

SCHATTEN CLASS AND NUCLEAR PSEUDO-DIFFERENTIAL OPERATORS ON HOMOGENEOUS SPACES OF COMPACT GROUPS

VISHVESH KUMAR AND SHYAM SWARUP MONDAL

ABSTRACT. Given a compact (Hausdorff) group G and a closed subgroup H of G , in this paper we present symbolic criteria for pseudo-differential operators on the compact homogeneous space G/H characterizing the Schatten-von Neumann classes $S_r(L^2(G/H))$ for all $0 < r \leq \infty$. We go on to provide a symbolic characterization for r -nuclear, $0 < r \leq 1$, pseudo-differential operators on $L^p(G/H)$ with applications to adjoint, product and trace formulae. The criteria here are given in terms of matrix-valued symbols defined on noncommutative analogue of phase space $G/H \times \widehat{G/H}$. Finally, we present an application of aforementioned results in the context of the heat kernels.

1. INTRODUCTION

The theory of pseudo-differential operators is one of the most significant tools in mathematics to study the problems of partial differential equations [23]. The study of pseudo-differential operators was initiated by Kohn and Nirenberg [29]. Ruzhansky and Turunen [38, 39] studied (global) pseudo-differential operators with matrix-valued symbols on compact (Lie) groups. They introduced symbol classes and symbolic calculus for matrix-valued symbols on compact Lie groups and presented plentiful applications of this global theory. After that, the theory of pseudo-differential operators with matrix-valued symbols on compact (Hausdorff) groups, compact homogeneous spaces, compact manifolds is broadly studied by several authors [3, 4, 5, 9, 11, 12, 13, 14, 15, 20, 30, 31, 33, 34, 38, 44] in many different contexts.

Let G be a compact (Hausdorff) group and let H be a closed subgroup of G . In this paper, we mainly address the following three problems: (i) To find criteria for pseudo-differential operators to be in r -Schatten-von Neumann class S_r of operators on $L^2(G/H)$ for $0 < r \leq \infty$; (ii) to find criteria for pseudo-differential operators from $L^{p_1}(G/H)$ into $L^{p_2}(G/H)$ to be r -nuclear, $0 < r \leq 1$, for $1 \leq p_1, p_2 < \infty$; and (iii) applications to find a trace formula and to provide criteria for the heat kernels to be nuclear on $L^p(G/H)$. In order to do this, we use the global quantization for compact homogeneous spaces as a non-commutative analogue of the Kohn-Nirenberg quantization of operators on \mathbb{R}^n .

Recently, several researchers started an extensive investigation to give criteria for operators belonging to r -Schatten-von Neumann class and to the class of r -nuclear operators in terms of their symbols with lower regularity [2, 13, 16, 41, 42]. Ruzhansky and Delgado

Date: May 30, 2022.

2010 Mathematics Subject Classification. Primary 35S05, 47G30; Secondary 43A85,

Key words and phrases. Pseudo-differential operators, Global quantization, Nuclear operators, Heat kernels, Schatten classes, Traces, Adjoints, Homogeneous spaces of compact groups.

[13, 15] successfully drop the regularity condition at least in their setting using matrix-valued symbols. Inspired by the work of Delgado and Ruzhansky, we present symbolic criteria for pseudo-differential operators on G/H to be r -Schatten-von Neumann class using the matrix-valued symbols defined on $G/H \times \widehat{G/H}$, a noncommutative analogue of phase space. It is well known that in the setting of Hilbert spaces, the class of r -nuclear operators agrees with the r -Schatten-von Neumann class of operators [36]. In general, for trace class operators on Hilbert spaces, the trace of an operator given by integration of its integral kernel over the diagonal is equal to the sum of its eigenvalues. However, this property fails in Banach spaces. The importance of r -nuclear operators lies in the seminal work of Grothendieck, who proved that, for $2/3$ -nuclear operators, the trace in L^p -spaces agrees with the sum of all the eigenvalues with multiplicities counted. Therefore, the notion of r -nuclear operators becomes useful. One of the interesting questions is to find good criteria for ensuring the r -nuclearity of operators on L^p -spaces. But this needs to be formulated differently than those on Hilbert spaces and has to take into account the impossibility of certain kernel formulations in view of Carleman's example [6] (also see [13]). In view of this, we will establish conditions imposed on symbols instead of kernels ensuring the r -nuclearity of the corresponding operators. The initiative of finding the necessary and sufficient conditions for pseudo-differential operators to be r -nuclear was started by Delgado and Wong [17]. The main tool used for such a characterization is a theorem of Delgado [12]. A multilinear version of this result was recently proved by the first author and D. Cardona to study the nuclearity of multilinear pseudo-differential operators on the lattice and the torus [4, 5]. Delgado and Ruzhansky [13] studied the L^p -nuclearity and traces of pseudo-differential operators on compact Lie groups using the global symbolic calculus developed by Ruzhansky and Turunen [38]. Later, they with their collaborators extended these results to more general compact homogeneous spaces and compact manifolds [12, 13, 14, 15]. On the other hand, Wong and his collaborators extended the results of [17] in the settings of abstract compact groups without differential structure [19, 20]. In particular, characterizations of nuclear operators in terms of decomposition of symbol via the Fourier transform were investigated by Ghaemi, Jamalpour Birgani and Wong [19] for \mathbb{S}^1 and for arbitrary compact groups [20].

The homogeneous spaces of abstract compact groups play an important role in mathematical physics, geometric analysis, constructive approximation, and coherent state transform, see [24, 25, 26, 27, 28, 29] and references therein. Pseudo-differential operators on homogeneous spaces of compact groups (without differential structure) studied by the first author [30] (see also [38]). We use the operator-valued Fourier transform on homogeneous spaces of compact groups developed by Ghani Farashahi [18]. Using this Fourier transform, we define global pseudo-differential operators on homogeneous spaces of compact groups and study the r -Schatten-von Neumann class of operators on $L^2(G/H)$ and r -nuclear operators on L^p -spaces on compact homogeneous spaces. Our results can be seen as a complement as well as a generalization of [13, 19].

We begin this paper by recapitulating some basic facts about the Fourier analysis on homogeneous spaces of compact groups from [18] in Section 2. Although, a parallel theory of homogeneous spaces of compact Lie groups can be found in the classical book of Vilenkin [43] and recent papers and books [1, 10, 16, 35]. Later in this section, we present a global

quantization (Ruzhansky and Turunen [38]) on homogeneous spaces of compact groups related to matrix-valued symbols. In Section 3, we give symbolic criteria of r -Schatten-von Neumann class of operators defined on $L^2(G/H)$. In Section 4, we start our investigation on r -nuclear operators. We begin this section by providing sufficient conditions for operators to be r -nuclear in terms of conditions on symbols. We also present a characterization of r -nuclear pseudo-differential operators on L^p -spaces for homogeneous spaces of compact groups. We calculate the nuclear trace of related pseudo-differential operators. In Section 5, we find symbols of the adjoint of r -nuclear pseudo-differential operators on homogeneous spaces of compact groups and provide a characterization for self-adjointness. We also compute symbol of the product of a nuclear operator and a bounded linear operator. We end this paper by presenting applications of our results in the context of the heat kernels.

2. FOURIER ANALYSIS AND THE GLOBAL QUANTIZATION ON HOMOGENEOUS SPACES OF COMPACT GROUPS

We begin this section by recalling some basic and important concepts of harmonic analysis on homogeneous spaces of compact (Hausdorff) groups from [18] which is almost similar to the theory given in [1] and [43] (see also [10, 16, 35] for homogeneous spaces of compact Lie groups).

Throughout the paper, we assume that G is a compact (Hausdorff) group with the normalized Haar measure dx and H is a closed subgroup of G with the probability Haar measure dh . The left coset space G/H can be seen as a homogeneous space with respect to the action of G on G/H given by left multiplication. Let $C(\Omega)$ denote the space of all continuous functions on a compact Hausdorff space Ω . Define $T_H : C(G) \rightarrow C(G/H)$ by

$$T_H(f)(xH) = \int_H f(xh) dh, \quad xH \in G/H.$$

Then T_H is an onto map. The homogeneous space G/H has a unique normalized G -invariant positive Radon measure μ such that the Weil formula

$$\int_{G/H} T_H(f)(xH) d\mu(xH) = \int_G f(x) dx$$

holds. The map T_H can be extended to $L^2(G/H, \mu)$ and is a partial isometry on $L^2(G/H)$ with $\langle T_H(f), T_H(g) \rangle_{L^2(G/H, \mu)} = \langle f, g \rangle_{L^2(G)}$ for all $f, g \in L^2(G)$.

Let (π, \mathcal{H}_π) be a continuous unitary representation of a compact group G on a Hilbert space \mathcal{H}_π . It is well-known that any irreducible representation (π, \mathcal{H}_π) of G is finite dimensional with the dimension d_π (say). Consider an operator-valued integral

$$T_H^\pi := \int_H \pi(h) dh$$

defined in the weak sense, i.e., $\langle T_H^\pi u, v \rangle = \int_H \langle \pi(h)u, v \rangle dh$, for all $u, v \in \mathcal{H}_\pi$. Note that, T_H^π is a bounded linear operator on \mathcal{H}_π with norm bounded by one. Further, T_H^π is a partial isometric orthogonal projection and T_H^π is an identity operator if and only if $\pi(h) = I$ for all $h \in H$ (see [18]).

Definition 2.1. Let H be a closed subgroup of a compact group G . Then the dual $\widehat{G/H}$ of G/H is a subset of \widehat{G} given by

$$\widehat{G/H} := \left\{ \pi \in \widehat{G} : T_H^\pi \neq 0 \right\} = \left\{ \pi \in \widehat{G} : \int_H \pi(h) dh \neq 0 \right\}.$$

We note here that the set $\widehat{G/H}$ is the set of all type 1 representations of G with respect to H which was denoted by \widehat{G}_0 in [35, 43].

Let $\pi \in \widehat{G/H}$. Then the functions $\pi_{\zeta, \xi}^H : G/H \rightarrow \mathbb{C}$ defined by

$$\pi_{\zeta, \xi}^H(xH) := \langle \pi(x) T_H^\pi \zeta, \xi \rangle, \quad xH \in G/H,$$

for $\xi, \zeta \in \mathcal{H}_\pi$, are called H -matrix elements of (π, \mathcal{H}_π) . If $\{e_1, e_2, \dots, e_{d_\pi}\}$ is an orthonormal basis for \mathcal{H}_π then we denote $\langle \pi(x) T_H^\pi e_i, e_j \rangle$ by $\pi_{ij}^H(xH)$. Using the orthogonality relation of matrix coefficients of G and the fact $T_H(\pi_{\zeta, \xi}) = \pi_{\zeta, \xi}^H$, we have

$$\langle \pi_{i,j}^H, \pi_{k,l}^H \rangle_{L^2(G/H, \mu)} = \frac{1}{d_\pi} \delta_{\pi\xi} \delta_{ik} \delta_{jl}.$$

Let $\varphi \in L^1(G/H, \mu)$ and $\pi \in \widehat{G/H}$. Then the group Fourier transform $\mathcal{F}_{G/H}(\varphi)$ of φ at π is a bounded linear operator defined by

$$(2.1) \quad \mathcal{F}_{G/H}(\varphi)(\pi) = \hat{\varphi}(\pi) := \int_{G/H} \varphi(xH) \Gamma_\pi(xH)^* d\mu(xH)$$

on the Hilbert space \mathcal{H}_π , where for $xH \in G/H$ the notation $\Gamma_\pi(xH)$ stands for a bounded linear operator on \mathcal{H}_π satisfying

$$\langle \zeta, \Gamma_\pi(xH) \xi \rangle = \langle \zeta, \pi(x) T_H^\pi \xi \rangle$$

for all $\zeta, \xi \in \mathcal{H}_\pi$. Note that, from the notation of $\Gamma_\pi(xH)$, the H -matrix coefficients $\pi_{i,j}^H(xH)$ are same as $\Gamma_\pi(xH)_{ij}$. Moreover, if $\varphi \in L^2(G/H)$ then $\hat{\varphi}(\pi)$ is a Hilbert-Schmidt operator on \mathcal{H}_π and satisfies the following Plancherel formula as stated in next theorem.

Theorem 2.2. For $\varphi \in L^2(G/H, \mu)$, we have

$$\sum_{[\pi] \in \widehat{G/H}} d_\pi \|\hat{\varphi}(\pi)\|_{S_2}^2 = \|\varphi\|_{L^2(G/H, \mu)}^2,$$

where $\|\cdot\|_{S_2}$ stands for the Hilbert-Schmidt norm on the space of all Hilbert-Schmidt operators on \mathcal{H}_π .

Theorem 2.3. For $\varphi \in L^2(G/H, \mu)$, the following Fourier inversion formula holds

$$(2.2) \quad \varphi(xH) = \sum_{[\pi] \in \widehat{G/H}} d_\pi \operatorname{Tr}[\hat{\varphi}(\pi) \pi(x) T_H^\pi], \quad \text{for } \mu - \text{a.e. } xH \in G/H.$$

We would like to record the following lemma whose proof is similar to [13, Lemma 2.5] by using the fact that the operator T_H^π is norm bounded by one.

Lemma 2.4. *Let G/H be a compact homogeneous space with the normalized measure μ and let $\pi \in \widehat{G/H}$. Then, for all $1 \leq i, j \leq d_\pi$, we have*

$$\|\Gamma_\pi(\cdot)_{ij}\|_{L^q(G/H)} \leq \begin{cases} d_\pi^{-\frac{1}{q}}, & \text{if } 2 \leq q \leq \infty, \\ d_\pi^{-\frac{1}{2}}, & \text{if } 1 \leq q \leq 2, \end{cases}$$

with the convention that for $q = \infty$ we have $d_\pi^{-\frac{1}{q}} = 1$.

Given a continuous linear operator $T : C(G/H) \rightarrow C(G/H)$, its *matrix-valued global symbol* $\sigma_T(xH, \pi) \in \mathbb{C}^{d_\pi \times d_\pi}$ is defined by

$$(2.3) \quad T_H^\pi \sigma_T(xH, \pi) = \pi(x)^*(T\Gamma_\pi)(xH),$$

where $T\Gamma_\pi$ stands for the action of T on the matrix components of $\Gamma_\pi(xH)$. Setting $(T\Gamma_\pi(xH))_{mn} = (T(\Gamma_{\pi_{mn}}))(xH)$, we have

$$(T_H^\pi \sigma_T(xH, \pi))_{mn} := \sum_{k=1}^{d_\pi} \overline{\pi_{km}(x)} (T\Gamma_\pi(xH))_{kn},$$

where $1 \leq m, n \leq d_\pi$.

Assume that σ_T is a matrix-valued global symbol for a continuous linear operator $T : C(G/H) \rightarrow C(G/H)$ as above. Then we can recover the operator T by using the Fourier inversion formula as follows:

$$\begin{aligned} Tf(xH) &= T \left(\sum_{[\pi] \in \widehat{G/H}} d_\pi \operatorname{Tr}(\pi(x) T_H^\pi \hat{f}(\pi)) \right) \\ &= \sum_{[\pi] \in \widehat{G/H}} d_\pi \operatorname{Tr}(T\Gamma_\pi(xH) \hat{f}(\pi)). \end{aligned}$$

Using (2.3) and the relation $\pi(x)T_H^\pi = \Gamma_\pi(xH)$, we get

$$(2.4) \quad Tf(xH) = \sum_{[\pi] \in \widehat{G/H}} d_\pi \operatorname{Tr}(\Gamma_\pi(xH) \sigma_T(xH, \pi) \hat{f}(\pi))$$

for all $f \in C(G/H)$, μ -a.e. $xH \in G/H$ and the sum is independent of the representation from each equivalence class $[\pi] \in \widehat{G/H}$. We will also write $T = \operatorname{Op}(\sigma_T)$ for the operator T given by the formula (2.4). The operator $T = \operatorname{Op}(\sigma_T)$ will be called *pseudo-differential operator* corresponding to matrix-valued symbol σ_T . For more details and consistent development of this quantization on compact Lie group and the corresponding symbolic calculus, we refer [38] and [39].

Remark 2.5. Let H be a closed normal subgroup of the compact group G and let μ be the normalized G -invariant measure over the left quotient group G/H associated to the Weil formula. Then μ is a Haar measure on the compact (quotient) group G/H and $\widehat{G/H} = H^\perp := \{\pi \in \widehat{G} : \pi(h) = I \text{ for all } h \in H\}$. Moreover, the Fourier transform (2.1), inverse Fourier transform (2.2) and pseudo-differential operator defined by (2.4) coincide with the

classical Fourier transform, inverse Fourier transform and pseudo-differential operator on the compact group G/H , respectively.

3. r -SCHATTEN-VON NEUMANN CLASS OF PSEUDO-DIFFERENTIAL OPERATORS ON $L^2(G/H)$

This section is devoted to the study of r -Schatten-von Neumann class of pseudo-differential operators on the Hilbert space $L^2(G/H)$. We begin this section with the definition of r -Schatten-von Neumann class of operators.

Let \mathcal{H} be a complex Hilbert space. A linear compact operator $A : \mathcal{H} \rightarrow \mathcal{H}$ belongs to the r -Schatten-von Neumann class $S_r(\mathcal{H})$ if

$$\sum_{n=1}^{\infty} (s_n(A))^r < \infty,$$

where $s_n(A)$ denote the singular values of A , i.e., the eigenvalues of $|A| = \sqrt{A^*A}$ with multiplicities counted.

For $1 \leq r < \infty$, the class $S_r(\mathcal{H})$ is a Banach space endowed with the norm

$$\|A\|_{S_r} = \left(\sum_{n=1}^{\infty} (s_n(A))^r \right)^{\frac{1}{r}}.$$

For $0 < r < 1$, $\|\cdot\|_{S_r}$ only defines a quasi-norm with respect to which $S_r(\mathcal{H})$ is complete. An operator belongs to the class $S_1(\mathcal{H})$ is known as a *Trace class* operator. Also, an operator belongs to $S_2(\mathcal{H})$ is known as a *Hilbert-Schmidt* operator.

Let $L^2(G/H \times \widehat{G/H})$ denotes the space of all matrix-valued functions σ_A on $G/H \times \widehat{G/H}$ such that

$$\|\sigma_A\|_{L^2(G/H \times \widehat{G/H})} = \left(\int_{G/H} \sum_{[\xi] \in \widehat{G/H}} d_{\xi} \|\sigma_A(xH, \xi) T_H^{\xi}\|_{S_2}^2 d\mu(xH) \right)^{\frac{1}{2}} < \infty.$$

The following theorem gives a characterization of Hilbert-Schmidt pseudo-differential operators on G/H . We remark here that the following theorem is already proved by the first author in [30] using a different method.

Theorem 3.1. *Let $T : L^2(G/H) \rightarrow L^2(G/H)$ be a continuous linear operator with matrix-valued symbol σ_T on $G/H \times \widehat{G/H}$. Then T is a Hilbert-Schmidt operator if and only if $\sigma_T \in L^2(G/H \times \widehat{G/H})$. Moreover, we have*

$$\|T\|_{S_2} = \|\sigma_T\|_{L^2(G/H \times \widehat{G/H})}.$$

Proof. For all $f \in L^2(G/H)$, we have

$$Tf(xH) = \sum_{[\xi] \in \widehat{G/H}} d_{\xi} \operatorname{Tr}(\Gamma_{\xi}(xH) \sigma_T(xH, \xi) \hat{f}(\xi))$$

$$\begin{aligned}
&= \int_{G/H} \sum_{[\xi] \in \widehat{G/H}} d_\xi \operatorname{Tr}(\Gamma_\xi(xH) \sigma_T(xH, \xi) \Gamma_\xi(wH)^*) f(wH) d\mu(wH) \\
&= \int_{G/H} K(xH, wH) f(wH) d\mu(wH),
\end{aligned}$$

where the kernel $K(xH, wH)$ is given by

$$K(xH, wH) = \sum_{[\xi] \in \widehat{G/H}} d_\xi \operatorname{Tr}(\Gamma_\xi(xH) \sigma_T(xH, \xi) \Gamma_\xi(wH)^*), \quad xH, wH \in G/H.$$

Then

$$\begin{aligned}
\|T\|_{\mathbb{S}_2}^2 &= \int_{G/H} \int_{G/H} |K(xH, yH)|^2 d\mu(xH) d\mu(yH) \\
&= \int_{G/H} \int_{G/H} |K(xH, xz^{-1}H)|^2 d\mu(xH) d\mu(zH).
\end{aligned}$$

Note that

$$\begin{aligned}
K(xH, xz^{-1}H) &= \sum_{[\xi] \in \widehat{G/H}} d_\xi \operatorname{Tr}(\Gamma_\xi(xH) \sigma_T(xH, \xi) \Gamma_\xi(xz^{-1}H)^*) \\
&= \sum_{[\xi] \in \widehat{G/H}} d_\xi \operatorname{Tr}(\Gamma_\xi(zH) \sigma_T(xH, \xi) T_H^\xi) \\
&= (\mathcal{F}^{-1} \tau(xH, \cdot))(zH),
\end{aligned}$$

where $\tau(xH, \xi) = \sigma_T(xH, \xi) T_H^\xi$. Therefore, using Plancherel's formula, we have

$$\begin{aligned}
\|T\|_{\mathbb{S}_2}^2 &= \int_{G/H} \int_{G/H} |K(xH, xz^{-1}H)|^2 d\mu(xH) d\mu(zH) \\
&= \int_{G/H} \int_{G/H} |\mathcal{F}^{-1} \tau(xH, \cdot)(zH)|^2 d\mu(xH) d\mu(zH) \\
&= \int_{G/H} \sum_{[\xi] \in \widehat{G/H}} d_\xi \|\tau(xH, \xi)\|_{\mathbb{S}_2}^2 d\mu(xH) \\
&= \int_{G/H} \sum_{[\xi] \in \widehat{G/H}} d_\xi \|\sigma_T(xH, \xi) T_H^\xi\|_{\mathbb{S}_2}^2 d\mu(xH) \\
&= \|\sigma_T\|_{L^2(G/H \times \widehat{G/H})}.
\end{aligned}$$

□

The following lemma is a consequence of the definition of r -Schatten-von Neumann class (see [15]).

Lemma 3.2. *Let $A : \mathcal{H} \rightarrow \mathcal{H}$ be a linear compact operator. Let $0 < r, t < \infty$. Then $A \in S_r(\mathcal{H})$ if and only if $|A|^{\frac{r}{t}} \in S_t(\mathcal{H})$. Moreover, $\|A\|_{S_r}^r = \| |A|^{\frac{r}{t}} \|_{S_t}^t$.*

The corollary below is the main result of this section which present a characterization of a pseudo-differential operator on $L^2(G/H)$ to be an r -Schatten-von Neumann class operator. The proof follows from Lemma 3.2 with $t = 2$ and Theorem 3.1.

Corollary 3.3. *Let $T : L^2(G/H) \rightarrow L^2(G/H)$ be a continuous linear operator with matrix-valued symbol σ_T on $G/H \times \widehat{G/H}$. Then $T \in S_r(L^2(G/H))$ if and only if*

$$\int_{G/H} \sum_{[\xi] \in \widehat{G/H}} d_\xi \|\sigma_{|T|^{\frac{r}{2}}}(xH, \xi) T_H^\xi\|_{S_2}^2 d\mu(xH) < \infty.$$

4. CHARACTERIZATIONS AND TRACES OF r -NUCLEAR, $0 < r \leq 1$, PSEUDO-DIFFERENTIAL OPERATORS ON $L^p(G/H)$

This section is devoted to the study of r -nuclear operators on Banach spaces $L^p(G/H)$. Here we present a symbolic characterization of r -nuclear operators and give a formula for the nuclear trace of such operators. We begin this section by recalling the basic notions of nuclear operators on Banach spaces.

Let $0 < r \leq 1$ and T be a bounded linear operator from a complex Banach space X into another complex Banach space Y such that there exist sequences $\{x'_n\}_{n=1}^\infty$ in the dual space X' of X and $\{y_n\}_{n=1}^\infty$ in Y such that $\sum_{n=1}^\infty \|x'_n\|_{X'}^r \|y_n\|_Y^r < \infty$ and

$$Tx = \sum_{n=1}^\infty x'_n(x) y_n, \quad x \in X.$$

Then we call $T : X \rightarrow Y$ a r -nuclear operator and if $X = Y$, then its nuclear trace $\text{Tr}(T)$ is given by

$$\text{Tr}(T) = \sum_{n=1}^\infty x'_n(y_n).$$

The definition of r -nuclear operators is independent of the choices of the sequences $\{x'_n\}_{n=1}^\infty$ and $\{y_n\}_{n=1}^\infty$. An 1-nuclear operators will be simply called a nuclear operator. The following theorem is a characterization of r -nuclear operators on σ -finite measure spaces [12].

Theorem 4.1. *Let $0 < r \leq 1$. Let (X_1, μ_1) and (X_2, μ_2) be two σ -finite measure spaces. Then a bounded linear operator $T : L^{p_1}(X_1, \mu_1) \rightarrow L^{p_2}(X_2, \mu_2)$, $1 \leq p_1, p_2 < \infty$, is r -nuclear if and only if there exist sequences $\{g_n\}_{n=1}^\infty$ in $L^{p_1'}(X_1, \mu_1)$ and $\{h_n\}_{n=1}^\infty$ in $L^{p_2}(X_2, \mu_2)$ such that for all $f \in L^{p_1}(X_1, \mu_1)$*

$$(Tf)(x) = \int_{X_1} K(x, y) f(y) d\mu_1(y), \quad x \in X_2,$$

where

$$K(x, y) = \sum_{n=1}^\infty h_n(x) g_n(y), \quad x \in X_2, y \in X_1,$$

and

$$\sum_{n=1}^\infty \|g_n\|_{L^{p_1'}(X_1, \mu_1)}^r \|h_n\|_{L^{p_2}(X_2, \mu_2)}^r < \infty.$$

Let $0 < r \leq 1$. Let (X, μ) be a σ -finite measure space. Let $T : L^p(X, \mu) \rightarrow L^p(X, \mu)$, $1 \leq p < \infty$, be a r -nuclear operator. Then by Theorem 4.1, we can find sequences $\{g_n\}_{n=1}^\infty$ in $L^{p'}(X, \mu)$ and $\{h_n\}_{n=1}^\infty$ in $L^p(X, \mu)$ such that

$$\sum_{n=1}^{\infty} \|g_n\|_{L^{p'}(X, \mu)}^r \|h_n\|_{L^p(X, \mu)}^r < \infty$$

and for all $f \in L^p(X, \mu)$, we have

$$(Tf)(x) = \int_X K(x, y) f(y) d\mu(y), \quad x \in X,$$

where

$$K(x, y) = \sum_{n=1}^{\infty} h_n(x) g_n(y), \quad x, y \in X,$$

and it satisfies

$$\int_X |K(x, y)| d\mu(y) \leq \sum_{n=1}^{\infty} \|g_n\|_{L^{p'}(X, \mu)}^r \|h_n\|_{L^p(X, \mu)}^r.$$

The nuclear trace $\text{Tr}(T)$ of $T : L^p(X, \mu) \rightarrow L^p(X, \mu)$ is given by

$$(4.1) \quad \text{Tr}(T) = \int_X K(x, x) d\mu(x).$$

Now, we present a characterization of r -nuclear pseudo-differential operators from $L^{p_1}(G/H)$ into $L^{p_2}(G/H)$.

Theorem 4.2. *Let $0 < r \leq 1$ and let $T : L^{p_1}(G/H) \rightarrow L^{p_2}(G/H)$, $1 \leq p_1, p_2 < \infty$, be a continuous linear operator with matrix-valued symbol σ_T on $G/H \times \widehat{G/H}$. Suppose that σ_T satisfies*

$$\sum_{[\xi] \in \widehat{G/H}} d_\xi^{2 + \frac{r}{p_1}} \|\sigma_T(\cdot, \xi)\|_{op(\ell^\infty, \ell^\infty)}^r \| \cdot \|_{L^{p_2}(G/H)}^r < \infty,$$

where $\tilde{p}_1 = \min\{2, p_1\}$. Then the operator T is r -nuclear.

Proof. Since the operator T can be written as

$$Tf(xH) = \int_{G/H} \sum_{[\xi] \in \widehat{G/H}} d_\xi \text{Tr}(\Gamma_\xi(xH) \sigma_T(xH, \xi) \Gamma_\xi(wH)^*) f(wH) d\mu(wH),$$

the kernel of T is given by

$$K(xH, wH) = \sum_{[\xi] \in \widehat{G/H}} d_\xi \text{Tr}(\Gamma_\xi(xH) \sigma_T(xH, \xi) \Gamma_\xi(wH)^*).$$

Now we write

$$\text{Tr}(\Gamma_\xi(xH) \sigma_T(xH, \xi) \Gamma_\xi(wH)^*) = \sum_{i,j=1}^{d_\xi} (\Gamma_\xi(xH) \sigma_T(xH, \xi))_{ij} \overline{\Gamma_\xi(wH)_{ij}},$$

and set that $h_{\xi,ij}(xH) = d_\xi (\Gamma_\xi(xH) \sigma_T(xH, \xi))_{ij}$ and $g_{\xi,ij}(wH) = (\Gamma_\xi(wH)^*)_{ji} = \overline{\Gamma_\xi(wH)_{ij}}$.

We observe that

$$(\Gamma_\xi(xH)\sigma_T(xH, \xi))_{ij} = \sum_{k=1}^{d_\xi} \Gamma_\xi(xH)_{ik} \sigma_T(xH, \xi)_{kj} = \sum_{k=1}^{d_\xi} (\sigma_T(xH, \xi))_{jk}^t \Gamma_\xi(xH)_{ik}.$$

By taking into account that $|\Gamma_\xi(xH)_{ik}| \leq 1$, we get

$$\begin{aligned} |(\Gamma_\xi(xH)\sigma_T(xH, \xi))_{ij}| &= \left| \sum_{k=1}^{d_\xi} (\sigma_T(xH, \xi))_{jk}^t \Gamma_\xi(xH)_{ik} \right| \\ &\leq \|\sigma_T(xH, \xi)^t\|_{op(\ell^\infty, \ell^\infty)} \|(\Gamma_\xi(xH)_{i1}, \Gamma_\xi(xH)_{i2}, \dots, \Gamma_\xi(xH)_{id_\xi})\|_{\ell^\infty} \\ &\leq \|\sigma_T(xH, \xi)^t\|_{op(\ell^\infty, \ell^\infty)}. \end{aligned}$$

Therefore,

$$\begin{aligned} \|h_{\xi, ij}(\cdot)\|_{L^{p_2}(G/H)}^r &= \|d_\xi (\Gamma_\xi(\cdot)\sigma_T(\cdot, \xi))_{ij}\|_{L^{p_2}(G/H)}^r \\ &\leq d_\xi^r \|\sigma_T(\cdot, \xi)^t\|_{op(\ell^\infty, \ell^\infty)} \|L^{p_2}(G/H). \end{aligned}$$

If p'_1 denotes the Lebesgue conjugate of p_1 , then we have $\frac{1}{p_1} + \frac{1}{q_1} = 1$, where $\tilde{q}_1 = \max\{2, p'_1\}$.

By Lemma 2.4, we have $\|\Gamma_\xi(\cdot)\|_{L^{p'_1}(G/H)}^r \leq d_\xi^{-\frac{r}{\tilde{p}'_1}}$. Thus

$$\begin{aligned} \sum_{[\xi], i, j} \|g_{\xi, ij}(\cdot)\|_{L^{p'_1}(G/H)}^r \|h_{\xi, ij}(\cdot)\|_{L^{p_2}(G/H)}^r &\leq \sum_{[\xi]} d_\xi^{-\frac{r}{\tilde{p}'_1}} d_\xi^r d_\xi^2 \|\sigma_T(\cdot, \xi)^t\|_{op(\ell^\infty, \ell^\infty)} \|L^{p_2}(G/H) \\ &\leq \sum_{[\xi]} d_\xi^{2+\frac{r}{\tilde{p}'_1}} \|\sigma_T(\cdot, \xi)^t\|_{op(\ell^\infty, \ell^\infty)} \|L^{p_2}(G/H) < \infty. \end{aligned}$$

Hence, by invoking Theorem 4.1, it follows that T is r -nuclear. \square

Next theorem gives a necessary and sufficient condition for an operator to be r -nuclear in terms of its symbolic decomposition.

Theorem 4.3. *Let $0 < r \leq 1$ and let $T : L^{p_1}(G/H) \rightarrow L^{p_2}(G/H)$, $1 \leq p_1, p_2 < \infty$, be a continuous linear operator with matrix-valued symbol σ_T on $G/H \times \widehat{G/H}$. Then T is r -nuclear if and only if there exist sequences $\{g_k\}_{k=1}^\infty \in L^{p'_1}(G/H)$ and $\{h_k\}_{k=1}^\infty \in L^{p_2}(G/H)$ such that*

$$\sum_{k=1}^\infty \|g_k\|_{L^{p'_1}(G/H)}^r \|h_k\|_{L^{p_2}(G/H)}^r < \infty$$

and

$$T_H^\xi \sigma_T(xH, \xi) = \xi(x)^* \sum_{k=1}^\infty h_k(xH) \widehat{g_k}(\xi)^*, \quad (xH, \xi) \in G/H \times \widehat{G/H}.$$

Proof. Suppose that $T : L^{p_1}(G/H) \rightarrow L^{p_2}(G/H)$ is r -nuclear for $1 \leq p_1, p_2 < \infty$. Then by Theorem 4.1, there exist sequences $\{g_k\}_{k=1}^\infty$ in $L^{p'_1}(G/H)$ and $\{h_k\}_{k=1}^\infty$ in $L^{p_2}(G/H)$ such

that

$$\sum_{k=1}^{\infty} \|g_k\|_{L^{p'_1}(G/H)}^r \|h_k\|_{L^{p_2}(G/H)}^r < \infty$$

and for all $f \in L^{p_1}(G/H)$, we have

$$\begin{aligned} (Tf)(xH) &= \sum_{[\pi] \in \widehat{G/H}} d_\pi \operatorname{Tr}(\Gamma_\pi(xH) \sigma_T(xH, \pi) \hat{f}(\pi)) \\ &= \sum_{[\pi] \in \widehat{G/H}} d_\pi \sum_{i,j=1}^{d_\pi} (\Gamma_\pi(xH) \sigma_T(xH, \pi))_{ij} \hat{f}(\pi)_{ji} \\ &= \int_{G/H} \sum_{[\pi] \in \widehat{G/H}} d_\pi \sum_{i,j=1}^{d_\pi} (\Gamma_\pi(xH) \sigma_T(xH, \pi))_{ij} \overline{\Gamma_\pi(wH)_{ij}} f(wH) d\mu(wH) \\ (4.2) \quad &= \int_{G/H} \left(\sum_{k=1}^{\infty} h_k(xH) g_k(wH) \right) f(wH) d\mu(wH) \end{aligned}$$

for all $xH \in G/H$. Let ξ be a fixed but arbitrary element in $\widehat{G/H}$. Then for $1 \leq m, n \leq d_\xi$, we define the function f on G/H by

$$f(wH) = \Gamma_\xi(wH)_{nm}, \quad wH \in G/H.$$

Since

$$\int_{G/H} \Gamma_\xi(wH)_{nm} \overline{\Gamma_\pi(wH)_{ij}} d\mu(wH) = \frac{1}{d_\xi}$$

if and only if $\pi = \xi$, $i = n$ and $j = m$, and is zero otherwise, it follows from (4.2) that

$$\begin{aligned} (\Gamma_\xi(xH) \sigma_T(xH, \xi))_{nm} &= \sum_{k=1}^{\infty} h_k(xH) \left(\int_{G/H} g_k(wH) \Gamma_\xi(wH)_{nm} d\mu(wH) \right) \\ &= \sum_{k=1}^{\infty} h_k(xH) \overline{\left(\widehat{g_k}(\xi) \right)_{mn}}. \end{aligned}$$

Therefore,

$$T_H^\xi \sigma_T(xH, \xi) = \xi(x)^* \sum_{k=1}^{\infty} h_k(xH) \widehat{g_k}(\xi)^*, \quad (xH, \xi) \in G/H \times \widehat{G/H}.$$

Conversely, suppose that there exist sequences $\{g_k\}_{k=1}^{\infty}$ in $L^{p'_1}(G/H)$ and $\{h_k\}_{k=1}^{\infty}$ in $L^{p_2}(G/H)$ such that

$$\sum_{k=1}^{\infty} \|g_k\|_{L^{p'_1}(G/H)}^r \|h_k\|_{L^{p_2}(G/H)}^r < \infty$$

and

$$T_H^\xi \sigma_T(xH, \xi) = \xi(x)^* \sum_{k=1}^{\infty} h_k(xH) \widehat{g_k}(\xi)^*, \quad (xH, \xi) \in G/H \times \widehat{G/H}.$$

Then, for all $f \in L^{p_1}(G/H)$, we have

$$\begin{aligned}
(Tf)(xH) &= \sum_{[\xi] \in \widehat{G/H}} d_\xi \operatorname{Tr}(\Gamma_\xi(xH) \sigma_T(xH, \xi) \hat{f}(\xi)) \\
&= \sum_{[\xi] \in \widehat{G/H}} d_\xi \sum_{m,n=1}^{d_\xi} (\Gamma_\xi(xH) \sigma_T(xH, \xi))_{nm} \hat{f}(\xi)_{mn} \\
&= \sum_{[\xi] \in \widehat{G/H}} d_\xi \sum_{m,n=1}^{d_\xi} \left(\sum_{k=1}^{\infty} h_k(xH) \widehat{g_k}(\xi)^*_{nm} \right) \hat{f}(\xi)_{mn} \\
&= \sum_{[\xi] \in \widehat{G/H}} d_\xi \sum_{m,n=1}^{d_\xi} \left(\sum_{k=1}^{\infty} h_k(xH) \overline{\widehat{g_k}(\xi)_{mn}} \right) \hat{f}(\xi)_{mn} \\
&= \sum_{[\xi] \in \widehat{G/H}} d_\xi \sum_{m,n=1}^{d_\xi} \sum_{k=1}^{\infty} h_k(xH) \left(\int_{G/H} g_k(wH) \Gamma_\xi(wH)_{nm} d\mu(wH) \right) \hat{f}(\xi)_{mn} \\
&= \int_{G/H} \left(\sum_{[\xi] \in \widehat{G/H}} d_\xi \sum_{m,n=1}^{d_\xi} \Gamma_\xi(wH)_{nm} \hat{f}(\xi)_{mn} \right) \sum_{k=1}^{\infty} h_k(xH) g_k(wH) d\mu(wH) \\
&= \int_{G/H} \left(\sum_{[\xi] \in \widehat{G/H}} d_\xi \operatorname{Tr}(\Gamma_\xi(wH) \hat{f}(\xi)) \right) \sum_{k=1}^{\infty} h_k(xH) g_k(wH) d\mu(wH) \\
&= \int_{G/H} \left(\sum_{k=1}^{\infty} h_k(xH) g_k(wH) \right) f(wH) d\mu(wH)
\end{aligned}$$

for all $xH \in G/H$. Therefore by Theorem 4.1, it follows that T is r -nuclear. \square

In the next theorem, we will give another characterization of r -nuclear operators from $L^{p_1}(G/H)$ into $L^{p_2}(G/H)$ in order to find the trace of r -nuclear operators from $L^p(G/H)$ into $L^p(G/H)$.

Theorem 4.4. *Let $0 < r \leq 1$ and let $T : L^{p_1}(G/H) \rightarrow L^{p_2}(G/H)$, $1 \leq p_1, p_2 < \infty$, be a continuous linear operator with matrix-valued symbol σ_T on $G/H \times \widehat{G/H}$. Then the operator T is r -nuclear if and only if there exist sequences $\{g_k\}_{k=1}^{\infty}$ in $L^{p'_1}(G/H)$ and $\{h_k\}_{k=1}^{\infty}$ in $L^{p_2}(G/H)$ such that*

$$\sum_{k=1}^{\infty} \|g_k\|_{L^{p'_1}(G/H)}^r \|h_k\|_{L^{p_2}(G/H)}^r < \infty$$

and

$$\sum_{[\xi] \in \widehat{G/H}} d_\xi \operatorname{Tr}(\Gamma_\xi(xH) \sigma_T(xH, \xi) \Gamma_\xi(yH)^*) = \sum_{k=1}^{\infty} h_k(xH) g_k(yH).$$

Proof. Suppose that $T : L^{p_1}(G/H) \rightarrow L^{p_2}(G/H)$ is a r -nuclear operator for $1 \leq p_1, p_2 < \infty$. Then by Theorem 4.3, there exist sequences $\{g_k\}_{k=1}^\infty$ in $L^{p'_1}(G/H)$ and $\{h_k\}_{k=1}^\infty$ in $L^{p_2}(G/H)$ such that

$$\sum_{k=1}^{\infty} \|g_k\|_{L^{p'_1}(G/H)}^r \|h_k\|_{L^{p_2}(G/H)}^r < \infty$$

and

$$(\Gamma_\xi(xH)\sigma_T(xH, \xi))_{nm} = \sum_{k=1}^{\infty} h_k(xH) \overline{\left(\widehat{g_k}(\xi)\right)_{mn}}, \quad (xH, \xi) \in G/H \times \widehat{G/H}$$

for all n, m with $1 \leq n, m \leq d_\xi$. Let $yH \in G/H$. Then

$$\begin{aligned} (\Gamma_\xi(xH)\sigma_T(xH, \xi))_{nm} \overline{\Gamma_\xi(yH)_{nm}} &= \sum_{k=1}^{\infty} h_k(xH) \overline{\left(\widehat{g_k}(\xi)\right)_{mn}} \overline{\Gamma_\xi(yH)_{nm}} \\ &= \int_{G/H} \Gamma_\xi(zH)_{nm} \overline{\Gamma_\xi(yH)_{nm}} \sum_{k=1}^{\infty} h_k(xH) g_k(zH) d\mu(zH). \end{aligned}$$

consequently

$$\begin{aligned} &\sum_{m,n=1}^{d_\xi} (\Gamma_\xi(xH)\sigma_T(xH, \xi))_{nm} \overline{\Gamma_\xi(yH)_{nm}} \\ &= \int_{G/H} \left(\sum_{m,n=1}^{d_\xi} \Gamma_\xi(zH)_{nm} \overline{\Gamma_\xi(yH)_{nm}} \right) \sum_{k=1}^{\infty} h_k(xH) g_k(zH) d\mu(zH). \end{aligned}$$

Therefore, for all $xH, yH \in G/H$, we get

$$\begin{aligned} &\sum_{[\xi] \in \widehat{G/H}} d_\xi \operatorname{Tr}(\Gamma_\xi(xH)\sigma_T(xH, \xi)\Gamma_\xi(yH)^*) \\ &= \int_{G/H} \left(\sum_{[\xi] \in \widehat{G/H}} d_\xi \operatorname{Tr}(\Gamma_\xi(zH)\Gamma_\xi(yH)^*) \right) \sum_{k=1}^{\infty} h_k(xH) g_k(zH) d\mu(zH) \\ &= \int_{G/H} \left(\sum_{[\xi] \in \widehat{G/H}} d_\xi \overline{\operatorname{Tr}(\Gamma_\xi(yH)\Gamma_\xi(zH)^*)} \right) \sum_{k=1}^{\infty} h_k(xH) g_k(zH) d\mu(zH) \\ &= \sum_{k=1}^{\infty} h_k(xH) \int_{G/H} \left(\sum_{[\xi] \in \widehat{G/H}} d_\xi \overline{\operatorname{Tr}(\Gamma_\xi(yH)\Gamma_\xi(zH)^*)} g_k(zH) \right) \\ &= \sum_{k=1}^{\infty} h_k(xH) \int_{G/H} \left(\sum_{[\xi] \in \widehat{G/H}} d_\xi \overline{\operatorname{Tr}(\Gamma_\xi(yH)\Gamma_\xi(zH)^* g_k(zH))} \right) \end{aligned}$$

$$\begin{aligned}
&= \sum_{k=1}^{\infty} h_k(xH) \overline{\left(\sum_{[\xi] \in \widehat{G/H}} d_{\xi} \operatorname{Tr} \left(\Gamma_{\xi}(yH) \widehat{g}(\xi) \right) \right)} = \sum_{k=1}^{\infty} h_k(xH) \overline{\left(\overline{g(yH)} \right)} \\
&= \sum_{k=1}^{\infty} h_k(xH) g_k(yH)
\end{aligned}$$

for all xH, yH in G/H .

Conversely, let $\{g_k\}_{k=1}^{\infty}$ and $\{h_k\}_{k=1}^{\infty}$ be sequences in $L^{p'_1}(G/H)$ and $L^{p_2}(G/H)$ such that

$$\sum_{k=1}^{\infty} \|g_k\|_{L^{p'_1}(G/H)}^r \|h_k\|_{L^{p_2}(G/H)}^r < \infty$$

and for all xH and yH in G/H , we have

$$\sum_{[\xi] \in \widehat{G/H}} d_{\xi} \operatorname{Tr}(\Gamma_{\xi}(xH) \sigma(xH, \xi) \Gamma_{\xi}(yH)^*) = \sum_{k=1}^{\infty} h_k(xH) g_k(yH).$$

Then, for all $f \in L^{p_1}(G/H)$, we get

$$\begin{aligned}
(T_{\sigma} f)(xH) &= \sum_{[\xi] \in \widehat{G/H}} d_{\xi} \operatorname{Tr}(\Gamma_{\xi}(xH) \sigma(xH, \xi) \hat{f}(\xi)) \\
&= \sum_{[\xi] \in \widehat{G/H}} d_{\xi} \sum_{m,n=1}^{d_{\xi}} (\Gamma_{\xi}(xH) \sigma(xH, \xi))_{mn} \hat{f}(\xi)_{nm} \\
&= \int_{G/H} \left(\sum_{[\xi] \in \widehat{G/H}} d_{\xi} \sum_{m,n=1}^{d_{\xi}} (\Gamma_{\xi}(xH) \sigma(xH, \xi))_{mn} \overline{\Gamma_{\xi}(yH)_{mn}} \right) f(yH) d\mu(yH) \\
&= \int_{G/H} \left(\sum_{[\xi] \in \widehat{G/H}} d_{\xi} \operatorname{Tr}(\Gamma_{\xi}(xH) \sigma(xH, \xi) \Gamma_{\xi}(yH)^*) \right) f(yH) d\mu(yH) \\
&= \int_{G/H} \left(\sum_{k=1}^{\infty} h_k(xH) g_k(yH) \right) f(yH) d\mu(yH)
\end{aligned}$$

for all $xH \in G/H$. This completes the proof of the theorem. \square

An immediate consequence of Theorem 4.4 gives the trace of a r -nuclear pseudo-differential operator on $L^p(G/H)$ for $1 \leq p < \infty$. Indeed, we have the following result.

Corollary 4.5. *Let $0 < r \leq 1$ and let $T : L^p(G/H) \rightarrow L^p(G/H)$, $1 \leq p < \infty$, be a r -nuclear operator with matrix-valued symbol σ_T on $G/H \times \widehat{G/H}$. Then the nuclear trace of*

T is given by

$$\mathrm{Tr}(T) = \int_{G/H} \sum_{[\xi] \in \widehat{G/H}} d_\xi \mathrm{Tr}(T_H^\xi \sigma_T(xH, \xi)) d\mu(xH).$$

Proof. Using trace formula (4.1) and Theorem 4.4, we have

$$\begin{aligned} \mathrm{Tr}(T) &= \int_{G/H} \sum_{k=1}^{\infty} h_k(xH) g_k(xH) d\mu(xH) \\ &= \int_{G/H} \sum_{[\xi] \in \widehat{G/H}} d_\xi \mathrm{Tr}(\Gamma_\xi(xH) \sigma_T(xH, \xi) \Gamma_\xi(xH)^*). \end{aligned}$$

Since T_H^ξ is an orthogonal projection, we obtain

$$\mathrm{Tr}(T) = \int_{G/H} \sum_{[\xi] \in \widehat{G/H}} d_\xi \mathrm{Tr}(T_H^\xi \sigma_T(xH, \xi)) d\mu(xH).$$

□

5. ADJOINT AND PRODUCT OF r -NUCLEAR PSEUDO-DIFFERENTIAL OPERATORS

In this section we give a formula for symbols of the adjoints of r -nuclear pseudo-differential operators from $L^{p_1}(G/H)$ into $L^{p_2}(G/H)$ for $1 \leq p_1, p_2 < \infty$, where G is a compact Hausdorff group and H be a closed subgroup of G .

Theorem 5.1. *Let $0 < r \leq 1$ and let $T : L^{p_1}(G/H) \rightarrow L^{p_2}(G/H)$, $1 \leq p_1, p_2 < \infty$, be a r -nuclear continuous linear operator with matrix-valued symbol σ_T on $G/H \times \widehat{G/H}$. Then the adjoint T^* of T is also a r -nuclear operator from $L^{p_2'}(G/H)$ into $L^{p_1'}(G/H)$ with symbol τ given by*

$$T_H^\xi \tau(xH, \xi) = \xi(x)^* \sum_{k=1}^{\infty} \overline{g_k}(xH) \widehat{h_k}(\xi)^*, \quad (xH, \xi) \in G/H \times \widehat{G/H},$$

where $\{g_k\}_{k=1}^{\infty}$ and $\{h_k\}_{k=1}^{\infty}$ are two sequences in $L^{p_1'}(G/H)$ and $L^{p_2}(G/H)$ respectively such that

$$\sum_{k=1}^{\infty} \|h_k\|_{L^{p_2}(G/H)}^r \|g_k\|_{L^{p_1'}(G/H)}^r < \infty.$$

Proof. For $f \in L^{p_1}(G/H)$ and $g \in L^{p_2'}(G/H)$, from the definition of the adjoint of an operator, we have

$$\int_{G/H} (Tf)(xH) \overline{g(xH)} d\mu(xH) = \int_{G/H} f(xH) \overline{(T^*g)(xH)} d\mu(xH).$$

Therefore,

$$\begin{aligned}
(5.1) \quad & \int_{G/H} \left(\int_{G/H} \sum_{[\xi] \in \widehat{G/H}} d_\xi \sum_{m,n=1}^{d_\xi} (\Gamma_\xi(xH) \sigma_T(xH, \xi))_{mn} \right. \\
& \quad \left. \times \overline{\Gamma_\xi(yH)_{mn}} f(yH) d\mu(yH) \right) \overline{g(xH)} d\mu(xH) \\
&= \overline{\int_{G/H} f(xH) \left(\int_{G/H} \sum_{[\xi] \in \widehat{G/H}} d_\xi \sum_{m,n=1}^{d_\xi} (\Gamma_\xi(xH) \tau(xH, \xi))_{mn} \right.} \\
& \quad \left. \times \overline{\Gamma_\xi(yH)_{mn}} g(yH) d\mu(yH) \right) d\mu(xH)}.
\end{aligned}$$

Let γ and η be elements in $\widehat{G/H}$. Then for $1 \leq i, j \leq d_\gamma$ and $1 \leq p, q \leq d_\eta$, we define the functions f and g on G/H by

$$f(xH) = \Gamma_\gamma(xH)_{ij}, \quad xH \in G/H,$$

and

$$g(xH) = \Gamma_\eta(xH)_{pq}, \quad xH \in G/H.$$

Thus, from the relation (5.1), it follows that

$$\begin{aligned}
& \int_{G/H} (\Gamma_\gamma(xH) \sigma_T(xH, \gamma))_{ij} \overline{\Gamma_\eta(xH)_{pq}} d\mu(xH) \\
&= \int_{G/H} \Gamma_\gamma(xH)_{ij} \overline{(\Gamma_\eta(xH) \tau(xH, \eta))_{pq}} d\mu(xH)
\end{aligned}$$

and so

$$\begin{aligned}
& \overline{\int_{G/H} (\Gamma_\gamma(xH) \sigma_T(xH, \gamma))_{ij} \overline{\Gamma_\eta(xH)_{pq}} d\mu(xH)} \\
&= \int_{G/H} (\Gamma_\eta(xH) \tau(xH, \eta))_{pq} \overline{\Gamma_\gamma(xH)_{ij}} d\mu(xH).
\end{aligned}$$

This implies that

$$(5.2) \quad \overline{((\Gamma_\gamma(\cdot) \sigma_T(\cdot, \gamma))_{ij})^\wedge(\eta))_{qp}} = ((\Gamma_\eta(\cdot) \tau(\cdot, \eta))_{pq})^\wedge(\gamma))_{ji}$$

for $1 \leq i, j \leq d_\gamma$, $1 \leq p, q \leq d_\eta$ and all γ and η in $\widehat{G/H}$. Since $T : L^{p_1}(G/H) \rightarrow L^{p_2}(G/H)$ is r -nuclear, from Theorem 4.3, there exist sequences $\{g_k\}_{k=1}^\infty$ in $L^{p_1'}(G/H)$ and $\{h_k\}_{k=1}^\infty$ in $L^{p_2}(G/H)$ such that

$$\sum_{k=1}^\infty \|h_k\|_{L^{p_2}(G/H)}^r \|g_k\|_{L^{p_1'}(G/H)}^r < \infty$$

and for all $(yH, \gamma) \in G/H \times \widehat{G/H}$, we have

$$(\Gamma_\gamma(yH)\sigma_T(yH, \gamma))_{ij} = \sum_{k=1}^{\infty} h_k(yH) \overline{(\widehat{g}_k(\gamma))_{ji}}.$$

Then

$$\begin{aligned} (\Gamma_\eta(xH)\tau(xH, \eta))_{pq} &= \sum_{[\gamma] \in \widehat{G/H}} d_\gamma \operatorname{Tr}[\Gamma_\gamma(xH)((\Gamma_\eta(\cdot)\tau(\cdot, \eta))_{pq})^\wedge(\gamma)] \\ &= \sum_{[\gamma] \in \widehat{G/H}} \sum_{i,j=1}^{d_\gamma} d_\gamma (\Gamma_\gamma(xH))_{ij} ((\Gamma_\eta(\cdot)\tau(\cdot, \eta))_{pq})^\wedge(\gamma)_{ji}. \end{aligned}$$

for all $(xH, \eta) \in G/H \times \widehat{G/H}$. Therefore, by (5.2), for all $(xH, \eta) \in G/H \times \widehat{G/H}$, we get

$$\begin{aligned} &(\Gamma_\eta(xH)\tau(xH, \eta))_{pq} \\ &= \sum_{[\gamma] \in \widehat{G/H}} \sum_{i,j=1}^{d_\gamma} d_\gamma (\Gamma_\gamma(xH))_{ij} \overline{((\Gamma_\gamma(\cdot)\sigma_T(\cdot, \gamma))_{ij})^\wedge(\eta)_{qp}} \\ &= \sum_{[\gamma] \in \widehat{G/H}} \sum_{i,j=1}^{d_\gamma} d_\gamma (\Gamma_\gamma(xH))_{ij} \int_{G/H} \overline{(\Gamma_\gamma(yH)\sigma_T(yH, \gamma))_{ij}} \Gamma_\eta(yH)_{pq} d\mu(yH) \\ &= \sum_{[\gamma] \in \widehat{G/H}} \sum_{i,j=1}^{d_\gamma} d_\gamma (\Gamma_\gamma(xH))_{ij} \int_{G/H} \sum_{k=1}^{\infty} \overline{h_k(yH)(\widehat{g}_k(\gamma))_{ji}} \Gamma_\eta(yH)_{pq} d\mu(yH) \\ &= \sum_{k=1}^{\infty} \left(\int_{G/H} \overline{h_k(yH)} \Gamma_\eta(yH)_{pq} d\mu(yH) \right) \sum_{[\gamma] \in \widehat{G/H}} \sum_{i,j=1}^{d_\gamma} d_\gamma (\Gamma_\gamma(xH))_{ij} (\widehat{g}_k(\gamma))_{ji} \\ &= \sum_{k=1}^{\infty} \overline{\widehat{h}_k(\eta)_{qp}} \sum_{[\gamma] \in \widehat{G/H}} \sum_{i,j=1}^{d_\gamma} d_\gamma (\Gamma_\gamma(xH))_{ij} (\widehat{g}_k(\gamma))_{ji} \\ &= \sum_{k=1}^{\infty} \overline{\widehat{h}_k(\eta)_{qp}} \sum_{[\gamma] \in \widehat{G/H}} d_\gamma \operatorname{Tr}(\Gamma_\gamma(xH) \widehat{g}_k(\gamma)) \\ &= \sum_{k=1}^{\infty} \overline{\widehat{h}_k(\eta)_{qp}} \widehat{g}_k(xH) = \sum_{k=1}^{\infty} \widehat{h}_k(\eta)^*_{pq} \widehat{g}_k(xH) \end{aligned}$$

for all $1 \leq p, q \leq d_\eta$. Thus, for all $(xH, \eta) \in G/H \times \widehat{G/H}$, we get

$$\Gamma_\eta(xH)\tau(xH, \eta) = \sum_{k=1}^{\infty} \widehat{h}_k(\eta)^* \widehat{g}_k(xH)$$

and hence

$$T_H^\eta \tau(xH, \eta) = \eta(x)^* \sum_{k=1}^{\infty} \widehat{h_k}(\eta)^* \overline{g_k}(xH).$$

□

As an application of Theorem 4.3 and Theorem 5.1, in the next corollary, we give a criterion for the self-adjointness of r -nuclear pseudo-differential operators.

Corollary 5.2. *Let $0 < r \leq 1$ and let $T : L^2(G/H) \rightarrow L^2(G/H)$ be a r -nuclear continuous linear operator with matrix-valued symbol σ_T on $G/H \times \widehat{G/H}$. Then T is self-adjoint if and only if there exist sequences $\{g_k\}_{k=1}^{\infty}$ and $\{h_k\}_{k=1}^{\infty}$ in $L^2(G/H)$ such that*

$$\sum_{k=1}^{\infty} \|h_k\|_{L^2(G/H)}^r \|g_k\|_{L^2(G/H)}^r < \infty,$$

$$\sum_{k=1}^{\infty} h_k(xH) \widehat{g_k}(\xi)^* = \sum_{k=1}^{\infty} \widehat{h_k}(\xi)^* \overline{g_k}(xH), \quad (xH, \xi) \in G/H \times \widehat{G/H},$$

and

$$T_H^\xi \sigma_T(xH, \xi) = \xi(x)^* \sum_{k=1}^{\infty} h_k(xH) \widehat{g_k}(\xi)^*, \quad (xH, \xi) \in G/H \times \widehat{G/H}.$$

We can give another formula for the adjoints of r -nuclear operators in terms of symbols. Indeed, we have the following theorem.

Theorem 5.3. *Let $0 < r \leq 1$. Let σ_T be a matrix-valued function on $G/H \times \widehat{G/H}$ such that the corresponding pseudo-differential operator $T : L^{p_1}(G/H) \rightarrow L^{p_2}(G/H)$ is r -nuclear for $1 \leq p_1, p_2 < \infty$. Then the symbol τ of the adjoint $T^* : L^{p'_2}(G/H) \rightarrow L^{p'_1}(G/H)$ is given by*

$$T_H^\xi \tau(xH, \xi) = \xi(x)^* \sum_{[\eta] \in \widehat{G/H}} d_\eta \int_{G/H} \text{Tr}[(\Gamma_\eta(yH) \sigma_T(yH, \eta))^* \Gamma_\eta(xH)] \Gamma_\xi(yH) d\mu(yH)$$

which is eventually same as

$$T_H^\xi \tau(xH, \xi) = \xi(x)^* \sum_{[\eta] \in \widehat{G/H}} d_\eta \left(\overline{\text{Tr}(\sigma_T(\cdot, \eta)^* \Gamma_\eta(\cdot)^* \Gamma_\eta(xH))}^\wedge(\xi) \right)^*$$

for all $(xH, \xi) \in G/H \times \widehat{G/H}$.

Proof. Suppose that $T : L^{p_1}(G/H) \rightarrow L^{p_2}(G/H)$ is r -nuclear operator for $1 \leq p_1, p_2 < \infty$. Then by Theorem 4.3, there exist sequences $\{g_k\}_{k=1}^{\infty}$ in $L^{p'_1}(G/H)$ and $\{h_k\}_{k=1}^{\infty}$ in $L^{p_2}(G/H)$ such that

$$\sum_{k=1}^{\infty} \|g_k\|_{L^{p'_1}(G/H)}^r \|h_k\|_{L^{p_2}(G/H)}^r < \infty$$

and for all $(yH, \eta) \in G/H \times \widehat{G/H}$, we have

$$\Gamma_\eta(yH)\sigma_T(yH, \eta) = \sum_{k=1}^{\infty} h_k(yH)\widehat{g_k}(\eta)^*$$

or

$$(\Gamma_\eta(yH)\sigma_T(yH, \eta))^* = \sum_{k=1}^{\infty} \overline{h_k(yH)}\widehat{g_k}(\eta).$$

Let $(xH, \xi) \in G/H \times \widehat{G/H}$. Then

$$\begin{aligned} & \int_{G/H} \text{Tr}[(\Gamma_\eta(yH)\sigma_T(yH, \eta))^*\Gamma_\eta(xH)]\Gamma_\xi(yH) \, d\mu(yH) \\ &= \int_{G/H} \text{Tr} \left[\sum_{k=1}^{\infty} \overline{h_k(yH)}\widehat{g_k}(\eta)\Gamma_\eta(xH) \right] \Gamma_\xi(yH) \, d\mu(yH) \\ &= \sum_{k=1}^{\infty} \text{Tr}(\widehat{g_k}(\eta)\Gamma_\eta(xH)) \int_{G/H} \overline{h_k(yH)}\Gamma_\xi(yH) \, d\mu(yH) \\ &= \sum_{k=1}^{\infty} \widehat{h_k}(\xi)^* \text{Tr}[\widehat{g_k}(\eta)\Gamma_\eta(xH)]. \end{aligned}$$

Then by Theorem 5.1, we obtain

$$\begin{aligned} & \sum_{[\eta] \in \widehat{G/H}} d_\eta \int_{G/H} \text{Tr}[(\Gamma_\eta(yH)\sigma_T(yH, \eta))^*\Gamma_\eta(xH)]\Gamma_\xi(yH) \, d\mu(yH) \\ &= \sum_{[\eta] \in \widehat{G/H}} d_\eta \left(\sum_{k=1}^{\infty} \widehat{h_k}(\xi)^* \text{Tr}[\widehat{g_k}(\eta)\Gamma_\eta(xH)] \right) \\ &= \sum_{k=1}^{\infty} \widehat{h_k}(\xi)^* \sum_{[\eta] \in \widehat{G/H}} d_\eta \text{Tr}[\widehat{g_k}(\eta)\Gamma_\eta(xH)] \\ &= \sum_{k=1}^{\infty} \widehat{h_k}(\xi)^* \overline{g_k}(xH) = \Gamma_\xi(xH)\tau(xH, \xi) \end{aligned}$$

for all $(xH, \xi) \in G/H \times \widehat{G/H}$. □

Another criterion for the self-adjointness of r -nuclear pseudo-differential operators on homogeneous space of compact groups is as follows.

Corollary 5.4. *Let $0 < r \leq 1$. Let σ_T be a matrix-valued function on $G/H \times \widehat{G/H}$ such that $T : L^2(G/H) \rightarrow L^2(G/H)$ is r -nuclear. Then $T : L^2(G/H) \rightarrow L^2(G/H)$ is self-adjoint if and only if*

$$T_H^\xi \sigma_T(xH, \xi) = \xi(x)^* \sum_{[\eta] \in \widehat{G/H}} d_\eta \left(\overline{\text{Tr}(\sigma_T(\cdot, \eta)^* \Gamma_\eta(\cdot)^* \Gamma_\eta(xH))}^\wedge(\xi) \right)^*$$

for all $(xH, \xi) \in G/H \times \widehat{G/H}$.

Next, we show that the product of a nuclear pseudo-differential operator on $L^p(G/H)$ with a bounded operator again a nuclear pseudo-differential operator on $L^p(G/H)$ for $1 \leq p < \infty$, where G is a compact (Hausdorff) group and H is a closed subgroup of G . We present a formula for the symbol of the product operator. The main theorem of this section is the following one.

Theorem 5.5. *Let $T : L^p(G/H) \rightarrow L^p(G/H)$, $1 \leq p < \infty$, be a nuclear operator with a matrix valued symbol σ_T and let $S : L^p(G/H) \rightarrow L^p(G/H)$ be a bounded linear operator with symbol σ_S . Then $ST : L^p(G/H) \rightarrow L^p(G/H)$ is a nuclear operator with symbol λ given by*

$$T_H^\xi \lambda(xH, \xi) = \xi(x)^* \sum_{k=1}^{\infty} h'_k(xH) \widehat{g_k}(\xi)^*$$

for all $(xH, \xi) \in G/H \times \widehat{G/H}$, where $\{g_k\}_{k=1}^{\infty}$ and $\{h_k\}_{k=1}^{\infty}$ are two sequences in $L^{p'}(G/H)$ and $L^p(G/H)$ respectively such that $\sum_{k=1}^{\infty} \|g_k\|_{L^{p'}(G/H)} \|h_k\|_{L^p(G/H)} < \infty$ with

$$h'_k(xH) = \sum_{[\eta] \in \widehat{G/H}} d_\eta \operatorname{Tr} \left[\Gamma_\eta(xH) \sigma_S(xH, \eta) \widehat{h_k}(\eta) \right], \quad xH \in G/H.$$

Proof. Since $T : L^p(G/H) \rightarrow L^p(G/H)$ is a nuclear pseudo-differential operator for $1 \leq p < \infty$, from Theorem 4.3, there exist sequences $\{g_k\}_{k=1}^{\infty} \in L^{p'}(G/H)$ and $\{h_k\}_{k=1}^{\infty} \in L^p(G/H)$ such that

$$\sum_{k=1}^{\infty} \|g_k\|_{L^{p'}(G/H)} \|h_k\|_{L^p(G/H)} < \infty$$

and

$$T_H^\xi \sigma_T(xH, \xi) = \xi(x)^* \sum_{k=1}^{\infty} h_k(xH) \widehat{g_k}(\xi)^*, \quad (xH, \xi) \in G/H \times \widehat{G/H}.$$

Let $f \in L^p(G/H)$. Then

$$\begin{aligned} (STf)(xH) &= \sum_{[\eta] \in \widehat{G/H}} d_\eta \operatorname{Tr}(\Gamma_\eta(xH) \sigma_S(xH, \eta) \widehat{Tf}(\eta)) \\ &= \sum_{[\eta] \in \widehat{G/H}} d_\eta \operatorname{Tr} \left[\Gamma_\eta(xH) \sigma_S(xH, \eta) \left(\int_{G/H} Tf(yH) \Gamma_\eta(yH)^* d\mu(yH) \right) \right] \\ &= \sum_{[\eta] \in \widehat{G/H}} d_\eta \operatorname{Tr} [\Gamma_\eta(xH) \sigma_S(xH, \eta) \\ &\quad \times \int_{G/H} \left(\sum_{[\xi] \in \widehat{G/H}} d_\xi \operatorname{Tr}(\Gamma_\xi(yH) \sigma_T(yH, \xi) \widehat{f}(\xi)) \right) \Gamma_\eta(yH)^* d\mu(yH) \end{aligned}$$

for all $xH \in G/H$. Using the nuclearity of T , we have

$$\begin{aligned}
(STf)(xH) &= \sum_{[\eta] \in \widehat{G/H}} d_\eta \operatorname{Tr} \left[\Gamma_\eta(xH) \sigma_S(xH, \eta) \int_{G/H} \sum_{[\xi] \in \widehat{G/H}} d_\xi \right. \\
&\quad \times \operatorname{Tr} \left(\sum_{k=1}^{\infty} h_k(yH) \widehat{g_k}(\xi)^* \widehat{f}(\xi) \right) \Gamma_\eta(yH)^* d\mu(yH) \Big] \\
&= \sum_{[\eta] \in \widehat{G/H}} d_\eta \operatorname{Tr} \left[\Gamma_\eta(xH) \sigma_S(xH, \eta) \sum_{[\xi] \in \widehat{G/H}} \sum_{k=1}^{\infty} d_\xi \operatorname{Tr} \left(\widehat{g_k}(\xi)^* \widehat{f}(\xi) \right) \right. \\
&\quad \times \int_{G/H} h_k(yH) \Gamma_\eta(yH)^* d\mu(yH) \Big] \\
&= \sum_{[\eta] \in \widehat{G/H}} d_\eta \operatorname{Tr} \left[\Gamma_\eta(xH) \sigma_S(xH, \eta) \sum_{[\xi] \in \widehat{G/H}} \sum_{k=1}^{\infty} d_\xi \widehat{h_k}(\eta) \operatorname{Tr} \left(\widehat{g_k}(\xi)^* \widehat{f}(\xi) \right) \right] \\
&= \sum_{[\eta] \in \widehat{G/H}} \sum_{k=1}^{\infty} \sum_{[\xi] \in \widehat{G/H}} d_\xi d_\eta \operatorname{Tr} \left[\Gamma_\eta(xH) \sigma_S(xH, \eta) \widehat{h_k}(\eta) \right] \operatorname{Tr} \left(\widehat{g_k}(\xi)^* \widehat{f}(\xi) \right) \\
&= \sum_{[\xi] \in \widehat{G/H}} d_\xi \operatorname{Tr} \left(\Gamma_\xi(xH) \lambda(xH, \xi) \widehat{f}(\xi) \right),
\end{aligned}$$

where

$$\begin{aligned}
T_H^\xi \lambda(xH, \xi) &= \xi(x)^* \sum_{k=1}^{\infty} \sum_{[\eta] \in \widehat{G/H}} d_\eta \operatorname{Tr} \left[\Gamma_\eta(xH) \sigma_S(xH, \eta) \widehat{h_k}(\eta) \right] \widehat{g_k}(\xi)^* \\
&= \xi(x)^* \sum_{k=1}^{\infty} h'_k(xH) \widehat{g_k}(\xi)^*
\end{aligned}$$

for all $(xH, \xi) \in G/H \times \widehat{G/H}$ and

$$h'_k(xH) = \sum_{[\eta] \in \widehat{G/H}} d_\eta \operatorname{Tr} \left[\Gamma_\eta(xH) \sigma_S(xH, \eta) \widehat{h_k}(\eta) \right], \quad xH \in G/H.$$

□

6. APPLICATIONS TO THE HEAT KERNELS ON G/H

In this section, we assume that G is a compact Lie group and H is a closed subgroup of G . Let \mathcal{L}_G be the Laplace-Beltrami operator (or the Casimir element of the universal

enveloping algebra) on G . For every $[\xi] \in \widehat{G}$, the matrix elements of ξ are the eigenfunctions of \mathcal{L}_G with same eigenvalue denoted by $-\lambda_{[\xi]}^2$. Therefore,

$$-\mathcal{L}_G \xi_{ij} = -\lambda_{[\xi]}^2 \xi_{ij} \quad \text{for all } 1 \leq i, j \leq d_\xi.$$

Let $-\mathcal{L}_{G/H} : C^\infty(G/H) \rightarrow C^\infty(G/H)$ be the differential operator on G/H obtained by $-\mathcal{L}_G$ acting on functions that are constant on cosets of G , i.e., such that $-\widetilde{\mathcal{L}_{G/H}} f = -\mathcal{L}_G \tilde{f}$ for $f \in C^\infty(G/H)$, where for $f \in C^\infty(G/H)$, $\tilde{f} \in C^\infty(G)$ is the lifting of f given by $\tilde{f}(x) = f(xH)$. The operator $-\mathcal{L}_{G/H}$ has the eigenspace $\Gamma_{\xi_{ij}}(xH)$ for $1 \leq i, j \leq d_\xi$ corresponding to the common eigenvalue $-\lambda_{[\xi]}^2$. For more details on $-\mathcal{L}_{G/H}$, see [35]. We make use the symbol of the heat kernel $e^{-t\mathcal{L}_{G/H}}$. Indeed, by taking into account $\sigma_{e^{-t\mathcal{L}_{G/H}}}(xH, \xi) = e^{-t|\xi|^2} T_H^\xi$, where $|\xi| = \lambda_{[\xi]}^2$, we have

$$\begin{aligned} e^{-t\mathcal{L}_{G/H}} f(xH) &= \sum_{[\xi] \in \widehat{G/H}} d_\xi \text{Tr}(\Gamma_\xi(xH) \sigma_{e^{-t\mathcal{L}_{G/H}}}(xH, \xi) \widehat{f}(\xi)) \\ &= \sum_{[\xi] \in \widehat{G/H}} d_\xi \text{Tr}(\Gamma_\xi(xH) e^{-t\lambda_{[\xi]}^2} T_H^\xi \widehat{f}(\xi)) \\ &= \sum_{[\xi] \in \widehat{G/H}} d_\xi e^{-t\lambda_{[\xi]}^2} \text{Tr}(\Gamma_\xi(xH) \widehat{f}(\xi)). \end{aligned}$$

Now, we show that the nuclearity of the heat kernel on L^p -spaces.

Theorem 6.1. *Let G be a compact Lie group and let H be closed subgroup of G . Then the heat kernel $e^{-t\mathcal{L}_{G/H}} : L^{p_1}(G/H) \rightarrow L^{p_2}(G/H)$ is nuclear for every $t > 0$ and all $1 \leq p_1, p_2 < \infty$. Moreover, if $0 < r \leq 1$, then $e^{-t\mathcal{L}_{G/H}} : L^p(G/H) \rightarrow L^p(G/H)$ is r -nuclear operator for every $t > 0$ and $1 \leq p < \infty$. In particular, on each $L^p(G/H)$, we have the following nuclear trace formula*

$$\text{Tr}(e^{-t\mathcal{L}_{G/H}}) = \sum_{[\xi] \in \widehat{G/H}} d_\xi e^{-t\lambda_{[\xi]}^2} \text{Tr}(T_H^\xi).$$

Proof. The kernel of $e^{-t\mathcal{L}_{G/H}}$ is given by

$$\begin{aligned} K_t(x, y) &= \sum_{[\xi] \in \widehat{G/H}} d_\xi e^{-t\lambda_{[\xi]}^2} \text{Tr}(\Gamma_\xi(xH) \Gamma(yH)^*) \\ &= \sum_{[\xi] \in \widehat{G/H}} d_\xi e^{-t\lambda_{[\xi]}^2} \text{Tr}(\Gamma_\xi(xH) \xi(y)^*) \end{aligned}$$

with

$$\text{Tr}(\Gamma_\xi(xH) \xi(y)^*) = \sum_{i,j}^{d_\xi} \Gamma_\xi(xH)_{ij} \overline{\xi(y)}_{ij}.$$

We set

$$h_{\xi,ij} = d_\xi e^{-t\lambda_{[\xi]}^2} \Gamma_\xi(xH)_{ij} \quad g_{\xi,ij} = \overline{\xi(y)}_{ij}.$$

Let p'_1 denotes the Lebesgue conjugate of p_1 and $\tilde{q}_1 = \max\{2, p'_1\}$. Then by Lemma 2.4, we get

$$\|g_{\xi,ij}\|_{L^{p'_1}(G/H)} = \|\bar{\xi}_{ij}\|_{L^{p'_1}(G/H)} \leq \|\Gamma_{\xi}(\cdot)_{ij}\|_{L^{p'_1}(G/H)} \leq d_{\xi}^{-\frac{1}{\tilde{q}_1}}.$$

Also, we have

$$\begin{aligned} \|h_{\xi,ij}\|_{L^{p_2}(G/H)} &= \|d_{\xi} e^{-t\lambda_{[\xi]}^2} \Gamma_{\xi}(xH)_{ij}\|_{L^{p_2}(G/H)} \\ &\leq \|d_{\xi} e^{-t\lambda_{[\xi]}^2} \|\Gamma_{\xi}(\cdot)\|_{op}\|_{L^{p_2}(G/H)} \leq d_{\xi} e^{-t\lambda_{[\xi]}^2}. \end{aligned}$$

Therefore,

$$\sum_{[\xi], i, j} \|g_{\xi,ij}(\cdot)\|_{L^{p'_1}(G/H)} \|h_{\xi,ij}(\cdot)\|_{L^{p_2}(G/H)} \leq \sum_{[\xi] \in \widehat{G/H}} d_{\xi}^2 d_{\xi}^{-\frac{1}{\tilde{q}_1}} e^{-t\lambda_{[\xi]}^2} < \infty,$$

where the last convergence follows from any of the Weyl formula, see, for example [10]. Therefore, $e^{-t\mathcal{L}_{G/H}}$ is a nuclear operator. Similarly one can prove r -nuclearity of $e^{-t\mathcal{L}_{G/H}}$.

By Corollary 4.5 and by using the fact that measure μ on G/H is normalized, the nuclear trace formula of $e^{-t\mathcal{L}_{G/H}}$ given by

$$\mathrm{Tr}(e^{-t\mathcal{L}_{G/H}}) = \int_{G/H} \sum_{[\xi] \in \widehat{G/H}} d_{\xi} \mathrm{Tr}(e^{-t\lambda_{[\xi]}^2} T_H^{\xi}) = \sum_{[\xi] \in \widehat{G/H}} d_{\xi} e^{-t\lambda_{[\xi]}^2} \mathrm{Tr}(T_H^{\xi}).$$

□

ACKNOWLEDGEMENT

The authors thank the referee for several suggestions leading to the improvement of the manuscript. Vishvesh Kumar is supported by the FWO Odysseus 1 grant G.0H94.18N: Analysis and Partial Differential Equations and Partial Differential Equations and by the Methusalem programme of the Ghent University Special Research Fund (BOF) (Grant number 01M01021). The authors thank Prof. Michael Ruzhansky for providing many valuable comments. Vishvesh Kumar thanks Duván Cardona for many fruitful discussions. Shyam Swarup Mondal thanks the Council of Scientific and Industrial Research, India, for providing financial support. He also thanks his supervisor Jitendriya Swain for his support and encouragement.

REFERENCES

- [1] D. Applebaum, *Probability on Compact Lie Groups*, With a foreword by Herbert Heyer. Probability Theory and Stochastic Modelling, vol. 70. Springer, Cham (2014).
- [2] E. Buzano and J. Toft, Schatten-von Neumann properties in the Weyl calculus, *J. Funct. Anal.* 259(12), 3080-3114 (2010).
- [3] D. Cardona, On the nuclear trace of Fourier Integral Operators, *Rev. Integr. temas Mat.* 37(2), 219-249 (2019).

- [4] D. Cardona and V. Kumar, Multilinear analysis for discrete and periodic pseudo-differential operators in L^p spaces, *Rev. Integr. temas Mat.* 36(2), 151-164 (2018).
- [5] D. Cardona and V. Kumar, L^p -boundedness and L^p -nuclearity of multilinear pseudo-differential operators on \mathbb{Z}^n and the torus \mathbb{T}^n , *J. Fourier Anal. Appl.* 25(6), 2973-3017 (2019).
- [6] T. Carleman, Über die Fourier koeffizienten einer stetigen Funktion, *Acta Math.* 41(1), 377-384 (1916).
- [7] Z. Chen and M. W. Wong, Traces of pseudo-differential operators on S^{n-1} , *J. Pseudo Differ. Oper. Appl.* 4(1), 13-24 (2013).
- [8] D. Connolly, Pseudo-differential operators on Homogeneous Spaces, *doctoral dissertation*, Imperial College London, (2013).
- [9] A. Dasgupta and V. Kumar, Hilbert-Schmidt and Trace Class Pseudo-differential Operators on the Abstract Heisenberg Group, *J. Math. Anal. Appl.* 486(12), 123936 (2020).
- [10] A. Dasgupta and M. Ruzhansky, Gevrey functions and ultra distributions on compact Lie groups and homogeneous spaces, *Bull. Sci. math.* 138(6), 756-782 (2014).
- [11] A. Dasgupta and M. W. Wong, Pseudo-differential operators on the affine group, *Pseudo-differential operators: groups, geometry and applications*, Trends Math., Birkhäuser/Springer, Cham, 1-14 (2017).
- [12] J. Delgado, The trace of nuclear operators on $L^p(\mu)$ for σ -finite Borel measures on second countable spaces, *Integral Equ. Oper. Theory* 68(1), 61-74 (2010).
- [13] J. Delgado and M. Ruzhansky, L^p -nuclearity, traces, and Grothendieck-Lidskii formula on compact Lie groups, *J. Math. Pures Appl.*(9) 102(1), 153-172 (2014).
- [14] J. Delgado, M. Ruzhansky and N. Tokmagambetov, Schatten classes, nuclearity and nonharmonic analysis on compact manifolds with boundary, *J. Math. Pures Appl.*(9) 107(6), 758-783 (2017).
- [15] J. Delgado and M. Ruzhansky, Schatten classes and traces on compact groups, *Math. Res. Lett.* 24(4), 979-1003 (2017).
- [16] J. Delgado and M. Ruzhansky, Fourier multipliers, symbols and nuclearity on compact manifolds, *J. Anal. Math.* 135(2), 757-800 (2018).
- [17] J. Delgado and M. W. Wong, L^p -nuclear pseudo-differential operators on \mathbb{Z} and \mathbb{S}^1 , *Proc. Amer. Math. Soc.* 141(11), 3935-3942 (2013).
- [18] A. Ghani Farashahi, Abstract operator-valued Fourier transforms over homogeneous spaces of compact groups, *Groups Geom. Dyn.* 11(4), 1437-1467 (2017).

- [19] M. B. Ghaemi, M. Jamalpourbirgani and M. W. Wong, Characterization of nuclear pseudo-differential operators on \mathbb{S}^1 with applications to adjoints and products, *J. Pseudo-Differ. Oper. Appl.* 8, 191-201 (2017).
- [20] M. B. Ghaemi, M. Jamalpourbirgani and M. W. Wong, Characterizations, adjoints and products of nuclear pseudo-differential operators on compact and Hausdorff groups, *Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys.* 79(4), 207-220 (2017).
- [21] A. Grothendieck, Produits Tensoriels Topologiques et Espaces Nucléaires, *Mem. Amer. Math. Soc.* 16, (1955).
- [22] A. Grothendieck, La théorie de Fredholm, *Bull. Soc. Math. France* 84, 319-384 (1956).
- [23] L. Hörmander, *The Analysis of Linear Partial Differential Operators. III*, Springer-Verlag, Berlin (1985).
- [24] A. A. Kirillov and A. D. Gvishiani, *Teoremy i zadachi funktsionalnogo analiza*, 2nd edn. Nauka, Moscow (1988).
- [25] V. V. Kisil, Relative convolutions I. Properties and applications, *Adv. Math.* 147(1), 35-73 (1999).
- [26] V. V. Kisil, *Geometry of Möbius transformations. Elliptic, parabolic and hyperbolic actions of $SL_2(\mathbb{R})$* , Imperial College Press, London (2012).
- [27] V. V. Kisil, Erlangen program at large: an overview, *In: Advances in applied analysis*, Trends Math., Birkhäuser/Springer Basel AG, Basel, 1-94 (2012).
- [28] V. V. Kisil, Calculus of operators: covariant transform and relative convolutions, *Banach J. Math. Anal.* 8(2), 156-184 (2014).
- [29] J. J. Kohn and L. Nirenberg, An algebra of pseudo-differential operators, *Comm. Pure Appl. Math.* 18, 269-305 (1965).
- [30] V. Kumar, Pseudo-differential operators on homogeneous spaces of compact and Hausdorff groups, *Forum Math.* 31(2), 275-282 (2019).
- [31] V. Kumar and M. W. Wong, C^* -algebras, H^* -algebras and trace ideals of pseudo-differential operators on locally compact, Hausdorff and abelian groups, *J. Pseudo-Differ. Oper. Appl.* 10(2), 269-283 (2019).
- [32] R. Lipsman, Non-abelian Fourier analysis, *Bull. Sci. Math.*(2) 98(4), 209-233 (1974).
- [33] S. Molahajloo and M. Pirhayati, Traces of pseudo-differential operators on compact and Hausdorff groups, *J. Pseudo-Differ. Oper. Appl.* 4(3), 361-369 (2013).
- [34] S. Molahajloo and K. L. Wong, Pseudo-differential operators on finite abelian groups, *J. Pseudo-Differ. Oper. Appl.* 6(1), 1-9 (2015).

- [35] E. Nursultanov, M. Ruzhansky and S. Tikhonov, Nikolskii inequality and Besov, Triebel-Lizorkin, Wiener and Beurling spaces on compact homogeneous manifolds, *Ann. Sc. Norm. Super. Pisa Cl. Sci.(5)* 16(3), 981-1017 (2016).
- [36] R. Oloff, p -normierte Operatorenideale, *Beiträge Anal.* (4), 105-108 (1972).
- [37] A. Pietsch, Grothendieck's concept of a p -nuclear operator, *Integral Equ. Oper. Theory* 7(2), 282-284 (1984).
- [38] M. Ruzhansky and V. Turunen, *Pseudo-differential Operators and Symmetries: Background Analysis and Advanced Topics*. Birkhäuser-Verlag, Basel (2010).
- [39] M. Ruzhansky and V. Turunen, Global quantization of pseudo-differential operators on compact Lie groups, $SU(2)$, 3-Sphere, and homogeneous spaces, *Int. Math. Res. Not. IMRN* (11), 2439-2496 (2013).
- [40] M. A. Shubin, *Pseudo-differential operators and spectral theory*, second edition, Springer-Verlag, Berlin (2001).
- [41] J. Toft, Schatten-von Neumann properties in the Weyl calculus and calculus of metrics on symplectic vector space, *Ann. Global Anal. Geom.* 30(2), 169-209 (2006).
- [42] J. Toft, Schatten properties for pseudo-differential on modulation spaces, In *Pseudo-differential operators*, volume 1949 of *Lecture Notes in Math.*, pages 175-202. Springer, Berlin (2008).
- [43] N. Ja. Vilenkin and A. U. Klimyk, *Representation of Lie Groups and Special Functions. Vol. 1. Simplest Lie Groups, Special Functions and Integral Transforms*, Kluwer Academic Publishers Group, Dordrecht (1991).
- [44] M. W. Wong, *An Introduction to Pseudo-Differential Operators*, 3rd edn., World Scientific, Singapore (2014).

VISHVESH KUMAR, PH. D.
 DEPARTMENT OF MATHEMATICS: ANALYSIS, LOGIC AND DISCRETE MATHEMATICS
 GHENT UNIVERSITY
 KRIJGSLAAN 281, BUILDING S8, B 9000 GHENT, BELGIUM .
Email address: vishveshmishra@gmail.com

SHYAM SWARUP MONDAL
 DEPARTMENT OF MATHEMATICS
 IIT GUWAHATI
 GUWAHATI, ASSAM, INDIA.
Email address: mondalshyam055@gmail.com