MICRO-LOCAL AND QUALITATIVE ANALYSIS OF THE FRACTIONAL ZENER WAVE EQUATION

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Abstract. This paper concerns the micro-local and qualitative analysis of the fractional Zener wave equation. The classical and Gevrey-type wave front sets of the fundamental solution are determined, and questions on dispersion, dissipation, wave propagation speed and wave packet shape are addressed.

1. Introduction

In this paper, we study the fractional Zener wave equation, which models the propagation of waves in a viscoelastic medium and which is given by

$$\frac{\partial^2}{\partial t^2}\,u(x,t) = \mathcal{L}^{-1}_{s\to t}\left(\frac{1+s^\alpha}{1+\tau s^\alpha}\right) *_t \frac{\partial^2}{\partial x^2}\,u(x,t), \qquad x\in\mathbb{R}, \quad t>0, \tag{1.1}$$

where $\mathcal{L}^{-1}$ denotes the inverse Laplace transform, while τ and α are constants satisfying $0<\tau<1$, $0<\alpha<1$.

Like the classical wave equation, the fractional Zener wave equation can be derived from a system of three basic equations of continuum mechanics. The equation of motion and the equation for the strain describing small local deformations are unchanged compared to the purely elastic case, while Hooke's law, the constitutive law for an elastic body, is replaced by a law which describes the mechanical properties of the linear *viscoelastic* body. For more details, we refer to [1, 3, 14]. The constitutive law yielding the fractional Zener wave equation (1.1) is the fractional Zener law, proposed in papers of Caputo and Mainardi [7, 8]. Some restrictions on the coefficients appearing in this constitutive equation, necessary for being physically consistent with the second law of thermodynamics and known as thermodynamical restrictions, are given in [4]. In [11], the system with the fractional Zener law instead of the Hooke's law was shown to be equivalent to the fractional Zener wave equation (1.1). The existence and uniqueness of solutions to the generalized Cauchy problem in the space of tempered distributions was proven under the assumption of the thermodynamical restrictions, a representation of the solution was given, and numerical examples were provided indicating some qualitative properties of the solutions, such as properties concerning shape and dissipation. In [16], energy dissipation was proven for a general class of fractional wave equations which includes the fractional Zener wave equation. The micro-local analysis for the fractional wave equation was initiated in [10], where an analogue of non-characteristic regularity was shown.

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The work we are to present continues the analysis of the fractional Zener wave equation, completing and extending the results on the wave front set given in [10]. We provide a complete description of the C^∞-wave front set of the fundamental solution of (1.1). In particular we show that the fundamental solution is smooth on the boundary of the forward light cone, in contrast with the classical wave equation. We also determine the wave front set with respect to Gevrey classes of functions, which assumes a finer notion of smoothness. As a consequence we prove that for the order of the Gevrey class sufficiently close to 1, the fundamental solution is singular at the boundary of the forward light cone.

Next, we perform a qualitative analysis of solutions in two cases. First, we investigate the response of the system when it is submitted to a forced harmonic oscillation at the origin. From this we detect the presence of dissipation and anomalous dispersion. Second, we investigate the evolution following a "delta impulse", i.e. the solution with initial conditions $u(x,0) = \delta(x)$, $\partial_t u(x,0) = 0$ (and zero force term). This solution consists of two wave packets moving in opposite directions. We will provide an accurate description of the "limiting shape" of this wave packet, and motivate the notion of a wave packet speed.

The content of the paper is organized as follows. The first section is introductory and provides motivation and mathematical preliminaries necessary for the work. In order to make the paper self-contained, the set up of the Cauchy problem for the fractional Zener wave equation under consideration with some details from previous works is given at the beginning of Section 2. We further discuss representation formulas and properties of the fundamental solution. The main result of the paper concerning the regularity of the fundamental solution, is given in Section 3. Theorem 3.1 and Theorem 3.3 describe the wave front set of the fundamental solution with respect to C^∞ and the Gevrey classes respectively. The proof of the latter theorem is long and quite technical. The main ideas are presented there, but the proof of a technical lemma is provided in Appendix A. Section 4 concerns the qualitative analysis and is divided into two subsection, treating the forced harmonic oscillation and the delta impulse respectively. Section 5 addresses the case when the viscoelastic medium is described by the *classical* Zener model (or the Standard Linear Solid (SLS) model) and shows that, as for the classical wave equation and in contrast to the fractional Zener wave equation, the fundamental solution is not smooth on the boundary of the light cone. Moreover, some qualitative difference between the two models in terms of dissipation is noted.

1.1. **Preliminaries.**

1.1.1. *Notations.* For Ω an open subset of $\mathbb{R}^n$, $\mathcal{D}(\Omega)$ denotes the space of compactly supported smooth functions on Ω. The space of distributions on Ω is denoted by $\mathcal{D}'(\Omega)$ and the space of compactly supported distributions is denoted by $\mathcal{E}'(\Omega)$. We denote with $\mathcal{S}(\mathbb{R}^n)$ the Schwartz space of rapidly decreasing smooth functions, and with $\mathcal{S}'(\mathbb{R}^n)$ the space of tempered distributions. Further, $\mathcal{D}'_+(\mathbb{R}) \subset \mathcal{D}'(\mathbb{R})$ and $\mathcal{S}'_+(\mathbb{R}) \subset \mathcal{S}'(\mathbb{R})$ are spaces of distributions supported on $[0,\infty)$, and $\mathcal{S}'(\mathbb{R}\times\mathbb{R}_+)$ is the space of distributions in $\mathcal{S}'(\mathbb{R}^2)$ vanishing on $\mathbb{R}\times(-\infty,0)$.

We use the following common notations for asymptotic relations:

$$
\begin{aligned}
&f(x) \lesssim g(x) \iff f(x) = O(g(x)) \iff \exists C>0\colon \big|f(x)\big| \le C g(x);\\
&f(x) \simeq g(x) \iff f(x) \lesssim g(x) \text{ and } g(x) \lesssim f(x);\\
&f(x) \sim g(x) \iff \frac{f(x)}{g(x)} \to 1.
\end{aligned}
$$

The denote the dependency of implicit O-constants on parameters, we will use subscripts, for example $\lesssim_n$.

1.1.2. *Integral transforms.* We define the Fourier transform for an integrable function $\varphi \in L^1(\mathbb{R})$ as

$$\mathcal{F}\varphi(\xi) = \hat{\varphi}(\xi) = \int_{-\infty}^{\infty} \varphi(x) \mathrm{e}^{-\mathrm{i}\xi x} \,\mathrm{d}x, \qquad \xi \in \mathbb{R},$$

while for $f \in \mathcal{S}'(\mathbb{R})$ the Fourier transform is given via $\langle \mathcal{F}f, \varphi\rangle = \langle f, \mathcal{F}\varphi\rangle$, $\varphi \in \mathcal{S}(\mathbb{R})$.

The Laplace transform of $f \in \mathcal{D}'_+(\mathbb{R})$ satisfying $\mathrm{e}^{-at} f \in \mathcal{S}'(\mathbb{R})$, for all $a > a_0 > 0$ is defined by

$$\mathcal{L}f(s) = \widetilde{f}(s) = \mathcal{F}(\mathrm{e}^{-at} f)(y), \qquad s = a + \mathrm{i}y,$$

$\mathcal{L}f$ being a holomorphic function in the half plane $\operatorname{Re} s > a_0$. If $f \in \mathcal{S}'(\mathbb{R}_+)$, then $\mathcal{L}f(s) = \langle f(t), \mathrm{e}^{-st}\rangle$. In particular, for $f \in L^1(\mathbb{R})$ with $f(t) = 0$, for $t < 0$, the Laplace transform is given by

$$\mathcal{L}f(s) = \int_0^{\infty} f(t) \mathrm{e}^{-st} \,\mathrm{d}t, \qquad \operatorname{Re} s \geq 0.$$

The inverse Laplace transform exists as distribution in $\mathcal{S}_+(\mathbb{R})$ for functions F holomorphic in the half plane $\operatorname{Re} s > a_0$ satisfying $\big|F(s)\big| \leq A\frac{(1+|s|)^m}{|\operatorname{Re} s|^k}$, $m, k \in \mathbb{N}$, $\operatorname{Re} s > 0$, and is given by

$$f(t) = \mathcal{L}^{-1}F(t) = \lim_{Y\to\infty} \frac{1}{2\pi\mathrm{i}} \int_{a-\mathrm{i}Y}^{a+\mathrm{i}Y} F(s)\mathrm{e}^{st} \,\mathrm{d}s, \qquad t > 0, \quad a > a_0, \tag{1.2}$$

whenever this limit exists.

If $f(x,t) \in \mathcal{S}'(\mathbb{R}\times\mathbb{R}_+)$, then the Laplace transform of f with respect to t is the distribution-valued function

$$\mathcal{L}_t f : \{s : \operatorname{Re} s > 0\} \to \mathcal{S}'(\mathbb{R}) : s \mapsto \big(\phi(x) \mapsto \langle f(x,t), \phi(x)\mathrm{e}^{-st}\rangle\big).$$

1.1.3. *Fractional derivatives.* The left Riemann-Liouville fractional derivative of order $\alpha \in [0,1)$ is defined for an absolutely continuous function, $f \in AC([0,a])$, on an interval $[0,a]$ with $a > 0$ by

$${}_0D_t^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t \frac{f(\zeta)}{(t-\zeta)^{\alpha}} \,\mathrm{d}\zeta, \quad t \in [0,a],$$

while the left Liouville-Weyl fractional derivative of order $\alpha \in [0,1)$ is defined for $f \in AC(\mathbb{R})$ with $f(-t) \lesssim 1/t$ for $t \to \infty$ by

$${}_{-\infty}D_t^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}t} \int_{-\infty}^t \frac{f(\zeta)}{(t-\zeta)^{\alpha}} \,\mathrm{d}\zeta, \quad t \in \mathbb{R},$$

where Γ denotes the Euler gamma function. In the spaces of distributions one introduces a family $\{\chi_+^{\alpha}\}_{\alpha\in\mathbb{C}} \in \mathcal{S}'_+(\mathbb{R})$:

$$\chi_+^{\alpha}(t) = \frac{1}{\Gamma(\alpha+1)} t_+^{\alpha}.$$

This family of distributions is initially defined for $\operatorname{Re}\alpha > -1$ as L^1_{loc} functions, but can be extended to every $\alpha \in \mathbb{C}$ by analytic continuation. When α is not a negative integer, $\langle \chi_+^{\alpha}(t), \varphi(t)\rangle$ can be evaluated using the Hadamard finite part:

$$\langle \chi_+^{\alpha}(t), \varphi(t)\rangle = \frac{1}{\Gamma(\alpha+1)} \operatorname{F.p.} \int_0^{\infty} t^{\alpha}\varphi(t) \,\mathrm{d}t, \quad \varphi \in \mathcal{S}(\mathbb{R}).$$

For more details concerning this family of distributions, we refer to [9, Section 3.2].

The convolution operator $f \mapsto \chi_+^{-\alpha-1} * f$ coincides with the left Riemann-Liouville and left Liouville-Weyl fractional derivative of order α, for $f \in AC([0,a])$ and $f \in AC(\mathbb{R})$ respectively.

The Fourier transform of the Liouville-Weyl fractional derivative of $f \in AC(\mathbb{R})$ and Laplace transform of the left Riemann-Liouville fractional derivative of $f \in AC([0,a])$ are given as follows:

$$\mathcal{F}[{}_{-\infty}D_t^\alpha f](\xi) = (\mathrm{i}\xi)^\alpha \mathcal{F}f(\xi), \qquad \mathcal{L}[{}_0D_t^\alpha f](s) = s^\alpha \mathcal{L}f(s).$$

For more details on fractional derivatives we refer to [15].

1.1.4. *Gevrey classes.* Let $\sigma \geq 0$ and let $\Omega \subseteq \mathbb{R}^n$ be open. A function $\varphi \in C^\infty(\Omega)$ belongs to the Gevrey class $G^\sigma(\Omega)$ of order σ if for every compact $K \subseteq \Omega$ there exists a constant $C = C_K > 0$ such that

$$\sup_{x\in K} \left|\partial^\beta \varphi(x)\right| \leq C^{1+|\beta|}(\beta!)^\sigma, \quad \text{for every multi-index } \beta \in \mathbb{N}^n.$$

The case $\sigma = 1$ corresponds to real analytic functions. When $\sigma > 1$, $G^\sigma(\Omega)$ contains compactly supported functions. For a distribution $u \in \mathcal{D}'(\Omega)$, $\operatorname{sing\,supp}_{G^\sigma} u$ is defined as the complement of the largest open set X where $u \in G^\sigma(X)$. Similarly as in the C^∞ case, one might perform a spectral analysis of the G^σ-singularities of u, by investigating the decay on cones of the Fourier transform of localizations of u. Since the space G^σ, $\sigma \leq 1$ does not contain compactly supported functions, one uses so-called analytic cut-off sequences to localize. This leads to the notion of the G^σ-wave front set of u, denoted as $\mathrm{WF}_{G^\sigma}(u)$. For more details we refer to [9, Section 8.4].

1.1.5. *The saddle point method.* To estimate certain integrals we will use the saddle point method, also known as the method of steepest descent. This method can be summarized as follows. Suppose f and g are holomorphic functions on a simply connected region Ω. Suppose that $a, b \in \Omega$, and suppose that z_0 is a simple saddle point of f, that is, a simple zero of f'. If there exists a path from a to b in Ω, which passes through z_0 in such a way that $\operatorname{Re} f(z)$ reaches its maximum at $z = z_0$, then

$$\int_a^b g(z)\mathrm{e}^{\lambda f(z)}\,\mathrm{d}z \sim g(z_0)\sqrt{\frac{2\pi}{-f''(z_0)\lambda}}\mathrm{e}^{\lambda f(z_0)}, \quad \text{as } \lambda \to \infty.$$

For an introduction to the saddle point method, we refer to [6, Chapter 5].

2. The Cauchy problem and the fundamental solution

To describe waves occurring in one-dimensional viscoelastic media one uses the system of basic equations of elasticity (see [1]), consisting of the equilibrium equation coming from Newton's second law: $\partial_x \sigma = \rho\, \partial_t^2 u$, the constitutive equation, describing the connection between stress and strain, and the strain measure for local small deformations, giving the connection between strain and displacement: $\varepsilon = \partial_x u$. Here, σ, u and ε denote stress, displacement, and strain respectively, being functions of $x \in \mathbb{R}$ and $t > 0$. The constant ρ denotes the density of the medium under consideration.

For purely elastic media, the constitutive equation is given by Hooke's law: $\sigma = E\varepsilon$, where E is a constant referred to as the Young modulus of elasticity. The classical wave equation

$\partial_t^2 u = c^2 \partial_x^2 u$ is then easily obtained from this system with $c = \sqrt{E/\rho}$ being a constant that can be physically interpreted as the wave speed.

For viscoelastic media one replaces Hooke's law by other constitutive equations, such as the Maxwell, Voight, or Zener models (see [14]). In the fractional models, one replaces the integer order derivatives by derivates of fractional order. The fractional Zener model is a generalization of the (classical) Zener model, also known as the Standard Linear Solid (SLS) model, and is given by

$$\sigma(x,t) + \tau_\sigma \, D^\alpha \sigma(x,t) = E[\varepsilon(x,t) + \tau_\varepsilon \, D^\alpha \varepsilon(x,t)], \qquad x \in \mathbb{R}, \quad t > 0, \tag{2.1}$$

where D^α denotes fractional differentiation of order α, $0 < \alpha < 1$. The case $\alpha = 1$, i.e. with ordinary derivatives, corresponds to the SLS model. In (2.1), τ_σ, τ_ε are constants referred to as the relaxation time and the retardation time respectively. They satisfy the thermodynamical restriction $\tau_\varepsilon > \tau_\sigma > 0$ following from the second law of thermodynamics [4]. The constant E is the Young modulus of elasticity.

In this paper, we analyze wave propagation in viscoelastic media modelled by the system of basic equations in which the fractional Zener law (2.1) instead of Hooke's law is used. In dimensionless form, this system is given by (see e.g. [11])

$$\begin{aligned} \frac{\partial}{\partial x}\,\sigma &= \frac{\partial^2}{\partial t^2}\,u, \\ \sigma + \tau \, {}_0D_t^\alpha \sigma &= \varepsilon + {}_0D_t^\alpha \varepsilon, \\ \varepsilon &= \frac{\partial}{\partial x}\,u; \end{aligned} \tag{2.2}$$

where as above, σ, u and ε are stress, displacement, and strain, respectively, considered as functions of $x \in \mathbb{R}$ and $t > 0$, $0 < \tau < 1$ is a constant and ${}_0D_t^\alpha$, $0 < \alpha < 1$, is the left Riemann-Liouville operator of fractional differentiation. In [11] it was shown that the wave equation corresponding to (2.2) is the fractional Zener wave equation (1.1). The convolution kernel $\mathcal{L}^{-1}\big((1+s^\alpha)/(1+\tau s^\alpha)\big)$ can be expressed using Mittag-Leffler functions (see [14, Appendix E]) as

$$\mathcal{L}^{-1}\left(\frac{1+s^\alpha}{1+\tau s^\alpha}\right) = \frac{1}{\tau}\delta(t) - \frac{1-\tau}{\tau^2} e_{\alpha,\alpha}(t;1/\tau) = \frac{1}{\tau}\delta(t) + \frac{1-\tau}{\tau}\,\frac{\mathrm{d}}{\mathrm{d}t}\,e_\alpha(t;1/\tau).$$

Further in [11], for given initial data u_0, $v_0 \in \mathcal{S}'(\mathbb{R})$, and force term $f \in \mathcal{S}'(\mathbb{R}\times\mathbb{R}_+)$, the generalized Cauchy problem for the fractional Zener wave equation

$$Pu(x,t) = f(x,t) + u_0(x)\delta'(t) + v_0(x)\delta(t), \tag{2.3}$$

with

$$P = \frac{\partial^2}{\partial t^2} - \mathcal{L}^{-1}\left(\frac{1+s^\alpha}{1+\tau s^\alpha}\right) *_t \frac{\partial^2}{\partial x^2},$$

was considered. It was shown that (2.3) has a unique solution expressed via convolution of the fundamental solution S with the Cauchy data:

$$u(x,t) = S(x,t) * \big(f(x,t) + u_0(x)\delta'(t) + v_0(x)\delta(t)\big). \tag{2.4}$$

The Laplace transform $\tilde{S}$ of the fundamental solution S with respect to t is calculated as

$$\tilde{S}(x,s) = \frac{1}{2s}\sqrt{\frac{1+\tau s^\alpha}{1+s^\alpha}}\exp\left(-|x|\,s\sqrt{\frac{1+\tau s^\alpha}{1+s^\alpha}}\right), \quad x \in \mathbb{R},\ \operatorname{Re} s > 0, \tag{2.5}$$

where the principal branch of the logarithm is used for the function s^α and the square root. Note that for fixed s, this is a continuous function of x. Denote by $l_\alpha(s)$ the function defined as

$$l_\alpha(s) = \sqrt{\frac{1+\tau s^\alpha}{1+s^\alpha}}, \quad \arg s \in [-\pi, \pi].$$

First we derive some properties of $l_\alpha(s)$.

Lemma 2.1. *The real and imaginary part of l_α satisfy*

$$\operatorname{Re} l_\alpha(s) > 0, \quad \operatorname{sgn} \operatorname{Im} l_\alpha(s) = -\operatorname{sgn} \operatorname{Im} s, \quad \arg s \in [-\pi, \pi]. \tag{2.6}$$

Its asymptotic behavior near the origin and infinity is given by

$$l_\alpha(s) = 1 - \frac{1-\tau}{2} s^\alpha + O(|s|^{2\alpha}), \qquad \text{as } |s| \to 0; \tag{2.7}$$

$$l_\alpha(s) = \sqrt{\tau}\left(1 + \frac{1}{2}\left(\frac{1}{\tau} - 1\right) s^{-\alpha} + O(|s|^{-2\alpha})\right), \qquad \text{as } |s| \to \infty. \tag{2.8}$$

In particular, there exist positive constants c_1 and c_2 such that

$$\operatorname{Im} l_\alpha(\mathrm{i}y) \le \begin{cases} -c_1 y^\alpha & \text{for } 0 \le y \le 1; \\ -c_2 y^{-\alpha} & \text{for } y \ge 1. \end{cases} \tag{2.9}$$

Proof. A straightforward calculation shows that, with $s = R\mathrm{e}^{\mathrm{i}\varphi}$, $-\pi \le \varphi \le \pi$,

$$\frac{1+\tau s^\alpha}{1+s^\alpha} = \frac{1 + \tau R^{2\alpha} + (1+\tau) R^\alpha \cos(\alpha\varphi) - \mathrm{i}(1-\tau) R^\alpha \sin(\alpha\varphi)}{1 + R^{2\alpha} + 2R^\alpha \cos(\alpha\varphi)}.$$

The denominator is real and positive. We see that $\frac{1+\tau s^\alpha}{1+s^\alpha} \in \mathbb{C} \setminus (-\infty, 0]$, so the real part of its square root is positive. Since taking the square root does not alter the sign of the imaginary part, the first claim of the lemma follows. The formulas (2.7) and (2.8) follow immediately from Taylor's formula, and upon writing $l_\alpha(s) = \sqrt{\tau}\sqrt{(1+\tau^{-1}s^{-\alpha})/(1+s^{-\alpha})}$ for large s. □

In [11], the following representation of S inside the forward cone $|x| < t/\sqrt{\tau}$ was given[1]

$$S(x,t) = \frac{1}{2} + \frac{1}{4\pi\mathrm{i}} \int_0^\infty \left(l_\alpha(q\mathrm{e}^{\mathrm{i}\pi}) \mathrm{e}^{|x| q l_\alpha(q\mathrm{e}^{\mathrm{i}\pi})} - l_\alpha(q\mathrm{e}^{-\mathrm{i}\pi}) \mathrm{e}^{|x| q l_\alpha(q\mathrm{e}^{-\mathrm{i}\pi})} \right) \frac{\mathrm{e}^{-qt}}{q} \, \mathrm{d}q. \tag{2.10}$$

Note that it follows from the asymptotic behavior of l_α that this integral converges absolutely whenever $|x| < t/\sqrt{\tau}$. The representation (2.10) was shown by Laplace inversion via formula (1.2), i.e. by calculation of the integral

$$\frac{1}{2\pi\mathrm{i}} \lim_{Y \to \infty} \int_{a-\mathrm{i}Y}^{a+\mathrm{i}Y} \frac{l_\alpha(s)}{2s} \mathrm{e}^{-|x| s l_\alpha(s) + ts} \, \mathrm{d}s, \quad a > 0, \tag{2.11}$$

There is some discrepancy in the literature (e.g. [2] and [11]) regarding claims about the support of S. We will clarify this here and prove[2] that S is supported in the forward cone $|x| \le t/\sqrt{\tau}$. We will also indicate how to deduce the representation (2.10), since this technique will be used multiple times throughout this paper.

[1] The constant right after the equality sign in [11, eq. (18)] should be 1/2 instead of 1.

[2] We note that support properties for a more general class of models were also proven in [12]

Proposition 2.2. *The fundamental solution S of (1.1) is supported in a forward cone:*

$$\operatorname{supp} S \subseteq \{(x,t) : |x| \le t/\sqrt{\tau}\}.$$

In the interior of this cone, S is given by (2.10).

Proof. Let x and t be such that $|x| > t/\sqrt{\tau}$. We show that $S(x,t) = 0$. Using Cauchy's formula, we may rewrite the integral (2.11) as an integral over the arc of the circle of radius $R = \sqrt{a^2+Y^2}$ and center 0, which connects the points $a - \mathrm{i}Y$ and $a + \mathrm{i}Y$. The polar angle varies between $-\varphi(R)$ and $\varphi(R)$, with $\varphi(R) = \arctan\sqrt{(R/a)^2 - 1}$. We get

$$S(x,t) = \lim_{R\to\infty} \frac{1}{4\pi} \int_{-\varphi(R)}^{\varphi(R)} l_\alpha(R\mathrm{e}^{\mathrm{i}\varphi}) \exp\big(R\mathrm{e}^{\mathrm{i}\varphi}(t - |x|\, l_\alpha(R\mathrm{e}^{\mathrm{i}\varphi}))\big)\, \mathrm{d}\varphi.$$

Using (2.6) and extending the range of integration to $[-\pi/2, \pi/2]$, we can bound the absolute value of S by

$$\big|S(x,t)\big| \lesssim \lim_{R\to\infty} \int_{-\pi/2}^{\pi/2} \exp\big(-(|x| \operatorname{Re} l_\alpha(R\mathrm{e}^{\mathrm{i}\varphi}) - t)R\cos\varphi\big)\, \mathrm{d}\varphi.$$

Let us write $\varepsilon = \sqrt{\tau}|x| - t > 0$. For R sufficiently large, $|x| \operatorname{Re} l_\alpha(R\mathrm{e}^{\mathrm{i}\varphi}) - t > \varepsilon/2$, since $\operatorname{Re} l_\alpha(R\mathrm{e}^{\mathrm{i}\varphi}) \to \sqrt{\tau}$ by (2.8). For such large R the integrand is bounded by $\mathrm{e}^{-(\varepsilon/2)R\cos\varphi}$, which converges pointwise to 0 and is bounded. From this it follows that the above integral converges to 0 when $R \to \infty$, by dominated convergence.

Suppose now that $|x| < t/\sqrt{\tau}$. Then $\operatorname{Re}(-|x|\, l_\alpha(s) + t) > 0$ for $|s|$ sufficiently large. We will shift the contour to the left, resulting in a Hankel contour encircling the branch cut $(-\infty, 0]$. For a small $\varepsilon > 0$, we set

$$\begin{aligned}
&\Gamma_1 = [a - \mathrm{i}Y, -\mathrm{i}Y]; && \Gamma_5 = [\varepsilon\mathrm{e}^{\mathrm{i}\pi}, Y\mathrm{e}^{\mathrm{i}\pi}];\\
&\Gamma_2 = \{Y\mathrm{e}^{\mathrm{i}\varphi} : -\pi/2 \ge \varphi \ge -\pi\}; && \Gamma_6 = \{Y\mathrm{e}^{\mathrm{i}\varphi} : \pi \ge \varphi \ge \pi/2\};\\
&\Gamma_3 = [Y\mathrm{e}^{-\mathrm{i}\pi}, \varepsilon\mathrm{e}^{-\mathrm{i}\pi}]; && \Gamma_7 = [\mathrm{i}Y, a + \mathrm{i}Y].\\
&\Gamma_4 = \{\varepsilon\mathrm{e}^{\mathrm{i}\varphi} : -\pi \le \varphi \le \pi\}; &&
\end{aligned}$$

By Cauchy's theorem, the contour integral in (2.11) equals $\frac{1}{2\pi\mathrm{i}} \int_{\cup_i \Gamma_i} \tilde{S}(x,s)\mathrm{e}^{ts}\, \mathrm{d}s$. On Γ_1 and Γ_7, $\tilde{S}(x,s)\mathrm{e}^{ts} \lesssim 1/Y$, so the integral over these pieces converges to zero as $Y \to \infty$. On Γ_2 and Γ_3, $\operatorname{Re}(-|x|\, sl_\alpha(s) + ts) \sim (t - \sqrt{\tau}|x|)(\cos\varphi)Y$. As before, the integrals over these contours tend to zero by dominated convergence, since now $t - \sqrt{\tau}|x| > 0$ and $\cos\varphi < 0$ (except at the boundary points $\varphi = \pm\pi/2$.) Since $\tilde{S}(x,s)\mathrm{e}^{ts} \sim 1/(2s)$ for $s \to 0$, the integral over Γ_4 converges to $\mathrm{i}\pi$ as $\varepsilon \to 0$. Finally, combining the integrals over Γ_3 and Γ_5 and letting $Y \to \infty$, $\varepsilon \to 0$, we get the absolutely convergent integral in (2.10). □

Remark 2.3. (i) Proposition 2.2 implies that the convolution in (2.4) is well-defined for arbitrary distributions u_0, $v_0 \in \mathcal{D}'(\mathbb{R})$, and therefore the result given in [11, Theorem 4.2] holds with u_0, $v_0 \in \mathcal{D}'(\mathbb{R})$.

(ii) In fact, the inverse Laplace integral (2.11) converges absolutely for every x, t with $x \neq 0$. Writing $s = a + \mathrm{i}y$, the approximation (2.8) shows that

$$\operatorname{Re} sl_\alpha(s) = \frac{\sqrt{\tau}}{2}\Big(\frac{1}{\tau} - 1\Big)\sin(\alpha\pi/2)|y|^{1-\alpha} + O(1 + |y|^{1-2\alpha}), \quad \text{as } |y| \to \infty, \tag{2.12}$$

locally uniformly in a. Recall that $0 < \alpha < 1$ and $0 < \tau < 1$, so for non-zero x, the exponential in the integral decays like $\mathrm{e}^{-c|x||y|^{1-\alpha}}$, where $c = \sqrt{\tau}/2(1/\tau - 1)\sin(\alpha\pi/2)$ is a

positive constant. This proves the absolute convergence. It is convenient to move the contour of integration to the line $\operatorname{Re} s = 0$. For this we consider the following contours for a small parameter $\varepsilon > 0$:

$$\Gamma_1 = [\mathrm{i}Y, a + \mathrm{i}Y]; \qquad \Gamma_4 = [-\mathrm{i}Y, -\mathrm{i}\varepsilon];$$
$$\Gamma_2 = [\mathrm{i}\varepsilon, \mathrm{i}Y]; \qquad \Gamma_5 = [-\mathrm{i}Y, a - \mathrm{i}Y].$$
$$\Gamma_3 = \{\varepsilon \mathrm{e}^{\mathrm{i}\varphi} : -\pi/2 \le \varphi \le \pi/2\};$$

The estimate (2.12) of $\operatorname{Re} s l_\alpha(s)$ immediately implies that the integrals of $\tilde{S}(x,s)\mathrm{e}^{ts}$ along the contours Γ_1 and Γ_5 tend to 0 as $Y \to \infty$, whenever $x \neq 0$. Since $\tilde{S}(x,s)\mathrm{e}^{ts} \sim 1/(2s)$ as $s \to 0$, the integral along Γ_3 tends to $\mathrm{i}\pi/2$ as $\varepsilon \to 0$. Combining the integrals over Γ_2 and Γ_4 and letting $\varepsilon \to 0$, $Y \to \infty$ yields the following representation for $S(x,t)$ when $x \neq 0$:

$$S(x,t) = \frac{1}{4} + \frac{1}{4\pi\mathrm{i}}\,\mathrm{p.\,v.} \int_{-\infty}^{\infty} l_\alpha(\mathrm{i}y) \exp\bigl(-|x|\,\mathrm{i}y l_\alpha(\mathrm{i}y) + \mathrm{i}ty\bigr) \frac{\mathrm{d}y}{y}, \tag{2.13}$$

where p. v. denotes the Cauchy principal value.

The integral in (2.10) does not converge for $|x| \ge t/\sqrt{\tau}$, so the representation (2.13) will be particularly useful for studying the behavior of S near the boundary of the cone. On the other hand, the integral in (2.13) does not converge absolutely for $x = 0$, so (2.10) will be useful for studying S at small values of x.

Combining both integral representations (2.10), (2.13) of the fundamental solution S allows us to give a complete description of its regularity.

3. Micro-local analysis of S

In this section, we provide the full micro-local analysis of the fundamental solution S with respect to C^∞ and G^σ, $\sigma \ge 1$, extending previous results in the literature. In [2], some regularity of S was shown, namely that the map $t \mapsto S(x,t)$ is smooth for fixed $x \neq 0$. The micro-local analysis of S was initiated in [10]. It reaches a form of non-characteristic regularity for solutions to the Cauchy problem (2.3) (see [9, Theorem 18.1.8]). Namely, [10, Theorem 3.2] states

$$\mathrm{WF}(u^+) \subseteq \Bigl\{(x,t;\xi,\eta) : x \in \mathbb{R}, t > 0, \xi \neq 0, \eta = 0 \ \text{ or } \ \eta^2 = \frac{1}{\tau}\xi^2\Bigr\},$$

where u^+ denotes the restriction of the solution to (2.3) to the forward time $t > 0$. This result suggests that apart from the frequencies $(\xi, 0)$, also the frequencies orthogonal to the boundary of the light cone could be singular frequencies. We will show that this is not the case: S is smooth on this boundary. This is in contrast with the classical wave equation, whose fundamental solution is singular at the boundary of the forward light cone, and for which the singular frequencies are those orthogonal to this boundary.

3.1. C^∞-regularity.

The following theorem provides the evaluation of the C^∞-wave front set of S. In particular, it will imply that S is smooth off the half line $x = 0$, $t \ge 0$.

Theorem 3.1. *The fundamental solution S is an L^1_{loc}-function which is continuous on $\mathbb{R}^2 \setminus \{(0,0)\}$. Its partial derivative with respect to x, $\partial_x S$, is discontinuous on the half-line $x = 0$, $t > 0$. Everywhere else, S is of class C^∞. In particular, for the wave front set we have*

$$\mathrm{WF}(S) = \{(0,0;\xi,\eta) : (\xi,\eta) \neq (0,0)\} \cup \{(0,t;\xi,0) : t > 0, \xi \neq 0\}.$$

Proof. The representations (2.10) and (2.13) imply the continuity of S in the open sets $|x| < t/\sqrt{\tau}$ and $x \neq 0$ respectively, showing that S coincides with a continuous function on $\mathbb{R}^2 \setminus \{(0,0)\}$. It is not possible that S contains linear combinations of $\delta^{(n)}(x)\delta^{(m)}(t)$, since these would show up in the Laplace transform $\tilde{S}$ as linear combinations of $s^m\delta^{(n)}(x)$, and they are not present in (2.5). Hence, S is an L^1_{loc}-function, continuous on $\mathbb{R}^2 \setminus \{(0,0)\}$.

Differentiating formula (2.10) with respect to x, and taking the limit for $x \to 0$ from the right and from the left, we see that

$$\begin{aligned}\frac{\partial S}{\partial x}(0^{\pm},t) &= \pm\frac{1}{4\pi\mathrm{i}}\int_0^\infty \left(\frac{1+\tau q^\alpha \mathrm{e}^{\mathrm{i}\alpha\pi}}{1+q^\alpha\mathrm{e}^{\mathrm{i}\alpha\pi}} - \frac{1+\tau q^\alpha \mathrm{e}^{-\mathrm{i}\alpha\pi}}{1+q^\alpha\mathrm{e}^{-\mathrm{i}\alpha\pi}}\right)\mathrm{e}^{-tq}\,\mathrm{d}q \\ &= \mp\frac{1}{2\pi}\int_0^\infty (1-\tau)\sin(\alpha\pi)\frac{q^\alpha}{1+2\cos(\alpha\pi)q^\alpha+q^{2\alpha}}\mathrm{e}^{-tq}\,\mathrm{d}q.\end{aligned}$$

The integrand of the last integral is positive when $q > 0$, so the integral is non-zero. This shows that $\partial_x S(x,t)$ is not continuous at $x = 0$.

Using the representation (2.13), we see that S is smooth at points (x,t) with $x \neq 0$. Indeed, differentiating under the integral yields an additional factor which is of polynomial growth. Since the exponential in the integrand decays like $\lesssim \mathrm{e}^{-c|x|\,|y|^{1-\alpha}}$, with $c = \sqrt{\tau}/2(1/\tau - 1)\sin(\alpha\pi/2)$ (see (2.12)), the integral remains convergent. Note that it is crucial here that $0 < \alpha < 1$ and $0 < \tau < 1$.

To compute the wave front set, we use (2.10). Differentiating under the integral, we see that $\partial_t^m S(x,t)$ is bounded on compact subsets of the cone $|x| < t/\sqrt{\tau}$, for each $m \in \mathbb{N}$. This implies that at a point $(0,t)$, $t > 0$, the "singular frequencies" can only be along the positive and negative $(\xi,0)$-direction. Since S is real-valued, its wave front set is symmetric about the origin in the frequency variables. Hence, both directions are present in the wave front set. Finally, since $PS(x,t) = \delta(x)\delta(t)$, and differential and convolution operators do not enlarge the wave front set, also $(0,0;\xi,\eta) \in \mathrm{WF}(S)$ for every $(\xi,\eta) \neq (0,0)$. □

From (2.10), one readily sees that S converges to $1/2$ for $t \to \infty$. Indeed, locally uniformly in x, the integral converges to 0 as $t \to \infty$ by dominated convergence: for $q \leq 1$, the integrand in (2.10) is dominated by $\mathrm{e}^{|x|q}O(q^{\alpha-1} + |x|\,q^\alpha)$ (which can be seen by Taylor approximation using (2.7)), while for $q \geq Q$, it is dominated by $\mathrm{e}^{(|x|\sup l_\alpha - T)q}$ if $t \geq T$.

We can also describe the behavior of S for $(x,t) \to (0,0)$.

Proposition 3.2. *Suppose $\lambda \in [0, 1/\sqrt{\tau}]$. Then*

$$\lim_{t\to 0^+} S(\pm\lambda t, t) = \frac{\sqrt{\tau}}{2}(1 - \lambda\sqrt{\tau}).$$

Proof. The statement for $\lambda = 1/\sqrt{\tau}$ is trivial, since $S(\pm t/\sqrt{\tau}, t) = 0$ for $t > 0$.

Suppose first that $\lambda = 0$. We compute the Laplace transform of $S(0,t) - (\sqrt{\tau}/2)H(t)$. We get

$$\mathcal{L}\Big(S(0,t) - \frac{\sqrt{\tau}}{2}H(t)\Big)(s) = \tilde{S}(0,s) - \frac{\sqrt{\tau}}{2s} = \frac{l_\alpha(s) - \sqrt{\tau}}{2s}.$$

By (2.8), $l_\alpha(s) = \sqrt{\tau} + O(|s|^{-\alpha})$ as $|s| \to \infty$. Therefore, the above Laplace transform decays as $\lesssim |s|^{-1-\alpha}$, so it is integrable on every vertical line $\mathrm{Re}\, s = a$, with $a > 0$. Hence, $S(0,t) - (\sqrt{\tau}/2)H(t)$ is a continuous function, which vanishes for $t \leq 0$.

For general $\lambda \in (0, 1/\sqrt{\tau})$, we apply the same strategy, namely determining the asymptotic behavior of the Laplace transform of $t \mapsto S(\lambda t, t)$. Let $s = 2 + \mathrm{i}y$. Using the Laplace inversion

for $S(x,t)$ (2.11) with $a=1$, we get

$$\mathcal{L}\{S(\lambda t,t)\}(s)=\frac{1}{4\pi\mathrm{i}}\int_0^\infty \mathrm{e}^{-st}\int_{1-\mathrm{i}\infty}^{1+\mathrm{i}\infty}\frac{l_\alpha(z)}{z}\exp\big(tz(1-\lambda l_\alpha(z))\big)\,\mathrm{d}z\,\mathrm{d}t.$$

We want to swap the order of integration here, using the Fubini-Tonelli theorem. This is allowed, since

$$\int_{1-\mathrm{i}\infty}^{1+\mathrm{i}\infty}\int_0^\infty\left|\frac{l_\alpha(z)}{z}\exp\big(t(z-\lambda z l_\alpha(z)-s)\big)\right|\mathrm{d}t|\mathrm{d}z| \\ =\int_{1-\mathrm{i}\infty}^{1+\mathrm{i}\infty}\left|\frac{l_\alpha(z)}{z}\right|\frac{1}{-1+\lambda\operatorname{Re}(zl_\alpha(z))+2}|\mathrm{d}z|<\infty$$

Here we used that $1-\lambda\operatorname{Re}(zl_\alpha(z))-2\le -1$ and that $\lambda\operatorname{Re}(zl_\alpha(z))\sim -\lambda c|\operatorname{Im} z|^{1-\alpha}$ for some $c>0$, by (2.12), so that the above integral converges absolutely. Swapping the order of integration and integrating with respect to t, we get

$$\mathcal{L}\{S(\lambda t,t)\}(s)=-\frac{1}{4\pi\mathrm{i}}\int_{1-\mathrm{i}\infty}^{1+\mathrm{i}\infty}\frac{l_\alpha(z)}{z\big(z(1-\lambda l_\alpha(z))-s\big)}\,\mathrm{d}z.$$

We will evaluate this integral via Cauchy's theorem. Let $y=\operatorname{Im} s$ be large but fixed. The integrand above decays like $\lesssim 1/|z|^2$, and has a unique pole in the right half plane $\operatorname{Re} z>1$, which we denote by $z(s)$ (this follows for example by applying the argument principle on the region enclosed by the line $[1-\mathrm{i}R,1+\mathrm{i}R]$ and the right semicircle with center 1 and radius $\sqrt{1+R^2}$, for sufficiently large R). We have

$$\big(1-\lambda l_\alpha(z(s))\big)z(s)-s=0,\quad z(s)=\frac{s}{1-\lambda\sqrt{\tau}}\big(1+O(|s|^{-\alpha})\big),$$

where we used (2.8). Applying Cauchy's theorem and (2.8) again, we get

$$\mathcal{L}\{S(\lambda t,t)\}(s)=\frac{l_\alpha(z(s))}{2z(s)}=\frac{\sqrt{\tau}(1-\lambda\sqrt{\tau})}{2s}\big(1+O(|s|^{-\alpha})\big),\qquad \text{for } |y|=|\operatorname{Im} s|\to\infty.$$

It follows that $S(\lambda t,t)-(\sqrt{\tau}/2)(1-\lambda\sqrt{\tau})H(t)$ is a continuous function, vanishing for $t\le 0$, since its Laplace transform, decaying like $\lesssim |s|^{-1-\alpha}$, is absolutely integrable on the line $\operatorname{Re} s=2$. □

3.2. **Gevrey regularity.** Using a finer notion of smoothness, namely by means of the Gevrey classes G^σ, more singularities become visible. In the following theorem, we describe the G^σ-regularity of S for every $\sigma\in[1,\infty)$. We see that for σ sufficiently close to 1, namely $1\le\sigma<1/(1-\alpha)$, the boundary of the forward light cone becomes singular.

Theorem 3.3. *On $\mathbb{R}^2\setminus\big(\{0\}\times[0,\infty)\big)$, S belongs to the Gevrey class $G^{\frac{1}{1-\alpha}}$. Furthermore, at points (x,t) with $|x|\ne t/\sqrt{\tau}$ and $x\ne 0$ it is real analytic. For the wave front set with respect to G^σ, we have the following equalities:*

$$\mathrm{WF}_{G^\sigma}(S)=\{(0,0;\xi,\eta):(\xi,\eta)\ne(0,0)\}\cup\{(0,t;\xi,0):t>0,\xi\ne0\}\qquad \text{if } \sigma\ge\frac{1}{1-\alpha};$$

$$\mathrm{WF}_{G^\sigma}(S)=\{(0,0;\xi,\eta):(\xi,\eta)\ne(0,0)\}\cup\{(0,t;\xi,0):t>0,\xi\ne0\} \\ \cup\{(x,t;\xi,\eta):t>0,(\xi,\eta)\ne(0,0),\ |x|=t/\sqrt{\tau},(x,t)\cdot(\xi,\eta)=0\}\quad \text{if } 1\le\sigma<\frac{1}{1-\alpha}.$$

Proof. The representation (2.10) readily implies that S is real analytic at points (x,t) with $x\neq 0, |x|<t/\sqrt{\tau}$. Indeed, the integral and its derivates with respect to x and t still converge when one replaces (x,t) by $(x+z_1,t+z_2)$, z_1, $z_2\in\mathbb{C}$ with $|z_1|$ and $|z_2|$ sufficiently small.

Let us now see that S is in the Gevrey class $G^{\frac{1}{1-\alpha}}$ for $x\neq 0$. Suppose that $x>0$. Differentiating under the integral sign in (2.13), we see that

$$\frac{\partial^n}{\partial x^n}\frac{\partial^m}{\partial t^m}S(x,t)=\frac{(-1)^n}{4\pi\mathrm{i}}\int_{-\infty}^{\infty}\big(l_\alpha(\mathrm{i}y)\big)^n(\mathrm{i}y)^{n+m}\exp\big(-\mathrm{i}xyl_\alpha(\mathrm{i}y)+t\mathrm{i}y\big)\frac{\mathrm{d}y}{y}.$$

Using (2.9), this is in absolute value bounded by

$$D_1^{n+1}\bigg(\int_0^1 y^{n+m-1}\mathrm{e}^{-c_1xy^{1+\alpha}}\,\mathrm{d}y+\int_1^\infty y^{n+m-1}\mathrm{e}^{-c_2xy^{1-\alpha}}\,\mathrm{d}y\bigg)\leq D_2^{n+m+1}\Big\{1+x^{-\frac{n+m}{1-\alpha}}\Gamma\Big(\frac{n+m}{1-\alpha}\Big)\Big\},$$

for some positive constants D_1, D_2, by changing variables $y'=c_2xy^{1-\alpha}$ in the second integral. For x in a closed subset F of $\mathbb{R}\setminus\{0\}$, this is bounded by $D_F^{n+m+1}(n!m!)^{\frac{1}{1-\alpha}}$, where D_F is a positive constant depending on F.

Next, we will show that for $\sigma<1/(1-\alpha)$, the boundary of the forward light cone is contained in $\operatorname{sing\,supp}_{G^\sigma}S$. For this it is convenient to perform the change of variables $u=\sqrt{\tau}x+t$, $v=\sqrt{\tau}x-t$, so that points with (x,t) satisfying $x>0$ and $x=t/\sqrt{\tau}$ have new coordinates $(u,0)$ with $u>0$. (To treat the boundary points with $x<0$, one considers an analogous change of variables, or one uses the symmetry $S(-x,t)=S(x,t)$.) We set

$$S^\natural(u,v)=S(x,t)=S\Big(\frac{u+v}{2\sqrt{\tau}},\frac{u-v}{2}\Big).$$

Let $u>0$ and $\sigma<1/(1-\alpha)$. We claim that $(u,0)\in\operatorname{sing\,supp}_{G^\sigma}S^\natural$. This is equivalent to the statement that for every neighborhood U of $(u,0)$ and for every $C>0$ there exists some $\beta\in\mathbb{N}^2$, and some point $(a,b)\in U$ for which

$$\Big|\partial^\beta S^\natural(a,b)\Big|>C^{1+|\beta|}(\beta!)^\sigma.$$

We will actually prove something stronger, namely that there exists a constant $C>0$ and a sequence $(v_m)_m$, $v_m>0$, $v_m\to 0$ (both C and v_m depending on u) so that

$$\Big|\frac{\partial^m S^\natural}{\partial v^m}(u,-v_m)\Big|\gtrsim C^m(m^m)^{\frac{1}{1-\alpha}},\tag{3.1}$$

provided that m is sufficiently large. Showing this estimate requires an intricate technical analysis of the inverse Laplace integral. First we write $\partial^m S^\natural/\partial v^m$ as a contour integral:

$$\frac{\partial^m S^\natural}{\partial v^m}(u,v)=\frac{(-1)^m}{2\pi}\operatorname{Im}\int_0^{+\mathrm{i}\infty}l_\alpha(s)\Big(\frac{s}{2}\Big)^m\Big(\frac{l_\alpha(s)}{\sqrt{\tau}}+1\Big)^m\exp\bigg(-\frac{us}{2}\Big(\frac{l_\alpha(s)}{\sqrt{\tau}}-1\Big)-\frac{vs}{2}\Big(\frac{l_\alpha(s)}{\sqrt{\tau}}+1\Big)\bigg)\frac{\mathrm{d}s}{s}.$$

Here, we integrate along the half-line $[0,+\mathrm{i}\infty)$, and we used the symmetry $\int_{-\mathrm{i}\infty}^0=-\overline{\int_0^{+\mathrm{i}\infty}}$ to write the inverse Laplace integral as the imaginary part of the integral over the part of the contour with $\operatorname{Im}s\geq 0$.

The first step is to perform a change of variables in the above integral, which will bring out the $(m^m)^{\frac{1}{1-\alpha}}$-behavior, and to substitute a well-chosen value for v, which will in some sense simplify the remaining integral. Namely, we will set

$$s=\mu m^{\frac{1}{1-\alpha}}w,\qquad\text{with }\mu=\bigg(\frac{u}{4}\Big(\frac{1}{\tau}-1\Big)\frac{1}{1-\alpha}\bigg)^{-\frac{1}{1-\alpha}},\qquad\text{and }v=-v_m=-\frac{\kappa}{\mu}m^{-\frac{\alpha}{1-\alpha}}.$$

Here, κ is some large number depending on m, whose value will be chosen later. For the moment, we only specify a fixed range for κ, say[3] $1000/\sin(\alpha\pi) \le \kappa^{1-\alpha} \le 2000/\sin(\alpha\pi)$.

With the above substitution we get

$$\frac{\partial^m S^\natural}{\partial v^m}(u,-v_m) = \frac{(-1)^m}{2\pi}\Big(\frac{\mu}{2}\Big)^m (m^m)^{\frac{1}{1-\alpha}} \operatorname{Im}\int_0^{+\mathrm{i}\infty} l_\alpha(\mu m^{\frac{1}{1-\alpha}}w) w^m \bigg(\frac{l_\alpha(\mu m^{\frac{1}{1-\alpha}}w)}{\sqrt{\tau}}+1\bigg)^m \exp\bigg\{-\frac{u\mu m^{\frac{1}{1-\alpha}}w}{2}\bigg(\frac{l_\alpha(\mu m^{\frac{1}{1-\alpha}}w)}{\sqrt{\tau}}-1\bigg)+\frac{m\kappa w}{2}\bigg(\frac{l_\alpha(\mu m^{\frac{1}{1-\alpha}}w)}{\sqrt{\tau}}+1\bigg)\bigg\}\frac{\mathrm{d}w}{w}.$$

Let us now consider the remainders

$$E_1(s) = \frac{l_\alpha(s)}{\sqrt{\tau}} - 1, \quad E_2(s) = \frac{l_\alpha(s)}{\sqrt{\tau}} - 1 - \frac{1}{2}\bigg(\frac{1}{\tau}-1\bigg)s^{-\alpha}, \tag{3.2}$$

then by (2.8) we have

$$E_1(s) \lesssim |s|^{-\alpha}, \quad \text{and} \quad E_2(s) \lesssim |s|^{-2\alpha}, \quad \text{as } |s|\to\infty. \tag{3.3}$$

The expression for $\partial_v^m S^\natural(u,-v_m)$ can be rewritten as

$$\frac{(-1)^m}{2\pi}\Big(\frac{\mu}{2}\Big)^m (m^m)^{\frac{1}{1-\alpha}} \operatorname{Im}\int_0^{+\mathrm{i}\infty} l_\alpha(\mu m^{\frac{1}{1-\alpha}}w) \exp\bigg\{m\bigg(\kappa w - (1-\alpha)w^{1-\alpha} + \log w + \log 2 + \bigg[\frac{\kappa w}{2}E_1(\mu m^{\frac{1}{1-\alpha}}w) - \frac{u\mu m^{\frac{\alpha}{1-\alpha}}w}{2}E_2(\mu m^{\frac{1}{1-\alpha}}w) + \log\Big(1+\frac{E_1(\mu m^{\frac{1}{1-\alpha}}w)}{2}\Big)\bigg]\bigg)\bigg\}\frac{\mathrm{d}w}{w}.$$

Denoting the terms in between the square brackets $[\dots]$ by $g_m(w)$, and setting

$$f(w) \coloneqq \kappa w - \frac{1}{1-\alpha}w^{1-\alpha} + \log w + \log 2,$$

we have

$$\frac{\partial^m S^\natural}{\partial v^m}(u,-v_m) = \frac{(-1)^m}{2\pi}\Big(\frac{\mu}{2}\Big)^m (m^m)^{\frac{1}{1-\alpha}} \operatorname{Im}\int_0^{+\mathrm{i}\infty} l_\alpha(\mu m^{\frac{1}{1-\alpha}}w) \exp\big(m(f(w)+g_m(w))\big)\frac{\mathrm{d}w}{w}.$$

Lemma 3.4. *For m sufficiently large, one can choose $\kappa = \kappa_m$ in the fixed range $1000/\sin(\alpha\pi) \le \kappa^{1-\alpha} \le 2000/\sin(\alpha\pi)$ such that*

$$\operatorname{Im}\int_0^{+\mathrm{i}\infty} l_\alpha(\mu m^{\frac{1}{1-\alpha}}w) \exp\big(m(f(w)+g_m(w))\big)\frac{\mathrm{d}w}{w} \gtrsim \frac{c^m}{\sqrt{m}}.$$

Here, c is a positive constant independent of m.

We will not give the proof here, since it is rather lengthy and technical. Instead we provide a proof in Appendix A. The main idea is the following. In view of the estimates (3.3), g is small for large m. Also $l_\alpha(\mu m^{\frac{1}{1-\alpha}}w) \to \sqrt{\tau}$ as $m\to\infty$. In some sense, the analysis reduces to the analysis of the simpler integral $\int(\sqrt{\tau}/w)\mathrm{e}^{mf(w)}\,\mathrm{d}w$, which can be estimated with the saddle point method. Indeed, the constant c is related to the value of $\mathrm{e}^{\operatorname{Re} f}$ at the saddle point w_0 of f. The purpose of the free parameter κ is to control the imaginary part of the integral. For each m, we will choose a κ_m so that the argument of $\int(\sqrt{\tau}/w)\mathrm{e}^{mf(w)}\,\mathrm{d}w$ is close to $\pi/2$.

[3]The value 1000 occurring here is somewhat arbitrary, we just require some large fixed number.

Assuming Lemma 3.4, we get that

$$\left|\frac{\partial^m S^\natural}{\partial v^m}(u,-v_m)\right| \gtrsim \frac{1}{\sqrt{m}}\Big(\frac{\mu c}{2}\Big)^m (m^m)^{\frac{1}{1-\alpha}},$$

from which (3.1) follows for any $C < \mu c/2$. This shows that $(u,0) \in \operatorname{sing\,supp}_{G^\sigma} S^\natural$, as soon as $\sigma < 1/(1-\alpha)$.

To determine the wave front set with respect to G^σ, we show that at the point $(u_0,0)$, $u_0 > 0$, $S^\natural$ is "real analytic in the $(u,0)$-direction". Indeed, taking partial derivatives with respect to u we get from (2.13) with $v \geq -u/2$

$$\frac{\partial^n S^\natural}{\partial u^n}(u,v) \lesssim \int_0^\infty \Big(\frac{y}{2}\Big)^n \left|\frac{l_\alpha(\mathrm{i}y)}{\sqrt{\tau}} - 1\right|^n \exp\Big(\frac{u}{4\sqrt{\tau}} y \operatorname{Im} l_\alpha(\mathrm{i}y)\Big)\frac{\mathrm{d}y}{y}.$$

Now by (2.8), $\left|l_\alpha(\mathrm{i}y)/\sqrt{\tau} - 1\right| \lesssim y^{-\alpha}$ for large y. Using this and (2.9), we get

$$\begin{aligned}\frac{\partial^n S^\natural}{\partial u^n}(u,v) &\lesssim D_1^n\bigg(\int_0^1 y^{n-1}\exp\Big(-\frac{uc_1}{4\sqrt{\tau}}y^{1+\alpha}\Big)\,\mathrm{d}y + \int_1^\infty (y^{1-\alpha})^n \exp\Big(-\frac{uc_2}{4\sqrt{\tau}}y^{1-\alpha}\Big)\frac{\mathrm{d}y}{y}\bigg)\\ &\leq D_1^n(1 + D_2^n n!).\end{aligned}$$

Here, $D_2 = D_2(u) = 4\sqrt{\tau}/(uc_2(1-\alpha))$ can be bounded uniformly on some neighborhood of $(u_0,0)$. This implies that at the point $(u_0,0)$, the "G^1-singular frequencies[4]" can only occur along the positive and negative $(0,\eta)$-direction. Since $WF_{G^\sigma}(S^\natural)$ is symmetric about the origin in the frequency variables, and since $(u_0,0) \in \operatorname{sing\,supp}_{G^\sigma} S^\natural$ for any $\sigma < 1/(1-\alpha)$, we get for such σ that $(u_0,0;\xi,\eta) \in WF_{G^\sigma}(S^\natural) \iff \xi = 0$ and $\eta \neq 0$.

A similar argument, now using representation (2.10), shows that at points $(0,t_0)$ with $t_0 > 0$, S is real analytic in the $(0,t)$-direction, so that the wave front set with respect to G^σ, $\sigma \geq 1$, can only contain directions orthogonal to the line $x = 0$ at points $(0,t_0)$. This completes the proof of the theorem. □

4. Qualitative analysis

In this section, we will discuss some qualitative aspects of the Fractional Zener wave equation and some of its solutions. First, we consider so-called pseudo-monochromatic waves as a means to study dispersion and dissipation. Next, we will analyse the solution of the Cauchy problem (2.3) with initial condition a delta concentrated at the origin. Studying this solution will allow us to define a meaningful notion of "wave speed" for this equation.

4.1. **Dispersion and dissipation.**

When studying dispersion, one investigates the relation between the phase velocity $V(\omega)$ and the frequency ω of a wave solution $A\mathrm{e}^{\mathrm{i}\omega(t-x/V(\omega))}$. In the absence of such purely monochromatic wave solutions of the homogeneous Cauchy problem, we will investigate the response when we submit the system to a forced oscillation with frequency ω: let u be the solution of (2.3) with initial conditions $u_0 = v_0 = 0$ and force term $f(x,t) = \delta(x)H(t)\cos(\omega t)$ for some $\omega > 0$. Let us first mention, for the sake of comparison,

[4] i.e. the points (ξ,η) with $(u_0,0;\xi,\eta) \in \mathrm{WF}_{G^1}(S^\natural)$

the solution u_{cl} to the *classical* wave equation with these Cauchy data:

$$\left(\frac{\partial^2}{\partial t^2} - \frac{1}{\tau}\frac{\partial^2}{\partial x^2}\right)u_{\mathrm{cl}}(x,t) = f(x,t), \quad f(x,t) = \delta(x)H(t)\cos(\omega t), \quad u_0(x) = v_0(x) = 0$$

$$\implies u_{\mathrm{cl}}(x,t) = H(t/\sqrt{\tau} - |x|)\frac{\sqrt{\tau}}{2\omega}\sin\bigl(\omega t - \sqrt{\tau}\omega|x|\bigr).$$

This solution represents two waves traveling in opposite directions. They have wave number k related to the frequency ω via the simple dispersion relation $k(\omega) = \sqrt{\tau}\omega$, and phase speed $V(\omega) = 1/\sqrt{\tau}$.

Let us now analyze the solution in the fractional Zener case. In view of Theorem 3.1, the solution $u = S * f$ is smooth for $x \neq 0$. It has Laplace transform

$$\tilde{u}(x,s) = \frac{l_\alpha(s)}{2}\mathrm{e}^{-|x|sl_\alpha(s)}\frac{1}{s^2+\omega^2}.$$

From Proposition 2.2 it follows that $u(x,t) = 0$ if $|x| > t/\sqrt{\tau}$. If $|x| < t/\sqrt{\tau}$, we transform the contour to the contour which encircles the negative real axis, like was done to deduce (2.10). However in this case, we get two contributions from the poles at $s = \pm\mathrm{i}\omega$, and no contribution from $s = 0$. We get

$$u(x,t) = H(t/\sqrt{\tau} - |x|)\bigl(u_{\mathrm{ss}}(x,t) + u_{\mathrm{ts}}(x,t)\bigr),$$

where, using the notation $l_\alpha(\mathrm{i}\omega) = a(\omega) - \mathrm{i}b(\omega) = \rho(\omega)\mathrm{e}^{-\mathrm{i}\phi(\omega)}$ with $\operatorname{sgn} b(\omega) = \operatorname{sgn}\phi(\omega) = \operatorname{sgn}\omega$,

$$\begin{aligned} u_{\mathrm{ss}}(x,t) &= \frac{l_\alpha(\mathrm{i}\omega)}{4\mathrm{i}\omega}\mathrm{e}^{-|x|\mathrm{i}\omega l_\alpha(\mathrm{i}\omega)+\mathrm{i}\omega t} - \frac{l_\alpha(-\mathrm{i}\omega)}{4\mathrm{i}\omega}\mathrm{e}^{|x|\mathrm{i}\omega l_\alpha(-\mathrm{i}\omega)-\mathrm{i}\omega t} \\ &= \frac{\rho(\omega)}{2\omega}\mathrm{e}^{-b(\omega)\omega|x|}\sin\bigl(\omega t - a(\omega)\omega|x| - \phi(\omega)\bigr); \end{aligned} \tag{4.1}$$

$$u_{\mathrm{ts}}(x,t) = \frac{1}{4\pi\mathrm{i}}\int_0^\infty \bigl(l_\alpha(q\mathrm{e}^{-\mathrm{i}\pi})\mathrm{e}^{|x|l_\alpha(q\mathrm{e}^{-\mathrm{i}\pi})q} - l_\alpha(q\mathrm{e}^{\mathrm{i}\pi})\mathrm{e}^{|x|l_\alpha(q\mathrm{e}^{\mathrm{i}\pi})q}\bigr)\frac{\mathrm{e}^{-tq}}{q^2+\omega^2}\,\mathrm{d}q.$$

We call u_{ss} the steady state, and u_{ts} the transient state. Indeed, from the above formula it is clear that $u_{\mathrm{ts}}(x,t) \to 0$ as $t \to \infty$, locally uniformly in x.

Following Mainardi [14, Section 4.3], we call the steady state (4.1) a "pseudo-monochromatic wave" with complex wave number k satisfying the dispersion relation $k(\omega) = \omega l_\alpha(\mathrm{i}\omega)$. It has phase velocity

$$V(\omega) = 1/a(\omega),$$

and has an amplitude which is exponentially decreasing in space, with attenuation coefficient $d(\omega) = b(\omega)\omega$. The exponential dampening in space indicates dissipation, quantified by the attenuation coefficient, which has the following asymptotics, following from Lemma 2.1:

$$\begin{aligned} d(\omega) &\sim \frac{1-\tau}{2}\sin(\alpha\pi/2)\omega^{1+\alpha}, && \omega \to 0, \\ d(\omega) &\sim \frac{\sqrt{\tau}}{2}(1/\tau - 1)\sin(\alpha\pi/2)\omega^{1-\alpha}, && \omega \to \infty. \end{aligned} \tag{4.2}$$

Since $V(\omega)$ is non-constant, there is some dispersion; however $V(\omega)$ is nearly constant, in the sense that it increases monotonically from 1 to $1/\sqrt{\tau}$ when ω increases from 0 to ∞.

The fact that the phase velocity $V(\omega)$ is increasing in ω, indicates that the dispersion is anomalous. One may define the group velocity as

$$U(\omega) = \Big(\frac{\mathrm{d}(\operatorname{Re} k)}{\mathrm{d}\omega}(\omega)\Big)^{-1} = \frac{1}{a(\omega) + \omega a'(\omega)}.$$

Note that $U(\omega) \geq V(\omega)$, with equality only for $\omega = 0$ and in the limit $\omega \to \infty$. In the presence of dissipation and anomalous dispersion, this notion of group velocity looses its physical interpretation as velocity of a wave packet. However, in the next subsection we will provide a natural definition for the wave packet speed.

4.2. **Shape of the wave packet.** We denote by $K(x,t)$ the solution of (2.3) with Cauchy data $u_0(x) = \delta(x)$, $v_0(x) = 0$, $f(x,t) = 0$; so $K(x,t) = \partial_t S(x,t)$. In this section we will accurately describe the shape of the "wave packet" $K(x,t)$, when t is sufficiently large.

The solution with general initial condition $u_0(x)$ and $v_0(x) = 0$ is given by $u(x,t) = K(x,t) *_x u_0(x)$. The evolution of a general wave packet with initial shape $u_0(x)$ can then be described using the analysis of K. Let us first list some simple properties of K.

Proposition 4.1. *For any fixed $t > 0$, the function $x \mapsto K(x,t)$ is an even continuous function of x, supported in $[-t/\sqrt{\tau}, t/\sqrt{\tau}]$, with integral 1.*

Proof. The continuity, evenness, and statement on the support all follow from properties of S. In order to compute the integral $\int_{-\infty}^{\infty} K(x,t)\,\mathrm{d}x$, we consider the representation of K as inverse Laplace transform:

$$K(x,t) = \frac{1}{4\pi \mathrm{i}} \int_{a-\mathrm{i}\infty}^{a+\mathrm{i}\infty} l_\alpha(s) \exp\big(-|x|\, s l_\alpha(s) + st\big)\,\mathrm{d}s, \quad x \neq 0. \tag{4.3}$$

We have

$$\int_{-\infty}^{\infty} K(x,t)\,\mathrm{d}x = 2 \lim_{\varepsilon \to 0^+} \int_{\varepsilon}^{\infty} K(x,t)\,\mathrm{d}x = \lim_{\varepsilon \to 0^+} \frac{1}{2\pi \mathrm{i}} \int_{a-\mathrm{i}\infty}^{a+\mathrm{i}\infty} \frac{1}{s} \exp\big(-\varepsilon s l_\alpha(s) + st\big)\,\mathrm{d}s = 1.$$

Here we introduced the parameter ε to be able to swap the order of integration, which is allowed for $\varepsilon > 0$ by the Fubini-Tonelli theorem. The last equality follows for example by shifting the contour to a Hankel contour, as in (2.10): if $\varepsilon < t/\sqrt{\tau}$, one may write the above integral as

$$\frac{1}{2\pi \mathrm{i}} \int_{a-\mathrm{i}\infty}^{a+\mathrm{i}\infty} \frac{1}{s} \exp\big(-\varepsilon s l_\alpha(s) + st\big)\,\mathrm{d}s = 1 + \frac{1}{2\pi \mathrm{i}} \int_0^{\infty} \big\{\exp\big(\varepsilon q l_\alpha(q\mathrm{e}^{\mathrm{i}\pi})\big) - \exp\big(\varepsilon q l_\alpha(q\mathrm{e}^{-\mathrm{i}\pi})\big)\big\} \frac{\mathrm{e}^{-tq}}{q}\,\mathrm{d}q.$$

The last integral converges to 0 as $\varepsilon \to 0$ by dominated convergence. Indeed, the integrand converges pointwise to 0, and for the dominating function, we can argue as follows. Suppose that $\varepsilon < t/(2\sqrt{\tau})$, and let Q be such that $q \geq Q \implies \operatorname{Re} l_\alpha(q\mathrm{e}^{\pm \mathrm{i}\pi}) \leq 4\sqrt{\tau}/3$. For $0 \leq q \leq Q$, we apply Taylor's theorem to see that the integrand is dominated by $O_Q(\mathrm{e}^{-tq})$. For $q \geq Q$, the integrand is dominated by $\mathrm{e}^{-tq/3}$. □

Let us write $K_+(x,t) = H(x)K(x,t)$. The wave packet K_+ for the parameter values $\alpha = \tau = 1/2$ is plotted at time instances $t = 1, 2, 3$ in Figure 1. We will interpret K_+ as a forward moving wave packet with speed 1. To see why this is justified, consider the rescaled version

$$\mathcal{K}_t(\lambda) \coloneqq tK_+(\lambda t, t).$$

For each $t > 0$, $\mathcal{K}_t$ is a function supported in $[0, 1/\sqrt{\tau}]$ with integral $1/2$.

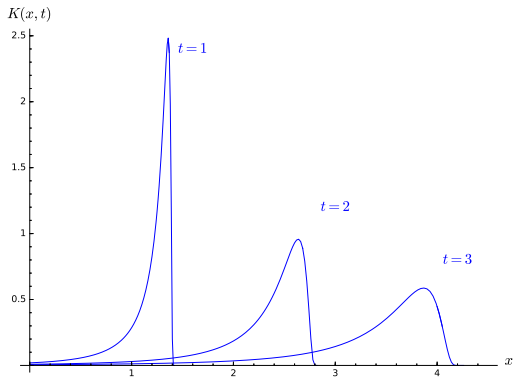

FIGURE 1. The wave packet $K(x,t)$, $x \in [0, 4.5]$, $t \in \{1, 2, 3\}$.

Proposition 4.2. *The function $\mathcal{K}_t(\lambda)$ converges to $(1/2)\delta(\lambda - 1)$ as $t \to \infty$ in the strong topology of $\mathcal{S}'(\mathbb{R})$.*

The proof consists essentially of justifying the following heuristic calculation. Sweeping technicalities such as exchanging limits, order of integration, etc. under the carpet, we get

$$\begin{aligned}\lim_{t\to\infty} \hat{\mathcal{K}}_t(\xi) &= \lim_{t\to\infty} \frac{t}{4\pi\mathrm{i}} \int_0^\infty \mathrm{e}^{-\mathrm{i}\xi\lambda} \int_{a-\mathrm{i}\infty}^{a+\mathrm{i}\infty} l_\alpha(s) \exp\bigl(-\lambda t s l_\alpha(s) + st\bigr)\,\mathrm{d}s\,\mathrm{d}\lambda \\ &= \lim_{t\to\infty} \frac{t}{4\pi\mathrm{i}} \int_{a-\mathrm{i}\infty}^{a+\mathrm{i}\infty} l_\alpha(s) \frac{\mathrm{e}^{ts}}{t s l_\alpha(s) + \mathrm{i}\xi}\,\mathrm{d}s = \lim_{t\to\infty} \frac{1}{4\pi\mathrm{i}} \int_{a'-\mathrm{i}\infty}^{a'+\mathrm{i}\infty} l_\alpha(s/t) \frac{\mathrm{e}^{s}}{s l_\alpha(s/t) + \mathrm{i}\xi}\,\mathrm{d}s \\ &= \frac{1}{4\pi\mathrm{i}} \int_{a'-\mathrm{i}\infty}^{a'+\mathrm{i}\infty} \frac{\mathrm{e}^{s}}{s + \mathrm{i}\xi}\,\mathrm{d}s = \frac{1}{2}\mathrm{e}^{-\mathrm{i}\xi} = \frac{1}{2}\mathcal{F}\{\delta(\lambda - 1)\}(\xi).\end{aligned}$$

Proof. We will show that $\hat{\mathcal{K}}_t(\xi) \to (1/2)\mathrm{e}^{-\mathrm{i}\xi}$, boundedly and locally uniformly in ξ. Then also $\hat{\mathcal{K}}_t(\xi) \to (1/2)\mathrm{e}^{-\mathrm{i}\xi}$ in the strong topology of $\mathcal{S}'(\mathbb{R})$. In order to avoid convergency issues of (4.3) when λ is close to 0, we first show that for some $\lambda_0 > 0$, $\int_0^{\lambda_0} \bigl|\mathcal{K}_t(\lambda)\bigr|\,\mathrm{d}\lambda \to 0$ as $t \to \infty$. Set

$$L \coloneqq \sup_{|\arg s| \le \pi} \bigl|l_\alpha(s)\bigr|, \quad \lambda_0 \coloneqq \frac{1}{2L}. \tag{4.4}$$

Note that $L \ge 1$, $\lambda_0 \le 1/2$. Using (2.10) and changing variables, we have

$$\mathcal{K}_t(\lambda) = \frac{-1}{4\pi\mathrm{i}} \int_0^\infty \Bigl\{ l_\alpha\Bigl(\frac{q}{t}\mathrm{e}^{\mathrm{i}\pi}\Bigr) \mathrm{e}^{\lambda l_\alpha(q\mathrm{e}^{\mathrm{i}\pi}/t)q} - l_\alpha\Bigl(\frac{q}{t}\mathrm{e}^{-\mathrm{i}\pi}\Bigr) \mathrm{e}^{\lambda l_\alpha(q\mathrm{e}^{-\mathrm{i}\pi}/t)q} \Bigr\} \mathrm{e}^{-q}\,\mathrm{d}q.$$

For $\lambda \le \lambda_0$, $\mathcal{K}_t(\lambda)$ is bounded by $\frac{2L}{4\pi}\int_0^\infty \mathrm{e}^{(1/2)q - q}\,\mathrm{d}q = L/\pi$, and converges pointwise to 0 as $t \to \infty$, since the above integrand is dominated by the integrable function $\mathrm{e}^{-(1/2)q}$, and converges pointwise to 0. By bounded convergence, it then follows that $\int_0^{\lambda_0} \mathcal{K}_t(\lambda)\mathrm{e}^{-\mathrm{i}\xi\lambda}\,\mathrm{d}\lambda$ converges to 0, uniformly in ξ.

To prove the claim, it then suffices to show that $\int_{\lambda_0}^\infty \mathcal{K}_t(\lambda)\mathrm{e}^{-\mathrm{i}\xi\lambda}\,\mathrm{d}\lambda$ converges boundedly and locally uniformly to $(1/2)\mathrm{e}^{-\mathrm{i}\xi}$. Suppose $\xi > 0$ (the case $\xi = 0$ follows from Proposition

4.1). We use representation (4.3) with some $a > 0$. Since

$$\int_{a-\mathrm{i}\infty}^{a+\mathrm{i}\infty}\int_{\lambda_0}^{\infty}\left|l_\alpha(s)\exp\bigl(-\lambda t s l_\alpha(s)+ts-\mathrm{i}\xi\lambda\bigr)\right|\mathrm{d}\lambda\,|\mathrm{d}s| \\ =\int_{a-\mathrm{i}\infty}^{a+\mathrm{i}\infty}\bigl|l_\alpha(s)\bigr|\,\frac{\exp\bigl(-\lambda_0 t\,\mathrm{Re}(sl_\alpha(s))+at\bigr)}{t\,\mathrm{Re}(sl_\alpha(s))}|\mathrm{d}s|<\infty,$$

we may interchange the order of integration by the Fubini-Tonelli theorem. We get

$$\int_{\lambda_0}^{\infty}\mathcal{K}_t(\lambda)\mathrm{e}^{-\mathrm{i}\xi\lambda}\,\mathrm{d}\lambda=\frac{1}{4\pi\mathrm{i}}\int_{a-\mathrm{i}\infty}^{a+\mathrm{i}\infty}\frac{l_\alpha(s)}{sl_\alpha(s)+\mathrm{i}\xi/t}\exp\bigl(-\lambda_0 t s l_\alpha(s)+ts-\mathrm{i}\xi\lambda_0\bigr)\,\mathrm{d}s.$$

We evaluate this integral by shifting the contour to a Hankel contour encircling the negative real axis, as was done to obtain (2.10). The integral has one singularity, namely the unique zero $s(\xi,t)$ of the function $sl_\alpha(s)+\mathrm{i}\xi/t$. Indeed, by applying the argument principle, one sees that this function has a unique zero in the set bounded by the line $[\mathrm{e}^{-\mathrm{i}\pi}R,R]$ and the semicircle with center 0 and radius R in the lower half plane, and no zeros in the set bounded by the line $[\mathrm{e}^{\mathrm{i}\pi}R,R]$ and the semicircle with center 0 and radius R in the upper half plane, provided that $\xi>0$ and R is sufficiently large. Since

$$\mathrm{Re}\,s(\xi,t)l_\alpha(s(\xi,t))=\mathrm{Re}\,s(\xi,t)\,\mathrm{Re}\,l_\alpha(s(\xi,t))-\mathrm{Im}\,s(\xi,t)\,\mathrm{Im}\,l_\alpha(s(\xi,t))=0,$$

$\mathrm{Re}\,l_\alpha(s)>0$ (see (2.6)), and $\mathrm{Im}\,s\,\mathrm{Im}\,l_\alpha(s)\le 0$, we have that $\mathrm{Re}\,s(\xi,t)\le 0$.

For large t, this zero satisfies the asymptotic $s(\xi,t)\sim-\mathrm{i}\xi/t$, as $t\to\infty$, locally uniformly in ξ. This can be seen by applying Rouché's theorem on the circle $\left|s+\mathrm{i}\xi/t\right|=|\xi|/t^{1+\alpha/2}$. By (2.7), we have for sufficiently large t that on this circle

$$\left|sl_\alpha(s)+\mathrm{i}\xi/t-(s+\mathrm{i}\xi/t)\right|\lesssim\frac{|\xi|^{1+\alpha}}{t^{1+\alpha}}<\frac{|\xi|}{t^{1+\alpha/2}}=|s+\mathrm{i}\xi t|\,.$$

Hence, $sl_\alpha(s)+\mathrm{i}\xi/t$ has the same number of zeros (i.e. 1) as $s+\mathrm{i}\xi/t$ inside this circle. We get the following representation for sufficiently large t:

$$\begin{aligned}&\int_{\lambda_0}^{\infty}\mathcal{K}_t(\lambda)\mathrm{e}^{-\mathrm{i}\xi\lambda}\,\mathrm{d}\lambda\\&=\frac{1}{2}l_\alpha(s(\xi,t))\exp\bigl(-\lambda_0 t s(\xi,t)l_\alpha(s(\xi,t))+ts(\xi,t)-\mathrm{i}\xi\lambda_0\bigr)\\&\quad+\frac{\mathrm{e}^{-\mathrm{i}\xi\lambda_0}}{4\pi\mathrm{i}}\int_0^{\infty}\left\{\frac{l_\alpha(q\mathrm{e}^{\mathrm{i}\pi})\mathrm{e}^{\lambda_0 tql_\alpha(q\mathrm{e}^{\mathrm{i}\pi})}}{\mathrm{i}\xi/t-ql_\alpha(q\mathrm{e}^{\mathrm{i}\pi})}-\frac{l_\alpha(q\mathrm{e}^{-\mathrm{i}\pi})\mathrm{e}^{\lambda_0 tql_\alpha(q\mathrm{e}^{-\mathrm{i}\pi})}}{\mathrm{i}\xi/t-ql_\alpha(q\mathrm{e}^{-\mathrm{i}\pi})}\right\}\mathrm{e}^{-tq}\,\mathrm{d}q\\&=\frac{1}{2}l_\alpha(s(\xi,t))\mathrm{e}^{ts(\xi,t)}+\frac{\mathrm{e}^{-\mathrm{i}\xi\lambda_0}}{4\pi\mathrm{i}}\int_0^{\infty}\left\{\frac{l_\alpha(\mathrm{e}^{\mathrm{i}\pi}q/t)\mathrm{e}^{\lambda_0 ql_\alpha(\mathrm{e}^{\mathrm{i}\pi}q/t)}}{\mathrm{i}\xi-ql_\alpha(\mathrm{e}^{\mathrm{i}\pi}q/t)}-\frac{l_\alpha(\mathrm{e}^{-\mathrm{i}\pi}q/t)\mathrm{e}^{\lambda_0 ql_\alpha(\mathrm{e}^{-\mathrm{i}\pi}q/t)}}{\mathrm{i}\xi-ql_\alpha(\mathrm{e}^{-\mathrm{i}\pi}q/t)}\right\}\mathrm{e}^{-q}\,\mathrm{d}q.\end{aligned}$$

Since $\mathrm{Re}\,s(\xi,t)\le 0$, the first term is bounded, and it converges locally uniformly to $(1/2)\mathrm{e}^{-\mathrm{i}\xi}$, in view of the asymptotic $s(\xi,t)\sim-\mathrm{i}\xi/t$. The integral converges uniformly to 0 as $t\to\infty$. Given $\varepsilon>0$, one can first find some Q so that

$$\int_Q^{\infty}2\frac{L\mathrm{e}^{\lambda_0 Lq}}{(\sqrt{\tau}/2)q}\mathrm{e}^{-q}\,\mathrm{d}q\le\frac{4L}{\sqrt{\tau}}\int_Q^{\infty}\frac{\mathrm{e}^{-(1/2)q}}{q}\,\mathrm{d}q\le\frac{\varepsilon}{2}.$$

On the interval $[0,Q]$, we will use a Taylor approximation. We get

$$\frac{l_\alpha(\mathrm{e}^{\pm\mathrm{i}\pi}q/t)\mathrm{e}^{\lambda_0 q l_\alpha(\mathrm{e}^{\pm\mathrm{i}\pi}q/t)}}{\mathrm{i}\xi - q l_\alpha(\mathrm{e}^{\pm\mathrm{i}\pi}q/t)} = \frac{\mathrm{e}^{\lambda_0 q}}{\mathrm{i}\xi - q}\biggl\{1 + O_Q\biggl(\frac{q^\alpha}{t^\alpha} + \frac{q^{1+\alpha}}{t^\alpha} + \frac{q^{1+\alpha}}{t^\alpha|\mathrm{i}\xi - q|}\biggr)\biggr\}.$$

Let then t be so large that

$$\left|\frac{l_\alpha(\mathrm{e}^{\mathrm{i}\pi}q/t)\mathrm{e}^{\lambda_0 q l_\alpha(\mathrm{e}^{\mathrm{i}\pi}q/t)}}{\mathrm{i}\xi - q l_\alpha(\mathrm{e}^{\mathrm{i}\pi}q/t)} - \frac{l_\alpha(\mathrm{e}^{-\mathrm{i}\pi}q/t)\mathrm{e}^{\lambda_0 q l_\alpha(\mathrm{e}^{-\mathrm{i}\pi}q/t)}}{\mathrm{i}\xi - q l_\alpha(\mathrm{e}^{-\mathrm{i}\pi}q/t)}\right| \leq \frac{\alpha\varepsilon}{2Q^\alpha}q^{\alpha-1}, \qquad \text{for } q \in [0,Q].$$

Then $\bigl|\int_0^Q \ldots\bigr| \leq \varepsilon/2$. We conclude that $\int_{\lambda_0}^\infty \mathcal{K}_t(\lambda)\mathrm{e}^{-\mathrm{i}\xi\lambda}\,\mathrm{d}\lambda$ converges boundedly and locally uniformly to $(1/2)\mathrm{e}^{-\mathrm{i}\xi}$, which finishes the proof of the proposition. □

This proposition gives some indication that K_+ is concentrated around $x = t$. This "concentration" around $x = t$ is however much less drastic than the concentration of $\mathcal{K}_t$ around $\lambda = 1$. It is for example not the case that $K_+(x,t) - (1/2)\delta(x-t) \to 0$. Actually, the wave packet will spread out in space, albeit on a scale smaller than $|x-t| \simeq t$. Namely, we will see that $K_+(x,t)$ can be described as a wave packet of height $\simeq t^{-\frac{1}{1+\alpha}}$ and width $\simeq t^{\frac{1}{1+\alpha}}$ centered around $x = t$. Let us first give a dispersion estimate for $K_+(x,t)$. For later use, we also bound the derivatives with respect to x.

Proposition 4.3. *For every $n \in \mathbb{N}$, we have the bound*

$$\left\|\frac{\partial^n K_+}{\partial x^n}\right\|_{L_x^\infty} \lesssim_n t^{-\frac{n+1}{1+\alpha}}.$$

Proof. Let L and λ_0 be as before, see (4.4). Suppose first that $0 \leq x \leq \lambda_0 t$. Then

$$\begin{aligned}\frac{\partial^n K_+}{\partial x^n}(x,t) &= \frac{1}{4\pi\mathrm{i}}\int_0^\infty q^n\Bigl(l_\alpha(q\mathrm{e}^{-\mathrm{i}\pi})^{n+1}\mathrm{e}^{xql_\alpha(q\mathrm{e}^{-\mathrm{i}\pi})} - l_\alpha(q\mathrm{e}^{\mathrm{i}\pi})^{n+1}\mathrm{e}^{xql_\alpha(q\mathrm{e}^{\mathrm{i}\pi})}\Bigr)\mathrm{e}^{-qt}\,\mathrm{d}q \\ &\lesssim_n \int_0^\infty q^n\mathrm{e}^{xLq-tq}\,\mathrm{d}q \leq \int_0^\infty q^n\mathrm{e}^{-(1/2)tq}\,\mathrm{d}q \lesssim_n t^{-n-1}.\end{aligned}$$

For $x \geq \lambda_0 t$, we use representation (4.3) and move the contour to the imaginary axis. We get

$$\frac{\partial^n K_+}{\partial x^n}(x,t) = \frac{1}{4\pi}\int_{-\infty}^\infty (-\mathrm{i}y)^n l_\alpha(\mathrm{i}y)^{n+1}\exp\bigl(-xiyl_\alpha(\mathrm{i}y)+tiy\bigr)\,\mathrm{d}y \lesssim_n \int_0^\infty y^n\exp\bigl(xy\,\mathrm{Im}\,l_\alpha(\mathrm{i}y)\bigr)\,\mathrm{d}y.$$

By (2.9), we get

$$\frac{\partial^n K_+}{\partial x^n}(x,t) \lesssim \int_0^1 y^n\exp\bigl(-\lambda_0 t c_1 y^{1+\alpha}\bigr)\,\mathrm{d}y + \int_1^\infty y^n\exp\bigl(-\lambda_0 t c_2 y^{1-\alpha}\bigr)\,\mathrm{d}y \lesssim_n t^{-\frac{n+1}{1+\alpha}}.$$

□

We will now give a precise description of the shape of the wave packet, in the limit $t \to \infty$. For this, we introduce the function

$$k_t(\nu) := t^{\frac{1}{1+\alpha}}K_+(t+\nu t^{\frac{1}{1+\alpha}}, t), \quad \nu \in \mathbb{R}.$$

Theorem 4.4. *There exists a function $k_\infty(\nu)$ with the property that $k_t(\nu) \to k_\infty(\nu)$ as $t \to \infty$, locally uniformly in ν. The function k_∞ has the following representation:*

$$k_\infty(\nu) = \frac{1}{4\pi}\int_{-\infty}^\infty \exp\Bigl(\frac{1-\tau}{2}(\mathrm{i}w)^{1+\alpha} - \mathrm{i}\nu w\Bigr)\,\mathrm{d}w. \tag{4.5}$$

Also, $\partial_\nu^n k_t(\nu) \to k_\infty^{(n)}(\nu)$ locally uniformly in ν, for every $n \in \mathbb{N}$.

Proof. We again use representation (4.3) on the imaginary axis. We split the range of integration into three parts as follows:

$$\begin{aligned} k_t(\nu) &= \frac{t^{\frac{1}{1+\alpha}}}{4\pi \mathrm{i}} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} l_\alpha(s) \exp\big(ts(1-l_\alpha(s)) - \nu t^{\frac{1}{1+\alpha}} s l_\alpha(s)\big)\,\mathrm{d}s \\ &= \frac{t^{\frac{1}{1+\alpha}}}{4\pi}\bigg(\int_{|y|\le t^{-\frac{1}{1+2\alpha}}} + \int_{t^{-\frac{1}{1+2\alpha}}\le|y|\le 1} + \int_{|y|\ge 1}\bigg) l_\alpha(\mathrm{i}y) \exp\big(t\mathrm{i}y(1-l_\alpha(\mathrm{i}y)) - \nu t^{\frac{1}{1+\alpha}} \mathrm{i}y l_\alpha(\mathrm{i}y)\big)\,\mathrm{d}y \\ &=: I_1 + I_2 + I_3. \end{aligned}$$

When $0 \le y \le 1$, we have $\operatorname{Im} l_\alpha(\mathrm{i}y) \le -c_1 y^\alpha$, see (2.9). By (2.7), we also have $\big|\operatorname{Im} l_\alpha(\mathrm{i}y)\big| \le \tilde{c}_1 y^\alpha$ for some $\tilde{c}_1 > 0$. This yields

$$\begin{aligned} I_2 &\lesssim t^{\frac{1}{1+\alpha}} \int_{t^{-\frac{1}{1+2\alpha}}}^1 \exp\big(-tc_1 y^{1+\alpha} + |\nu|\, t^{\frac{1}{1+\alpha}} \tilde{c}_1 y^{1+\alpha}\big)\,\mathrm{d}y \\ &\lesssim t^{\frac{1}{1+\alpha}} \int_{t^{-\frac{1}{1+2\alpha}}}^1 \exp\big(-t(c_1/2) y^{1+\alpha}\big)\,\mathrm{d}y \lesssim \int_{(c_1/2)^{\frac{1}{1+\alpha}} t^{\frac{\alpha}{(1+\alpha)(1+2\alpha)}}}^{(tc_1/2)^{\frac{1}{1+\alpha}}} \mathrm{e}^{-w^{1+\alpha}}\,\mathrm{d}w \to 0, \end{aligned}$$

as $t \to \infty$. Here we used that $\tilde{c}_1 |\nu|\, t^{\frac{1}{1+\alpha}} \le c_1 t/2$ for sufficiently large t, uniformly for ν in compact sets.

When $y \ge 1$, we have $\operatorname{Im} l_\alpha(\mathrm{i}y) \le -c_2 y^{-\alpha}$ and $\big|\operatorname{Im} l_\alpha(\mathrm{i}y)\big| \le \tilde{c}_2 y^{-\alpha}$ for some $\tilde{c}_2 > 0$ (see again (2.9) and (2.8)). This implies that

$$\begin{aligned} I_3 &\lesssim t^{\frac{1}{1+\alpha}} \int_1^\infty \exp\big(-tc_2 y^{1-\alpha} + |\nu|\, t^{\frac{1}{1+\alpha}} \tilde{c}_2 y^{1-\alpha}\big)\,\mathrm{d}y \\ &\lesssim t^{\frac{1}{1+\alpha}} \int_1^\infty \exp\big(-t(c_2/2) y^{1-\alpha}\big)\,\mathrm{d}y \lesssim t^{\frac{1}{1+\alpha}-\frac{1}{1-\alpha}} \int_{(tc_2/2)^{\frac{1}{1-\alpha}}}^\infty \mathrm{e}^{-w^{1-\alpha}}\,\mathrm{d}w \to 0, \end{aligned}$$

as $t \to \infty$, uniformly for ν in compact sets. Hence, in the limit $t \to \infty$, only the contribution from I_1 remains.

To treat I_1, we will approximate l_α using (2.7). For the remainders, we write

$$\tilde{E}_1(s) = l_\alpha(s) - 1, \quad \tilde{E}_2(s) \coloneqq l_\alpha(s) - \Big(1 - \frac{1-\tau}{2} s^\alpha\Big).$$

In the integral I_1, change variables to $w = t^{\frac{1}{1+\alpha}} y$ and approximate $l_\alpha(\mathrm{i}y)$ by $1 - (1/2)(1-\tau)(\mathrm{i}y)^\alpha$. This gives

$$I_1 = \frac{1}{4\pi} \int_{-t^\beta}^{t^\beta} l_\alpha(t^{-\frac{1}{1+\alpha}}\mathrm{i}w) \exp\Big(\frac{1-\tau}{2}(\mathrm{i}w)^{1+\alpha} - \mathrm{i}\nu w - t^{\frac{\alpha}{1+\alpha}} \mathrm{i}w \tilde{E}_2(t^{-\frac{1}{1+\alpha}}\mathrm{i}w) - \nu \mathrm{i}w \tilde{E}_1(t^{-\frac{1}{1+\alpha}}\mathrm{i}w)\Big)\,\mathrm{d}w.$$

Here, $\beta = \frac{\alpha}{(1+\alpha)(1+2\alpha)}$. Since $\tilde{E}_1(s) \lesssim |s|^\alpha$ and $\tilde{E}_2(s) \lesssim |s|^{2\alpha}$ for $s \to 0$, the integrand converges pointwise to the function $\exp\big((1/2)(1-\tau)(\mathrm{i}w)^{1+\alpha} - \mathrm{i}\nu w\big)$, uniformly for ν in compact sets. Furthermore, on the interval $[-t^\beta, t^\beta]$,

$$\nu \mathrm{i}w \tilde{E}_1(t^{-\frac{1}{1+\alpha}}\mathrm{i}w) \lesssim t^{-\frac{\alpha^2}{(1+\alpha)(1+2\alpha)}} \lesssim 1, \quad t^{\frac{\alpha}{1+\alpha}} \mathrm{i}w \tilde{E}_2(t^{-\frac{1}{1+\alpha}}\mathrm{i}w) \lesssim 1,$$

so the integrand is dominated by the integrable function $\exp\bigl(-(1/2)(1-\tau)\sin(\alpha\pi/2)|w|^{1+\alpha}\bigr)$. By dominated convergence,

$$I_1 \to \frac{1}{4\pi}\int_{-\infty}^{\infty} \exp\Bigl(\frac{1-\tau}{2}(\mathrm{i}w)^{1+\alpha} - \mathrm{i}\nu w\Bigr)\,\mathrm{d}w =: k_\infty(\nu),$$

as $t \to \infty$, uniformly for ν in compact sets.

The proof for $\partial_\nu^n k_t(\nu)$ is completely analogous. The corresponding integrals I_2 and I_3 tend to zero, while the integral I_1 converges to

$$\frac{1}{4\pi}\int_{-\infty}^{\infty} (-\mathrm{i}w)^n \exp\Bigl(\frac{1-\tau}{2}(\mathrm{i}w)^{1+\alpha} - \mathrm{i}\nu w\Bigr)\,\mathrm{d}w = \frac{\mathrm{d}^n}{\mathrm{d}\nu^n}\,k_\infty(\nu).$$

□

In Figure 2 we compare the shape of the wave packet at some large time with the function $k_\infty(\nu)$. We choose again parameter values $\alpha = \tau = 1/2$. On the left we show K at time $t = 100$ scaled by factor $t^{\frac{1}{1+\alpha}} = 100^{\frac{2}{3}} \approx 21.54$, for $x \in [80, 120]$. On the right we show a plot of the function k_∞ for $\nu \in [-1, 1]$. In Figure 3 we plot k_∞ in a larger range.

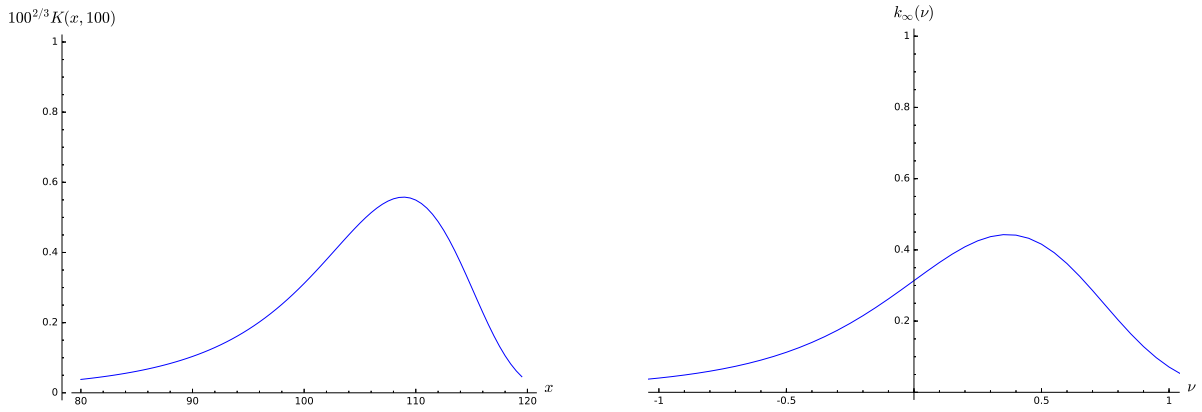

FIGURE 2. Comparison between $100^{\frac{1}{1+\alpha}}K(x, 100)$ and $k_\infty(\nu)$.

Let us list some properties of the function $k_\infty(\nu)$.

- From the representation (4.5), it follows immediately that k_∞ belongs to the Gevrey class $G^{\frac{1}{1+\alpha}}(\mathbb{R})$. In particular, it is an entire function.
- The function k_∞ is real valued, since it can be written as

$$k_\infty(\nu) = \frac{1}{2\pi}\,\mathrm{Re}\int_0^{\infty} \exp\Bigl(\frac{1-\tau}{2}(\mathrm{i}w)^{1+\alpha} - \mathrm{i}\nu w\Bigr)\,\mathrm{d}w.$$

  The integral from 0 to ∞ above also defines an entire function. It belongs to a general class of Fourier-Laplace transforms which was investigated in [5]. In that paper, a complete asymptotic analysis of such functions was performed on half-lines emanating from the origin. Taking the first term from the asymptotic series [5, Equation (4.1)] with non-vanishing real part, we see that

$$k_\infty(\nu) \sim \frac{\sin(\alpha\pi)}{2\pi}\Bigl(\frac{1-\tau}{2}\Bigr)^{\frac{1}{1+\alpha}} \frac{\Gamma(2+\alpha)}{|\nu|^{2+\alpha}}, \quad \text{as } \nu \to -\infty. \tag{4.6}$$

For $\nu \to +\infty$, every term in the asymptotic series [5, Equation (4.1)] is purely imaginary, so this only tells us that $k_\infty(\nu) \lesssim_n \nu^{-n}$, as $\nu \to \infty$, for every n. However, using the saddle-point method, one can determine its precise asymptotic. We restrict ourselves here to just sketching the method. Suppose $\nu > 0$. In (4.5), change variables $w = \nu^{\frac{1}{\alpha}} y$ to get

$$k_\infty(\nu) = \frac{\nu^{\frac{1}{\alpha}}}{4\pi} \int_{-\infty}^{\infty} \exp\Big\{ \nu^{1+\frac{1}{\alpha}} \Big(\frac{1-\tau}{2} (\mathrm{i}y)^{1+\alpha} - \mathrm{i}y \Big) \Big\} \, \mathrm{d}y.$$

The function $((1-\tau)/2) z^{1+\alpha} - z$ is holomorphic on $\mathbb{C} \setminus (-\infty, 0]$ and has a unique saddle point $z_0 = \big((1-\tau)(1+\alpha)/2\big)^{-\frac{1}{\alpha}}$. One can shift the contour of integration to a contour passing through this saddle point via the "steepest path." Applying the saddle point method then yields the following asymptotic:

$$k_\infty(\nu) \sim \frac{1}{4} \sqrt{\frac{2 z_0}{\alpha \pi}} \nu^{\frac{1}{2\alpha} - \frac{1}{2}} \exp\Big(-\frac{\alpha z_0}{1+\alpha} \nu^{1+\frac{1}{\alpha}} \Big), \quad \text{as } \nu \to \infty. \tag{4.7}$$

- Another interesting property is that k_∞ is a close cousin of the Gaussian and Airy functions. Indeed, renormalizing by setting

$$k(\nu) \coloneqq \left(\frac{(1-\tau)(1+\alpha)}{2} \right)^{\frac{1}{1+\alpha}} k_\infty \left(-\left(\frac{(1-\tau)(1+\alpha)}{2} \right)^{\frac{1}{1+\alpha}} \nu \right),$$

we have that k satisfies the fractional ordinary differential equation

$${}_{-\infty}D_\nu^\alpha k(\nu) + \nu k(\nu) = 0.$$

This follows immediately by taking Fourier transforms, since $\hat{k}(\xi) = \exp\big(\frac{1}{1+\alpha}(\mathrm{i}\xi)^{1+\alpha}\big)$.

- From the previous property we can deduce that k and hence also k_∞ is everywhere positive. From both asymptotics (4.6) and (4.7) we see that k is eventually positive and so it has (at most) finitely many zeros. Suppose that it does have zeros. Let ν_0 be the smallest one. Then $k(\nu) > 0$ for every $\nu < \nu_0$. From the above differential equation, it follows that

$${}_{-\infty}D_\nu^\alpha k(\nu_0) = \frac{1}{\Gamma(-\alpha)} \, \mathrm{F.p.} \int_{-\infty}^{\nu_0} k(\nu) (\nu_0 - \nu)^{-\alpha-1} \, \mathrm{d}\nu = 0.$$

Here we used the Hadamard finite part to compute the α-th order fractional derivative. However, if ν_0 is a zero of k, then $k(\nu)(\nu_0 - \nu)^{-\alpha-1}$ is integrable. Hence we get

$$0 = \int_{-\infty}^{\nu_0} k(\nu) (\nu_0 - \nu)^{-\alpha-1} \, \mathrm{d}\nu > 0,$$

a contradiction. We conclude that k has no zeros, so it is everywhere positive.

Remark 4.5. Theorem 4.4 can be rephrased by saying that $K_+(t + \nu t^{\frac{1}{1+\alpha}}, t) \sim k_\infty(\nu) t^{-\frac{1}{1+\alpha}}$, as $t \to \infty$, locally uniformly in ν. In particular we have

$$K_+(t,t) \sim k_\infty(0) t^{-\frac{1}{1+\alpha}} = \frac{1}{2\pi(1+\alpha)} \sin\Big(\frac{\pi}{1+\alpha}\Big) \Gamma\Big(\frac{1}{1+\alpha}\Big) \Big(\frac{2}{1-\tau}\Big)^{\frac{1}{1+\alpha}} t^{-\frac{1}{1+\alpha}}.$$

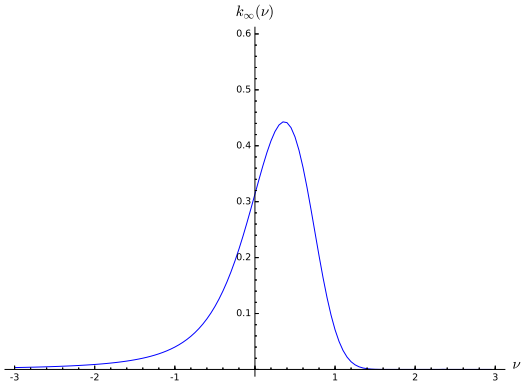

FIGURE 3. The function k_∞ for $\nu \in [-3, 3]$.

It is also possible to determine the asymptotics of $K_+(\lambda t, t)$ with $\lambda \neq 1$. If $0 \leq \lambda < 1$, we have the power decay

$$K_+(\lambda t, t) \sim \frac{\sin(\alpha\pi)(1-\tau)(1+2\lambda)}{4\pi(1-\lambda)^{2+\alpha}}\Gamma(1+\alpha)t^{-1-\alpha}.$$

This asymptotic relation holds uniformly for $\lambda \in [0, \lambda_1]$, for any $\lambda_1 < 1$. If $1 < \lambda < 1/\sqrt{\tau}$, we have exponential decay. Set $f(s) = s - \lambda s l_\alpha(s)$, and denote by a_λ the unique positive real zero of the function $f'(s)$. Then

$$K_+(\lambda t, t) \sim \frac{l_\alpha(a_\lambda)}{4}\sqrt{\frac{2}{\pi f''(a_\lambda)}}\frac{\mathrm{e}^{f(a_\lambda)t}}{\sqrt{t}}.$$

We remark that $f(a_\lambda) < 0$ and $f''(a_\lambda) > 0$, and that $f(a_\lambda) \to -\infty$ if $\lambda \to 1/\sqrt{\tau}$.

Both of these asymptotic relations can be obtained via the method of steepest descent, but we omit the details.

In view of the preceding discussion, it is natural to consider K_+ as a dispersive wave packet with speed 1 and wave front speed $1/\sqrt{\tau}$. In previous works on (fractional) wave equations, several ways of assigning a velocity to waves in dissipative media are used. See for example [13]. If one defines the maximum position, the center of gravity, and the center of mass of the wave respectively as

$$\begin{aligned} x^{\max}(t) &= \operatorname{argmax}_x K_+(x,t); \\ x^g(t) &= \frac{\int_0^\infty x K_+(x,t)\,\mathrm{d}x}{\int_0^\infty K_+(x,t)\,\mathrm{d}x}; \\ x^m(t) &= \frac{\int_0^\infty x K_+^2(x,t)\,\mathrm{d}x}{\int_0^\infty K_+^2(x,t)\,\mathrm{d}x}; \end{aligned}$$

one can define associated velocities as the instantaneous or average propagation speed of these points. In our case, it would appear that

$$x^{\max}(t) \sim t, \quad x^g(t) \sim t, \quad x^m(t) \sim t,$$

so the associated velocities would all be asymptotically equal to 1.

Remark 4.6. Note that (1.1) resulted from a reduction to dimensionless quantities (i.e. (2.2)). For the model in its original form, so with Newton's second law in the form $\partial_x \sigma = \rho \partial_t^2 u$, including the density constant ρ, and with the fractional Zener constitutive law in the form (2.1), the wave front speed and wave packet speed are given by $\sqrt{E\tau_\varepsilon/(\rho\tau_\sigma)}$ and $\sqrt{E/\rho}$ respectively. These speeds can be related to the limiting values of the material functions of the body. We have

$$\begin{aligned}\text{wave front speed } &= \sqrt{\frac{E\tau_\varepsilon}{\rho\tau_\sigma}} = \frac{1}{\sqrt{\rho J_g}} = \sqrt{\frac{G_g}{\rho}},\\ \text{wave packet speed } &= \sqrt{\frac{E}{\rho}} = \frac{1}{\sqrt{\rho J_e}} = \sqrt{\frac{G_e}{\rho}}.\end{aligned}$$

Here, J_g and G_g are the glass compliance and glass modulus, related to the instantaneous behavior of the material, and J_e and G_e are the equilibrium compliance and equilibrium modulus, related to the equilibrium behavior of the material, see e.g. [14, Chapter 2]. In dimensionless form $G_g = 1/J_g = 1/\tau$ and $G_e = 1/J_e = 1$. For a more general class of materials (including the fractional Zener model) they are calculated and presented in [16, Table 1].

Finally, let us describe the shape of the solution with initial conditions $u(x,0) = u_0(x)$, $\partial_t u(x,0) = 0$, given by $u(x,t) = K(x,t) *_x u_0(x)$.

Theorem 4.7. *Suppose $u_0 \in \mathcal{S}$ and suppose that $\int u_0(x)\,\mathrm{d}x \neq 0$. Then*

$$\|u\|_{L^\infty_x} \lesssim t^{-\frac{1}{1+\alpha}} \left\|u_0\right\|_{L^1_x},$$

and $u(x,t) = K_+(x,t) *_x u_0(x) + K_+(-x,t) *_x u_0(x) =: u_+(x,t) + u_-(x,t)$, where

$$t^{\frac{1}{1+\alpha}} u_+(t + \nu t^{\frac{1}{1+\alpha}}, t) \to A k_\infty(\nu), \quad t^{\frac{1}{1+\alpha}} u_-(-t - \nu t^{\frac{1}{1+\alpha}}, t) \to A k_\infty(\nu), \quad A = \int_{-\infty}^{\infty} u_0(x)\,\mathrm{d}x,$$

as $t \to \infty$, locally uniformly in ν.

Proof. By Proposition 4.3,

$$u(x,t) = \int_{x-t/\sqrt{\tau}}^{x+t/\sqrt{\tau}} K(x-y,t) u_0(y)\,\mathrm{d}y \lesssim t^{-\frac{1}{1+\alpha}} \int_{-\infty}^{\infty} |u_0(x)|\,\mathrm{d}x.$$

Set now $x = t + \nu t^{\frac{1}{1+\alpha}}$. Rewriting the convolution in terms of $k_t(\nu)$ gives

$$u_+(x,t) = \int K_+(x-y,t) u_0(y)\,\mathrm{d}y = t^{-\frac{1}{1+\alpha}} \int k_t(\nu - y t^{-\frac{1}{1+\alpha}}) u_0(y)\,\mathrm{d}y.$$

Applying Theorem 4.4 and dominated convergence, we see that locally uniformly in ν

$$t^{\frac{1}{1+\alpha}} u_+(t + \nu t^{\frac{1}{1+\alpha}}, t) \to \int k_\infty(\nu) u_0(y)\,\mathrm{d}y = \left(\int_{-\infty}^{\infty} u_0(y)\,\mathrm{d}y\right) k_\infty(\nu).$$

The proof for u_- is analogous. □

Remark 4.8. If $\int u_0(x)\,\mathrm{d}x = 0$ and $\int x u_0(x)\,\mathrm{d}x \neq 0$, than u_0 has a primitive $u_0^{(-1)}$ in $\mathcal{S}$. We can integrate by parts in the convolution:

$$u(x,t) = \int_{x-t/\sqrt{\tau}}^{x+t/\sqrt{\tau}} K(x-y,t) u_0(y)\,\mathrm{d}y = \int_{x-t/\sqrt{\tau}}^{x+t/\sqrt{\tau}} \frac{\partial K}{\partial x}(x-y,t) u_0^{(-1)}(y)\,\mathrm{d}y,$$

since the boundary terms vanish. Similarly as in the above proof, using Proposition 4.3 and Theorem 4.4, now defining $u_+(x,t) = \partial_x K_+(x,t) *_x u_0^{(-1)}(x)$ and $u_-(x,t) = \partial_x K_+(-x,t) *_x u_0^{(-1)}(x)$,

$$\|u\|_{L_x^\infty} \lesssim t^{-\frac{2}{1+\alpha}} \|u_0^{(-1)}\|_{L_x^1},$$
$$t^{\frac{2}{1+\alpha}} u_+(t+\nu t^{\frac{1}{1+\alpha}},t) \to \left(\int_{-\infty}^{\infty} u_0^{(-1)}(x)\,\mathrm{d}x\right) k'_\infty(\nu) = \left(-\int_{-\infty}^{\infty} x u_0(x)\,\mathrm{d}x\right) k'_\infty(\nu),$$

and similarly for u_-.

If more moments of u_0 vanish, then integrating by parts introduces extra boundary terms, by the non-differentiability of K at $x=0$. Suppose $n \geq 2$ is the smallest integer such that $\int x^n u_0(x)\,\mathrm{d}x \neq 0$. Denoting the j-th order primitive, $j \leq n$, of u_0 in $\mathcal{S}$ by $u_0^{(-j)}$, and setting $m = 2\lfloor n/2 \rfloor$, we get

$$u(x,t) = 2u_0^{(-2)}(x)\,\frac{\partial K}{\partial x}\,(0^+,t) + \cdots + 2u_0^{(-m)}(x)\,\frac{\partial^{m-1} K}{\partial x^{m-1}}\,(0^+,t) + \int_{x-t/\sqrt{\tau}}^{x+t/\sqrt{\tau}} \frac{\partial^n K}{\partial x^n}\,(x-y,t) u_0^{(-n)}(y)\,\mathrm{d}y.$$

It is possible to estimate $\partial_x^j K(0^+,t)$ for large t; using these estimates, one might then estimate the L_x^∞-norm of u by a linear combination of the L_x^∞-norms of $u_0^{(-2)}, \ldots, u_0^{(-m)}$ and the L_x^1-norm of $u_0^{(-n)}$, where the coefficients are negative powers of t depending on the order of the respective primitive of u_0 and α.

5. The case $\alpha = 1$

We now briefly discuss the case $\alpha = 1$, which is known as the (classical) Zener model, or the Standard Linear Solid (SLS) model. The SLS wave equation is

$$\frac{\partial^2}{\partial t^2}\,u(x,t) = \mathcal{L}_{s\to t}^{-1}\left(\frac{1+s}{1+\tau s}\right) *_t \frac{\partial^2}{\partial x^2}\,u(x,t), \qquad x \in \mathbb{R}, \quad t > 0, \tag{5.1}$$

so (1.1) with $\alpha = 1$. The fundamental solution S of (5.1) is again supported in the forward cone $|x| \leq t/\sqrt{t}$, but we will see that it is not smooth on the boundary of this cone, in contrast with the case $0 < \alpha < 1$. The Laplace transform $\tilde{S}$ of S is now given by

$$\tilde{S}(x,s) = \frac{1}{2s}\sqrt{\frac{1+\tau s}{1+s}}\exp\left(-|x|\,s\sqrt{\frac{1+\tau s}{1+s}}\right).$$

Note that this function has analytic continuation to $\mathbb{C} \setminus \big(\{0\} \cup [-1/\tau, -1]\big)$. The point $s=0$ is a simple pole of $\tilde{S}$; the line segment $[-1/\tau, -1]$ is a branch cut.

Theorem 5.1. *The fundamental solution S of (5.1) is discontinuous at the boundary of the forward light cone. More precise, it has the following form:*

$$S(x,t) = \frac{\sqrt{\tau}}{2}\exp\left(-\frac{\sqrt{\tau}}{2}\left(\frac{1}{\tau}-1\right)|x|\right) H(t-\sqrt{\tau}|x|) + E(x,t),$$

where E is a continuous function supported in the forward cone $\{(x,t) : |x| \leq t/\sqrt{\tau}\}$.

Proof. The fact that $S(x,t)$, and hence also $E(x,t) \coloneqq S(x,t) - (\sqrt{\tau}/2)\mathrm{e}^{-\frac{\sqrt{\tau}}{2}(1/\tau - 1)|x|} H(t-\sqrt{\tau}|x|)$, is supported in the cone $|x| \leq t/\sqrt{\tau}$ can be proven in a similar fashion as in Proposition 2.2. (In fact, that proof does not require that $\alpha < 1$.)

To show that E is continuous, we first note that, similarly as in (2.8),

$$l_1(s) \coloneqq \sqrt{\frac{1+\tau s}{1+s}} = \sqrt{\tau}\Big(1+\frac{1}{2}\Big(\frac{1}{\tau}-1\Big)s^{-1}+O(|s|^{-2})\Big), \quad \text{as } |s|\to\infty. \tag{5.2}$$

We have

$$\begin{aligned}\mathcal{L}\{E(x,t)\}(s) &= \frac{1}{2s}l_1(s)\exp\big(-|x|\,sl_1(s)\big) - \frac{\sqrt{\tau}}{2s}\exp\Big(-\sqrt{\tau}|x|\,s - \frac{\sqrt{\tau}}{2}\Big(\frac{1}{\tau}-1\Big)|x|\Big)\\ &= \frac{\sqrt{\tau}}{2s}\exp\Big(-\sqrt{\tau}|x|\,s - \frac{\sqrt{\tau}}{2}\Big(\frac{1}{\tau}-1\Big)|x|\Big)\Big(\big(1+O(1/|s|)\big)\exp\big(O(|x/s|)\big)-1\Big)\\ &= \frac{\sqrt{\tau}}{2s}\exp\Big(-\sqrt{\tau}|x|\,s - \frac{\sqrt{\tau}}{2}\Big(\frac{1}{\tau}-1\Big)|x|\Big)\cdot O\Big(\frac{1+|x|}{|s|}\Big), \quad \text{as } |s|\to\infty.\end{aligned}$$

Thus, we see that $\tilde{E}(x,s) = (\mathcal{L}E)(x,s)$ is absolutely integrable on vertical lines $\operatorname{Re} s = a$, $a>0$, so $E(x,t) = \frac{1}{2\pi\mathrm{i}}\int_{a-\mathrm{i}\infty}^{a+\mathrm{i}\infty}\tilde{E}(x,s)\mathrm{e}^{ts}\,\mathrm{d}s$ is a continuous function of x and t. □

Actually one can show that

$$\begin{aligned}\mathrm{WF}(S) = {}&\{(0,0;\xi,\eta) : (\xi,\eta)\neq(0,0)\}\cup\{(0,t;\xi,0) : t>0, \xi\neq 0\}\\ &\cup\{(x,t;\xi,\eta) : t>0, (\xi,\eta)\neq(0,0),\ |x| = t/\sqrt{\tau}, (x,t)\cdot(\xi,\eta)=0\}.\end{aligned}$$

For $|x|<t/\sqrt{\tau}$, one shifts the contour in the inverse Laplace transform to the left to obtain

$$S(x,t) = \frac{1}{2} + \frac{1}{4\pi\mathrm{i}}\int_\Gamma \frac{l_1(s)}{s}\exp\big(-|x|\,sl_1(s)+ts\big)\,\mathrm{d}s,$$

where Γ is a (finite) contour encircling the branch cut $[-1/\tau,-1]$ in the counterclockwise direction. This shows that S is analytic on the set $0<|x|<t/\sqrt{\tau}$. By differentiating the above formula with respect to x and using the residue theorem, one gets that

$$\frac{\partial S}{\partial x}(0^+,t) = -\frac{1-\tau}{2}\mathrm{e}^{-t}, \quad \frac{\partial S}{\partial x}(0^-,t) = \frac{1-\tau}{2}\mathrm{e}^{-t}, \quad t>0.$$

As in the proof of Theorem 3.3, S is analytic in the $(0,t)$-direction at points $(0,t_0)$, $t_0>0$. For $x\neq 0$, consider $S^{(-1)}(x,t) = \int_0^t S(x,t_1)\,\mathrm{d}t_1$. Changing variables as in the proof of Theorem 3.3 and differentiating with respect to $u=\sqrt{\tau}x+t$, we get

$$\frac{\partial^n (S^{(-1)})^\natural}{\partial u^n}(u,v) = \frac{1}{4\pi\mathrm{i}}\int_{a-\mathrm{i}\infty}^{a+\mathrm{i}\infty}\frac{l_1(s)}{s^2}\Big\{\frac{s}{2}\Big(1-\frac{l_1(s)}{\sqrt{\tau}}\Big)\Big\}^n\exp\Big\{\frac{us}{2}\Big(1-\frac{l_1(s)}{\sqrt{\tau}}\Big)-\frac{vs}{2}\Big(1+\frac{l_1(s)}{\sqrt{\tau}}\Big)\Big\}\,\mathrm{d}s.$$

In view of (5.2), this integral converges absolutely for every n, and is bounded even by D^{n+1} for some $D>0$, showing the $(S^{(-1)})^\natural$ is real analytic in the $(u,0)$-direction at points $(u_0,0)$, $u_0>0$. We omit further details.

We can again investigate the response in this model to a forced oscillation at the origin, starting at $t=0$. As before, we set $f(x,t)=\delta(x)H(t)\cos(\omega t)$, with $\omega>0$, and $u_0=v_0=0$. Then the solution u of the initial value problem (2.3) has Laplace transform

$$\tilde{u}(x,s) = \frac{l_1(s)}{2}\exp\big(-|x|\,sl_1(s)\big)\frac{1}{s^2+\omega^2}.$$

Notice that this Laplace transform is integrable on vertical lines $\operatorname{Re} s = a$, so u is continuous (although it is not of class C^1). Again the solution has support inside the cone $t\geq\sqrt{\tau}|x|$.

For x and t with $t > \sqrt{\tau}|x|$, we can move the contour in the Inverse Laplace transform to the left to get

$$u(x,t) = H(t/\sqrt{\tau} - |x|)\big(u_{\rm ss}(x,t) + u_{\rm ts}(x,t)\big),$$

with

$$\begin{aligned} u_{\rm ss}(x,t) &= \frac{l_1(\mathrm{i}\omega)}{4\mathrm{i}\omega}\exp\big(-|x|\,\mathrm{i}\omega l_1(\mathrm{i}\omega) + \mathrm{i}\omega t\big) - \frac{l_1(-\mathrm{i}\omega)}{4\mathrm{i}\omega}\exp\big(|x|\,\mathrm{i}\omega l_1(-\mathrm{i}\omega) - \mathrm{i}\omega t\big) \\ &= \frac{\rho_1(\omega)}{2\omega}\mathrm{e}^{-b_1(\omega)\omega|x|}\sin\big(\omega t - a_1(\omega)\omega|x| - \phi_1(\omega)\big); \\ u_{\rm ts}(x,t) &= \frac{1}{4\pi\mathrm{i}}\int_\Gamma l_1(s)\exp\big(-|x|\,sl_1(s) + ts\big)\frac{1}{s^2+\omega^2}\,\mathrm{d}s; \end{aligned}$$

where again Γ is a (finite) closed contour encircling the branch cut $[-1/\tau, -1]$ in the counterclockwise direction, and $l_1(\mathrm{i}\omega) = a_1(\omega) - \mathrm{i}b_1(\omega) = \rho_1(\omega)\mathrm{e}^{-\mathrm{i}\phi_1(\omega)}$, as before. The transient state $u_{\rm ts}$ converges to 0 as $t \to \infty$, locally uniformly in x. For fixed ω, the steady state $u_{\rm ss}$ is formally identical to the steady states in the fractional Zener model (4.1). We have the complex dispersion relation $k_1(\omega) = \omega l_1(\mathrm{i}\omega)$, and phase velocity $V_1(\omega) = 1/a_1(\omega)$. However, there is a qualitative difference between the two models in terms of the dependency of the dissipation on the frequency ω. In the SLS model (corresponding to $\alpha = 1$), the attenuation coefficient $d_1(\omega) = b_1(\omega)\omega$ has asymptotic behavior

$$d_1(\omega) \sim \frac{\sqrt{\tau}}{2}\Big(\frac{1}{\tau} - 1\Big), \quad \text{as } \omega \to \infty,$$

contrasting (4.2), which shows that the attenuation coefficient in the fractional Zener model ($0 < \alpha < 1$) grows to ∞ as $\omega \to \infty$. In the SLS model, two pseudo-monochromatic waves with different frequencies have roughly the same amount of spatial dampening, while in the fractional Zener model, the wave with the higher frequency will experience more dampening than the wave with the lower frequency.

Appendix A. Proof of Lemma 3.4

In this appendix we provide a proof Lemma 3.4. We use the same notations as in the proof of Theorem 3.3. Set

$$\begin{aligned} h_m(w) &\coloneqq f(w) + g_m(w), \quad \text{where} \\ f(w) &= \kappa w - \frac{1}{1-\alpha}w^{1-\alpha} + \log w + \log 2, \\ g_m(w) &= \frac{\kappa w}{2}E_1(\mu m^{\frac{1}{1-\alpha}}w) - \frac{u\mu m^{\frac{\alpha}{1-\alpha}}w}{2}E_2(\mu m^{\frac{1}{1-\alpha}}w) + \log\Big(1 + \frac{E_1(\mu m^{\frac{1}{1-\alpha}}w)}{2}\Big). \end{aligned}$$

Here, μ is a fixed constant, E_1 and E_2 are "remainder functions" given by (3.2), and κ is a number in a fixed range, namely

$$\kappa \in I \coloneqq \big[\big(1000/\sin(\alpha\pi)\big)^{\frac{1}{1-\alpha}}, \big(2000/\sin(\alpha\pi)\big)^{\frac{1}{1-\alpha}}\big]. \tag{A.1}$$

We need to show that we can choose $\kappa = \kappa_m$ in the interval I in such a way that

$$\operatorname{Im}\int_0^{+\mathrm{i}\infty} l_\alpha(\mu m^{\frac{1}{1-\alpha}}w)\mathrm{e}^{mh_m(w)}\frac{\mathrm{d}w}{w} \gtrsim \frac{c^m}{\sqrt{m}}. \tag{A.2}$$

for some $c > 0$ independent from m.

First we show necessary estimates uniformly for $\kappa \in I$, and later we show the existence of κ_m so that (A.2) holds. In order to prove (A.2), we will use the saddle point method. We view the phase function h_m as a small perturbation of f. Indeed, by the bounds (3.3) we get

$$g_m(w) \lesssim m^{-\frac{\alpha}{1-\alpha}} |w|^{1-\alpha}, \quad m \to \infty, \tag{A.3}$$

provided that $|w|$ is greater than some fixed $\varepsilon > 0$. In particular, g_m, as well as its derivatives, converges locally uniformly to 0 on the set $|w| > \varepsilon$. This convergence is also uniform with respect to $\kappa \in I$. We want the quantity g_m and its derivatives to be very small near the saddle point, so we also bound $|w|$ from above, and consider the fixed range $\varepsilon \leq |w| \leq W$. We will perform the saddle point analysis in this range, and show that the parts of the integral with $|w| \leq \varepsilon$ and $|w| \geq W$ are negligible with respect to the contribution from the saddle point. We now set

$$\varepsilon = \frac{1}{1 + 1/\sqrt{\tau}} \left(\frac{\sin(\alpha\pi)}{2000} \right)^{\frac{4}{1-\alpha}}, \quad W = \frac{8}{1-\alpha}. \tag{A.4}$$

It will be useful to consider f and g_m as holomorphic functions on the set $\Omega = \{w : \varepsilon \leq |w| \leq W, 0 \leq \arg w \leq 3\pi/2\}$ by analytic continuation[5]. If m is sufficiently large, then the zero and the pole of the function $(1 + \tau(\mu m^{\frac{1}{1-\alpha}} w)^\alpha)/(1 + (\mu m^{\frac{1}{1-\alpha}} w)^\alpha)$ have modulus strictly smaller than ε, so that g_m is well defined and holomorphic in Ω. We let m be so large that

$$\left| g_m^{(i)}(w) \right| < \frac{\sin^2(\alpha\pi)}{1000^2}, \quad \text{for } w \in \Omega, \quad \kappa \in I, \quad i = 0, 1, 2, 3. \tag{A.5}$$

Next we focus our attention on f. We have

$$f'(w) = \kappa - w^{-\alpha} + \frac{1}{w}.$$

To solve the saddle point equation $f'(w) = 0$, it is convenient to solve for $z := 1/w$. The equation $\kappa - z^\alpha + z = 0$ will have a solution z_0 near $\kappa \mathrm{e}^{-\mathrm{i}\pi}$. Indeed, noting that $|z^\alpha| < |\kappa + z|$ on $\partial B(\kappa \mathrm{e}^{-\mathrm{i}\pi}, \kappa/2)$, it follows from Rouché's theorem that $\kappa - z^\alpha + z$ has a unique zero in the disc $B(\kappa \mathrm{e}^{-\mathrm{i}\pi}, \kappa/2)$. Next, we will deduce a precise estimate for z_0. We want to keep all the constants in the error terms explicit. For this, we will use ζ to denote some complex number with $|\zeta| \leq 1$. At each next occurrence, this number might have a different value than the previous occurrence, but we will use the same notation ζ each time. We have

$$\begin{aligned} z_0 &= \mathrm{e}^{-\mathrm{i}\pi} \kappa \frac{1}{1 - z_0^{\alpha-1}} = \mathrm{e}^{-\mathrm{i}\pi} \kappa \big(1 + z_0^{\alpha-1} + 8\zeta |z_0|^{2\alpha-2}\big) = \mathrm{e}^{-\mathrm{i}\pi} \kappa \Big(1 + z_0^{\alpha-1} + \frac{16 \sin^2(\alpha\pi)}{1000^2} \zeta\Big) \\ &= \mathrm{e}^{-\mathrm{i}\pi} \kappa \Big(1 + \mathrm{e}^{\mathrm{i}\pi(1-\alpha)} \kappa^{\alpha-1} \Big(1 + \frac{6 \sin(\alpha\pi)}{1000} \zeta\Big) + \frac{16 \sin^2(\alpha\pi)}{1000^2} \zeta\Big) \\ &= \mathrm{e}^{-\mathrm{i}\pi} \kappa \Big(1 + \mathrm{e}^{\mathrm{i}\pi(1-\alpha)} \kappa^{\alpha-1} + \frac{22 \sin^2(\alpha\pi)}{1000^2} \zeta\Big). \end{aligned}$$

[5]Note that this continuation is different from the one to $\arg s \in [-\pi, -\pi/2]$, which appeared in (the derivation of) (2.10). They are situated on different sheets of the Riemann surface of the logarithm.

Here we used Taylor's theorem with explicit error terms, the a priori estimate $|z_0| > \kappa/2$, and the bound $\kappa^{\frac{1}{1-\alpha}} \geq 1000/\sin(\alpha\pi)$. For $w_0 = 1/z_0$ we get

$$w_0 = \frac{\mathrm{e}^{\mathrm{i}\pi}}{\kappa}\Big(1 - \mathrm{e}^{\mathrm{i}\pi(1-\alpha)}\kappa^{\alpha-1} + \frac{54\sin^2(\alpha\pi)}{1000^2}\zeta\Big), \tag{A.6}$$
$$\arg(w_0) = \pi - \frac{2\sin(\alpha\pi)}{998}\xi, \quad \text{for some } 0 < \xi < 1.$$

Let us denote by w_m the saddle point of $h_m = f + g_m$. By Hurwitz's theorem, we may assume that

$$w_m = w_0\Big(1 + \frac{\sin^2(\alpha\pi)}{1000^2}\zeta\Big) = \frac{\mathrm{e}^{\mathrm{i}\pi}}{\kappa}\Big(1 - \mathrm{e}^{\mathrm{i}\pi(1-\alpha)}\kappa^{\alpha-1} + \frac{56\sin^2(\alpha\pi)}{1000^2}\zeta\Big) \tag{A.7}$$

for sufficiently large m. We will let the contour pass through w_m via the steepest path.

Lemma A.1. *There exists some $\delta > 0$, and a contour Γ, the path of steepest descent, which connects two (nearly) opposing points c and d on the circle $|w - w_m| = \delta$. This path passes through w_m, $\operatorname{Im} h_m$ is constant along it, while $\operatorname{Re} h_m$ reaches its maximum at w_m. The tangent vector along Γ has its argument in the range $(3\pi/4, 5\pi/4)$.*

The proof will show that we may take $\delta = (27/680)\kappa^{-1}$.

Proof. The idea is to approximate $h_m(w) - h_m(w_m)$ by the quadratic function $(h_m''(w_m)/2)(w - w_m)^2$. By Taylor's theorem, we have on a small neighborhood of w_m

$$h_m(w) - h_m(w_m) = \frac{h_m''(w_m)}{2}(w - w_m)^2(1 + \eta_m(w)),$$

where $\eta_m(w)$ is a holomorphic function satisfying

$$\big|\eta_m(w)\big| \leq \left(\frac{2}{3!\big|h_m''(w_m)\big|}\max_{z\in[w_m,w]}\big|h_m'''(z)\big|\right)|w - w_m|\,.$$

The derivatives of h_m can be approximated by those of f. We have

$$f''(w) = -\frac{1}{w^2}\big(1 - \alpha w^{1-\alpha}\big), \quad f'''(w) = \frac{2}{w^3}\big(1 - \alpha(\alpha+1)w^{1-\alpha}\big),$$

so that by (A.5) and (A.7),

$$\big|h_m''(w_m)\big| \geq \big|f''(w_m)\big| - \frac{\sin^2(\alpha\pi)}{1000^2} \geq \kappa^2\Big(\frac{998}{1000}\Big)^2\Big(1 - \frac{2}{1000}\Big) - \frac{\sin^2(\alpha\pi)}{1000^2} \geq \frac{9\kappa^2}{10}.$$

Also for $\big|w - \mathrm{e}^{\mathrm{i}\pi}/\kappa\big| \leq 1/(2\kappa)$,

$$\big|h_m'''(w)\big| \leq \frac{2}{1/(2\kappa)^3}\big(1 + 2(3/(2\kappa))^{1-\alpha}\big) + \frac{\sin^2(\alpha\pi)}{1000^2} \leq 17\kappa^3.$$

Hence we get

$$\big|\eta_m(w)\big| \leq (170/27)\kappa|w - w_m|\,, \quad \text{for } \big|w - \mathrm{e}^{\mathrm{i}\pi}/\kappa\big| \leq (1/2)\kappa^{-1}. \tag{A.8}$$

We now set $\delta = (27/680)\kappa^{-1}$, then for $|w - w_m| \leq \delta$,

$$h_m(w) - h_m(w_m) = \frac{h_m''(w_m)}{2}(w - w_m)^2(1 + \eta_m(w)), \quad \big|\eta_m(w)\big| \leq \frac{1}{4}, \tag{A.9}$$
$$=: \big(\psi_m(w)\big)^2.$$

Here[6], $\psi_m(w) = -\sqrt{h_m''(w_m)/2}(w-w_m)\sqrt{1+\eta_m(w)}$, where $\sqrt{\ }$ denotes the principal branch of the square root. We claim that this is a holomorphic bijection from the closed disk $\overline{B}(w_m,\delta)$ onto some compact neighborhood F of zero. This will follow if we show that its derivative does not vary too much. We have

$$\psi_m'(w) = -\sqrt{\frac{h_m''(w_m)}{2}}\bigg(\sqrt{1+\eta_m(w)} + (w-w_m)\frac{\eta_m'(w)}{2\sqrt{1+\eta_m(w)}}\bigg).$$

Estimating η_m' by Cauchy's formula, if $|w-w_m| \le \delta$, then

$$\big|\eta_m'(w)\big| \le \frac{1}{2\pi}\left|\int_{\partial B(w_m,12\delta)} \frac{\eta_m(z)}{(z-w)^2}\,\mathrm{d}z\right| \le \frac{1}{2\pi}\cdot\frac{1}{(11\delta)^2}\cdot 3\cdot(24\delta\pi) = \frac{36}{121\delta}.$$

Here we used that $|z-w| \ge 11\delta$, and that $\big|\eta_m(z)\big| \le 3$ on the circle $|z-w_m| = 12\delta$, by (A.8). Hence, if $|w-w_m| \le \delta$, then

$$\psi_m'(w) = -\sqrt{\frac{h_m''(w_m)}{2}}\Big(1+\frac{\zeta_w}{2}\Big), \quad \text{for some } \zeta_w \text{ with } |\zeta_w| \le 1.$$

In particular, if $w, w' \in \overline{B}(w_m,\delta)$, $w \ne w'$, then $\psi_m(w') - \psi_m(w) = \int_w^{w'} \psi_m'(z)\,\mathrm{d}z \ne 0$, showing that ψ_m is injective.

We now set $\Gamma \coloneqq \psi_m^{-1}(L)$, where $L = [\mathrm{i}a, \mathrm{i}b]$ is the maximal line segment along the imaginary axis, which contains the origin and lies completely within F. This line segment connects the boundary points $\mathrm{i}a, \mathrm{i}b \in \partial F$ via the imaginary axis, passing through the origin, and Γ is a path which connects the points $c \coloneqq \psi_m^{-1}(\mathrm{i}a)$ and $d \coloneqq \psi_m^{-1}(\mathrm{i}b)$ on the circle $\partial B(w_m,\delta)$ via a path passing through the saddle point w_m. For points $w \in \Gamma$, clearly $\operatorname{Im} h_m(w) = \operatorname{Im} h_m(w_m)$, and $\operatorname{Re} h_m(w) \le \operatorname{Re} h_m(w_m)$, with equality only when $w = w_m$.

Let us now locate the points c and d on the circle a bit more, using (A.9). Writing $h_m''(w_m) = \big|h_m''(w_m)\big|\,\mathrm{e}^{\mathrm{i}\varphi_m}$, similar calculations as before show that

$$\varphi_m = \pi - \frac{8\xi}{992}, \quad \text{for some } \xi \text{ with } 0 < \xi < 1.$$

Setting $c = w_m + \delta\mathrm{e}^{\mathrm{i}\theta_c}$, $d = w_m + \delta\mathrm{e}^{\mathrm{i}\theta_d}$, we get the following relations for $\theta = \theta_c, \theta_d$, by taking real and imaginary part in (A.9):

$$\begin{aligned}
0 &> \frac{\big|h_m''(w_m)\big|}{2}\delta^2\big(\cos(\varphi_m+2\theta)(1+\operatorname{Re}\eta_m(\delta\mathrm{e}^{\mathrm{i}\theta})) - \sin(\varphi_m+2\theta)\operatorname{Im}\eta_m(\delta\mathrm{e}^{\mathrm{i}\theta})\big),\\
0 &= \frac{\big|h_m''(w_m)\big|}{2}\delta^2\big(\cos(\varphi_m+2\theta)\operatorname{Im}\eta_m(\delta\mathrm{e}^{\mathrm{i}\theta}) + \sin(\varphi_m+2\theta)(1+\operatorname{Re}\eta_m(\delta\mathrm{e}^{\mathrm{i}\theta}))\big).
\end{aligned}$$

The first inequality implies that $\theta \in (-3\pi/8, 3\pi/8) \cup (5\pi/8, 11\pi/8)$ (if not, then using the estimate of φ_m and the bound on η_m, one shows that the right-hand side would be positive). Using this initial localization, we can narrow the range down using the second equality, to $\theta \in (-\pi/8, \pi/8) \cup (7\pi/8, 9\pi/8)$ say. Actually, $\theta_c \in (-\pi/8, \pi/8)$, while $\theta_d \in (7\pi/8, 9\pi/8)$. This will follow from the following estimate for the argument of the tangent vector along Γ.

[6]The minus sign here is introduced for convenience. With this minus sign, the steepest path defined later on will have the desired orientation.

We have the following parametrization of Γ: $\gamma : [a,b] \to \Gamma : y \mapsto \psi_m^{-1}(\mathrm{i}y)$. We have

$$\gamma'(y) = \frac{\mathrm{i}}{\psi_m'(\psi_m^{-1}(\mathrm{i}y))} = -\mathrm{i}\sqrt{\frac{2}{h_m''(w_m)}}\Big(1+\frac{\zeta_y}{2}\Big),$$

for some ζ_y with $\big|\zeta_y\big| \le 1$. Using the bounds on φ_m, we see that this tangent vector has its argument in the range $(5\pi/6 - 4/992, 7\pi/6 + 4/992)$. □

We will now use the obtained information to estimate $\int_\Gamma l_\alpha(\mu m^{\frac{1}{1-\alpha}}w)w^{-1}\mathrm{e}^{mh_m(w)}\,\mathrm{d}w$, with Γ as in the above lemma. First we reparametrize Γ with arc length:

$$\tilde\gamma : [\tilde a, \tilde b] \to \Gamma : u \mapsto \tilde\gamma(u), \quad \tilde\gamma(0) = w_m, \quad \big|\tilde\gamma'(u)\big| = 1.$$

From the lemma, we have $\big|\arg\tilde\gamma'(u) - \pi\big| < \pi/4$. If m is sufficiently large, then $\big|\arg l_\alpha(\mu m^{\frac{1}{1-\alpha}}w)\big| \le 1/1000$ and $\big|\arg w^{-1} + \pi\big| \le \arctan\frac{2/1000+27/680}{998/1000-27/680} \le 1/20$, for w on the steepest path Γ. We have

$$\begin{aligned}\int_\Gamma l_\alpha(\mu m^{\frac{1}{1-\alpha}}w)\mathrm{e}^{mh_m(w)}\frac{\mathrm{d}w}{w} &= \mathrm{e}^{mh_m(w_m)}\int_{\tilde a}^{\tilde b} l_\alpha(\mu m^{\frac{1}{1-\alpha}}\tilde\gamma(u))\exp\big\{m\big(h_m(\tilde\gamma(u)) - h_m(w_m)\big)\big\}\frac{\tilde\gamma'(u)}{\tilde\gamma(u)}\,\mathrm{d}u\\ &=: \mathrm{e}^{mh_m(w_m)}R\mathrm{e}^{\mathrm{i}\phi}.\end{aligned}$$

Here, $|\phi| \le \pi/4 + 1/1000 + 1/20$, and $R \ge \cos(\pi/4 + 1/1000 + 1/20)\int_{\tilde a}^{\tilde b}|\dots|\,\mathrm{d}u$. Using (A.9) and the estimates $l_\alpha \gtrsim 1$, $1/\tilde\gamma(u) \gtrsim \kappa$, and $\big|h_m''(w_m)\big| \simeq \kappa^2$, we get

$$\begin{aligned}R &\gtrsim \kappa\int_{\tilde a}^{\tilde b}\exp\bigg\{m\bigg(\frac{h_m''(w_m)}{2}(\tilde\gamma(u)-w_m)^2(1+\eta_m(\tilde\gamma(u)))\bigg)\bigg\}\,\mathrm{d}u\\ &\gtrsim \kappa\int_{\tilde a}^{\tilde b}\exp\bigg\{-m\frac{5\big|h_m''(w_m)\big|}{8}u^2\bigg\}\,\mathrm{d}u \gtrsim \kappa\cdot\frac{1}{\sqrt{m\big|h_m''(w_m)\big|}} \gtrsim \frac{1}{\sqrt m}, \quad \text{as } m\to\infty.\end{aligned}$$

Here we used that the exponent is real and non-positive along Γ, so that

$$\begin{aligned}\frac{h_m''(w_m)}{2}(\tilde\gamma(u)-w_m)^2(1+\eta_m(\tilde\gamma(u))) &= -\frac{\big|h_m''(w_m)\big|}{2}\big|\tilde\gamma(u)-w_m\big|^2\big|1+\eta_m(\tilde\gamma(u))\big|\\ &\ge -\frac{\big|h_m''(w_m)\big|}{2}\cdot u^2\cdot\frac54.\end{aligned}$$

The last inequality follows from the fact that u is arc length: $\big|\tilde\gamma(u)-w_m\big| = \big|\tilde\gamma(u)-\tilde\gamma(0)\big| \le |u|$. For the same reason $|\tilde b - \tilde a| = \mathrm{length}(\Gamma) \gtrsim 1/\kappa$, so that the integral above can be transformed to the integral of e^{-t^2} over an interval containing 0 of length $\gtrsim 1$. We can conclude that

$$\bigg|\int_\Gamma l_\alpha(\mu m^{\frac{1}{1-\alpha}}w)\mathrm{e}^{mh_m(w)}\frac{\mathrm{d}w}{w}\bigg| \gtrsim \frac{\mathrm{e}^{m\,\mathrm{Re}\,h_m(w_m)}}{\sqrt m} \tag{A.10}$$

$$\gtrsim \frac{c^m}{\sqrt m}, \quad c = \bigg(\frac{\sin(\alpha\pi)}{2000}\bigg)^{\frac{2}{1-\alpha}}. \tag{A.11}$$

The value for c arises from the following (rough) lower bound for $\operatorname{Re} h_m(w_m)$: using $1/(2\kappa) \le |w_m| \le 2/\kappa$ and $\kappa \in I$ we get

$$\begin{aligned}\operatorname{Re} h_m(w_m) &= \operatorname{Re}\Big(\kappa w_m - \frac{1}{1-\alpha} w_m^{1-\alpha} + \log w_m + \log 2 + g_m(w_m)\Big)\\ &\ge -2 - \frac{2\sin(\alpha\pi)}{1000(1-\alpha)} - \frac{1}{1-\alpha}\log\frac{2000}{\sin(\alpha\pi)} - \frac{1}{1000^2}\\ &\ge -\frac{2}{1-\alpha}\log\frac{2000}{\sin(\alpha\pi)}.\end{aligned}$$

To control the phase of $\int_\Gamma$, we need a precise estimate of $\operatorname{Im} h_m(w_m)$. Using (A.7), we get (now using ξ for a *real* number satisfying $|\xi| \le 1$, with a possibly different value at each occurrence)

$$\begin{aligned}\operatorname{Im} w_m &= \frac{1}{\kappa}\Big(\sin(\alpha\pi)\kappa^{\alpha-1} + \frac{56\sin^2(\alpha\pi)}{1000^2}\xi\Big);\\ \operatorname{Im} w_m^{1-\alpha} &= \sin(\alpha\pi)\kappa^{\alpha-1}\Big(1 + \frac{4\sin(\alpha\pi)}{1000}\xi\Big) = \sin(\alpha\pi)\kappa^{\alpha-1} + \frac{4\sin^2(\alpha\pi)}{1000^2}\xi;\\ \arg w_m &= \pi - \arctan\left(\frac{\sin(\alpha\pi)\kappa^{\alpha-1} + \frac{56\sin^2(\alpha\pi)}{1000^2}\xi}{1 + \frac{2\sin(\alpha\pi)}{1000}\xi}\right) = \pi - \sin(\alpha\pi)\kappa^{\alpha-1} + \frac{63\sin^2(\alpha\pi)}{1000^2}\xi.\end{aligned}$$

This implies that

$$\begin{aligned}\operatorname{Im} h_m(w_m) &= \kappa \operatorname{Im} w_m - \frac{1}{1-\alpha}\operatorname{Im} w_m^{1-\alpha} + \arg w_m + \operatorname{Im} g_m(w_m)\\ &= \pi - \frac{\sin(\alpha\pi)}{1-\alpha}\Big(\kappa^{\alpha-1} + \frac{124\sin(\alpha\pi)}{1000^2}\xi\Big),\end{aligned}$$

where we also used (A.5) to bound $\operatorname{Im} g_m(w_m)$. Now that we have such a precise estimate for $\operatorname{Im} h_m(w_m)$, we will demonstrate how to choose $\kappa \in I$. We have

$$\begin{aligned}\big(\operatorname{Im} h_m(w_m)\big)\big|_{\kappa^{1-\alpha}=\frac{1000}{\sin(\alpha\pi)}} &= \pi - \frac{\sin(\alpha\pi)}{1-\alpha}\Big(\frac{\sin(\alpha\pi)}{1000} + \frac{124\sin(\alpha\pi)}{1000^2}\xi\Big)\\ < \big(\operatorname{Im} h_m(w_m)\big)\big|_{\kappa^{1-\alpha}=\frac{2000}{\sin(\alpha\pi)}} &= \pi - \frac{\sin(\alpha\pi)}{1-\alpha}\Big(\frac{\sin(\alpha\pi)}{2000} + \frac{124\sin(\alpha\pi)}{1000^2}\xi'\Big).\end{aligned}$$

We now note that the value of w_m depends continuously on κ, and so also $h_m(w_m)$ depends continuously on κ. Hence, for each sufficiently large m, we may choose $\kappa = \kappa_m \in I$ in such a way that $m\big(\operatorname{Im} h_m(w_m)\big) \in \pi/2 + 2\pi\mathbb{Z}$. This guarantees that

$$\operatorname{Im}\int_\Gamma l_\alpha(\mu m^{\frac{1}{1-\alpha}} w)\mathrm{e}^{m h_m(w)}\frac{\mathrm{d}w}{w} = \operatorname{Im}\big(\mathrm{e}^{m h_m(w_m)} R\mathrm{e}^{\mathrm{i}\phi}\big) \gtrsim \frac{c^m}{\sqrt{m}}.$$

Finally, we have to deform the complete contour $[0, +\mathrm{i}\infty)$ to a contour containing Γ, and show that the contribution from the other pieces is negligible. We do this in several steps. First, we set $\Upsilon_1 = [0, \mathrm{i}\varepsilon]$. For points w with $|w| < \varepsilon$, the asymptotic estimates (3.3) on the remainder functions E_1 and E_2 cannot be used. Writing the integrand in its original form,

that is before introducing the functions E_1, E_2, f, and g, we get

$$\begin{aligned}\int_{\Upsilon_1} l_\alpha(\mu m^{\frac{1}{1-\alpha}}w)\mathrm{e}^{mh_m(w)}\frac{\mathrm{d}w}{w} &= \int_{\Upsilon_1} l_\alpha(\mu m^{\frac{1}{1-\alpha}}w)w^m\Big(\frac{l_\alpha(\mu m^{\frac{1}{1-\alpha}}w)}{\sqrt{\tau}}+1\Big)^m\\ &\qquad \exp\Big\{-\frac{u\mu m^{\frac{1}{1-\alpha}}w}{2}\Big(\frac{l_\alpha(\mu m^{\frac{1}{1-\alpha}}w)}{\sqrt{\tau}}-1\Big)+\frac{m\kappa w}{2}\Big(\frac{l_\alpha(\mu m^{\frac{1}{1-\alpha}}w)}{\sqrt{\tau}}+1\Big)\Big\}\frac{\mathrm{d}w}{w}\\ &\lesssim \varepsilon^m\Big(\frac{1}{\sqrt{\tau}}+1\Big)^m = c^{2m}.\end{aligned}$$

Here we used that $\big|l_\alpha(s)\big|\le 1$ and $\operatorname{Im} l_\alpha(s)\le 0$ for $s\in \mathrm{i}\mathbb{R}_+$, and the definitions of ε and c, (A.4) and (A.11). Since $c<1$, this is of strictly lower order than the contribution from the integral over Γ.

Next we set $\Upsilon_2 \coloneqq \{\varepsilon \mathrm{e}^{\mathrm{i}\varphi} : \pi/2\le\varphi\le\theta_m\}$, were $\theta_m = \arg w_m$. We have

$$\begin{aligned}\operatorname{Re} h_m(\varepsilon\mathrm{e}^{\mathrm{i}\varphi}) &= \kappa\varepsilon\cos\varphi - \frac{1}{1-\alpha}\varepsilon^{1-\alpha}\cos((1-\alpha)\varphi)+\log\varepsilon+\log 2+\operatorname{Re} g(\varepsilon\mathrm{e}^{\mathrm{i}\varphi})\\ &\le \frac{1}{1-\alpha}+\frac{1}{1000^2}-\frac{4}{1-\alpha}\log\frac{2000}{\sin(\alpha\pi)}\le\frac{3}{2}\log c.\end{aligned}$$

Hence,

$$\int_{\Upsilon_2} l_\alpha(\mu m^{\frac{1}{1-\alpha}}w)\mathrm{e}^{mh_m(w)}\frac{\mathrm{d}w}{w}\lesssim c^{3m/2},$$

which is negligible.

Next we set $\Upsilon_3 \coloneqq \{r\mathrm{e}^{\mathrm{i}\theta_m} : \varepsilon\le r\le r_1\}$, where r_1 is such that this line segment connects $\varepsilon\mathrm{e}^{\mathrm{i}\theta_m}$ to the circle $|w-w_m|=\delta$, so $r_1 = |w_m|-\delta$. Note that $r_1\approx \frac{653}{680}\kappa^{-1}\ge \frac{1}{2}\big(\sin(\alpha\pi)/2000\big)^{\frac{1}{1-\alpha}} > \varepsilon$. Consider the function $r\mapsto \operatorname{Re} f(r\mathrm{e}^{\mathrm{i}\theta_m})$. This function is non-decreasing for $r\in[\varepsilon,r_1]$. Indeed, using that $r\le r_1\le\frac{682-27}{680}\kappa^{-1}$ we get

$$\begin{aligned}\frac{\partial}{\partial r}\operatorname{Re} f(r\mathrm{e}^{\mathrm{i}\theta_m}) &= \frac{\partial}{\partial r}\left(\kappa r\cos(\theta_m)-\frac{1}{1-\alpha}r^{1-\alpha}\cos((1-\alpha)\theta_m)+\log r+\log 2\right)\\ &= \kappa\cos(\theta_m)-\cos((1-\alpha)\theta_m)r^{-\alpha}+\frac{1}{r}\\ &\ge \kappa\Big(\frac{680}{655}+\cos\theta_m-\frac{2}{1000}\Big)>0.\end{aligned}$$

Therefore, $\operatorname{Re} h_m(r\mathrm{e}^{\mathrm{i}\theta_m})\le \operatorname{Re} h_m(r_1\mathrm{e}^{\mathrm{i}\theta_m})+2/1000^2$. Comparing $h_m(r_1\mathrm{e}^{\mathrm{i}\theta_m})$ to $h_m(w_m)$, using the notations and estimates from Lemma A.1, we get

$$\begin{aligned}&\operatorname{Re} h_m(r\mathrm{e}^{\mathrm{i}\theta_m})-\operatorname{Re} h_m(w_m)\\ &\qquad\le \frac{\big|h_m''(w_m)\big|}{2}\delta^2\Big(\cos(\varphi_m+2\theta_m)\big((1+\operatorname{Re}\eta_m(r_1\mathrm{e}^{\mathrm{i}\theta_m})\big)+\big|\operatorname{Im}\eta_m(r_1\mathrm{e}^{\mathrm{i}\theta_m})\big|\Big)+\frac{2}{1000^2}\\ &\qquad\le \frac{9/10}{2}\cdot\frac{27^2}{680^2}\big((3/4)\cos(3\pi/4-8/992)+1/4\big)+\frac{2}{1000^2}\le -\frac{1}{10000}.\end{aligned}$$

We can conclude that

$$\int_{\Upsilon_3} l_\alpha(\mu m^{\frac{1}{1-\alpha}}w)\mathrm{e}^{mh_m(w)}\frac{\mathrm{d}w}{w}\lesssim\frac{\mathrm{e}^{m\operatorname{Re} h_m(w_m)}}{\mathrm{e}^{m/10000}},$$

which is negligible compared to (A.10).

We now let Υ_4 be the arc of the circle $|w-w_m|=\delta$ which connects $r_1\mathrm{e}^{\mathrm{i}\theta_m}$ to the initial point c of Γ. Similarly, let Υ_5 be the arc of the circle which connects the end point d of Γ to the point $r_2\mathrm{e}^{\mathrm{i}\theta_m}$, where $r_2 \coloneqq |w_m|+\delta$. Since these arcs lie in the sectors $\arg(w-w_m)\in(-\pi/8,\pi/8)$ and $(7\pi/8,9\pi/8)$ respectively, the same estimate as before holds:

$$\int_{\Upsilon_4\cup\Upsilon_5} l_\alpha(\mu m^{\frac{1}{1-\alpha}}w)\mathrm{e}^{mh_m(w)}\frac{\mathrm{d}w}{w} \lesssim \frac{\mathrm{e}^{m\operatorname{Re}h_m(w_m)}}{\mathrm{e}^{m/10000}}.$$

Next, we set $\Upsilon_6 \coloneqq \{r\mathrm{e}^{\mathrm{i}\theta_m} : r_2 \le r \le W\}$, with W as in (A.4). This line segment is treated similarly as the line Υ_3. We now use that the function $r\mapsto \operatorname{Re} f(r\mathrm{e}^{\mathrm{i}\theta_m})$ is *non-increasing* in the range $r_2\le r\le W$, as apparent from a similar calculation. If $r\ge r_2\ge \frac{678+27}{680}\kappa^{-1}$, then

$$\frac{\partial}{\partial r}\operatorname{Re} f(r\mathrm{e}^{\mathrm{i}\theta_m}) \le \kappa\Big(\frac{680}{705}+\cos\theta_m+\frac{1}{1000}\Big)<0,$$

since $|\theta_m-\pi|\le 2/1000$, so $|\cos\theta_m+1|\le 2/1000$. Similarly as for Υ_3, we conclude that

$$\int_{\Upsilon_6} l_\alpha(\mu m^{\frac{1}{1-\alpha}}w)\mathrm{e}^{mh_m(w)}\frac{\mathrm{d}w}{w} \lesssim \kappa W\frac{\mathrm{e}^{m\operatorname{Re}h_m(w_m)}}{\mathrm{e}^{m/10000}}.$$

Finally we set $\Upsilon_7 \coloneqq \{r\mathrm{e}^{\mathrm{i}\theta_m} : r\ge W\}$. Using estimate (A.3), we get for $r\ge W$:

$$\begin{aligned}\operatorname{Re}h(r\mathrm{e}^{\mathrm{i}\theta_m}) &= \kappa r\cos\theta_m - \frac{1}{1-\alpha}r^{1-\alpha}\cos((1-\alpha)\theta_m)+\log r+\log 2+\operatorname{Re}g_m(r\mathrm{e}^{\mathrm{i}\theta_m})\\ &= r\bigg(\kappa\cos\theta_m - \frac{\cos((1-\alpha)\theta_m)}{1-\alpha}r^{-\alpha}+\frac{\log r+\log 2}{r}+O\big(m^{-\frac{\alpha}{1-\alpha}}r^{-\alpha}\big)\bigg)\le -\frac{\kappa}{2}r.\end{aligned}$$

Hence,

$$\int_{\Upsilon_7} l_\alpha(\mu m^{\frac{1}{1-\alpha}}w)\mathrm{e}^{mh_m(w)}\frac{\mathrm{d}w}{w} \lesssim \int_W^\infty \mathrm{e}^{-m\kappa r/2}\,\mathrm{d}r = \frac{2}{m\kappa}\mathrm{e}^{-m\kappa W/2}.$$

This is of lower order than the contribution from the integral over Γ, since $\mathrm{e}^{-\kappa W/2}<c$, by the definitions of W (A.4) and c (A.11), and the fact that $\kappa\in I$ (A.1). We may thus conclude that

$$\begin{aligned}\operatorname{Im}\int_0^{\mathrm{i}\infty} l_\alpha(\mu m^{\frac{1}{1-\alpha}}w)\mathrm{e}^{mh_m(w)}\frac{\mathrm{d}w}{w} &= \operatorname{Im}\int_{\Gamma\cup\bigcup_i\Upsilon_i} l_\alpha(\mu m^{\frac{1}{1-\alpha}}w)\mathrm{e}^{mh_m(w)}\frac{\mathrm{d}w}{w}\\ &\gtrsim \operatorname{Im}\int_\Gamma l_\alpha(\mu m^{\frac{1}{1-\alpha}}w)\mathrm{e}^{mh_m(w)}\frac{\mathrm{d}w}{w} \gtrsim \frac{c^m}{\sqrt{m}},\end{aligned}$$

as claimed earlier.

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