum_{m_1, \dots, m_n=0}^{\infty} \frac{(b_1)_{m_1} \cdots (b_n)_{m_n}}{(c)_{m_1+\dots+m_n}} \frac{x_1^{m_1}}{m_1!} \cdots \frac{x_n^{m_n}}{m_n!}.$$

It is the confluent form of the Lauricella function F_D and satisfies

$$\Phi_2^{(n)}[b_1, \dots, b_n; c; x_1, \dots, x_n] = \lim_{|a| \rightarrow \infty} F_D \left[a, b_1, \dots, b_n; c; \frac{x_1}{a}, \dots, \frac{x_n}{a} \right],$$

see [25] (Section 1.4, formula (10)). When $c - \sum_{j=1}^n b_j$ and each b_j , $j = 1, 2, \dots, n$ are positive numbers, $\Phi_2^{(n)}$ has the following integral expression,

$$(2.4) \quad \begin{aligned} & \Phi_2^{(n)}(b_1, \dots, b_n; c; x_1, \dots, x_n) \\ &= C_b^{(c)} \int_{T^n} e^{\sum_{j=1}^n x_j t_j} \left(1 - \sum_{j=1}^n t_j \right)^{c - \sum_{j=1}^n b_j - 1} \prod_{j=1}^n t_j^{b_j - 1} dt_1 \dots dt_n \end{aligned}$$

where $C_b^{(c)} = \frac{\Gamma(c)}{\Gamma(c - \sum_{j=1}^n b_j) \prod_{j=1}^n \Gamma(b_j)}$ and T^n is the open unit simplex in $\mathbb{R}^n$ given by

$$T^n = \left\{ (t_1, \dots, t_n) : t_j > 0, j = 1, \dots, n, \sum_{j=1}^n t_j < 1 \right\}.$$

We refer to [5, 18] for more details on these functions.

3. GENERALIZED BESSEL FUNCTION OF TYPE A_{n-1}

The limit relation (2.3) of the integral kernels in the rational and trigonometric setting together with (2.4) leads to an explicit expression for the generalized Bessel function of type A_{n-1} .

Theorem 3.1. *Assume $\kappa \geq 0$, $\nu \in \mathbb{C}$ and $x \in \mathbb{V} \subset \mathbb{R}^n$. Then the generalized Bessel function for the root system A_{n-1} is given by*

$$\begin{aligned} J_\kappa(\lambda, x) &= \Phi_2^{(n)}[\kappa, \dots, \kappa, n\kappa; \nu x_1, \dots, \nu x_n] \\ &= e^{\nu x_n} \Phi_2^{(n-1)}[\kappa, \dots, \kappa, n\kappa; \nu(x_1 - x_n), \dots, \nu(x_{n-1} - x_n)] \end{aligned}$$

where $\lambda = \left(-\frac{\nu}{n}, \dots, -\frac{\nu}{n}, \frac{(n-1)\nu}{n} \right)$.

Proof. For $x \in \mathbb{V}$, we adopt the rational limit relation (2.3) to the explicit expression of the Heckman-Opdam hypergeometric functions of Theorem 2.1. This yields

$$\begin{aligned}
& J_\kappa(\lambda, x) \\
&= \lim_{m \rightarrow \infty} F\left(m\lambda + \rho(\kappa), \frac{x}{m}\right) \\
&= \lim_{m \rightarrow \infty} \left(e^{\frac{x_1 - x_n}{m}} \cdots e^{\frac{x_{n-1} - x_n}{m}} \right)^{-\frac{m\nu}{n}} \\
&\quad \times F_D\left(-m\nu, \kappa, \dots, \kappa, n\kappa; 1 - e^{\frac{x_1 - x_n}{m}}, \dots, 1 - e^{\frac{x_{n-1} - x_n}{m}}\right) \\
&= (y_1 \cdots y_{n-1})^{-\frac{\nu}{n}} \lim_{m \rightarrow \infty} F_D\left(-m\nu, \kappa, \dots, \kappa, n\kappa; 1 - e^{\frac{x_1 - x_n}{m}}, \dots, 1 - e^{\frac{x_{n-1} - x_n}{m}}\right).
\end{aligned}$$

By the limit relation (2.4) and the fact

$$\lim_{m \rightarrow \infty} \frac{1 - e^{(x_j - x_n)/m}}{(x_n - x_j)/m} = 1,$$

we obtain

$$\begin{aligned}
& J_\kappa(\lambda, x) \\
&= (y_1 \cdots y_{n-1})^{-\frac{\nu}{n}} \lim_{m \rightarrow \infty} F_D\left(-m\nu, \kappa, \dots, \kappa, n\kappa; 1 - e^{\frac{x_1 - x_n}{m}}, \dots, 1 - e^{\frac{x_{n-1} - x_n}{m}}\right) \\
&= e^{-\frac{\nu}{n} \sum_{j=1}^{n-1} (x_j - x_n)} \Phi_2^{(n-1)}[\kappa, \dots, \kappa, n\kappa; \nu(x_1 - x_n), \dots, \nu(x_{n-1} - x_n)].
\end{aligned}$$

Here $\Phi_2^{(n-1)}$ is the second class of Humbert functions, see Section 2.3. Note that as $x \in \mathbb{V}$, we have $\sum_{j=1}^{n-1} (x_j - x_n) = -nx_n$. Therefore, for $x \in \mathbb{V}$,

$$\begin{aligned}
J_\kappa(\lambda, x) &= e^{\nu x_n} \Phi_2^{(n-1)}[\kappa, \dots, \kappa, n\kappa; \nu(x_1 - x_n), \dots, \nu(x_{n-1} - x_n)] \\
&= \Phi_2^{(n)}[\kappa, \dots, \kappa, n\kappa; \nu x_1, \dots, \nu x_{n-1}, \nu x_n],
\end{aligned}$$

where the last identity is obtained by the Laplace transform of $\Phi_2^{(n)}$, see [7, 10]. $\square$

If we take $\nu = 1$, we get the following corollary.

Corollary 3.2. For $\lambda = \left(-\frac{1}{n}, \dots, -\frac{1}{n}, \frac{n-1}{n}\right)$, $x \in \mathbb{V}$, the generalized Bessel function $J_\kappa(\lambda, x)$ of type A_{n-1} is given by

$$\begin{aligned}
J_\kappa(\lambda, x) &= e^{x_n} \Phi_2^{(n-1)}[\kappa, \dots, \kappa, n\kappa; x_1 - x_n, \dots, x_{n-1} - x_n] \\
&= c_\kappa \int_{T^{n-1}} e^{\sum_{j=1}^n x_j t_j} \prod_{j=1}^n t_j^{\kappa-1} dt_1 \cdots dt_{n-1},
\end{aligned}$$

where $t_n = 1 - \sum_{j=1}^{n-1} t_j$ and $c_\kappa = \Gamma(n\kappa)/(\Gamma(\kappa))^n$.

Alternatively, the generalized Bessel function is defined as the symmetric analogue of the Dunkl kernel,

$$J_\kappa(\lambda, x) = \frac{1}{n!} \sum_{\sigma \in S_n} E_\kappa(x, \lambda\sigma) = \frac{1}{n!} \sum_{\sigma \in S_n} V_\kappa \left[e^{\langle \cdot, \lambda\sigma \rangle} \right] (x).$$

Furthermore, when $\lambda = \left(-\frac{1}{n}, \dots, -\frac{1}{n}, \frac{n-1}{n}\right)$, the condition $x \in \mathbb{V}$ yields $\langle x, \lambda \rangle = x_n$ and the exponential becomes

$$e^{\langle x, \lambda \rangle} = e^{x_n} = e^{\langle x, e_n \rangle},$$

which only depends on the component x_n , here $e_n = (0, 0, \dots, 1)$. Combining this with Corollary 3.2, for any $x \in \mathbb{V}$, we have

$$\begin{aligned} J_\kappa(\lambda, x) &= \frac{1}{n!} \sum_{\sigma \in S_n} V_\kappa \left[e^{\langle \cdot, e_n \sigma \rangle} \right] (x) \\ &= c_\kappa \int_{T^{n-1}} e^{\sum_{j=1}^n x_j t_j} \prod_{j=1}^n t_j^{\kappa-1} dt_1 \dots dt_{n-1}. \end{aligned}$$

Moreover, since the intertwining operator is homogenous, i.e. $V_\kappa : \mathcal{P}_m^n \rightarrow \mathcal{P}_m^n$, and the generalized Bessel functions are analytic, we have

$$(3.1) \quad V_\kappa \left[\sum_{\sigma \in S_n} \langle \cdot, e_n \sigma \rangle^m \right] (x) = n! c_\kappa \int_{T^{n-1}} \left(\sum_{j=1}^n x_j t_j \right)^m \prod_{j=1}^n t_j^{\kappa-1} dt_1 \dots dt_{n-1}$$

for $x \in \mathbb{V}$. By analytic continuation, it is seen that the integral expression also works for all $x \in \mathbb{R}^n$.

The above formula (3.1) further leads to an explicit expression of the intertwining operator for general functions by a limit discussion.

Theorem 3.3. *For $x \in \mathbb{R}^n$ and a function $f(x_j)$ in a single component, define a S_n -invariant function by $F(x) = \sum_{\sigma \in S_n} f(x_j \sigma)$. Then the intertwining operator acting on $F(x)$ is given by*

$$V_\kappa F(x) = n! c_\kappa \int_{T^{n-1}} f(x_1 t_1 + x_2 t_2 + \dots + x_n t_n) \prod_{j=1}^n t_j^{\kappa-1} dt_1 \dots dt_{n-1}.$$

Remark 3.4. This can also be verified by checking the intertwining relations of (2.1) directly in a similar way as in [28].

Corollary 3.5. *For $1 \leq \ell \leq n$, $x \in \mathbb{R}^n$ and $e_\ell = e_n(\ell, n)$, the generalized Bessel function $J_\kappa(e_\ell, x)$ is given by*

$$J_\kappa(e_\ell, x) = c_\kappa \int_{T^{n-1}} e^{\langle x, t \rangle} \prod_{j=1}^n t_j^{\kappa-1} dt_1 \dots dt_{n-1}.$$

Proof. Since

$$J_\kappa(e_\ell, x) = V_\kappa \left[\frac{1}{n!} \sum_{\sigma \in S_n} e^{\langle \cdot, e_\ell \sigma \rangle} \right] (x),$$

we put $f(x_\ell) = \frac{1}{n!} e^{x_\ell} = \frac{1}{n!} e^{\langle x, e_\ell \rangle}$ in Theorem 3.3 and then obtain the formula. $\square$

Remark 3.6. The same formula was obtained in [28], Corollary 2.4.

4. DUNKL KERNEL AND INTERTWINING OPERATOR OF TYPE A_{n-1}

It was routine to use the shift principle of [19] to derive the Dunkl kernel from the generalized Bessel function, see e.g. [2]. For our purpose, there exists a simpler way to achieve this goal. However, we still start from computing for the root system A_2 using the shift principle to show how it works. Denote by $W(\lambda)$ the alternating polynomial associated to A_2

$$W(\lambda) = (\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3).$$

Theorem 4.1. *For the root system A_2 , $x \in \mathbb{R}^3$ and $e_3 = (0, 0, 1)$, the Dunkl kernel can be expressed as*

$$\begin{aligned} E_\kappa(x, e_3) &= V_\kappa \left(e^{\langle \cdot, e_3 \rangle} \right) (x) = e^{x_3} \Phi_2^{(2)}[\kappa, \kappa, 3\kappa + 1; x_1 - x_3, x_2 - x_3] \\ &= \Phi_2^{(3)}[\kappa, \kappa, \kappa, 3\kappa + 1; x_1, x_2, x_3]. \end{aligned}$$

Proof. Recall that the shift principle implies (see Proposition 1.4 in [15])

$$(4.1) \quad \sum_{\sigma \in S_3} \det(\sigma) E_\kappa(x\sigma, \lambda) = \gamma_\kappa W(x) W(\lambda) J_{\kappa+1}(x, \lambda)$$

where γ_κ is a normalizing constant which will not be explicitly used here. Combining (4.1) with (2.2), we have

$$(4.2) \quad \begin{aligned} &E_\kappa(x, \lambda) + E_\kappa(x, \lambda\sigma) + E_\kappa(x, \lambda\sigma^2) \\ &= \frac{1}{2} (\gamma_\kappa W(\lambda) W(x) J_{\kappa+1}(x, \lambda) + 6J_\kappa(x, \lambda)) \end{aligned}$$

where $\sigma = (1, 3)(1, 2)$, here (i, j) is the transposition exchanging the i th and j th coordinates of $x \in \mathbb{R}^n$. This relation (4.2) has been used to derive an integral expression for the Dunkl kernel of type A_2 in [2].

Now, we act with the Dunkl operator $D_j^{(x)}$ on both sides of (4.2) with $\lambda = e_3 = (0, 0, 1)$. Using the relations

$$D_j^{(x)} E_\kappa(x, \lambda) = \lambda_j E_\kappa(x, \lambda), \quad j = 1, 2, 3,$$

this yields

$$\begin{aligned} E_\kappa(x, e_3) &= D_3 E_\kappa(x, e_3) \\ &= D_3 (E_\kappa(x, e_3) + E_\kappa(x, e_3\sigma) + E_\kappa(x, e_3\sigma^2)) \\ &= \frac{1}{2} D_3 (\gamma_\kappa W(e_3) W(x) J_{\kappa+1}(x, e_3) + 6J_\kappa(x, e_3)) \\ &= 3\partial_3 J_\kappa(x, e_3) \\ &= 3\partial_3 c_\kappa \int_{T^2} e^{(x_1 t_1 + x_2 t_2 + x_3 t_3)} (t_1 t_2 t_3)^{\kappa-1} dt_1 dt_2 \\ &= 3c_\kappa \int_{T^2} e^{(x_1 t_1 + x_2 t_2 + x_3 t_3)} t_3 (t_1 t_2 t_3)^{\kappa-1} dt_1 dt_2 \\ &= e^{x_3} \Phi_2^{(2)}[\kappa, \kappa, 3\kappa + 1; x_1 - x_3, x_2 - x_3] \\ &= \Phi_2^{(3)}[\kappa, \kappa, \kappa, 3\kappa + 1; x_1, x_2, x_3]. \end{aligned}$$

Here the third identity holds by the fact $W(e_3) = 0$ and that $J_\kappa(x, e_3)$ is S_3 -invariant in the variable x . □

In the following, we consider the general symmetric group S_n . Recalling the representation (2.2), for $1 \leq \ell \leq n$, the generalized Bessel function of type A_{n-1} can also be expressed as

$$(4.3) \quad J_\kappa(e_\ell, x) = J_\kappa(e_1, x) = \frac{1}{n} \sum_{j=1}^n E_\kappa(e_1, x(1, j)).$$

Hence, $E_\kappa(x, e_\ell)$ can be obtained by acting with $D_m^{(x)}$ on both sides of (4.3) using the relations

$$D_j^{(x)} E_\kappa(x, \lambda) = \lambda_j E_\kappa(x, \lambda), \quad j = 1, 2, \dots, n.$$

Similar as Theorem 4.1, we then have a result for A_{n-1} ,

Theorem 4.2. *For root system A_{n-1} , $x \in \mathbb{R}^n$, the Dunkl kernel admits*

$$\begin{aligned} E_\kappa(x, e_\ell) &= V_\kappa \left(e^{\langle \cdot, e_\ell \rangle} \right) (x) \\ &= e^{x_n} \Phi_2^{(n-1)}(\kappa, \dots, \underbrace{\kappa+1}_\ell, \dots, \kappa; n\kappa+1; x_1 - x_n, \dots, x_{n-1} - x_n) \\ &= nc_\kappa \int_{T^{n-1}} e^{\sum_{j=1}^n x_j t_j} t_\ell \prod_{j=1}^n t_j^{\kappa-1} dt_1 \dots dt_{n-1}, \end{aligned}$$

where $t_n = 1 - \sum_{j=1}^{n-1} t_j$ and $c_\kappa = \Gamma(n\kappa)/(\Gamma(\kappa)^n)$.

Remark 4.3. This expression is first given in [28], Corollary 2.4.

Remark 4.4. Our results depend heavily on Theorem 2.1 of [23], which only works for degenerated parameters. It is not clear to us how to obtain a full classical hypergeometric expression for the Heckman-Opdam function $F_\kappa(\lambda, x)$ at present. Hence, this approach can not lead to a full expression for the intertwining operator as in the dihedral case. On the other side, these results both for the dihedral and symmetric groups show that the Dunkl kernel at some special lines (corresponding to the degenerated parameters) behaves much simpler than in other points. For general root systems and those special lines, the Dunkl kernel and the intertwining operator may have a close relationship with the Humbert functions as well.

Since the intertwining operator maps polynomials of degree m to polynomials of the same degree, we have

$$V_\kappa(x_\ell^m) = nc_\kappa \int_{T^{n-1}} (x_1 t_1 + x_2 t_2 + \dots + x_n t_n)^m t_\ell \prod_{j=1}^n t_j^{\kappa-1} dt_1 dt_2 \dots dt_{n-1}.$$

This leads to an explicit expression for the intertwining operator when the function is of a single component, which was obtained recently by Xu in [28] Theorem 2.1.

Theorem 4.5. *Let $f : \mathbb{R} \rightarrow \mathbb{R}$. For $1 \leq \ell \leq n$, define $F(x_1, x_2, \dots, x_n) = f(x_\ell)$. Then the intertwining operator acting on $F(x)$ is given by*

$$V_\kappa F(x) = c_\kappa^{(n)} \int_{T^{n-1}} f(x_1 t_1 + x_2 t_2 + \dots + x_n t_n) t_\ell \prod_{j=1}^n t_j^{\kappa-1} dt_1 dt_2 \dots dt_{n-1}$$

where $c_\kappa^{(n)} = nc_\kappa = \Gamma(n\kappa+1)/(\kappa\Gamma(\kappa)^n)$.

5. CONCLUSION

In this note, we have expressed the generalized Bessel function and Dunkl kernel of type A_{n-1} in terms of the Humbert function $\Phi_2^{(n)}$, with one variable fixed. A new proof of Xu's integral formula for the intertwining operator was developed by these formulas. The same approach will also lead to explicit expressions for the trigonometric Dunkl intertwining operator associated to the dihedral and symmetric groups.

REFERENCES

- [1] B. Amri. *Note on Bessel functions of type A_{N-1}* . Integral Transf. Spec. Funct. **25** (2014), 448-461.
- [2] B. Amri. *On the integral representations for Dunkl kernels of type A_2* . J. Lie Theory. **26** (2016), 1163-1175.
- [3] J.-P. Anker. *An introduction to Dunkl theory and its analytic aspects*. Analytic, Algebraic and Geometric Aspects of Differential Equations, 3-58. Trends Math., Birkhäuser, Chem, 2017.
- [4] S. Ben Saïd, B. Ørsted. *Bessel functions for root systems via the trigonometric setting*. Int. Math. Res. Not. **9** (2005), 551-585.
- [5] J. F. Chamayou, J. Wesolowski. *Lauricella and Humbert functions through probabilistic tools*. Integral Transf. Spec. Funct. **20** (2009), 529-538.
- [6] D. Constales, H. De Bie, P. Lian. *Explicit formulas for the Dunkl dihedral kernel and the (κ, a) -generalized Fourier kernel*. J. Math. Anal. Appl. **460** (2018), 900-926.
- [7] H. De Bie, P. Lian. *The Dunkl kernel and intertwining operator for dihedral groups*. J. Funct. Anal. (2021), <https://doi.org/10.1016/j.jfa.2021.108932>.
- [8] M. de Jeu. *The Dunkl transform*. Invent. Math. **113** (1993), 147-162.
- [9] M. de Jeu. *Paley-Wiener theorems for the Dunkl transform*. Trans. Amer. Math. Soc. **358** (2006), 4225-4250.
- [10] L. Deleaval, N. Demni. *On a Neumann-type series of modified Bessel functions*. Proc. Amer. Math. Soc. **146** (2018), no. 5, 2149-2161.
- [11] L. Deleaval, N. Demni. *Generalized Bessel functions of dihedral-type: expression as a series of confluent Horn functions and Laplace-type integral representation*. Ramanujan J. **54** (2020), 197-217.
- [12] L. Deleaval, N. Demni, H. Youssfi. *Dunkl kernel associated with dihedral groups*. J. Math. Anal. Appl. **432** (2015), 928-944.
- [13] C. F. Dunkl. *Differential-difference operators associated to reflection groups*. Trans. Amer. Math. Soc. **311** (1989), no.1, 167-183.
- [14] C. F. Dunkl. *Integral kernels with reflection group invariance*. Can. J. Math. **43** (1991), 1213-1227.
- [15] C. F. Dunkl. *Intertwining operators associated to the group S_3* . Trans. Amer. Math. Soc. **347** (1995), 3347-3374.
- [16] C. F. Dunkl. *Polynomials associated with dihedral groups*. SIGMA: Symmetry Integr. Geom. Methods Appl. **3** (2007), 52.
- [17] C. F. Dunkl. *Intertwining operators and polynomials associated with the symmetric group*. Monatsh. Math. **126** (1998), 181-209.
- [18] P. Humbert. *The confluent hypergeometric functions of two variables*. Proc. Roy. Soc. Edinburgh. **41** (1920), 73-82.
- [19] E. M. Opdam. *Dunkl operators, Bessel functions, and the discriminant of a finite Coxeter group*. Compos. Math. **85** (1993), 333-373.
- [20] M. Rösler, M. Voit. *Positivity of Dunkl's intertwining operator via the trigonometric setting*. Int. Math. Res. Not. **63** (2004), 3379-3389.
- [21] P. Sawyer. *A Laplace-type representation of the generalized spherical functions associated to the root systems of type A*. Mediterr. J. Math. **14** (2017), 147.
- [22] J. Sekiguchi. *Zonal spherical functions on $SL(3, \mathbb{R})$ (in Japanese)*. RIMS Kokyuroku **266** (1976), 259-274.
- [23] N. Shimeno, Y. Tamaoka. *The hypergeometric function for the root system of type A with a certain degenerate parameter*. Tsukuba J. Math. **42** (2018), 155-172.
- [24] Y. Tamaoka. *Jack polynomials and Lauricella hypergeometric series (in Japanese)*. Master Dissertation, Kwansei Gakuin University, 2018.
- [25] H. M. Srivastava, P. W. Karlsson. *Multiple Gaussian Hypergeometric Series*, Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, Chichester, Brisbane and Toronto, 1985.
- [26] Y. Xu. *Intertwining operator and h -harmonics associated with reflection group*. Can. J. Math. **50** (1998), 193-209.
- [27] Y. Xu. *Intertwining operators associated to dihedral groups*. Constr. Approx. **52** (2020), 395-422.

- [28] Y. Xu. *Intertwining operator associated to symmetric groups and summability on the unit sphere*. arXiv:2004.08727.

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