$L^2 - L^p$ ESTIMATES AND HILBERT–SCHMIDT PSEUDO DIFFERENTIAL OPERATORS ON HEISENBERG MOTION GROUP

VISHVESH KUMAR AND SHYAM SWARUP MONDAL

Abstract. In this paper, we study some operator theoretical properties of pseudo-differential operators with operator-valued symbols on the Heisenberg motion group. Specifically, we investigate L^2-L^p boundedness of pseudo-differential operators on the Heisenberg motion group for the range $2 \leq p \leq \infty$. We also provide a necessary and sufficient condition on the operator-valued symbols in terms of λ-Weyl transforms such that the corresponding pseudo-differential operators on the Heisenberg motion group are in the class of Hilbert–Schmidt operators. As a consequence, we obtain a characterization of the trace class pseudo-differential operators on the Heisenberg motion group and provide a trace formula for these trace class operators.

1. Introduction

The theory of pseudo-differential operators was originated with the works of Kohn and Nirenberg [20] and Hörmander [19]. The study of pseudo-differential operators plays an important role in modern mathematics due to its applications in various areas of harmonic analysis, geometry, PDE, mathematical physics, time-frequency analysis, imagin,g and computations, see [19, 29, 16] and references therein.

Let σ be a measurable function on $\mathbb{R}^n \times \mathbb{R}^n$. Then the classical (global) pseudo-differential operator T_σ on $\mathbb{R}^n$ associated with the symbol σ is defined by

$$\left(T_\sigma f\right)(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \sigma(x,\xi) \hat{f}(\xi) d\xi, \quad x \in \mathbb{R}^n,$$

for all f in the Schwartz space $\mathcal{S}\left(\mathbb{R}^n\right)$, provided that the integral exists. Here $\hat{f}$ denotes the Euclidean Fourier transform of f and is defined by

$$\hat{f}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x) dx, \quad \xi \in \mathbb{R}^n.$$

The formation of a pseudo-differential operator on $\mathbb{R}^n$ is mainly based on the Fourier inversion formula for the Fourier transform and can be done by inserting a symbol on the phase space $\mathbb{R}^n \times \mathbb{R}^n$ in the Fourier inversion formula. To extend pseudo-differential operators to other settings, one observes that the second $\mathbb{R}^n$ in the Cartesian product $\mathbb{R}^n \times \mathbb{R}^n$ is the dual of the additive group $\mathbb{R}^n$. These observations allow us to extend the definition of pseudo-differential operators to other groups G, provided we have an explicit formula for the dual of G and an explicit Fourier inversion formula on G. Using this approach, the global theory of pseudo-differential operators on other classes of groups, such as $\mathbb{S}^1, \mathbb{Z}$, affine groups, compact (Lie) groups, homogeneous spaces of compact (Lie) groups, Heisenberg groups, graded Lie groups, step two nilpotent Lie groups, and locally compact type I groups has been widely studied by several researchers [8, 15, 16, 4, 21, 34, 23, 6, 5].

The global theory of pseudo-differential operators on the Heisenberg group was developed by Ruzhansky and Fischer in [15]. Later, the authors introduced the theory of pseudo-differential operators in a more general settings, for example on graded Lie groups [16]. Further, the global

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quantization on unimodular type I locally compact groups and on nilpotent Lie groups were studied by Ruzhansky and Mantoiu in [24] and [25], respectively.

The boundedness and the operator theoretical properties of pseudo-differential play a phenomenal role in the study of partial differential equations and spectral theory. There is a vast literature available on these topics, which is difficult to mention; we refer to recent papers [8, 9, 4, 21, 5, 11, 3]. In particular, to mention the operator theoretical properties such as the belongingness of pseudo-differential operators in the class of Hilbert-Schmidt and more general Schatten class of operators on compact groups and manifold, we refer to [34, 26, 12, 13, 14, 21]. These types of results have also been extended to non-compact (non-abelian) groups by several researchers. Dasgupta and Wong [7] obtained necessary and sufficient conditions on symbols such that the corresponding pseudo-differential operators on the Heisenberg group belong to the Hilbert-Schmidt class, which was further extended by Dasgupta and the first author for abstract Heisenberg groups [8]. Such type of properties for pseudo-differential on H-type groups and on the Affine groups are given in [35] and [10], respectively. Recently, trace class and Hilbert-Schmidt pseudo differential operators on step two nilpotent Lie groups were discussed by the authors in [22]. Motivated by these previous studies as well as the recent developments on λ-Weyl transform on the Heisenberg motion group [18, 17], in this paper, we study and extend some of the aforementioned results to the setting of the Heisenberg motion group. The λ-Weyl transform plays an important role in the proof of our results.

In this paper, we study some operator theoretical properties of pseudo-differential operators on the Heisenberg motion group $G = \mathbb{H}^n \rtimes K$, where $\mathbb{H}^n$ is the Heisenberg group and $K = U(n)$, the group of $n \times n$ complex unitary matrices. One of the main goal of this note is to study $L^2 - L^p, 2 \leq p \leq \infty$, estimates of pseudo-differential operator on the Heisenberg motion group G. We also provide sufficient and necessary conditions on the symbol τ such that the corresponding pseudo-differential operator T_τ on G is a Hilbert–Schmidt operator. We give a characterization of the trace class pseudo-differential operators on G and provide a trace formula for these trace class operators. The main results of this paper are as follows:

Theorem 1.1. *Let τ be a operator-valued symbol on $G \times G'$ such that*

$$\sum_{\sigma \in \hat{K}} d_\sigma \int_{\mathbb{R}\setminus\{0\}} \int_G \|\tau(z,t,k,\lambda,\sigma)\|_{S_{p'}}^p \; |\lambda|^n \, d\lambda \, dzdtdk < \infty$$

for $2 \leq p < \infty$ with p' being the Lebesgue conjugate of p. Then the pseudo-differential operator $T_\tau : L^2(G) \to L^p(G)$ is a bounded operator. Moreover,

$$\|T_\tau\|_{B(L^2(G),L^p(G))} \leq \left[\sum_{\sigma \in \hat{K}} d_\sigma \int_{\mathbb{R}\setminus\{0\}} \left\{ \int_G \|\tau(z,t,k,\lambda,\sigma)\|_{S_{p'}}^2 \; dzdtdk \right\}^{\frac{2}{p}} |\lambda|^n d\lambda \right]^{1/2}.$$

The proof of this result can be found in Section 3. We extend the above result for $p = \infty$ as follows.

Theorem 1.2. *Let τ be a operator-valued symbol on $G \times G'$ such that*

$$\sum_{\sigma \in \hat{K}} d_\sigma \int_{\mathbb{R}\setminus\{0\}} \| \|\tau(\cdot,\cdot,\cdot,\lambda,\sigma)\|_{S_1} \|_{L^\infty(G)}^2 \; |\lambda|^n \, d\lambda < \infty.$$

Then the pseudo-differential operator $T_\tau : L^2(G) \to L^\infty(G)$ is a bounded operator. Further, we have

$$\|T_\tau\|_{B(L^2(G),L^\infty(G))} \leq \left[\sum_{\sigma \in \hat{K}} d_\sigma \int_{\mathbb{R}\setminus\{0\}} \| \|\tau(\cdot,\cdot,\cdot,\lambda,\sigma)\|_{S_1} \|_{L^\infty(G)}^2 |\lambda|^n d\lambda \right]^{1/2}.$$

See Section 3 for the proof of the above theorem. Next, by deriving an equality (Proposition 2.4) for the trace class λ-Weyl transform with symbol in $L^2(\mathbb{C}^n \times K)$, we prove the following result.

Theorem 1.3. *Let τ be a symbol such that it satisfies the hypotheses of Theorem 3.4. Then the corresponding pseudo-differential operator T_τ is a Hilbert–Schmidt operator if and only if*

$$\tau(z,t,k,\lambda,\sigma)=\rho_\sigma^\lambda(z,t,k)W_\sigma^\lambda\left(\alpha(z,t,k)^{-\lambda}\right),$$

where $(z,t,k,\lambda,\sigma)\in G\times G'$ and $\alpha: G\to L^2(G)$ is a weakly continuous mapping such that it satisfies

(i) $\displaystyle\int_G \|\alpha(z,t,k)(\cdot,\cdot,\cdot,\cdot)\|_{L^2(G)}\,dzdkdt<\infty,$

(ii) $\displaystyle\sup_{(z,t,k,t)\in G\times G'\times\mathbb{R}^*}\|\mathcal{F}\alpha(z,t,k)(\cdot,\lambda,\cdot)\|_{L^2(G^\times)}<\infty,$

(iii) $\displaystyle\int_{\mathbb{R}\setminus\{0\}}\|\mathcal{F}\alpha(z,t,k)(\cdot,\lambda,\cdot)\|_{L^2(G^\times)}|\lambda|^n d\lambda<\infty,\quad \forall(z,t,k)\in G.$

The proof of this theorem can be found in Section 4.

Apart from the introduction, this paper is organized as follows: In Section 2, we recall basic harmonic analysis on the Heisenberg motion group G and define the pseudo-differential operators on G. We note here that our quantization can be seen as a particular case of the quantization defined by Ruzhansky and Mantoiu in [24]. We also derive some properties of the λ-Weyl transform on G. In Section 3, we investigate L^2-L^p estimates of pseudo-differential operators on G for $2\le p\le\infty$. We also prove that if two symbols under some suitable conditions give rise to same pseudo-differential operator then the symbols must be same. In Section 4, we obtain a necessary and sufficient condition on the symbol τ such that the corresponding pseudo-differential operator T_τ on G is a Hilbert–Schmidt operator. We present a characterization for the trace class pseudo-differential operators on the Heisenberg motion group G and find a trace formula for these trace class operators. Finally, we end this paper by providing a concluding remark and future work aspects.

2. Preliminaries

2.1. **Harmonic analysis on the Heisenberg motion group.** In this subsection, we recall some basics of harmonic analysis on the Heisenberg motion group to make the paper self-contained. A complete account of representation theory of the Heisenberg motion group can be found in [31, 28, 2, 18, 27]. However, we mainly adopt the notation and terminology given in [18].

We first recall Heisenberg group $\mathbb{H}^n$. The Heisenberg group $\mathbb{H}^n=\mathbb{C}^n\times\mathbb{R}$ is a step two nilpotent Lie group equipped with the group law

$$(z,t)\cdot(w,s)=\left(z+w,t+s+\frac{1}{2}\operatorname{Im}(z\cdot\bar{w})\right),\quad (z,t),(w,s)\in\mathbb{H}^n,$$

By Stone–von Neumann theorem, the set of infinite dimensional irreducible unitary representations of $\mathbb{H}^n$ can be parameterized by $\mathbb{R}^*=\mathbb{R}\setminus\{0\}$. For each $\lambda\in\mathbb{R}^*$, the Schrödinger representation π_λ of $\mathbb{H}^n$ is defined by

$$\pi_\lambda(z,t)\varphi(\xi)=e^{i\lambda t}e^{i\lambda\left(x\cdot\xi+\frac{1}{2}x\cdot y\right)}\varphi(\xi+y),\quad z=x+iy,$$

where $\varphi\in L^2(\mathbb{R}^n)$. For a more detailed study on the Heisenberg group, we refer to [32, 31, 15, 16]. The group $U(n)$ of $n\times n$ complex unitary matrices acts on $\mathbb{H}^n$ by the automorphisms

$$\sigma(z,t)=(\sigma z,t),\quad \sigma\in U(n).$$

The Heisenberg motion group G is then the semi direct product of $\mathbb{H}^n$ with the unitary group $K=U(n)$ by the group law

$$(z,t,k_1)\cdot(w,s,k_2)=\left(z+k_1w,t+s-\frac{1}{2}\operatorname{Im}(k_1w\cdot\bar{z}),k_1k_2\right).$$

The functions on $\mathbb{H}^n$ can be viewed as right K-invariant functions on the Heisenberg motion group G. Since a right K-invariant function on G can be thought as a function on $\mathbb{H}^n$, the Haar measure on G is given by $dg = dzdtdk$, where $dzdt$ and dk are the normalized Haar measure on $\mathbb{H}^n$ and K respectively. For $k \in K$, we have another set of representations of the Heisenberg group $\mathbb{H}^n$ by $\pi_{\lambda,k}(z,t) = \pi_\lambda(kz,t)$. Since $\pi_{\lambda,k}$ agrees with π_λ on the center of $\mathbb{H}^n$, it follows from Stone–Von Neumann theorem for the Schrödinger representation that $\pi_{\lambda,k}$ is equivalent to π_λ. This implies that there exists an intertwining operator $\mu_\lambda(k)$ satisfying

$$\pi_\lambda(kz,t) = \mu_\lambda(k)\pi_\lambda(z,t)\mu_\lambda(k)^*.$$

The operator valued function μ_λ can be chosen so that it becomes a unitary representation of K on $L^2(\mathbb{R}^n)$ and is known as metaplectic representation [1]. In general, the metaplectic representation is a projective representation of the symplectic group but if one restricts the metaplectic representation to $U(n)$, then the constants can be redefined so that it becomes a unitary representation of $U(n)$. Let $(\sigma, \mathcal{H}_\sigma)$ be an irreducible unitary representation of K and $\mathcal{H}_\sigma = \operatorname{span}\{e_j^\sigma : 1 \le j \le d_\sigma\}$. For $k \in K$, the matrix coefficients of the representation $\sigma \in \hat{K}$ are given by

$$\varphi_{ij}^\sigma(k) = \left\langle \sigma(k)e_j^\sigma, e_i^\sigma \right\rangle, \ i,j = 1,\dots,d_\sigma.$$

Consider the functions

$$\phi_\alpha^\lambda(x) = |\lambda|^{\frac{n}{4}}\phi_\alpha(\sqrt{|\lambda|}x), \quad \alpha \in \mathbb{N}^n,$$

where ϕ_α's are the Hermite functions on $\mathbb{R}^n$. Then for each $\lambda \in \mathbb{R}^*$, the set $\left\{\phi_\alpha^\lambda : \alpha \in \mathbb{N}^n\right\}$ forms an orthonormal basis for $L^2(\mathbb{R}^n)$. Let $P_m^\lambda = \operatorname{span}\left\{\phi_\alpha^\lambda : |\alpha| = m\right\}$. Then μ_λ becomes an irreducible unitary representation of K on P_m^λ. The action of μ_λ can be realized on P_m^λ by the following

$$\mu_\lambda(k)\phi_\gamma^\lambda = \sum_{|\alpha|=|\gamma|} \xi_{\alpha\gamma}^\lambda(k)\phi_\alpha^\lambda,$$

where $\xi_{\alpha\gamma}^\lambda$'s are the matrix coefficients of $\mu_\lambda(k)$. Define a bilinear form $\phi_\alpha^\lambda \otimes e_j^\sigma$ on $L^2(\mathbb{R}^n) \times \mathcal{H}_\sigma$ by $\phi_\alpha^\lambda \otimes e_j^\sigma = \phi_\alpha^\lambda e_j^\sigma$. Then the set $\left\{\phi_\alpha^\lambda \otimes e_j^\sigma : \alpha \in \mathbb{N}^n, 1 \le j \le d_\sigma\right\}$ forms an orthonormal basis for $L^2(\mathbb{R}^n) \otimes \mathcal{H}_\sigma$. Let us write $\mathcal{H}_\sigma^2$ for the space $L^2(\mathbb{R}^n) \otimes \mathcal{H}_\sigma$.

For $\lambda \neq 0$, a representation ρ_σ^λ of G on the space $\mathcal{H}_\sigma^2$ defined by the following way

$$\rho_\sigma^\lambda(z,t,k) = \pi_\lambda(z,t)\mu_\lambda(k) \otimes \sigma(k), \quad (z,t,k) \in G,$$

Then ρ_σ^λ are all possible irreducible unitary representations of G those participate in the Plancherel formula [28]. We denote the partial dual of the group G as

$$G' \cong \mathbb{R}^* \times \hat{K}.$$

The group Fourier transform of $f \in L^1(G)$ is defined by

$$\hat{f}(\lambda,\sigma) = \int_K \int_{\mathbb{R}} \int_{\mathbb{C}^n} f(z,t,k)\rho_\sigma^\lambda(z,t,k)\, dzdtdk,$$

where $(\lambda,\sigma) \in \mathbb{R}^* \times \hat{K}$. Then $\hat{f}(\lambda,\sigma)$ is a bounded linear operator on $\mathcal{H}_\sigma^2$. Let

$$f^\lambda(z,k) = \int_{\mathbb{R}} f(z,t,k)e^{i\lambda t}\, dt \tag{1}$$

be the inverse Fourier transform of the function f in the t variable. Then the group Fourier transform of f can be expressed as

$$\hat{f}(\lambda,\sigma) = \int_K \int_{\mathbb{C}^n} f^\lambda(z,k)\rho_\sigma^\lambda(z,k)\, dzdk, \tag{2}$$

where $\rho_\sigma^\lambda(z,k) = \rho_\sigma^\lambda(z,0,k)$. Moreover, $\hat{f}(\lambda,\sigma)$ is a Hilbert–Schmidt operator on $\mathcal{H}_\sigma^2$ and its satisfies the following versions of Plancherel formula

$$\int_K \int_{\mathbb{H}^n} |f(z,t,k)|^2 \, dzdtdk = (2\pi)^{-n} \sum_{\sigma\in\hat{K}} d_\sigma \int_{\mathbb{R}\setminus\{0\}} \|\hat{f}(\lambda,\sigma)\|_{S_2}^2 |\lambda|^n \, d\lambda$$

for $f \in L^2(G)$, where S_2 represent the space of all Hilbert-Schmidt operators on $\mathcal{H}_\sigma^2$ (see Section 2.2 below for more details on S_2).

For $f \in L^1 \cap L^2(G)$, the following Fourier inversion formula on G holds

$$f(z,t,k) = (2\pi)^{-n} \sum_{\sigma\in\hat{K}} d_\sigma \int_{\mathbb{R}\setminus\{0\}} \operatorname{tr}\left((\rho_\sigma^\lambda)^*(z,t,k)\hat{f}(\lambda,\sigma)\right) |\lambda|^n \, d\lambda, \quad (z,t,k) \in G.$$

Let $B\left(\mathcal{H}_\sigma^2\right)$ denote the C^*-algebra of all bounded linear operators on $\mathcal{H}_\sigma^2$. The operator-valued symbol or simply a symbol τ is a mapping $\tau : G \times G' \to B(\mathcal{H}_\sigma^2)$. Then, we define the pseudo-differential operator $T_\tau : L^2(G) \to L^2(G)$ corresponding to the symbol τ by

$$(T_\tau f)(z,t,k) = (2\pi)^{-n} \sum_{\sigma\in\hat{K}} d_\sigma \int_{\mathbb{R}\setminus\{0\}} \operatorname{tr}\left((\rho_\sigma^\lambda)^*(z,t,k)\tau(z,t,k,\lambda,\sigma)\hat{f}(\lambda,\sigma)\right) |\lambda|^n d\lambda \tag{3}$$

for all $f \in \mathcal{S}(G)$ and $(z,t,k) \in G$.

2.2. **Schatten–von Neumann classes.** If $\mathcal{X}$ is a complex Hilbert space, a linear compact operator $T : \mathcal{X} \to \mathcal{X}$ belongs to the r-Schatten–von Neumann class $S_r(\mathcal{X})$ if

$$\sum_{n=1}^{\infty} (s_n(T))^r < \infty,$$

where $s_n(T)$ denote the singular values of T, i.e. the eigenvalues of $|T| = \sqrt{T^*T}$ with multiplicities counted. For $1 \le r < \infty$, the class $S_r(\mathcal{X})$ is a Banach space endowed with the norm

$$\|T\|_{S_r} = \left(\sum_{n=1}^{\infty} (s_n(T))^r\right)^{\frac{1}{r}}.$$

For $0 < r < 1$, the $\|\cdot\|_{S_r}$ as above only defines a quasi-norm with respect to which $S_r(\mathcal{X})$ is complete. An operator belonging to the class $S_1(\mathcal{X})$ (and $S_2(\mathcal{X})$) is known as *Trace class* operator (and *Hilbert–Schmidt* operator).

Another equivalent definition of Hilbert-Schmidt operators also given in terms of orthonormal basis by the following. Let $\mathcal{H}$ be a complex and separable Hilbert space with respect to the inner product $\langle\cdot,\cdot\rangle_{\mathcal{H}}$. Also let $\{\phi_k, k = 1,2,..\}$ be an orthonormal basis for the Hilbert space $\mathcal{H}$. Then an operator $A \in \mathcal{B}(\mathcal{H})$ is a Hilbert–Schmidt operator if for any orthonormal basis $\{\phi_k\}_{k=1}^\infty$ of $\mathcal{H}$ we have $\sum_k \|T\phi_k\|_{\mathcal{H}} < \infty$. In this case, the Hilbert–Schmidt norm on S_2 is defined by

$$\|A\|_{S_2} = \left(\sum_{k=1}^{\infty} \langle A\psi_k, A\psi_k\rangle_{\mathcal{H}}\right)^{\frac{1}{2}}.$$

For $1 \le p \le q \le \infty$, from the definition of the Schatten-von Neumann classes, it follows that $S_p \subseteq S_q$ and consequently, for all $A \in S_p$, we have

$$\|A\|_{S_q} \le \|A\|_{S_p}.$$

2.3. **λ-Weyl transform.** In this subsection, we recall some basic definitions and important properties of λ-Weyl transform associated to a symbol on $L^2(G^\times)$, where $G^\times = \mathbb{C}^n \times K$. The authors in [18] studied unboundedness properties of λ-Weyl transforms on G. For a detailed study on λ-Weyl transforms on the Heisenberg motion group, we refer to [2, 17, 28]. We also refer to the book of Wong [33] for Weyl transform on $\mathbb{R}^n$.

For $(\lambda, \sigma) \in G' \cong \mathbb{R}^* \times \hat{K}$, the λ-Weyl transform W_σ^λ on $L^1(G^\times)$ is defined by (see [2])

$$W_\sigma^\lambda(F) = \int_K \int_{\mathbb{C}^n} F(z,k)\rho_\sigma^\lambda(z,k)\, dzdk,$$

where $\rho_\sigma^\lambda(z,k) = \rho_\sigma^\lambda(z,0,k)$. Then by the definition of group Fourier transform (2), we have

$$\hat{f}(\lambda,\sigma) = W_\sigma^\lambda(f^\lambda), \tag{4}$$

where f^λ is defined in (1). Since $\hat{f}(\lambda,\sigma)$ is a bounded linear operator on $\mathcal{H}_\sigma^2$, $W_\sigma^\lambda(F)$ is a bounded operator if $F \in L^1(G^\times)$. On the other hand, if $F \in L^2(G^\times)$, then $W_\sigma^\lambda(F)$ becomes a Hilbert–Schmidt operator satisfying the following Plancherel formula (see [17])

$$\int_K \int_{\mathbb{C}^n} |F(z,k)|^2 dzdk = (2\pi)^{-n}|\lambda|^n \sum_{\sigma\in\hat{K}} d_\sigma \left\| W_\sigma^\lambda(F)\right\|_{S_2}^2. \tag{5}$$

For $F_1, F_2 \in L^1 \cap L^2(G^\times)$, the λ-twisted convolutions of F_1 and F_2 is defined by

$$F_1 \times_\lambda F_2(g) = \int_{G^\times} F_1\left(gg'^{-1}\right) F_2\left(g'\right) e^{-\frac{i}{2}\lambda\, \mathrm{Im}(kw\cdot\bar{z})} dg',$$

where $g = (z,k)$ and $g' = (w,s) \in G^\times$. For $\lambda = 1$, the λ-twisted convolutions called twisted convolutions and denote it by $F_1 \times F_2$. By Proposition 3.1 of [2], we have the following properties related to the λ-Weyl transform W_σ^λ.

Theorem 2.1. *Let $F_1, F_2 \in L^1 \cap L^2(G^\times)$. Then*
(a) $W_\sigma^\lambda(F_1)^* = W_\sigma^\lambda(F_1^*)$, where $F_1^*(z,k) = \overline{F_1((z,k)^{-1})}$,
(b) $W_\sigma^\lambda(F_1 \times_\lambda F_2) = W_\sigma^\lambda(F_1) W_\sigma^\lambda(F_2)$.

Using the Plancherel formula (5) and part (b) of Theorem 2.1 with the appropriate use of Cauchy-Schwarz inequality, the space $\left(L^2(G^\times), \times_\lambda\right)$ is a Banach $*$-algebra. Further, using the relation (4) and Plancherel formula (5), the λ-Weyl transform W_σ^λ is a isomorphism from $L^2(G^\times)$ onto the space of all Hilbert–Schmidt operators on $\mathcal{H}_\sigma^2$ denoted by $S_2\left(\mathcal{H}_\sigma^2\right)$. Therefore, for any $A \in S_2\left(\mathcal{H}_\sigma^2\right)$, there exists a unique $F \in L^2(G^\times)$ such that $A = W_\sigma^\lambda(F)$.

Let us define a set

$$Z_\lambda := \{F \in L^2(G^\times) : \exists\ F_1, F_2 \in L^2(G^\times) \text{ such that } F = F_1 \times_\lambda F_2\ \}.$$

Now we present the following theorems on the characterization of trace class λ-Weyl transform on G.

Theorem 2.2. *Let $W_\sigma^\lambda(F)$ be the λ-Weyl transform associated with $F \in L^2(G^\times)$. Then $W_\sigma^\lambda(F)$ is a trace class operator if and only if $F \in Z_\lambda$.*

Theorem 2.3. *Z_λ is a dense subspace of $L^2(G^\times)$.*

The proof of Theorem 2.2 and Theorem 2.3 follows similar line given in section 4 of [8]. The following estimate will be used in Section 3 to prove one of our main results.

Proposition 2.4. *Let $F = F_1 \times_\lambda F_2$ for some $F_1, F_2 \in L^2(G^\times)$. Then*

$$\sum_{\sigma\in\hat{K}} d_\sigma \operatorname{tr}\left(W_\sigma^\lambda(F)\right) = (2\pi)^n|\lambda|^{-n} \int_{G^\times} F_2(z,k)F_1((-k^{-1}z, k^{-1}))\, dzdk.$$

Proof. Since $F = F_1 \times_\lambda F_2$, from part (b) of Theorem 2.1, we get

$$W_\sigma^\lambda(F) = W_\sigma^\lambda(F_1)W_\sigma^\lambda(F_2) = W_\sigma^\lambda(F_1 \times_\lambda F_2).$$

Since $W_\sigma^\lambda(F)$ is a trace class operator and hence a Hilbert-Schmidt operator on $\mathcal{H}_\sigma^2$. Let $\{\phi_k : k \in \mathbb{N}\}$ is an orthonormal basis for $\mathcal{H}_\sigma^2$. Then using part (a) of Theorem 2.1, we obtain

$$\begin{aligned}
\sum_{\sigma\in\hat{K}} d_\sigma \operatorname{tr}\left(W_\sigma^\lambda(F)\right) &= \sum_{\sigma\in\hat{K}} d_\sigma \sum_{k\in\mathbb{N}} \left\langle W_\sigma^\lambda(F)\phi_k, \phi_k\right\rangle \\
&= \sum_{\sigma\in\hat{K}} d_\sigma \sum_{k\in\mathbb{N}} \left\langle W_\sigma^\lambda(F_1)W_\sigma^\lambda(F_2)\phi_k, \phi_k\right\rangle \\
&= \sum_{\sigma\in\hat{K}} d_\sigma \sum_{k\in\mathbb{N}} \left\langle W_\sigma^\lambda(F_2)\phi_k, W_\sigma^\lambda(F_1)^*\phi_k\right\rangle \\
&= \sum_{\sigma\in\hat{K}} d_\sigma \sum_{k\in\mathbb{N}} \left\langle W_\sigma^\lambda(F_2)\phi_k, W_\sigma^\lambda(F_1^*)\phi_k\right\rangle \\
&= \sum_{\sigma\in\hat{K}} d_\sigma \left\langle W_\sigma^\lambda(F_2), W_\sigma^\lambda(F_1^*)\right\rangle_{S_2} \\
&= (2\pi)^n|\lambda|^{-n} \left\langle F_2, F_1^*\right\rangle_{L^2(G^\times)}.
\end{aligned}$$

Then

$$\begin{aligned}
\sum_{\sigma\in\hat{K}} d_\sigma \operatorname{tr}\left(W_\sigma^\lambda(F)\right) &= (2\pi)^n|\lambda|^{-n} \int_{G^\times} F_2(z,k)F_1((z,k)^{-1})dzdk \\
&= (2\pi)^n|\lambda|^{-n} \int_{G^\times} F_2(z,k)F_1((-k^{-1}z, k^{-1}))dzdk.
\end{aligned}$$

$\square$

3. Boundedness

This section is devoted to study the $L^2 - L^p$ boundedness of pseudo-differential operators on the Heisenberg motion group G. We also prove that if two symbols with some additional conditions give arise to same pseudo-differential operator then symbol must be same.

The following theorem is about the L^2-boundedness of pseudo-differential operators on G. In fact, a more general result in terms of Schatten–von Neumann class follows from Corollary 3.18 of [24]. Indeed, we have the following property.

Theorem 3.1. *Let $1 \leq p \leq 2$ with Lebesgue conjugate p' and let $\tau : G \times G' \to S_p$ be a operator-valued symbol such that*

$$\sum_{\sigma\in\hat{K}} d_\sigma \int_{\mathbb{R}\setminus\{0\}} \int_G \|\tau(z,t,k,\lambda,\sigma)\|_{S_p}^p \; |\lambda|^n \, d\lambda \, dzdtdk < \infty.$$

Then the pseudo-differential operator $T_\tau : L^2(G) \to L^2(G)$ is in the p'-Schatten class $S_{p'}(G)$. In particular, the pseudo-differential operator $T_\tau : L^2(G) \to L^2(G)$ is a bounded operator.

In the next, we investigate a more general result, $L^2 - L^p$-estimates for pseudo-differential operators on the Heisenberg motion group G for $2 \leq p < \infty$.

Theorem 3.2. *Let τ be a operator-valued symbol on $G \times G'$ such that*

$$\sum_{\sigma\in\hat{K}} d_\sigma \int_{\mathbb{R}\setminus\{0\}} \int_G \|\tau(z,t,k,\lambda,\sigma)\|_{S_{p'}}^p \; |\lambda|^n \, d\lambda \, dzdtdk < \infty$$

for $2 \leq p < \infty$ and p' be the Lebesgue conjugate of p. Then the pseudo-differential operator $T_\tau : L^2(G) \to L^p(G)$ is a bounded operator. Moreover,

$$\|T_\tau\|_{B(L^2(G),L^p(G))} \leq \left[\sum_{\sigma\in\hat{K}} d_\sigma \int_{\mathbb{R}\setminus\{0\}} \left\{\int_G \|\tau(z,t,k,\lambda,\sigma)\|^2_{S_{p'}}\ dzdtdk\right\}^{\frac{2}{p}} |\lambda|^n d\lambda\right]^{1/2}.$$

Proof. Let $f \in L^2(G)$. Then by Minkowski's integral inequality, Hölder's inequality and Plancherel theorem, we have

$$\begin{aligned}
\|T_\tau f\|_{L^p(G)} &= \left\{\int_G |(T_\tau f)(z,t,k)|^p\, dzdtdk\right\}^{1/p} \\
&= (2\pi)^{-n}\left\{\int_G \left|\sum_{\sigma\in\hat{K}} d_\sigma \int_{\mathbb{R}\setminus\{0\}} \operatorname{tr}\left((\rho^\lambda_\sigma)^*(z,t,k)\tau(z,t,k,\lambda,\sigma)\hat{f}(\lambda,\sigma)\right)|\lambda|^n d\lambda\right|^p dzdtdk\right\}^{1/p} \\
&\leq (2\pi)^{-n}\sum_{\sigma\in\hat{K}} d_\sigma \int_{\mathbb{R}\setminus\{0\}} \left\{\int_G \left|\operatorname{tr}\left((\rho^\lambda_\sigma)^*(z,t,k)\tau(z,t,k,\lambda,\sigma)\hat{f}(\lambda,\sigma)\right)\right|^p dzdtdk\right\}^{1/p} |\lambda|^n d\lambda \\
&\leq (2\pi)^{-n}\sum_{\sigma\in\hat{K}} d_\sigma \int_{\mathbb{R}\setminus\{0\}} \left\{\int_G \|\tau(z,t,k,\lambda,\sigma)\|^p_{S_{p'}}\|\hat{f}(\lambda,\sigma)\|^p_{S_p}\ dzdtdk\right\}^{1/p} |\lambda|^n d\lambda \\
&= (2\pi)^{-n}\sum_{\sigma\in\hat{K}} d_\sigma \int_{\mathbb{R}\setminus\{0\}} \|\hat{f}(\lambda,\sigma)\|_{S_p}\left\{\int_G \|\tau(z,t,k,\lambda,\sigma)\|^p_{S_{p'}}\ dzdtdk\right\}^{1/p} |\lambda|^n d\lambda \\
&= (2\pi)^{-n}\sum_{\sigma\in\hat{K}} d_\sigma \int_{\mathbb{R}\setminus\{0\}} \|\hat{f}(\lambda,\sigma)\|_{S_2}\left\{\int_G \|\tau(z,t,k,\lambda,\sigma)\|^p_{S_{p'}}\ dzdtdk\right\}^{1/p} |\lambda|^n d\lambda \\
&\leq (2\pi)^{-n}\left\{\sum_{\sigma\in\hat{K}} d_\sigma \int_{\mathbb{R}\setminus\{0\}} \|\hat{f}(\lambda,\sigma)\|^2_{S_2}|\lambda|^n d\lambda\right\}^{\frac{1}{2}} \\
&\quad\times\left[\sum_{\sigma\in\hat{K}} d_\sigma \int_{\mathbb{R}\setminus\{0\}} \left\{\int_G \|\tau(z,t,k,\lambda,\sigma)\|^2_{S_{p'}}\ dzdtdk\right\}^{\frac{2}{p}} |\lambda|^n d\lambda\right]^{1/2} \\
&= \|f\|_{L^2(G)}\left[\sum_{\sigma\in\hat{K}} d_\sigma \int_{\mathbb{R}\setminus\{0\}} \left\{\int_G \|\tau(z,t,k,\lambda,\sigma)\|^2_{S_{p'}}\ dzdtdk\right\}^{\frac{2}{p}} |\lambda|^n d\lambda\right]^{1/2}.
\end{aligned}$$

This shows that $T_\tau : L^2(G) \to L^p(G)$ is a bounded operator. Moreover, we get

$$\|T_\tau\|_{B(L^2(G),L^p(G))} \leq \left[\sum_{\sigma\in\hat{K}} d_\sigma \int_{\mathbb{R}\setminus\{0\}} \left\{\int_G \|\tau(z,t,k,\lambda,\sigma)\|^2_{S_{p'}}\ dzdtdk\right\}^{\frac{2}{p}} |\lambda|^n d\lambda\right]^{1/2}.$$

□

In the next, we study the remaining case when $p = \infty$.

Theorem 3.3. *Let τ be a operator-valued symbol on $G \times G'$ such that*

$$\sum_{\sigma\in\hat{K}} d_\sigma \int_{\mathbb{R}\setminus\{0\}} \|\|\tau(\cdot,\cdot,\cdot,\lambda,\sigma)\|_{S_1}\|^2_{L^\infty(G)}\ |\lambda|^n\, d\lambda < \infty.$$

Then the pseudo-differential operator $T_\tau : L^2(G) \to L^\infty(G)$ is a bounded operator. Further, we have

$$\|T_\tau\|_{B(L^2(G),L^\infty(G))} \leq \left[\sum_{\sigma\in\hat{K}} d_\sigma \int_{\mathbb{R}\setminus\{0\}} \big\|\|\tau(\cdot,\cdot,\cdot,\lambda,\sigma)\|_{S_1}\big\|^2_{L^\infty(G)} |\lambda|^n d\lambda\right]^{1/2}.$$

Proof. Let $f \in L^2(G)$. Then, using Minkowski's integral inequality, Hölder's inequality and Plancherel theorem, we have

$$\begin{aligned}
\|T_\tau f\|_{L^\infty(G)} &= (2\pi)^{-n} \left\|\sum_{\sigma\in\hat{K}} d_\sigma \int_{\mathbb{R}\setminus\{0\}} \mathrm{tr}\left((\rho_\sigma^\lambda)^*(\cdot,\cdot,\cdot)\tau(\cdot,\cdot,\cdot,\lambda,\sigma)\hat{f}(\lambda,\sigma)\right)|\lambda|^n d\lambda\right\|_{L^\infty(G)} \\
&\leq (2\pi)^{-n} \sum_{\sigma\in\hat{K}} d_\sigma \int_{\mathbb{R}\setminus\{0\}} \left\|\mathrm{tr}\left((\rho_\sigma^\lambda)^*(\cdot,\cdot,\cdot)\tau(\cdot,\cdot,\cdot,\lambda,\sigma)\hat{f}(\lambda,\sigma)\right)\right\|_{L^\infty(G)} |\lambda|^n d\lambda \\
&\leq (2\pi)^{-n} \sum_{\sigma\in\hat{K}} d_\sigma \int_{\mathbb{R}\setminus\{0\}} \big\|\|\tau(\cdot,\cdot,\cdot,\lambda,\sigma)\|_{S_1}\|\hat{f}(\lambda,\sigma)\|_{S_\infty}\big\|_{L^\infty(G)} |\lambda|^n d\lambda \\
&= (2\pi)^{-n} \sum_{\sigma\in\hat{K}} d_\sigma \int_{\mathbb{R}\setminus\{0\}} \|\hat{f}(\lambda,\sigma)\|_{S_\infty}\big\|\|\tau(\cdot,\cdot,\cdot,\lambda,\sigma)\|_{S_1}\big\|_{L^\infty(G)} |\lambda|^n d\lambda \\
&\leq (2\pi)^{-n} \sum_{\sigma\in\hat{K}} d_\sigma \int_{\mathbb{R}\setminus\{0\}} \|\hat{f}(\lambda,\sigma)\|_{S_2}\big\|\|\tau(\cdot,\cdot,\cdot,\lambda,\sigma)\|_{S_1}\big\|_{L^\infty(G)} |\lambda|^n d\lambda \\
&\leq (2\pi)^{-n} \left\{\sum_{\sigma\in\hat{K}} d_\sigma \int_{\mathbb{R}\setminus\{0\}} \|\hat{f}(\lambda,\sigma)\|^2_{S_2} |\lambda|^n d\lambda\right\}^{\frac{1}{2}} \\
&\qquad \times \left[\sum_{\sigma\in\hat{K}} d_\sigma \int_{\mathbb{R}\setminus\{0\}} \big\|\|\tau(\cdot,\cdot,\cdot,\lambda,\sigma)\|_{S_1}\big\|^2_{L^\infty(G)} |\lambda|^n d\lambda\right]^{1/2} \\
&\leq \|f\|_{L^2(G)} \left[\sum_{\sigma\in\hat{K}} d_\sigma \int_{\mathbb{R}\setminus\{0\}} \big\|\|\tau(\cdot,\cdot,\cdot,\lambda,\sigma)\|_{S_1}\big\|^2_{L^\infty(G)} |\lambda|^n d\lambda\right]^{1/2}.
\end{aligned}$$

This shows that $T_\tau : L^2(G) \to L^\infty(G)$ is a bounded operator. Moreover, we have

$$\|T_\tau\|_{B(L^2(G),L^\infty(G))} \leq \left[\sum_{\sigma\in\hat{K}} d_\sigma \int_{\mathbb{R}\setminus\{0\}} \big\|\|\tau(\cdot,\cdot,\cdot,\lambda,\sigma)\|_{S_1}\big\|^2_{L^\infty(G)} |\lambda|^n d\lambda\right]^{1/2}.$$

□

In the next theorem we show that if two symbols with some conditions give arise to same pseudo-differential operator then symbols must be same.

Theorem 3.4. *Let $\tau : G \times G' \to S_2$ be a symbol such that it satisfies the following properties:*

(i) $\displaystyle\sum_{\sigma\in\hat{K}} d_\sigma \int_{\mathbb{R}\setminus\{0\}} \int_G \|\tau(z,t,k,\lambda,\sigma)\|^2_{S_2}\ |\lambda|^n\ d\lambda dz dt dk < \infty.$

(ii) $\displaystyle\sum_{\sigma\in\hat{K}} d_\sigma \int_{\mathbb{R}\setminus\{0\}} \|\tau(z,t,k,\lambda,\sigma)\|_{S_2}\ |\lambda|^n d\lambda < \infty, \quad (z,t,k) \in G$

(iii) $\displaystyle\sup_{(z,t,k,\lambda,\sigma)\in G\times G'} \|\tau(z,t,k,\lambda,\sigma)\|_{S_2} < \infty,$

(iv) the mapping $G \times G' \ni (z,t,k,\lambda,\sigma) \mapsto (\rho_\sigma^\lambda)^*(z,t,k)\tau(z,t,k,\lambda,\sigma) \in S_2$ is weakly continuous.

Then, $T_\tau f = 0$ for all f in $L^2(G)$ if and only if $\tau(z,t,k,\lambda,\sigma) = 0$ for almost all $(z,t,k,\lambda,\sigma) \in G \times G'$.

Proof. First assume that $T_\tau f = 0$ for all f in $L^2(G)$. For $(z,t,k) \in G$, let us consider the function $f_{(z,t,k)} \in L^2(G)$ by

$$\widehat{f_{(z,t,k)}}(\lambda,\sigma) = \tau(z,t,k,\lambda,\sigma)^* \rho_\sigma^\lambda(z,t,k)$$

for all $(\lambda,\sigma) \in G'$. Therefore, for all $(z',t',k') \in G$, we have

$$\begin{aligned}
&\left(T_\tau f_{(z,t,k)}\right)(z',t',k') \\
&= (2\pi)^{-n} \sum_{\sigma \in \hat{K}} d_\sigma \int_{\mathbb{R}\setminus\{0\}} \operatorname{tr}\left((\rho_\sigma^\lambda)^*(z',t',k')\tau(z',t',k',\lambda,\sigma)\widehat{f_{(z,t,k)}}(\lambda,\sigma)\right) |\lambda|^n d\lambda \\
&= (2\pi)^{-n} \sum_{\sigma \in \hat{K}} d_\sigma \int_{\mathbb{R}\setminus\{0\}} \operatorname{tr}\Big[(\rho_\sigma^\lambda)^*(z',t',k')\tau(z',t',k',\lambda,\sigma) \\
&\qquad \times \tau(z,t,k,\lambda,\sigma)^* \rho_\sigma^\lambda(z,t,k)\Big] |\lambda|^n d\lambda.
\end{aligned}$$

Take $(z_0,t_0,k_0) \in G$. By the weakly continuous mapping property (iv), we have

$$\begin{aligned}
&\operatorname{tr}\left((\rho_\sigma^\lambda)^*(z',t',k')\tau(z',t',k',\lambda,\sigma)\tau(z,t,k,\lambda,\sigma)^*\rho_\sigma^\lambda(z,t,k)\right) \\
&\to \operatorname{tr}\left((\rho_\sigma^\lambda)^*(z_0,t_0,k_0)\tau(z_0,t_0,k_0,\lambda,\sigma)\tau(z,t,k,\lambda,\sigma)^*\rho_\sigma^\lambda(z,t,k)\right)
\end{aligned}$$

as $(z',t',k') \to (z_0,t_0,k_0)$ in G. Now using the property (iii), there exists a constant C such that for all $(z',t',k',\lambda,\sigma) \in G \times G'$, we have

$$\begin{aligned}
&\left|\operatorname{tr}\left((\rho_\sigma^\lambda)^*(z',t',k')\tau(z',t',k',\lambda,\sigma)\tau(z,t,k,\lambda,\sigma)^*\rho_\sigma^\lambda(z,t,k)\right)\right| \\
&\leq C\|\tau(z,t,k,\lambda,\sigma)\|_{S_2}.
\end{aligned}$$

Since

$$\sum_{\sigma \in \hat{K}} d_\sigma \int_{\mathbb{R}\setminus\{0\}} \|\tau(z,t,k,\lambda,\sigma)\|_{S_2} |\lambda|^n d\lambda < \infty$$

for all $(z,t,k) \in G$, by Lebesgue's dominated convergence theorem, we have

$$\begin{aligned}
&\sum_{\sigma \in \hat{K}} d_\sigma \int_{\mathbb{R}\setminus\{0\}} \operatorname{tr}\left((\rho_\sigma^\lambda)^*(z',t',k')\tau(z',t',k',\lambda,\sigma)\widehat{f_{(z,t,k)}}(\lambda,\sigma)\right) |\lambda|^n d\lambda \\
&\to \sum_{\sigma \in \hat{K}} d_\sigma \int_{\mathbb{R}\setminus\{0\}} \operatorname{tr}\left((\rho_\sigma^\lambda)^*(z_0,t_0,k_0)\tau(z_0,t_0,k_0,\lambda,\sigma)\widehat{f_{(z,t,k)}}(\lambda,\sigma)\right) |\lambda|^n d\lambda
\end{aligned}$$

as $(z',t',k') \to (z_0,t_0,k_0)$ in G. This shows that $T_\tau f_{(x,y,z,t)}$ is continuous on G. Letting $(z_0,t_0,k_0) = (z,t,k)$ and using the property that $T_\tau f = 0$ for all f in $L^2(G)$, we get

$$\begin{aligned}
&\left(T_\tau f_{(z,t,k)}\right)(z,t,k) \\
&= (2\pi)^{-n} \sum_{\sigma \in \hat{K}} d_\sigma \int_{\mathbb{R}\setminus\{0\}} \operatorname{tr}\left(\tau(z,t,k,\lambda,\sigma)\tau(z,t,k,\lambda,\sigma)^*\right) |\lambda|^n d\lambda \\
&= (2\pi)^{-n} \sum_{\sigma \in \hat{K}} d_\sigma \int_{\mathbb{R}\setminus\{0\}} \|\tau(z,t,k,\lambda,\sigma)\|_{S_2} |\lambda|^n d\lambda = 0
\end{aligned}$$

This implies that $\|\tau(z,t,k,\lambda,\sigma)\|_{S_2} = 0$ for almost all $(\lambda,\sigma) \in G'$. Thus the symbol $\tau(z,t,k,\lambda,\sigma) = 0$ for almost all $(z,t,k,\lambda,\sigma) \in G \times G'$.

Conversely, if $\tau(z,t,k,\lambda,\sigma) = 0$ for almost all $(z,t,k,\lambda,\sigma) \in G \times G'$, then from the definition (3) of T_τ, it is obvious that $T_\tau f = 0$ for all f in $L^2(G)$. ☐

4. Hilbert–Schmidt operators

In this section, we characterize the Hilbert–Schmidt pseudo-differential operators in terms of their corresponding symbols. We obtain a necessary and sufficient condition on the operator valued symbols τ such that the corresponding pseudo-differential operator T_τ on the Heisenberg motion group G are in the class of Hilbert–Schmidt operators.

Note that f^λ defined in (1) is the inverse Fourier transform of f in t variable or Fourier transform of f with respect to the center of G. Therefore, one can write f^λ in the following form:

$$f^\lambda(z,k) = \left(\mathcal{F}^{-1}f\right)(z,\lambda,k) = (\mathcal{F}f)(z,-\lambda,k),$$

where $\mathcal{F}$ denote the Fourier transform with respect to center of G. Now we are ready to state and prove the following result.

Theorem 4.1. *Let τ be a symbol such that it satisfies the hypotheses of Theorem 3.4. Then the corresponding pseudo-differential operator T_τ is a Hilbert–Schmidt operator if and only if*

$$\tau(z,t,k,\lambda,\sigma) = \rho_\sigma^\lambda(z,t,k) W_\sigma^\lambda\left(\alpha(z,t,k)^{-\lambda}\right),$$

where $(z,t,k,\lambda,\sigma) \in G \times G'$ and $\alpha : G \to L^2(G)$ is a weakly continuous mapping such that it satisfies

(i) $\displaystyle\int_G \|\alpha(z,t,k)(\cdot,\cdot,\cdot,\cdot)\|_{L^2(G)}\, dzdkdt < \infty,$

(ii) $\displaystyle\sup_{(z,t,k,t)\in G\times G'\times\mathbb{R}^*} \|\mathcal{F}\alpha(z,t,k)(\cdot,\lambda,\cdot)\|_{L^2(G^\times)} < \infty,$

(iii) $\displaystyle\int_{\mathbb{R}\setminus\{0\}} \|\mathcal{F}\alpha(z,t,k)(\cdot,\lambda,\cdot)\|_{L^2(G^\times)}\, |\lambda|^n d\lambda < \infty, \quad (z,t,k) \in G.$

Proof. Let $f \in \mathcal{S}(G)$. Then for all $(z,t,k) \in G$

$$T_\tau(z,t,k) = (2\pi)^{-n} \sum_{\sigma\in\hat{K}} d_\sigma \int_{\mathbb{R}\setminus\{0\}} \operatorname{tr}\left((\rho_\sigma^\lambda)^*(z,t,k)\ \tau(z,t,k,\lambda,\sigma)\ \hat{f}(\lambda,\sigma)\right) |\lambda|^n d\lambda.$$

Using the expression of τ and the fact that $\hat{f}(\lambda,\sigma) = W_\sigma^\lambda(f^\lambda)$, we get

$$\begin{aligned} T_\tau(z,t,k) &= (2\pi)^{-n} \sum_{\sigma\in\hat{K}} d_\sigma \int_{\mathbb{R}\setminus\{0\}} \operatorname{tr}\left((\rho_\sigma^\lambda)^*(z,t,k)\ \rho_\sigma^\lambda(z,t,k)\ W_\sigma^\lambda(\alpha(z,t,k)^{-\lambda})\ W_\sigma^\lambda(f^\lambda)\right) |\lambda|^n d\lambda \\ &= (2\pi)^{-n} \sum_{\sigma\in\hat{K}} d_\sigma \int_{\mathbb{R}\setminus\{0\}} \operatorname{tr}\left(W_\sigma^\lambda(\alpha(z,t,k)^{-\lambda})\ W_\sigma^\lambda(f^\lambda)\right) |\lambda|^n d\lambda \\ &= (2\pi)^{-n} \sum_{\sigma\in\hat{K}} d_\sigma \int_{\mathbb{R}\setminus\{0\}} \operatorname{tr}\left(W_\sigma^\lambda\left(\alpha(z,t,k)^{-\lambda} \times_\lambda f^\lambda\right)\right) |\lambda|^n d\lambda. \end{aligned}$$

Now, using Theorem 2.4, for all $(z,t,k) \in G$, we obtain

$$\begin{aligned} T_\tau(z,t,k) &= \int_{\mathbb{R}\setminus\{0\}} \int_{G^\times} \alpha(z,t,k)^{-\lambda}(-k_1^{-1}z_1, k_1^{-1})\ f^\lambda(z_1,k_1)\ dz_1dk_1d\lambda \\ &= \int_{\mathbb{R}\setminus\{0\}} \int_{G^\times} (\mathcal{F}\alpha(z,t,k))(-k_1^{-1}z_1,\lambda,k_1^{-1})\ (\mathcal{F}f)(z_1,-\lambda,k_1)\ dz_1dk_1d\lambda \\ &= \int_{\mathbb{R}\setminus\{0\}} \int_{G^\times} \alpha(z,t,k)(-k_1^{-1}z_1,t_1,k_1^{-1})\ f(z_1,t_1,k_1)\ dz_1dk_1dt_1. \end{aligned}$$

This shows that T_τ is an almost everywhere integral operator with kernel

$$K(z,t,k,z_1,t_1,k_1) = \alpha(z,t,k)(-k_1^{-1}z_1,t_1,k_1^{-1}), \tag{6}$$

where $(z,t,k),(z_1,t_1,k_1)\in G$. Using Fubini's theorem and Plancherel theorem, we get

$$\begin{aligned}&\int_G\int_G |K(z,t,k,z_1,t_1,k_1)|^2\, dzdtdk\, dz_1dt_1dk_1\\ &=\int_G\int_G \left|\alpha(z,t,k)(-k_1^{-1}z_1,t_1,k_1^{-1})\right|^2 dzdtdk\, dz_1dt_1dk_1\\ &=\int_G \|\alpha(z,t,k)(\cdot,\cdot,\cdot,\cdot)\|_{L^2(G)}\ dzdkdt<\infty.\end{aligned}$$

Therefore, $T_\tau : L^2(G)\to L^2(G)$ is a Hilbert–Schmidt operator.

Conversely, suppose that $T_\tau : L^2(G)\to L^2(G)$ is a Hilbert–Schmidt operator. Then there exists a function $\alpha\in L^2(G\times G)$ such that for all $f\in L^2(G)$, we have

$$T_\tau(z,t,k)=\int_G \alpha(z,t,k)(z_1,t_1,k_1)f(z_1,t_1,k_1)dt_1,\qquad (z,t,k)\in G.$$

Let $\alpha : G\to L^2(G)$ be the mapping defined by

$$\alpha(z,t,k)(z_1,t_1,k_1)=\alpha(z,t,k,z_1,t_1,k_1),$$

where $(z,t,k),(z_1,t_1,k_1)\in G$. Then, from (5), we get

$$\sum_{\sigma\in\hat{K}} d_\sigma\|\tau(z,t,k,\lambda,\sigma)\|_{S_2}=(2\pi)^n|\lambda|^{-n}\,\|\mathcal{F}\alpha(z,t,k)(\cdot,\lambda,\cdot)\|_{L^2(G^\times)}\,.$$

Now, reversing the argument for sufficiency and using Theorem 3.4, we get the converse part and this completes the proof the the theorem. $\square$

As an immediate consequence of the Theorem 4.1 above, in the following corollary, we present a result related to trace class pseudo-differential operators on G and find its trace formula.

Corollary 4.2. *Let $\alpha\in L^2(G\times G)$ such that*

$$\int_G |\alpha(z,t,k)(z,t,k)|\, dzdtdk<\infty.$$

Let $\tau : G\times G'\to B(\mathcal{H}_\sigma^2)$ be the symbol defined as in Theorem 4.1. Then $T_\tau : L^2(G)\to L^2(G)$ is a trace class operator. Moreover, its trace is given by

$$\operatorname{tr}(T_\tau)=\int_G \alpha(z,t,k)(-k^{-1}z,t,k^{-1})\, dzdtdk$$

Proof. The proof of Corollary 4.2 follows from the formula (6) on the kernel of the pseudo-differential operator in the proof of the Theorem 4.1. $\square$

Next we obtain a necessary and sufficient condition on the symbol τ so that the corresponding pseudo-differential operator T_τ is a trace class operator and we derive the trace formula of the operator T_τ. Indeed, we have the following result.

Corollary 4.3. *Let $\tau : G\times G'\to S_2$ be a symbol such that it satisfying the conditions of Theorem 3.4. Then T_τ is a trace class operator if and only if*

$$\tau(z,t,k,\lambda,\sigma)=\rho_\sigma^\lambda(z,t,k)W_\sigma^\lambda\left(\alpha(z,t,k)^{-\lambda}\right),$$

where $(z,t,k,\lambda,\sigma)\in G\times G'$, $\alpha : G\to L^2(G)$ is a mapping such that it satisfies the conditions of Theorem 4.1 and

$$\alpha(z,t,k)(z_1,t_1,k_1)=\int_G \alpha_1(z,t,k)(z_2,t_2,k_2)\alpha_2(z_2,t_2,k_2)(z_1,t_1,k_1)dz_2dt_2dk_2,$$

for all $(z,t,k),(z_1,t_1,k_1)\in G$, here $\alpha_1 : G\to L^2(G)$ and $\alpha_2 : G\to L^2(G)$ satisfies the conditions

$$\int_G \|\alpha_1(z,t,k)\|^2_{L^2(G)}\, dqdt<\infty,\qquad \int_G \|\alpha_2(z,t,k)\|^2_{L^2(G)}\, dqdt<\infty.$$

Moreover, if $T_\tau : L^2(G) \to L^2(G)$ is a trace class operator, then we have the trace formula

$$\begin{aligned}\operatorname{tr}(T_\tau) &= \int_G \alpha(z,t,k)(z,t,k)\, dzdtdk \\ &= \int_G \int_G \int_G \alpha_1(z,t,k)(z_2,t_2,k_2)\alpha_2(z_2,t_2,k_2)(z,t,k)\, dzdtdk\, dz_2dt_2dk_2.\end{aligned}$$

Proof. The proof follows from the expression of the kernel of pseudo-differential operators in Theorem 4.1 and the fact that every trace class operator can be written as a product of two Hilbert–Schmidt operators. □

5. Discussion and Conclusions

This article takes up the L^2-L^p boundedness problem of pseudo-differential operators on the Heisenberg motion group for the range $2 \leq p \leq \infty$. Using the λ-Weyl transform, we provided a necessary and sufficient condition on the operator-valued symbols such that the associated pseudo-differential operators on the Heisenberg motion group are in the class of Hilbert–Schmidt operators. Further, it will be interesting to investigate $L^p - L^p$ or $L^p - L^q$ boundedness of pseudo-differential operators on the Heisenberg motion group for the range $1 \leq p, q \leq \infty$.

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7. Data availability statement

The authors confirm that the data supporting the findings of this study are available within the article and its supplementary materials.

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Vishvesh Kumar
Department of Mathematics: Analysis, Logic and Discrete Mathematics
Ghent University
Krijgslaan 281, Building S8, B 9000 Ghent, Belgium .
Email address: `vishveshmishra@gmail.com`

Shyam Swarup Mondal
Department of Mathematics
IIT Guwahati
Guwahati, Assam, India.
Email address: `mondalshyam055@gmail.com`