Recent advances in holography have led to the formulation of fluid-gravity duality, a remarkable connection between the hydrodynamics of certain strongly coupled media and dynamics of higher dimensional black holes. This paper introduces a correspondence between phenomenologically defined entropy currents in relativistic hydrodynamics and “generalized horizons” of near-equilibrium black objects in a dual gravitational description. A general formula is given, expressing the divergence of the entropy current in terms of geometric objects which appear naturally in the gravity dual geometry. The proposed definition is explicitly covariant with respect to boundary diffeomorphisms and reproduces known results when evaluated for the event horizon.

I. INTRODUCTION

The holographic representation of field theories developed following the original formulation of the AdS/CFT correspondence [1] has led to an effective description of strongly coupled media in, as well as out of, equilibrium. The best understood example of the latter type is provided by fluid-gravity duality [2], which maps the hydrodynamics of strongly coupled holographic plasmas to the long-wavelength dynamics of black branes appearing as classical solutions of string or M theory. This description has found remarkable applications to the physics of QCD plasma experimentally studied at the Relativistic Heavy Ion Collider and the LHC.

From the hydrodynamics framework comes the notion of an entropy current: a phenomenological generalization of equilibrium entropy. Such a current is constructed in the hydrodynamic gradient expansion by requiring that it reproduces thermodynamic entropy in equilibrium situations and has non-negative divergence when evaluated on solutions of the hydrodynamic equations. However, these requirements are not sufficient to specify a unique current and beyond leading order in gradients there appears to be a family of currents [3].

The aim of this paper is to provide the dual interpretation of ambiguities affecting the local rate of entropy production within the framework of fluid-gravity duality. This issue was first raised in [4], where a definition of the hydrodynamic entropy current was given in terms of the area form on the black brane event horizon. A key element of that proposal is that the divergence of such an entropy current is guaranteed to be non-negative by the area increase theorem. The event horizon is an IR concept, and in order to associate points on the event horizon with points at the boundary, a so-called bulk-boundary map is needed. A choice of the bulk-boundary map clearly will not affect the non-negativity of the entropy current’s divergence, but it can modify the value of the divergence. The ambiguity in the divergence of the entropy current constructed in [4] was entirely due to nonuniqueness of such a bulk-boundary map.

There is however a more fundamental source of ambiguity, which comes from the fact that, apart from the event horizon, there are other hypersurfaces for which an area increase theorem holds. For example, in the gravity literature much attention has been devoted to the study of so-called quasilocal horizons, since in contrast to the event horizon they may be defined without determining the full causal structure of spacetime (see [5,6] for reviews). In static situations one of the quasilocal horizons typically coincides with the event horizon. In dynamical cases these notions differ, but in the near-equilibrium regime relevant to fluid-gravity duality they are expected to be close [7,8]. It is thus natural to expect that quasilocal horizons lead to different hydrodynamic entropy currents in the boundary field theory [4,9]. Furthermore, other bulk surfaces may also play a role in the phenomenological construction of
the entropy current in the dual field theory. In particular, timelike surfaces (often considered in the context of the membrane paradigm) located outside (but close to) the event horizon may also lead to acceptable entropy currents, as discussed in [10] for the case of boost-invariant hydrodynamics [11,12].

This paper examines the asymptotically AdS gravity dual to second order conformal fluid dynamics in arbitrary dimension [13] and demonstrates a correspondence between boundary entropy currents and bulk hypersurfaces which satisfy generalized area theorems and asymptote to the event horizon at late time. Such hypersurfaces (“generalized horizons”) provide a new source of ambiguity in dual hydrodynamic entropy currents.

II. GRAVITATIONAL NOTIONS OF ENTROPY

In its simplest form, the bulk counterpart of a hydrodynamic entropy current should be a gravitational concept which in equilibrium reduces to the standard event horizon and in dynamical situations obeys a suitable generalization of the second law of thermodynamics. The latter requirement should be equivalent to the non-negativity of the divergence of a dual entropy current and via the bulk-boundary map should relate features of the bulk construction to coefficients appearing in the gradient expansion of a boundary entropy current. The event horizon provides one concrete realization of this notion [4]: the equilibrium limit is trivially satisfied and the generalized second law of thermodynamics is provided by the famous area theorem stating that the area of spatial sections of the event horizon is nondecreasing under the evolution along the generators of the horizon.

This example suggests that within the realm of two-derivative gravity, the minimal structure needed to accommodate the various boundary entropy currents in a single framework is a hypersurface \( \Delta \) along with an associated tangent vector field \( v \): \( \Delta \) is a generalization of the event horizon and \( v \) is an analogue of its generators. If this vector field satisfies the Frobenius condition, then there is a slicing of the horizon into a family of codimension-two spatial hypersurfaces (“horizon sections”) orthogonal to \( v \) with area density \( \sqrt{\kappa} \). The generalized second law of thermodynamics will take the form of the following area increase theorem:

\[
\theta_{(v)} = \frac{1}{\sqrt{\kappa}} \mathcal{L}_v \sqrt{\kappa} \geq 0.
\] (1)

The relevant hypersurfaces \( \Delta \) are those whose equilibrium limit coincides with the event horizon and for which (1) holds by virtue of the dynamics of general relativity.

Such generalized area theorems follow from the properties of light rays propagating in directions normal to the “horizon sections.” The normal space to these sections is spanned by \( v \) along with a vector normal to \( \Delta \), denoted here by \( m \). The normalization can be chosen so that \( m \) is exact and \( m^2 + v^2 = 0 \). In the following section it is shown that for each \( \Delta \), (1) is equal to the divergence of some boundary entropy current. Thus the gravitational construction introduced here parallels the phenomenological definition of the boundary entropy current.

Introducing the ingoing and outgoing future-pointing null vectors normal to “horizon sections” reveals the connection with the framework of quasilocal horizons [5,6]. Those vectors, denoted, respectively, by \( \ell \) and \( n \) and conventionally normalized so as \( \ell \cdot n = -1 \), are given by linear combinations of \( v \) and \( m \):

\[
v = \ell - Cn \quad \text{and} \quad m = \ell + Cn.
\] (2)

The scalar \( C \) is called the evolution parameter. In terms of \( \ell \) and \( n \) the inequality (1) defining \( \Delta \) can be equivalently written in a more familiar way in terms of expansions in directions \( \ell \) and \( n \),

\[
\frac{1}{\sqrt{\kappa}} \mathcal{L}_v \sqrt{\kappa} = \theta_{(\ell)} - C \theta_{(n)} \geq 0,
\] (3)

where the expansion for a vector field \( X \) is defined by

\[
\theta_{(X)} = q_{ab} \nabla_a X_b = \mathcal{L}_X \log \sqrt{\kappa}
\] (4)

and \( q \) is the induced metric on the leaves:

\[
q_{ab} = g_{ab} + \ell_a n_b + n_a \ell_b.
\] (5)

Two well known choices of \( \Delta \) which guarantee (1) are the event horizon, for which \( C = 0 \) and \( \theta_{(\ell)} \geq 0 \), and a future outer trapping horizon,\(^1\) for which \( C \geq 0 \), \( \theta_{(\ell)} = 0 \) and \( \theta_{(n)} < 0 \). The latter is often referred to as an apparent horizon, and this terminology will be used in the following.

The essence of the proposal introduced in this paper is that (1) gives a characterization of bulk hypersurfaces which is dual to the notion of entropy current provided that the hypersurfaces in question asymptote to the event horizon. Dynamic event and apparent horizons might then be regarded as particular realizations of this idea, though from the perspective of the phenomenological definition, none of the entropy currents defined this way is distinguished (in line with the arguments made in [10]).

III. GENERAL HYDRODYNAMIC ENTROPY CURRENT AND ITS GRAVITY DUAL

Fluid-gravity duality provides a setting in which the idea outlined in the previous section is realized. For simplicity, this paper focuses on uncharged conformal fluid dynamics in \( d \) dimensions and its gravity dual. This asymptotically AdS geometry has been explicitly constructed up to second order in gradients in [13]. The form of the metric is

\[
ds^2 = -2u_\mu dx^\mu (dr + \mathcal{V}_a dx^a) + G_{\mu \nu} dx^\mu dx^\nu,
\] (6)

\(^1\)One also requires that \( \mathcal{L}_v \theta_{(\ell)} < 0 \), so that there be fully trapped surfaces near \( \Delta \) [5,6].
where \( r \) is the radial direction in the bulk and \( \mu = 0, \ldots, d - 1 \). The components of \( G_{\alpha \beta} \) of the metric (6) depend on boundary directions \( x^\mu \) only via \( u^\mu \) and \( T \), which have the interpretation of local velocity and temperature of the dual fluid, as well as the boundary metric \( g_{\mu \nu} \). The geometry is constructed systematically in the gradient expansion in derivatives of \( u^\mu \), \( T \) and \( g_{\mu \nu} \). The leading order solution to Einstein’s equations is a locally boosted and dilated (by \( b = d/4\pi T \)) black brane

\[
\mathcal{V}^{(0)}_{\mu} = \frac{1}{2} r^2 \left( 1 - \frac{1}{(rb)^d} \right) u_\mu \quad \text{and} \quad G^{(0)}_{\mu \nu} = r^2 P_{\mu \nu}, \quad (7)
\]

where \( P_{\mu \nu} = g_{\mu \nu} + u_\mu u_\nu \) is the projector to the local rest frame of the fluid. An important simplifying property of the solution obtained in [13] is the manifest realization of Weyl covariance [14], which is a symmetry of conformal hydrodynamics (up to the Weyl anomaly which appears in even dimensions at order \( d \) in the gradient expansion [15]). This symmetry severely restricts the structure and number of allowed gradient terms.

The near-equilibrium nature of the geometry (6) manifests itself in the fact that the (future) event horizon of the leading order solution (7) is located at \( r = 1/b \). This suggests considering more general hypersurfaces \( \Delta \) defined as level sets of a function \( S(r, x) = rb(x) + g(x) \), where \( g \) is a Weyl-invariant function which can be written as a linear combination of Weyl-invariant scalars available at each order of the gradient expansion.

The covector normal to \( \Delta \) is

\[
m = dS. \quad (8)
\]

On \( \Delta \) one can introduce coordinates \( y^\mu \) (\( \mu = 0, \ldots, d - 1 \)), such that \( S(r(y), x(y)) = \text{const} \), which implies that tangent vectors to the horizon are of the form

\[
v = v^\mu \frac{\partial x^\beta}{\partial y^\mu} \left\{ \frac{\partial S}{\partial x^\beta} - \left( \frac{\partial S}{\partial r} \right)^{-1} \frac{\partial S}{\partial x^\beta} \frac{\partial}{\partial r} \right\}. \quad (9)
\]

It is very convenient and physically well motivated to choose the coordinates on the horizon \( y^\mu = x^\mu \). The position of the horizon can then be expressed in the form \( r = r_H(x) \). This choice, made already in [4], uses bulk causality and connects boundary with bulk points along ingoing null geodesics emanating from the boundary in the direction specified locally by \( u^\mu \) [4]. In the gauge (6), this bulk-boundary map acts trivially relating points at different radial position and the same \( x^\mu \) coordinates. It is clear that beyond leading order in the gradient expansion such a bulk-boundary map can be modified in two ways: by changing the direction of null geodesics emanating from the boundary and by combining a given bulk-boundary map with boundary diffeomorphisms [4]. Up to second order in gradients only the latter freedom is present and will modify the consequences of the choice \( y^\mu = x^\mu \) precisely in the same way as discussed in [4].

Vvectors tangent to the horizon have a specific \( r \) component (9), which ensures that \( m \cdot v = 0 \). Thus, choosing the \( \mu \) components of the tangent vector \( v \) determines it fully. These components are constructed as a linear combination of all the available Weyl-invariant vectors made out of the hydrodynamic observables and boundary curvature. The coefficients appearing in this gradient expansion of \( v \) are constrained by the imposed normalization

\[
m^2 + v^2 = 0 \quad (10)
\]

and the requirement of hypersurface orthogonality. Up to second order [16] these coefficients are completely fixed and so the only quantities defining the gravitational construction are those appearing in \( r_H(x) \). This is consistent with what is known about two important special cases: if \( \Delta \) is the event horizon, then \( v \) is tangent to its generators and in this sense is determined (up to normalization) by \( r_H(x) = r_{EH}(x) \); if \( \Delta \) is a spacelike apparent horizon\(^2\) there is a unique foliation [17] and so \( v \) is again specified up to a normalization by \( r_H(x) = r_{AH}(x) \).

The key fact which leads to the general formula for the hydrodynamic entropy current is that the expansion \( \theta(v) \) is simply the divergence of \( v \):

\[
\theta(v) = \nabla_a v^a. \quad (11)
\]

To show this, note that the definition of \( \theta(v) \) [in the form (4)] can be rewritten in terms of \( m \) and \( v \) as

\[
\theta(v) = \nabla_a v^a + \frac{1}{C} (m^a m^b - v^a v^b) \nabla_a v_b. \quad (12)
\]

The second term above can be shown to vanish by applying two identities which follow by differentiating the normalization condition (10) and the fact that \( m \) is exact (8).

Since \( v \) is tangent to the horizon, its divergence in the adopted coordinate system can be written in a way which makes it trivial to map the relation (11) to the boundary. To this end, note that (11) can be expressed in the form

\[
\theta(v) = \frac{1}{\sqrt{-G}} \frac{\partial}{\partial x^\alpha} (\sqrt{-G} v^\alpha), \quad (13)
\]

where \( G \) is the determinant of the bulk metric. This relation is to be evaluated on \( \Delta \). The derivatives with respect to the bulk coordinates \( x^\mu \) can be expressed in terms of derivatives with respect to the coordinates \( y^\mu \) intrinsic to \( \Delta \), and \( v_r \) can be eliminated using (9), which leads to

\[
\theta(v) = b \frac{1}{\sqrt{-G}} \frac{\partial}{\partial y^\mu} \left( \frac{1}{b} \sqrt{-G} u^\mu \right). \quad (14)
\]

Finally, this can be expressed in terms of the Levi-Civita connection associated with the boundary metric \( g \) as

\(^2\)This should properly be called a future outer trapping horizon.
where $G_N$ is a $(d+1)$-dimensional Newton’s constant introduced to reproduce thermodynamic entropy in the leading order and AdS radius is set to 1, as in (7). The point of this definition is that it follows from (15) that

$$\nabla_{\mu} J^\mu = \frac{1}{4G_N} \frac{1}{b} \sqrt{g} \theta(v).$$

Thus the divergence of the current (16) is indeed non-negative if and only if the hypersurface $\Delta$ satisfies (1). Note that one can always shift the entropy current (16) by a vector whose divergence vanishes without changing (17). This possibility follows from the freedom to shift $v \rightarrow v + w$ in (11), where $w$ is tangent to $\Delta$ and $\nabla_{\mu} w^\mu = 0$.

Formula (16) [together with (1)] is the main result reported in this paper. It provides a map between hypersurfaces in the IR region of fluid-gravity duality geometry and phenomenologically defined entropy currents. It is explicitly covariant under boundary diffeomorphisms and applies to both apparent and event horizons, as well as other (timelike or spacelike) hypersurfaces, provided they satisfy $\theta_v \equiv 0$ (or equivalently $\nabla_{\mu} J^\mu \equiv 0$).

### IV. Entropy Currents and Dual Hypersurfaces at Second Order

The geometry of fluid-gravity duality has been explicitly constructed only up to second order in gradients. To that order the defining condition for the hypersurface $\Delta$ contains three parameters multiplying the three independent hydrodynamic scalars

$$r_{i1} = \frac{1}{b} + b (h_1 \sigma_{\mu \nu} \sigma^{\mu \nu} + h_2 \omega_{\mu \nu} \omega^{\mu \nu} + h_3 \mathcal{R}),$$

where $\sigma$, $\omega$ and $\mathcal{R}$ are the shear tensor, vorticity and boundary Ricci scalar, as defined in [13]. The vector $v$ is completely fixed [16] by the self-consistency of the bulk construction discussed earlier; up to second order

$$v^\mu = b u^\mu + b^3 \mu_{\mu \nu} \left( \frac{2}{d(d-2)} D^\rho \sigma_{\nu \rho} + \frac{1}{d-2} D^\rho \omega_{\nu \rho} \right),$$

where $\mathcal{D}$ is the Weyl-covariant derivative (see [13] for details). The $v^\rho$ component is fixed by requiring that $v$ is tangent to $\Delta$ [see (9)].

The current (16) dual to $r_{i1}(x)$ given in (19) reads

$$J^\mu = \frac{1}{4G_N} b^{1-d} (u^\mu + b^2 \mu_{\mu \nu} (j^1_{\mu} D^\mu \sigma_{\nu \rho} + j^2_{\mu} D^\mu \omega_{\nu \rho}) + b^2 (j^1_\mu \sigma_{\alpha \beta} \sigma^{\alpha \beta} + j^2_\mu \omega_{\alpha \beta} \omega^{\alpha \beta} + j^3_\mu \mathcal{R}) u^\mu),$$

where $j^1_\mu$ and $j^2_\mu$ are fixed and equal,

$$j^1_\mu = \frac{2}{d(d-2)}$$

$$j^2_\mu = \frac{2}{d(d-2)}$$

while the $j^3_\mu$’s depend on the choice of $h_i$’s,

$$j^3_1 = (d-1)h_1 - K_1(1),$$

$$j^3_2 = (d-1)h_2 + \frac{1}{2}$$

and

$$j^3_3 = (d-1)h_3.$$ (23)

Here $K_1$ [as well as $K_2$ in Eq. (24)] is a function defined in [13]. The current obeys a generalized second law of thermodynamics if and only if the dual surface $\Delta$ satisfies (1).

To get a feeling for what this entails, it is interesting to compare the entropy currents which follow from taking the $h_i$’s computed for the event and apparent horizon [the details of this calculation will appear in [16]]. For the event horizon one finds

$$j^1_1 = -K_1(1) + \frac{1}{d} K_2(1) + \frac{2(d^2 + d - 4)}{(d-2)d^2},$$

$$j^2_2 = \frac{(2-3d)}{2(d-2)}$$

and

$$j^3_3 = -\frac{1}{d(d-2)},$$

which matches the result of [13]. One can similarly evaluate the entropy current on the apparent horizon. One again finds (21) with the coefficients given by (22) and (24), apart from $j^1_1$, which differs from its value on the event horizon by $-\frac{4}{d^2}$. Note that $j^1_1$ is the only unconstrained parameter in the analysis presented in [3] which appears in the divergence of the current. Changing $j^1_1$ coincides with modifying the bulk-boundary map given by $\nu^\mu = x^\mu$ in the gauge (6) by including boundary diffeomorphisms [4]. It is natural to speculate that changing $j^1_1$ continuously within a given bulk-boundary map might be related to a one-parameter family of second order entropy currents. These would correspond to a set of timelike, spacelike or null bulk surfaces satisfying area theorem at the level of the gradient expansion in line with the proposal of [10].

Note finally, that the freedom (18) (which does not change the divergence of the entropy current) translates to a shift of $j^2_2, j^1_1$ with $\delta j^2_2 + \delta j^1_1 = 0$. This is the same as the ambiguity noted in [4].

### V. Conclusions

This paper proposes a direct link between phenomenologically defined entropy currents in holographic field theories and generalized horizons in the dual gravity.
background within the near-equilibrium regime. These geometrical objects are defined by the area theorem and the requirement of asymptoting to the event horizon. Such a definition matches precisely the defining condition of the hydrodynamic entropy current and embraces in a common framework both the event and apparent horizons within fluid-gravity duality. The geometric formula for the dual entropy current is manifestly covariant in the boundary sense and in the case of the event horizon leads to results which agree with [4,13].

It is further stipulated that at second order in the gradient expansion there is a one-parameter family of entropy currents arising in this way. This freedom arises solely from the freedom of choosing different $r_H(x)$ to define a dual entropy current and needs to be supplemented with the known freedom of defining a bulk-boundary map, as well as adding terms with vanishing divergence. Apart from these, the surface itself is the only geometric object relevant for the dual entropy current: it is conjectured (and verified to second order of the gradient expansion) that the foliation of the horizon is uniquely fixed within the gradient expansion. This conclusion is consistent with the results of the analysis presented in [3].

From the point of view of the phenomenological definition, none of the entropy currents introduced here is distinguished. Perhaps extending the requirements (by, e.g., causality) will rule out some of them. Note, in particular, that the entropy current defined by the apparent horizon in a given bulk-boundary map is equivalent (up to second order in the gradient expansion) to the entropy current defined by the event horizon in another bulk-boundary map. Since the latter is known to be acausal, it is clear that there should be an interplay between causality of dual entropy currents and the choice of bulk-boundary map. It would be fascinating to investigate this issue further, especially since the apparent horizon in the fluid-gravity duality seems to be free of foliation-dependence. It would also be very interesting to generalize the construction presented in this paper to cases of gravity duals to charged [18,19] and nonconformal fluid dynamics [20].

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