Mapping between Morita-equivalent string-net states with a constant depth quantum circuit

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We construct a constant depth quantum circuit that maps between Morita-equivalent string-net models. Due to its constant depth and unitarity, the circuit cannot alter the topological order, which demonstrates that Morita-equivalent string nets are in the same phase. The circuit is constructed from an invertible bimodule category connecting the two input fusion categories of the relevant string-net models, acting as a generalized Fourier transform for fusion categories. The circuit not only acts on the ground state subspace but also acts unitarily on the full Hilbert space when supplemented with ancillas.

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I. INTRODUCTION

Ever since their conception, the string-net models as originally proposed by Levin and Wen [1] and their subsequent generalizations [2–7] have provided a rich playground for studying microscopic realizations of nonchiral topologically ordered phases of matter in $2+1$ dimensions. Taking a unitary fusion category (UFC) $\mathcal{D}$ [8] as input, these exactly solvable models allow for the explicit realization of several key features of topologically ordered systems, such as ground-state degeneracies that depend on the topology of the space and anyonic quasiparticle excitations that satisfy nontrivial braiding statistics. From the category-theoretical side, the different ground states and excitations are described by the monoidal center $Z(\mathcal{D})$ of the input UFC $\mathcal{D}$ [2], which is itself a unitary modular fusion category (UMFC) and describes the topological order. The fact that the UMFC describing the topological order is always the center of some other UFC is the reason why string-net models are only able to describe nonchiral topological order with gappable boundaries; exactly solvable lattice models for chiral topological order have proven more challenging to obtain [9].

An important observation is that the process of going to the center $Z(\mathcal{D})$ of a UFC $\mathcal{D}$ is not an injective operation, or, put differently, that distinct UFCs $\mathcal{D}_1$ and $\mathcal{D}_2$ can have the same center $Z(\mathcal{D}_1) \simeq Z(\mathcal{D}_2)$ [2]. This has led to the conjecture that two such string-net models based on different $\mathcal{D}_1$ and $\mathcal{D}_2$ belong to the same topological phase, despite the fact that microscopically they can look very different. UFCs with the same center are said to be Morita equivalent [10], and the collection of UFCs Morita equivalent to $\mathcal{D}$ is called its Morita class. A direct consequence of Morita equivalence is that the topological excitations or anyons of two Morita-equivalent string-net models are in one-to-one correspondence with each other. Kitaev and Kong demonstrated the existence of an invertible domain wall between two Morita-equivalent string-net models through which anyons can move freely without condensing on the domain wall [2]. Mathematically, the existence of such an invertible domain wall is guaranteed by the fact that, for any two Morita-equivalent UFCs $\mathcal{D}_1$ and $\mathcal{D}_2$, there exists an invertible $(\mathcal{D}_1, \mathcal{D}_2)$-bimodule category $\mathcal{M}$ [8]. The central goal of this paper is to show that this implies the existence of a constant depth quantum circuit which maps the different string-net states onto each other.

From a quantum information point of view, it has been understood that two states are in the same phase if there exists a constant depth quantum circuit that is able to map between the two [11–13]. In this work, we provide such a constant depth circuit for the case of Morita-equivalent string nets $\mathcal{D}_1$ and $\mathcal{D}_2$ that maps the ground states as well as the excitations of one model to those of the other. For the ground state, the quantum circuit can be understood as a generalized version of the Hamiltonian, using the invertible $(\mathcal{D}_1, \mathcal{D}_2)$-bimodule category $\mathcal{M}$ to intertwine the Hamiltonian of the string-net $\mathcal{D}_1$ to the Hamiltonian of the string-net $\mathcal{D}_2$. Our construction generalizes a previous mapping, which was obtained from Kitaev's quantum double models [14] for a group G to string-net models Rep(G) [15,16], to the general case of two Morita-equivalent string nets.

Outline. We begin by providing a brief review of string nets, focusing mainly on a generalization known as the extended string-net models [17]. The reason for this is that these generalizations allow for a more unified treatment of the vertex and plaquette excitations, the two types of excitations in a string-net model. They are understood in the diagrammatic language as idempotents of the tube algebra associated with the string-net UFC $\mathcal{D}$ [3], which is well known to provide a characterization of the center $Z(\mathcal{D})$. Next, we discuss how an invertible $(\mathcal{D}_1, \mathcal{D}_2)$-bimodule category can be used to provide a map between the tube algebras of the UFCs $\mathcal{D}_1$ and $\mathcal{D}_2$, and thereby show how excitations from one model are mapped to excitations of the other. Taking all this together, we are able to explicitly construct a constant depth unitary quantum circuit which we explain in detail.

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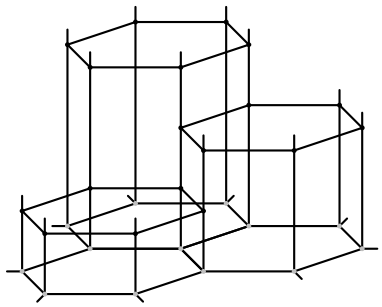


FIG. 1. An illustration of how our circuit acts on the three sublattices of the hexagonal lattice.

For the case of a hexagonal lattice, this circuit is depth three, which follows from the fact that our circuit sequentially acts on three sublattices. This is depicted in Fig. 1 below. The action of the circuit on the three sublattices is worked out in the main text. Each of these local unitaries acts on 18 degrees of freedom associated with a plaquette. We show that the unitary map on the three sublattices is respectively constructed from the different associators of the $(\mathcal{D}_1, \mathcal{D}_2)$-bimodule category $\mathcal{M}$, which are the solutions to a set of coupled pentagon equations [8,18]. In this way our construction gives a very concrete quantum information perspective on the categorical notion of Morita equivalence. We also show how the quantum circuit can be generalized to act on the full (extended) string-net Hilbert space and obtain a very similar circuit as in the absence of excitations, the details of which are relegated to the Appendix.

II. STRING NETS

In this section we give a brief review of Levin and Wen's string-net models on the hexagonal lattice. First, we review the ordinary string-net model, herein closely following the original work of Levin and Wen [1], and then we turn to the extended string nets [17]. These can be regarded as a generalization of the ordinary string nets that provide a more natural and convenient setting in which to study and classify excited states.

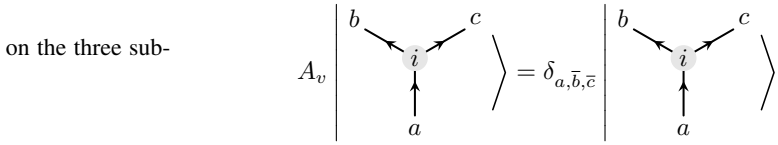
A. Levin and Wen's string nets

A string-net model is defined by the data of some given input unitary fusion category $\mathcal{D}$. This input data contains a set of simple objects (string types) that we denote $a, b, c, \ldots$ with corresponding fusion rules $a \otimes b \simeq \bigoplus_c N_{ab}^c c$, corresponding quantum dimensions $d_a, d_b, d_c, \ldots$, and an isomorphism $F : (a \otimes b) \otimes c \to a \otimes (b \otimes c)$. Given some oriented trivalent lattice, such as the hexagonal lattice in this work, the Hilbert space of the string net is defined by configurations of the string types living on the edges of the lattice. Furthermore, for every string type a there exists a corresponding conjugate string type $\overline{a}$ such that $\overline{\overline{a}} \equiv a$ and $\overline{a}$ can be interpreted as a living on the same edge with the opposite orientation. The quantum dimensions obey $d_{\overline{a}} = d_a$. There is a unique self-conjugate vacuum string $\mathbf{1}$, $\overline{\mathbf{1}} \equiv \mathbf{1}$.

The string-net Hamiltonian consists of two terms:

$$H = -\sum_v A_v - \sum_p B_p. \tag{1}$$

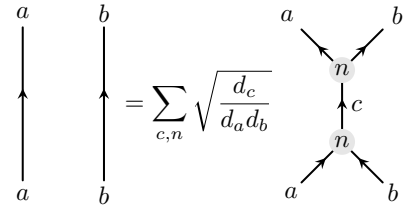
Herein the two sums in the Hamiltonian are over the vertices and plaquettes of the lattice, respectively. The sum over s is a sum over all string types of the input UFC. D is the total quantum dimension defined as $D = (\sum_i d_i^2)^{1/2}$. The vertex operators A_v act on the vertices according to

A_v  $= \delta_{a,\overline{b},\overline{c}}$ (2)

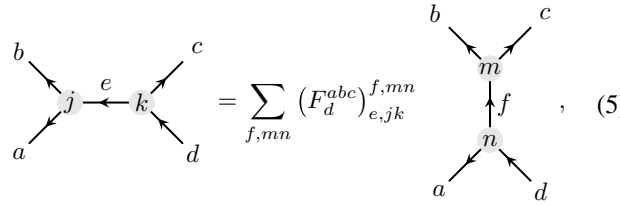
where $\delta_{a,b,c}$ is 1 whenever the fusion of $a \otimes b$ contains $\overline{c}$ and is 0 else. The plaquette operators B_p are defined as

$$B_p = \frac{1}{D}\sum_a d_a B_p^a, \tag{3}$$

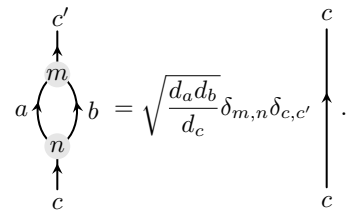
where the terms B_a^s acts on a plaquette by inserting a clockwise-oriented string of type a in the plaquette and fusing it to the lattice. This is done by making repeated use of the resolution of the identity for UFCs, namely,

 (4)

the F moves of the given input UFC,

 (5)

and the "bubble pop" identity

 (6)

The matrix elements of this operator are worked out in detail in Refs. [1,6]. Every operator B_p^a is a 18-body operator in the sense that its action explicitly depends on the six string types living on the edges of the plaquette p, the six multiplicities and the six edges adjacent to the plaquette, but acts diagonally on the latter. It can be shown that both A_v and B_p are Hermitian projection operators, $A_v^2 = A_v$ and $B_p^2 = B_p$, that moreover all commute. Hence, the string-net condensed renormalization

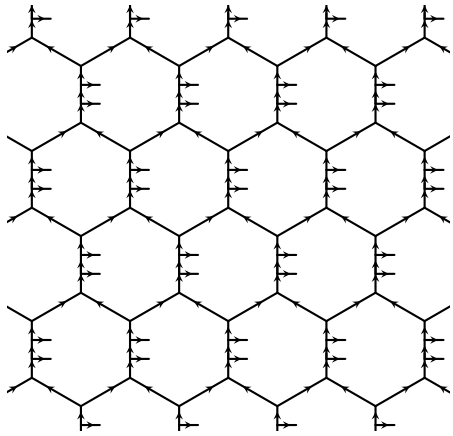


FIG. 2. We adapt the convention that all arrows on the edges point upward and the open legs are attached to the lattice as depicted where every open edge corresponds to the closest vertex of the hexagonal lattice. Multiplicity degrees of freedom are suppressed.

group fixed-point ground states are the simultaneous eigenvectors of all A_v and B_p. Quasiparticle excitations on top of these ground states are obtained by violating at least one of the projector constraints. These excitations are gapped and necessarily come in pairs. Vertices—or electric excitations—are obtained by violating the A_v constraints, whereas plaquettes—or magnetic excitations—violate the B_p constraints. These excitations are treated in a unified way in the language of extended string nets which we now revisit.

B. Extended string nets

The extended string-net models [17] are very similar to the ordinary string-net models except for a few key differences which we discuss in this section.

Given a hexagonal lattice we associate with every vertex an additional open edge that is attached to an edge that is connected to the vertex under consideration. These open edges also carry string degrees of freedom. The choice of which open edge belongs to which vertex is to some extent arbitrary and the open edges can even be moved around the plaquette by a series of F moves but we will adapt the convention denoted in Fig. 2. Furthermore, we demand the fusion rules to be satisfied on every vertex, $\delta_{a,b,c} = 1$. Finally, the Hamiltonian of the extended string net is of the same form as the Hamiltonian of the ordinary string-net model:

$$H = -\sum_v \mathcal{A}_v - \sum_p \mathcal{B}_p, \tag{7}$$

where the $\mathcal{A}_v$ and the $\mathcal{B}_p$ are again mutually commuting Hermitian projectors. $\mathcal{A}_v$ acts on vertices according to

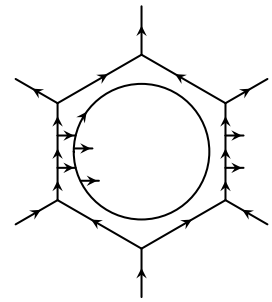
$$\mathcal{A}_v \left| \begin{array}{c} b \quad c \\ \searrow \; \swarrow \\ i \\ a_2 \uparrow \\ j \rightarrow d \\ \uparrow \\ a_1 \end{array} \right\rangle = \delta_{d,\mathbf{1}} \left| \begin{array}{c} b \quad c \\ \searrow \; \swarrow \\ i \\ a_2 \uparrow \\ j \rightarrow d \\ \uparrow \\ a_1 \end{array} \right\rangle. \tag{8}$$

The action of $\mathcal{B}_p$ on the plaquette is explained in detail in Ref. [17] and reduces for p_1, $p_2 = \mathbf{1}$ to the action of the plaquette operator B_p of the ordinary model, where p_1, p_2 denote the open edges of the plaquette p.

The extended string-net model thus deals with vertex excitations in the ordinary string-net model by considering extra degrees of freedom that signal the presence of vertex projector violations.

C. Tube algebra

In the extended string-net models, vertex and plaquette excitations are both characterized by the *tube algebra* [3,17]. The action of a tube $\mathcal{T}_i$ on a plaquette can graphically be depicted as the insertion of the following diagram in a plaquette:

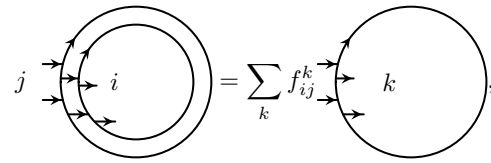


, (9)

and fusing it to the boundary; here, i runs over all possible configurations that give a nonzero tube. The set of tubes is closed under multiplication,

$$\mathcal{T}_i \mathcal{T}_j = \sum_k f_{ij}^k \mathcal{T}_k, \tag{10}$$

defined by stacking:



(11)

as well as Hermitian conjugation [19,20]:

$$(\mathcal{T}_i)^\dagger = \sum_j u_i^j \mathcal{T}_j. \tag{12}$$

This turns the tube algebra into a finite $C^\star$ algebra, for which the Artin-Wedderburn theorem dictates that one can find an isomorphism to a direct sum of simple matrix algebras. Explicitly, this implies the existence of simple idempotents p_{ii}^a and nilpotents p_{ij}^a, $i \neq j$ that satisfy

$$p_{ij}^a = \sum_k x_{ij}^{a,k} \mathcal{T}_k, \quad p_{ij}^a p_{kl}^b = \delta_{ab} \delta_{jk} p_{il}^a, \quad \left(p_{ij}^a\right)^\dagger = p_{ji}^a. \tag{13}$$

The elementary excitations of the extended string-net model correspond to the simple idempotents p_{ii}^a of the tube algebra, each of which corresponds to a specific combination of one plaquette excitation and two vertex excitations. These

simple idempotents have to be grouped into central idempotents P_a satisfying

$$P_a = \sum_{i=0}^{n_a-1} p_{ii}^a, \quad [P_a, T_i] = 0, \quad \sum_a P_a = \mathbb{1}, \tag{14}$$

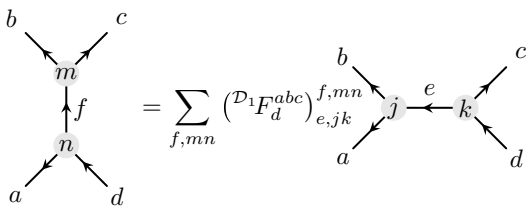
where n_a denotes the dimension of the central idempotent P_a. These central idempotents then correspond to the irreducible representations of the tube algebra and are in one-to-one correspondence with the simple objects of the monoidal center $Z(\mathcal{D})$.

III. BIMODULE CATEGORIES

We consider two string-net models $\mathcal{D}_1$ and $\mathcal{D}_2$ that are supposedly in the same phase; this requires that $\mathcal{D}_1$ and $\mathcal{D}_2$ are Morita equivalent, $Z(\mathcal{D}_1) \simeq Z(\mathcal{D}_2)$. From a categorical perspective, this implies the existence of an invertible $(\mathcal{D}_1, \mathcal{D}_2)$-bimodule category $\mathcal{M}$. In this section, we briefly review some properties of a bimodule category, mainly to fix notation,[1] and show how this bimodule category provides a map between the excitations of the two string-net models.

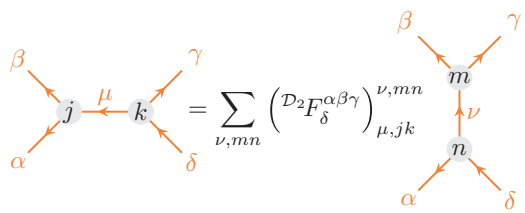
A. Bimodule categories

We start with a spherical UFC $\mathcal{D}_1$, with objects $a, b, c, \ldots \in \mathcal{D}_1$ and an associator expressed in a basis of simple objects as



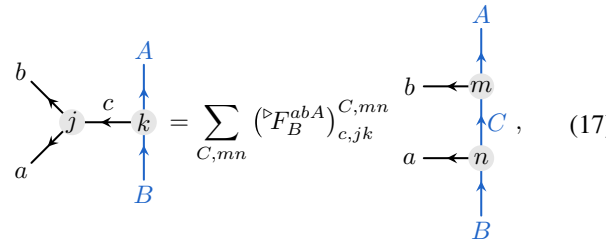
$$= \sum_{f,mn} \left({}^{\mathcal{D}_1}F_d^{abc}\right)_{e,jk}^{f,mn}, \tag{15}$$

where in the diagrams we color lines labeled by objects in $\mathcal{D}_1$ in black for the remainder of this work. We also have a second spherical UFC $\mathcal{D}_2$, with objects $\alpha, \beta, \gamma, \ldots \in \mathcal{D}_2$ and an associator given by



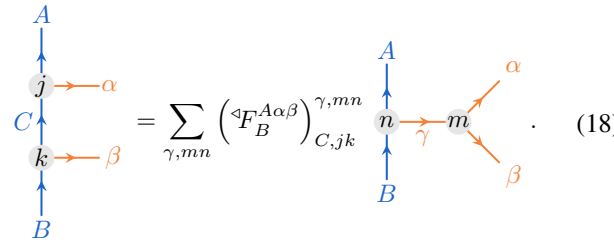
$$= \sum_{\nu,mn} \left({}^{\mathcal{D}_2}F_\delta^{\alpha\beta\gamma}\right)_{\mu,jk}^{\nu,mn}, \tag{16}$$

with lines labeled by objects in $\mathcal{D}_2$ drawn in orange. A left $\mathcal{D}_1$-module category $\mathcal{M}$ is a category $\mathcal{M}$ with objects $A, B, C, \ldots \in \mathcal{M}$, a left-action of $\mathcal{D}_1$ on $\mathcal{M}$ given by $a \triangleright A \simeq \bigoplus_B N_{aA}^B B$ and an isomorphism ${}^{\triangleright}F : (a \otimes b) \triangleright A \to a \triangleright (b \triangleright A)$, which can graphically be expressed as



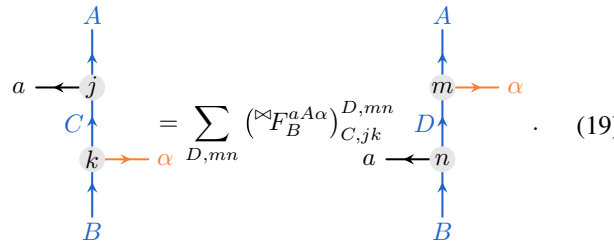
$$= \sum_{C,mn} \left({}^{\triangleright}F_B^{abA}\right)_{c,jk}^{C,mn}, \tag{17}$$

where lines labeled by objects in $\mathcal{M}$ are blue. Similarly, a right $\mathcal{D}_2$-module category $\mathcal{M}$ has a right-action of $\mathcal{D}_2$ on $\mathcal{M}$ given by $A \triangleleft \alpha \simeq \bigoplus_B N_{A\alpha}^B B$ and an isomorphism ${}^{\triangleleft}F : (A \triangleleft \alpha) \triangleleft \beta \to A \triangleleft (\alpha \otimes \beta)$, graphically expressed as



$$= \sum_{\gamma,mn} \left({}^{\triangleleft}F_B^{A\alpha\beta}\right)_{C,jk}^{\gamma,mn}. \tag{18}$$

A $(\mathcal{D}_1, \mathcal{D}_2)$-bimodule category $\mathcal{M}$ is then defined as a left-$\mathcal{D}_1$ and right-$\mathcal{D}_2$ module category $\mathcal{M}$ that additionally is equipped with an isomorphism ${}^{\triangleright\triangleleft}F : (a \triangleright A) \triangleleft \alpha \to a \triangleright (A \triangleleft \alpha)$, graphically depicted as



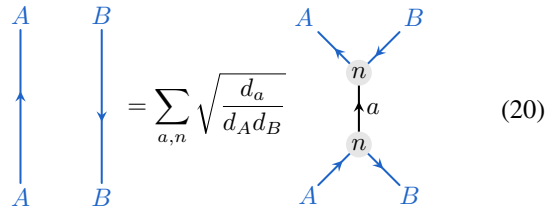
$$= \sum_{D,mn} \left({}^{\triangleright\triangleleft}F_B^{aA\alpha}\right)_{C,jk}^{D,mn}. \tag{19}$$

Physically, these bimodule categories arise most evidently in the study of domain walls between different string-net models [2], where the various associativity conditions discussed above amount to the generalizations of the bulk F moves. For a general domain between Morita-equivalent string-net models, the process of pushing excitations through such a domain wall is not reversible, because several excitations can condense on the domain wall. In the special case where excitations can freely move through the domain wall, we are dealing with an *invertible* bimodule category. For every two Morita-equivalent UFCs $\mathcal{D}_1$ and $\mathcal{D}_2$ one can always write down an invertible $(\mathcal{D}_1, \mathcal{D}_2)$-bimodule category, and we will use this particular bimodule category in the remainder of this work.

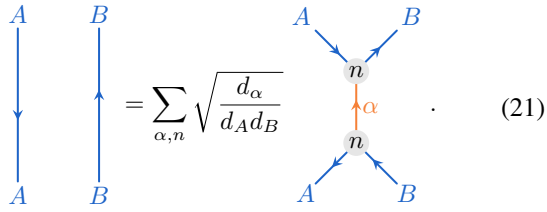
The $(\mathcal{D}_1, \mathcal{D}_2)$-bimodule category $\mathcal{M}$ does not have an intrinsic duality, but one can define a $(\mathcal{D}_2, \mathcal{D}_1)$-bimodule category denoted by $\mathcal{M}^{\text{op}}$ or $\overline{\mathcal{M}}$ to contain the dual objects of $\mathcal{M}$. Crucially, this allows us to impose a generalized notion of sphericality [2,21], which we need to perform certain diagrammatical manipulations. Concretely, invertibility of $\mathcal{M}$ can be used to generalize the resolution of the identity (4) of UFCs to a resolution of the identity which allows for the

[1] The ${}^{\mathcal{D}_1}F$, ${}^{\triangleright}F$, ${}^{\triangleright\triangleleft}F$, ${}^{\triangleleft}F$, and ${}^{\mathcal{D}_2}F$ symbols in this work are denoted as ${}^0F$, ${}^1F$, ${}^2F$, ${}^3F$, and ${}^4F$ symbols in Ref. [18], respectively.

creation of a $\mathcal{D}_1$ or $\mathcal{D}_2$ line from the fusion of an $\mathcal{M}$ line with an $\mathcal{M}^{\text{op}}$ line:



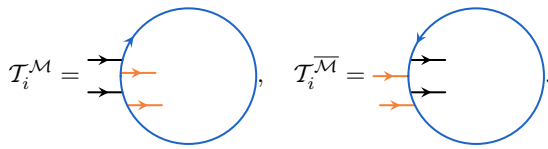
(20)



(21)

B. Tube algebra bimodules

Using the bimodule categories $\mathcal{M}$ and $\overline{\mathcal{M}}$ we can write down bimodule tubes for the tube algebras $\mathcal{T}^{\mathcal{D}_1}$ and $\mathcal{T}^{\mathcal{D}_2}$ of $\mathcal{D}_1$ and $\mathcal{D}_2$, respectively. These bimodule tube algebras, denoted by $\mathcal{T}^{\mathcal{M}}$ and $\mathcal{T}^{\overline{\mathcal{M}}}$, are generated by



(22)

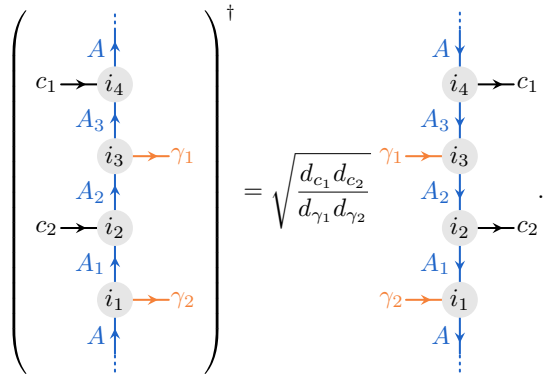
They satisfy

$$\mathcal{T}_i^{\mathcal{M}} \mathcal{T}_j^{\mathcal{D}_1} = \sum_k a_{ij}^k \mathcal{T}_k^{\mathcal{M}}, \quad \mathcal{T}_i^{\mathcal{D}_2} \mathcal{T}_j^{\mathcal{M}} = \sum_k b_{ij}^k \mathcal{T}_k^{\mathcal{M}}, \tag{23}$$

$$\mathcal{T}_i^{\mathcal{D}_1} \mathcal{T}_j^{\overline{\mathcal{M}}} = \sum_k c_{ij}^k \mathcal{T}_k^{\overline{\mathcal{M}}}, \quad \mathcal{T}_i^{\overline{\mathcal{M}}} \mathcal{T}_j^{\mathcal{D}_1} = \sum_k d_{ij}^k \mathcal{T}_k^{\overline{\mathcal{M}}}, \tag{24}$$

$$\mathcal{T}_i^{\overline{\mathcal{M}}} \mathcal{T}_j^{\mathcal{M}} = \sum_k e_{ij}^k \mathcal{T}_k^{\mathcal{D}_1}, \quad \mathcal{T}_i^{\mathcal{M}} \mathcal{T}_j^{\overline{\mathcal{M}}} = \sum_k g_{ij}^k \mathcal{T}_k^{\mathcal{D}_2}, \tag{25}$$

where the last line requires the invertibility of the bimodule category, manifested in the identities (20) and (21). These bimodule tubes are mapped into one another under Hermitian conjugation:



(26)

This Hermitian conjugation is defined such that the operators obtained from inserting $\mathcal{T}_i^{\mathcal{M}}$ and $(\mathcal{T}_i^{\mathcal{M}})^\dagger$ in a plaquette expressed as matrices acting on the relevant degrees of freedom are related via the usual Hermitian conjugation. Using these properties, we can define a big $C^\star$ algebra generated by $\{\mathcal{T}_i^{\mathcal{D}_1}, \mathcal{T}_j^{\mathcal{M}}, \mathcal{T}_k^{\overline{\mathcal{M}}}, \mathcal{T}_l^{\mathcal{D}_2}\}$, where we take undefined tube multiplications to be zero. This big $C^\star$ algebra contains the $C^\star$ algebras $\mathcal{T}^{\mathcal{D}_1}$ and $\mathcal{T}^{\mathcal{D}_2}$ as subalgebras. Using the Artin-Wedderburn theorem for this big (finite) $C^\star$ algebra, there exists an isomorphism to a direct sum of simple matrix algebras, the number of which equals the number of central idempotents of $\mathcal{T}^{\mathcal{D}_1}$ and $\mathcal{T}^{\mathcal{D}_2}$. In addition to the simple idempotents and nilpotents of $\mathcal{T}^{\mathcal{D}_1}$ and $\mathcal{T}^{\mathcal{D}_2}$, we can define simple bimodules:

$$p_{ij}^{a,\mathcal{M}} = \sum_k y_{ij}^{a,k} \mathcal{T}_k^{\mathcal{M}}, \quad p_{ji}^{a,\overline{\mathcal{M}}} = \sum_k z_{ji}^{a,k} \mathcal{T}_k^{\overline{\mathcal{M}}}, \tag{27}$$

$$\left(p_{ij}^{a,\mathcal{M}}\right)^\dagger = p_{ji}^{a,\overline{\mathcal{M}}}, \tag{28}$$

with $0 \leqslant i < n_a^{\mathcal{D}_2}$ and $0 \leqslant j < n_a^{\mathcal{D}_1}$, such that they satisfy

$$\begin{aligned}
p_{ij}^{a,\mathcal{D}_1} p_{kl}^{b,\mathcal{D}_1} &= \delta_{ab}\delta_{jk} p_{il}^{a,\mathcal{D}_1}, & p_{ij}^{a,\mathcal{D}_2} p_{kl}^{b,\mathcal{D}_2} &= \delta_{ab}\delta_{jk} p_{il}^{a,\mathcal{D}_2}, \\
p_{ij}^{a,\mathcal{M}} p_{kl}^{b,\mathcal{D}_1} &= \delta_{ab}\delta_{jk} p_{il}^{a,\mathcal{M}}, & p_{ij}^{a,\mathcal{D}_2} p_{kl}^{b,\mathcal{M}} &= \delta_{ab}\delta_{jk} p_{il}^{a,\mathcal{M}}, \\
p_{ij}^{a,\mathcal{D}_1} p_{kl}^{b,\overline{\mathcal{M}}} &= \delta_{ab}\delta_{jk} p_{il}^{a,\overline{\mathcal{M}}}, & p_{ij}^{a,\overline{\mathcal{M}}} p_{kl}^{b,\mathcal{D}_2} &= \delta_{ab}\delta_{jk} p_{il}^{a,\overline{\mathcal{M}}}, \\
p_{ij}^{a,\overline{\mathcal{M}}} p_{kl}^{b,\mathcal{M}} &= \delta_{ab}\delta_{jk} p_{il}^{a,\mathcal{D}_1}, & p_{ij}^{a,\mathcal{M}} p_{kl}^{b,\overline{\mathcal{M}}} &= \delta_{ab}\delta_{jk} p_{il}^{a,\mathcal{D}_2},
\end{aligned} \tag{29}$$

where from now on the range of the indices i, j, k, l should be inferred from the context. The simple bimodules will be the essential building blocks of the quantum circuit we aim to construct, providing a map between the ground states and excitations of the string-net model $\mathcal{D}_1$ to the ground states and excitations of the string-net model $\mathcal{D}_2$.

IV. QUANTUM CIRCUIT

Before we construct our quantum circuit, we have to address the point that the total dimension of the Hilbert space need not match between the string-net model $\mathcal{D}_1$ and $\mathcal{D}_2$; the fact that they are Morita equivalent only ensures that number of ground states on some closed manifold is the same. The number of distinct excitations which form a complete basis for the Hilbert space will in general differ between the two models, which is reflected in their tube algebras; since we have $Z(\mathcal{D}_1) \simeq Z(\mathcal{D}_2)$, the number of central idempotents $P_a^{\mathcal{D}_1}$ and $P_a^{\mathcal{D}_2}$ is the same, but their dimensions $n_a^{\mathcal{D}_1}$ and $n_a^{\mathcal{D}_2}$ can differ. This discrepancy in the dimensions of the Hilbert spaces poses an obstacle if we want to construct a unitary quantum circuit that maps the full Hilbert space of one model to the other.

To solve this problem, we will place an ancillary qudit $|i\rangle$ of dimension $n^{\mathcal{D}_2} = \max_a n_a^{\mathcal{D}_2}$ in each plaquette of the string-net model $\mathcal{D}_1$. The original Hilbert space of the string-net model is obtained by fixing these ancillas to $|0\rangle$; we denote states in this Hilbert space as $|\psi^{\mathcal{D}_1}, 0\rangle$. We can then define the following operator:

$$V_{\mathcal{D}_1}^{\mathcal{D}_2} := \left(\sum_{a,ij} p_{ij}^{a,\mathcal{M}} \otimes |j\rangle\langle i|\right)^{\otimes N}, \tag{30}$$

with N being the number of plaquettes in the system, where $p_{ij}^{a,\mathcal{M}}$ acts on a plaquette as defined before, and $|j\rangle\langle i|$ acts on the corresponding ancilla. The Hermitian conjugate of this

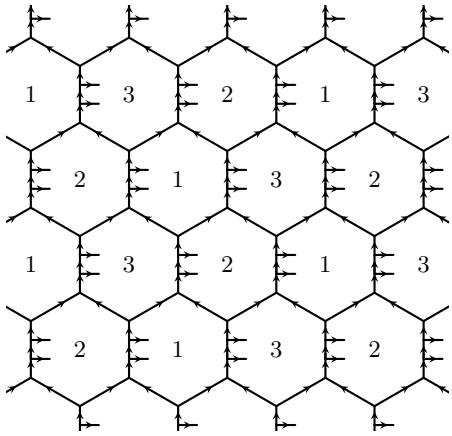


FIG. 3. Division of the hexagonal lattice into three sublattices. We build the quantum circuit by acting on the plaquettes in the order denoted in the figure.

operator is given by

$$\left(V_{\mathcal{D}_1}^{\mathcal{D}_2}\right)^\dagger = \left(\sum_{a,ij} p_{ij}^{a,\overline{\mathcal{M}}} \otimes |j\rangle\langle i|\right)^{\otimes N}, \tag{31}$$

and the product of $V_{\mathcal{D}_1}^{\mathcal{D}_2}$ with its Hermitian conjugate is

$$\begin{aligned}\left(V_{\mathcal{D}_1}^{\mathcal{D}_2}\right)^\dagger V_{\mathcal{D}_1}^{\mathcal{D}_2} &= \left(\sum_{ab,ii'jj'} p_{j'i'}^{a,\overline{\mathcal{M}}} p_{ij}^{a,\mathcal{M}} \otimes |i'\rangle\langle j'|j\rangle\langle i|\right)^{\otimes N} \\ &= \left(\sum_{a,ij} p_{jj}^{a,\mathcal{D}_1} \otimes |i\rangle\langle i|\right)^{\otimes N} \\ &= \left(\sum_a P_a^{\mathcal{D}_1} \otimes \mathbb{1}_{n_a^{\mathcal{D}_2}}\right)^{\otimes N}. \end{aligned} \tag{32}$$

In general this is a projector implying $V_{\mathcal{D}_1}^{\mathcal{D}_2}$ is an isometry, but restricting to the original Hilbert space where all ancillas are in the $|0\rangle$ state we get

$$\begin{aligned}\left(V_{\mathcal{D}_1}^{\mathcal{D}_2}\right)^\dagger V_{\mathcal{D}_1}^{\mathcal{D}_2} \left|\psi^{\mathcal{D}_1}, 0\right\rangle &= \left(\sum_a P_a^{\mathcal{D}_1} \otimes \mathbb{1}_{n_a^{\mathcal{D}_2}}\right)^{\otimes N} \left|\psi^{\mathcal{D}_1}, 0\right\rangle \\ &= \left|\psi^{\mathcal{D}_1}, 0\right\rangle, \end{aligned} \tag{33}$$

meaning that the isometry $V_{\mathcal{D}_1}^{\mathcal{D}_2}$ is in fact a unitary on the original Hilbert space of the string-net model.

A constant depth quantum circuit is obtained by applying the isometry $V_{\mathcal{D}_1}^{\mathcal{D}_2}$ in such a way that it is a constant depth operator. For the hexagonal lattice, this can be achieved by dividing the lattice into three sublattices as shown in Fig. 3 and acting sequentially on the first, second, and third sublattice. This implementation of the isometry $V_{\mathcal{D}_1}^{\mathcal{D}_2}$ turns it into a depth-three quantum circuit. We now discuss this quantum circuit in more detail, first for the ground states, and then for a generic excited state in the original string-net Hilbert space spanned by $|\psi^{\mathcal{D}_1}, 0\rangle$.

A. Ground state

The ground states of the string-net model $\mathcal{D}_1$, which we will denote as $|\psi_{\mathrm{GS}}^{\mathcal{D}_1}, 0\rangle$, are defined as the eigenvectors of the ground-state projector $P_{\mathrm{GS}}^{\mathcal{D}_1}$:

$$\begin{aligned} P_{\mathrm{GS}}^{\mathcal{D}_1} \left|\psi_{\mathrm{GS}}^{\mathcal{D}_1}, 0\right\rangle &= \left(p_{00}^{0,\mathcal{D}_1} \otimes |0\rangle\langle 0|\right)^{\otimes N} \left|\psi_{\mathrm{GS}}^{\mathcal{D}_1}, 0\right\rangle \\ &= \left|\psi_{\mathrm{GS}}^{\mathcal{D}_1}, 0\right\rangle, \end{aligned} \tag{34}$$

projecting each plaquette onto the simple idempotent $p_{00}^{0,\mathcal{D}_1}$ which corresponds to the vacuum; it is equal to the plaquette term of the Hamiltonian,

$$p_{00}^{0,\mathcal{D}_1} = \frac{1}{D} \sum_a d_a \mathcal{B}_p^{a,\mathcal{D}_1}, \tag{35}$$

which also projects onto the ground state of the vertex term by fixing the extended string-net legs to the trivial string. At first glance, one would expect the simple idempotent $p_{00}^{0,\mathcal{D}_1}$ corresponding to the vacuum to be central; the reason this is not the case is that in the extended string-net model, the simple idempotents keep track of one plaquette and two vertex excitations. If the two vertices are excited with a particle-antiparticle pair then together they are in the vacuum, and the corresponding simple idempotent $p_{ii}^{0,\mathcal{D}_1}$, $i \neq 0$ will also be contained in the central idempotent $P_{\mathbb{1}}^{\mathcal{D}_1}$ corresponding to the vacuum in $Z(\mathcal{D}_1)$.

Acting with our quantum circuit on the ground state, we obtain

$$V_{\mathcal{D}_1}^{\mathcal{D}_2} \left|\psi_{\mathrm{GS}}^{\mathcal{D}_1}, 0\right\rangle = \left(p_{00}^{0,\mathcal{M}} \otimes |0\rangle\langle 0|\right)^{\otimes N} \left|\psi_{\mathrm{GS}}^{\mathcal{D}_1}, 0\right\rangle. \tag{36}$$

One can readily verify that

$$\begin{aligned} P_{\mathrm{GS}}^{\mathcal{D}_2} V_{\mathcal{D}_1}^{\mathcal{D}_2} \left|\psi_{\mathrm{GS}}^{\mathcal{D}_1}, 0\right\rangle &= \left(p_{00}^{0,\mathcal{M}} \otimes |0\rangle\langle 0|\right)^{\otimes N} \left|\psi_{\mathrm{GS}}^{\mathcal{D}_1}, 0\right\rangle \\ &= \left|\psi_{\mathrm{GS}}^{\mathcal{D}_2}, 0\right\rangle, \end{aligned} \tag{37}$$

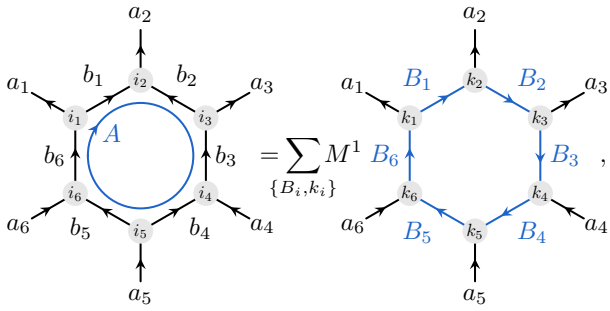
showing that $V_{\mathcal{D}_1}^{\mathcal{D}_2}$ does indeed provide a unitary map between the ground states of the string-net model $\mathcal{D}_1$ and $\mathcal{D}_2$. The operator $p_{00}^{0,\mathcal{M}}$ can be interpreted as a generalized plaquette term, in the sense that

$$p_{00}^{0,\mathcal{M}} = \frac{1}{D} \sum_A d_A \mathcal{B}_p^{A,\mathcal{M}}, \tag{38}$$

i.e., it is the result of fusing a weighted sum of all loops labeled by $A \in \mathcal{M}$ with the plaquette. When applied to the string-net configuration where all edges are fixed to the trivial label (a product state), $(p_{00}^{0,\mathcal{M}})^{\otimes N}$ projects onto the ground state of the string-net model $\mathcal{D}_2$. This operator can be used to derive a projected entangled pair state (PEPS) representation of the string-net ground state, generalizing the constructions in Refs. [22,23] to the tensor network representations obtained in Ref. [18].

More explicitly, the application of $\mathcal{B}_p^{A,\mathcal{M}}$, $\mathcal{B}_p^{D,\mathcal{M}}$, and $\mathcal{B}_p^{G,\mathcal{M}}$ to the three sublattices in the order depicted in Fig. 3 can be worked out by a series of recouplings involving the bimodule

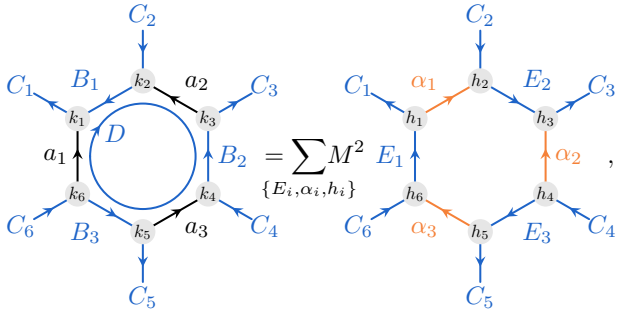
F symbols. For a plaquette of the first kind, we get



where the matrix M^1 depends on $\{a_i, A\}$, which we suppress for convenience, and its components read

$$\begin{aligned}
&\left(M^1\right)^{\{B_i,k_i\}}_{\{b_i,i_i\}} \\
&\equiv \sum_{\{j_i\}} \sqrt{\frac{d_{b_1} d_{b_2} d_{b_4} d_{b_5} d_{B_3} d_{B_6}}{d_{a_2} d_{a_5} d_{b_3} d_{b_6} d_A^2}} \left({}^{\triangleright}F^{a_1 b_1 A}_{B_6}\right)^{B_1, j_1 k_1}_{b_6, i_1 j_6} \\
&\times \left({}_{\triangleright}F^{b_1 b_2 B_2}_{B_1}\right)^{A, j_1 j_2}_{a_2, i_2 k_2} \left({}^{\triangleright}F^{b_2 a_3 B_3}_{A}\right)^{B_2, k_3 j_2}_{b_3, i_3 j_3} \left({}_{\triangleright}F^{b_4 a_4 B_3}_{A}\right)^{B_4, k_4 j_4}_{b_3, i_4 j_3} \\
&\times \left({}^{\triangleright}F^{b_5 b_4 B_4}_{B_5}\right)^{A, j_4 j_5}_{a_5, i_5 k_5} \left({}_{\triangleright}F^{a_6 b_5 A}_{B_6}\right)^{B_5, j_5 k_6}_{b_6, i_6 j_6}. \qquad (39)
\end{aligned}$$

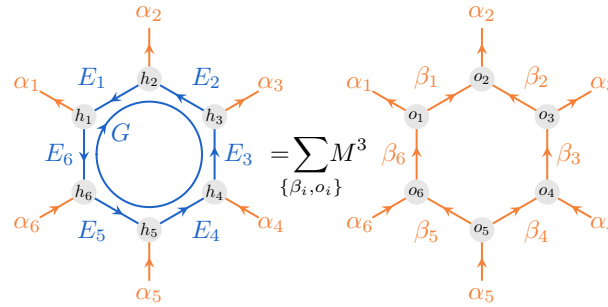
For a plaquette of the second kind, we get



where the matrix M^2 depends on $\{C_i, D\}$ and its components read

$$\begin{aligned}
&\left(M^2\right)^{\{E_i,\alpha_i,h_i\}}_{\{a_i,B_i,k_i\}} \\
&\equiv \sum_{\{l_i\}} \sqrt{\frac{d_{a_1} d_{a_2} d_{a_3} d_{B_2} d_{E_1} d_{\alpha_1} d_{\alpha_2} d_{\alpha_3}}{d_{C_1} d_{C_3} d_{C_4} d_{C_6} d_D^4}} \left({}_{\triangleright\triangleleft}F^{a_1 B_1 \alpha_1}_{E_1}\right)^{D, l_2 l_1}_{C_1, k_1 h_1} \\
&\times \left({}_{\triangleright\triangleleft}F^{a_2 C_2 \alpha_1}_{D}\right)^{E_2, h_2 l_3}_{B_1, k_2 l_2} \left({}_{\triangleright\triangleleft}F^{a_2 E_2 \alpha_2}_{B_2}\right)^{C_3, h_3 k_3}_{D, l_3 l_4} \left({}^{\triangleright\triangleleft}F^{a_3 E_3 \alpha_2}_{B_2}\right)^{C_4, h_4 k_4}_{D, l_5 l_4} \\
&\times \left({}^{\triangleright\triangleleft}F^{a_3 C_5 \alpha_3}_{D}\right)^{E_3, h_5 l_5}_{B_3, k_5 l_6} \left({}^{\triangleright\triangleleft}F^{a_1 B_3 \alpha_3}_{E_1}\right)^{D, l_6 l_1}_{C_6, k_6 h_6}. \qquad (40)
\end{aligned}$$

Finally, for the third kind of plaquette we get



where the matrix M^3 depends on $\{\alpha_i, G\}$ and its components read

$$\begin{aligned}
&\left(M^3\right)^{\{\beta_i,o_i\}}_{\{E_i,h_i\}} \\
&\equiv \sum_{\{n_i\}} \sqrt{\frac{d_{\beta_1} d_{\beta_2} d_{\beta_4} d_{\beta_5} d_{E_3} d_{E_6}}{d_{\alpha_2} d_{\alpha_5} d_{\beta_3} d_{\beta_6} d_G^2}} \left({}^{\triangleleft}F^{E_6 \alpha_1 \beta_1}_{G}\right)^{\beta_6, o_1 n_6}_{E_1, h_1 n_1} \\
&\times \left({}_{\triangleleft}F^{E_1 \beta_1 \beta_2}_{E_2}\right)^{\alpha_2, o_2 h_2}_{G, n_1 n_2} \left({}^{\triangleleft}F^{G \beta_2 \alpha_3}_{E_3}\right)^{\beta_3, o_3 n_3}_{E_2, n_2 h_3} \left({}_{\triangleleft}F^{G \beta_4 \alpha_4}_{E_3}\right)^{\beta_3, o_4 n_3}_{E_4, n_4 h_4} \\
&\times \left({}^{\triangleleft}F^{E_5 \beta_5 \beta_4}_{E_4}\right)^{\alpha_5, o_5 h_5}_{G, n_5 n_4} \left({}_{\triangleleft}F^{E_6 \alpha_6 \beta_5}_{G}\right)^{\beta_6, o_6 n_6}_{E_5, h_6 n_5}. \qquad (41)
\end{aligned}$$

The explicit expression for the operator $(p_{00}^{0,\mathcal{M}})^{\otimes N}$ is then obtained by taking the appropriate linear combinations of the matrices M^i and acting on their respective sublattices in the order described above. While the local Hilbert space of the two string-net models looks different, they have the same number of ground states, and therefore $(p_{00}^{0,\mathcal{M}})^{\otimes N}$ is represented as a square matrix. To show that it is also unitary, we note that

$$\left(p_{00}^{0,\mathcal{M}}\right)^\dagger p_{00}^{0,\mathcal{M}} = p_{00}^{0,\mathcal{D}_1}, \qquad (42)$$

implying that the Hermitian conjugate of the matrix representation of $p_{00}^{0,\mathcal{M}}$ should correspond to the matrix representation of $(p_{00}^{0,\mathcal{M}})^\dagger$. By virtue of Eq. (26),

$$\left(p_{00}^{0,\mathcal{M}}\right)^\dagger = \frac{1}{D} \sum_A d_A \mathcal{B}_p^{A,\overline{\mathcal{M}}}, \qquad (43)$$

which implies that the matrix representation of $(p_{00}^{0,\mathcal{M}})^\dagger$ is obtained as a linear combination of matrices $\overline{M}^i$ obtained in similar fashion as the matrices M^i. Using the fact that the different F symbols themselves are unitary matrices since we are dealing with unitary bimodule categories, one can indeed show that $(M^i)^\dagger = \overline{M}^i$ through a cumbersome but straightforward computation. Taking everything together, this implies that $(p_{00}^{0,\mathcal{M}})^{\otimes N}$ is represented as a unitary matrix on the ground state.

B. Excited states

When applied to a state with excitations, the mapping provided by our circuit is less obvious since, as stressed before, the dimension of the Hilbert spaces does not match between the string-net model $\mathcal{D}_1$ and $\mathcal{D}_2$. We compensated for this by introducing an ancillary qudit in each plaquette, that we initialize to be in the $|0\rangle$ state. Looking at the action of $V_{\mathcal{D}_1}^{\mathcal{D}_2}$

on the space spanned by $|\psi^{\mathcal{D}_1}, 0\rangle$, we find

$$V_{\mathcal{D}_1}^{\mathcal{D}_2}|\psi^{\mathcal{D}_1}, 0\rangle = \left(\sum_{a,j} p_{0j}^{a,\mathcal{M}} \otimes |j\rangle\langle 0|\right)^{\otimes N} |\psi^{\mathcal{D}_1}, 0\rangle. \quad (44)$$

Acting on any plaquette of the resulting state with a simple idempotent $p_{ii}^{a,\mathcal{D}_2}$ only gives a nonzero result if $i = 0$, meaning that for every central idempotent P_a we only get the excitation corresponding to p_{00}^a. Additionally, this state is no longer part of the original Hilbert space spanned by $|\psi^{\mathcal{D}_2}\rangle$, as the ancilla $|j\rangle$ is now entangled with the other degrees of freedom of the string-net model $\mathcal{D}_2$.

There are a number of ways to disentangle these ancillas, but due to the mismatch in Hilbert space it is impossible to do so in a unique or unitary way. One option is to act with the operator $V_{\mathcal{D}_2}^{\mathcal{D}_2}$, which gives

$$\begin{aligned} V_{\mathcal{D}_2}^{\mathcal{D}_2} V_{\mathcal{D}_1}^{\mathcal{D}_2} |\psi^{\mathcal{D}_1}, 0\rangle &= \left(\sum_a \sum_{i=0}^{m_a-1} p_{ii}^{a,\mathcal{M}} \otimes |0\rangle\langle 0|\right)^{\otimes N} |\psi^{\mathcal{D}_1}, 0\rangle \\ &= |\psi^{\mathcal{D}_2}, 0\rangle, \end{aligned} \quad (45)$$

where $m_a = \min(n_a^{\mathcal{D}_1}, n_a^{\mathcal{D}_2})$. This state is again part of the original Hilbert space, and the ancilla can be discarded. If $m_a = n_a^{\mathcal{D}_1}$, not all excitations of type $a \in Z(\mathcal{D}_2)$ in the string-net model $\mathcal{D}_2$ are in the image of this map, while if $m_a = n_a^{\mathcal{D}_2}$ some excitations of the string-net model $\mathcal{D}_1$ are projected out, all depending on the choice of enumeration of the simple idempotents. A different approach might be to make sure the energy of the state is conserved when mapping back to the original Hilbert space; we leave the exploration of the different possibilities to future work.

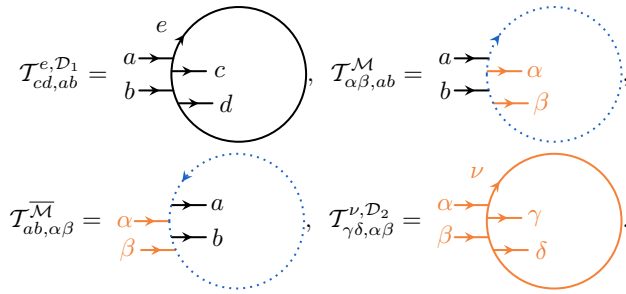
An explicit expression generalizing the matrices M^i to the case of excitations is given in the Appendix. Unitarity of the full circuit can again be shown from unitarity of the F symbols and the fact that the Hermitian conjugate of the bimodule tubes defined in Eq. (26) coincides with the Hermitian conjugate of the matrix representation of the corresponding operator, by definition.

V. EXAMPLES

To illustrate the simple idempotents and simple bimodules that build up our circuit, we write them down for the case of a constant depth circuit that maps the toric code to itself. While this example is almost trivial, it provides enough insight to understand the more general case. We also present the required categorical data for the generic case of a quantum double.

A. Toric code to toric code

The toric code [14] is built from the fusion category $\mathcal{D}_1 = \mathrm{Vec}_{\mathbb{Z}_2}$, the category of $\mathbb{Z}_2$ graded vector spaces with two simple objects $\{0, 1\}$ that fuse according to $\mathbb{Z}_2$. It only has one nontrivial module category $\mathcal{M} = \mathrm{Vec}$, the category of finite-dimensional vector spaces with a single simple object $\mathbb{C}$ from which one can construct an invertible bimodule category to $\mathcal{D}_2 = \mathrm{Rep}(\mathbb{Z}_2) \simeq \mathrm{Vec}_{\mathbb{Z}_2}$. We can use this bimodule category to construct a constant depth quantum circuit that maps the toric code to itself, but is not completely trivial. The relevant tubes of this bimodule category can be written as



The central idempotents with their corresponding simple idempotents and nilpotents of the tube algebra of $\mathcal{D}_1$ are labeled by the four excitations in the toric code. These are the vacuum $\mathbb{1}$:

$$\begin{aligned} p_{00}^{\mathbb{1},\mathcal{D}_1} &= \tfrac{1}{2}(\mathcal{T}_{00,00}^{0,\mathcal{D}_1} + \mathcal{T}_{00,00}^{1,\mathcal{D}_1}), & p_{01}^{\mathbb{1},\mathcal{D}_1} &= \tfrac{1}{2}(\mathcal{T}_{00,11}^{0,\mathcal{D}_1} + \mathcal{T}_{00,11}^{1,\mathcal{D}_1}), \\ p_{10}^{\mathbb{1},\mathcal{D}_1} &= \tfrac{1}{2}(\mathcal{T}_{11,00}^{0,\mathcal{D}_1} + \mathcal{T}_{11,00}^{1,\mathcal{D}_1}), & p_{11}^{\mathbb{1},\mathcal{D}_1} &= \tfrac{1}{2}(\mathcal{T}_{11,11}^{0,\mathcal{D}_1} + \mathcal{T}_{11,11}^{1,\mathcal{D}_1}), \end{aligned}$$

the electric excitation e:

$$\begin{aligned} p_{00}^{e,\mathcal{D}_1} &= \tfrac{1}{2}(\mathcal{T}_{01,01}^{0,\mathcal{D}_1} + \mathcal{T}_{01,01}^{1,\mathcal{D}_1}), & p_{01}^{e,\mathcal{D}_1} &= \tfrac{1}{2}(\mathcal{T}_{01,10}^{0,\mathcal{D}_1} + \mathcal{T}_{01,10}^{1,\mathcal{D}_1}), \\ p_{10}^{e,\mathcal{D}_1} &= \tfrac{1}{2}(\mathcal{T}_{10,01}^{0,\mathcal{D}_1} + \mathcal{T}_{10,01}^{1,\mathcal{D}_1}), & p_{11}^{e,\mathcal{D}_1} &= \tfrac{1}{2}(\mathcal{T}_{10,10}^{0,\mathcal{D}_1} + \mathcal{T}_{10,10}^{1,\mathcal{D}_1}), \end{aligned}$$

the magnetic excitation m:

$$\begin{aligned} p_{00}^{m,\mathcal{D}_1} &= \tfrac{1}{2}(\mathcal{T}_{00,00}^{0,\mathcal{D}_1} - \mathcal{T}_{00,00}^{1,\mathcal{D}_1}), & p_{01}^{m,\mathcal{D}_1} &= \tfrac{1}{2}(\mathcal{T}_{00,11}^{0,\mathcal{D}_1} - \mathcal{T}_{00,11}^{1,\mathcal{D}_1}), \\ p_{10}^{m,\mathcal{D}_1} &= \tfrac{1}{2}(\mathcal{T}_{11,00}^{0,\mathcal{D}_1} - \mathcal{T}_{11,00}^{1,\mathcal{D}_1}), & p_{11}^{m,\mathcal{D}_1} &= \tfrac{1}{2}(\mathcal{T}_{11,11}^{0,\mathcal{D}_1} - \mathcal{T}_{11,11}^{1,\mathcal{D}_1}), \end{aligned}$$

and the fermion f:

$$\begin{aligned} p_{00}^{f,\mathcal{D}_1} &= \tfrac{1}{2}(\mathcal{T}_{01,01}^{0,\mathcal{D}_1} - \mathcal{T}_{01,01}^{1,\mathcal{D}_1}), & p_{01}^{f,\mathcal{D}_1} &= \tfrac{1}{2}(\mathcal{T}_{01,10}^{0,\mathcal{D}_1} - \mathcal{T}_{01,10}^{1,\mathcal{D}_1}), \\ p_{10}^{f,\mathcal{D}_1} &= \tfrac{1}{2}(\mathcal{T}_{10,01}^{0,\mathcal{D}_1} - \mathcal{T}_{10,01}^{1,\mathcal{D}_1}), & p_{11}^{f,\mathcal{D}_1} &= \tfrac{1}{2}(\mathcal{T}_{10,10}^{0,\mathcal{D}_1} - \mathcal{T}_{10,10}^{1,\mathcal{D}_1}). \end{aligned}$$

These central idempotents are all built from two simple idempotents, reflecting the fact that these tube algebras describe the excitations of two vertices and one plaquette. For instance, the simple idempotent $p_{00}^{\mathbb{1},\mathcal{D}_1}$ corresponds to the true vacuum, where neither the vertices nor the plaquette is excited. The other simple idempotent $p_{11}^{\mathbb{1},\mathcal{D}_1}$ represents the case where both vertices are excited; together, they are in the vacuum, and therefore they also contribute to the vacuum central idempotent. The other central idempotents follow a similar pattern. The simple idempotents and nilpotents of the tube algebra of $\mathcal{D}_2$ are related to those of the tube algebra of $\mathcal{D}_1$ as follows:

$$\begin{aligned} p_{ij}^{\mathbb{1},\mathcal{D}_2} &= p_{ij}^{\mathbb{1},\mathcal{D}_1}, & p_{ij}^{e,\mathcal{D}_2} &= p_{ij}^{m,\mathcal{D}_1}, \\ p_{ij}^{m,\mathcal{D}_2} &= p_{ij}^{e,\mathcal{D}_1}, & p_{ij}^{f,\mathcal{D}_2} &= p_{ij}^{f,\mathcal{D}_1}. \end{aligned}$$

We note that the electric and magnetic excitations are swapped, which is a manifestation of the *e-m* duality in the toric code. From these idempotents and nilpotents, one can compute the simple bimodules of $\mathcal{M}$. We find

$$\begin{aligned} p_{00}^{\mathbb{1},\mathcal{M}} &= \tfrac{1}{\sqrt{2}}\mathcal{T}_{00,00}^{\mathcal{M}}, & p_{01}^{\mathbb{1},\mathcal{M}} &= \tfrac{1}{\sqrt{2}}\mathcal{T}_{00,11}^{\mathcal{M}}, \\ p_{10}^{\mathbb{1},\mathcal{M}} &= \tfrac{1}{\sqrt{2}}\mathcal{T}_{11,00}^{\mathcal{M}}, & p_{11}^{\mathbb{1},\mathcal{M}} &= \tfrac{-1}{\sqrt{2}}\mathcal{T}_{11,11}^{\mathcal{M}}, \end{aligned}$$

for the vacuum,

$$p_{00}^{e,\mathcal{M}} = \tfrac{1}{\sqrt{2}}\mathcal{T}_{00,01}^{\mathcal{M}}, \quad p_{01}^{e,\mathcal{M}} = \tfrac{1}{\sqrt{2}}\mathcal{T}_{00,10}^{\mathcal{M}},$$
$$p_{10}^{e,\mathcal{M}} = \tfrac{-1}{\sqrt{2}}\mathcal{T}_{11,01}^{\mathcal{M}}, \quad p_{11}^{e,\mathcal{M}} = \tfrac{1}{\sqrt{2}}\mathcal{T}_{11,10}^{\mathcal{M}},$$

for the electric excitation,

$$p_{00}^{m,\mathcal{M}} = \tfrac{1}{\sqrt{2}}\mathcal{T}_{01,00}^{\mathcal{M}}, \quad p_{01}^{m,\mathcal{M}} = \tfrac{1}{\sqrt{2}}\mathcal{T}_{01,11}^{\mathcal{M}},$$
$$p_{10}^{m,\mathcal{M}} = \tfrac{1}{\sqrt{2}}\mathcal{T}_{10,00}^{\mathcal{M}}, \quad p_{11}^{m,\mathcal{M}} = \tfrac{-1}{\sqrt{2}}\mathcal{T}_{10,11}^{\mathcal{M}},$$

for the magnetic excitation and

$$p_{00}^{f,\mathcal{M}} = \tfrac{1}{\sqrt{2}}\mathcal{T}_{01,01}^{\mathcal{M}}, \quad p_{01}^{f,\mathcal{M}} = \tfrac{1}{\sqrt{2}}\mathcal{T}_{01,10}^{\mathcal{M}},$$
$$p_{10}^{f,\mathcal{M}} = \tfrac{-1}{\sqrt{2}}\mathcal{T}_{10,01}^{\mathcal{M}}, \quad p_{11}^{f,\mathcal{M}} = \tfrac{1}{\sqrt{2}}\mathcal{T}_{10,10}^{\mathcal{M}},$$

for the fermion. The simple bimodules of $\overline{\mathcal{M}}$ are obtained by Hermitian conjugation of the simple bimodules of $\mathcal{M}$:

$$p_{ji}^{\mathbb{1},\overline{\mathcal{M}}} = \left(p_{ij}^{\mathbb{1},\mathcal{M}}\right)^{\dagger}, \quad p_{ji}^{e,\overline{\mathcal{M}}} = \left(p_{ij}^{e,\mathcal{M}}\right)^{\dagger},$$
$$p_{ji}^{m,\overline{\mathcal{M}}} = \left(p_{ij}^{m,\mathcal{M}}\right)^{\dagger}, \quad p_{ji}^{f,\overline{\mathcal{M}}} = \left(p_{ij}^{f,\mathcal{M}}\right)^{\dagger}.$$

B. Ground states of Kitaev's quantum doubles to string nets

The previous example of the toric code can now be generalized to a circuit mapping between quantum doubles and string-net ground states. In those examples the group $\mathbb{Z}_2$ is replaced by an arbitrary finite group G. This then amounts to choosing the UFC $\mathcal{D}_1$ to be $\mathcal{D}_1 = \mathrm{Vec}_G$, the category of all finite-dimensional G-graded vector spaces which is the input category of Kitaev's quantum doubles [14]. The simple objects of Vec_G are in one-to-one correspondence with the group elements of G such that the fusion of elements in Vec_G is then equivalent to the group multiplication in G. The ground states of these quantum doubles can then be mapped to string nets with input category $\mathcal{D}_2 = \mathrm{Rep}(G)$, the category of all representations of G in which the simple objects are the irreducible representations. The invertible bimodule category connecting $\mathcal{D}_1$ and $\mathcal{D}_2$ is $\mathcal{M} = \mathrm{Vec}$, as in the case of the toric code. A crucial difference with the previous example is that for a generic group, there is no monoidal equivalence between Vec_G and $\mathrm{Rep}(G)$. The data of this bimodule category is as follows [18]:

(i) $({}^{\mathcal{D}_1}F^{g_1,g_2,g_3}_{g_{123}})^{g_{23},11}_{g_{12},11} = \omega(g_1,g_2,g_3)$ where ω is a three-cocycle belonging to the trivial class. One can choose a gauge of the ${}^{\mathcal{D}_1}F$ symbol in which this cocycle is identically one, $\omega(g_1,g_2,g_3) \equiv 1, \forall\, g_1,g_2,g_3 \in G$.

(ii) Writing $({}^{\triangleright}F^{g_1 g_2 \mathbb{C}}_{\mathbb{C}})^{\mathbb{C},11}_{g_{12},11} = \psi(g_1,g_2)$, it can be shown that by virtue of the pentagon equation between ${}^{\mathcal{D}_1}F$ and ${}^{\triangleright}F$, ψ is a two-cocycle classified by $H^2(G,U(1))$.

(iii) $({}^{\triangleright\triangleleft}F^{g\mathbb{C}\alpha}_{\mathbb{C}})^{\mathbb{C},j1}_{\mathbb{C},1i} = D^{\alpha}(g)^{j}_{i}$ where $D^{\alpha}(g)$ is the matrix representation of the group element $g \in \mathrm{G}$ in the irreducible representation α and where i, j are d_α-dimensional indices in that representation.

(iv) $({}^{\triangleleft}F^{\mathbb{C}\alpha_1\alpha_2}_{\mathbb{C}})^{\alpha_3,ki_3}_{\mathbb{C},i_1 i_2} = C^{\alpha_1\alpha_2\alpha_3}_{i_1 i_2 i_3,k}$ are the Clebsch-Gordan coefficients of the group. In this notation k denotes the possible number of ways in which α_1 and α_2 can fuse to α_3.

(v) The ${}^{\mathcal{D}_2}F$-symbols can then be recognized as the Racah W coefficients which are equal to the $6j$ symbols of G. These $6j$ symbols serve as the input of the $\mathrm{Rep}(G)$ string net.

Substituting these F symbols in the expressions of the quantum circuit then results in a circuit that allows one to map the ground states of all of Kitaev's quantum double models to corresponding string nets. A similar circuit was found in Refs. [15,16]. In their map the $\mathrm{Rep}(G)$ degrees of freedom where introduced by means of a Fourier transform on G. In our circuit these irreps appear through the ${}^{\triangleright\triangleleft}F$ symbol of the bimodel category. We also recover the $6j$ symbols that appeared in their construction in the form of the ${}^{\mathcal{D}_2}F$ symbols. We note that the quantum doubles as defined in Refs. [16,22] are defined on a triangular lattice and are shown to be equivalent to the string-net model on the hexagonal lattice; our construction does not change the lattice, and a direct comparison is therefore somewhat cumbersome.

VI. CONCLUSION AND OUTLOOK

In this work, we have constructed a constant depth quantum circuit that is able to map between Morita-equivalent ground states. Crucially, the circuit is unitary on the full Hilbert space, meaning that it can be implemented in a truly local way without performing a measurement as would be required to construct the string-net state from a product state. We have shown that the mapping is exact for the ground states, which implies that they are in the same phase as they can be adiabatically connected without closing a gap.

One immediate application of our circuit exploits the recent insight that string-net models can be used to construct partition functions of classical statistical mechanics models [24,25]. From the tensor network point of view, this is understood as the strange correlator [26], which boils down to taking the overlap of the string-net ground-state PEPS with some unentangled product state to obtain a tensor network representation of a classical partition function. The existence of a quantum circuit between Morita-equivalent string nets allows one to construct different lattice models with the same partition functions. We plan to explore this further in future work.

A natural generalization would be to no longer restrict to closed manifolds but also include the case of boundaries to the vacuum or domain walls between different string-net models. In general this requires the use of a four-object bicategory containing four fusion categories and six invertible bimodule categories [18], the recoupling theory of which has not been written down explicitly in the literature.

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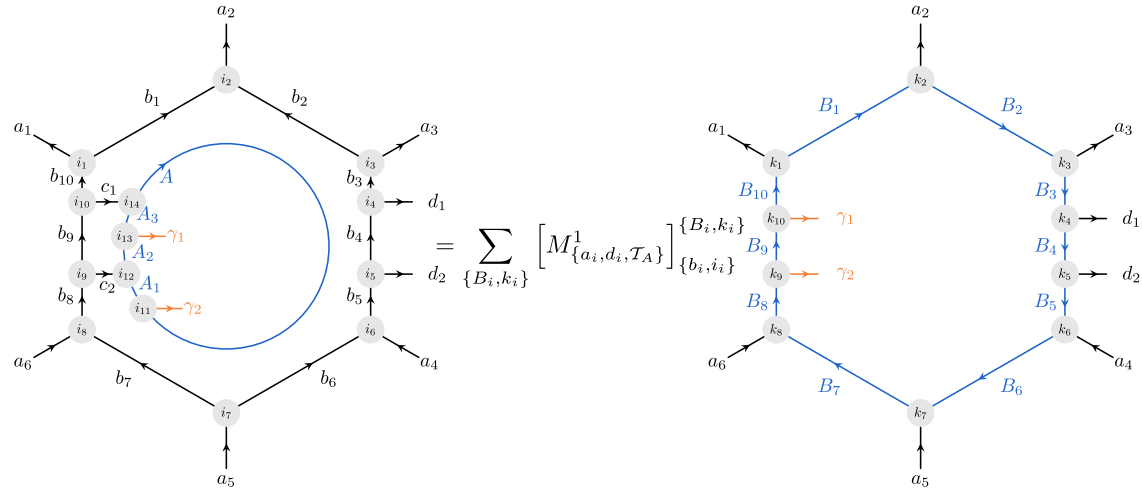
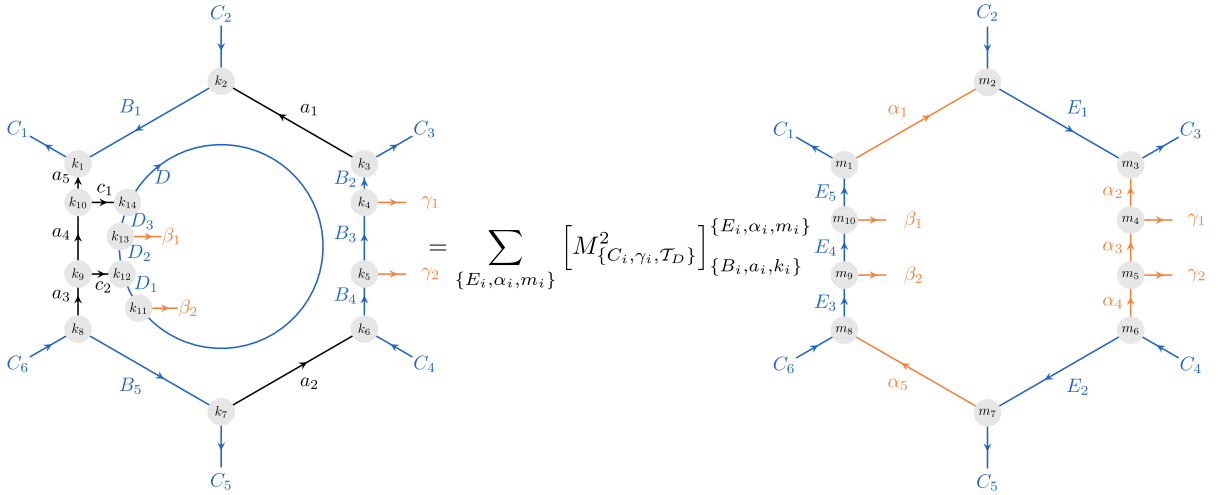
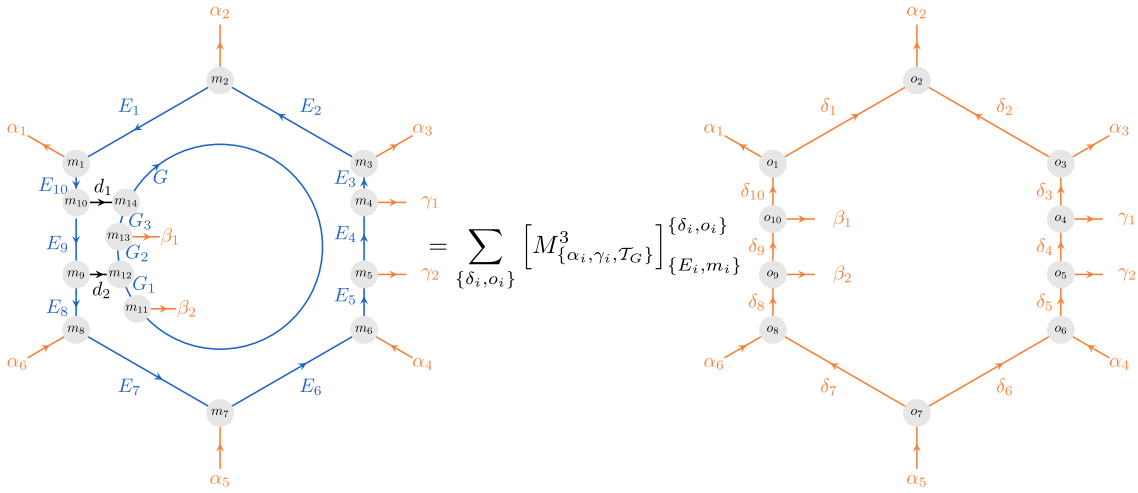
APPENDIX: THE QUANTUM CIRCUIT FOR THE EXTENDED STRING NET



$$\left[M^1_{\{a_i,d_i,\mathcal{T}_A\}}\right]^{\{B_i,k_i\}}_{\{b_i,i_i\}} \equiv \sum_{\{j_i\}} \left(\frac{d_{b_1}d_{b_2}d_{b_6}d_{b_7}d_{c_1}d_{c_2}d_{A_1}d_{A_3}d_{B_5}d_{B_8}}{d_A^3 d_{a_2}d_{a_5}d_{b_5}d_{b_8}d_{A_2}}\right)^{1/2} \left({}^{\triangleright}F^{b_{10}c_1A_3}_{B_{10}}\right)^{A_1,i_{14}j_{12}}_{b_9,i_{10}j_{10}} \left({}^{\bowtie}F^{b_9A_3\gamma_1}_{B_9}\right)^{A_2,i_{13}j_{10}}_{B_{10},j_{11}k_{10}}$$
$$\times \left({}^{\triangleright}F^{b_9c_2A_1}_{B_9}\right)^{A_2,i_{12}j_{10}}_{b_8,i_9j_9} \left({}^{\bowtie}F^{b_8A_1\gamma_2}_{B_8}\right)^{A_3,i_{11}j_8}_{B_9,j_9k_9} \left({}^{\triangleright}F^{a_1b_1A}_{B_{10}}\right)^{B_1,j_1k_1}_{b_{10},i_1j_{12}} \left({}^{\triangleright}F^{b_1b_2B_2}_{B_1}\right)^{A,j_1j_2}_{a_2,i_2k_2} \left({}^{\triangleright}F^{b_2a_3B_3}_{A}\right)^{B_2,k_3j_2}_{b_3,i_3j_3}$$
$$\times \left({}^{\triangleright}F^{b_4d_2B_5}_{A}\right)^{B_4,k_5j_4}_{b_5,i_5j_5} \left({}^{\triangleright}F^{b_3d_1B_4}_{A}\right)^{B_3,k_4j_3}_{b_4,i_4j_4} \left({}^{\triangleright}F^{b_6a_4B_5}_{A}\right)^{B_6,k_6j_6}_{b_5,i_6j_5} \left({}^{\triangleright}F^{b_7b_6B_6}_{B_7}\right)^{A,j_6j_7}_{a_5,i_7k_7} \left({}^{\triangleright}F^{a_6b_7A}_{B_8}\right)^{B_7,j_7k_8}_{b_8,i_8j_8}$$



$$\left[M^2_{\{C_i,\gamma_i,\mathcal{T}_D\}}\right]^{\{E_i,\alpha_i,m_i\}}_{\{B_i,a_i,k_i\}} \equiv \sum_{\{l_i\}} \left(\frac{d_{a_1}d_{a_2}d_{a_5}d_{c_1}d_{c_2}d_{D_1}d_{D_3}d_{B_4}d_{E_3}d_{E_4}d_{\alpha_1}d_{\alpha_2}d_{\alpha_5}}{d_D^5 d_{C_1}d_{C_3}d_{C_4}d_{C_6}d_{D_2}d_{E_4}}\right)^{1/2} \left({}^{\triangleright}F^{a_5c_1D_3}_{E_5}\right)^{D,k_{14}l_{12}}_{a_4,k_{10}l_{11}} \left({}^{\bowtie}F^{a_4D_3\beta_1}_{E_4}\right)^{D_2,k_{13}l_{10}}_{E_5,l_{11}m_{10}}$$
$$\times \left({}^{\triangleright}F^{a_4c_2D_1}_{E_4}\right)^{D_2,k_{12}l_{10}}_{a_3,k_9l_9} \left({}^{\bowtie}F^{a_3D_1\beta_2}_{E_3}\right)^{D,k_{11}l_8}_{E_4,l_9m_9} \left({}^{\bowtie}F^{a_5B_1\alpha_1}_{E_5}\right)^{D,l_1l_{12}}_{C_1,k_1m_1} \left({}^{\bowtie}F^{a_1C_2\alpha_1}_{D}\right)^{E_1,m_2l_2}_{B_1,k_2l_1} \left({}^{\bowtie}F^{a_1E_1\alpha_2}_{B_2}\right)^{C_3,m_3k_3}_{D,l_2l_3}$$
$$\times \left({}^{\triangleleft}F^{D\alpha_2\gamma_1}_{B_3}\right)^{\alpha_3,m_4l_4}_{B_2,l_3k_4} \left({}^{\triangleleft}F^{D\alpha_3\gamma_2}_{B_4}\right)^{\alpha_4,m_5l_5}_{B_3,l_4k_5} \left({}^{\bowtie}F^{a_2E_2\alpha_4}_{B_4}\right)^{C_4,m_6k_6}_{D,l_6l_5} \left({}^{\bowtie}F^{a_2C_5\alpha_5}_{D}\right)^{E_2,m_7l_6}_{B_5,k_7l_7} \left({}^{\bowtie}F^{a_3B_5\alpha_5}_{E_3}\right)^{D,l_7l_8}_{C_6,k_8m_8}$$



$$\left[M^3_{\{\alpha_i,\gamma_i,\mathcal{T}_G\}}\right]^{\{\delta_i,o_i\}}_{\{E_i,m_i\}} \equiv \sum_{\{n_i\}} \left(\frac{d_{d_1} d_{d_2} d_{E_1} d_{E_2} d_{E_3} d_{E_5} d_{E_6} d_{E_7} d^2_{E_8} d_{E_{10}} d_{G_2} d_{\delta_1} d_{\delta_2} d_{\delta_6} d_{\delta_7}}{d^2_G d_{G_1} d_{G_3} d_{\alpha_2} d_{\alpha_5} d_{\delta_8}}\right)^{1/2} \left(\bowtie F^{d_1 E_9 \delta_{10}}_{G}\right)^{G_3,n_9 m_{14}}_{E_{10},m_{10}n_{10}}$$
$$\times \left(\triangleleft F^{E_9\delta_{10}\beta_1}_{G_2}\right)^{\delta_9,o_{10}n_8}_{G_3,n_9 m_{13}} \left(\bowtie F^{d_2 E_8 \delta_9}_{G_2}\right)^{G_1,n_{11}m_{12}}_{E_9,m_9 n_{12}} \left(\triangleleft F^{E_8\delta_9\beta_{12}}_{G}\right)^{\delta_8,o_9 n_{10}}_{G_1,n_{11}m_{11}} \left(\triangleleft F^{E_{10}\alpha_1\delta_1}_{G}\right)^{\delta_{10},o_1 n_{12}}_{E_1,m_1 n_1} \left(\triangleleft F^{E_1\delta_1\delta_2}_{E_2}\right)^{\alpha_2,o_2 m_2}_{G,n_1 n_2}$$
$$\times \left(\triangleleft F^{G\delta_2\alpha_3}_{E_3}\right)^{\delta_3,o_3 n_3}_{E_2,n_2 m_3} \left(\triangleleft F^{G\delta_3\gamma_1}_{E_4}\right)^{\delta_4,o_4 n_4}_{E_3,n_3 m_4} \left(\triangleleft F^{G\delta_4\gamma_2}_{E_5}\right)^{\delta_5,o_5 n_5}_{E_4,n_4 m_5} \left(\triangleleft F^{G\delta_6\alpha_4}_{E_5}\right)^{\delta_5,o_6 n_5}_{E_6,n_6 m_6} \left(\triangleleft F^{E_7\delta_7\delta_6}_{E_6}\right)^{\alpha_5,o_7 m_7}_{G,n_7 n_6} \left(\triangleleft F^{E_8\alpha_6\delta_7}_{G}\right)^{\delta_8,o_8 n_8}_{E_7,m_8 n_7}$$

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