HARDY-LITTLEWOOD INEQUALITY AND L^p-L^q FOURIER MULTIPLIERS ON COMPACT HYPERGROUPS

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Abstract. This paper deals with the inequalities comparing the norm of a function on a compact hypergroup and the norm of its Fourier coefficients. We prove the classical Paley inequality in the setting of compact hypergroups which further gives the Hardy-Littlewood and Hausdorff-Young-Paley inequalities in the noncommutative context. We establish Hörmander's L^p-L^q Fourier multiplier theorem on compact hypergroups for $1 < p \leq 2 \leq q < \infty$ as an application of the Hausdorff-Young-Paley inequality. We examine our results for the hypergroups constructed from the conjugacy classes of compact Lie groups and for a class of countable compact hypergroups.

1. Introduction

The inequalities which involve functions and their Fourier coefficients played a pivotal role in Fourier analysis as well as in its applications to several different areas. This paper contributes to some of the classical inequalities of this nature, namely, Hardy-Littlewood inequality, Paley inequality and Hausdorff-Young-Paley inequality, and their applications to the theory of Fourier multiplier in the non-commutative setting. The first inequality we consider is the Hardy-Littlewood inequality proved by Hardy and Littlewood for the torus $\mathbb{T}$ ([23]). They proved that for each $1 \leq p \leq 2$ there exists a constant $C_p > 0$ such that

$$\left(\sum_{n \in \mathbb{Z}} |\widehat{f}(n)|^p \, (1+|n|)^{p-2} \right)^{\frac{1}{p}} \leq C_p \|f\|_{L^p(\mathbb{T})}, \quad f \in L^p(\mathbb{T}).$$

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Hewitt and Ross [24] extended this inequality to compact abelian groups using the structure theory of groups. Recently, the second author with his coauthors explored the non-commutative version of the Hardy-Littlewood inequality in the setting of compact homogeneous spaces [1, 3] and compact quantum groups [2] (see also [49]). The Hardy-Littlewood inequality also has an application to Sobolev embedding theorems and to the boundedness of Fourier multipliers [49, 10, 3]. Compact Riemannian symmetric spaces can be viewed as homogeneous spaces of compact Lie groups. It is well-known that the spherical analysis on Riemannian symmetric spaces is interconnected with the analysis on the double coset spaces which are special examples of hypergroups for which a convolution structure can be defined on the space of all bounded Borel measures. Our goal is to investigate the Hardy-Littlewood, Paley and Hausdorff-Young-Paley inequalities and their applications to the boundedness of Fourier multipliers in the context of compact hypergroups. The results of this paper are not only applicable to compact double coset spaces but also to the large class of other examples, for instance, the space of group orbits, space of conjugacy classes of compact (Lie) groups and countable compact hypergroups [11]. In particular, the results of this paper are also true for several interesting examples including Jacobi hypergroups with Jacobi polynomials as characters [20], compact hypergroup structure on the fundamental alcove with Heckman-Opdam polynomials as characters [38], and multivariant disk hypergroups [39, 8]

Hewitt and Ross [24] used structure theory of compact abelian groups and in [3], the authors used the eigenvalue counting formula for the Laplace operator on compact manifolds to derive the Hardy-Littlewood inequality. When working with compact hypergroups, we do not have such luxury. In this case, we obtain the following Hardy-Littlewood inequality (see Theorem 3.5):

Theorem 1.1. *Let $1 < p \leq 2$ and let K be a compact hypergroup and $\widehat{K}$ the set of inequivalent continuous representations π of K. We denote by k_π the hyperdimension of π and assume that a sequence $\{\mu_\pi\}_{\pi \in \widehat{K}}$ grows sufficiently fast, that is,*

$$\sum_{\pi \in \widehat{K}} \frac{k_\pi^2}{|\mu_\pi|^\beta} < \infty \quad \textit{for some}\, \beta \geq 0.$$

Then we have

$$\sum_{\pi \in \widehat{K}} k_\pi^2 |\mu_\pi|^{\beta(p-2)} \left(\frac{\|\widehat{f}(\pi)\|_{\mathrm{HS}}}{\sqrt{k_\pi}} \right)^p \lesssim \|f\|_{L^p(K)}.$$

When K is the hypergroup of conjugacy classes of the compact Lie group SU(2), Theorem 1.1 gives the following Hardy-Littlewood inequality for the commutative hypergroup Conj(SU)(2). This is also a natural analogue of the Hardy-Littlewood inequality for $\mathbb{T}$ (see Theorem 5.2):

Theorem 1.2. *If* $1 < p \leq 2$ *and* $f \in L^p(\mathrm{Conj}(\mathrm{SU})(2))$, *then we have*

$$\sum_{l \in \frac{1}{2}\mathbb{N}_0} (2l+1)^{5p-8} |\widehat{f}(l)|^p \leq C_p \|f\|_{L^p(\mathrm{Conj}(\mathrm{SU})(2))}. \tag{1}$$

The inequality (1) can be interpreted in the following form similar to the Hardy-Littlewood inequality on $\mathbb{T}$:

$$\sum_{l \in \frac{1}{2}\mathbb{N}_0} (2l+1)^{5(p-2)} (2l+1)^2 |\widehat{f}(l)|^p \leq C_p \|f\|_{L^p(\mathrm{Conj}(\mathrm{SU})(2))}. \tag{2}$$

In contrast to the case of $\mathbb{T}$, an extra term $(2l+1)^2$ appears in (2). But this is natural as the Plancherel measure ω on $\frac{1}{2}\mathbb{N}_0$, the dual of Conj(SU)(2), is given by $\omega(l) = (2l+1)^2$ for $l \in \frac{1}{2}\mathbb{N}_0$ while for $\mathbb{T}$, the Plancherel measure of the dual group $\mathbb{Z}$ is the counting measure.

Corollary 1.3. *If* $2 \leq p < \infty$ *and* $\sum_{l \in \frac{1}{2}\mathbb{N}_0} (2l+1)^{5p-8} |\widehat{f}(l)|^p < \infty$ *then*

$$f \in L^p(\mathrm{Conj}(\mathrm{SU})(2)).$$

Moreover, we have

$$\|f\|_{L^p(\mathrm{Conj}(\mathrm{SU})(2))} \leq C_p \sum_{l \in \frac{1}{2}\mathbb{N}_0} (2l+1)^{5p-8} |\widehat{f}(l)|^p.$$

For $p = 2$, Theorem 1.2 and Corollary 1.3 boil down to the Plancherel theorem for the hypergroup Conj(SU)(2). Therefore, these follow the philosophy of Hardy and Littlewood [23] who argue that the Hardy-Littlewood inequality is a suitable extension of the Plancherel theorem in the case of $\mathbb{T}$.

Another set of interesting examples of commutative infinite hypergroups which we will investigate is the family of countable compact hypergroups studied by Dunkl and Ramirez [16]. Recently, in [28, 29] the first author with Singh and Ross studied classification results of such classes of hypergroups arising from the discrete semigroups and investigated applications of these results to the Ramsey theory [30]. Interestingly, the property of being countable infinite and compact simultaneously is a purely hypergroups theoretical property as any infinite compact group can never be countable. We also obtain the following analogue of the Hardy-Littlewood inequality for this class of hypergroups H_a (see Section 5.2 for the definition).

The Hardy-Littlewood inequality is obtained by the following Paley inequality for compact hypergroups (see Theorem 3.1):

Theorem 1.4. *Let K be a compact hypergroup and let $1 < p \leq 2$. If $\varphi(\pi)$ is a positive sequence over $\widehat{K}$ such that the quantity*

$$M_\varphi := \sup_{y>0} y \sum_{\substack{\pi \in \widehat{K} \\ \varphi(\pi) \geq y}} k_\pi^2$$

is finite, then we have

$$\left(\sum_{\pi \in \widehat{K}} k_\pi^2 \left(\frac{\|\widehat{f}(\pi)\|_{\mathrm{HS}}}{\sqrt{k_\pi}} \right)^p \varphi(\pi)^{2-p} \right)^{\frac{1}{p}} \lesssim M_\varphi^{\frac{2-p}{p}} \|f\|_{L^p(K)}.$$

The Paley inequality describes the growth of the Fourier transform of a function in terms of its L^p-norm. Interpolating the Paley inequality with the Hausdorff-Young inequality one can obtain the following Hörmander's version of the Hausdorff-Young-Paley inequality,

$$\left(\int_{\mathbb{R}^n} |(\mathscr{F} f)(\xi) \phi(\xi)^{\frac{1}{r} - \frac{1}{p'}}|^r \, d\xi \right)^{\frac{1}{r}} \leq \|f\|_{L^p(\mathbb{R}^n)}, \;\; 1 < p \leq r \leq p' < \infty, \;\; 1 < p < 2.$$

Also, as a consequence of the Hausdorff-Young-Paley inequality, Hörmander [26, page 106] proves that the condition

$$\sup_{t>0} t^b |\{\xi \in \mathbb{R}^n : m(\xi) \geq t\}| < \infty, \quad \frac{1}{p} - \frac{1}{q} = \frac{1}{b},$$

where $1 < p \leq 2 \leq q < \infty$ and $1 < b < \infty$, implies the existence of a bounded extension of a Fourier multiplier T_m with symbol m from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$. Recently, the second author and R. Akylzhanov extended Hörmander's classical results to unimodular locally compact groups and to homogeneous spaces [3, 4]. In [4], the key idea behind the extension of Hörmander's theorem is the reformulation of this theorem as follows:

$$\|T_m\|_{L^p(\mathbb{R}^n)\to L^q(\mathbb{R}^n)} \lesssim \sup_{s>0} s\left(\int_{\{\xi\in\mathbb{R}^n:\, m(\xi)\geq s\}} d\xi\right)^{\frac{1}{p}-\frac{1}{q}} \simeq \|m\|_{L^{r,\infty}(\mathbb{R}^n)} \simeq \|T_m\|_{L^{r,\infty}(\mathrm{VN}(\mathbb{R}^n))},$$

where $\frac{1}{r}=\frac{1}{p}-\frac{1}{q}$, $\|m\|_{L^{r,\infty}(\mathbb{R}^n)}$ is the Lorentz norm of m, and $\|T_m\|_{L^{r,\infty}(\mathrm{VN}(\mathbb{R}^n))}$ is the norm of the operator T_m in the Lorentz space on the group von Neumann algebra $\mathrm{VN}(\mathbb{R}^n)$ of $\mathbb{R}^n$. Then one can use the Lorentz spaces and group von Neumann algebra techniques for extending it to general locally compact unimodular groups. The unimodularity assumption has its own advantages such as the existence of the canonical trace on the group von Neumann algebra and, consequently, the Plancherel formula and the Hausdorff-Young inequality. It was also pointed out that the unimodularity can in principle be avoided by using the Tomita-Takesaki modular theory and the Haagerup reduction technique.

By interpolating the Hausdorff-Young inequality and the Paley inequality we get the following Hausdorff-Young-Paley inequality for compact hypergroups (see Theorem 3.8):

Theorem 1.5. *Let K be a compact hypergroup and let $1<p\leq b\leq p'<\infty$. If a positive sequence $\varphi(\pi),\pi\in\widehat{K}$, satisfies the condition*

$$M_\varphi := \sup_{y>0} y \sum_{\substack{\pi\in\widehat{K}\\ \varphi(\pi)\geq y}} k_\pi^2 < \infty,$$

then we have

$$\left(\sum_{\pi\in\widehat{K}} k_\pi^2 \left(\frac{\|\widehat{f}(\pi)\|_{\mathrm{HS}}}{\sqrt{k_\pi}}\varphi(\pi)^{\frac{1}{b}-\frac{1}{p'}}\right)^b\right)^{\frac{1}{b}} \lesssim M_\varphi^{\frac{1}{b}-\frac{1}{p'}}\|f\|_{L^p(K)}.$$

As a consequence of the Hausdorff-Young-Paley inequality we prove the L^p-L^q boundedness of Fourier multipliers on compact hypergroups (see Theorem 4.1) as a natural analogue of Hörmander's theorem (see [26]).

Theorem 1.6. *Let K be a compact hypergroup and let $1<p\leq 2\leq q<\infty$. Let A be a left Fourier multiplier with symbol σ_A. Then we have*

$$\|A\|_{L^p(K)\to L^q(K)} \lesssim \sup_{y>0} y\left(\sum_{\substack{\pi\in\widehat{K}\\ \|\sigma_A(\pi)\|_{op}\geq y}} k_\pi^2\right)^{\frac{1}{p}-\frac{1}{q}}.$$

The organisation of the paper is as follows. In the next section, we discuss the basics of the Fourier analysis on compact hypergroups. Section 3 is the heart of the paper where we shall prove the Paley, Hardy-Littlewood and Hausdorff-Young-Paley inequalities for compact hypergroups. Section 4 is devoted to establishing the Hörmander multiplier theorem for compact hypergroups. In the last section, we discuss our results for countable compact hypergroups and for the hypergroups arising from conjugacy classes of compact Lie groups.

Throughout the paper, we denote by $\mathbb{N}$ the set of natural numbers and set $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For notational convenience, we take empty sums to be zero. We shall also use the notation $P \lesssim Q$ to indicate $P \leq cQ$ for a suitable constant $c > 0$.

2. Preliminaries

For the basics of compact hypergroups one can refer to standard books, monographs and research papers [15, 27, 11, 40, 41, 45, 46]. In [27], Jewett refers to hypergroups as convos. However we mention here certain results we need.

2.1. Definitions and representations of compact hypergroups.

We begin this section with the definition of a compact hypergroup.

Definition 2.1. A compact *hypergroup* is a non empty compact Hausdorff space K with a weakly continuous, associative convolution $*$ on the Banach space $M(K)$ of all bounded regular Borel measures on K such that $(M(K), *)$ becomes a Banach algebra and the following properties hold:

(i) For any $x, y \in K$, the convolution $\delta_x * \delta_y$ is a probability measure with compact support, where δ_x is the point mass measure at x. Also, the mapping $(x, y) \mapsto \text{supp}(\delta_x * \delta_y)$ is continuous from $K \times K$ to the space $\mathcal{C}(K)$ of all nonempty compact subsets of K equipped with the Michael (Vietoris) topology (see [36] for details).

(ii) There exists a unique element $e \in K$ such that $\delta_x * \delta_e = \delta_e * \delta_x = \delta_x$ for every $x \in K$.

(iii) There is a homeomophism $x \mapsto \check{x}$ on K of order two which induces an involution on $M(K)$ where $\check{\mu}(E) = \mu(\check{E})$ with $\check{E}$ defined as $\check{E} := \{\check{x} : x \in E\}$ for any Borel set E, and $e \in \text{supp}(\delta_x * \delta_y)$ if and only if $x = \check{y}$.

Note that the weak continuity assures that the convolution of bounded measures on a hypergroup is uniquely determined by the convolution of point measures. A compact hypergroup is called a commutative compact hypergroup if the convolution is commutative. A compact hypergroup K is called *hermitian* if the involution on K is the identity map, i.e., $\check{x} = x$ for all $x \in K$. Note that a hermitian hypergroup is commutative. Every compact group is a trivial example of a compact hypergroup. Other essential and non-trivial examples are double coset hypergroups $G//H$ asing from a Gelfand pair (G, H) for a compact group G and a closed subgroup H [27], conjugacy classes of compact Lie groups [46, 11], countable compact hypergroups [16, 11], Jacobi hypergroups [19, 11], hypergroup joins [47] of compact hypergroups by finite hypergroups [5, 11].

A *left Haar measure* λ on K is a non-zero positive Radon measure such that

$$\int_K f(x * y) d\lambda(y) = \int_K f(y)\, d\lambda(y) \quad (\forall x \in K,\ f \in C_c(K)),$$

where we used the notation $f(x * y) = (\delta_x * \delta_y)(f)$. It is well known that a Haar measure is unique if it exists [27]. Throughout this article, a left Haar measure is simply called a Haar measure. We would like to make a remark here that it is still not known if a general hypergroup has a Haar measure but several important class of hypergroups including commutative hypergroups, compact hypergroups, discrete hypergroups, nilpotent hypergroups possess a Haar measure [27, 11, 48, 6].

An *irreducible representation* π of K is an irreducible $*$- algebra representation of $M(K)$ into $\mathcal{L}(\mathcal{H}_\pi)$, the algebra of all bounded linear operators on some Hilbert space $\mathcal{H}_\pi$, such that

(i) $\pi(\delta_e) = I$ and
(ii) for every $u, v \in \mathcal{H}_\pi$, the mapping $\mu \mapsto \langle \pi(\mu)u, v\rangle$ is continuous from $M(K)^+$ to $\mathbb{C}$, where $M(K)^+$ is the set of all those measures in $M(K)$ which are non-negative and is equipped with the weak (cone) topology.

In [27], it was also included in the definition of a representation that π must be norm decreasing, that is, $\|\pi(\mu)\|_{\text{op}} \leq \|\mu\|$, where $\|\cdot\|_{\text{op}}$ denotes the operator norm on $\mathcal{L}(\mathcal{H}_\pi)$, but it follows as a consequence of the above definition. For any $x \in K$, we also write $\pi(\delta_x)$ as $\pi(x)$. Therefore, we get $\|\pi(x)\|_{\text{op}} \leq \|\delta_x\| = 1$.

2.2. **Fourier analysis on compact hypergroups.** Let K be a compact hypergroup with the normalized Haar measure λ and let $\widehat{K}$ be the set of irreducible inequivalent continuous representations of K. Throughout this paper we will assume that K is metrizable which is equivalent to the condition that $\widehat{K}$ is countable [17]. The set $\widehat{K}$ equipped with the discrete topology is called the dual space of K. Vrem [46] showed that every irreducible representation $(\pi, \mathcal{H}_\pi)$ of a compact hypergroup is finite dimensional. For any $\pi \in \widehat{K}$, the map $x \mapsto \langle \pi(x)u, v\rangle$ for $u, v \in \mathcal{H}_\pi$ is called a matrix coefficient function and is denoted by $\pi_{u,v}$. Let $\pi(x) = [\pi_{i,j}]_{d_\pi \times d_\pi}$ be the matrix representation of any $(\pi, \mathcal{H}_\pi)$ of dimension d_π with respect to an orthonormal basis $\{e_i\}_{i=1}^{d_\pi}$ of $\mathcal{H}_\pi$. For each $\pi \in \widehat{K}$ there exists a constant $k_\pi \geq d_\pi$ such that for each pair π, π' we have

$$\int_K \pi_{i,j}(x)\overline{\pi'_{m,l}(x)}\, d\lambda(x) = \begin{cases} \frac{1}{k_\pi} & \text{when}\, i = m, j = l, \text{ and}\, \pi = \pi', \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

If K is a compact group then $k_\pi = d_\pi$ [46, Theorem 2.6]. The constant k_π is called the hyperdimension of the representation π (see [5]). The function $x \mapsto \chi_\pi(x) =: \mathrm{Tr}(\pi(x))$ is called the (hypergroup) *character* and it is a continuous function. The following relation for characters can be derived from the orthogonality relation (3) of matrix coefficients

$$\int_K \chi_\pi(x)\overline{\chi_{\pi'}(x)} d\lambda(x) = \begin{cases} \frac{d_\pi}{k_\pi} & \text{if } \pi = \pi', \\ 0 & \text{otherwise,} \end{cases} \tag{4}$$

for all $\pi, \pi' \in \widehat{K}$. Therefore, $\|\chi_\pi\|^2_{L^2(K)} = \frac{d_\pi}{k_\pi}$.

We introduce the ℓ^p Schatten space $\ell^p_{\text{sch}}(\widehat{K})$, which is defined in Hewitt and Ross [25] and studied by Vrem [45]. Let Σ be the space of matrix coefficients, that is,

$$\Sigma(K) = \{\sigma : \pi \mapsto \sigma(\pi) \in \mathbb{C}^{d_\pi \times d_\pi} : \pi \in \widehat{K}\} = \prod_{\pi \in \widehat{K}} \mathbb{C}^{d_\pi \times d_\pi}. \tag{5}$$

Then $\ell^p_{\text{sch}}(\widehat{K})$ is defined as the set of all $\sigma \in \Sigma(K)$ with the finite

$$\|\sigma\|_{\ell^p_{\text{sch}}(\widehat{K})} := \left(\sum_{\pi \in \widehat{K}} k_\pi \|\sigma(\pi)\|^p_{S^p} \right)^{\frac{1}{p}}, \ 1 \leq p < \infty, \tag{6}$$

and

$$\|\sigma\|_{\ell^\infty_{\text{sch}}(\widehat{K})} := \sup_{\pi \in \widehat{K}} \|\sigma(\pi)\|_{\mathcal{L}(\mathcal{H}_\pi)},$$

where the Schatten p-norm $\|\sigma\|_{S^p}$ of a matrix $\sigma \in \mathbb{C}^{d_\pi \times d_\pi}$ with its singular numbers s_j is defined as $\|\sigma\|_{S^p} := \left(\sum_{j=1}^{d_\pi} s_j^p\right)^{\frac{1}{p}}$.

We denote by $\Sigma_c(\widehat{K})$ the set of all $\sigma \in \Sigma(K)$ such that $\#\{\pi \in \widehat{K} : \sigma(\pi) \neq 0\} < \infty$ and by $\Sigma_0(K)$ the set of all $\sigma \in \Sigma(K)$ such that $\#\{\pi \in \widehat{K} : \|\sigma(\pi)\|_{\mathcal{L}(\mathcal{H}_\pi)} \geq \epsilon\} < \infty$ for all $\epsilon > 0$. For each $\pi \in \widehat{K}$, the Fourier transform $\widehat{f}$ of $f \in L^1(K)$ is defined as

$$\widehat{f}(\pi) = \int_K f(x)\bar{\pi}(x)\, d\lambda(x),$$

where $\bar{\pi}(x) = \pi(\check{x})$ is the conjugate representation of π. Vrem [46] proved that the map $f \mapsto \widehat{f}$ is a norm decreasing $*$-isomorphism of $L^1(K)$ onto a dense subalgebra of $\Sigma_0(K)$. For $f \in L^2(K)$, we have

$$f = \sum_{\pi \in \widehat{K}} k_\pi \sum_{i,j=1}^{d_\pi} \widehat{f}(\pi)_{i,j}\pi_{i,j} \tag{7}$$

and the series converges in $L^2(K)$, see [46, Corollary 2.10]. Hence, we have the following Plancherel identity

$$\|f\|_2^2 = \sum_{\pi \in \widehat{K}} k_\pi \sum_{i,j=1}^{d_\pi} |\widehat{f}(\pi)_{i,j}|^2 = \sum_{\pi \in \widehat{K}} k_\pi \|\widehat{f}(\pi)\|_{\mathrm{HS}}^2 = \|\widehat{f}\|^2_{\ell^2_{\mathrm{sch}}(\widehat{K})}.$$

The following Hausdorff-Young inequality holds for Fourier transform on compact hypergroups ([45]).

Theorem 2.2. *Let $1 < p \leq 2$ with $\frac{1}{p} + \frac{1}{p'} = 1$. For any $f \in L^p(K)$ we have the following inequality*

$$\left(\sum_{\pi \in \widehat{K}} k_\pi \|\widehat{f}(\pi)\|_{S^p}^{p'}\right)^{\frac{1}{p'}} = \|\widehat{f}\|_{\ell^{p'}_{sch}(\widehat{K})} \leq \|f\|_{L^p(K)}. \tag{8}$$

Recently, the first author with R. Sarma [31] also obtained a Hausdorff-Young inequality using different norm which was useful to study the Hausdorff-Young inequality for Orlicz spaces [31]. We will discuss it in the next section in more detail.

2.3. **Commutative compact hypergroups.** In this section we assume that a compact hypergroup K is commutative. Then every representation of K is one dimensional. The dual space of K defined as follows

$$\widehat{K} = \left\{\chi \in C^b(K) : \chi \neq 0,\, \chi(\check{m}) = \overline{\chi(m)},\, (\delta_m * \delta_n)(\chi) = \chi(m)\chi(n) \text{ for all } m, n \in K\right\}.$$

An element in $\widehat{K}$ will be called a *character*. We equip $\widehat{K}$ with the uniform convergence on compact sets. In the case of a compact hypergroups K, the dual space $\widehat{K}$ is discrete. In general, $\widehat{K}$ may not have a dual hypergroup structure with respect to the pointwise product [27, Example 9.1 C] but it holds for most "natural" hypergroups including the conjugacy classes of compact groups. Then the Fourier transform on $L^1(K, \lambda)$ is defined by

$$\widehat{f}(\chi) := \int_K f(x)\, \overline{\chi(x)}\, d\lambda(x), \quad \chi \in \widehat{K}.$$

The Fourier transform is injective and there exists a Radon measure ω on $\widehat{K}$, called the Plancherel measure on $\widehat{K}$ such that the map $f \mapsto \widehat{f}$ extends to an isometric isomorphism from $L^2(K, d\lambda)$ onto $L^2(\widehat{K}, d\omega)$, that is,

$$\sum_{\chi \in \widehat{K}} |\widehat{f}(\chi)|^2 d\omega(\chi) = \int_K |f(x)|^2\, d\lambda(x). \tag{9}$$

In this case, the Fourier series of f given by (7) takes the form

$$f = \sum_{\chi \in \widehat{K}} k_\chi\, \widehat{f}(\chi)\, \chi. \tag{10}$$

It follows from the orthogonality relation of characters (4) that the set $\{k_\chi^{\frac{1}{2}} \chi\}_{\chi \in \widehat{K}}$ forms an orthonormal basis of $L^2(K, d\lambda)$. It is also known that $\omega(\chi) = k_\chi$ for each $\chi \in \widehat{K}$ (see [5, Proposition 1.2]). If K is a compact commutative group then $k_\chi = d_\chi = 1$ for all $\chi \in \widehat{K}$; and therefore Plancherel measure on $\widehat{K}$ is constant 1.

3. Hausdorff-Young-Paley and Hardy-Littlewood inequalities on compact hypergroups

In this section, we will study the Paley inequality, the Hausdorff-Young-Paley inequality and the Hardy-Littlewood inequality for compact hypergroups. By abusing the notation, the measure of a set E with respect to measure ν will be denoted by $\nu(E)$, $\nu(\{E\})$, or by $\nu\{E\}$. At times, we will denote $L^p(K, \lambda)$ by $L^p(K)$ for simplicity.

3.1. **Paley inequality on compact hypergroups.** In this subsection, we prove the Paley inequality for compact hypergroups. The Paley inequality is an important inequality in itself but also plays a vital role in obtaining the Hardy-Littlewood inequality and the Hausdorff-Young-Paley inequality for compact hypergroups. We follow the method of [3].

Theorem 3.1. *Let K be a compact hypergroup and let $1 < p \leq 2$. Let $\varphi : \widehat{K} \to (0, \infty)$ be a function such that*

$$M_\varphi := \sup_{y>0} y \sum_{\substack{\pi \in \widehat{K} \\ \varphi(\pi) \geq y}} k_\pi^2 < \infty. \tag{11}$$

Then, for all $f \in L^p(K)$, we have

$$\left(\sum_{\pi \in \widehat{K}} k_\pi^2 \left(\frac{\|\widehat{f}(\pi)\|_{\mathrm{HS}}}{\sqrt{k_\pi}} \right)^p \varphi(\pi)^{2-p} \right)^{\frac{1}{p}} \lesssim M_\varphi^{\frac{2-p}{p}} \|f\|_{L^p(K)}. \tag{12}$$

Proof. Let us consider the measure on ν on the dual space $\widehat{K}$ of K given by

$$\nu(\{\pi\}) = \varphi(\pi)^2 k_\pi^2, \quad \pi \in \widehat{K}.$$

Define the space $L^p(\widehat{K}, \nu), 1 \leq p < \infty$, as the space of all real or complex sequences $a : \pi \mapsto a_\pi$ such that

$$\|a\|_{L^p(\widehat{K},\nu)} = \left(\sum_{\pi \in \widehat{K}} |a_\pi|^p \nu(\pi) \right)^{\frac{1}{p}} < \infty.$$

We will show that the sublinear operator $A : L^p(K, \lambda) \to L^p(\widehat{K}, \nu)$ defined by

$$Af := \left(\frac{\|\widehat{f}(\pi)\|_{\mathrm{HS}}}{\sqrt{k_\pi}\, \varphi(\pi)} \right)_{\pi \in \widehat{K}}$$

is well defined and bounded for $1 < p \leq 2$. In other words, we will get the following estimate which will eventually give us the required estimate (12),

$$\|Af\|_{L^p(\widehat{K},\nu)} = \left(\sum_{\pi \in \widehat{K}} \left(\frac{\|\widehat{f}(\pi)\|_{\mathrm{HS}}}{\sqrt{k_\pi} \varphi(\pi)} \right)^p \nu(\pi) \right)^{\frac{1}{p}} \lesssim M_\varphi^{\frac{2-p}{p}} \|f\|_{L^p(K)}, \tag{13}$$

where $M_\varphi := \sup_{y>0} y \sum_{\substack{\pi \in \widehat{K} \\ \varphi(\pi) \geq y}} k_\pi^2$. To prove the above estimate (13) it is enough to show that A is of weak type $(1,1)$ and of weak type $(2,2)$, thanks to Marcinkiewicz interpolation theorem. In fact, we show that, with the distribution function $\nu_{\widehat{K}}$, that

$$\nu_{\widehat{K}}(y; Af) \leq \frac{M_1 \|f\|_{L^1(K)}}{y} \quad \text{with the norm } M_1 = M_\varphi, \tag{14}$$

$$\nu_{\widehat{K}}(y; Af) \leq \left(\frac{M_2 \|f\|_{L^2(K)}}{y} \right)^2 \quad \text{with the norm } M_2 = 1, \tag{15}$$

where $\nu_{\widehat{K}}(y; Af)$ is defined by $\nu_{\widehat{K}}(y; Af) := \sum_{\substack{\pi \in \widehat{K} \\ |(Af)(\pi)| \geq y}} \nu(\pi), \ \ y > 0.$

First, we show that A is of weak type $(1,1)$ with norm $M_1 = M_\varphi$; more precisely we show that

$$\nu_{\widehat{K}}(y; Af) = \nu \left\{ \pi \in \widehat{K} : \frac{\|\widehat{f}(\pi)\|_{\mathrm{HS}}}{\sqrt{k_\pi}\varphi(\pi)} > y \right\} \lesssim \frac{M_\varphi \|f\|_{L^1(K)}}{y}, \tag{16}$$

where $\nu \left\{ \pi \in \widehat{K} : \frac{\|\widehat{f}(\pi)\|_{\mathrm{HS}}}{\sqrt{k_\pi}\varphi(\pi)} > y \right\}$ can be interpreted as the sum of $\nu(\pi)$ taken over those $\pi \in \widehat{K}$ such that $\frac{\|\widehat{f}(\pi)\|_{\mathrm{HS}}}{\sqrt{k_\pi}\varphi(\pi)} > y$. By the defintion of the Fourier transform and the fact that π is a norm decreasing $*$-homomorphism, i.e., $\|\pi(\check{x})\|_{\mathrm{op}} \leq 1$ for all $x \in K$, we have

$$\|\widehat{f}(\pi)\|_{\mathrm{HS}} \leq \|f\|_{L^1(K)} \|\pi(\check{x})\|_{\mathrm{HS}} \leq \|f\|_{L^1(K)} \sqrt{d_\pi} \|\pi(\check{x})\|_{\mathrm{op}} \leq \sqrt{d_\pi} \|f\|_{L^1(K)}.$$

Therefore, by using $d_\pi \leq k_\pi$, we get

$$y < \frac{\|\widehat{f}(\pi)\|_{\mathrm{HS}}}{\sqrt{k_\pi}\varphi(\pi)} \leq \frac{\sqrt{d_\pi}\|f\|_{L^1(K)}}{\sqrt{k_\pi}\varphi(\pi)} \leq \frac{\|f\|_{L^1(K)}}{\varphi(\pi)}.$$

This inequality yields that

$$\left\{ \pi \in \widehat{K} : \frac{\|\widehat{f}(\pi)\|_{\mathrm{HS}}}{\sqrt{k_\pi}\varphi(\pi)} > y \right\} \subset \left\{ \pi \in \widehat{K} : \frac{\|f\|_{L^1(K)}}{\varphi(\pi)} > y \right\}$$

for any $y > 0$. So

$$\nu \left\{ \pi \in \widehat{K} : \frac{\|\widehat{f}(\pi)\|_{\mathrm{HS}}}{\sqrt{k_\pi}\varphi(\pi)} > y \right\} \leq \nu \left\{ \pi \in \widehat{K} : \frac{\|f\|_{L^1(K)}}{\varphi(\pi)} > y \right\}.$$

Setting $w = \frac{\|f\|_{L^1(K)}}{y}$, we have

$$\nu \left\{ \pi \in \widehat{K} : \frac{\|\widehat{f}(\pi)\|_{\mathrm{HS}}}{\sqrt{k_\pi}\varphi(\pi)} > y \right\} \leq \sum_{\substack{\pi \in \widehat{K} \\ \varphi(\pi) \leq w}} \varphi(\pi)^2 k_\pi^2.$$

We claim that

$$\sum_{\substack{\pi \in \widehat{K} \\ \varphi(\pi) \leq w}} \varphi(\pi)^2 k_\pi^2 \lesssim M_\varphi w. \tag{17}$$

In fact, we have

$$\sum_{\substack{\pi \in \widehat{K} \\ \varphi(\pi) \leq w}} \varphi(\pi)^2 k_\pi^2 = \sum_{\substack{\pi \in \widehat{K} \\ \varphi(\pi) \leq w}} k_\pi^2 \int_0^{\varphi^2(\pi)} d\tau.$$

By interchanging sum and integration we have

$$\sum_{\substack{\pi \in \widehat{K} \\ \varphi(\pi) \leq w}} k_\pi^2 \int_0^{\varphi^2(\pi)} d\tau = \int_0^{w^2} d\tau \sum_{\substack{\pi \in \widehat{K} \\ \tau^{\frac{1}{2}} \leq \varphi(\pi) \leq w}} k_\pi^2.$$

Next, by making substitution $\tau = t^2$, we have

$$\int_0^{w^2} d\tau \sum_{\substack{\pi\in\widehat{K}\\ \tau^{\frac{1}{2}}\leq\varphi(\pi)\leq w}} k_\pi^2 = 2\int_0^w dt\, t \sum_{\substack{\pi\in\widehat{K}\\ t\leq\varphi(\pi)\leq w}} k_\pi^2 \leq 2\int_0^w dt\, t \sum_{\substack{\pi\in\widehat{K}\\ t\leq\varphi(\pi)}} k_\pi^2.$$

Since

$$t \sum_{\substack{\pi\in\widehat{K}\\ t\leq\varphi(\pi)}} k_\pi^2 \leq \sup_{t>0} t \sum_{\substack{\pi\in\widehat{K}\\ t\leq\varphi(\pi)}} k_\pi^2 = M_\varphi$$

is finite by the assumption, we get

$$2\int_0^w dt\, t \sum_{\substack{\pi\in\widehat{K}\\ t\leq\varphi(\pi)}} k_\pi^2 \lesssim M_\varphi w.$$

Therefore, we get the required estimate (16)

$$\nu_{\widehat{K}}(y; Af) = \nu\left\{\pi\in\widehat{K} : \frac{\|\widehat{f}(\pi)\|_{\mathrm{HS}}}{\sqrt{k_\pi}\varphi(\pi)} > y\right\} \lesssim \frac{M_\varphi\|f\|_{L^1(K)}}{y}.$$

Now, we will prove that A is of weak type $(2,2)$, that is, the equality (15). By using Plancherel's identity we get

$$\begin{aligned} y^2\nu_{\widehat{K}}(y; Af) \leq \|Af\|^2_{L^2(\widehat{K},\nu)} &= \sum_{\pi\in\widehat{K}} k_\pi^2 \left(\frac{\|\widehat{f}(\pi)\|_{\mathrm{HS}}}{\sqrt{k_\pi}\varphi(\pi)}\right)^2 \varphi(\pi)^2 \\ &= \sum_{\pi\in\widehat{K}} k_\pi \|\widehat{f}(\pi)\|^2_{\mathrm{HS}} = \|f\|^2_{L^2(K)}. \end{aligned}$$

Thus A is of weak type $(2,2)$ with norm $M_2 \leq 1$. Thus we have proved (15) and (14). Thus, by using the Marcinkiewicz interpolation theorem with $p_1 = 1$, $p_2 = 2$ and $\frac{1}{p} = 1 - \theta + \frac{\theta}{2}$ we now obtain

$$\left(\sum_{\pi\in\widehat{K}} \left(\frac{\|\widehat{f}(\pi)\|_{\mathrm{HS}}}{\sqrt{k_\pi}\varphi(\pi)}\right)^p \varphi(\pi)^2 k_\pi^2\right)^{\frac{1}{p}} = \|Af\|_{L^p(\widehat{K},\nu)} \lesssim M_\varphi^{\frac{2-p}{p}} \|f\|_{L^p(K)}.$$

This completes the proof. □

Remark 1. One may notice that instead of the Schatten p-norm we used the Hilbert-Schmidt norm in Theorem 3.1. This is because the Hilbert-Schmidt norm gives sharp inequality in the Paley inequality as already noticed in [3] for compact homogeneous spaces and in [49] for compact quantum groups. We will see this for compact hypergroups from the discussion below.

Now, we will define and discuss an another important family of Lebesgue spaces $\ell^p(\widehat{K})$ on $\widehat{K}$ defined by using the Hilbert-Schmidt norm $\|\cdot\|_{\mathrm{HS}}$ instead of Schatten p-norm $\|\cdot\|_{S^p}$ on the space of $(d_\pi \times d_\pi)$-dimensional matrices.

The space $\ell^p(\widehat{K}) \subset \Sigma(K)$ is the set of all $\sigma \in \Sigma(K)$ with finite sum

$$\|\sigma\|_{\ell^p(\widehat{K})} := \left(\sum_{\pi \in \widehat{K}} k_\pi^{(2-\frac{p}{2})} \|\sigma(\pi)\|_{\mathrm{HS}}^p \right)^{\frac{1}{p}}, \quad 1 \leq p < \infty, \tag{18}$$

and

$$\|\sigma\|_{\ell^\infty(\widehat{K})} := \sup_{\pi \in \widehat{K}} k_\pi^{-\frac{1}{2}} \|\sigma(\pi)\|_{\mathrm{HS}}.$$

Remark 2. These ℓ^p-spaces $\ell^p(\widehat{K})$ were introduced in [42, Chapter 10] in the context of compact Lie groups. Recently, these spaces have been studied in more details by the second author and his collaborators [42, 1, 2, 32, 17, 13]. In particular, it was shown in [13] that the space $\ell^p(\widehat{G})$ and the Hausdorff-Young inequality for it become useful for investigating convergence properties of the Fourier series and the characterisation of Gevrey-Roumieu ultradifferentiable functions and Gevrey-Beurling ultradifferentiable functions on compact homogeneous manifolds.

The following proposition presents the relation between both norms on Lebesgue spaces on $\widehat{K}$.

Proposition 3.2. *For $1 \leq p \leq 2$, we have the following continuous embeddings as well as the estimates: $\ell^p(\widehat{K}) \hookrightarrow \ell^p_{sch}(\widehat{K})$ and $\|\sigma\|_{\ell^p_{sch}(\widehat{K})} \leq \|\sigma\|_{\ell^p(\widehat{K})}$ for all $\sigma \in \Sigma(K)$. For $2 \leq p \leq \infty$, we have $\ell^p_{sch}(\widehat{K}) \hookrightarrow \ell^p(\widehat{K})$ and $\|\sigma\|_{\ell^p(\widehat{K})} \leq \|\sigma\|_{\ell^p_{sch}(\widehat{K})}$ for all $\sigma \in \Sigma(K)$.*

Proof. For $p=2$, since $\|\cdot\|_{S^2} = \|\cdot\|_{\mathrm{HS}}$, the assertion is obvious. Let $1 \leq p < 2$. Since $\sigma(\pi) \in \mathbb{C}^{d_\pi \times d_\pi}$, denoting s_j its singular number, by the Hölder inequality we have

$$\|\sigma(\pi)\|_{S^p}^p = \sum_{j=1}^{d_\pi} s_j^p \leq \left(\sum_{j=1}^{d_\pi} 1 \right)^{\frac{2-p}{2}} \left(\sum_{j=1}^{d_\pi} s_j^{p\frac{2}{p}} \right)^{\frac{p}{2}} = d_\pi^{\frac{2-p}{2}} \|\sigma(\pi)\|_{\mathrm{HS}}^p. \tag{19}$$

Consequently, it follows that

$$\|\sigma\|_{\ell^p_{sch}(\widehat{K})}^p = \sum_{\pi \in \widehat{K}} k_\pi \|\sigma(\pi)\|_{S^p}^p \leq \sum_{\pi \in \widehat{K}} k_\pi d_\pi^{\frac{2-p}{2}} \|\sigma(\pi)\|_{\mathrm{HS}}^p \leq \sum_{\pi \in \widehat{K}} k_\pi k_\pi^{\frac{2-p}{2}} \|\sigma(\pi)\|_{\mathrm{HS}}^p = \|\sigma\|_{\ell^p(\widehat{K})}^p.$$

Now, for $2 < p < \infty$, we have

$$\|\sigma(\pi)\|_{\mathrm{HS}}^2 = \sum_{j=1}^{d_\pi} s_j^2 \leq \left(\sum_{j=1}^{d_\pi} 1\right)^{\frac{p-2}{p}} \left(\sum_{j=1}^{d_\pi} s_j^{2\frac{p}{2}}\right)^{\frac{2}{p}} = d_\pi^{\frac{p-2}{p}} \|\sigma(\pi)\|_{S^p}^2, \tag{20}$$

and thus

$$\|\sigma(\pi)\|_{\mathrm{HS}} \leq d_\pi^{\frac{p-2}{2p}} \|\sigma(\pi)\|_{S^p}.$$

Therefore, we have

$$\|\sigma\|_{\ell^p(\widehat{K})} = \sum_{\pi\in\widehat{k}} k_\pi^{(2-\frac{p}{2})} \|\sigma(\pi)\|_{\mathrm{HS}}^p \leq \sum_{\pi\in\widehat{K}} k_\pi^{(2-\frac{p}{2})} d_\pi^{\frac{p-2}{2}} \|\sigma(\pi)\|_{S^p}^p \leq \sum_{\pi\in\widehat{k}} k_\pi \|\sigma(\pi)\|_{S^p}^p = \|\sigma\|_{\ell^p_{sch}(\widehat{K})}^p.$$

Finally, for $p=\infty$, the inequality

$$\|\sigma(\pi)\|_{\mathrm{HS}} \leq k_\pi^{\frac{1}{2}} \|\sigma(\pi)\|_{\mathcal{L}(\mathcal{H}_\pi)}$$

implies

$$\|\sigma\|_{\ell^\infty(\widehat{K})} = \sup_{\pi\in\widehat{K}} k_\pi^{-\frac{1}{2}} \|\sigma(\pi)\|_{\mathrm{HS}} \leq \sup_{\pi\in\widehat{K}} \|\sigma(\pi)\|_{\mathcal{L}(\mathcal{H}_\pi)} = \|\sigma\|_{\ell^\infty_{\mathrm{sch}}(\widehat{K})},$$

completing the proof. □

The following Hausdorff-Young inequality for Fourier transform on compact hypergroups was recently obtained by the first author and R. Sarma [31].

Theorem 3.3. *Let $1 \leq p \leq 2$ with $\frac{1}{p}+\frac{1}{p'}=1$. For any $f \in L^p(K)$ we have the following inequality*

$$\left(\sum_{\pi\in\widehat{K}} k_\pi^{2-\frac{p'}{2}} \|\widehat{f}(\pi)\|_{\mathrm{HS}}^{p'}\right)^{\frac{1}{p'}} = \|\widehat{f}\|_{\ell^{p'}(\widehat{K})} \leq \|f\|_{L^p(K)}. \tag{21}$$

In the view of Proposition 3.2 one can see that the Hausdorff-Young inequality (8) using the Schatten p-norm is sharper than the inequality (21). In [31], Theorem 3.3 is further used to define Orlicz space on dual of compact hypergroups and to obtained the Hausdorff-Young inequality for Orlicz spaces on compact hypergroup.

The Paley inequality can be reduced to the familiar form using Schatten p-norms. The proof of it is immediate from the inequality (19) and the fact that $d_\pi \leq k_\pi$.

Corollary 3.4. *Let K be a compact hypergroup and let $1 < p \leq 2$. If $\varphi : \widehat{K} \to (0, \infty)$ is a function satisfying condition (11) of Theorem 3.1 then there exist a universal constant $C = C(p)$ such that*

$$\left(\sum_{\pi \in \widehat{G}} k_\pi \|\widehat{f}(\pi)\|_{S^p}^p \, \varphi(\pi)^{2-p} \right)^{\frac{1}{p}} \leq C \|f\|_{L^p(K)}. \tag{22}$$

3.2. **Hardy-Littlewood inequality on compact hypergroups.** In this section, we apply the Paley inequality to get the Hardy-Littlewood inequality on compact hypergroups. This approach has been recently employed to prove the Hardy-Littlewood inequality in the context of the compact Lie group SU(2) in [1], compact homogeneous manifolds in [3], and compact quantum groups in [2, 49]. The philosophy to derive the Hardy-Littlewood inequality is to choose a function φ suitably, so that the condition (11) of Theorem 3.1 is satisfied. For a Laplacian Δ_G on a compact Lie group G, we have that for a fixed $\xi \in \widehat{G}$, all $\xi_{ij}, 1 \leq i, j \leq d_\xi$, are eigenfunctions of $-\Delta_G$ with the same eigenvalue, which we denote by $|\xi|^2$, so that we have

$$-\Delta_G \xi_{ij}(x) = |\xi|^2 \xi_{ij}(x) \quad 1 \leq i, j \leq d_\xi.$$

We denote $\langle \xi \rangle := (1+|\xi|^2)^{1/2}$, which is the eigenvalue of the operator $(1-\Delta_G)^{\frac{1}{2}}$. In the case of a compact Lie group G of dimension n, in [3] the authors took $\varphi(\pi) = \langle \pi \rangle^{-n}$. Although, for SU(2) this was proved by repeating the proof of the Paley inequality and estimating the bound explicitly ([1]). In the case of compact quantum groups, the proof of this inequality has been achieved by using the geometric information of compact quantum groups like spectral triples [2] and the natural length function on the dual of compact quantum groups [49]. Since the compact hypergroups in general are not equipped with any geometric or differential structure, we prove the following the Hardy-Littlewood inequality for compact hypergroups.

Theorem 3.5. *Let $1 < p \leq 2$ and let K be a compact hypergroup. Assume that a positive function $\pi \mapsto \mu_\pi$ on $\widehat{K}$ grows sufficiently fast, that is,*

$$\sum_{\pi \in \widehat{K}} \frac{k_\pi^2}{|\mu_\pi|^\beta} < \infty \;\; \textit{for some}\, \beta \geq 0. \tag{23}$$

Then we have

$$\sum_{\pi\in\widehat{K}} k_\pi^2 |\mu_\pi|^{\beta(p-2)} \left(\frac{\|\widehat{f}(\pi)\|_{\mathrm{HS}}}{\sqrt{k_\pi}}\right)^p \lesssim \|f\|_{L^p(K)}. \tag{24}$$

Proof. By the assumption, we know that

$$C := \sum_{\pi\in\widehat{K}} \frac{k_\pi^2}{|\mu_\pi|^\beta} < \infty.$$

Then we have

$$C \geq \sum_{\substack{\pi\in\widehat{K}\\ |\mu_\pi|^\beta \leq \frac{1}{t}}} \frac{k_\pi^2}{|\mu_\pi|^\beta} \geq t \sum_{\substack{\pi\in\widehat{K}\\ |\mu_\pi|^\beta \leq \frac{1}{t}}} k_\pi^2 = t \sum_{\substack{\pi\in\widehat{K}\\ \frac{1}{|\mu_\pi|^\beta} \geq t}} k_\pi^2,$$

and consequently we have

$$\sup_{t>0} t \sum_{\substack{\pi\in\widehat{K}\\ \frac{1}{|\mu_\pi|^\beta} \geq t}} k_\pi^2 \leq C < \infty.$$

Then, as an application of Theorem 3.1 with $\varphi(\pi) = \frac{1}{|\mu_\pi|^\beta}$, $\pi \in \widehat{K}$, we get the required estimate (24). $\square$

In the case when K is abelian, Theorem 3.5 takes the following form.

Theorem 3.6. *Let $1 < p \leq 2$ and let K be a compact abelian hypergroup. Assume that a positive function $\chi \mapsto \mu_\chi$ on $\widehat{K}$ satisfies the condition*

$$\sum_{\chi\in\widehat{K}} \frac{k_\chi^2}{|\mu_\chi|^\beta} < \infty \quad \textit{for some } \beta \geq 0. \tag{25}$$

Then we have

$$\sum_{\chi\in\widehat{K}} k_\chi^{2-\frac{p}{2}} |\mu_\chi|^{\beta(p-2)} |\widehat{f}(\chi)|^p \lesssim \|f\|_{L^p(K)}. \tag{26}$$

Remark 3. We would like to note here that in the case when K is a compact Lie group, the natural choices of $\pi \mapsto \mu_\pi$ is $\pi \mapsto \langle\pi\rangle$. But for this choice of μ_π the quantity $\sum_{\pi\in\widehat{K}} \frac{d_\pi^2}{\langle\pi\rangle^\beta}$ in this case, is not finite for $\beta = n := \dim(G)$, as proved by the second author and Dasgupta [13]. So this does not give the Hardy-Littlewood inequality for compact Lie groups, in particular, for $\mathbb{T}^n$ ([3]). Surprisingly, the quantity $\sum_{\pi\in\widehat{K}} \frac{k_\pi^2}{|\mu_\pi|^\beta}$ is finite with a natural choice of $\pi \mapsto \mu_\pi$ and β for (pure) hypergroups including conjugacy classes of compact Lie groups and countable compact hypergroups as shown in the last section and consequently, provides the Hardy-Littlewood inequality for these compact hypergroups.

3.3. **Hausdorff-Young-Paley inequality on compact hypergroups.** In this subsection, we prove the Hausdorff-Young-Paley inequality for compact hypergroups. The Hausdorff-Young-Paley inequality is an important inequality in itself but it serves as an essential tool to prove an L^p-L^q Fourier multiplier theorem for compact hypergroups.

The following theorem obtained by Bergh and Lofstrom [9] is useful in the proof of the Hausdorff-Young-Paley inequality.

Theorem 3.7. *Let $d\mu_0(x) = \omega_0(x)d\mu(x)$, $d\mu_1(x) = \omega_1(x)d\mu(x)$. Suppose that $0 < p_0, p_1 < \infty$. If a continuous linear operator A admits bounded extensions, $A : L^p(Y,\mu) \to L^{p_0}(\omega_0)$ and $A : L^p(Y,\mu) \to L^{p_1}(\omega_1)$, then there exists a bounded extension $A : L^p(Y,\mu) \to L^b(\tilde{\omega})$ of A, where $0 < \theta < 1$, $\frac{1}{b} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ and $\tilde{\omega} = \omega_0^{\frac{b(1-\theta)}{p_0}} \omega_1^{\frac{b\theta}{p_1}}$.*

Now, we are ready to state the Hausdorff-Young-Paley inequality for compact hypergroups, and we follow the idea of [3] for the proof.

Theorem 3.8. *Let K be a compact hypergroup and let $1 < p \leq b \leq p' < \infty$, where p' is the Lebesgue conjugate of p. If a function $\varphi : \widehat{K} \to (0,\infty)$ satisfies the condition*

$$M_\varphi := \sup_{y>0} y \sum_{\substack{\pi\in\widehat{K} \\ \varphi(\pi)\geq y}} k_\pi^2 < \infty \tag{27}$$

then we have

$$\left(\sum_{\pi\in\widehat{G}} k_\pi^2 \left(\frac{\|\widehat{f}(\pi)\|_{\mathrm{HS}}}{\sqrt{k_\pi}} \varphi(\pi)^{\frac{1}{b}-\frac{1}{p'}}\right)^b\right)^{\frac{1}{b}} \lesssim M_\varphi^{\frac{1}{b}-\frac{1}{p'}} \|f\|_{L^p(K)}. \tag{28}$$

Proof. We consider a sublinear operator $A : L^p(K) \to \ell^p(\widehat{K}, \tilde{\omega})$ which takes a function f to its Fourier coefficient $\widehat{f}(\pi) \in \mathbb{C}^{d_\pi\times d_\pi}$ divided by $\sqrt{k_\pi}$, that is,

$$f \mapsto Af := \left\{\frac{\widehat{f}(\pi)}{\sqrt{k_\pi}}\right\}_{\pi\in\widehat{K}}.$$

Here the space $\ell^p(\widehat{K}, \tilde{\omega})$ is the set of all $\sigma \in \Sigma(K)$ with finite

$$\|a\|_{\ell^p(\widehat{K},\tilde{\omega})} := \left(\sum_{\pi\in\widehat{K}} \|a(\pi)\|_{\mathrm{HS}}^p \, \tilde{\omega}(\pi)\right)^{\frac{1}{p}},$$

and $\tilde{\omega}$ is a scalar sequence on $\widehat{K}$. Then the desired result follows from Theorem 3.7 if we consider the left hand side of the inequalities (12) and (21) as $\ell^p(\widehat{K}, \tilde{\omega}_i)$-norm of Af, where the weights ω_i, $i = 1, 2$ are given by $\omega_0(\pi) = k_\pi^2 \varphi(\pi)^{2-p}$ and $\omega_1(\pi) = k_\pi^2$, $\pi \in \widehat{K}$, respectively. $\square$

4. L^p-L^q-boundedness of Fourier multipliers on compact hypergroups

In this section, we prove the L^p-L^q boundedness of Fourier multipliers on compact hypergroups as a natural analogue of Hörmander's theorem (see [26]). We will apply the Hausdorff-Young-Paley inequality in Theorem 3.8 to provide a sufficient condition for the L^p-L^q boundedness of Fourier multipliers for the range $1 < p \leq 2 \leq q < \infty$. This approach was developed by the second author with R. Akylzhanov to prove the L^p-L^q boundedness of Fourier multipliers on locally compact groups [4] by using the von-Neumann algebra machinery. In [37], this theorem was proved for the torus $\mathbb{T}$ by using a different method. We begin this section by recalling the definition of Fourier multipliers on compact hypergroups.

An operator A which is invariant under the left translations will be called a left Fourier multiplier. The left invariant operators can be characterised using the Fourier transform [45, 43]. Indeed, if A is a left Fourier multiplier then there exists a function $\sigma_A : \widehat{K} \to \mathbb{C}^{d_\pi \times d_\pi}$, known as the symbol associated with A, such that

$$\widehat{Af}(\pi) = \sigma_A(\pi)\widehat{f}(\pi), \quad \pi \in \widehat{K},$$

for all suitable functions f on K. In the next result, we show that if the symbol σ_A of a Fourier multipliers A defined on $C_c(K)$ satisfies certain Hörmander's condition, then A can be extended as a bounded linear operator from $L^p(K)$ to $L^q(K)$ for the range $1 < p \leq 2 \leq q < \infty$. The Plancherel formula provides a condition on symbol σ_A for the L^2-L^2-boundedness of Fourier multiplier A. Indeed, we have $\|A\|_{L^2(K)\to L^2(K)} \leq \|\sigma_A\|_{\ell^\infty(\widehat{K})}$. Therefore, we restrict ourselves to the case when p and q are both not equal to 2. For the proof we follow the idea of [3].

Theorem 4.1. *Let K be a compact hypergroup and let $1 < p \leq 2 \leq q < \infty$ with p and q both not equal to 2. Let A be a left Fourier multiplier with symbol σ_A. Then we have*

$$\|A\|_{L^p(K)\to L^q(K)} \lesssim \sup_{y>0} y \left(\sum_{\substack{\pi \in \widehat{K} \\ \|\sigma_A(\pi)\|_{op} \geq y}} k_\pi^2 \right)^{\frac{1}{p}-\frac{1}{q}}. \tag{29}$$

Proof. Let us first consider the case when $p \leq q'$ (where $\frac{1}{q}+\frac{1}{q'}=1$). Since $q' \leq 2$, for $f \in C_c(K)$, the Hausdorff-Young inequality gives

$$\|Af\|_{L^q(K)} \leq \|\widehat{Af}\|_{\ell^{q'}(\widehat{K})} = \|\sigma_A \widehat{f}\|_{\ell^{q'}(\widehat{K})} = \left(\sum_{\pi\in\widehat{K}} k_\pi^2 \left(\frac{\|\sigma_A(\pi)\widehat{f}(\pi)\|_{\mathrm{HS}}}{\sqrt{k_\pi}} \right)^{q'} \right)^{\frac{1}{q'}} \tag{30}$$

$$\leq \left(\sum_{\pi\in\widehat{K}} k_\pi^2 \|\sigma_A(\pi)\|_{\mathrm{op}}^{q'} \left(\frac{\|\widehat{f}(\pi)\|_{\mathrm{HS}}}{\sqrt{k_\pi}} \right)^{q'} \right)^{\frac{1}{q'}}. \tag{31}$$

The case $q' \leq p = (p')'$ can be reduced to the case $p \leq q'$ as follows. The L^p-duality (see [1, Theorem 4.2]) yields

$$\|A\|_{L^p(K)\to L^q(K)} = \|A^*\|_{L^{q'}(K)\to L^{p'}(K)}.$$

Also, the symbol $\sigma_{A^*}(\pi)$ of the adjoint operator A^* is equal to σ_A^*, i.e.,

$$\sigma_{A^*}(\pi) = \sigma_A(\pi)^*, \quad \pi \in \widehat{K},$$

and its operator norm $\|\sigma_{A^*}(\pi)\|_{\mathrm{op}}$ is equal to $\|\sigma_A(\pi)\|_{\mathrm{op}}$.
We set $\sigma(\pi) = \|\sigma_A(\pi)\|_{\mathrm{op}}^r I_{d_\pi}, \pi \in \widehat{K}$, where $\frac{1}{r} = \frac{q-p}{pq}$, and it is easy to see that

$$\|\sigma(\pi)\|_{\mathrm{op}} = \|\sigma_A(\pi)\|_{\mathrm{op}}^r.$$

Now, its time to apply Theorem 3.8 with $\varphi(\pi) = \|\sigma(\pi)\|_{\mathrm{op}},\ \pi \in \widehat{K}$, and $b = q'$. Since the assumption of Theorem 3.8 is satisfied and $\frac{1}{q'} - \frac{1}{p'} = \frac{1}{p} - \frac{1}{q} = \frac{1}{r}$, we obtain

$$\left(\sum_{\pi\in\widehat{K}} k_\pi^2 \|\sigma_A(\pi)\|_{\mathrm{op}}^{q'} \left(\frac{\|\widehat{f}(\pi)\|_{\mathrm{HS}}}{\sqrt{k_\pi}} \right)^{q'} \right)^{\frac{1}{q'}} \lesssim \left(\sup_{y>0} y \sum_{\substack{\pi\in\widehat{K} \\ \|\sigma(\pi)\|_{op}\geq y}} k_\pi^2 \right)^{\frac{1}{r}} \|f\|_{L^p(K)},\ f \in L^p(K). \tag{32}$$

Further, it can be easily checked that

$$\left(\sup_{y>0} y \sum_{\substack{\pi\in\widehat{K} \\ \|\sigma(\pi)\|_{\mathrm{op}}\geq y}} k_\pi^2 \right)^{\frac{1}{r}} = \left(\sup_{y>0} y \sum_{\substack{\pi\in\widehat{K} \\ \|\sigma_A(\pi)\|_{\mathrm{op}}^r\geq y}} k_\pi^2 \right)^{\frac{1}{r}} = \left(\sup_{y>0} y^r \sum_{\substack{\pi\in\widehat{K} \\ \|\sigma_A(\pi)\|_{\mathrm{op}}\geq y}} k_\pi^2 \right)^{\frac{1}{r}}$$

$$= \sup_{y>0} y \left(\sum_{\substack{\pi\in\widehat{K} \\ \|\sigma_A(\pi)\|_{op}\geq y}} k_\pi^2 \right)^{\frac{1}{r}}.$$

Therefore,

$$\|Af\|_{L^q(K)} \lesssim \sup_{y>0} y \left(\sum_{\substack{\pi\in\widehat{K} \\ \|\sigma_A(\pi)\|_{\mathrm{op}}\geq y}} k_\pi^2 \right)^{\frac{1}{r}} \|f\|_{L^p(K)}$$

and hence

$$\|A\|_{L^p(K)\to L^q(K)} \lesssim \sup_{y>0} y \left(\sum_{\substack{\pi\in\widehat{K} \\ \|\sigma_A(\pi)\|_{\mathrm{op}}\geq y}} k_\pi^2 \right)^{\frac{1}{p}-\frac{1}{q}},$$

which completes the proof. □

Remark 4. Recall that if $\omega(M) := \sum_{\pi\in M} k_\pi^2,\ \ M \subseteq \widehat{K}$, is the Plancherel measure on $\widehat{K}$ then we can interpret the condition (29) in a similar form as in Hörmander's theorem for $\mathbb{R}^n$ ([26]) as follows:

$$\|A\|_{L^p(K)\to L^q(K)} \leq \sup_{s>0} \left\{ s\ \omega\{\pi\in\widehat{K} \,:\, \|\sigma_A(\pi)\|_{\mathrm{op}} > s\} \right\}^{\frac{1}{p}-\frac{1}{q}}. \tag{33}$$

Corollary 4.2. *Let* $1<p,q<\infty$ *and suppose that* A *is a Fourier multiplier with symbol* σ_A *on a compact hypergroup* K*. If* $1<p,q\leq 2$*, then*

$$\|A\|_{L^p(K)\to L^q(K)} \lesssim \sup_{y>0} y \left(\sum_{\substack{\pi\in\widehat{K} \\ \|\sigma_A(\pi)\|_{op}\geq y}} k_\pi^2 \right)^{\frac{1}{p}-\frac{1}{2}},$$

while for $2\leq p,q<\infty$ *we have*

$$\|A\|_{L^p(K)\to L^q(K)} \lesssim \sup_{y>0} y \left(\sum_{\substack{\pi\in\widehat{K} \\ \|\sigma_A(\pi)\|_{op}\geq y}} k_\pi^2 \right)^{\frac{1}{q'}-\frac{1}{2}}.$$

Proof. Let us assume that $1<p,q\leq 2$. Using the compactness of K, we have $\|A\|_{L^p(K)\to L^q(K)} \lesssim \|A\|_{L^p(K)\to L^2(K)}$ and therefore, Theorem 4.1 gives

$$\|A\|_{L^p(K)\to L^q(K)} \lesssim \|A\|_{L^p(K)\to L^2(K)} \lesssim \sup_{y>0} y \left(\sum_{\substack{\pi\in\widehat{K} \\ \|\sigma_A(\pi)\|_{op}\geq y}} k_\pi^2 \right)^{\frac{1}{p}-\frac{1}{2}}.$$

Now, let us assume that $2 \leq p, q < \infty$. Then $1 < p', q' \leq 2$, and using the first part of the proof we deduce

$$\|A\|_{L^p(K)\to L^q(K)} = \|A^*\|_{L^{q'}(K)\to L^{p'}(K)} \lesssim \sup_{y>0} y \left(\sum_{\substack{\pi\in\widehat{K} \\ \|\sigma_A(\pi)\|_{op}\geq y}} k_\pi^2 \right)^{\frac{1}{q'}-\frac{1}{2}}.$$

Thus, we finish the proof. $\square$

5. Examples of hypergroups

In this section we discuss the results obtained in previous sections and prove some new results for two important classes of hypergroups, namely, the conjugacy classes of the compact non-abelian Lie group SU(2) and countable compact hypergroups introduced and studied by Dunkl and Ramirez in [16].

5.1. **Conjugacy classes of compact Lie groups.** Let G be a compact non-abelian (Lie) group. Denote the set of all conjugacy classes of G by $\mathrm{Conj}(G)$, that is, $\mathrm{Conj}(G) := \{C_x : x \in G\}$, where for each $x \in G$ the conjugacy class C_x of x is given by $C_x := \{yxy^{-1} : y \in G\}$. The set $\mathrm{Conj}(G)$ equipped with the topology induced by the natural map $q : x \mapsto C_x$, is a compact Hausdorff space. $\mathrm{Conj}(G)$ becomes a commutative hypergroup [27, Section 8] with respect to the convolution defined, for $x, y \in G$, by

$$\delta_{C_x} * \delta_{C_y} = \int_G \int_G \delta_{C_{txt^{-1}sys^{-1}}}\, dt\, ds.$$

For $\pi \in \widehat{G}$ let d_π denote the dimension of π and ψ_π the trace of π. Then ψ_π is called the character of π, but the hypergroup character χ_π of $\mathrm{Conj}(G)$ is defined as $\chi_\pi \circ q = d_\pi^{-1}\psi_\pi$, where q is the natural map $x \mapsto C_x$. Then the dual $\widehat{\mathrm{Conj}(G)}$ of the commutative hypergroup $\mathrm{Conj}(G)$ is given by: $\widehat{\mathrm{Conj}(G)} := \{\chi_\pi : \pi \in \widehat{G}\}$. In fact, the map $\pi \mapsto d_\pi^2 \psi_\pi$ is a bijection between $\widehat{G}$ and $\widehat{\mathrm{Conj}(G)}$. The Haar measure ω of $\widehat{\mathrm{Conj}(G)}$ is induced from the one on G and thus, $\omega(\chi_\pi) := k_{\chi_\pi} = d_\pi^2$.

In the sequel of the paper we will consider the case when $G = \mathrm{SU}(2)$, the compact group of all 2×2 special unitary matrices. The representation theory of SU(2) is well established. One can refer to [25, 44, 42] for more details. Conj(SU(2)) is identified with

$[0,1]$ where t in $[0,1]$ corresponds to the conjugacy class containing the matrix

$$\begin{bmatrix} \exp(i\pi t) & 0 \\ 0 & \exp(-i\pi t) \end{bmatrix},$$

(see [27, 15.4]). The dual of SU(2) can be represented by

$$\{\pi_l \in \mathrm{Hom}(\mathrm{SU}(2), \mathrm{U}(2l+1)) : l \in \frac{1}{2}\mathbb{N}_0\},$$

where $\mathrm{U}(d)$ is the $d \times d$ unitary matrix group. The number $l \in \frac{1}{2}\mathbb{N}_0$ is called the quantum number. The character ψ_l, defined as the trace of π_l, is given by

$$\psi_l(t) = \frac{\sin(2l+1)\pi t}{\sin \pi t}.$$

Therefore, since $d_\pi = 2l+1$, the set $\widehat{\mathrm{Conj}(\mathrm{SU}(2))}$ of hypergroup characters is given by $\{(2l+1)^{-1}\psi_l : l \in \frac{1}{2}\mathbb{N}_0\}$ and $k_{\chi_l} = (2l+1)^2$.

The Paley inequality in Theorem 3.1 takes the following form in the setting of the compact abelian hypergroup Conj(SU(2)).

Theorem 5.1. *Let $1 < p \leq 2$ and let $\{\varphi(l)\}_{l \in \frac{1}{2}\mathbb{N}_0}$ be a positive sequence such that*

$$M_\varphi := \sup_{y>0} y \sum_{\substack{l \in \frac{1}{2}\mathbb{N}_0 \\ \varphi(l) \geq y}} (2l+1)^4 < \infty.$$

Then we have

$$\sum_{l \in \frac{1}{2}\mathbb{N}_0} (2l+1)^{4-p} \widehat{f}(l) \varphi(l)^{2-p} \lesssim M_\varphi^{2-p} \|f\|^p_{L^p(Conj(SU(2)))}.$$

We have the following Hardy-Littlewood inequality.

Theorem 5.2. *If $1 < p \leq 2$ and $f \in L^p(\mathrm{Conj}(\mathrm{SU})(2))$, then there exists a universal constant $C = C(p)$ such that*

$$\sum_{l \in \frac{1}{2}\mathbb{N}_0} (2l+1)^{5p-8} |\widehat{f}(l)|^p \leq C \|f\|_{L^p(\mathrm{Conj}(\mathrm{SU})(2))}. \tag{34}$$

Proof. Take $\beta = 3 = \dim(\mathrm{SU}(2))$ and $\{\mu_{\chi_\pi}\}_{\pi \in \widehat{\mathrm{Conj}(\mathrm{SU})(2)}} := \{(2l+1)^2\}_{l \in \frac{1}{2}\mathbb{N}_0}$. Then the condition (25) turns out to be

$$\sum_{l \in \frac{1}{2}\mathbb{N}_0} \frac{(2l+1)^4}{(|(2l+1)^2|)^3} = \sum_{l \in \frac{1}{2}\mathbb{N}_0} \frac{1}{(2l+1)^2} = \frac{\pi^2}{6}$$

which is finite. Therefore, (34) follows from Theorem 3.6. □

Remark 5. We would like to recall here the Hardy-Littlewood inequality on the compact Lie group SU(2) obtained by the second author and R. Akylzhanov in [1], which says that for $1 < p \leq 2$ and $f \in L^p(\mathrm{SU}(2))$ we have

$$\sum_{l \in \frac{1}{2}\mathbb{N}_0} (2l+1)^{\frac{5}{2}p-4} \|\widehat{f}(l)\|_{\mathrm{HS}} \leq C_p \|f\|_{L^p(\mathrm{SU}(2))}.$$

In view of this inequality, the Hardy-Littlewood inequality for the compact commutative hypergroup Conj(SU(2)) above is a suitable analogue because in Conj(SU(2)) the dimension $(2l+1)$ of the representation π_l is replaced by hyperdimension $(2l+1)^2$ of π_l and the Fourier transform $\widehat{f}$ at $l \in \frac{1}{2}\mathbb{N}_0$ is scalar and thus $\|\widehat{f}(l)\|_{\mathrm{HS}}$ is just $|\widehat{f}(l)|$.

Using the duality, we get the following corollary.

Corollary 5.3. *If* $2 \leq p < \infty$ *and* $\sum_{l \in \frac{1}{2}\mathbb{N}_0} (2l+1)^{5p-8} |\widehat{f}(l)|^p < \infty$ *then*

$$f \in L^p(\mathrm{Conj}(\mathrm{SU})(2)).$$

Moreover, we have

$$\|f\|_{L^p(\mathrm{Conj}(\mathrm{SU})(2))} \leq C(p) \sum_{l \in \frac{1}{2}\mathbb{N}_0} (2l+1)^{5p-8} |\widehat{f}(l)|^p.$$

Proof. Using the duality of L^p-spaces, we have

$$\|f\|_{L^p(\mathrm{Conj}(\mathrm{SU})(2))} = \sup_{\substack{g \in L^{p'}(\mathrm{Conj}(\mathrm{SU})(2)) \\ \|g\|_{L^{p'}(\mathrm{Conj}(\mathrm{SU})(2))} \leq 1}} \left| \int_{\mathrm{Conj}(\mathrm{SU})(2)} f(x)\, \overline{g(x)}\, d\lambda(x) \right|.$$

Now, by the Plancherel identity (9), we get

$$\int_{\mathrm{Conj}(\mathrm{SU})(2)} f(x) \overline{g(x)}\, d\lambda(x) = \sum_{l \in \frac{1}{2}\mathbb{N}_0} (2l+1)^2 \widehat{f}(l)\, \overline{\widehat{g}(l)}.$$

By noting that $(2l+1)^2 = (2l+1)^{2\left(\frac{5}{2} - \frac{4}{p} + \frac{5}{2} - \frac{4}{p'}\right)}$ and applying the Hölder inequality, for any $g \in L^{p'}(\mathrm{Conj}(\mathrm{SU})(2))$, we have

$$\begin{aligned} \left| \sum_{l \in \frac{1}{2}\mathbb{N}_0} (2l+1)^2 \widehat{f}(l)\, \overline{\widehat{g}(l)} \right| &\leq \sum_{l \in \frac{1}{2}\mathbb{N}_0} (2l+1)^{5-\frac{8}{p}} |\widehat{f}(l)| (2l+1)^{5-\frac{8}{p}} |\widehat{g}(l)| \\ &\leq \left(\sum_{l \in \frac{1}{2}\mathbb{N}_0} (2l+1)^{5p-8} |\widehat{f}(l)|^p \right)^{\frac{1}{p}} \left(\sum_{l \in \frac{1}{2}\mathbb{N}_0} (2l+1)^{5p'-8} |\widehat{g}(l)|^{p'} \right)^{\frac{1}{p'}} \end{aligned}$$

$$\leq C(p) \left(\sum_{l \in \frac{1}{2}\mathbb{N}_0} (2l+1)^{5p-8} |\widehat{f}(l)|^p \right)^{\frac{1}{p}} \|g\|_{L^{p'}(\mathrm{Conj(SU)(2)})},$$

where we used Theorem 5.2 in the last inequality. Therefore, by (9) we have

$$\left| \int_{\mathrm{Conj(SU)(2)}} f(x)\, \overline{g(x)}\, d\lambda(x) \right| \leq C(p) \left(\sum_{l \in \frac{1}{2}\mathbb{N}_0} (2l+1)^{5p-8} |\widehat{f}(l)|^p \right)^{\frac{1}{p}} \|g\|_{L^{p'}(\mathrm{Conj(SU)(2)})}.$$

Thus, by taking supremum over all $g \in L^{p'}(\mathrm{Conj(SU)(2)})$ with $\|g\|_{L^{p'}(\mathrm{Conj(SU)(2)})} \leq 1$, we get

$$\|f\|_{L^p(\mathrm{Conj(SU)(2)})} \leq C(p) \left(\sum_{l \in \frac{1}{2}\mathbb{N}_0} (2l+1)^{5p-8} |\widehat{f}(l)|^p \right)^{\frac{1}{p}},$$

which completes the proof. □

5.2. **Countable compact hypergroups.** Dunkl and Ramirez [16] studied an interesting class of countable hypergroups. Let $\mathbb{N}_0^* = \{0, 1, 2, \ldots, \infty\}$ be the one-point compactification of $\mathbb{N}_0$. They defined a convolution structure $*$ on $\mathbb{N}_0^*$ for every $0 < a \leq \frac{1}{2}$, which makes $\mathbb{N}_0^*$ a (hermitian) countable compact hypergroup H_a. For a prime p, let Δ_p be the ring of p-adic integers and $\mathcal{W}$ be its group of units, that is, $\{x = x_0 + x_1 p + \ldots + x_n p^n + \ldots \in \Delta_p : x_j = 0, 1, \ldots, p-1 \text{ for } j \geq 0 \text{ and } x_0 \neq 0\}$. $\mathcal{W}$ acts on Δ_p by multiplication and H_a for $a = \frac{1}{p}$, derives its structure from the $\mathcal{W}$-orbits in Δ_p. In fact, the convolution is given as follows: for $m, n \in \mathbb{N}_0$, define

$$\delta_m * \delta_n = \delta_{\min\{m,n\}} \quad \text{if } m \neq n,$$

$\delta_m * \delta_\infty = \delta_\infty * \delta_m = \delta_m$, $\delta_\infty * \delta_\infty = \delta_\infty$, and for $m = n$,

$$\delta_m * \delta_m(t) = \begin{cases} 0 & t < m, \\ \frac{1-2a}{1-a} & t = m, \\ a^k & t = m+k > m, \\ 0 & t = \infty. \end{cases}$$

The Haar measure λ on H_a is given by

$$\lambda(\{k\}) = a^k(1-a) \quad \text{for } k < \infty, \quad \lambda(\{\infty\}) = 0.$$

The dual space $\widehat{H_a}$ of H_a is given by $\{\chi_n : n \in \mathbb{N}_0\}$, where, for $k \in H_a$,

$$\chi_n(k) = \begin{cases} 0 & \text{if } k < n-1, \\ \frac{a}{a-1} & \text{if } k = n-1, \\ 1 & \text{if } k \geq n \ \text{(or } k = \infty). \end{cases}$$

Then the convolution '$*$' on $\mathbb{N}_0$ is identified with the one on $\widehat{H_a}$, which is

$$\begin{aligned} \delta_{\chi_m} * \delta_{\chi_n} &= \delta_{\chi_{\max\{m,n\}}} \ \text{for } \ m \neq n, \\ \delta_{\chi_0} * \delta_{\chi_0} &= \delta_{\chi_0}, \quad \delta_{\chi_1} * \delta_{\chi_1} = \frac{a}{1-a}\delta_{\chi_0} + \frac{1-2a}{1-a}\delta_{\chi_1}, \\ \delta_{\chi_n} * \delta_{\chi_n} &= \frac{a^n}{1-a}\delta_{\chi_0} + \sum_{k=1}^{n-1} a^{n-k}\delta_{\chi_k} + \frac{1-2a}{1-a}\delta_{\chi_n} \quad \text{for}\, n \geq 2. \end{aligned}$$

Then $\widehat{H_a}$ turns into a hermitian discrete hypergroup. We see that $k_{\chi_\pi} = a^{-n}(1-a)$ and the Plancherel measure ω on $\widehat{H}_a$ is given by

$$\omega(\chi_0) = 1 \quad \text{and} \quad \omega(\chi_n) = (1-a)a^{-n} \quad \text{for } n \geq 1.$$

The Paley inequality for the Dunkl-Ramirez hypergroup H_a is then given by the following theorem.

Theorem 5.4. *Let $1 < p \leq 2$ and let $\{\varphi(n)\}_{n\in\mathbb{N}_0}$ be a positive sequence such that*

$$M_\varphi := \sup_{y>0} y \sum_{\substack{n\in\mathbb{N} \\ \varphi(n)\geq y}} (1-a)^2 a^{-2n} + \varphi(0) < \infty.$$

Then we have

$$\sum_{n\in\mathbb{N}} (a^{-n}(1-a))^{2-\frac{p}{2}} \widehat{f}(n)\varphi(n)^{2-p} \lesssim M_\varphi^{2-p} \|f\|^p_{L^p(H_a)}.$$

We have the following Hardy-Littlewood inequality for H_a.

Theorem 5.5. *If $1 < p \leq 2$ then there exists a constant $C = C(p)$ such that*

$$f(0) + \sum_{n\in\mathbb{N}} ((1-a)a^{-n})^{p(\frac{5}{2}-\frac{4}{p})} |\widehat{f}(n)|^p \leq C\|f\|_{L^p(H_a)}. \tag{35}$$

Proof. We apply Theorem 3.6 to get the inequality (35) above. The condition (25) for $\beta = 3$ by choosing the sequence $\{\mu_{\chi_n}\}_{n\in\mathbb{N}} := \{(1-a)a^{-n}\}_{n\in\mathbb{N}}$ with $\mu_{\chi_0} = 1$ turns out to be

$$\sum_{n\in\mathbb{N}_0} \frac{k_{\chi_n}^2}{|\mu_{\chi_n}|^\beta} = \sum_{n\in\mathbb{N}_0} \frac{(1-a)^2 a^{-2n}}{(1-a)^3 a^{-3n}} = \frac{1}{1-a}\sum_{n\in\mathbb{N}_0} a^n = \frac{1}{(1-a)^2},$$

which is finite. Therefore, (35) follows from Theorem 3.6. $\square$

The proof of the following corollary is similar to Corollary 5.3 in the previous subsection.

Corollary 5.6. *If $2 \leq p < \infty$ and $f(0) + \sum_{n\in\mathbb{N}}((1-a)a^{-n})^{p(\frac{5}{2}-\frac{4}{p})}|\widehat{f}(n)|^p < \infty$, then*

$$f \in L^p(H_a).$$

Moreover, we have

$$\|f\|_{L^p(H_a)} \leq C_p \left(f(0) + \sum_{n\in\mathbb{N}}((1-a)a^{-n})^{p(\frac{5}{2}-\frac{4}{p})}|\widehat{f}(n)|^p \right).$$

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