Bridging exceptional geometries and algebras using inner ideals of Lie algebras

Jeroen Meulewaeter

Promotoren:
Prof. Dr. Tom De Medts
Prof. Dr. Hendrik Van Maldeghem

Vakgroep Wiskunde: Algebra en Meetkunde
Faculteit Wetenschappen
Universiteit Gent

Proefschrift voorgelegd aan de Faculteit Wetenschappen tot het behalen van de graad van Doctor in de wetenschappen: wiskunde

FACULTEIT WETENSCHAPPEN

UNIVERSITEIT GENT

fwo

Contents

Acknowledgements

After four years of research, I am very happy to present my doctoral dissertation. I am also very grateful to a lot of people who crossed my path the last few years and who were crucial in the completion of this project.

Tom, mijn promotor, hartelijk bedankt voor de afgelopen vier jaar. Je deur stond altijd open voor vragen en levendige discussies, en ik apprecieerde hoe je alles met een open blik benaderde. Je advies over allerhande zaken was cruciaal voor het schrijven van deze thesis. Het was een voorrecht om de afgelopen jaren met je samen te werken en ik zal er altijd met veel plezier aan terugdenken! Hendrik, mijn copromotor, bedankt voor de vele inzichten over incidentiemeetkunden, de magic square seminaries, en de vele buildings-conferenties.

During the first half of my PhD journey, before something changed the world, I had the pleasure to go on several research visits. Thank you, Hans Cuypers, Cristina Draper, Alberto Elduque and Antonio Fernández López, for your hospitality and for all the interesting discussions.

Many thanks to the members of my examination committee for suggesting various improvements to a previous version of this manuscript, and for the interesting discussion during my private defence.

Dit werk zou niet tot stand kunnen komen zijn zonder de financiële ondersteuning van het Fonds Wetenschappelijk Onderzoek (FWO). Ik ben hen hier dan ook zeer erkentelijk voor.

Ik wil ook graag al mijn bureaugenoten en dichte collega's bedanken voor de aangename (lunch)pauzes waar we over van alles en nog wat van gedachten wisselden. Bedankt Jens, Michiel, Simon, Maarten, Jeroen, Anneleen, Paulien, Magali, Jozefien, Lins, Sam,...

Ten slotte wil ik graag mijn vrienden en familie, en in het bijzonder mijn ouders en broer, bedanken voor de nodige afleiding en voor hun steun de afgelopen jaren, ook (en vooral) wanneer niet alles verliep zoals gepland.

Jeroen Meulewaeter, December 2021

Introduction

Historical context

Structurable algebras

In 1978, Bruce Allison [All78] introduced a class of non-associative algebras called *structurable algebras*, which includes the class of Jordan algebras. Any structurable algebra $\mathcal{A}$ has an involution and hence a subspace $\mathcal{S}$ of skew elements with respect to this involution. The Jordan algebras are precisely the structurable algebras with trivial involution. Another example of structurable algebras are associative algebras with involution. Structurable algebras have been classified by Bruce Allison in [All78] if the characteristic equals 0, and by Oleg Smirnov in [Smi92] if the characteristic is at least 7 (and he also discovered a structurable algebra of dimension 35 which was missing in the original classification, see [Smi90]).

In [All79], Bruce Allison describes how to construct a 5-graded Lie algebra starting from a structurable algebra. The ends of this grading are isomorphic to $\mathcal{S}$. If $\mathcal{S} = 0$, i.e., we are considering a Jordan algebra, we actually get a 3-graded Lie algebra. This construction of a 3-graded Lie algebra starting from a Jordan algebra is due to Jacques Tits, Max Koecher and Issai Kantor, see [Tit62, Koe67, Kan64]. We call this construction of a graded Lie algebra out of a structurable algebra the Tits–Kantor–Koecher-construction, or, in short, the TKK-construction. It is shown in [All79] that all isotropic Lie algebras in characteristic 0 are obtained by applying the TKK-construction to a structurable algebra, in particular one obtains the exceptional Lie algebras. For example, the TKK-construction applied to a *Brown algebra*, a 56-dimensional structurable algebra of skew-dimension one, yields a Lie algebra of type E_8.

Each linear algebraic group has an associated Lie algebra. If the algebraic group is isotropic and the underlying field has characteristic not 2 or 3, then the Lie algebra (or, more precisely, its derived algebra) also arises via the TKK-construction starting from a structurable algebra, often in more than one way [Sta20, Theorem 5.9].

Inner ideals

An important concept in this thesis is that of an *inner ideal* of a Lie algebra. A subspace I of a Lie algebra L is called an inner ideal if $[I,[I,L]] \leq I$. Inner ideals in Lie algebras have been introduced by John Faulkner in [Fau73] and further investigated by Georgia Benkart in her PhD thesis [Ben74]; see also [Ben77, Ben76]. If the Lie algebra is simple and defined over an algebraically closed field of characteristic 0, then its inner ideals have been studied in detail in [DFLGGL12]. Moreover, under the same assumptions, John Faulkner has connected these inner ideals to geometries; see [Fau73]. In the recent book [FL19] inner ideals in Lie algebras also play a crucial role.

In [CI06], Arjeh Cohen and Gabor Ivanyos have introduced *extremal geometries* associated to Lie algebras. An element of a Lie algebra is called *extremal* if it spans a one-dimensional inner ideal (if the characteristic equals 2, some additional conditions have to be satisfied), and the corresponding extremal geometry has as point set the set of all those one-dimensional inner ideals. Under some conditions, these extremal geometries have the structure of so-called root shadow spaces of spherical buildings, see [CI06, CI07]. More recently, Hans Cuypers, Yael Fleischmann, Kieran Roberts and Sergey Shpectorov [CRS15, CF17, CF18] investigated how simple Lie algebras generated by extremal elements are characterized by their extremal geometry.

The concept of an inner ideal also exists in Jordan theory. (In fact, it was introduced in Jordan algebras before it was introduced in Lie algebras.) The inner ideals of Jordan algebras have been studied (and in many cases classified) in [McC71] and they can also be used to describe (exceptional) geometries, see [Fau70]. In [Gar01], Skip Garibaldi shows that some of the inner ideals of a (split) Brown algebra are related to a building of type E_7.

Spherical buildings

Spherical buildings have been introduced by Jacques Tits [Tit74] as a tool to study isotropic simple linear algebraic groups over arbitrary fields. The spherical buildings associated with such algebraic groups always satisfy the so-called *Moufang property*, which says that the automorphism group of such a building is highly transitive (in a very precise way). If the rank of the building (which coincides with the relative rank of the algebraic group) is 1, then the building is called a *Moufang set*; if it is 2, then the building is called a *Moufang poly-*

gon. Depending on the relative type, the Moufang polygon will be a *Moufang triangle* (relative type A_2), *Moufang quadrangle* (relative type B_2 or BC_2) or a *Moufang hexagon* (relative type G_2). The Moufang polygons have been classified and investigated in detail in [TW02].

All known examples of so-called proper Moufang sets with abelian root groups arise from (quadratic) Jordan division algebras, see [DMW06, DMS08, Grü15]. More generally, all known examples of proper Moufang sets with (abelian or non-abelian) root groups without elements of order 2 or 3 arise from structurable division algebras, see [BDMS19].

The algebraic structures coordinatizing Moufang triangles (i.e., Moufang planes), are the alternative division algebras. The algebraic structures coordinatizing Moufang hexagons are the anisotropic cubic norm structures (i.e., cubic Jordan division algebras).

The classification of the Moufang quadrangles in [TW02] separates them into several different classes, and each class has its own corresponding algebraic structure. In order to handle most of these quadrangles in a uniform manner, and in particular the exceptional ones, the notion of a *quadrangular algebra* was introduced in [Wei06]. Recently, in [MW19], the notion of a quadrangular algebra was extended to allow for isotropic quadrangular algebras. In [BDM13, BDM15], Lien Boelaert and Tom De Medts made two different connections between structurable algebras and quadrangular algebras (of exceptional type). In the former paper, the associated structurable algebras have skew-dimension one, whereas in the latter paper, the associated structurable algebras are tensor products of composition algebras. At that time it was unclear how to connect these two different constructions.

Outline

In Chapter 1 we introduce the algebraic structures needed in this thesis and discuss corresponding geometric structures, namely Moufang polygons and Moufang sets.

We start Chapter 2 by introducing a very specific type of geometries, namely the *root filtration spaces*, which were classified in [CI07]. In Section 2.2 we consider subgeometries of these root filtration spaces which are fixed by an involution and show, under some additional conditions, that these subgeometries form a *polar space*. In the next section we explain the construction of a root filtration space in certain Lie algebras L generated by so-called pure extremal elements. More precisely we construct an extremal geometry $\Gamma = \Gamma(L)$, whose point and line set we denote by $\mathcal{E} = \mathcal{E}(L)$ and $\mathcal{F} = \mathcal{F}(L)$, respectively. The set $\mathcal{E}$ coincides with the set of 1-dimensional spaces spanned by the pure extremal elements of L, and the line set $\mathcal{F}$ consists of certain 2-dimensional subspaces of L, incidence is defined by containment. If $\mathcal{F} \neq \emptyset$, then the extremal geometry has the structure of a root filtration space. This construction was first introduced by Arjeh Cohen and Gabor Ivanyos [CI06]. In Section 2.4, we consider the case $\mathcal{F} = \emptyset$. If we assume that there exist so-called *symplectic* pairs of extremal elements and the Lie algebra is not symplectic, then we can link the extremal points of L to the points of this polar space fixed by an involution from Section 2.2. In the final Section 2.5 we extend the notion of an extremal geometry to that of an inner ideal geometry and show that this geometry is either a root filtration space (and it coincides with the extremal geometry), or it is a polar space, or it is just a set without lines.

In Chapter 3 we construct Moufang sets, Moufang triangles and Moufang hexagons using inner ideals of Lie algebras obtained from structurable algebras via the Tits–Kantor–Koecher construction. The three different types of structurable algebras we use are, respectively:

- structurable division algebras; see Sections 3.2 and 3.3;
- algebras $D \oplus D^{\text{op}}$ for some alternative division algebra D, equipped with the exchange involution; see Section 3.4;
- matrix structurable algebras $M(J, 1)$ for some cubic Jordan division algebra J, see Section 3.5.

In each case, we also determine the root groups directly in terms of the structurable algebra.

In Chapter 4 we recover certain algebraic structures, namely structurable algebras, cubic norm structures and quadrangular algebras, if we make suitable assumptions on the extremal geometry. The first section of this chapter collects some properties of the 5-gradings associated with certain pairs of extremal elements. The ends of these 5-gradings, i.e. the (-2)- and 2-part, are one-dimensional. Using this 5-grading, we show in Section 4.2 that if the characteristic does not equal 2 or 3 then any non-symplectic simple Lie algebra generated by its extremal elements is obtained by applying the TKK-construction to a skew-dimension one structurable algebra. In particular, if the extremal geometry contains lines, this structurable algebra is isotopic to a so-called matrix structurable algebra. (Such a structurable algebra is often related to a cubic norm structure.) We end this section by showing that if the inner ideal geometry mentioned before contains no lines, then the extremal points, together with appropriate root groups, form a Moufang set.

In Sections 4.3, 4.4 and 4.6 we extend some of the results of Section 4.2 to fields of characteristic 2 and 3. More precisely, we show in Section 4.3 that if L is a simple Lie algebra generated by its pure extremal elements such that $\mathcal{F}(L) \neq \emptyset$, then the 1-part of the earlier mentioned 5-grading is linearly spanned by its extremal elements. We can then use this to show the existence of automorphisms related to this grading. Using a descent argument, we obtain some statements in larger generality. In Section 4.4 we use these automorphisms and two different 5-gradings on L to recover a cubic norm structure, if the extremal geometry contains lines and is not of so-called type $A_{n,\{1,n\}}$. We also show that the extremal geometry is a Moufang hexagon if and only if J is an anisotropic cubic norm structure, and we determine the root groups explicitly. We can use this to show that a finite-dimensional simple Lie algebra L generated by its pure extremal elements with $\mathcal{F}(L) \neq \emptyset$ is determined by its extremal geometry. In the final subsection of Section 4.4 we focus on the case when the extremal geometry is of type $A_{n,\{1,n\}}$.

In Section 4.5 we consider a simple Lie algebra L over the field k generated by its pure extremal elements which contains symplectic pairs and such that there exists a Galois extension k'/k of degree at most 2 such that the extremal geometry of $L \otimes_k k'$ contains lines. We do *not* necessarily assume that $\mathcal{F}(L) = \emptyset$. Again using two different 5-gradings and automorphisms obtained from Section 4.3, we can recover a quadrangular algebra if the characteristic of the field is not 2. If $\mathcal{F}(L) = \emptyset$, then it follows by Chapter 2 that the inner ideal geometry forms a polar space. We show that if the inner ideal geometry is a polar space of rank 2, i.e. a generalized quadrangle, that the corresponding quadrangular algebra is anisotropic. Then we proceed to show that this quadrangle is precisely the Moufang quadrangle associated with this anisotropic quadrangular algebra.

In the final section of Chapter 4 we consider the case that L is a simple Lie algebra generated by pure extremal elements with $\mathcal{F}(L) = \emptyset$ and there are no symplectic pairs. Then, under some mild additional assumptions, the set $\mathcal{E}(L)$ together with appropriate root groups, which we obtain by Section 4.3, forms a Moufang set.

Chapter 2 is based on the first 7 sections of [CM21]. Chapter 3 is based on [DMM20]. Section 4.2 is based on the last section of [CM21]. The other sections of Chapter 4 are not published (yet).

CHAPTER 1

Preliminaries

In this chapter we introduce most of the algebraic structures which we will use in the other chapters. More precisely, we introduce composition algebras, Jordan algebras, cubic norm structures, quadrangular algebras, Lie algebras, structurable algebras and Kantor pairs. We also introduce the Tits–Kantor–Koecher construction, which associates a Lie algebra to a structrable algebra and we discuss the notion of an inner ideal in a Lie algebra.

In the second part of this chapter we relate some of these algebraic structures to geometric structures, more precisely to Moufang polygons. We end this chapter by introducing the notion of a Moufang set.

In this thesis k always denotes a field. Unless mentioned otherwise, an algebra or vector space is assumed to be defined over k.

SECTION 1.1

Algebraic structures

1.1.1 Quadratic forms

Definition 1.1.1. Let M be a vector space over the field k. A *quadratic form* Q on M is a map $Q : M \to k$ such that

- $Q(\lambda m) = \lambda^2 Q(m)$ for all $\lambda \in k$ and $m \in M$;

- the map $T : M \times M \to k$ defined by

$$T(m, n) = Q(m + n) - Q(m) - Q(n)$$

for all $m, n \in M$ is bilinear.

We call Q

- *anisotropic* if $Q(m) = 0$ implies $m = 0$;
- *non-degenerate* if $M^\perp := \{m \in M \mid T(m, M) = 0\} = 0$;
- *non-singular* if it is either non-degenerate or $\dim(M^\perp) = 1$ and $Q(M^\perp) \neq 0$;
- *regular* if $\{m \in M \mid Q(m) = 0 = T(m, M)\} = 0$.

A *basepoint* of Q is an element $1 \in M$ such that $Q(1) = 1$.

Using this basepoint we can define an involution $\sigma : M \to M$ by

$$m^\sigma = T(m, 1)1 - m, \tag{1.1}$$

for all $m \in M$. Note that

$$T(m, n) = T(m^\sigma, n^\sigma), \tag{1.2}$$

for all $m, n \in M$.

1.1.2 Composition algebras

The octonions are a well-known class of non-associative[1] algebras. They are contained in a larger class of algebras, namely the *compositions algebras*.

Definition 1.1.2. A composition algebra is a unital k-algebra C such that there exists a non-singular quadratic form Q which is *multiplicative*, i.e. $Q(xy) = Q(x)Q(y)$ for all $x, y \in C$. We also call Q *the norm* of C.

Let σ be the involution associated with Q, then $cc^\sigma = Q(c)1$, for all $c \in C$.

By the Generalized Hurwitz Theorem, see for example [Jac58] (for $\mathrm{char}(k) \neq 2$) and [vdBS59], a composition algebra has dimension 1, 2, 4 or 8 over k. The algebras of dimension 4 and 8 are called *quaternions* (or quaternion algebras) and *octonions* (or octonion algebras), respectively. These two types of algebras can be obtained from a smaller composition algebra and a scalar by a doubling procedure, the so-called *Cayley-Dickson doubling process.*

Construction 1.1.3. Let A be an associative composition algebra with corresponding quadratic form Q_A. Denote the involution associated with Q_A by σ_A. Consider $\lambda \in k$ arbitrary. Then $C := A \oplus A\mathbf{k}$, with multiplication given by

$$(a_1 + a_2\mathbf{k})(b_1 + b_2\mathbf{k}) = (a_1b_1 + \lambda b_2^{\sigma_A} a_2) + (b_2a_1 + a_2b_1^{\sigma_A})\mathbf{k}$$

[1] In general, we take *non-associative* to mean *not necessarily associative*. In this case however, the octonions are actually *not* associative.

and quadratic form given by $Q(a_1 + a_2\mathbf{k}) = Q_A(a_1) - \lambda Q_A(a_2)$, for $a_1, a_2, b_1, b_2 \in A$ arbitrary, is a composition algebra. The associated involution σ_C sends $a_1 + a_2\mathbf{k}$ to $a_1^{\sigma_A} - a_2\mathbf{k}$. We denote this composition algebra by $\mathrm{CD}(A, \lambda)$.

Let C be a composition algebra over k, by the Generalized Hurwitz Theorem

- if $\dim(C) = 1$, then $C = k$ and $Q(x) = x^2$.
- if $\dim(C) = 2$, then $C = k1 \oplus k\mathbf{i}$ with $\mathbf{i}^2 = \mathbf{i} + \mu 1$ for certain $\mu \in k$ with $4\mu + 1 \neq 0$. The norm is given by $Q(\lambda_1 1 + \lambda_2\mathbf{i}) = \lambda_1^2 - \mu\lambda_2^2 + 2\lambda_1\lambda_2$ for all $\lambda_1, \lambda_2 \in k$. In particular, every separable quadratic field extension, together with the usual norm as quadratic form, is an example of a 2-dimensional composition algebra.
- if $\dim(C) = 4$, then there exist a non-zero scalar $\lambda \in k$ and a 2-dimensional composition algebra A such that $C \cong \mathrm{CD}(A, \lambda)$.
- if $\dim(C) = 8$, then there exist a non-zero scalar $\lambda \in k$ and a 4-dimensional composition algebra A such that $C \cong \mathrm{CD}(A, \lambda)$.

Remark 1.1.4. The classical Hamiltonians $\mathbb{H}$ are isomorphic with $\mathrm{CD}(\mathbb{C}, -1)$, and the classical octonions $\mathbb{O}$ with $\mathrm{CD}(\mathbb{H}, -1)$.

Quaternion algebras are not commutative but are still associative. Octonion algebras are not commutative nor associative. All composition algebras are examples of *alternative algebras*.

Definition 1.1.5. We call an algebra F *alternative* if $(yx)x = y(xx)$ and $(xx)y = x(xy)$ for all $x, y \in F$. We call an alternative algebra F *division* if for every non-zero $x \in F$ there exists a (unique) $x^{-1} \in F$ such that $x^{-1}(xy) = y = (yx)x^{-1}$ for all $y \in F$.

In particular we defined division composition algebras. Alternatively, a composition algebra C is division if, and only if, its associated quadratic form Q is anisotropic. Also note that the only division composition algebras of dimension 2 are separable quadratic field extensions.

The Bruck-Kleinfeld theorem states that any simple alternative division algebra is either a field, a skew-field or an octonion division algebra (over its center), see [BK51, Kle51].

1.1.3 Jordan algebras

Definition 1.1.6. Let k be a field of characteristic not 2. A *Jordan algebra* over k is a commutative unital k-algebra J such that $(a^2b)a = a^2(ba)$ for all $a, b \in J$.

We can define a so-called U-operator on J for every element $x \in J$ by setting

$$U_x(y) = 2x(xy) - x^2y$$

for all $y \in J$.

Assume in the following examples $\mathrm{char}(k) \neq 2$.

Example 1.1.7. Consider a unital associative k-algebra A. Then $A^+ = (A, \circ)$, where $a \circ b = \frac{1}{2}(ab + ba)$ for all $a, b \in A$, is a Jordan algebra over k. Note that $U_a(b) = aba$ for $a, b \in A$.

Example 1.1.8. Let $Q : J \to k$ be quadratic form with basepoint c. Let T be the bilinear form associated with Q. Then $(J, \cdot)$, where

$$j \cdot l = \frac{1}{2}(T(j,c)l + T(l,c)j - T(j,l)c),$$

for all $j, l \in J$, is a Jordan algebra. In this case we have $U_j(k) = T(j, l^\sigma)j - Q(j)l^\sigma$ for all $j, l \in J$, and where σ is the standard involution associated with Q. We call this Jordan algebra a *Jordan algebra of quadratic type* and denote it by $J(Q, c)$.

After some time, Kevin McCrimmon proposed a definition for Jordan algebras in all characteristics. As is clear from the preceding examples defining the multiplication involves dividing by 2, while defining the U-operator does not. So the uniform definition basically forgets the multiplication and only detects the U-operators, more precisely:

Definition 1.1.9. A (quadratic) Jordan algebra over k is a k-vector space J together with linear maps $U_x : J \to J$ for every $x \in J$ satisfying the following identities strictly, i.e. under any field extension,

- $U_{\lambda x} = \lambda^2 U_x$ for all $\lambda \in k$ and $x \in J$;
- there exists $1 \in J$ such that $U_1 = \mathrm{id}$;
- $U_{U_x(y)} = U_x U_y U_x$ for all $x, y \in J$;
- $U_x V_{y,x} = V_{x,y} U_x$ for all $x, y \in J$, where $V_{x,y} : J \to J$ is defined by $V_{x,y}(z) = (U_{x+z} - U_x - U_z)(y)$ for all $z \in J$.

We call a Jordan algebra *non-degenerate* if it has no non-zero *zero divisors*, i.e. $x \in J$ such that $U_x = 0$.

In characteristic not 2, the two definitions are equivalent.

Also note that any associative algebra A can be given the structure of a (quadratic) Jordan algebra, by defining the U-operator as in Example 1.1.7. We denote this algebra by A^+ as well. Similarly for Example 1.1.8.

Definition 1.1.10. An *inner ideal* of a Jordan algebra J, with maps U_x for $x \in J$, is a a subspace I of J such that $U_i(J) \leq I$ for all $i \in I$.

Example 1.1.11. If $I \leq A$ such that $IAI \leq I$, then I is an an inner ideal of A^+.

Example 1.1.12. The proper inner ideals of $J(Q, c)$ are precisely the isotropic subspaces of J, i.e. the subspaces I of J such that $Q(I) = 0$.

Remark 1.1.13. Inner ideals of a Jordan algebra often yield a geometric structure. For example, the 1- and 2-dimensional isotropic subspaces of the quadratic form Q, so the 1- and 2-dimensional inner ideals of $J(Q, c)$, together with inclusion as incidence, are an example of a so-called *polar space*. See Definition 1.2.17 for the definition of a polar space.

Definition 1.1.14. A Jordan algebra is called *special* if it is a subalgebra of A^+, for a certain associative algebra A, and it is called *exceptional* otherwise.

All finite-dimensional exceptional simple Jordan algebras are so-called cubic Jordan algebras of dimension 27. We also call these exceptional Jordan algebras *Albert algebras*.

We now discuss (non-degenerate) cubic Jordan algebras, although for our purposes later on we do this in the disguise of *cubic norm structures*, see for example [McC69]. (The definition in *loc. cit.* is slightly different since identities have to hold strictly.) See [McC70, Theorem 1] for the connection with Jordan algebras.

In this part on cubic norm structures, we extract the relevant definitions and statements from [TW02, Chapter 15].

Definition 1.1.15. A *cubic norm structure* is a set

$$(J, k, N, \sharp, T, \times, 1),$$

where k is a field, J is a vector space over k, $N : J \to k$ is a map called the *norm*, $\sharp : J \to J$ is a map called the *adjoint*, $T : J \times J \to k$ is a symmetric bilinear form called the *trace*, $\times : J \times J \to J$ is a symmetric bilinear map, and 1 is a non-zero element of J called the *identity* such that for all $\lambda \in k$, and all $a, b, c \in J$ we have:

(i) $(\lambda a)^\sharp = \lambda^2 a^\sharp$;
(ii) $N(\lambda a) = \lambda^3 N(a)$;
(iii) $T(a, b \times c) = T(a \times b, c)$;
(iv) $(a + b)^\sharp = a^\sharp + a \times b + b^\sharp$;
(v) $N(a + b) = N(a) + T(a^\sharp, b) + T(a, b^\sharp) + N(b)$;
(vi) $T(a, a^\sharp) = 3N(a)$;
(vii) $(a^\sharp)^\sharp = N(a)a$;
(viii) $a^\sharp \times (a \times b) = N(a)b + T(a^\sharp, b)a$;
(ix) $a^\sharp \times b^\sharp + (a \times b)^\sharp = T(a^\sharp, b)b + T(a, b^\sharp)a$;
(x) $1^\sharp = 1$;
(xi) $a = T(a, 1)1 - 1 \times a$.

We call this cubic norm structure *non-degenerate* if $0 = \{a \in J \mid N(a) = 0 = T(a, J) = T(a^\sharp, J)\}$.

Definition 1.1.16. We call $a \in J$ *invertible* if $N(a) \neq 0$. We denote the set of all invertible elements of J by $J^\times$. If all non-zero elements of J are invertible, we call J *anisotropic*. If there does exist a non-zero non-invertible element in J, we call J *isotropic*.

Lemma 1.1.17 ([TW02, (15.16)])**.** *Let $(J, k, N, \sharp, T, \times, 1)$ be an anisotropic cubic norm structure. Then the functions T, N, $\times$ and the identity element 1 are uniquely determined by the function $\sharp$.*

Hence we will often denote an anisotropic cubic norm structure $(J, k, N, \sharp, T, \times, 1)$ by $(J, k, \sharp)$ or even $(J, \sharp)$ if it is clear over which field J is defined.

Definition 1.1.18. Let $(J, k, N, \sharp, T, \times, 1)$ and $(J', k, N', \sharp', T', \times', 1')$ be two cubic norm structures. A vector space isomorphism $\varphi : J \to J'$ is an *isomorphism* from $(J, k, N, \sharp, T, \times, 1)$ to $(J', k, N', \sharp', T', \times', 1')$ if $\varphi \circ \sharp = \sharp' \circ \varphi$ and $\varphi(1) = 1'$. By Lemma 1.1.17 the last condition is automatic if the cubic norm structures are anisotropic.

Lemma 1.1.19 ([TW02, (15.18)])**.** *In Definition 1.1.15, condition (iii) is a consequence of (iv) and (v). If $|k| > 3$, then conditions (vi), (viii) and (ix) are a consequence of (i), (ii), (iv), (v) and (vii).*

The U-operator of the Jordan algebra corresponding to the cubic norm structure is given by

$$U_x(y) = T(y, x)x - y \times x^\sharp,$$

with $x, y \in J$.

Definition 1.1.20. Let $(J, k, N, \sharp, T, \times, 1)$ be an anisotropic cubic norm structure. Consider $d \in J^\times$ and let $\sharp_d$ be given by

$$j^{\sharp_d} = N(d)^{-1}(T(d, j^\sharp)d - j^\sharp \times d^\sharp), \tag{1.3}$$

for all $j \in J$. Then $(J, k, \sharp_d)$ is a cubic norm structure, see [TW02, (29.36)]. Note $j^{\sharp_d} = N(d)^{-1}U_d(j^\sharp)$.

We call two anisotropic cubic norm structures $(J, k, \sharp)$ and $(J', k, \sharp')$ *isotopic* if there exists an isomorphism from $(J', k, \sharp')$ to $(J, k, \sharp_d)$ for a certain $d \in J^\times$.

We now give some examples of both isotropic and anisotropic cubic norm structures.

Example 1.1.21. Let l/k be a field extension such that $l^3 \subseteq k$. So either $l = k$ or $\mathrm{char}(k) = 3$ and l/k is a purely inseparable extension. Then, by setting $a^\sharp = a^2$, $N(a) = a^3$, $a \times b = 2ab$, and $T(a, b) = 3ab$, for all $a, b \in l$, it is straightforward to check that all conditions of a cubic norm structure are satisfied.

Example 1.1.22 ([TW02, (15.5),(15.7),(15.26)-(15.28)])**.** Let l/k be a cubic Galois extension. Consider $\mathrm{id} \neq \sigma \in \mathrm{Gal}(l/k)$ and $\lambda \in k^\times$. Let D be the subring of $\mathrm{Mat}_3(l)$ consisting of matrices of the form

$$\begin{pmatrix} a & b & c \\ \lambda c^\sigma & a^\sigma & b^\sigma \\ \lambda b^{\sigma^2} & \lambda c^{\sigma^2} & a^{\sigma^2} \end{pmatrix},$$

with $a, b, c \in l$. Note that l can be identified with the diagonal matrices in D. By considering the subfield k of l, D is an algebra over k of dimension 9. We call such an algebra D a *cyclic algebra of degree three.*

Let $N : D \to k$, $\sharp : D \to D$ and $T' : D \to k$ be the restriction to D of the determinant, adjoint map and trace of a matrix in $\mathrm{Mat}_3(l)$. Define the bilinear form $T : D \times D \to k$ by $T(d, e) = T'(de)$ for all $d, e \in D$.

If $\lambda \notin N(l)$, then $(D, k, N, \sharp, T, \times, 1)$ is an anisotropic cubic norm structure. (The function $\times$ can be obtained from the other maps by property (iv).)

If $\lambda \in N(l)$, then there is a k-linear isomorphism from D to $\mathrm{Mat}_3(k)$ with respect to which T and N correspond to the trace and determinant of $\mathrm{Mat}_3(k)$. This is an example of an isotropic cubic norm structure.

Now we discuss a class of isotropic cubic norm structures, which are obtained from composition algebras.

Example 1.1.23 ([PR86, Example 2.3])**.** Let C be a composition algebra. Denote its standard involution by $\overline{}$. Let $H_3(C)$ be the vector space consisting of the matrices of the form

$$x = \begin{pmatrix} \lambda_1 & c_1 & c_2 \\ \overline{c_1} & \lambda_2 & c_3 \\ \overline{c_2} & \overline{c_3} & \lambda_3 \end{pmatrix},$$

with $\lambda_1, \lambda_2, \lambda_3 \in k$ and $c_1, c_2, c_3 \in C$. These are precisely the 3×3-matrices over C which are fixed by the involution $X \mapsto \overline{X}^T$. Let E_{ij} be the 3×3-matrix with all entries equal to 0, except for the (i, j)-entry, which equals 1. Denote the norm of the composition algebra by n and set let t be its trace, i.e. $t(c) = T(c, 1)$, where T is the bilinear form associated with n. We can give this vector space the structure of a cubic norm structure by setting

$$\begin{aligned} N(x) &= \lambda_1\lambda_2\lambda_3 + t(c_1c_2c_3) - \sum_{i=1}^{3} \lambda_i n(c_i); \\ x^\sharp &= \sum_{k=1}^{3} (\lambda_i\lambda_j - n(c_k))E_{kk} + (\overline{c_i c_j} - \lambda_k c_k)E_{ij} + (c_i c_j - \lambda_k \overline{c_k})E_{ji}; \\ 1 &= E_{11} + E_{22} + E_{33}; \\ T(x, y) &= \sum_{i=1}^{3} \lambda_i \mu_i + t(\overline{c_i} d_i), \end{aligned}$$

with $(i\ j\ k)$ a cyclic permuatation of $(1\ 2\ 3)$ and x and y are arbitrary elements of $H_3(C)$. (For y we use scalars μ_i and elements d_i of C.) The cross product can be deduced from $\sharp$ by using property (iv) of a cubic norm structure. Note that this cubic norm structure is isotropic, since $N(E_{ij}) = 0$ for $i, j \in \{1, 2, 3\}$ with $i \neq j$.

Remark 1.1.24. As is shown in [TW02, Chapter 30], any anisotropic cubic norm structure $(J, k, \sharp)$ is either coming from a purely inseparable field extension as in Example 1.1.21 or $\dim_k(J) \in \{1, 3, 9, 27\}$.

An easier example of an isotropic cubic norm structure is the following.

Example 1.1.25 ([PR86, Example 2.2])**.** Let $Q : M \to k$ be a quadratic form with basepoint 1. Consider the vector space $J = k \oplus M$. We can give this vector space the structure of a cubic norm structure by setting $N((\lambda, m)) = \lambda Q(m)$, $(\lambda, m)^\sharp = (Q(m), \lambda m^\sigma)$, $1 = (1, 1)$ and $T((\lambda, m), (\mu, n)) = \lambda\mu + T_Q(m, n^\sigma)$ for all $\lambda, \mu \in k$ and $m, n \in M$, and with σ the standard involution on M associated with Q, and T_Q the bilinear form associated with Q. This cubic norm structure is isotropic since $N((1, 0)) = 0$.

Remark 1.1.26. By [Rac72, Theorem 1] the only non-degenerate isotropic cubic norm structures are as in Example 1.1.25, with Q regular, or are as in Example 1.1.23, where we allow $C = 0$, or a generalization thereof, see [PR86, Example 2.3].

1.1.4 Quadrangular algebras

In this section we introduce quadrangular algebras. These algebras were introduced by Richard Weiss in [Wei06]. We start by giving its definition, and then show that if the characteristic is not 2, some conditions are automatically satisfied. Then we discuss some examples of quadrangular algebras.

Definition 1.1.27. A quadrangular algebra is a set

$$(k, M, Q, 1, X, \cdot, h, \theta),$$

where

- k is a field;
- M is a vector space over k;
- $Q : M \to k$ is a regular quadratic form (with associated bilinear form T, as in Definition 1.1.1);
- 1 is a basepoint for Q, i.e. $Q(1) = 1$ (with associated involution σ, as in Definition 1.1.1);
- X is a vector space over k;
- $\cdot : X \times M \to X : (x, v) \mapsto x \cdot v$ is a bilinear map;
- $h : X \times X \to M$ is a bilinear map;
- $\theta : X \times M \to M$ is a map;

such that

(i) $x \cdot 1 = x$ for all $x \in X$;
(ii) $(x \cdot m) \cdot m^\sigma = Q(m)x$ for all $x \in X$ and $m \in M$;
(iii) $h(x, y \cdot m) = h(y, x \cdot m) + T(h(x, y), 1)m$ for all $x, y \in X$ and $m \in M$;

(iv) $T(h(x \cdot m, y), 1) = T(h(x, y), m)$ for all $x, y \in X$ and $m \in M$;
(v) For each $x \in X$, the map $m \mapsto \theta(x, m)$ is linear;
(vi) $\theta(\lambda x, m) = \lambda^2 \theta(x, m)$ for all $x \in X$, $m \in M$ and $\lambda \in k$;
(vii) There exists a function $g : X \times X \to k$ such that

$$\theta(x + y, m) = \theta(x, m) + \theta(y, m) + h(x, y \cdot m) - g(x, y)m,$$

for all $x, y \in X$ and $m \in M$;
(viii) There exists a function $\phi : X \times M \to k$ such that

$$\begin{aligned}\theta(x \cdot m, n) = \theta(x, n^\sigma)^\sigma Q(m) - T(n, m^\sigma)\theta(x, m)^\sigma \\ + T(\theta(x, m), n^\sigma)m^\sigma + \phi(x, m)n,\end{aligned}$$

for all $x \in X$ and $m, n \in M$;
(ix) $x \cdot \theta(x, m) = (x \cdot \theta(x, 1)) \cdot m$ for all $x \in X$ and $m \in M$.

Notation 1.1.28. We set $\pi(x) = \theta(x, 1)$ for all $x \in X$.

Definition 1.1.29. We call a quadrangular algebra *anisotropic* if Q is anisotropic and $\pi(x)$ is a multiple of 1 if and only if $x = 0$.

Remark 1.1.30. The existence of 1 implies $M \neq 0$. Note however, contrary to the common convention, that we allow $X = 0$. In other words, we view quadratic forms as special (degenerate) cases of the class of quadrangular algebras.

Now we proceed to show that if $\mathrm{char}(k) \neq 2$ one still obtains a quadrangular algebra if one does not assume certain properties.

Lemma 1.1.31 ([Wei06, Remark 4.8])**.** *Let $(k, M, Q, 1, X, \cdot, h)$ be a set with the same assumptions on its elements as in the first part of Definition 1.1.27. Assume moreover* $\mathrm{char}(k) \neq 2$. *Set*

$$\theta(x, m) = \frac{1}{2} h(x, x \cdot m),$$

for all $x \in X$ and $m \in M$. Then $(k, M, Q, 1, X, \cdot, h, \theta)$ is a quadrangular algebra if (i) to (iv) and (ix) of Definition 1.1.27 are satisfied.

Proof. Properties (v) and (vi) are satisfied since h and $\cdot$ are both bilinear.

Set $g(x, y) = \frac{1}{2} T(h(x, y), 1)$ for all $x, y \in X$. Then using (iii)

$$\begin{aligned}\theta(x + y, m) &= \frac{1}{2} h(x, x \cdot m) + \frac{1}{2} h(y, y \cdot m) + \frac{1}{2} h(x, y \cdot m) + \frac{1}{2} h(y, x \cdot m) \\ &= \theta(x, m) + \theta(y, m) + h(x, y \cdot m) - \frac{1}{2} T(h(x, y), 1)m\end{aligned}$$

for all $x, y \in X$ and $m \in M$, which shows (vii).

Consider $x \in X$ and $m, n \in M$ arbitrary. By property (ii) and the definition of σ,

$$\begin{aligned} Q(m+n)x &= (x \cdot (m+n)) \cdot (m^\sigma + n^\sigma) \\ &= Q(m)x + Q(n)x - (x \cdot n) \cdot m - (x \cdot m) \cdot n \\ &\quad + T(m,1)x \cdot n + T(n,1)x \cdot m, \end{aligned}$$

and hence

$$(x \cdot m) \cdot n = -(x \cdot n^\sigma) \cdot m^\sigma + T(n, m^\sigma)x. \tag{1.4}$$

Also note that property (iii) for $m = 1$ together with property (i) yields

$$h(y,x) = -h(x,y)^\sigma \tag{1.5}$$

for all $x, y \in X$. Property (iii) for $m = 1$ and $y = x$ yields $T(h(x,x),1) = 0$.

We are now ready to show property (viii). For ease of notation in the following equations, we will write xm instead of $x \cdot m$. Consider $x \in X$ and $m, n \in M$ arbitrary. By twice applying property (1.4) and (1.5)

$$\begin{aligned} 2\theta(xm, n) &= h(xm, (xm)n) = -h(xm, (xn^\sigma)m^\sigma) + T(n, m^\sigma)h(xm, x) \\ &= h((xn^\sigma)m^\sigma, xm)^\sigma - T(n, m^\sigma)h(x, xm)^\sigma. \end{aligned} \tag{1.6}$$

Now, property (ii) to (iv) yield

$$\begin{aligned} h((xn^\sigma)m^\sigma, xm) &= h(x, ((xn^\sigma)m^\sigma)m) + T(h((xn^\sigma)m^\sigma, x), 1)m \\ &= Q(m)h(x, xn^\sigma) + T(h((xn^\sigma)m^\sigma, x), 1)m. \end{aligned} \tag{1.7}$$

Combining (1.6) and (1.7) gives

$$\begin{aligned} 2\theta(xm, n) &= h(x, xn^\sigma)^\sigma Q(m) - T(n, m^\sigma)h(x, xm)^\sigma + T(h((xn^\sigma)m^\sigma, x), 1)m^\sigma \\ &= 2\theta(x, n^\sigma)^\sigma Q(m) - 2T(n, m^\sigma)\theta(x, m)^\sigma + T(h((xn^\sigma)m^\sigma, x), 1)m^\sigma \end{aligned}$$

By $T(h(x,x),1) = 0$, (1.2), (1.4) and (1.5) and property (iv), we get

$$\begin{aligned} T(h((xn^\sigma)m^\sigma, x), 1) &= -T(h((xm)n), x), 1) = -T(h(xm, x), n) \\ &= T(2\theta(x, m)^\sigma, n) = 2T(\theta(x, m), n^\sigma). \end{aligned}$$

If we let $\phi : X \times M \to k$ be the 0-map, then property (viii) is satisfied. □

We now discuss some classes of quadrangular algebras. We start with a more classical class, and then mention a more exceptional class.

Definition 1.1.32. A *standard pseudo-quadratic space* is a set

$$(M, \sigma, X, h, \pi),$$

where

(i) M is a skew field;
(ii) σ is an involution of M, i.e. an anti-automorphism of M of order 2;
(iii) X is right vector space over M;
(iv) $h : X \times X \to M$ is a *skew-hermitian form* on X, i.e.
- $h(x, ym) = h(x, y)m$;
- $h(x, y)^\sigma = -h(x, y)$,

for all $x, y \in X$ and $m \in M$;
(v) $\pi : X \to M$ satisfies
- $\pi(x + y) \equiv \pi(x) + \pi(y) + h(x, y) \pmod{M_\sigma}$;
- $\pi(xm) = m^\sigma \pi(x)m$,

for all $x, y \in X$ and $m \in M$, where $M_\sigma = \{m + m^\sigma \mid m \in M\}$.

We call a standard pseudo-quadratic space *anisotropic* if $\pi(x)$ is contained in M_σ if and only if $x = 0$.

Lemma 1.1.33 ([Wei06, Proposition 1.18],[MW19, Notation 4.16])**.** *Let C be an associative composition algebra over k with associated quadratic form Q and standard involution σ, which we assume to be non-trivial. Consider any standard pseudo-quadratic space (C, σ, X, h, π) and set $\theta(x, c) = \pi(x)c$ for all $x \in X$ and $c \in C$. Then $(k, C, Q, 1, X, \text{scalar multiplication}, h, \theta)$ is a quadrangular algebra.*

Remark 1.1.34. Note that the quadrangular algebra obtained in Lemma 1.1.33 is anisotropic if, and only if, the associated standard pseudo-quadratic space is anisotropic and C is division.

We now briefly discuss the quadrangular algebras of type E_6, E_7 and E_8.

Definition 1.1.35. Let M be a k-vector space. We call an anisotropic quadratic form $Q : M \to k$ with basepoint of type E_6, E_7 or E_8 if there exists a separable quadratic field extension l/k with norm N such that, respectively,

- $M \cong l^3$, there exists a basis $\{v_1, v_2, v_3\}$ of M over l and scalars s_1, s_2 and s_3 such that
$$Q(\lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3) = s_1 N(\lambda_1) + s_2 N(\lambda_2) + s_3 N(\lambda_3),$$
for all $\lambda_1, \lambda_2, \lambda_3 \in l$.
- $M \cong l^4$, there exists a basis $\{v_1, v_2, v_3, v_4\}$ of M over l and scalars s_1, s_2, s_3 and s_4 such that
$$Q(\sum_{i=1}^{4} \lambda_i v_i) = \sum_{i=1}^{4} s_i N(\lambda_i)$$
for all $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in l$ and $s_1 s_2 s_3 s_4 \notin N(l)$.
- $M \cong l^6$, there exists a basis $\{v_1, v_2, v_3, v_4, v_5, v_6\}$ of M over l and scalars s_1, s_2, s_3, s_4, s_5 and s_6 such that
$$Q(\sum_{i=1}^{6} \lambda_i v_i) = \sum_{i=1}^{6} s_i N(\lambda_i)$$
for all $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6 \in l$ and $-s_1 s_2 s_3 s_4 s_5 s_6 \in N(l)$.

Example 1.1.36. Consider an octonion division algebra C over k, with norm Q. Then there exists a separable quadratic field extension l/k with norm N and scalars λ and μ such that $C \cong \mathrm{CD}(\mathrm{CD}(l,\lambda),\mu)$. Then, by construction, $C \cong l\mathbf{v}_1 \oplus l\mathbf{v}_2 \oplus l\mathbf{v}_3 \oplus l\mathbf{v}_4$ with

$$Q(\lambda_1\mathbf{v}_1 + \lambda_2\mathbf{v}_2 + \lambda_3\mathbf{v}_3 + \lambda_4\mathbf{v}_4) = N(\lambda_1) - \lambda N(\lambda_2) - \mu N(\lambda_3) + \lambda\mu N(\lambda_4),$$

for any $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in l$. Now since C is division, Q is anisotropic, and, in particular, Q restricted to $l\mathbf{v}_1 \oplus l\mathbf{v}_2 \oplus l\mathbf{v}_3$ is anisotropic. If we set $s_1 = 1$, $s_2 = -\lambda$, $s_3 = -\mu$, we see that this restriction is a quadratic form of type E_6.

We only defined a quadratic form and not the corresponding vector space X or any of the other maps in the definition of a quadrangular algebra. Since these constructions are quite involved we only mention that for every quadratic form $Q : M \to k$ with basepoint 1 of type E_6, E_7 or E_8 it is possible to construct a vector space X and maps $\cdot$, h, θ such that $(k, M, Q, 1, X, \cdot, h, \theta)$ is an anisotropic quadrangular algebra. We call this quadrangular algebra of type E_6, E_7 or E_8, respectively. The construction can be found in [TW02, Chapter 13]. For another approach to define quadrangular algebras of type E_6, E_7 or E_8, using structurable algebras, see [BDM15]. The vector space X has dimension 8, 16 or 32 over k, if Q is of type E_6, E_7 or E_8, respectively. If $k = \mathbb{R}$, there exists a quadrangular algebra of type E_6 by Example 1.1.36, but as is shown in [TW02, (12.38)] there do not exist quadrangular algebras of type E_7 or E_8 over $\mathbb{R}$.

The examples of anisotropic quadrangular algebras discussed before are actually all anisotropic quadrangular algebras if the characteristic does not equal 2. If the characteristic is equal to 2, there are three other classes of quadrangular algebras, and one of these classes corresponds to Moufang quadrangles of type F_4, see [Wei06].

Theorem 1.1.37 ([Wei06, 2.3, 2.4, 3.2 and 3.14])**.** *Let k be a field of characteristic not 2 and let $(k, M, Q, 1, X, \cdot, h, \theta)$ be an anisotropic quadrangular algebra. Then either this quadrangular algebra is coming from an anisotropic standard pseudo-quadratic space as in Lemma 1.1.33, where moreover the composition algebra is division, or it is a quadrangular algebra of type E_6, E_7 or E_8.*

1.1.5 Lie algebras

Definition 1.1.38. A k-vector space L together with a bilinear operation $L \times L \to L : (l, m) \mapsto [l, m]$ (called the *Lie bracket*) is a *Lie algebra* (over k) if

- $[x, x] = 0$,
- $[z, [x, y]] + [x, [y, z]] + [y, [z, x]] = 0$,

for all $x, y, z \in L$. The second identity is called the *Jacobi identity*.

Definition 1.1.39. Let L and L' be two Lie algebras defined over the same field

k. We call a linear map $\varphi : L \to L'$ a *morphism* if $\varphi([l,m]) = [\varphi(l), \varphi(m)]$ for all $l, m \in L$. If $L = L'$, and φ is an isomorphism, we call it an *automorphism* of L.

A *derivation* of a Lie algebra L is a linear map D from L to itself such that $D([l,m]) = [D(l), m] + [l, D(m)]$ for all $l, m \in L$. For each $l \in L$, the map $\mathrm{ad}(l) : L \to L : m \mapsto [l,m]$ is a derivation of L by the Jacobi identity. We call such a derivation an *inner derivation*.

Definition 1.1.40. A subspace I of a Lie algebra L is an *ideal* if $[L, I] \leq I$. We call a Lie algebra *simple* if it does not have proper non-trivial ideals.

Definition 1.1.41. A $\mathbb{Z}$-grading of a Lie algebra L is a vector space decomposition $L = \bigoplus_{i \in \mathbb{Z}} L_i$ such that $[L_i, L_j] \leq L_{i+j}$ for all $i, j \in \mathbb{Z}$. If n is a natural number such that $L_i = 0$ for all $i \in \mathbb{Z}$ such that $|i| > n$ while $L_{-n} \oplus L_n \neq 0$, then we call this grading a $(2n+1)$-grading. We call L_{-n} and L_n the *ends* of this grading. The *i-component* of $x \in L$ is the image of the projection of x onto L_i. We also set $L_{\leq i} = \bigoplus_{j \leq i} L_j$ and $L_{\geq i} = \bigoplus_{j \geq i} L_j$.

Definition 1.1.42. If $L = L_{-n} \oplus L_{-n+1} \oplus \ldots \oplus L_{n-1} \oplus L_n$ is a $(2n+1)$-grading of a Lie algebra L, the grading derivation is the derivation ζ of L given by

$$\zeta(x) = i \cdot x,$$

for all $x \in L_i$, with i between $-n$ and n. If L contains an element ζ such that ad_ζ is the grading derivation we call ζ the grading derivation of L by abuse of language.

If A is an associative algebra, then A together with the Lie bracket $[a,b] = ab - ba$ for all $a, b \in A$, is a Lie algebra.

Later on, we will encounter a lot of of 5-graded Lie algebras, namely the ones obtained by the Tits–Kantor–Koecher–construction applied to a structurable algebra.

1.1.6 Structurable algebras

Structurable algebras have been introduced by Bruce Allison in [All78] as a generalization of Jordan algebras. It is precisely in this context that we will study them. Each Jordan algebra gives rise, via the Tits–Kantor–Koecher (TKK) construction, to a Lie algebra equipped with a 3-grading. This construction has been generalized to structurable algebras, giving rise to Lie algebras equipped with a 5-grading; see Definition 1.1.64 below.

In this subsection, and, more generally, when we consider structurable algebras, we always assume algebras to be finite-dimensional and defined over a field of characteristic not 2 or 3.

The material in this subsection is based on [BDMS19, Chapter 2] and we refer to that paper for a more detailed exposition.

Definition 1.1.43. Let $\mathcal{A}$ be a unital k-algebra equipped with an involution $\sigma\colon x \mapsto \overline{x}$, i.e., a k-linear map of order at most 2 satisfying $\overline{x.y} = \overline{y}.\overline{x}$ for all $x, y \in \mathcal{A}$. Let

$$V_{x,y}(z) := (x\overline{y})z + (z\overline{y})x - (z\overline{x})y$$

for all $x, y, z \in \mathcal{A}$. If

$$[V_{x,y}, V_{z,w}] = V_{V_{x,y}(z),w} - V_{z,V_{y,x}(w)}$$

for all $x, y, z, w \in \mathcal{A}$ (where the left hand side denotes the Lie bracket of the two operators) then we call $\mathcal{A}$ a *structurable algebra*.

For all $x, y, z \in \mathcal{A}$, we write $U_{x,y}z := V_{x,z}y$ and $U_x y := U_{x,x}y$.

Definition 1.1.44. Let $\mathcal{A}$ be a structurable algebra; then $\mathcal{A} = \mathcal{H} \oplus \mathcal{S}$, with

$$\mathcal{H} = \{h \in \mathcal{A} \mid \overline{h} = h\} \text{ and } \mathcal{S} = \{s \in \mathcal{A} \mid \overline{s} = -s\}.$$

The elements of $\mathcal{H}$ are called *hermitian elements*, the elements of $\mathcal{S}$ are called *skew elements*. The dimension of $\mathcal{S}$ is called the *skew-dimension* of $\mathcal{A}$.

Definition 1.1.45. Let $\mathcal{A}$ be a structurable algebra. An element $u \in \mathcal{A}$ is called *conjugate invertible* (or simply *invertible*) if there exists an element $\hat{u} \in \mathcal{A}$ such that

$$V_{u,\hat{u}} = \mathrm{id}, \text{ or equivalently, } V_{\hat{u},u} = \mathrm{id}\,.$$

If u is conjugate invertible, then the element $\hat{u}$ is uniquely determined, and is called the *conjugate inverse* of u. Moreover, if u is conjugate invertible, the operator U_u is invertible; see [AH81, Section 6].

Definition 1.1.46. An *ideal* of $\mathcal{A}$ is a two-sided ideal stabilized by the involution, and $\mathcal{A}$ is *simple* if its only ideals are $\{0\}$ and $\mathcal{A}$. The center of $\mathcal{A}$ is defined by

$$Z(\mathcal{A}) = \{z \in \mathcal{H} \mid [z, \mathcal{A}] = [z, \mathcal{A}, \mathcal{A}] = [\mathcal{A}, z, \mathcal{A}] = [\mathcal{A}, \mathcal{A}, z] = 0\},$$

and $\mathcal{A}$ is *central* if its center equals $k1$.

Definition 1.1.47. The following map, called the *skewer map*, plays an important role in the theory of structurable algebras:

$$\psi\colon \mathcal{A} \times \mathcal{A} \to \mathcal{S}\colon (x, y) \mapsto x\overline{y} - y\overline{x}.$$

Definition 1.1.48. For each $x \in \mathcal{A}$, we let L_x and R_x denote the left and right multiplication by x, respectively. For each $A \in \mathrm{End}_k(\mathcal{A})$, we define two new k-linear operators

$$A^\epsilon := A - L_{A(1)+\overline{A(1)}}\,,$$
$$A^\delta := A + R_{\overline{A(1)}}\,.$$

One can verify, see [All79, Equations (4) and (8)], that

$$V^{\epsilon}_{x,y} = -V_{y,x} \quad \text{and} \tag{1.8}$$
$$V^{\delta}_{x,y}(s) = -\psi(x, sy) \tag{1.9}$$

for all $x, y \in \mathcal{A}$ and $s \in \mathcal{S}$ and that

$$(L_r L_t)^{\epsilon} = -L_t L_r \quad \text{and} \tag{1.10}$$
$$(L_r L_t)^{\delta}(s) = s(tr) + r(ts) \tag{1.11}$$

for all $r, s, t \in \mathcal{S}$.

Definition 1.1.49. By the definition of a structurable algebra, the subspace

$$\operatorname{Instrl}(\mathcal{A}) := \operatorname{span}\{V_{x,y} \mid x, y \in \mathcal{A}\}$$

is a Lie subalgebra of $\operatorname{End}_k(\mathcal{A})$, which is called the inner structure Lie algebra.

We recall the notion of derivations in structurable algebras.

Definition 1.1.50. A *derivation* of $\mathcal{A}$ is a k-linear map $D\colon \mathcal{A} \to \mathcal{A}$ such that $D(ab) = D(a)b + aD(b)$ and $D(\overline{a}) = \overline{D(a)}$, for all $a, b \in \mathcal{A}$. In particular, we have $D(\mathcal{S}) \subseteq \mathcal{S}$ and $D(\mathcal{H}) \subseteq \mathcal{H}$.

The Lie algebra of all derivations of $\mathcal{A}$ is denoted by $\operatorname{Der}(\mathcal{A})$.

In the following lemma we use the notation $T_x := V_{x,1}$.

Lemma 1.1.51 ([All79, page 1840])**.** *We have*

$$\operatorname{Instrl}(\mathcal{A}) = \{T_x \mid x \in \mathcal{A}\} \oplus \operatorname{Inder}(\mathcal{A}),$$

with $\operatorname{Inder}(\mathcal{A})$ *a certain Lie subalgebra of* $\operatorname{Der}(\mathcal{A})$ *(namely the Lie subalgebra of inner derivations, but we will not need this explicitly).*

We will now prove some elementary lemmas on structurable algebras.

Lemma 1.1.52. *We have* $V_{a,b}(c) = V_{c,b}(a) + \psi(a, c)b$ *for all* $a, b, c \in \mathcal{A}$*. Moreover, we have* $V_{a,a} = L_{a\overline{a}} = T_{a\overline{a}}$ *for all* $a \in \mathcal{A}$*.*

Proof. This is immediate from the definitions. □

Lemma 1.1.53. *The skewer map* ψ *is non-degenerate.*

Proof. This is precisely [AF84, Lemma 2.2]. Although it is only stated for skew-dimension one structurable algebras, it continues to hold for arbitrary central simple structurable algebras; this follows from the remarks regarding the *multiplication algebra* on page 189 of *loc. cit.* □

Lemma 1.1.54. *Let $\mathcal{A}$ be a central simple structurable algebra with $\mathcal{S} \neq 0$. Then $\mathcal{S}\mathcal{A} = \mathcal{A}$.*

Proof. By the proof of [AF84, Lemma 2.1(a)]. (Although it is stated for $\dim(\mathcal{S}) = 1$, it holds more generally.) □

Lemma 1.1.55. *Let $a, b \in \mathcal{A}$ such that a is conjugate invertible and let $V \in \mathrm{Instrl}(\mathcal{A})$. Then*

$$V_{a,b} = 0 \iff b = 0 \iff V_{b,a} = 0$$

and

$$V(a) = 0 \iff V^{\epsilon}(\hat{a}) = 0.$$

Proof. The first claim follows from $U_a(b) = V_{a,b}(a) = 0$ and the fact that U_a is invertible. The second claim follows immediately by $V^{\epsilon}_{a,b} = -V_{b,a}$. The last claim follows from

$$0 = [V, \mathrm{id}] = [V, V_{a,\hat{a}}] = V_{V(a),\hat{a}} + V_{a,V^{\epsilon}(\hat{a})}$$

and the previous claims. □

Lemma 1.1.56. *If $V, W \in \mathrm{Instrl}(\mathcal{A})$ satisfy $V^{\delta}(s) = 0 = W^{\delta}(s)$ for some conjugate invertible $s \in \mathcal{S}$, then*

$$WV = 0 \iff W^{\epsilon}V^{\epsilon} = 0.$$

Proof. By [AF84, (1.10)] we have $V(sx) = V^{\delta}(s)x + sV^{\epsilon}(x) = sV^{\epsilon}(x)$ and hence $(WV)(sx) = s(W^{\epsilon}V^{\epsilon})(x)$, for any $x \in \mathcal{A}$. Since s is conjugate invertible we get the desired equivalence. □

Lemma 1.1.57. *If $a \in \mathcal{A}$ and $s \in \mathcal{S}$ are conjugate invertible, then $V_{a,sa} \neq 0$.*

Proof. If $V_{a,sa} = 0$, then $0 = V_{a,sa}(a) = U_a L_s(a)$. Since a and s are conjugate invertible, the operators U_a and L_s are invertible; hence $a = 0$, a contradiction. □

Later on, in Chapter 4, we need the notion of isotopic structurable algebras.

Definition 1.1.58. Two structurable algebras $\mathcal{A}$ and $\mathcal{A}'$ over k are *isotopic* if there exists a vector space isomorphism $\psi : \mathcal{A} \to \mathcal{A}'$ such that there exists $\chi \in \mathrm{Hom}_k(\mathcal{A}, \mathcal{A}')$ such that

$$\psi(V_{x,y}z) = V_{\psi(x),\chi(y)}\psi(z), \forall x, y, z \in \mathcal{A}\,.$$

We now turn to examples of structurable algebras.

Example 1.1.59. The *central simple* structurable algebras have been classified and are usually listed in 6 (non-disjoint) classes. We will need the following three classes:

(i) The Jordan algebras are precisely the structurable algebras with trivial involution. They have skew-dimension 0.

(ii) The structurable algebras of skew-dimension 1 form a separate class and have peculiar features. They all arise as *forms* of structurable matrix algebras (see Definition 1.1.60 and Corollary 1.1.62 below); see [DMM20] for an explicit construction of these algebras and for a recent overview of the theory.

(iii) If C_i is a composition algebra over k with standard involution σ_i, for $i = 1, 2$, then the k-algebra $C_1 \otimes_k C_2$, equipped with the involution

$$\overline{\cdot} = \sigma := \sigma_1 \otimes \sigma_2,$$

is a structurable algebra. It has skew-dimension $\dim_k C_1 + \dim_k C_2 - 2$.

The 3 other classes are the central simple associative algebras with involution, structurable algebras constructed from a non-degenerate hermitian form over a central simple associative algebra with involution and an exceptional 35-dimensional algebra (the *Smirnov algebra*, which always gives rise to a split Lie algebra of type E_7 via the TKK-construction). See [BDMS19, Section 2.3] for a more detailed overview.

Definition 1.1.60. Let J be a Jordan algebra over a field k, let $T\colon J \times J \to k$ be a symmetric bilinear form, let $\times\colon J \times J \to J$ be a symmetric bilinear map, and let $N\colon J \to k$ be a cubic form such that one of the following holds:

- J is a cubic Jordan algebra with a non-degenerate form N, with basepoint 1, trace form T, and (Freudenthal) cross product $\times$; see Definition 1.1.15.
- J is a Jordan algebra of a non-degenerate quadratic form q with basepoint 1, and T is the linearization of q. In this case, N and $\times$ are the zero maps. See Example 1.1.8.
- $J = 0$, and the maps N, T and $\times$ are the zero maps. (In this case, J is not unital.)

Fix a constant $\eta \in k^\times$. We now define the *structurable matrix algebra* $M(J, \eta)$ as follows. Let

$$\mathcal{A} = \left\{ \begin{pmatrix} k_1 & j_1 \\ j_2 & k_2 \end{pmatrix} \mid k_1, k_2 \in k, j_1, j_2 \in J \right\},$$

and define the multiplication and the involution by the formulae

$$\begin{pmatrix} k_1 & j_1 \\ j_2 & k_2 \end{pmatrix} \begin{pmatrix} k_1' & j_1' \\ j_2' & k_2' \end{pmatrix} = \begin{pmatrix} k_1 k_1' + \eta T(j_1, j_2') & k_1 j_1' + k_2' j_1 + \eta (j_2 \times j_2') \\ k_1' j_2 + k_2 j_2' + j_1 \times j_1' & k_2 k_2' + \eta T(j_2, j_1') \end{pmatrix},$$

$$\overline{\begin{pmatrix} k_1 & j_1 \\ j_2 & k_2 \end{pmatrix}} = \begin{pmatrix} k_2 & j_1 \\ j_2 & k_1 \end{pmatrix},$$

for all $k_1, k_2, k_1', k_2' \in k$ and $j_1, j_2, j_1', j_2' \in J$. It is shown in [All78, Section 8.v] and [AF84, Section 4] that $M(J, \eta)$ is a central simple structurable algebra.

The following proposition relates all structurable algebras of skew-dimension one to these structurable matrix algebras.

Proposition 1.1.61 ([AF84, Proposition 4.5])**.** *Let $\mathcal{A}$ be a structurable algebra of skew-dimension one. Consider an arbitrary non-zero element $s_0 \in \mathcal{S}$. Then $s_0^2 = \mu 1$, with $\mu \in k^\times$, and $\mathcal{A}$ is isomorphic to a structurable matrix algebra $M(J, \eta)$ if and only if μ is a square in k.*

Corollary 1.1.62. *Let $\mathcal{A}$ be a structurable algebra of skew-dimension one. Then there exists a field extension ℓ/k of degree at most 2 such that $\mathcal{A} \otimes_k \ell$ is isomorphic to a structurable matrix algebra over ℓ.*

Remark 1.1.63. In [DM19], the structurable algebras of skew-dimension one are constructed explicitly, either in terms of *hermitian cubic norm structures*, or equivalently, in terms of (ordinary) cubic norm structures equipped with a semilinear self-adjoint autotopy.

Examples of structurable algebras of skew-dimension one that are not isomorphic to structurable matrix algebras can be obtained by applying a generalized Cayley–Dickson process to a certain class of Jordan algebras, see [AF84].

1.1.7 Tits-Kantor-Koecher construction

We are now ready to introduce the TKK construction for structurable algebras. We again assume $\mathrm{char}(k) \neq 2, 3$, and all algebras are finite-dimensional in this subsection.

Definition 1.1.64. Consider two copies $\mathcal{A}_+$ and $\mathcal{A}_-$ of $\mathcal{A}$ with corresponding isomorphisms $\mathcal{A} \to \mathcal{A}_+ : a \mapsto a_+$ and $\mathcal{A} \to \mathcal{A}_- : a \mapsto a_-$, and let $\mathcal{S}_+ \subset \mathcal{A}_+$ and $\mathcal{S}_- \subset \mathcal{A}_-$ be the corresponding subspaces of skew elements. Define the vector space

$$K(\mathcal{A}) = \mathcal{S}_- \oplus \mathcal{A}_- \oplus \mathrm{Instrl}(\mathcal{A}) \oplus \mathcal{A}_+ \oplus \mathcal{S}_+ .$$

As in [All79, §3], we define a Lie algebra on $K(\mathcal{A})$ as the unique extension of the Lie algebra on $\mathrm{Instrl}(\mathcal{A})$ satisfying

$$\begin{aligned}
&[V, a_+] := (Va)_+ \in \mathcal{A}_+ && [V, a_-] := (V^\epsilon a)_- \in \mathcal{A}_- \\
&[V, s_+] := (V^\delta s)_+ \in \mathcal{S}_+ && [V, s_-] := (V^{\epsilon\delta} s)_- \in \mathcal{S}_- \\[1ex]
&[s_+, a_+] := 0 && [s_-, a_-] := 0 \\
&[s_+, a_-] := (sa)_+ \in \mathcal{A}_+ && [s_-, a_+] := (sa)_- \in \mathcal{A}_- \\[1ex]
&[a_+, b_-] := V_{a,b} \in \mathrm{Instrl}(\mathcal{A}) && \\
&[a_+, b_+] := \psi(a, b)_+ \in \mathcal{S}_+ && [a_-, b_-] := \psi(a, b)_- \in \mathcal{S}_- \\[1ex]
&[s_+, t_+] := 0 && [s_-, t_-] := 0 \\
&[s_+, t_-] := L_s L_t \in \mathrm{Instrl}(\mathcal{A}) &&
\end{aligned}$$

for all $a, b \in \mathcal{A}$, $s, t \in \mathcal{S}$ and $V \in \mathrm{Instrl}(\mathcal{A})$.

From the definition of the Lie bracket we clearly see that the Lie algebra $K(\mathcal{A})$ has a 5-grading given by $K(\mathcal{A})_j = 0$ for all $|j| > 2$ and

$$K(\mathcal{A})_{-2} = \mathcal{S}_-, \quad K(\mathcal{A})_{-1} = \mathcal{A}_-, \quad K(\mathcal{A})_0 = \mathrm{Instrl}(\mathcal{A}),$$
$$K(\mathcal{A})_1 = \mathcal{A}_+, \quad K(\mathcal{A})_2 = \mathcal{S}_+ .$$

If $\mathcal{A}$ is a Jordan algebra, then $\mathcal{S} = 0$. So in this case $K(\mathcal{A}) = \mathcal{A}_- \oplus \mathrm{Instrl}(\mathcal{A}) \oplus \mathcal{A}_+$ is a 3-graded Lie algebra.

1.1.8 Kantor pairs

We again assume $\mathrm{char}(k) \neq 2, 3$, and all algebras are finite-dimensional in this subsection.

Definition 1.1.65. A *Kantor pair* is a pair of k-vector spaces $P = (P^-, P^+)$ together with two trilinear maps

$$\{\cdot,\cdot,\cdot\}^\sigma : P^\sigma \times P^{-\sigma} \times P^\sigma \to P^\sigma, \ \sigma \in \{+, -\},$$

such that

$$[V_{x,y}, V_{a,b}] = V_{V_{x,y}a,b} - V_{a,V_{y,x}b},$$
$$K_{x,a}V_{y,z} + V_{z,y}K_{x,a} = K_{K_{x,a}y,z},$$

with $x, a, z \in P^\sigma$ and $y, b \in P^{-\sigma}$. The involved operators are defined as follows: $V_{x,y}z = \{x, y, z\}^\sigma$ and $K_{a,b}c = \{a, c, b\}^\sigma - \{b, c, a\}^\sigma$ for all $x, z, a, b \in P^\sigma$ and $y, c \in P^{-\sigma}$.

Kantor pairs are a generalization of structurable algebras: if $\mathcal{A}_+$ and $\mathcal{A}_-$ are two isomorphic copies of a structurable algebra $\mathcal{A}$, then $(\mathcal{A}_-, \mathcal{A}_+)$ is a Kantor pair, with $\{x, y, z\}^\sigma = 2V_{x,y}z_\sigma$, for $x, z \in \mathcal{A}_\sigma$ and $y \in \mathcal{A}_{-\sigma}$.

Kantor pairs are also a generalization of so-called Jordan pairs; Jordan pairs precisely correspond to the case that $K_{x,y} = 0$ for all $x, y \in P^\sigma$.

1.1.9 Inner ideals

We now turn to inner ideals of Lie algebras and in particular of the Lie algebras $K(\mathcal{A})$ arising from structurable algebras through the TKK construction.

Definition 1.1.66. Let L be a Lie algebra. An *inner ideal* of L is a subspace I of L satisfying $[I, [I, L]] \leq I$.

If the characteristic is not 2 and I is 1-dimensional, we call any non-zero element of I an *extremal element*. See Definition 2.3.1 for the definition of an extremal element in characteristic 2.

We call an inner ideal *singular* if all its non-zero elements are extremal elements. Singular inner ideals are also called *point spaces*. We call an inner ideal *trivial* if is equal to 0, and we call it *proper* if it is not equal to L. We call an inner ideal I *abelian* if $[I, I] = 0$.

Definition 1.1.67. Let L be a Lie algebra of characteristic not 2. An element x of L is called an *absolute zero divisor*[2] if $[x, [x, L]] = 0$. A Lie algebra is *non-degenerate* if it has no non-trivial absolute zero divisors.

Assume for the rest of this section $\mathrm{char}(k) \neq 2, 3$ and all algebras are finite-dimensional in this subsection.

If D is a map from a Lie algebra L to itself such that $D^n = 0$ for certain $n \in \mathbb{N}$ and $(n-1)! \in k^\times$, then $\exp(D) = \sum_{i=0}^{n-1} \frac{1}{i!} D^i$.

Definition 1.1.68. Let L be a 5-graded Lie algebra over k. We say that L is *algebraic* if for any $(x, s) \in L_{\sigma 1} \oplus L_{\sigma 2}$ (with $\sigma \in \{+, -\}$), the endomorphism $\exp(\mathrm{ad}(x + s))$ of L is a Lie algebra automorphism. We say that a structurable algebra $\mathcal{A}$ over k is *algebraic* if $K(\mathcal{A})$ is algebraic in the above sense.

If the characteristic is not 5 this property follows easily, as we will see in the following lemma. If the characteristic equals 5, this is quite tricky, see [BDMS19, Sta20] and Chapter 4.

Lemma 1.1.69 ([BDMS19, Lemma 3.1.7])**.** *If* $\mathrm{char}(k) \neq 2, 3, 5$, *then any 5-graded Lie algebra is algebraic.*

Proof. A straightforward calculation shows

$$\exp(\mathrm{ad}(x + s)) = \exp(\mathrm{ad}(x)) \exp(\mathrm{ad}(s)),$$

for all $(x, s) \in L_\sigma \oplus L_{2\sigma}$. Hence it suffices to show that $\exp(\mathrm{ad}(x)) \in \mathrm{Aut}(L)$ for all $0 \neq x \in L_j$, $j \neq 0$. Let D be any derivation of L. For any $l, m \in L$ and $6 \geq i \geq 0$, we have, by induction on i, that

$$\frac{1}{i!} D^i([l, m]) = \sum_{p+q=i} \left[\frac{1}{p!} D^p(l), \frac{1}{q!} D^q(m) \right]. \tag{1.12}$$

Consider $x \in L_j$, $l \in L_r$ and $m \in L_t$, with $0 \neq j$. For any $i \in \mathbb{N}$, the $(ij + r + t)$-component of $[\exp(\mathrm{ad}(x))(l), \exp(\mathrm{ad}(x))(m)]$ equals

$$\sum_{p+q=i, p,q \leq 4} \left[\frac{1}{p!} \mathrm{ad}(x)^p(l), \frac{1}{q!} \mathrm{ad}(x)^q(m) \right], \tag{1.13}$$

[2] In some papers, e.g. [CI06], these elements are called *sandwich elements*.

where we used $\mathrm{ad}^5(x)(L_a) \le L_{5j+a} = 0$ for any $a \in \mathbb{Z}$. If $i \le 6$, then (1.12) for $D = \mathrm{ad}(x)$ shows that (1.13) is equal to the $(ij+r+t)$-component of $\exp(\mathrm{ad}(x))([l,m])$. If $i > 6$, then $|ij| > 6$ while $|r+t| \le 4$, which implies $L_{ij+r+t} = 0$. □

The next lemma is mentioned in (the proof of) [GGLN11, Theorem 3.1(1)].

Lemma 1.1.70. *Let $L = L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus L_2$ be a 5-graded Lie algebra with $L_0 = [L_1, L_{-1}]$. Suppose that I is a subspace closed under all Lie algebra endomorphisms $\exp(\mathrm{ad}(l))$, with $l \in L_i, i \ne 0$. Then I is an ideal of L.*

Proof. Let $l \in L_i, i \neq 0$, be arbitrary. Consider $x \in I$. By assumption, I contains

$$\exp(\mathrm{ad}(l))(x) - \exp(\mathrm{ad}(-l))(x) = 2[l,x] + \frac{1}{3}[l,[l,[l,x]]].$$

If we replace l by $2l$ in this expression, we get that I contains $4[l,x]+\frac{8}{3}[l,[l,[l,x]]]$ as well. Therefore, I contains $12[l,x]$ and hence also $[l,x]$. In particular, $[L_i,x] \le I$ for each $i \ne 0$.

Next, let $l \in L_1$ and $l' \in L_{-1}$. Then

$$[[l,l'],x] = -[[l',x],l] - [[x,l],l'] \in I.$$

Since $L_0 = [L_1, L_{-1}]$, this implies $[L,x] \le I$. Hence I is an ideal of L. □

Theorem 1.1.71 ([Sta20, Theorem 2.13, Theorem 3.4])**.** *Any central simple structurable algebra over k is algebraic.*

The next corollary is similar to [GGLN11, Theorem 3.1.(1)]. (Notice, however, that the characteristic 5 case is excluded in that paper).

Corollary 1.1.72. *Let $\mathcal{A}$ be a central simple structurable algebra over k. Then $K(\mathcal{A})$ is a non-degenerate Lie algebra.*

Proof. $L := K(\mathcal{A})$ is a central simple 5-graded Lie algebra such that $L_0 = [L_1, L_{-1}]$ (see [All79, §5]). Moreover, by definition of algebraicity of $\mathcal{A}$ and Theorem 1.1.71, all endomorphisms $\exp(\mathrm{ad}(l))$ of L, with $l \in L_i, i \neq 0$, are actually automorphisms of L. Let I be the subspace spanned by all absolute zero divisors. It is clear that I is closed under any Lie algebra automorphism. By Lemma 1.1.70, I is an ideal. Since L is simple, $I = L$ or $I = 0$. If $I = L$, then L is generated by (a finite set of) absolute zero divisors, and it is thus nilpotent by [Zel80], which is impossible since L is non-abelian and simple. □

Corollary 1.1.73. *Let $\mathcal{A}$ be a central simple structurable algebra over k. Then any proper inner ideal of $K(\mathcal{A})$ is abelian.*

Proof. Since $K(\mathcal{A})$ is a finite-dimensional non-degenerate simple Lie algebra, this follows from [Ben77, Lemma 1.13]. □

The following result from [Ben77] will be useful in the proof of Lemma 3.1.8.

Lemma 1.1.74 ([Ben77, Lemma 1.8],[DFLGGL08, Lemma 1.11])**.** *If I is an inner ideal of a Lie algebra L and $x \in L$ an element element such that $\mathrm{ad}_x^3 = 0$, then $[x, [x, I]]$ is an inner ideal.*

We will now define some subgroups of $\mathrm{GL}_k(L)$.

Definition 1.1.75 ([BDMS19, Lemma 3.2.7])**.** Consider $L := K(\mathcal{A})$, with $\mathcal{A}$ a central simple structurable algebra. Let $E_\sigma(\mathcal{A})$ denote the subgroup of $\mathrm{Aut}(L)$ consisting of the automorphisms[3]

$$e_\sigma(a, s) := \exp(\mathrm{ad}(a_\sigma + s_\sigma)),$$

and

$$e_\sigma(a, s)e_\sigma(b, t) = e_\sigma\big(a + b, s + t + \tfrac{1}{2}\psi(a, b)\big) = e_\sigma(a, t) \circ e_\sigma(b, s) \tag{1.14}$$

as multiplication, with $a, b \in \mathcal{A}$, $s, t \in \mathcal{S}$, $\sigma \in \{+, -\}$ arbitrary. Let $E(\mathcal{A})$ be the subgroup of $\mathrm{Aut}(L)$ generated by $E_+(\mathcal{A})$ and $E_-(\mathcal{A})$.

The explicit computations in Lemmas 1.1.55 to 1.1.57 and 1.1.76 will be needed later.

Lemma 1.1.76. *Let $a \in \mathcal{A}, s, t \in \mathcal{S}$. Then*

$$\begin{aligned}
e_+(a, s)(t_-) = t_- + (-ta)_- + (L_sL_t - \tfrac{1}{2}V_{a,ta}) + (-s(ta) + \tfrac{1}{6}U_a(ta))_+ \\
+ (-s(ts) - \tfrac{1}{2}\psi(a, s(ta)) + \tfrac{1}{24}\psi(a, U_a(ta)))_+ \, , \\
e_-(a, s)(t_+) = t_+ + (-ta)_+ + (-L_tL_s + \tfrac{1}{2}V_{ta,a}) + (-s(ta) + \tfrac{1}{6}U_a(ta))_- \\
+ (-s(ts) - \tfrac{1}{2}\psi(a, s(ta)) + \tfrac{1}{24}\psi(a, U_a(ta)))_- \, .
\end{aligned}$$

Proof. Using (1.9) and (1.11), we compute that

$$\begin{aligned}
\mathrm{ad}(a_+ + s_+)(t_-) &= (-ta)_- + L_sL_t \\
\mathrm{ad}(a_+ + s_+)^2(t_-) &= [a_+ + s_+, \mathrm{ad}(a_+ + s_+)(t_-)] \\
&= -V_{a,ta} + (-2s(ta))_+ + (-(L_sL_t)^\delta(s))_+ \\
&= -V_{a,ta} + (-2s(ta))_+ + (-2s(ts))_+ \\
\mathrm{ad}(a_+ + s_+)^3(t_-) &= (V_{a,ta}(a)_+ - 2\psi(a, s(ta))_+) - \psi(a, s(ta))_+ \\
&= U_a(ta)_+ - 3\psi(a, s(ta))_+ \\
\mathrm{ad}(a_+ + s_+)^4(t_-) &= \psi(a, U_a(ta))_+.
\end{aligned}$$

These identities together with the definition of $e_+(a, s)$ yield the result. The computations for $e_-(a, s)(t_+)$ are similar, using (1.10). $\square$

[3]By Theorem 1.1.71, these are indeed automorphisms.

SECTION 1.2

Geometries

1.2.1 Moufang polygons

In this subsection we state the necessary preliminaries regarding generalized polygons and Moufang polygons.

We first recall some definitions from incidence geometry.

Definition 1.2.1. A point-line geometry is a triple $\Gamma = (\mathbf{P}, \mathbf{L}, I)$, with

- $\mathbf{P}$ a nonempty set whose elements are called *points*,
- $\mathbf{L}$ a possibly empty set, disjoint from $\mathbf{P}$, whose elements are called *lines*,
- I a subset of $\mathbf{P} \times \mathbf{L}$, called the *incidence relation*,

such that for every $l \in \mathbf{L}$ there are at least two $p \in \mathbf{P}$ for which $(p, l) \in I$. If $(p, l) \in I$, we will say that p is contained in l, that p lies on l, that l contains p, or that l goes through p. We also denote this by $p \in l$. Two points p and p' are called *collinear* if there is some line incident with both p and p'. For every point p, $p^\perp$ denotes the set of all points collinear with or equal to p.

An isomorphism from a point-line geometry $\Gamma = (\mathbf{P}, \mathbf{L}, I)$ to a point-line geometry $\Gamma' = (\mathbf{P}', \mathbf{L}', I')$ is a pair (α, β), where $\alpha : \mathbf{P} \to \mathbf{P}'$ and $\beta : \mathbf{L} \to \mathbf{L}'$ are bijections such that $(p, l) \in I$ if, and only if, $(\alpha(p), \beta(l)) \in I'$, for all $p \in \mathbf{P}$ and $l \in \mathbf{L}$.

Definition 1.2.2. Let $\Gamma = (\mathbf{P}, \mathbf{L}, I)$ be a point-line geometry. We call it a *partial linear space* if two distinct points lie on at most one line. Assume now that Γ is a partial linear space. We call $X \subseteq \mathbf{P}$ a *subspace* if for all $x, y \in X$ with x and y on a (necessarily unique) line l, all points of l are contained in X. As the intersection of subspaces is again a subspace, we can define the subspace *generated* by a subset X of the point set to be the intersection of all subspaces containing X. We call a subspace *singular* if any two distinct points of this subspace are collinear. Subspaces are often identified with the partial linear space induced on them by the lines they contain. If every line has at least 3 points we call a point-line geometry *thick*. A point-line geometry $\Gamma' = (\mathbf{P}', \mathbf{L}, I')$ is a subgeometry of Γ if $\mathbf{P}' \subseteq \mathbf{P}$, $\mathbf{L}' \subseteq \mathbf{L}$ and $(p', l') \in I'$ if and only if $(p', l') \in I$, for any $p' \in \mathbf{P}'$ and $l' \in \mathbf{L}'$.

Remark 1.2.3. Let $\Gamma = (\mathbf{P}, \mathbf{L}, I)$ be a partial linear space and set $S(l) = \{p \in \mathbf{P} \mid (p, l) \in I\}$. Since $S(l) \neq S(l')$ for two distinct $l, l' \in \mathbf{L}$, Γ is isomorphic to $(\mathbf{P}, \{S(l) \mid l \in \mathbf{L}\}, I')$, where $(p, S(l)) \in I'$ if $p \in S(l)$. In other words, if we are looking at a partial linear space, then we can identify the lines with subsets of the point set and containment as incidence. So, for ease of notation, we can and sometimes will suppress I.

Definition 1.2.4. The *rank* of a singular subspace is the length of a maximal chain of non-trivial subspaces. The *rank* of a partial linear space is the supremum of all ranks of maximal singular subspaces.

Definition 1.2.5. A *geometric hyperplane*, or just hyperplane, is a subspace meeting each line non-trivially.

We now introduce generalized polygons.

Definition 1.2.6. Consider $n \in \mathbb{N}\backslash\{0,1,2\}$. A *generalized n-gon* is a partial linear space $\mathcal{T} = (\mathbf{P}, \mathbf{L}, I)$ such that:

- $\mathcal{T}$ does not contain ordinary m-gons as subgeometries, for $m < n$;
- For any $x, y \in \mathbf{P} \cup \mathbf{L}$, there exists a subgeometry containing both x and y isomorphic to an ordinary n-gon.

We call a point-line geometry a *generalized polygon* if it is a generalized n-gon for some $n \geq 3$. Generalized 3-, 4- and 6-gons are also called *generalized triangles, quadrangles* and *hexagons*, respectively.

We will often encounter generalized hexagons which have the property that any point lies on precisely two lines, and we will call such a generalized hexagon a thin generalized hexagon.

Definition 1.2.7. If $\mathcal{T} = (\mathbf{P}, \mathbf{L}, I)$ is a generalized n-gon, then we denote its incidence graph by $\tilde{\mathcal{T}}$. This is the bipartite graph with $\mathbf{P} \cup \mathbf{L}$ as vertices and $\{p, l\}$ as edges, where $p \in \mathbf{P}$ and $l \in \mathbf{L}$ with $p \in l$. Note that $\tilde{\mathcal{T}}$ has diameter n and girth $2n$. If $(x_0, \ldots, x_n)$ is a path of length n in $\tilde{\mathcal{T}}$, then we define the subgroup

$$U(x_0, \ldots, x_n)$$

of $\mathrm{Aut}(\tilde{\mathcal{T}})$ to be the pointwise stabilizer in $\tilde{\mathcal{T}}$ of all vertices at distance ≤ 1 from x_1, x_2, $\ldots$, or x_{n-1} and we call this a *root group*. We call a graph *thick* if every vertex is contained in at least 3 edges.

Remark 1.2.8. If G is a group, then we define the $g^h := hgh^{-1}$ for all $g, h \in G$.

Notation 1.2.9. If we fix a cycle $(x_0, \ldots, x_{2n-1}, x_{2n})$ of length $2n$ in $\tilde{\mathcal{T}}$, then we set

$$U_i = U(x_i, \ldots, x_{n+i}),$$

with $1 \leq i \leq n$.

Definition 1.2.10. We call a generalized n-gon $\mathcal{T} = (\mathbf{P}, \mathbf{L}, I)$ *Moufang* if $\tilde{\mathcal{T}}$ is thick and for any path $(x_0, \ldots, x_n)$ of length n in $\tilde{\mathcal{T}}$, the root group $U(x_0, \ldots, x_n)$ acts transitively on the set of all neighbors of x_0 distinct from x_1.

Moufang n-gons only exist for $n = 3, 4, 6$ or 8, by a famous result by Jacques Tits, see [Tit76, Tit79] (and also [Wei79]).

Lemma 1.2.11 ([TW02, (3.7)])**.** *Let Γ be a generalized n-gon, G its automorphism group and $(x_0, \ldots, x_n)$ an n-path. Consider neighbors x, y of x_0, distinct from x_1. Then there is at most one element of $U := U(x_0, \ldots, x_n)$ mapping x on to y. In particular, Γ is Moufang if and only if Γ is thick and U acts sharply transitively on the set of neighbors of x_0 distinct from x_1 (for all n-paths).*

Lemma 1.2.12. *Consider a Moufang n-gon Γ with root groups U_i as in Notation 1.2.9. The root group U_i acts sharply transitively on all neighbors of x_{n+i} distinct from x_{n+i-1}.*

Proof. Clearly $(x_{n+i}, x_{n+i-1}, \ldots, x_i)$ is also a path of length n. Since Γ is Moufang, Lemma 1.2.11 implies that

$$U(x_{n+i}, \ldots, x_i) = U(x_i, \ldots, x_{n+i}) = U_i$$

acts sharply transitively on all neighbors of x_{n+i} distinct from x_{n+i-1}. $\square$

Theorem 1.2.13 ([TW02, (5.5), (5.6)])**.** *Consider $u_i \in U_i$ and $u_j \in U_j$ arbitrary, where $i, j \in \{1, \ldots, n\}$ with $i + 1 < j$, Then there exist elements $u_{i+1} \in U_{i+1}, \ldots, u_{j-1} \in U_{j-1}$ such that $[u_i, u_j] = u_{i+1} \ldots u_{j-1}$. Moreover, $[U_i, U_{i+1}] = 1$ holds.*

Theorem 1.2.14 ([TW02, Chapter 7])**.** *A Moufang n-gon Γ is completely determined by its root groups $U_1, \ldots, U_n$ together with the commutator relations, i.e. the description of the commutators of elements in two root groups.*

So in the next few subsections we will describe the root groups of various Moufang polygons. Later on, we will describe more concrete models for these Moufang polygons, and then we need these commutator relations. In some of the cases we describe, we indicate to which algebraic groups these correspond. We will now briefly indicate this correspondence.

Linear algebraic groups are matrix groups defined by polynomials; a typical example is the group SL_n of $(n \times n)$-matrices of determinant 1. Semisimple connected linear algebraic groups over an algebraically closed field are classified by their Dynkin diagram, which is either of type A_n, B_n, C_n, D_n or of type G_2, F_4, E_6, E_7, E_8. The former list of Dynkin diagrams are called *classical* and the latter list of diagrams are called *exceptional*. (Sometimes, D_4 is also considered exceptional.)

If the field is not algebraically closed, the situation is more complex. Then the groups can be classified according to their *Tits index*, which is a Dynkin diagram together with some circled vertices representing distinguished orbits. The number of distinguished orbits equals the relative rank of the group, and the number of vertices of the Dynkin diagram equals the absolute rank. We list in Table 1.1 the Tits indices of the semisimple connected linear algebraic groups of relative rank 2 whose Dynkin diagram is exceptional. See part 2.3, Table I and Table II of [Tit66].

Using the parabolic subgroups of a linear algebraic group it is possible to define a geometric structure out of this algebraic group, more precisely one can construct a *spherical building*. (See chapter 39 and page 480 of [TW02] for more information on buildings and the aforementioned construction.) This building has the same rank as the algebraic group. Not all spherical buildings are obtained by this construction, but a lot of them are, see Chapter 41 of [TW02]. Spherical buildings of rank 2 are equivalent with generalized polygons, see (39.39) and (39.40) of *loc. cit.* If a building is coming from a linear algebraic group as above, then we call this algebraic group the algebraic group corresponding to this building. In order to refer to the Tits indices in Table 1.1, we use the notation ${}^gX^t_{n,r}$ for this algebraic group, where n is its absolute rank, r its relative rank, $X \in \{A, B, C, D, E, F, G\}$, and g and t are as in [Tit66, pp. 54]. (We omit g if $g = 1$.)

1.2.2 Moufang triangles

First note that since any generalized triangle is equivalent with a projective plane, a Moufang triangle is equivalent with a Moufang plane. A Moufang triangle is coordinatized by an alternative division algebra. More precisely the following holds.

Theorem 1.2.15 ([TW02, (17.2) and (16.1)]). *Let Γ be a Moufang triangle, with root groups U_1, U_2 and U_3. Then there exists an alternative division algebra F and isomorphisms $(F, +) \to U_i : f \mapsto x_i(f)$, for all $i = 1, 2, 3$, such that $[U_1, U_2] = 1 = [U_2, U_3]$ and*

$$[x_1(f), x_3(g)] = x_2(fg),$$

for all $f, g \in F$.

The most exceptional Moufang triangle is the one corresponding to an octonion division algebra. Its associated linear algebraic group is of type $E^{28}_{6,2}$ and has index as described in Table 1.1.

1.2.3 Moufang quadrangles coming from quadrangular algebras

The following definition of a generalized quadrangle is equivalent with the one given in Definition 1.2.6.

Definition 1.2.16. A generalized quadrangle is a partial linear space $(\mathbf{P}, \mathbf{L}, I)$ such that

- there exist two disjoint lines,
- for every line l and any point p not on l, there exists a unique point on l collinear with p.

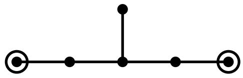
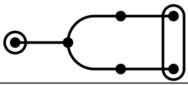
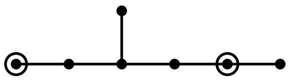
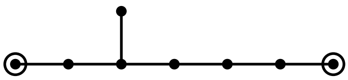
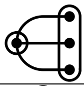
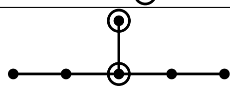
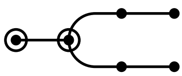
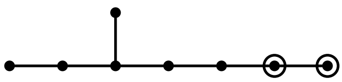
index	diagram	corresponding polygon
$E_{6,2}^{28}$		Moufang triangle
${}^2E_{6,2}^{16'}$		Moufang quadrangle
$E_{7,2}^{31}$		Moufang quadrangle
$E_{8,2}^{66}$		Moufang quadrangle
$G_{2,2}^{0}$		Moufang hexagon
${}^{3,6}D_{4,2}^{2}$		Moufang hexagon
$E_{6,2}^{16}$		Moufang hexagon
${}^2E_{6,2}^{16''}$		Moufang hexagon
$E_{8,2}^{78}$		Moufang hexagon

Table 1.1: Exceptional Tits indices of relative rank two

With this definition it is obvious that generalized quadrangles belong to the larger class of so-called *polar spaces*.

Definition 1.2.17. A *polar space* is a partial linear space satisfying the *one-or-all* axiom:

a point is collinear with one or all points of a line.

Note that a point p is collinear with itself, if there is a line containing p.

A polar space is called *degenerate* if it contains a point p collinear with all other points, and *non-degenerate* otherwise.

In a non-degenerate polar space, singular subspaces are projective spaces.

Example 1.2.18. Examples of polar spaces are obtained from polarities. Let $\perp$ be a polarity of a projective space $\mathbb{P}$, then let $\mathcal{E}$ be the set of *absolute points*, i.e. the points p with $p \in p^\perp$ and as lines the *absolute lines*, i.e, the lines l where for each point $p \in l$ we have $l \subseteq p^\perp$. This polar space is non-degenerate if $p^\perp$ is never the full projective space.

We now discuss a specific example of such a polarity. Consider a non-degenerate quadratic form $Q : M \to k$, $\mathrm{char}(k) \neq 2$, and let T be the associated bilinear form. Let $\mathbb{P}$ be the projective space of M. Then $\perp$ which sends the projective point $\langle m \rangle$ onto the hyperplane $\{\langle n \rangle \in \mathbb{P} \mid T(n, m) = 0\}$ is a polarity. Then $\mathcal{E} = \{\langle m \rangle \mid Q(m) = 0\}$, and the set of absolute lines coincides with $\{\langle m, n \rangle \mid Q(\langle m, n \rangle) = 0 \text{ and } \langle m \rangle \neq \langle n \rangle\}$.

We now return to Moufang quadrangles. In order to describe the Moufang quadrangle associated with an anisotropic quadrangular algebra, we need the following definition.

Definition 1.2.19. Consider an anisotropic quadrangular algebra $(k, M, Q, 1, X, \cdot, h, \theta)$. We define the operation $\circ$ on $X \oplus k$ as follows

$$(x, \lambda) \circ (y, \mu) = (x + y, \lambda + \mu + g(y, x)),$$

for all $x, y \in X$ and $\lambda, \mu \in k$, with g as in Definition 1.1.27(vii). This defines a group structure on $X \oplus k$, see [Wei06, Proposition 11.10].

Theorem 1.2.20. *Consider an anisotropic quadrangular algebra* $(k, M, Q, 1, X, \cdot, h, \theta)$. *Then there exists a Moufang quadrangle* Γ, *with root groups* U_1, U_2, U_3 *and* U_4, *for which there exist isomorphisms* $(X \oplus k, \circ) \to U_i : (x, \lambda) \mapsto x_i(x, \lambda)$ *for* $i = 1, 3$ *and isomorphisms* $(M, +) \to U_i : m \mapsto x_i(m)$ *for* $i = 2, 4$ *such that* $[U_1, U_2] = [U_2, U_3] = [U_3, U_4] = 1$ *and*

$$\begin{aligned}
[x_1(x, \lambda), x_3(y, \mu)^{-1}] &= x_2(h(x, y)), \\
[x_2(m), x_4(n)^{-1}] &= x_3(0, T(m, n)), \\
[x_1(x, \lambda), x_4(m)^{-1}] &= x_2(\theta(x, m) + \lambda m) x_3(x \cdot m, \lambda Q(m) + \phi(x, m)),
\end{aligned}$$

for all $x, y \in X$, $m, n \in M$ *and* $\lambda, \mu \in k$, *where* T *is the bilinear form associated with* Q, *and* ϕ *as in Definition 1.1.27*(viii).

Proof. If $X \neq 0$ this is precisely [Wei06, Theorem 11.11(ii)]. If $X = 0$ this is precisely [TW02, (16.3)]. □

If $\mathrm{char}(k) \neq 2$, then by Theorem 1.1.37 an anisotropic quadrangular algebra is either coming from an anisotropic standard pseudo-quadratic space over a division associative composition algebra or it is of type E_6, E_7 or E_8. In the former case the corresponding algebraic group is of classical type. In the latter case, the corresponding algebraic group is of type ${}^2E_{6,2}^{16'}$, $E_{7,2}^{31}$ or $E_{8,2}^{66}$, respectively. See Table 1.1 for the corresponding Tits indices. If $\mathrm{char}(k) = 2$ there are more classes of quadrangular algebras.

1.2.4 Moufang hexagons

A Moufang hexagon is coordinatized by an anisotropic cubic norm structure. More precisely the following holds.

Theorem 1.2.21 ([Wei06, (17.5) and (16.8)])**.** *Let Γ be a Moufang hexagon, with root groups U_1, U_2, U_3, U_4, U_5 and U_6. After potentially re-labeling the path in Notation 1.2.9, there exists a field k, an anisotropic cubic norm structure $(J, k, \sharp)$*[4]*, isomorphisms $(k, +) \to U_i : \lambda \mapsto x_i(\lambda)$ for all $i = 2, 4, 6$ and isomorphisms $(J, +) \to U_i : j \mapsto x_i(j)$ for all $i = 1, 3, 5$ such that $[U_1, U_2] = [U_1, U_4] = [U_2, U_3] = [U_2, U_4] = [U_2, U_5] = [U_3, U_4] = [U_3, U_6] = [U_4, U_5] = [U_4, U_6] = [U_5, U_6] = 1$ and*

$$\begin{aligned}
[x_1(a), x_3(b)] &= x_2(T(a, b)),\\
[x_3(a), x_5(b)] &= x_4(T(a, b)),\\
[x_1(a), x_5(b)] &= x_2(-T(a^\sharp, b))x_3(a \times b)x_4(T(a, b^\sharp)),\\
[x_2(\lambda), x_6(\mu)] &= x_4(\lambda\mu),\\
[x_1(a), x_6(\lambda)] &= x_2(-\lambda N(a))x_3(\lambda a^\sharp)x_4(\lambda^2 N(a))x_5(-\lambda a),
\end{aligned}$$

for all $a, b \in J$ and $\lambda, \mu \in k$.

As noted in Remark 1.1.24 the cubic norm structure is either coming from a purely inseparable field extension as in Example 1.1.21 or $\dim_k(J) \in \{1, 3, 9, 27\}$. If $\dim_k(J) = 1$, then the hexagon is the split Cayley hexagon and the corresponding algebraic group is the split group $G^0_{2,2}$. If $\dim_k(J) = 3$, then the corresponding algebraic group is either of type ${}^3D^2_{4,2}$ or ${}^6D^2_{4,2}$, depending on the precise construction of J. If $\dim_k(J) = 9$, and J is constructed as in Example 1.1.22, then the corresponding algebraic group is of type $E^{16}_{6,2}$. There also exists 9-dimensional twisted forms of such an algebra which also form an anisotropic cubic norm structure, corresponding to the algebraic group ${}^2E^{16''}_{6,2}$. If $\dim_k(J) = 27$, i.e. J is an Albert division algebra, then the corresponding algebraic group is of type $E^{78}_{8,2}$. The Tits indices of all these groups are described in Table 1.1.

Notation 1.2.22. We denote the Moufang hexagon with root groups as in Theorem 1.2.21 by $\Gamma(J, k, \sharp)$, for any anisotropic cubic norm structure $(J, k, \sharp)$.

Theorem 1.2.23. *Let $(J, k, \sharp)$ and $(J', k, \sharp')$ be two arbitrary anisotropic cubic norm structures. Then $\Gamma(J, k, \sharp)$ is isomorphic with $\Gamma(J', k, \sharp')$ if, and only if, $(J, k, \sharp)$ and $(J', k, \sharp')$ are isotopic.*

Proof. By (7.5), (8.9), (8.10) and (35.13) of [TW02]. □

[4] Recall Lemma 1.1.17.

1.2.5 Moufang sets

We briefly recall the definition of a Moufang set. We refer to [DMS09] for a detailed introduction to the subject.

Definition 1.2.24. Let X be a set (with $|X| \geq 3$) and let $\{U_x \mid x \in X\}$ be a collection of subgroups of $\mathrm{Sym}(X)$. The data $(X, \{U_x\}_{x \in X})$ is a *Moufang set* if the following two properties are satisfied:

- For each $x \in X$, U_x fixes x and acts sharply transitively on $X \backslash \{x\}$.
- For each $g \in G^+ := \langle U_x \mid x \in X \rangle \leq \mathrm{Sym}(X)$ and each $y \in X$ we have $U_y^g = U_{g(y)}$.[5]

The group G^+ is called the *little projective group* of the Moufang set, and the groups U_x are called the *root groups*.

Each group G acting sharply doubly transitively on a set X gives rise to a Moufang set (with $U_x = \mathrm{Stab}_G(x)$ and $G^+ = G$). A Moufang set is called *proper* if the action of G^+ on X is *not* sharply doubly transitive.

All known examples of proper Moufang sets with abelian root groups arise from (quadratic) Jordan division algebras [DMW06, DMS08, Grü15]. More generally, all known examples of proper Moufang sets with (abelian or non-abelian) root groups without elements of order 2 or 3 arise from structurable division algebras [BDMS19]. (There are infinite families of counterexamples over fields of characteristic 2 and 3, but also those examples are still of algebraic nature.) See also [BDMS19, DM20] for a detailed overview of examples of Moufang sets.

[5] In the theory of Moufang sets, one usually denotes group actions on the right. Since we always denote Lie algebra automorphisms on the left, we chose to denote group actions for Moufang sets on the left in order to avoid confusion notation later on.

CHAPTER 2

Extremal geometries and polar spaces

This chapter is organised as follows. In Section 2.1 we discuss root filtration spaces. Most of the results presented there are based on the work of Arjeh Cohen and Gabor Ivanyos [CI07], we also provide some examples. Section 2.2 is devoted to subgeometries of root filtration spaces fixed by some involutive automorphism. In particular we show that such subgeometries carry the structure of a polar space. Lie algebras generated by extremal elements will be considered in Sections 2.3, 2.4 and 2.5. In these sections we study the extremal geometry of a Lie algebra generated by extremal elements. Building upon the work of Arjeh Cohen et al. [CI06] we obtain a proof that the inner ideal geometry, which we define in Section 2.5, of such a Lie algebra containing two linearly independent but commuting elements is a root shadow space of a spherical building of rank at least 2. More precisely, we obtain that if L is a simple Lie algebra generated by its pure extremal elements, then its inner ideal geometry is either a root filtration space, a polar space or just a set (then there are no lines in the geometry).

SECTION 2.1

Root filtration spaces

In this section we consider root filtration spaces.

Definition 2.1.1. A thick partial linear space $\Gamma = (\mathcal{E}, \mathcal{F})$ is a *root filtration space with filtration* $\mathcal{E}_i$, $-2 \leq i \leq 2$, if the sets $\mathcal{E}_i$, with $-2 \leq i \leq 2$, provide a partition of $\mathcal{E} \times \mathcal{E}$ into five symmetric relations satisfying the following for all $x, y, z \in \mathcal{E}$:

(A) The relation $\mathcal{E}_{-2}$ is equality.
(B) The relation $\mathcal{E}_{-1}$ is collinearity of distinct points.
(C) For each $(x, y) \in \mathcal{E}_1$, there is a unique point, denoted by $[x, y]$, such that if $z \in \mathcal{E}_i(x) \cap \mathcal{E}_j(y)$, then $[x, y] \in \mathcal{E}_{\leq i+j}(z)$.
(D) If $(x, y) \in \mathcal{E}_2$, then $\mathcal{E}_{\leq 0}(x) \cap \mathcal{E}_{\leq -1}(y) = \emptyset$.
(E) The subsets $\mathcal{E}_{\leq i}(x)$ are subspaces of Γ.
(F) The subset $\mathcal{E}_{\leq 1}(x)$ is a geometric hyperplane.
(G) $\mathcal{E}_2(x)$ is non-empty.
(H) Γ is connected.

Here $\mathcal{E}_{\leq i}$ is the union of all $\mathcal{E}_j$ with $-2 \leq j \leq i$. Moreover, for $x \in \mathcal{E}$ the set $\mathcal{E}_i(x)$ is the set of all $y \in \mathcal{E}$ with $(x, y) \in \mathcal{E}_i$, and, similarly, $\mathcal{E}_{\leq i}(x)$ is the set of all $y \in \mathcal{E}$ with $(x, y) \in \mathcal{E}_{\leq i}$.

A point pair x, y is called *collinear* if $(x, y) \in \mathcal{E}_{-1}$, *polar* if $(x, y) \in \mathcal{E}_0$, *special* if $(x, y) \in \mathcal{E}_1$, and *hyperbolic* if $(x, y) \in \mathcal{E}_2$.

Remark 2.1.2. Note that by conditions (G) and (H) the set $\mathcal{E}_{-1}(x)$ is non-empty for each point $x \in \mathcal{E}$, and, moreover, by (D) and (F) even $\mathcal{E}_1(x)$ is non-empty. The fact that these sets are non-empty is sometimes referred to as the root filtration space Γ being *non-degenerate*.

Root filtration spaces have been studied by Arjeh Cohen and Gabor Ivanyos [CI07]. The main result is that a root filtration space $\Gamma = (\mathcal{E}, \mathcal{F})$ in which singular subspaces have finite rank are so-called *root shadow spaces* of spherical buildings. Consider a building of type X_n with corresponding Dynkin diagram Y_n. (If the building is of type BC_n, choose B_n.[1]) Let J be the set of nodes adjacent to the node extending the Dynkin diagram Y_n to an affine diagram. Then the shadow space $X_{n,J}$ with as the points the flags whose types are the nodes from the set J is called a root shadow space. If $J = \{j\}$ we denote $X_{n,J}$ by $X_{n,j}$. For a detailed discussion of root shadow spaces, the reader is referred to [BC13, Section 11.6] or [Shu11, Chapter 17]. Below we discuss some examples of root shadow spaces. See also [Fle15, 4.2.3] for a more precise connection with spherical buildings.

Example 2.1.3. Let V be a vector space over a skew field of dimension $n + 1$ at least 3 and $\mathbb{P}(V)$ be the corresponding projective geometry. Take for $\mathcal{E}$ the set of incident point-hyperplane pairs of $\mathbb{P}(V)$. Lines in $\mathcal{F}$ consist of all point-hyperplane pairs (p, H), where either H is running through the set of all hyperplanes containing a codimension 2 subspace K on p, or p is running through a 2-dimensional subspace L inside H. Then Γ is a root shadow space of type $A_{n,\{1,n\}}$, where $n + 1$ is the dimension of V.

The space Γ admits the following filtration. Let $x = (p, H)$ and $y = (q, K)$ be incident point-hyperplane pairs in $\mathcal{E}$. Then

[1] The type of a building is a Coxeter diagram, and we denote the Coxeter diagram corresponding to the Dynkin diagrams B_n and C_n by BC_n to avoid confusion.

- $(x, y) \in \mathcal{E}_{-2} \quad \Leftrightarrow \quad x = y$;
- $(x, y) \in \mathcal{E}_{-1} \quad \Leftrightarrow \quad x \neq y$ and $p = q$ or $H = K$;
- $(x, y) \in \mathcal{E}_{0} \quad \Leftrightarrow \quad p \neq q$, $H \neq K$, $p \in K$ and $q \in H$;
- $(x, y) \in \mathcal{E}_{1} \quad \Leftrightarrow \quad p \in K$ but $q \notin H$, or $p \notin K$ but $q \in H$;
- $(x, y) \in \mathcal{E}_{2} \quad \Leftrightarrow \quad p \notin K$ and $q \notin H$.

Suppose W^* is a subspace of V^* with the property that for any two linearly independent $v, v' \in V$ there is a $\phi \in W^*$ with $\phi(v) \neq 0$ but $\phi(v') = 0$. We say W^* *separates* points of the projective space of V. If we allow the dimension of V to be infinite, then we can construct more examples by choosing our hyperplanes as kernels of elements inside a subspace W^* of V^* separating the points of the projective space of V. This geometry is then denoted by $\Gamma(V, W^*)$.

Example 2.1.4. Let $\mathcal{E}$ be the set of lines of a non-degenerate polar space of rank at least 3. Take for $\mathcal{F}$ the sets of the form $\{\ell \in \mathcal{E} \mid p \in \ell, \ell \subset \pi\}$, where (p, π) runs over the set of point-singular plane pairs with $p \in \pi$. Then $(\mathcal{E}, \mathcal{F})$ is a root shadow space of type $BC_{n,2}$ (or $D_{n,2}$, depending on the polar space). It is a root filtration space with filtration defined as follows:

- $(l, m) \in \mathcal{E}_{-2} \Leftrightarrow l = m$;
- $(l, m) \in \mathcal{E}_{-1} \Leftrightarrow l, m$ are contained in a singular plane;
- $(l, m) \in \mathcal{E}_{0} \Leftrightarrow l, m$ do intersect but are not contained in some singular subspace, or l, m do not intersect but are contained in singular subspace;
- $(l, m) \in \mathcal{E}_{1} \Leftrightarrow l$ contains a unique point collinear with all points of m;
- $(l, m) \in \mathcal{E}_{2} \Leftrightarrow$ each point on l is collinear with a unique point of m.

If $(l, m) \in \mathcal{E}_2$, we also call the lines l and m *opposite*.

The main result of Arjeh Cohen and Gabor Ivanyos [CI07] reads as follows.

Theorem 2.1.5 ([CI07, Theorem 1])**.** *Suppose that Γ is a root filtration space of finite rank. Then Γ is isomorphic to a root shadow space of type $A_{n,\{1,n\}}$ $(n \geq 2)$, $BC_{n,2}$ $(n \geq 3)$, $D_{n,2}$ $(n \geq 4)$, $E_{6,2}$, $E_{7,1}$, $E_{8,8}$, $F_{4,1}$ or $G_{2,2}$.*

The key to this result is the construction of so-called symplecta. Arjeh Cohen and Gabor Ivanyos showed:

Proposition 2.1.6 ([CI07])**.** *Let Γ be a root filtration space. Suppose the relation $\mathcal{E}_0$ is non-empty. Then Γ contains a collection $\mathcal{S}$ of subspaces such that every pair of points x, y with $(x, y) \in \mathcal{E}_0$ is contained in a unique element $S \in \mathcal{S}$.*

Moreover, for each $S \in \mathcal{S}$ we have:

(a) *S is a non-degenerate polar space of rank at least 2.*
(b) *If x is collinear to two non-collinear points of S, then it is contained in S.*
(c) *For all points $x, y \in S$ we have $(x, y) \in \mathcal{E}_{\leq 0}$.*
(d) *For each point x the set of points in $\mathcal{E}_{\leq -1}(x) \cap S$ is either empty or contains a line.*

Proof. By Theorem 13 of [CI07] we find that Γ contains a collection of symplecta satisfying (a)–(d), or every line is contained in a unique maximal singular subspace.

In the latter case, Theorem 29, Corollary 18, Proposition 23 and Theorem 35 of [CI07] imply that Γ is either isomorphic to $\Gamma(V, W^*)$ for some vector space V and subspace W^* of its dual, see Example 2.1.3, or to the root filtration space of lines in a polar space of rank 3 as described in Example 2.1.4. Also in these cases we find a collection of symplecta if the dimension of V is at least 4, see Example 2.1.7 and Example 2.1.8 below. One easily checks properties (a)–(d). $\square$

The elements of $\mathcal{S}$ are called *symplecta*.

Example 2.1.7. In the root filtration space $\Gamma = (\mathcal{E}, \mathcal{F})$ of Example 2.1.3, two point-hyperplane pairs (p, H) and (q, K) of $\mathcal{E}$ are in relation $\mathcal{E}_0$ if and only if $p \neq q$ and $H \neq K$, but $p, q \in H \cap K$. The unique symplecton containing (p, H) and (q, K) is the set of point-hyperplane pairs (r, L) where r is on the line through p and q and L contains $H \cap K$. This symplecton carries the structure of a generalized quadrangle, i.e. a rank 2 polar space.

Example 2.1.8. The symplecta of the root filtration space $(\mathcal{E}, \mathcal{F})$ of lines in a polar space (P, L) as in Example 2.1.4 come in two types.

The first are the symplecta determined by two intersecting lines of L which are not in a singular subspace. It consists of all the lines on the intersection point. Suppose S is a symplecton of this type and consists of all lines in L on a fixed point $p \in P$. Then any line $\ell \in L \setminus S$ is in relation $\mathcal{E}_{-1}$ with none of the lines of S, or the line is in a singular plane with p and ℓ is in relation $\mathcal{E}_{-1}$ with all lines on S which are inside this plane. These lines form an element of $\mathcal{F}$, showing (d) of Proposition 2.1.6 holds true in this case.

A symplecton of the second type is the set of all lines in a singular subspace which is a projective 3-space. (Note that this is a polar space.) In case the polar space has rank ≤ 3, this class of symplecta is empty.

We provide some extra information on root filtration spaces that will be used in the next sections and Chapter 4.

The following two lemmas can also be found in [CI06] and [CI07] and are concerned with a root filtration space $\Gamma = (\mathcal{E}, \mathcal{F})$.

Lemma 2.1.9. *Let $x, y \in \mathcal{E}$.*

(a) *If $(x, y) \in \mathcal{E}_1$, then there is a unique point z in $\mathcal{E}_{-1}(x) \cap \mathcal{E}_{-1}(y)$.*
(b) *If $(x, y) \in \mathcal{E}_1$, then there is a point z in $\mathcal{E}_2(x) \cap \mathcal{E}_{-1}(y)$.*
(c) *If $(x, y) \in \mathcal{E}_0$, then there is a point z in $\mathcal{E}_0(x) \cap \mathcal{E}_2(y)$.*
(d) *If $(x, y) \in \mathcal{E}_0$, then there is a point z in $\mathcal{E}_{-1}(x) \cap \mathcal{E}_1(y)$.*
(e) *If $(x, y) \in \mathcal{E}_2$, then there is a point z in $\mathcal{E}_{-1}(x) \cap \mathcal{E}_1(y)$.*
(f) *If $(x, y) \in \mathcal{E}_{-1}$, then there is a point z in $\mathcal{E}_{-1}(x) \cap \mathcal{E}_1(y)$.*

Proof. Statement (a) is [CI06, Lemma 1(ii)]. Statement (b) follows from [CI06, Lemma 1(v), Lemma 4] and (c) is [CI07, Lemma 8(ii)]. Statement (d) follows from (c). Indeed, let v be a point in $\mathcal{E}_0(x) \cap \mathcal{E}_2(y)$, then a common neighbor of z of x and v is, by property (D), in $\mathcal{E}_1(y)$. Statement (e) follows from Condition (F) and (D) of Definition 2.1.1. The final statement follows by [CI06, Lemma 4]. □

Lemma 2.1.10 ([CI06, Lemma 1(vi)], Pentagon Property)**.** *Let $x_1, \ldots, x_5$ be five points forming a pentagon, i.e., x_i and x_j are collinear if and only if $i - j \equiv 1 \pmod 5$. Then $(x_i, x_{i+2}) \in \mathcal{E}_0$ for $i \in \{1, 2, 3\}$.*

We analyze the relation between a point and a symplecton.

Lemma 2.1.11. *Consider a point x and a symplecton S of Γ with $\mathcal{E}_{\leqslant -1}(x) \cap S = \emptyset$. Then either $S \subseteq \mathcal{E}_{\leqslant 0}(x)$ or $A := \mathcal{E}_{\leqslant 0}(x) \cap S$ is a non-empty singular subspace of S. In the latter case, any point of S not in A but collinear with a point of A is in $\mathcal{E}_1(x)$ and all other points of S not in A are in $\mathcal{E}_2(x)$.*

Proof. Fix a point x and symplecton S such that x is not collinear to any point of S. Either S is contained in $\mathcal{E}_{\leqslant 0}(x)$ or we can find $y \in S$ which is contained in $\mathcal{E}_1(x) \cup \mathcal{E}_2(x)$. Suppose we are in the latter case. If $y \in \mathcal{E}_2(x)$, then, as S is of rank at least 2, we can find a line on y inside S. This line meets the geometric hyperplane $\mathcal{E}_{\leq 1}(x)$ in a point which, by condition (D), is in $\mathcal{E}_1(x)$. So we may assume $y \in S$ to be inside $\mathcal{E}_1(x)$. Let $z = [x, y]$ be the unique common neighbor of x and y. By Proposition 2.1.6(d), the subspace $\mathcal{E}_{-1}(z) \cap S$ of S is singular and contains a line L.

Now by Lemma 2.1.9 there is a point v collinear with x and in $\mathcal{E}_2(y)$. The line L meets the geometric hyperplane $\mathcal{E}_{\leq 1}(v)$ in a point u. If the points $v, x, z, u, [v, u]$ form a pentagon, Lemma 2.1.10 implies that $(x, u) \in \mathcal{E}_0$. Assume they do not form a pentagon. Note that $v \in \mathcal{E}_1(z)$ since $v \in \mathcal{E}_2(y)$ and hence z cannot be collinear to v or $[v, u]$, otherwise $(v, z) \in \mathcal{E}_0$ in the latter case. So x and $[v, u]$ are collinear, which also implies that $(x, u) \in \mathcal{E}_0$.

This shows that each point of $S \cap \mathcal{E}_1(x)$ is on a line of S meeting $\mathcal{E}_0(x) \cap S$ in a point. In particular $A := \mathcal{E}_{\leqslant 0}(x) \cap S$ is a non-empty subspace of S.

Now let $d \in A$ be arbitrary and let $e \in S \backslash A$ be any point collinear with d. If T is the unique symplecton containing x and d, then e is collinear with a point, and hence with a line of T. As T is a polar space, we can find a point p on this line collinear with x, and x and e are at distance 2. As $e \notin A$, we have $(x, e) \in \mathcal{E}_1$. Moreover, $p = [x, e]$ is collinear with both d and e and hence with all the points of the line through d and e.

It remains to show that A is singular. Assume there are two non-collinear points i, j in A. Then let k be a point collinear with both i and j. Assume $k \notin A$. In the previous paragraph we showed that there exists a neighbor p of x collinear with the line ik, and similarly a neighbor q collinear to all points of the line jk. Since these two neighbors of x can not coincide (using that $\mathcal{E}_{-1}(h) \cap S$ is always a singular subspace for any point h not in S), we get $(x, k) \in \mathcal{E}_0$. Hence

$k \in A$, a contradiction with the assumption $k \notin A$. So, all common neighbors in S of i and j are contained in A. This shows that A is a convex non-singular subspace of S. Hence $A = S$, which contradicts our initial assumption. □

We analyze this somewhat further.

Lemma 2.1.12. *Suppose S is a symplecton and x is a point such that $\mathcal{E}_2(x) \cap S$ is non-empty. Then there is a unique point $y \in S \cap \mathcal{E}_0(x)$ and on every line on y inside S there is a singular plane containing a unique point collinear with x.*

Proof. By Lemma 2.1.11, $S \cap \mathcal{E}_0(x)$ is a non-empty singular subspace. If y is any element of this intersection, all lines on this point are contained in $\mathcal{E}_{\leq 1}(x)$ by this lemma. But then the fact that S is a non-degenerate polar space implies that $\mathcal{E}_0(x) \cap S$ consists of precisely one point, which we call y.

Now assume T is the symplecton on x and y. Then every point $z \in S$ collinear with y is collinear to all the points of a line ℓ on y in T. The line ℓ, and hence the plane on z and ℓ contains a unique point collinear to x. □

We also obtain the following consequence of Lemma 2.1.11.

Corollary 2.1.13. *Consider a symplecton S and a point x not contained in S such that $A := \mathcal{E}_{-1}(x) \cap S$ and $\mathcal{E}_1(x) \cap S$ are non-empty. Then A is a line, all points of S collinear with A are contained in $\mathcal{E}_0(x)$ and all other points of S not on A are contained in $\mathcal{E}_1(x)$.*

Proof. By part (b) and (d) of Proposition 2.1.6, A is a singular subspace which contains a line L.

If $L \neq A$, then A contains a plane and hence every point of the symplecton S is collinear to a line of A and hence any point of S has at least 2 common neighbors x, so the points of S not in A are contained in $\mathcal{E}_0(x)$, contradicting $\mathcal{E}_1(x) \neq \emptyset$. We obtain $L = A$.

If a point of S is collinear with all points of the line L, then this point is contained in $\mathcal{E}_0(x)$ since all points on L are common neighbors with x.

Let z be a point in $\mathcal{E}_1(x) \cap S$. Let q be the unique neighbor of z and x. Note $q \in L$. By Lemma 2.1.9(b) there is a point y collinear with x and such that $(z, y) \in \mathcal{E}_2$. By Lemma 2.1.12 there is a unique point p contained in S and symplectic with y. By Lemma 2.1.11 and $(q, y) \in \mathcal{E}_1$, q is collinear with p. Again by Lemma 2.1.12 there exist a plane through pq containing the unique common neighbor of q and y, i.e. x. In particular x is collinear with pq, and thus $L = pq$.

Suppose $s \in S$ is not collinear with p. Then $s \in \mathcal{E}_2(y)$. Suppose $s \in \mathcal{E}_{\leq 0}(x)$. Together with $(x, y) \in \mathcal{E}_{-1}$ this contradicts Property (D) of a root filtration space. Since s is collinear with a point of L (S is a polar space), we get $s \in \mathcal{E}_1(x)$.

Now assume that $s \in S$ is collinear with p but not with all points of the line L. We want to show $s \in \mathcal{E}_1(x)$. Suppose this is not the case, by definition of

$A = L$, we get $s \in \mathcal{E}_0(x)$. Let s' be any common neighbor of x and s not equal to p. Since the only point of L with which s is collinear is p, s' is not contained in S. Since S forms a polar space, we can find a line M through s containing a point t not collinear with p. By the previous paragraph $t \in \mathcal{E}_1(x)$ and the common neighbor l of x and t lies on L. We now show that x, s', s, t, l, x forms an ordinary pentagon. By $t \in \mathcal{E}_1(x)$ and $s \in \mathcal{E}_0(x)$, s and t are not collinear with x. Since s is collinear to p but no other point of L and t is not collinear to p, s and l are not collinear. If s' is collinear to l then s' would be contained in S since s and l are not collinear. If s' is collinear with t, then t and x have two common neighbors, so $t \in \mathcal{E}_0(x)$, a contradiction. So we can apply Lemma 2.1.10 to the ordinary pentagon x, s', s, t, l, x and obtain in particular $t \in \mathcal{E}_0(x)$, a contradiction. Hence every point of S collinear with p but not with all points of L is contained in $\mathcal{E}_1(x)$. $\square$

Corollary 2.1.14. *If $\mathcal{E}_0 \neq \emptyset$, then every line is contained in a symplecton.*

Proof. We first show that every point is contained in a symplecton. By assumption, $\mathcal{E}_0 \neq \emptyset$, and hence Proposition 2.1.6 implies that there exists a symplecton S. Consider any point $x \in \mathcal{E}$. If $x \in S$ there is nothing to show. If $x \notin S$ is collinear to a point of S or S contains a point opposite to x, then Corollary 2.1.13 and Lemma 2.1.11 imply, respectively, that there exists a symplecton containing x. Now assume that x is at distance 2 with a point $s \in S$. Let y be a common neighbor of x and s. Then by the previous arguments, there exists a symplecton T containing y. Now x is collinear to a point of the symplecton T, and, as before, this implies that x itself is contained in a symplecton.

Consider any line L of the root filtration space. Let x be a point of L and S a symplecton containing this point. If L is contained in S, there is nothing to show. So we may assume that there exists a point y of L not contained in S. Since it is collinear with a point of S, it is collinear with a line M of S containing x, by Proposition 2.1.6(d). Since $y^\perp \cap S$ is a singular subspace by Proposition 2.1.6(b), we can find $s \in S$ collinear with M but not with y. Hence y and s are contained in a symplecton T, and x is a common neighbor of s and y. We can conclude that T contains $L = xy$. $\square$

If X is a subset of $\mathcal{E}$, then we call X connected, if the graph on X induced by collinearity (i.e., the relation $\mathcal{E}_{-1}$) is connected.

We conclude this section with two lemmas on the connectedness of some complements of subspaces. The following lemma is well known.

Lemma 2.1.15. *Let S be a thick non-degenerate polar space of rank at least 2. If H is a proper geometric hyperplane of S, then $S \setminus H$ is connected.*

Proof. Let x, y be non-collinear points of S not in H. We will show that there is a path of collinear points from x to y outside H.

If ℓ is a line on x, then ℓ contains a unique point z collinear to y. Clearly we can assume that z is in H. Now pick a point x' on ℓ distinct from x and z. Moreover let y' be a point on the line through y and z different from y and z. Let m be a line on x' different from ℓ and not in a singular subspace with ℓ. Then both y and y' are collinear with points $u, v \in m$ different from x'. If $u = v$, then $u = v$ is also collinear to z. Since $u = v$ is collinear to x', this contradicts ℓ and m not to be in a singular subspace. As at least one of u or v is not in H, we find that at least one of x, x', u, y or x, x', v, y', y is a path from x to y outside H. □

Lemma 2.1.16. *Suppose $\mathcal{E}_0$ is non-empty. Let $x \in \mathcal{E}$. The complement of the geometric hyperplane $\mathcal{E}_{\leqslant 1}(x)$ in $\mathcal{E}$ is connected.*

Proof. Let $\mathcal{S}$ be as in Proposition 2.1.6. Let $x \in \mathcal{E}$ and set $H = \mathcal{E}_{\leq 1}(x)$. Suppose $y, z \in \mathcal{E} \setminus H$. We will show that y and z are connected by a path in the collinearity graph, which does not contain points from H. Clearly we can assume that y and z are non-collinear. If $z \in \mathcal{E}_0(y)$, then y and z are contained in a symplecton S and we can apply Lemma 2.1.15 to see that y and z are connected by a path outside H.

Assume now that z is a point not contained in H but inside $\mathcal{E}_1(y)$. Clearly, we can assume that the unique neighbor c of y and z is contained in H. Consider a symplecton S containing the line cz, which exists by Corollary 2.1.14. Then there exists a line ℓ in S such that $\ell \subseteq \mathcal{E}_{\leq -1}(y)$. If ℓ meets H in a single point, then, again using Lemma 2.1.15, we find a path from y via a point of ℓ to z without using points from H.

Thus we can assume that ℓ is in H. Now $A := \mathcal{E}_0(x) \cap S$ is a non-empty singular subspace contained in H. By Lemma 2.1.12, we find that A consists of a single point, call it a. Note $a \notin \mathcal{E}_{-1}(y)$ and hence $a \notin l$. Since $\ell \leq H \cap S$, Lemma 2.1.11 implies that a and ℓ are contained in a singular plane π of S. In particular, S has rank at least 3 and we can find a point $d \in S$ which is collinear to all points of ℓ but not to a. But then $d \in \mathcal{E}_{\leq 0}(y)$ and $d \notin H$. By the above, we can find paths from y to d and from d to z not containing any point from H.

Finally assume now that y is at distance 3 with z. Let (z, a, b, y) be any path of length 3 in Γ from z to y. If a or b are not in H, we can apply the above to the pairs a, y or b, z, respectively and find a path from y to z. So assume $a, b \in H$.

By Lemma 2.1.9, we can find a neighbor c of z at distance 3 with b. Since H is a geometric hyperplane the line cz intersects H in a unique point. Since $\mathcal{E}_{\leqslant 1}(b)$ is a geometric hyperplane as well and b and z are at distance 2, we may assume $c \notin H$. Since $\mathcal{E}_{\leqslant 1}(c)$ is a geometric hyperplane, the line by contains a point d at distance (at most) 2 from c. If $d = b$, then c and b are at distance (at most) 2, a contradiction. Since $b \in H$ and $y \notin H$, $d \notin H$. Note that d is at distance 2 from c. By the above we find a path outside H from y to z via d and c. □

Corollary 2.1.17. *Suppose $\mathcal{E}_0$ is non-empty. Let $x \in \mathcal{E}$. Then Γ is generated by $\mathcal{E}_{\leq 1}(x) \cup \{z\}$, for each point $z \in \mathcal{E}_2(x)$.*

SECTION 2.2

Subgeometries of root filtration spaces fixed by involutions

We proceed with the notation of the previous section. So, suppose $\Gamma = (\mathcal{E}, \mathcal{F})$ is a (non-degenerate) root filtration space with filtration $\mathcal{E}_i$, $-2 \leq i \leq 2$.

Definition 2.2.1. Let σ be an involution in the automorphism group of Γ. We define the geometry $\Gamma_\sigma = (\mathcal{E}_\sigma, \mathcal{S}_\sigma)$ to be the point-line geometry with as point set the set $\mathcal{E}_\sigma$ of points of Γ fixed by σ, and as line set $\mathcal{S}_\sigma$, the set of symplecta of Γ fixed (as a set of points) by σ and containing at least two non-collinear points of $\mathcal{E}_\sigma$. Incidence is defined by inclusion.

Notice that a line $S \in \mathcal{S}_\sigma$ can be identified with the subsets $\mathcal{E}_\sigma \cap S$, as any two of its non-collinear points determine the symplecton S uniquely. This also implies that we can consider Γ_σ to be a partial linear space.

Example 2.2.2. Let V be a vector space over a skew field of dimension $n+1$ at least 3. Let Γ be the root filtration space of type $A_{n,\{1,n\}}$ described in Example 2.1.3. Let σ be a non-degenerate polarity on $\mathbb{P}(V)$. Then σ induces an automorphism of order 2 on Γ, which we also denote by σ.

The points fixed by σ are the point-hyperplane pairs (p, p^σ) with $p \in p^\sigma$. So, the points in $\mathcal{E}_\sigma$ can be identified with the points of $\mathbb{P}(V)$ that are absolute with respect to σ, i.e. points p with $p \in p^\sigma$.

As σ maps a line of Γ consisting of point-hyperplane pairs with a fixed point to a line consisting of point-hyperplane pairs in which the hyperplane is fixed, we find that σ does not fix any line of Γ.

A symplecton S of Γ fixed set-wise by σ meets $\mathcal{E}_\sigma$ in the point-hyperplane pairs (p, p^σ) where p is absolute and running over an absolute line of $\mathbb{P}(V)$, which is a line which is contained in p^σ for any point p on it.

The geometry Γ_σ is isomorphic to the polar space defined by σ on $\mathbb{P}(V)$, i.e. the point-line geometry of absolute points and lines with respect to σ.

This example generalizes to the following:

Proposition 2.2.3. *Let σ be an involution in the automorphism group of Γ. Suppose that σ does not fix any line of Γ. Then Γ_σ satisfies the one-or-all-axiom: if $x \in \mathcal{E}_\sigma$ and $S \in \mathcal{S}_\sigma$, then x is collinear in Γ_σ with one or all points $y \in \mathcal{E}_\sigma$ that are contained in S.*

Proof. Suppose that σ fixes a singular subspace V of Γ. For any $x \in V$, we have $x^\sigma \in V$. Hence, if $x^\sigma \neq x$, then the line xx^σ is fixed by σ. On the other hand, if

$x^\sigma = x$ for all $x \in V$, then any line in V is (point-wise) fixed by σ. So we obtain that the only singular subspaces of Γ fixed by σ are points. I.e., the points of Γ_σ.

Consider an arbitrary point x of Γ_σ and an arbitrary symplecton S of Γ_σ not containing x.

Suppose first that x is collinear with a point of S. Then, by Proposition 2.1.6, $\mathcal{E}_{-1}(x) \cap S$ is a singular subspace V of S containing a line. Now $(\mathcal{E}_{-1}(x) \cap S)^\sigma = \mathcal{E}_{-1}(x^\sigma) \cap S^\sigma = \mathcal{E}_{-1}(x) \cap S$. Hence $\mathcal{E}_{-1}(x) \cap S$ should be a point by the previous paragraph, a contradiction.

This implies that any point of S has distance at least 2 with x. Then either all points of S are in $\mathcal{E}_0(x)$ or $A := \mathcal{E}_{\leqslant 0}(x) \cap S$ is a singular subspace, by Lemma 2.1.11. Assume the former. Then any point y of S fixed by σ is contained in a unique symplecton with x. Since x and y are fixed by σ this symplecton is also fixed. I.e., x and y lie on a line in Γ_σ. Since the points in S fixed by σ are precisely the points of the line S in Γ_σ, we find x to be collinear with all points of the line S in Γ_σ.

Assume now that A is a singular subspace. Since x and S are fixed by σ, the subspace A is fixed by σ as well. Hence, as follows from the first part of this proof, A consists of a unique point y. Moreover, y is fixed by σ. In particular, in Γ_σ the point x is collinear with the unique point y of S. □

The above proposition indeed implies that the fixed point geometry Γ_σ is a polar space. However, in general this polar space may be thin (with just two points on a line) or degenerate (i.e., there is a point collinear to all other points). Moreover, it can also happen that Γ_σ contains no lines, or even no points.

SECTION 2.3

Lie algebras generated by extremal elements

Now we turn our attention to Lie algebras. In particular, Lie algebras generated by extremal elements.

In this section we provide some definitions and collect some results on extremal elements, mainly from [CI06].

Definition 2.3.1. Let L be a Lie algebra over the field k. A non-zero element $x \in L$ is called *extremal* if there is a map $g_x : L \to k$, called the *extremal form* on x, such that

$$[x, [x, y]] = 2g_x(y)x, \tag{2.1}$$

and moreover

$$\big[[x, y], [x, z]\big] = g_x\big([y, z]\big)x + g_x(z)[x, y] - g_x(y)[x, z], \tag{2.2}$$

and

$$\big[x, [y, [x, z]]\big] = g_x\big([y, z]\big)x - g_x(z)[x, y] - g_x(y)[x, z], \tag{2.3}$$

for every $y, z \in L$.

The last two identities are called the *Premet identities*. If the characteristic of k is not 2, then the Premet identities follow from equation (2.1), see [CI06, Definition 14]. Also, using the Jacobi identity, (2.2) and (2.3) are equivalent if (2.1) holds.

As a consequence, $\big[x, [x, L]\big] \subseteq kx$ for any extremal element $x \in L$. We call $x \in L$ a *sandwich* or *absolute zero divisor* if $[x, [x, y]] = 0$ and $[x, [y, [x, z]]] = 0$ for every $y, z \in L$. So, a sandwich is an element x for which the extremal form g_x can be chosen to be identically zero. We introduce the convention that *g_x is identically zero whenever x is a sandwich in L.* An extremal element is called *pure* if it is not a sandwich.

We denote the set of extremal elements of a Lie algebra L by $E(L)$ or, if L is clear from the context, by E. Accordingly, we denote the set $\{kx | x \in E(L)\}$ of *extremal points* in the projective space on L by $\mathcal{E}(L)$ or $\mathcal{E}$.

We continue with some examples.

Example 2.3.2. Let V be a vector space over a field k with dual space V^*. Suppose W^* is a subspace of V^* separating the points of the projective space of V.

On $V \otimes W^*$ we can define a Lie bracket by linear extension of the following product for pure tensors $v \otimes \phi$ and $w \otimes \psi$:

$$[v \otimes \phi, w \otimes \psi] = (v \otimes \psi)\phi(w) - (w \otimes \phi)\psi(v).$$

The Lie algebra thus obtained will be denoted by $\mathfrak{g}(V \otimes W^*)$.

A pure tensor $v \otimes \phi$ is called singular if $\phi(v) = 0$. Let g be the k-bilinear form on $V \otimes W^*$ defined by

$$g(v \otimes \phi, w \otimes \psi) = -\psi(v)\phi(w)$$

for $v \otimes \phi, w \otimes \psi \in V \otimes W^*$. Then for all singular pure tensors $v \otimes \phi$ and tensors $w \otimes \psi$ we have

$$\begin{aligned}
\big[v \otimes \phi, [v \otimes \phi, w \otimes \psi]\big] &= \big[v \otimes \phi, \phi(w)v \otimes \psi - \psi(v)w \otimes \phi\big] \\
&= -\psi(v)\phi(w)v \otimes \phi - \psi(v)\phi(w)v \otimes \phi \\
&= -2\psi(v)\phi(w)v \otimes \phi \\
&= 2g(v \otimes \phi, w \otimes \psi)v \otimes \phi.
\end{aligned}$$

In characteristic $\neq 2$ this implies that the singular pure tensors are extremal elements in $\mathfrak{g}(V \otimes W^*)$ with extremal form at the singular tensor $v \otimes \phi$ given by $g(v \otimes \phi, \cdot)$. This also holds true in characteristic 2. (It is straightforward, but somewhat tedious, to check that the Premet identities also hold.) The subalgebra of $\mathfrak{g}(V \otimes W^*)$ generated by the singular tensors is denoted by $\mathfrak{s}(V \otimes W^*)$.

An element $v \otimes \phi \in V \otimes V^*$ acts linearly on V by

$$(v \otimes \phi)(w) = \phi(w)v$$

for all $w \in V$. This provides an isomorphism between $\mathfrak{g}(V \otimes V^*)$ and the finitary general linear Lie algebra $\mathfrak{fgl}(V)$. (A linear map is finitary if its kernel has finite codimension.) Under this isomorphism the subalgebra $\mathfrak{s}(V \otimes V^*)$ is mapped isomorphically to $\mathfrak{fsl}(V)$, the finitary special Lie algebra, or $\mathfrak{sl}(V)$ in case V is finite-dimensional.

In the rest of this chapter we often identify the Lie algebra $\mathfrak{g}(V \otimes V^*)$ as well as its subalgebras with the finitary general linear Lie algebra $\mathfrak{fgl}(V)$, or the corresponding subalgebras.

Note that $V \otimes W^*$ carries the structure of an associative algebra whose product is defined as the linear expansion of the product

$$(v \otimes \phi)(w \otimes \psi) = (v \otimes \psi)\phi(w).$$

The associated Lie algebra is then $\mathfrak{g}(V \otimes W^*)$. A pure tensor $v \otimes \phi$ being singular is equivalent to its square being zero in the associative algebra. So, if a is such a singular pure tensor, then $[a, [a, b]] = -2aba$ which is a scalar multiple of a, for all $b \in \mathfrak{g}(V \otimes W^*)$.

Example 2.3.3. Let σ be a field automorphism of order ≤ 2 of the field k, and V be a vector space over k. Assume that the characteristic of k is different from 2. Moreover, suppose f is an antisymmetric σ-sesquilinear form on V, linear in the second coordinate. So, for all $u, v, w \in V$ and $\lambda, \mu \in k$ we have:

$$f(v, w) = -f(w, v)^\sigma,$$

$$f(v, \lambda w + \mu u) = f(v, w)\lambda + f(v, u)\mu.$$

Then for each vector $v \in V$ the map $f_v : V \to k$, with $f_v(w) = f(v, w)$ for all $w \in V$ is an element of V^*. By $S_f(V \otimes V^*)$ or just S_f we denote the subspace of $V \otimes V^*$ spanned by the pure tensors $v \otimes f_v$, with $v \in V$. (We say it is spanned by the f-symmetric elements.)

The space $S_f(V \otimes V^*)$ is closed under the Lie bracket:

$$\begin{aligned}
[v \otimes f_v, w \otimes f_w] &= (v \otimes f_w)f(v, w) - (w \otimes f_v)f(w, v) \\
&= f(v, w)v \otimes f_w + w \otimes f_{f(v,w)v} \\
&= (f(v, w)v + w) \otimes f_{f(v,w)v+w} - f(v, w)v \otimes f_{f(v,w)v} - w \otimes f_w.
\end{aligned}$$

The corresponding Lie subalgebra of $\mathfrak{g}(V \otimes V^*)$ will be denoted by $\mathfrak{s}_f(V \otimes V^*)$ or, for short, $\mathfrak{s}_f$.

If f is a non-degenerate symplectic form, then $\mathfrak{s}_f$ is simple and can be identified with the finitary Lie algebra $\mathfrak{fsp}(V, f)$, i.e., the Lie subalgebra of $\mathfrak{fgl}(V)$ of finitary linear transformations $t : V \to V$ satisfying $f(t(v), w) = -f(v, t(w))$ for all $v, w \in V$. Moreover, its set of extremal elements is the set of all elements $v \otimes f_v$, where $0 \neq v \in V$. See [CF17, Theorem 3.1, Propositions 3.5 and 3.6].

If σ is non-trivial and f a non-degenerate skew-Hermitian form with positive Witt index, then the elements $v \otimes f_v$, where $0 \neq v \in V$ with $f(v, v) = 0$ generate a subalgebra of $\mathfrak{s}(V \otimes V^*)$, which can be identified with the finitary special unitary Lie algebra $\mathfrak{fsu}(V, f)$. The extremal elements in this subalgebra are the elements $v \otimes f_v$, where $0 \neq v \in V$ and $f(v, v) = 0$. See for example [CO19].

In view of the last remark in the previous example, we can also view $\mathfrak{s}_f(V \otimes V^*)$ as the Lie algebra of skew elements in $\mathfrak{g}(V \otimes V^*)$ with respect to the involution on the associative algebra $V \otimes V^*$ induced by $v \otimes f_w \mapsto -w \otimes f_v$.

Remark 2.3.4. The *long roots elements* of a so-called *Chevalley Lie algebra* are extremal elements, see Proposition 3.4.2 of [Fle15]. (See Definition 3.3.2 of *loc. cit.* for the definition of a long root element.) Since we will not need it, we do not include a precise description of Chevalley Lie algebras (or root systems), and instead refer to Chapter 3 of *loc. cit.*

We will now revise some general theory on extremal elements. For $x \in E$ and $\lambda \in k$ we define the map $\exp(x, \lambda) : L \to L$ by

$$\exp(x, \lambda)(y) = y + \lambda[x, y] + \lambda^2 g_x(y)x,$$

for all $y \in L$.

Since the automorphisms obtained from the following proposition are quite essential, especially in Chapter 4, we include a proof.

Proposition 2.3.5 ([CI06, Lemma 15])**.** *Let $x \in E$ be pure and $\lambda \in k$. Then $\exp(x, \lambda)$ is an automorphism of L.*

Proof. Note that the map $\exp(x, \lambda)$ coincides with $\exp(\lambda x, 1)$. We can thus assume $\lambda = 1$. Consider $y, z \in L$ arbitrary. By the Jacobi identity and the defining properties of an extremal element, (2.1) and (2.2), we get

$$\begin{aligned}
[\exp(x,1)(y), \exp(x,1)(z)] &= [y + [x, y] + g_x(y)x, z + [x, z] + g_x(z)x] \\
&= [y, z] + ([[x, y], z] + [y, [x, z]]) + g_x(y)g_x(z)[x, x] \\
&\quad + ([[x, y], [x, z]] + g_x(y)[x, z] - g_x(z)[x, y]) \\
&\quad + (g_x(y)[x, [x, z]] - g_x(z)[x, [x, y]]) \\
&= [y, z] + [x, [y, z]] + g_x([y, z])x \\
&\quad + (2g_x(y)g_x(z) - 2g_x(z)g_x(y))x \\
&= \exp(x, 1)([y, z]).
\end{aligned}$$

This shows that $\exp(x, 1)$ is Lie algebra endomorphism.

Since, by (2.1), $\exp(x, \mu)\exp(x, \nu) = \exp(x, \mu + \nu)$ for all $\mu, \nu \in k$, we get in particular $\exp(x, -1)\exp(x, 1) = \mathrm{id}$. Hence $\exp(x, 1)$ is an automorphism. □

Let $x \in E$ be pure. By $\mathrm{Exp}(x)$ we denote the set $\{\exp(x, \lambda) \mid \lambda \in k\}$. Since, for $\lambda, \mu \in k$, we have $\exp(x, \lambda)\exp(x, \mu) = \exp(x, \lambda + \mu)$, we find that $\mathrm{Exp}(x)$ is a subgroup of $\mathrm{Aut}(L)$ isomorphic to the additive group of k.

Clearly, $\mathrm{Exp}(x) = \mathrm{Exp}(\lambda x)$ for $\lambda \in k^\times$. Therefore we can define $\mathrm{Exp}(\langle x\rangle)$ to be equal to $\mathrm{Exp}(x)$. We also set $\exp(x) = \exp(x, 1)$.

Proposition 2.3.6 ([CI06, Proposition 20])**.** *Suppose that L is generated by its extremal elements (as a Lie algebra). The extremal elements span the vector space L. Moreover, there is a bilinear form $g : L \times L \to k$, such that for all $x, y \in E$ we have $g(x, y) = g_x(y)$. The form g is symmetric and associates with the Lie product $[\cdot, \cdot]$ on L.*

The form g is called the *extremal form* on L. As the form g is associative, its radical $\mathrm{Rad}(g) := \{u \in L | g_u(z) = 0, \forall\ z \in L\}$ is an ideal in L. Notice that the extremal form f from [CSUW01] satisfies $f = 2g$.

Proposition 2.3.7 ([CI06, Lemma 21, 24, 25 and 27])**.** *Suppose that L is generated by its extremal elements and $x, y \in E$ are pure. Then we have one of the following:*

(a) $kx = ky$;
(b) $[x, y] = 0$ *and* $\lambda x + \mu y \in E \cup \{0\}$ *for all* $\lambda, \mu \in k$;
(c) $[x, y] = 0$ *and* $\lambda x + \mu y \in E$ *only if* $\lambda = 0$ *or* $\mu = 0$;
(d) $z := [x, y] \in E$, *and* x, z *and* y, z *are as in case* (b);
(e) *the subalgebra generated by x and y is isomorphic to* $\mathfrak{sl}_2(k)$.

Moreover, $g(x, y) \neq 0$ if and only if the subalgebra generated by x and y is isomorphic to $\mathfrak{sl}_2(k)$.

Based on the previous proposition, we define the following relations on $E \times E$.

Definition 2.3.8. For $x, y \in E$ extremal elements we define

$$(x, y) \in \begin{cases} E_{-2}, & \Longleftrightarrow kx = ky, \\ E_{-1}, & \Longleftrightarrow [x, y] = 0, (x, y) \notin E_{-2} \text{ and } kx + ky \subseteq E \cup \{0\}, \\ E_0, & \Longleftrightarrow [x, y] = 0 \text{ and } (x, y) \notin E_{-2} \cup E_{-1}, \\ E_1, & \Longleftrightarrow [x, y] \neq 0 \text{ and } g(x, y) = 0, \\ E_2, & \Longleftrightarrow g(x, y) \neq 0. \end{cases}$$

In light of Theorem 2.3.14 we call two extremal elements (x, y) collinear, symplectic, special or hyperbolic if (x, y) is contained in E_{-1}, E_0, E_1 or E_2, respectively. For the corresponding extremal points $\langle x\rangle, \langle y\rangle$, we define

$$(\langle x\rangle, \langle y\rangle) \in \mathcal{E}_i \Longleftrightarrow (x, y) \in E_i.$$

Let $x \in E$. Then $y \in E_i(x)$ denotes that $(x, y) \in E_i$. By $E_{\leq i}(x)$ we denote the set $\bigcup_{-2 \leq j \leq i} E_j(x)$. Similarly, if $x \in \mathcal{E}$, then $\mathcal{E}_i(x)$ consists of all y with $(x, y) \in \mathcal{E}_i$, and $\mathcal{E}_{\leq i}(x)$ denotes $\bigcup_{-2 \leq j \leq i} \mathcal{E}_j(x)$.

Proposition 2.3.9. *Suppose L is a Lie algebra defined over a field k, generated by its set of extremal elements E. Suppose moreover that L contains a pure extremal element. Then if L is simple, the extremal form g is non-degenerate and $(\mathcal{E}, \mathcal{E}_2)$ is connected. The converse also holds if* $\mathrm{char}(k) \neq 2$.

Proof. First assume that L is simple. As L contains a pure extremal element, the form g is non-trivial and its radical $\mathrm{Rad}(g)$ is not L. This radical is an ideal of L, so L being simple implies this radical to be trivial and g to be non-degenerate.

Since sandwiches are contained in the radical of g, all elements of E are pure. Now by Theorem 28 of [CI06] we find $(\mathcal{E}, \mathcal{E}_2)$ to be connected.

Assume from now on $\mathrm{char}(k) \neq 2$, g to be non-degenerate and $(\mathcal{E}, \mathcal{E}_2)$ to be connected. Let I be a non-zero ideal of L. Then we can find an extremal element x and $i \in I$ such that $g(x, i) \neq 0$. But then $[x, [x, i]] = 2g(x, i)x \in I$, which implies that $\langle x \rangle$ is in I.

Applying the same argument to $(x, y) \in \mathcal{E}_2$ instead of (i, x) we find that all neighbors of $\langle x \rangle$ in the graph $(\mathcal{E}, \mathcal{E}_2)$ are in I. By connectedness of the latter graph we even find all elements of $\mathcal{E}$ in I, implying $I = L$, and proving L to be simple. □

The case that g is trivial and all extremal elements are sandwich elements is covered by the next result:

Proposition 2.3.10 ([ZK90])**.** *If L is a finite-dimensional Lie algebra generated by its sandwich elements, then L is nilpotent.*

Corollary 2.3.11. *If L is a simple finite-dimensional Lie algebra defined over a field of characteristic different from 2 generated by its extremal elements, then L does not contain a sandwich element.*

Proof. If L does contain a pure element, then Proposition 2.3.9 shows that L does not contain sandwich elements. If L does not contain pure elements, then L is generated by its sandwich elements and, by Proposition 2.3.10, we find L to be nilpotent. □

Definition 2.3.12. Let L be a Lie algebra. The *associated extremal geometry* is the incidence geometry with as points

$$\mathcal{E} = \mathcal{E}(L) := \{\langle x \rangle \mid x \in E = E(L)\},$$

as lines

$$\mathcal{F} = \mathcal{F}(L) := \{\langle x, y \rangle \mid x, y \in E, (x, y) \in E_{-1}\},$$

and inclusion as incidence relation. We denote this geometry by $\Gamma(L)$, or in case it is clear what L is, by Γ.

The *rank* of $\Gamma(L)$ is the maximal dimension (as a linear subspace of L) of a subspace X of $\Gamma(L)$ in which any two points are collinear.

If x and y are two extremal points with $(x, y) \in \mathcal{E}_2$, then they generate an $\mathfrak{sl}_2$-subalgebra. This subalgebra is generated by any two of its extremal points. The intersection of $\mathcal{E}$ with this subalgebra is called the $\mathfrak{sl}_2$-*line* on x and y.

Example 2.3.13. Let V be a vector space over a field k of dimension at least 3 and W^* a subspace of V^* separating the points of the projective space of V. Let $\mathcal{E}$ be the set of extremal points in $\mathfrak{s}(V \otimes W^*)$, which is, as follows from [Fle15, Corollary 3.4.11], the set of elements $\langle v \otimes \phi \rangle$ where $v \in V$ and $\phi \in W^*$ with $\phi(v) = 0$. Then each extremal point $\langle v \otimes \phi \rangle$ corresponds to an incident point-hyperplane pair $(\langle v \rangle, \ker(\phi))$ of $\mathbb{P}(V)$. The extremal geometry with point set $\mathcal{E}$ is isomorphic to the geometry $\Gamma(V, W^*)$ described in Example 2.1.3. In particular, if V has dimension $n + 1 < \infty$, then the extremal geometry is a root shadow space of type $A_{n,\{1,n\}}$.

Theorem 2.3.14 ([CI06, Theorem 28]). *Suppose L is a simple Lie algebra generated by its set of pure extremal elements E. If the set $\mathcal{E}_{-1}$ is not empty, then $(\mathcal{E}, \mathcal{F})$ is a root filtration space.*

The following lemma characterizes collinearity.

Lemma 2.3.15 ([CI07, Lemma 27]). *Suppose L is a generated by its set of pure extremal elements E. Let $x, y \in E$ be linearly independent and pure. Then $(x, y) \in E_{-1} \Leftrightarrow$ there are $\lambda, \mu \in k^\times$ with $\lambda x + \mu y \in E$.*

Combining Theorem 2.1.5 and the main result of Cuypers-Roberts-Shpectorov [CRS15, Theorem 1.1] and Cuypers-Fleischmann [CF18, Theorem 1.1] we obtain:

Theorem 2.3.16. *Suppose that the extremal geometry Γ of a simple Lie algebra L, generated by its set of pure extremal elements, has finite rank.*

If $\mathcal{E}_{-1} \neq \emptyset$, we find Γ to be isomorphic to a root shadow space of type $A_{n,\{1,n\}}$ $(n \geq 2)$, $BC_{n,2}$ $(n \geq 3)$, $D_{n,2}$ $(n \geq 4)$, $E_{6,2}$, $E_{7,1}$, $E_{8,8}$, $F_{4,1}$ or $G_{2,2}$.

Furthermore, if both $\mathcal{E}_{-1}$ and $\mathcal{E}_0$ are non-empty, then L is determined, up to isomorphism, by its extremal geometry Γ.

Note that the labeling of the Coxeter diagrams follows [Bou68].

Theorem 2.3.17. *Assume that L is generated by its pure extremal elements, and that $\mathcal{F} \neq \emptyset$. If there is a line in $\mathcal{F}$ which is a maximal singular subspace of the extremal geometry $(\mathcal{E}, \mathcal{F})$, then $(\mathcal{E}, \mathcal{F})$ is a generalized hexagon.*

Proof. This is [CI07, Corollary 18] combined with [CI06, Theorem 28(ii)]. □

Remark 2.3.18. The only case in which $\mathcal{E}_{-1} \neq \emptyset$ while $\mathcal{E}_0 = \emptyset$, is when the extremal geometry is a root shadow space of type $G_{2,2}$. Indeed, if $\mathcal{E}_{-1} \neq \emptyset$ and $\mathcal{E}_0 = \emptyset$, then combining Theorem 13 and Theorem 15 of [CI07] yields that we either have a root filtration space of type $G_{2.2}$ or a root filtration space as in Example 2.1.3 or Example 2.1.4, but in these last two cases $\mathcal{E}_0$ is non-empty, see Examples 2.1.7 and 2.1.8.

In this last theorem of this section we show that the automorphism group works transitive on extremal points.

Theorem 2.3.19. *Let L be simple Lie algebra generated by its set E of pure extremal elements. Then the automorphism group of L acts transitively on $\mathcal{E}$.*

Proof. Let $x, y \in E$ with $g(x,y) = g(y,x) \neq 0$. Then, after replacing y by a scalar multiple, we can assume $g(x,y) = g(y,x) = 1$. But then $\exp(x,1)y = y + [x,y] + x = \exp(y,-1)x$. So, $\exp(y,1)\exp(x,1)$ maps y to x. This implies that for each $(x,y) \in \mathcal{E}_2$ there is an automorphism of L that maps x to y.

As L is simple, we find the graph $(\mathcal{E}, \mathcal{E}_2)$ to be connected by Proposition 2.3.9, and hence by the above the automorphism group of L to be transitive on $\mathcal{E}$. $\square$

In the next lemma, $N_L(x)$ denotes the normalizer of x, i.e. $N_L(x) = \{l \in L \mid [x,l] \in \langle x \rangle\}$, similarly for $N_L(y)$. In this chapter we will use the next lemma only once, in Chapter 4 it is a key lemma.

Lemma 2.3.20 ([CI06, Proposition 22])**.** *Suppose that there exist extremal elements $x, y \in L$ such that $g_x(y) = 1$. Then L has a $\mathbb{Z}$-grading*

$$L = L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus L_2,$$

with $L_{-2} = \langle x \rangle$, $L_2 = \langle y \rangle$, $L_0 = N_L(x) \cap N_L(y)$, $L_1 = [y,U]$ and $L_{-1} = [x,U]$, where

$$U = \{u \in L \mid g_x(u) = g_y(u) = g_x([y,u]) = 0\}.$$

Moreover, L_i is contained in the i-eigenspace of $\mathrm{ad}_{[x,y]}$, and ad_x defines a linear isomorphism from L_1 tot L_{-1} with inverse $-\mathrm{ad}_y$.

Example 2.3.21. If $L = \mathfrak{sl}_n$, the special linear Lie algebra of traceless $n \times n$-matrices, then $x = E_{1n}$ and $y = -E_{n1}$ are hyperbolic extremal elements with $g_x(y) = 1$. (With E_{ij} the $n \times n$-matrix with all entries equal to 0, except for the (i,j)-entry, which equals 1.) The 5-grading corresponding to x and y as above is: $L_{-1} = \oplus_{i=2}^{n-1}(\langle E_{1i} \rangle \oplus \langle E_{in} \rangle)$, $L_1 = \oplus_{i=2}^{n-1}(\langle E_{i1} \rangle \oplus \langle E_{ni} \rangle)$ and $L_0 = \sum_{i,j=2, i\neq j}^{n-1} \langle E_{ij} \rangle \oplus \{\lambda_1 E_{11} + \lambda_2 E_{22} + \cdots + \lambda_n E_{nn} \mid \lambda_1 + \cdots + \lambda_n = 0\}$. The above claims can, for example, be verified using Example 2.3.2, and Lemma 4.1.3 can also be used to simplify some calculations.

SECTION 2.4

Lie algebras generated by extremal elements with no lines

Theorem 2.3.16 excludes Lie algebras generated by pure extremal elements in which $\mathcal{E}_{-1} = \emptyset$ (and hence also $\mathcal{E}_1$ is empty, by Proposition 2.3.7(d)). In this section we will study the Lie algebras generated by pure extremal elements with $\mathcal{E}_{-1} = \emptyset$ somewhat closer.

From now on we assume L to be a simple Lie algebra over a field k generated by its set of pure extremal elements such that $\mathcal{E} \times \mathcal{E} = \mathcal{E}_{-2} \cup \mathcal{E}_0 \cup \mathcal{E}_2$. Moreover, we sometimes have to assume the field k to be of characteristic different from 2. By Proposition 2.3.7 we get:

Lemma 2.4.1. *Any two extremal elements in $\mathcal{E}$ either commute or generate an $\mathfrak{sl}_2(k)$.*

Examples of Lie algebras generated by extremal elements that either commute or generate an $\mathfrak{sl}_2$ are the finitary symplectic and special unitary ones described in Example 2.3.3.

In [CF17, Theorem 1.1] we find the following characterization of the symplectic Lie algebras.

Theorem 2.4.2. *Let L be a simple Lie algebra over the field k of characteristic $\neq 2$ and generated by its set of pure extremal elements. Assume the following:*

(a) *any two extremal elements x and y in L either commute or generate an $\mathfrak{sl}_2(k)$;*
(b) *for any three extremal elements x, y, z in L with $[x, y] \neq 0$, there is an extremal element u in the subalgebra $\langle x, y\rangle$ commuting with z.*

Then L is isomorphic to $\mathfrak{fsp}(V, f)$ for some non-degenerate symplectic space (V, f).

Moreover, under this isomorphism the extremal elements in L are mapped to rank 1 elements in $\mathfrak{fsp}(V, f)$.

In the proof of the above theorem, as provided in [CF17], particular Lie subalgebras generated by three extremal elements play an important role. These subalgebras are also of importance in our setting. They are described in the following example.

Example 2.4.3. Let (V, f) be a non-degenerate symplectic space containing three linearly independent u, v, w with $f(u, v) = 1, f(v, w) = 1$ and $f(u, w) = 0$.

Then the subalgebra of $\mathfrak{s}_f$ generated by the three extremal elements $u \otimes f_u$, $v \otimes f_v$, and $w \otimes f_w$ is 6-dimensional and we denote it by $\mathfrak{sp}_3(k)$. It contains a 1-dimensional center spanned by $(u+w)\otimes f_{u+w}$. Notice that this center is non-trivial, as there is a vector $x \in V$ with $f(u+w,x)=1$. Modulo this center, $\mathfrak{sp}_3(k)$ is isomorphic to an extension of $\mathfrak{sl}_2$ by its natural 2-dimensional module, which we denote by $\mathfrak{psp}_3(k)$.

The group $\langle \mathrm{Exp}(\langle u\otimes f_u\rangle), \mathrm{Exp}(\langle v\otimes f_v\rangle), \mathrm{Exp}(\langle w\otimes f_w\rangle)\rangle$ acts transitively on the non-central extremal points of $(\mathfrak{p})\mathfrak{sp}_3(k)$. Moreover, for any two non-central commuting extremal points x and y of $(\mathfrak{p})\mathfrak{sp}_3(k)$ we find that every extremal point z commuting with x also commutes with y. A maximal set of pairwise commuting non-central extremal points is called a *transversal* of $(\mathfrak{p})\mathfrak{sp}_3(k)$. The transversal on $\langle u\otimes f_u\rangle$ consists of all extremal points $\langle x\otimes f_x\rangle$, where $x\in\langle u,w\rangle$ but not in $\langle u-w\rangle$. In particular, it contains $|k|$ extremal points. Notice that this transversal is contained in the linear span of $u\otimes f_u$, $w\otimes f_w$ and $[u\otimes f_u,[w\otimes f_w,z]]$ for any non-central z of $(\mathfrak{p})\mathfrak{sp}_3(k)$ not in the transversal. It intersects each $\mathfrak{sl}_2$-line spanned by two non-central points of $(\mathfrak{p})\mathfrak{sp}_3(k)$ in exactly one extremal point.

We note that the (non-central) extremal points and $\mathfrak{sl}_2$-lines in $(\mathfrak{p})\mathfrak{sp}_3(k)$ form a dual affine plane, i.e. a projective plane from which a point and all the lines on this point are removed. If ∞ denotes the removed point, then the union of $\{\infty\}$ and any transversal is a removed line.

All these properties can be checked easily. For more details, see [CF17, Example 3.8].

Any triple (x,y,z) of distinct extremal elements from E with $[x,z]=0$, and $[x,y]\neq 0\neq [y,z]$ is called a *symplectic triple.* A symplectic triple of extremal points is a triple (x,y,z) such that there exist $x_1\in x$, $y_1\in y$ and $z_1\in z$ such that (x_1,y_1,z_1) is a symplectic triple of extremal elements.

The following theorem is Proposition 4.2 of [CF17]. Although it is stated in *loc. cit.* for $\mathrm{char}(k)\neq 2$, it holds in that case as well, using the Premet identities.

Proposition 2.4.4. *A symplectic triple (x,y,z) of extremal elements of the Lie algebra L generates either a subalgebra isomorphic to $\mathfrak{sp}_3(k)$, in which case it is of dimension 6, or to $\mathfrak{psp}_3(k)$ of dimension 5.*

Under this isomorphism x, y and z are mapped onto scalar multiples of pure tensors of $\mathfrak{sp}_3(k)$ or $\mathfrak{psp}_3(k)$, respectively.

For each $x\in\mathcal{E}$ we denote by $x^\perp$ the set $\mathcal{E}_{\leq 0}(x)$.

Lemma 2.4.5. *Consider a point z on the transversal on x and y in the subalgebra generated by a symplectic triple (x,u,y). Then $x^\perp\cap y^\perp\subseteq z^\perp$.*

Proof. By Example 2.4.3, x,y,z are contained in the linear span of x,y and $[x,[y,u]]$. Now if $v\in x^\perp\cap y^\perp$, then by associativity of g we have $g(v,[x,[y,u]]) = g([v,x],[y,u]) = g(0,[y,u]) = 0$. So $g(v,z)=0$ and $v\in z^\perp$. □

Lemma 2.4.6. *Assume* $\mathrm{char}(k)\neq 2$. *Let $x,y,z\in E$ be linearly independent and such that $g(x,y)$, $g(x,z)$ and $g(y,z)$ are all non-zero. If there is no extremal* *element $u \in \langle x, y\rangle$ commuting with z, then there exists a quadratic extension $\hat{k}$ of k such that $L \otimes_k \hat{k}$ contains extremal lines.*

Proof. Suppose x, y and $z \in E$ such that $g(x,y)$, $g(x,z)$ and $g(y,z)$ are all non-zero. Moreover, assume that there is no extremal element $u \in \langle x, y\rangle$ commuting with z. Note that the elements $u_\lambda = g(x,y)x + \lambda^2 y + \lambda[x,y] = \exp(x, 1/\lambda)(\lambda^2 y)$, where $\lambda \in k$, are extremal. Now

$$g(z, u_\lambda) = g(x,y)g(z,x) + \lambda^2 g(z,y) + \lambda g(z, [x,y])$$

either takes the value 0 for some value $\lambda = \lambda_1 \in k$ and we find that u_{λ_1} does not commute with z but $g(z, u_{\lambda_1}) = 0$ (case (d) of Proposition 2.3.7), which implies $\mathcal{E}_{-1} \neq \emptyset$, a contradiction, or, as the characteristic of k is different from 2, we find two distinct elements λ_1 and λ_2 in a quadratic extension $\hat{k}$ of k with $g(z, u_{\lambda_1}) = g(z, u_{\lambda_2}) = 0$.

Suppose we are in the latter case. Then inside $L \otimes_k \hat{k}$ we find the following. As $\langle u_{\lambda_1}, u_{\lambda_2}\rangle$ contains $\langle x, y\rangle$, the element z cannot commute with both u_{λ_1} and u_{λ_2}, which then implies that z and at least one of u_{λ_1} and u_{λ_2} are in relation (d) of Proposition 2.3.7. But then $L \otimes_k \hat{k}$ contains extremal lines. □

Now we are in a position to prove the main result of this section.

Theorem 2.4.7. *Let L be a simple Lie algebra over the field $k \neq \mathbb{F}_2$ generated by its set of pure extremal elements such that any two extremal elements x and y in L either commute or generate an $\mathfrak{sl}_2(k)$.*

If $\mathrm{char}(k) \neq 2$, then either L is isomorphic to the finitary symplectic Lie algebra $\mathfrak{fsp}(V, f)$ for some non-degenerate symplectic space (V, f), or there is a quadratic Galois extension $\hat{k}$ of k such that the Lie algebra $\hat{L} := L \otimes_k \hat{k}$ is a simple Lie algebra generated by its pure extremal elements and with its extremal geometry $\Gamma := \Gamma(\hat{L})$ being a root filtration space.

In the latter case, also if $\mathrm{char}(k) = 2$, the extremal points of the Lie algebra L form the point set of the geometry Γ_σ, which is a non-degenerate thick polar space. Here σ denotes the automorphism of Γ induced by the unique field automorphism of order 2 of the extension $\hat{k}$ of k.

Proof. Assume $\mathrm{char}(k) \neq 2$. By Theorem 2.4.2 we either have that L is a finitary symplectic Lie algebra as in the theorem, or by Lemma 2.4.6 there is a quadratic field extension $\hat{k}$ of k such that the Lie algebra $\hat{L} := L \otimes_k \hat{k}$ is generated by its extremal elements and its extremal geometry contains lines. From now on we consider the latter case, and allow $\mathrm{char}(k) = 2$ again.

Let σ denote the automorphism of Γ induced by the unique field automorphism of order 2 of the extension $\hat{k}$ of k. Then σ induces an automorphism of $\hat{L}$, also denoted by σ, that acts on $x \otimes \lambda$ by $(x \otimes \lambda)^\sigma = x \otimes \lambda^\sigma$. Note that σ does not fix a line in $\hat{L}$, otherwise the extremal geometry in L would contain lines (using

Lemma 2.5.9). We identify L and its elements with the subalgebra and elements of $\hat{L}$ fixed by σ.

We claim that $\hat{L}$ is simple. Let I be a non-trivial ideal of $\hat{L}$, and suppose $0 \neq i \in I$. Then, as the extremal elements in E linearly span L, we find that over $\hat{k}$ they also span $\hat{L}$. So, we can express i as $x_1 \otimes \lambda_1 + \cdots + x_k \otimes \lambda_k$ where $x_j \in E$ are linearly independent and $0 \neq \lambda_j \in \hat{k}$. After replacing i with a scalar multiple, we can assume $\lambda_1 + \lambda_1^\sigma \neq 0$. Then $i + i^\sigma$ is a non-trivial element in $I + I^\sigma$ fixed by σ. But then the subspace spanned by the elements in $I + I^\sigma$ fixed by σ forms a non-trivial ideal of L, which by simplicity of L equals L. This implies that $I + I^\sigma = \hat{L}$. Since both I and I^σ are ideals, $I \cap I^\sigma$ is an ideal which is stabilized by σ. Then Lemma 2.5.9, together with the simplicity of L implies that if $I \neq \hat{L}$, then $I \cap I^\sigma = 0$. So we may assume $L = I \oplus I^\sigma$. Since $[I, I^\sigma] \leq I \cap I^\sigma$, $[I, I^\sigma] = 0$ and thus $[\hat{L}, \hat{L}] = [I, I] \oplus [I^\sigma, I^\sigma]$. Since L is simple, $L = [L, L]$ and thus $[I, I] = I$. Let $\hat{g}$ be the extremal form on $\hat{L}$. For $x \in E$ we can find (up to switching I and I^σ) an element $y \in I$ with $\hat{g}(x, y) \neq 0$. By $I = [I, I]$, we can find $z_1, z_2 \in I$ such that $[z_1, z_2] = y$. Then (2.2) implies $x = g_x([z_1, z_2])x \in I$. Since $x \in E$ was arbitrary, E is contained in I. As E generates $\hat{L}$ we find $I = \hat{L}$, proving simplicity of $\hat{L}$.

Clearly σ also induces an automorphism (again denoted by σ) on the extremal geometry Γ of $\hat{L}$. The set $\mathcal{E}$ is the point set of Γ_σ. The relation $\perp$ denotes the relation of being equal or collinear in Γ_σ. By Proposition 2.2.3 we know that Γ_σ is a polar space. Moreover, as for each point $x \in \mathcal{E}$, the set $\mathcal{E}_2(x)$ is non-empty, no point is collinear to all and hence Γ_σ is non-degenerate. But in a non-degenerate polar space a line on two collinear points x, y equals $(x^\perp \cap y^\perp)^\perp$. Now as follows from Lemma 2.4.5, any transversal on x, y is contained in the line on x and y. As transversals contain $|k| \geq 3$ points, the line on x and y is thick, provided there is at least one transversal on x and y. This follows from the following observations.

Let $x = x_1, \ldots, x_n = y$ be a shortest path from x to y in the $\mathcal{E}_2$-graph. Such path exists by Proposition 2.3.9. If $n > 3$, then, according to Example 2.4.3 and Proposition 2.4.4, $x_1^\perp$ meets the $\mathfrak{sl}_2$-line on x_2 and x_3 in x_3, while $x_4^\perp$ meets this line in x_2. As such $\mathfrak{sl}_2$-line contains at least 3 points, we find a point on the line which is in relation $\mathcal{E}_2$ with both x_1 and x_4, contradicting that we have a shortest path from x to y. In particular we find that there is a symplectic triple on x and y and thus a transversal. $\square$

Example 2.4.8. Suppose L is a Lie algebra as in the hypothesis of Theorem 2.4.7, but not isomorphic to a finitary symplectic Lie algebra. If moreover, the extremal geometry Γ of $\hat{L} := L \otimes_k \hat{k}$ is isomorphic to $\Gamma(V, W^*)$ for some $\hat{k}$-vector space V and subspace W^* of V^* separating the points of the projective space of V, then it can be shown that the involution σ induces a Hermitian polarity on $\mathbb{P}(V)$ and that Γ_σ is isomorphic to the polar space of absolute points with respect to this polarity. See also Example 2.2.2. In this particular case we can identify the Lie algebra $\hat{L}$, up to a center, with the Lie algebra $\mathfrak{s}(V \otimes W^*)$ and L, up to a center, with the corresponding finitary unitary Lie algebra $\mathfrak{fsu}(V, h)$, where h is a Hermitian form with associated involution σ. This has been worked out by Marc Oostendorp and Hans Cuypers in [CO19].

SECTION 2.5

The geometry of inner ideals

In this section we assume that L is a simple Lie algebra over a field k generated by its set of pure extremal elements E. Its extremal geometry is denoted by Γ.

In the first few lemmas we allow the field k to be of characteristic 2.

Recall the definition of an inner ideal.

Definition 2.5.1. An *inner ideal* of L is a linear subspace I such that

$$[I, [I, L]] \subseteq I.$$

Extremal points are inner ideals.

An *inner line ideal* is an inner ideal I, properly contained in L, and containing at least two extremal points, which is minimal with respect to this property.

Definition 2.5.2. The *inner ideal geometry* of a Lie algebra L is the point-line geometry with $\mathcal{E}(L)$ as point set, and as lines the intersections of $\mathcal{E}(L)$ with the inner line ideals of L.

Notice that by the minimality of inner line ideals, this geometry is a partial linear space.

If $X \subset \mathcal{E}$ then we denote by L_X the Lie subalgebra of L generated by X.

Lemma 2.5.3. *Suppose ℓ is a line of Γ. Then the 2-dimensional subalgebra L_ℓ is an inner line ideal of L.*

Proof. Let $x, y \in E$ be two linearly independent elements from E such that $\langle x \rangle$ and $\langle y \rangle$ are points of ℓ. Then $[x, y] = 0$, and $x + y \in E$. So, for all $z \in L$ we have

$$[x + y, [x + y, z]] = [x, [x, z]] + 2[x, [y, z]] + [y, [y, z]]$$

and thus, if the characteristic is not 2,

$$[x, [y, z]] \in L_\ell.$$

This shows L_ℓ to be an inner ideal, which clearly is an inner line ideal.

Now assume that the characteristic is 2.[2] If $z \in \mathcal{E}_{\leq 0}(\langle x\rangle)$, then $[x,[y,z]] = [y,[x,z]] = 0$, by the Jacobi identity and $[x,y] = 0 = [x,z]$. If $z \in \mathcal{E}_1(\langle x\rangle)$, then $[x,[y,z]] = [y,[x,z]]$ as before. Since $[x,z] \in \mathcal{E}$ is a neighbor of $\langle x\rangle$, $\langle y\rangle$ and $[x,z]$ are at distance at most 2. If $[x,z] \in \mathcal{E}_{\leq 0}(\langle y\rangle)$ we get $[y,[x,z]] = 0$. If $\langle y\rangle$ and $[x,z]$ are special, then clearly $\langle x\rangle = [y,[x,z]]$. We conclude $[x,[y,z]] \leq \langle x\rangle$ for all $z \in \mathcal{E}_{\leq 1}(\langle x\rangle)$. Similarly, $[x,[y,z]] = [y,[x,z]]] \leq \langle y\rangle$ for all $z \in \mathcal{E}_{\leq 1}(\langle y\rangle)$. By Corollary 2.1.17 and Lemma 2.1.9(e) we get that $[x,[y,L]] \leq \langle x,y\rangle = L_\ell$, which concludes the proof. □

Lemma 2.5.4. *Suppose Γ contains lines. Then every inner line ideal equals L_ℓ for some line ℓ of Γ.*

Proof. Suppose I is an inner line ideal containing two points $x, y \in \mathcal{E}$. If $(x,y) \in \mathcal{E}_{-1}$, then by minimality $I = L_\ell$, where ℓ is the line on x and y.

If $(x,y) \in \mathcal{E}_1$, then let z be an element in $\mathcal{E}_2(x)$ collinear with y. Then $[x,y] = [x,[[x,y],z]] = [x,[y,[z,x]]]$ is in I, using the Jacobi identity and $[y,z] = 0$. The inner line ideal L_ℓ, where ℓ is the line through x and $[x,y]$, is thus contained in I. This contradicts the minimality of I.

If $(x,y) \in \mathcal{E}_0$, we can find $z \in \mathcal{E}_1(x) \cap \mathcal{E}_{-1}(y)$ by Lemma 2.1.9. Let S be the symplecton through x and y. Then $\mathcal{E}_{-1}(z) \cap S$ is a line ℓ through y. Hence $[x,z]$ is the unique point of ℓ collinear with x. By Lemma 2.1.9(e) there exists an extremal point u such that $[u,y] = z$. Hence I contains $[x,[y,u]] = [x,z] \in \ell$ and thus $L_l \leq I$, again contradicting the minimality.

Finally assume that $(x,y) \in \mathcal{E}_2$. Then there is a path x,u,z,y from x to y and $[u,y] = z$ and $[x,z] = u$. So $u = [x,[y,u]]$ is in I, which again leads to a contradiction. □

Lemma 2.5.5. *Suppose S is a symplecton of Γ. Then L_S is an inner ideal. Moreover, if I is an inner ideal containing two non-collinear points of S then $L_S \subseteq I$.*

Proof. Suppose $x, y \in \mathcal{E}$ are points of S. If x,y are collinear we find for each point $z \in \mathcal{E}$ that $[x,[y,z]] \subseteq L_S$. If $(x,y) \in \mathcal{E}_0$, then for each point $z \in \mathcal{E}$ we have, as $[x,y] = 0$,

$$[x,[y,z]] = -[y,[z,x]] = [y,[x,z]].$$

So, for all $z \in \mathcal{E}_{\leq 0}(x) \cup \mathcal{E}_{\leq 0}(y)$ we find

$$[x,[y,z]] = 0.$$

If $z \in \mathcal{E}_1(x)$, then $[x,z]$ is a point at distance 1 from x and $\mathcal{E}_{-1}([x,z]) \cap S$ contains a line ℓ. Then either $[y,[x,z]] = 0$, or $[x,z] \in \mathcal{E}_1(y)$ and $[y,[x,z]]$ is a

[2] The proof is actually valid for all characteristics but the argument in the previous paragraph is more elegant if the characteristic is not 2.

point on ℓ and thus in S. By symmetry of the argument we now have for all $z \in \mathcal{E}_{\leq 1}(x) \cup \mathcal{E}_{\leq 1}(y)$ that

$$[x, [y, z]] \subseteq L_S.$$

As $\mathcal{E}_0(y)$ contains a point z in $\mathcal{E}_2(x)$, by Lemma 2.1.9, and Γ and hence also L is generated by $\mathcal{E}_{\leq 1}(x) \cup \{z\}$, see Corollary 2.1.17, we find that

$$[x, [y, L]] \subseteq L_S.$$

Suppose I is an inner ideal containing the two non-collinear points x and y of S. By Lemma 2.1.9 there is a point $u \in \mathcal{E}_0(x) \cap \mathcal{E}_2(y)$.

Then Lemma 2.1.12 implies that each point $z \in S$ collinear with x and y has a common neighbor v with x and u. But then $v \in \mathcal{E}_1(y)$. Note that by Lemma 2.1.9(e) there exists $u \in \mathcal{E}$ collinear with v and special with x, hence $[x, u] = v$. Together with $[y, v] = z$ this yields $z = [y, v] = [y, [x, u]] \in I$. So any common neighbor of x and y is in I. Repeating this argument, we find that all points of S are in I and I contains L_S. □

Corollary 2.5.6. *Let P be a convex subspace of Γ such that for all $(x, y) \in P \times P$ we have $(x, y) \in \mathcal{E}_{\leq 0}$. Then L_P is an inner ideal.*

Lemma 2.5.7. *If I is an inner ideal containing two points x and y with $(x, y) \in \mathcal{E}_2$, then $I = L$.*

Proof. Suppose x, y are in I with $(x, y) \in E_2$. We may assume $g_x(y) = 1$. Now consider the 5-grading on L as in Lemma 2.3.20. Then I contains $[y, [x, L_1]] = L_1$ and similarly L_{-1}. Now note $[u, [v, [x, y]]] = -[u, v]$ for all $u \in L_{-1}$ and $v \in L_1$. Note that $L_{-2} \oplus L_{-1} \oplus ([L_{-1}, L_1] \oplus [L_{-2}, L_2]) \oplus L_1 \oplus L_2$ is an ideal of L and hence it equals L. We get $I = L$. □

From now on we assume the characteristic of k to be different from 2.

In the remaining lemmas we consider the case that there are no lines in Γ. As we have seen in the previous section, we either find L to be isomorphic to a finitary symplectic Lie algebra, or there is a quadratic field extension $\hat{k}$ of k such that $\hat{L} := L \otimes \hat{k}$ is a simple Lie algebra generated by its set of extremal elements $\hat{E}$ and the extremal geometry $\hat{\Gamma}$ contains lines.

We first handle the symplectic case.

Lemma 2.5.8. *Let L be $\mathfrak{s}_f(V \otimes V^*)$, where (V, f) is a non-degenerate symplectic space. If I is an inner line ideal, then there is a singular 2-dimensional subspace U of V such that I is the subspace spanned by the elements $v \otimes f_v$, with $0 \neq v \in U$.*

Proof. Suppose I contains $v \otimes f_v$ and $w \otimes f_w$ with v, w linearly independent. If $f(v, w) \neq 0$, then Lemma 2.5.7 applies. So assume that $f(v, w) = 0$. Consider an extremal element $u \otimes f_u$. Then

$$\begin{aligned}[v \otimes f_v, [w \otimes f_w, u \otimes f_u]] &= f(w,u)[v \otimes f_v, w \otimes f_u + u \otimes f_w] \\ &= f(w,u)f(v,u)\big(w \otimes f_v + v \otimes f_w\big) \\ &= f(v,u)f(w,u)\big((v+w) \otimes f_{v+w} - v \otimes f_v - w \otimes f_w\big).\end{aligned}$$

As the latter element is in $\langle t \otimes f_t \mid t \in \langle v, w\rangle\rangle$, the proposition follows immediately. (Note that there exist $u \in V$ such that $f(u,v) \neq 0 \neq f(u,w)$.) □

Denote by σ the field automorphism of order 2 of $\hat{k}$ fixing k as well as the induced automorphisms of $\hat{L}$ and its extremal geometry $\hat{\Gamma}$, such that $\Gamma = \hat{\Gamma}_\sigma$.

Lemma 2.5.9. *For any subspace I of $\hat{L}$ fixed by σ there exists a subspace J of L such that $J \otimes \hat{k} = I$.*

Proof. Let $\mathcal{B}$ be a basis of L. Consider $0 \neq a \in I$ such that a^σ is a $\hat{k}$-multiple of a. We can write a uniquely as $b_1 \otimes \lambda_1 + \cdots + b_n \otimes \lambda_n$, for non-zero $\lambda_1, \ldots, \lambda_n \in \hat{k}$ and $b_1, \ldots, b_n \in \mathcal{B}$, with all b_i different from each other. By considering $a' = \lambda_1^{-1} a$ instead of a and using the fact that a'^σ should be a multiple of a', we get $a'^\sigma = a'$ and $a' = x \otimes 1$, for some $x \in L$.

Consider $a \in I$ such that a^σ is not a multiple of a. Then $v = a + a^\sigma$ is non-zero and fixed by σ. Now consider an element $\lambda \in \hat{k}$ not fixed by σ. Set $w = \lambda a + \lambda^\sigma a^\sigma$, then $w^\sigma = w$. Moreover w is not a multiple of v by construction. Hence the subspace $\langle a, a^\sigma\rangle$ contains linearly independent elements $x_1 \otimes 1$ and $x_2 \otimes 1$, for some $x_1, x_2 \in L$.

Let $\mathcal{B}_I$ be a basis of I. If for $b \in \mathcal{B}_I$ we find b^σ to be a multiple of itself, we can replace it by a $b' \in \langle b\rangle$ which is fixed by σ by the first paragraph. If it is not a multiple of b then we can find an element of b' in $\langle b, b^\sigma\rangle$ which is fixed by σ and is linearly independent of the other basis elements by the second paragraph. Applying this procedure for any basis element, we see that we can assume that any b is fixed by σ, and the k-subspace $J = \langle b \mid b \in \mathcal{B}_I\rangle$ satisfies $I = J \otimes \hat{k}$. □

Corollary 2.5.10. *If I is an inner line ideal of L, then I meets $\mathcal{E}$ in the set of all the points of a line of $\hat{\Gamma}_\sigma$.*

Proof. Let x and y be two points in I. If $(x,y) \in \mathcal{E}_0$, then the minimal inner ideal of $\hat{L}$ containing x, y is the subspace J of $\hat{L}$ spanned by all the extremal points of $\hat{L}$ inside the symplecton on x and y, by Lemma 2.5.5. Note that J is fixed by σ. But then Lemma 2.5.9 implies that I meets $\mathcal{E}$ in the points of S fixed by σ.

If $(x,y) \in \mathcal{E}_2$, then by Lemma 2.5.7 we find that the minimal inner ideal of $\hat{L}$ containing x and y is $\hat{L}$, which implies $I = L$, contradicting that I is a proper inner ideal of L. □

Combining all results from the above we find the following.

Theorem 2.5.11. *Suppose L is a simple Lie algebra generated by pure extremal elements over a field of characteristic not 2.*

Then we have one of the following:

(a) *The extremal geometry Γ of L contains lines and equals the inner ideal geometry, which is then a root filtration space.*
(b) *The extremal geometry Γ of L contains no lines, but L contains two commuting, but linearly independent extremal elements; the inner ideal geometry is a non-degenerate polar space of rank at least 2.*
(c) *The Lie algebra L does not contain commuting, but linearly independent extremal elements, and the inner ideal geometry has no lines.*

Proof. If the extremal geometry Γ of L does contain lines, then Lemmas 2.5.3 and 2.5.4 and Theorem 2.3.14 imply that its inner ideal geometry equals Γ, which is then a root filtration space.

So, we can assume that Γ contains no lines. Then, by Theorem 2.4.7, either L is isomorphic to a symplectic Lie algebra or there is a quadratic field extension $\hat{k}$ of k such that the extremal geometry $\hat{\Gamma}$ of $\hat{L} := L \otimes_k \hat{k}$ does have lines.

If L is isomorphic to $\mathfrak{fsp}(V, f)$ for some non-degenerate symplectic space (V, f) of dimension at least 2, then Lemma 2.5.8 shows that the inner ideal geometry is isomorphic to the non-degenerate polar space defined by (V, f). If the dimension is 2, then $L \simeq \mathfrak{sl}_2$ has no inner line ideals. If the dimension is at least 3 (and hence at least 4), then the inner ideal geometry is isomorphic to the symplectic polar space.

If L is not a symplectic Lie algebra, then the point sets of Γ and $\hat{\Gamma}_\sigma$ can be identified, where σ is the automorphism induced on $\hat{\Gamma}$ by the unique field automorphism of $\hat{k}$ of order 2 fixing k. Applying Corollary 2.5.10 we find that the inner ideal geometry is isomorphic with $\hat{\Gamma}_\sigma$, which is a non-degenerate polar space by Theorem 2.4.7 containing lines if and only if L contains two distinct but commuting extremal points. $\square$

CHAPTER 3

From structurable algebras to Moufang buildings

We can summarize the main results of this chapter as follows.

Main Theorem 3.0.1. *Let $\mathcal{A}$ be a structurable algebra and let $\mathcal{G}$ be the poset of all proper non-trivial inner ideals of the TKK Lie algebra associated to $\mathcal{A}$.*

(i) *If $\mathcal{A}$ is a structurable division algebra, then $\mathcal{G}$ forms a Moufang set (Theorems 3.2.9 and 3.3.4).*
(ii) *If $\mathcal{A} = D \oplus D^{\mathrm{op}}$ for some alternative division algebra D, equipped with the exchange involution, then $\mathcal{G}$ forms the incidence graph of a dual double of a Moufang triangle (Theorem 3.4.19).*
(iii) *If $\mathcal{A} = M(J, 1)$ for some cubic Jordan division algebra J, then $\mathcal{G}$ forms the incidence graph of a Moufang hexagon (Theorems 3.5.11 and 3.5.15).*

In each case, we also describe the root groups of these geometries in terms of the associated structurable algebras.

Notice that in part (ii) of our main theorem, we obtain the Moufang triangles only via their dual double (which is a thin generalized hexagon). Indeed, it is impossible to construct Moufang triangles directly as the geometry $\mathcal{G}$ for some structurable algebra. See Remark 3.4.1 below.

In the previous chapter we discussed extremal geometries, and also extended the concept to allow for polar spaces. However, if there are no one-dimensional inner ideals, then the extremal geometry is empty. In these cases, one can take the *minimal proper inner ideals* as points. Only some of our constructions are examples of extremal geometries: in Sections 3.1, 3.2 and 3.4 we describe geometries which are not extremal geometries.

The correspondence between inner ideals of a Jordan algebra and inner ideals of its TKK Lie algebra is obvious. Conversely, by [FL19, Section 12.5] any abelian inner ideal of a Lie algebra is *Jordan isomorphic* to an inner ideal of a Jordan algebra. In [Gar01], Skip Garibaldi defines inner ideals in structurable algebras of skew-dimension one[1], and this is also what we will use in Section 3.5. To the best of our knowledge, inner ideals in structurable algebras in general have not yet been considered in the literature, but we can do without such a notion in Sections 3.2 to 3.4. Our rather technical proof of Proposition 3.5.4 below suggests that it is not obvious to show that inner ideals of an arbitrary structurable algebra of skew-dimension one correspond to inner ideals of its TKK Lie algebra. (We only prove this for a restricted class of structurable algebras of skew-dimension one.) For arbitrary structurable algebras (not necessarily of skew-dimension one), it seems likely that a more restrictive definition than the one of Skip Garibaldi will be needed, but we have not pursued this.

Recently, some deep connections between structurable algebras and low rank geometries have been discovered. More precisely, Lien Boelaert, Tom De Medts and Anastasia Stavrova have established a strong relationship between structurable division algebras and Moufang sets [BDMS19]. In Sections 3.2 and 3.3, we translate this result to the setting of inner ideals of Lie algebras of relative rank one. This settles the case for rank 1 geometries.

We now briefly discuss the organization of this chapter.

In Section 3.1 we prove, among other things, that we can reduce the situation to inner ideals containing $\mathcal{S}_+$. We then apply this reduction theorem in all other sections, except in Section 3.2.

We then proceed to prove our main theorem. We prove (i) in Sections 3.2 and 3.3, we prove (ii) in Section 3.4 and we finally prove (iii) in Section 3.5. In each case, we include the description of the root groups in terms of the corresponding structurable algebra.

Assumption 3.0.2. We assume in this chapter that all our algebras are finite-dimensional over a field k of characteristic different from 2 and 3. Except for Lie algebras, all algebras in this chapter are assumed to be unital, but not necessarily associative.

[1] Interestingly, Garibaldi shows in [Gar01, §7] that some of the inner ideals of a Brown algebra are related to a building of type E_7, which is a result in the spirit of the current paper.

SECTION 3.1

Reduction to inner ideals containing $\mathcal{S}_+$

In this section, we will provide a tool to reduce our study of inner ideals of $K(\mathcal{A})$ to those containing the 2-component $\mathcal{S}_+$. We will need one technical condition (Assumption 3.1.10), but as we will see, this assumption will be satisfied in all situations we are interested in; see Theorems 3.4.9 and 3.5.3 below.

Assumption 3.1.1. Throughout this section, we will assume that $\mathcal{A}$ is a central simple structurable algebra over k such that $\mathcal{S} \neq 0$ and *all non-zero elements of* $\mathcal{S}$ *are conjugate invertible.*

Remark 3.1.2. Recall that by Corollary 1.1.73, all proper inner ideals of $K(\mathcal{A})$ are abelian. We will use this fact repeatedly in this section without explicitly mentioning it.

Remark 3.1.3. It will be obvious that if we replace $\mathcal{S}_+$ by $\mathcal{S}_-$ and $\mathcal{A}_+$ by $\mathcal{A}_-$ in any statement in this section, the statement remains valid.

Remark 3.1.4. Let $\mathcal{B}$ be an arbitrary central simple structurable algebra. In many cases, we can find a central simple structurable algebra $\mathcal{A}$ satisfying Assumption 3.1.1 such that $K(\mathcal{B}) \cong K(\mathcal{A})$. In other words, Assumption 3.1.1 ensures that we choose a 'good' model of the Lie algebra.

We now make a few comments about when we can and when we cannot rechoose the structurable algebra such that Assumption 3.1.1 is satisfied.

Assume that $K(\mathcal{B})$ contains (at least) one pure extremal element. Then, along similar lines as the proof of Proposition 4.2.2, it follows that $K(\mathcal{B})$ is generated by extremal elements. Hence Theorem 4.2.7 shows that in this case, $K(\mathcal{B}) \cong K(\mathcal{A})$ for a certain skew-dimension one structurable algebra $\mathcal{A}$ if and only if $K(\mathcal{B})$ is not a symplectic Lie algebra.

The condition for $K(\mathcal{B})$ to contain a pure extremal element is often satisfied. For example, every possible non-division central simple structurable algebra $\mathcal{B}$ such that $K(\mathcal{B})$ is of *exceptional* type satisfies this condition. (Notice that Assumption 3.1.1 is of course satisfied if $\mathcal{B}$ is a division algebra.) On the other hand, the previous paragraph shows that for a symplectic Lie algebra, we *cannot* find a model such that Assumption 3.1.1 is satisfied. However, in that case we recover a Kantor pair and one could, in principle, try to generalize the theory in this section to Kantor pairs to resolve this issue.

Remark 3.1.5. Recall that the *i-component* of $x \in L$ is the image of the projection of x onto L_i, where $L_i = K(\mathcal{A})_i$.

Lemma 3.1.6. *If I is an inner ideal of $K(\mathcal{A})$ containing s_+, with $s \in \mathcal{S}$ non-zero, then $\mathcal{S}_+ \leq I$.*

Proof. Note that I contains $[s_+, [s_+, t_-]] = -2sts_+$ for any $t \in \mathcal{S}$, using (1.11). Since s is conjugate invertible, we get $\mathcal{S}_+ \leq I$. $\square$

Our goal is to show that any proper non-trivial inner ideal of $L := K(\mathcal{A})$ can be mapped by an element of $E(\mathcal{A})$ to an inner ideal containing the set of skew elements $\mathcal{S}_+$.

Lemma 3.1.7. *Let I be a non-trivial inner ideal of L. Then there exists an automorphism in $E(\mathcal{A})$ mapping I to an inner ideal containing an element with non-zero 2-component.*

Proof. By Lemma 1.1.70 the subspace $\sum_{\varphi \in E(\mathcal{A})} \varphi(I)$ is a non-zero ideal of L. Consider $0 \neq s \in \mathcal{S}$ arbitrary, since L is non-degenerate, $[s_+, [s_+, \varphi(I)]] \neq 0$ for some $\varphi \in E(\mathcal{A})$. This implies that $\varphi(I)$ is an inner ideal containing an element with non-zero 2-component. $\square$

Lemma 3.1.8. *Let I be a minimal inner ideal of L containing an element with non-zero 2-component. Then $I \cap (L_{-2} \cup L_{-1} \cup L_0 \cup L_1) = 0$.*

Proof. By assumption I contains an element x with non-zero 2-component $s_+ \in \mathcal{S}_+$. By Lemma 1.1.74, $[x, [x, \mathcal{S}_-]] \leq I$ is an inner ideal. Now note that the 2-component of $[x, [x, t_-]]$ equals $-2sts_+$, which is non-zero if $t \in \mathcal{S}$ is non-zero. Hence $I = [x, [x, \mathcal{S}_-]]$ and $I \cap (L_{-2} \cup L_{-1} \cup L_0 \cup L_1) = 0$. $\square$

Lemma 3.1.9. *Let I be a minimal inner ideal of L containing an element with non-zero 2-component. Then there exists an element of $E_-(\mathcal{A})$ mapping I to $\mathcal{S}_+$.*

Proof. By assumption, I contains an element $x := t_- + b_- + V + a_+ + s_+$, with $a, b \in \mathcal{A}$, $V \in \mathrm{Instrl}(\mathcal{A})$, $s, t \in \mathcal{S}$, with $s \neq 0$. Then I also contains

$$\begin{aligned}
[x, [x, -\hat{s}_-]] &= [x, -V^{\epsilon\delta}(\hat{s})_- + (\hat{s}a)_- + \mathrm{id}] \\
&= \big(-(V^{\epsilon\delta})^2(\hat{s})_- + (V^{\epsilon\delta}(\hat{s})a)_- - L_s L_{V^{\epsilon\delta}(\hat{s})}\big) \\
&\qquad + \big(\psi(b, \hat{s}a)_- + V^{\epsilon}(\hat{s}a)_- + V_{a,\hat{s}a} - a_+\big) \\
&\qquad + \big(2t_- + b_- - a_+ - 2s_+\big) \\
&= \big(2t + \psi(b, \hat{s}a) - (V^{\epsilon\delta})^2(\hat{s})\big)_- + \big(b + V^{\epsilon}(\hat{s}a) + V^{\epsilon\delta}(\hat{s})a\big)_- \\
&\qquad + \big(V_{a,\hat{s}a} - L_s L_{V^{\epsilon\delta}(\hat{s})}\big) - 2a_+ - 2s_+.
\end{aligned}$$

By adding twice the element x, we find that I contains

$$\begin{aligned}
y := \big(4t + \psi(b, \hat{s}a) - (V^{\epsilon\delta})^2(\hat{s})\big)_- &+ \big(3b + V^{\epsilon}(\hat{s}a) + V^{\epsilon\delta}(\hat{s})a\big)_- \\
&+ \big(2V + V_{a,\hat{s}a} - L_s L_{V^{\epsilon\delta}(\hat{s})}\big).
\end{aligned}$$

Since $y \in L_{-2} \oplus L_{-1} \oplus L_0$, Lemma 3.1.8 now implies that $y = 0$. Set

$$s' = -\tfrac{1}{2} V^{\epsilon\delta}(\hat{s}) \quad \text{and} \quad a' = \hat{s}a.$$

Expressing that the 0-component of y is 0 yields

$$V = \tfrac{1}{2}(-V_{a,\hat{s}a} + L_s L_{V^{\epsilon\delta}(\hat{s})}) = \tfrac{1}{2}V_{sa',a'} - L_s L_{s'}, \tag{3.1}$$

using $a = -sa'$. Now we use this equation, the formulas in Definition 1.1.48 and the fact that the (-1)-component of y equals 0 to get

$$\begin{aligned} 3b &= -V^{\epsilon}(\hat{s}a) - V^{\epsilon\delta}(\hat{s})a \\ &= \big(\tfrac{1}{2}V_{a',sa'}(a') - s'(sa')\big) + \big(-\tfrac{1}{2}\psi(a', \hat{s}(sa')) - s'(s\hat{s}) - \hat{s}(ss')\big)a \\ &= \tfrac{1}{2}V_{a',sa'}(a') - 3s'(sa'), \end{aligned}$$

hence

$$b = \tfrac{1}{6}V_{a',sa'}(a') - s'(sa'). \tag{3.2}$$

Finally, expressing that the (-2)-component of y equals 0 yields, using (3.1) and (3.2), that

$$\begin{aligned} 4t &= -2V^{\epsilon\delta}(s') + \psi(a', b) \\ &= -\psi(a', s'(sa')) - 4s'(ss') + \psi\big(a', \tfrac{1}{6}V_{a',sa'}(a') - s'(sa')\big) \\ &= -4s'(ss') - 2\psi(a', s'(sa')) + \tfrac{1}{6}\psi(a', U_{a'}(sa')). \end{aligned}$$

Hence

$$t = -s'(ss') - \tfrac{1}{2}\psi(a', s'(sa')) + \tfrac{1}{24}\psi(a', U_{a'}(sa')). \tag{3.3}$$

By (3.1) to (3.3) and Lemma 1.1.76, we conclude that $x = e_-(a', s')(s_+)$.

Hence $J := e_-(a', s')^{-1}(I)$ is an inner ideal containing $s_+ \in \mathcal{S}_+$. Since s is non-zero and I is minimal, Lemma 3.1.6 implies $\mathcal{S}_+ = J$. ☐

Assumption 3.1.10. Consider the following (technical) assumption:

> If $V \in \operatorname{Instrl}(\mathcal{A})$ is such that $V^{\delta}(\mathcal{S}) = 0$ and $V^2 = 0$, then there exists some $W \in \operatorname{Instrl}(\mathcal{A})$ such that $U := [V, [V, W]] \in \operatorname{Instrl}(\mathcal{A})$ satisfies $U^2 \neq 0$. In particular, $U \notin \langle V \rangle$.

We will always explicitly mention when we make this assumption.

Remark 3.1.11. Our motivation for Assumption 3.1.10 is that it basically ensures that we cannot get a generalized quadrangle as the point-line geometry we will obtain in the next sections. We now explain this in some detail.

Assume that there are two types of non-trivial proper inner ideals in $K(\mathcal{A})$, say of dimension m and n, with $m < n$, and assume that the point-line space with as points the m-dimensional inner ideals and as lines the n-dimensional inner ideals is a generalized quadrangle.

Since every non-zero element of $\mathcal{S}$ is conjugate invertible, $\mathcal{S}_+$ is a minimal inner ideal by Lemma 3.1.6, hence $m = \dim(\mathcal{S})$. So $\mathcal{S}_+$ and $\mathcal{S}_-$ are two points of the generalized quadrangle. Since $[s_+, \hat{s}_-] = L_s L_{\hat{s}} = -\operatorname{id} \neq 0$, these two points are not collinear (since collinear points lie in a common n-dimensional inner ideal, which is abelian). Hence they should be at distance 2 and have a common neighbor, say I. Then I is an inner ideal of $K(\mathcal{A})$ of dimension m. Since $\mathcal{S}_+$ and I are collinear, they are contained in a common proper inner ideal, which is abelian, hence $[\mathcal{S}_+, I] = 0$. Using that all non-zero elements of $\mathcal{S}$ are conjugate invertible, we get $I \leq \mathrm{Instrl}(\mathcal{A}) \oplus \mathcal{A}_+ \oplus \mathcal{S}_+$. Similarly, the fact that $\mathcal{S}_-$ and I are collinear implies $I \leq \mathcal{S}_- \oplus \mathcal{A}_- \oplus \mathrm{Instrl}(\mathcal{A})$ and hence $I \leq \mathrm{Instrl}(\mathcal{A})$. Consider any non-zero $V \in I$. Since $I \leq \mathrm{Instrl}(\mathcal{A})$, $[V, [V, a_+]] = V^2(a)_+$ for all $a \in \mathcal{A}$ implies $V^2 = 0$. Using $[I, \mathcal{S}_+] = 0$ we get $V^\delta(\mathcal{S}) = 0$.

However, for each $W \in \mathrm{Instrl}(\mathcal{A})$, we now have $U := [V, [V, W]] \in I$, so by the previous observations $U^2 = 0$. Hence $\mathcal{A}$ does not satisfy Assumption 3.1.10.

Lemma 3.1.12. *Let I be a proper inner ideal properly containing $\mathcal{S}_+$. Then $I = I_0 \oplus I_1 \oplus \mathcal{S}_+$ for some inner ideal $I_0 \leq \mathrm{Instrl}(\mathcal{A})$ and some subspace $0 \neq I_1 \leq \mathcal{A}_+$. If, in addition, $\mathcal{A}$ satisfies Assumption 3.1.10, then $I_0 = 0$.*

Proof. Consider an arbitrary element $x := u_- + b_- + V + a_+ + t_+ \in I$. Choose some $0 \neq s \in \mathcal{S}$; then by assumption, $s_+ \in I$. Since I is abelian, $[s_+, x] = 0$, which implies $[s_+, b_-] = 0$ and $[s_+, u_-] = 0$. Since s is conjugate invertible, $(sb)_+ = [s_+, b_-] = 0$ implies $b = 0$ and $L_s L_u = [s_+, u_-] = 0$ implies $u = 0$. Hence $I \leq L_0 \oplus L_1 \oplus L_2$.

Note that $a_+ + 2t_+ = -[x, \mathrm{id}] = [x, [s_+, \hat{s}_-]] \in I$. Together with $t_+ \in I$ and $V + a_+ + t_+ = x \in I$ this implies $V \in I$ and $a_+ \in I$. Hence $I = I_0 \oplus I_1 \oplus \mathcal{S}_+$, with $I_0 \leq \mathrm{Instrl}(\mathcal{A})$ and $I_1 \leq \mathcal{A}_+$.

Consider V and W in I_0 arbitrary. Since I is abelian $[V, \mathcal{S}_+] = V^\delta(\mathcal{S}) = 0$. Moreover, since $I \leq L_0 \oplus L_1 \oplus L_2$, $[V, [W, L_i]] = 0$ for $i = -1, -2$. Hence Lemma 1.1.56 implies $[V, [W, L_1]] = 0$ and clearly $[V, [W, L_0]] \leq I_0$. Hence I_0 is indeed an inner ideal.

Consider $0 \neq V \in I_0$, then $V(sa)_+ = [V, [s_+, a_-]] \in I$ for all $a \in \mathcal{A}$ and $0 \neq s \in \mathcal{S}$. Since $V \neq 0$ and s is conjugate invertible, this implies $I_1 \neq 0$.

Now assume that $\mathcal{A}$ satisfies Assumption 3.1.10. Consider $0 \neq V \in I_0$. Note that the fact that I is abelian implies $V^\delta(\mathcal{S}) = 0$. For each $b \in \mathcal{A}$, $(V^\epsilon)^2(b)_- = [V, [V, b_-]] \in I \leq L_0 \oplus L_1 \oplus L_2$ must be 0; hence $(V^\epsilon)^2 = 0$. By Lemma 1.1.56, we get $V^2 = 0$ and by Assumption 3.1.10, there exists $W \in \mathrm{Instrl}(\mathcal{A})$ such that $U := [V, [V, W]]$ satisfies $U^2 \neq 0$. Hence $U \in I_0$. The previous argument shows that $U^\delta(\mathcal{S}) = 0$ and $(U^\epsilon)^2 = 0$ and hence, by Lemma 1.1.56, $U^2 = 0$, a contradiction. We conclude that $I \leq L_1 \oplus L_2 = \mathcal{A}_+ \oplus \mathcal{S}_+$. $\square$

Remark 3.1.13. If $\mathrm{char}(k) \neq 5$ (and still $\mathrm{char}(k) \neq 2, 3$), then [FL19, Proposition 5.29] shows a claim similar to Lemma 3.1.9. Under the same condition on the characteristic of the field, Proposition 11.56 of *loc. cit.* shows the same as Lemma 3.1.12 and moreover shows that if $I_0 = 0$ then I_1 is an inner ideal.

Theorem 3.1.14. *Let $\mathcal{A}$ be a central simple structurable algebra satisfying Assumption 3.1.1 and let I be a proper non-trivial inner ideal of $K(\mathcal{A})$.*

(i) *There exists an element of* $E(\mathcal{A})$ *mapping* I *to an inner ideal* J *such that* $\mathcal{S}_+ \leq J \leq \mathrm{Instrl}(\mathcal{A}) \oplus \mathcal{A}_+ \oplus \mathcal{S}_+$.

(ii) *If, in addition,* $\mathcal{A}$ *satisfies Assumption 3.1.10, then* $\mathcal{S}_+ \leq J \leq \mathcal{A}_+ \oplus \mathcal{S}_+$.

Proof. Let J be a minimal inner ideal contained in I. By Lemma 3.1.7, we can assume that J contains an element with a non-zero 2-component. By Lemma 3.1.9, we may then assume that J equals $\mathcal{S}_+$. Now Lemma 3.1.12 concludes the proof. □

The next result will not be used in this thesis, but it is a useful fact requiring only very little effort to prove at this point.

Proposition 3.1.15. *Let* $\mathcal{A}$ *be a central simple structurable algebra satisfying Assumption 3.1.1. Any proper non-trivial inner ideal of* $K(\mathcal{A})$ *is linearly spanned by its minimal inner ideals.*

Proof. We will prove this claim by induction on $\dim(I)$. When $\dim(I) = \dim(\mathcal{S})$, I itself is a minimal inner ideal by Theorem 3.1.14 and then the claim is obvious. Assume $\dim(I) > \dim(\mathcal{S}_+)$. By Theorem 3.1.14, we may assume that $\mathcal{S}_+ \leq I$. By Lemma 3.1.12, $I = I_0 \oplus I_1 \oplus \mathcal{S}_+$ for some inner ideal $I_0 \leq \mathrm{Instrl}(\mathcal{A})$ and some subspace $0 \neq I_1 \leq \mathcal{A}_+$. By induction, I_0 is spanned by its minimal inner ideals (where we allow $I_0 = 0$).

Let $a_+ \in I_1$ be arbitrary and let $s \in \mathcal{S} \setminus \{0\}$. Then $[a_+, [a_+, s_-]] = -V_{a,sa}$ is contained in I_0 and hence $[V_{a,sa}, [a_+, s_-]] = V_{sa,a}(sa)_- \in L_{-1}$ is contained in I and is thus 0. Together with Lemma 1.1.76 this gives $e_-(sa)(\hat{s}_+) = -\frac{1}{2}V_{a,sa} + a_+ + \hat{s}_+$. This is an element of the minimal inner ideal $e_-(sa)(\mathcal{S}_+)$. Since $\hat{s}_+$ is an element of a minimal inner ideal and $V_{a,sa} \in I_0$ is contained in the linear span of minimal inner ideals, a_+ is also contained in the linear span of minimal inner ideals. □

As was pointed out to us by Antonio Fernández López, this actually holds in a larger generality:

Proposition 3.1.16. *Let* $\mathcal{A}$ *be a central simple structurable algebra. Any proper non-trivial inner ideal of* $K(\mathcal{A})$ *is linearly spanned by its minimal inner ideals.*

Proof. By Corollaries 1.1.72 and 1.1.73, the Lie algebra $L := K(\mathcal{A})$ is non-degenerate and any proper inner ideal I of L is abelian. Hence any element $a \in I$ is ad-nilpotent of index 3. By [FL19, Theorem 8.43], there is a Jordan algebra $J := L_a$ attached to a. Moreover, J is non-degenerate by Proposition 8.51 of *loc. cit.* Since it is also finite-dimensional, the Capacity Existence Theorem [McC04, II.20.1.3] implies that J is unital. So by [FL19, Proposition 8.61], we have $a \in [a, [a, L]]$. Note that for each minimal inner ideal B of L, $[a, [a, B]]$ is either 0 or a minimal inner ideal of L by Corollary 4.20 of *loc. cit.* By the first line of the proof of Lemma 3.1.7, L is linearly spanned by its minimal inner ideals, so we see that a lies in the linear span of minimal inner ideals. □

Note, however, that by [Ben74, Lemma 2.8], $K(\mathcal{A})$ has a 5-grading such that the Jordan pair associated to the ends of the 5-grading is a division Jordan pair. So in the case that this particular 5-grading corresponds to a structurable algebra, one obtains that any proper non-trivial inner ideal is spanned by the minimal ones.

SECTION 3.2

Moufang sets with abelian root groups

We are now prepared to begin our investigation of the geometry of proper non-trivial inner ideals in specific situations. In Sections 3.2 and 3.3, we will deal with the case of structurable *division* algebras and we will show that this gives rise to Moufang sets. The case of Jordan division algebras will give rise to Moufang sets with abelian root groups (Section 3.2) whereas the structurable division algebras of skew-dimension > 0 will give rise to Moufang sets with non-abelian root groups (Section 3.3).

Throughout this section, we will assume that $\mathcal{A}$ is a Jordan division algebra and we denote $\mathcal{A}$ by J. We also write $a^{-1} := \hat{a}$, as is usual in Jordan theory. Note that J is commutative and hence $V_{x,y}z = V_{z,y}x$ for all $x, y, z \in J$. Set $L = K(J)$.

Remark 3.2.1. It is obvious, as in the previous section, that if we replace J_+ by J_- in any statement in this section, the statement remains valid.

Lemma 3.2.2. *Let I be an inner ideal of L. If $I \cap J_+ \neq 0$, then $J_+ \leq I$.*

Proof. Consider $0 \neq a_+ \in I \cap J_+$. The inner ideal I contains

$$[a_+, [a_+, b_-]] = [a_+, V_{a,b}] = -(V_{a,b}a)_+ = -(U_a b)_+,$$

for every $b \in J$. Since J is a Jordan division algebra, U_a is invertible and hence this implies $J_+ \leq I$. □

Lemma 3.2.3. *Let $x, y \in J \setminus \{0\}$. Then $V_{x,y}^2 \neq 0$ and $(V_{x,y}^\epsilon)^2 \neq 0$.*

Proof. Since J is a division algebra, the elements x and y are invertible. Then $V_{x,y}(y^{-1}) = V_{y^{-1},y}(x) = x$ and hence $(V_{x,y})^2(y^{-1}) = V_{x,y}(x) = U_x(y)$ which is non-zero since U_x is invertible; therefore $V_{x,y}^2 \neq 0$. Since $V_{x,y}^\epsilon = -V_{y,x}$, we also have $(V_{x,y}^\epsilon)^2 \neq 0$. □

Lemma 3.2.4. *Let I be an inner ideal of L with $J_- \oplus J_+ \leq I$. Then $I = L$.*

Proof. Consider $x, y \in J$ arbitrary. Then I contains

$$[y_-, [x_+, \mathrm{id}]] = [x_+, y_-] = V_{x,y}.$$

Hence $\mathrm{Instrl}(J) \leq I$ and our claim follows. $\square$

Lemma 3.2.5. *Let $0 \neq V \in \mathrm{Instrl}(J)$ be arbitrary. The only inner ideal of L containing V is L itself.*

Proof. Note that $V \neq 0$ implies $V^\epsilon \neq 0$. Suppose that I is an inner ideal containing V. For each $a \in J$, we then have

$$I \ni [V, [V, a_+]] = [V, V(a)_+] = V^2(a)_+.$$

Similarly, I also contains $(V^\epsilon)^2(a)_-$, for all $a \in J$. We distinguish the following cases:

- $V^2 \neq 0 \neq (V^\epsilon)^2$: By Lemma 3.2.2 we get $J_- \oplus J_+ \leq I$.
- $V^2 \neq 0 = (V^\epsilon)^2$: By Lemma 3.2.2 we get $J_+ \leq I$. So I contains x_+ and V and thus
$$[x_+, [V, x_-^{-1}]] = [x_+, V^\epsilon(x^{-1})_-] = V_{x,V^\epsilon(x^{-1})},$$
for any $0 \neq x \in J$. By Lemma 3.2.3 there exists an $x \in J$ such that $V_{x,V^\epsilon(x^{-1})} \neq 0$. By the first case and Lemma 3.2.3 we get $J_- \oplus J_+ \leq I$.
- $V^2 = 0 \neq (V^\epsilon)^2$: Similarly as in the previous case we get $J_- \oplus J_+ \leq I$.
- $V^2 = 0 = (V^\epsilon)^2$: For any $x, y \in J$ the inner ideal I contains
$$[V, [V, V_{x,y}]] = [V, V_{V(x),y} + V_{x,V^\epsilon(y)}] = 2V_{V(x),V^\epsilon(y)}.$$
By $V \neq 0 \neq V^\epsilon$ and Lemma 3.2.3 we get that $0 \neq V_{a,b} \in I$ for some $a, b \in J$. Again, by the first case and Lemma 3.2.3 we get $J_- \oplus J_+ \leq I$.

So, in any case, we get $J_- \oplus J_+ \leq I$. Hence, by Lemma 3.2.4 we have $I = L$. $\square$

Lemma 3.2.6. *Let I be a proper inner ideal containing $a_+ + V$, where $V \neq 0, a \in J$. Then $I = L$.*

Proof. If $a = 0$, the claim follows from Lemma 3.2.5. Note that I contains

$$[V + a_+, [V + a_+, \mathrm{id}]] = [V + a_+, -a_+] = (-V(a))_+.$$

Suppose $V(a) \neq 0$, we get by Lemma 3.2.2 that $J_+ \leq I$ and hence $0 \neq V \in I$ and thus, by Lemma 3.2.5, $I = L$. So we may assume $V(a) = 0$. By Lemma 1.1.55 we get $V^\epsilon(a^{-1}) = 0$. Hence, I contains

$$[V + a_+, [V + a_+, a_-^{-1}]] = [V + a_+, V^\epsilon(a^{-1})_- + \mathrm{id}] = -a_+,$$

and thus, as in the previous case, $J_+ \leq I$ and $I = L$. $\square$

Corollary 3.2.7. *There are only two inner ideals of L containing J_+, namely J_+ and L.*

Proof. Let I be an inner ideal containing J_+. If $J_- \cap I \neq 0$ we get $J_- \leq I$ by Lemma 3.2.2 and Lemma 3.2.4 yields $I = L$. If $J_- \cap I = 0$ and $I \neq J_+$, I contains an element $a_- + V$ with $V \neq 0$ and Lemma 3.2.6 yields $I = L$, a contradiction. □

Proposition 3.2.8. *Let I be a proper non-trivial inner ideal of L with $I \neq J_+$. Then there exists a unique $x \in J$ such that*

$$I = e_+(x)(J_-) = \left\{ b_- + V_{x,b} - \tfrac{1}{2} V_{x,b}(x)_+ \mid b \in J \right\}.$$

Moreover, for each $x \in J$, $e_+(x)(J_-)$ is an inner ideal.

Proof. Since J_- is clearly an inner ideal of L, we get that $e_+(x)(J_-)$ is an inner ideal for all $x \in J$, using $e_+(x) \in \mathrm{Aut}(L)$.

Consider an arbitrary proper non-trivial inner ideal $I \neq J_+$ of L and let $0 \neq b_- + V + a_+ \in I$ be arbitrary. If $b = 0$, then Lemma 3.2.6 yields $V = 0$. But this is also impossible since I then contains J_+ and is, by Corollary 3.2.7, equal to L. So $b \neq 0$ and I contains

$$\begin{aligned}
[b_- + V + a_+, [b_- + V + a_+, b_+^{-1}]] &= [b_- + V + a_+, -\mathrm{id} + V(b^{-1})_+] \\
&= -b_- - V_{V(b^{-1}),b} + (V^2(b^{-1}) + a)_+.
\end{aligned}$$

Hence, I contains $W + c_+ := (V - V_{V(b^{-1}),b}) + (V^2(b^{-1}) + 2a)_+$. If $W = 0$ and $c \neq 0$, Lemma 3.2.2 and Corollary 3.2.7 imply $I = L$, a contradiction. If $W \neq 0$, then Lemma 3.2.6 implies $I = L$ which yields a contradiction. Hence $W = 0$ and $c = 0$ and thus $V = V_{V(b^{-1}),b}$ and

$$a = -\tfrac{1}{2} V_{V(b^{-1}),b}(V(b^{-1})).$$

Hence $(e_+(V(b^{-1})))^{-1}(I) \cap J_- \neq 0$ and by Corollary 3.2.7 and Lemma 3.2.2, we get $(e_+(V(b^{-1})))^{-1}(I) = J_-$, or equivalently, $I = e_+(V(b^{-1}))(J_-)$.

The uniqueness claim follows since $V_{x,b} = V_{y,b}$ implies $V_{x-y,b} = 0$ and hence $x = y$, by Lemma 3.2.3. □

Theorem 3.2.9. *Let J be a Jordan division algebra. Then the set of all proper non-trivial inner ideals of $L = K(J)$ forms a Moufang set, with root groups*

$$\begin{aligned}
U_{J_+} &= E_+(J); \\
U_{e_+(j)(J_-)} &= E_-(J)^{e_+(j)}, \quad \text{for all } j \in J.
\end{aligned}$$

This Moufang set is isomorphic to the Moufang set $\mathbb{M}(J)$ as defined in [DMW06, §4].

Proof. Let X be the set of proper non-trivial inner ideals. By Proposition 3.2.8 we see that U_{J_+} acts sharply transitively on the set of all proper non-trivial inner ideals different from J_+. Similarly, U_{J_-} acts sharply transitively on the set of all proper non-trivial inner ideals different from J_-. Clearly $E_-(J)^{e_+(j)}$ fixes $e_+(j)(J_-)$ and acts sharply transitively on the set of all other proper non-trivial inner ideals, for $j \in J$ arbitrary. By definition of the root groups, we have $G^+ = \langle E_-(J), E_+(J) \rangle$. So in order to prove $U_x^g = U_{x.g}$ for all $g \in G^+$ and $x \in X$ it suffices to show this for all $g \in E_-(J)$ and $x \in X$, since it is clear for all $g \in E_+(J)$ by construction. By the last equation in [BDMS19, Theorem 5.1.1] we find, for each $0 \neq x \in J$, an element $y \in J$ such that $U_{J_-}^{e_+(x)} = U_{J_+}^{e_-(y)}$. Since $U_{J_+}^{e_-(y)}$ fixes $e_-(y)(J_+)$ we get $U_{e_-(y)(J_+)} = U_{J_-}^{e_+(x)} = U_{J_+}^{e_-(y)}$. Now it is clear that $U_x^g = U_{x.g}$ for all $g \in E_-(J)$ and $x \in X$ as well. The last claim now follows since the corresponding abstract rank one groups coincide; see [BDMS19, Lemma 1.1.12 and §5.1]. $\square$

SECTION 3.3

Moufang sets with non-abelian root groups

We continue with our investigation of structurable division algebras. Throughout this section, $\mathcal{A}$ is a central simple structurable division algebra with $\mathcal{S} \neq 0$. Notice that Assumption 3.1.1 is trivially satisfied. Set $L = K(\mathcal{A})$.

Lemma 3.3.1. *$\mathcal{A}$ satisfies Assumption 3.1.10.*

Proof. Consider $0 \neq V \in \mathrm{Instrl}(\mathcal{A})$ arbitrary with $V^2 = 0$ and $V^\delta(\mathcal{S}) = 0$. By Lemma 1.1.56 we get $(V^\epsilon)^2 = 0$. Consider $a \in \mathcal{A}$ arbitrary, with $a \neq 0$. Then, using $V^2(a) = 0$ and $(V^\epsilon)^2(\hat{a}) = 0$, we get

$$0 = [V, [V, \mathrm{id}]] = [V, [V, V_{a,\hat{a}}]] = 2V_{V(a),V^\epsilon(\hat{a})}.$$

Since $\mathcal{A}$ is division, Lemma 1.1.55 implies $V(a) = 0$ or $V^\epsilon(\hat{a}) = 0$. By the same lemma we get $V = 0$, a contradiction. Hence Assumption 3.1.10 is trivially satisfied. $\square$

Lemma 3.3.2. *The only inner ideal strictly containing $\mathcal{S}_+$ is L itself.*

Proof. Let I be a proper inner ideal containing $\mathcal{S}_+$ properly. By Lemma 3.1.12 there exists $0 \neq a_+ \in I$. Thus I also contains $[a_+, [a_+, b_-]] = U_a(b)_+$. Since $0 \neq a$ is conjugate invertible U_a is invertible, we get $\mathcal{A}_+ \leq I$. Hence $\psi(\mathcal{A}, \mathcal{A}) = [\mathcal{A}_+, \mathcal{A}_+] \leq [I, I] = 0$, we get a contradiction by Lemma 1.1.53. $\square$

Proposition 3.3.3. *If I is an inner ideal of L distinct from $\mathcal{S}_+$, then $I = e_+(a, s)(\mathcal{S}_-)$, for unique $a \in \mathcal{A}$, $s \in \mathcal{S}$.*

Proof. Consider $0 \neq a \in \mathcal{A}$ and $0 \neq s \in \mathcal{S}$ arbitrary. Since $\mathcal{A}$ is division, [BDMS19, Lemma 3.3.4(ii)] shows that $\psi(a, U_a(sa)) \neq 0$. By Theorem 3.1.14 and Lemmas 3.3.1 and 3.3.2 it suffices to show $E(\mathcal{A})(\mathcal{S}_+) = \{\mathcal{S}_+\} \cup E_+(\mathcal{A})(\mathcal{S}_-) =: S$. Since $E_+(\mathcal{A})(S) = S$ it suffices to show $E_-(\mathcal{A})(S) = S$.

Consider $a \in \mathcal{A}$ and $s \in \mathcal{S}$ arbitrary with $(a, s) \neq (0, 0)$. We show that $e_-(a, s)(\mathcal{S}_+) \in E_+(\mathcal{A})(\mathcal{S}_-)$. By Lemma 3.1.9 it suffices to show that $e_-(a, s)(\mathcal{S}_+)$ has an element with non-zero (-2)-component. Recall Lemma 1.1.76 for a precise description of the elements inside $e_-(a, s)(\mathcal{S}_+)$. Assume first $s = 0$, then $e_-(a, 0)(t_+)$ has (-2)-component $\frac{1}{24}\psi(a, U_a(ta))$ which is non-zero as soon as t is non-zero, as noted before. Assume now $a = 0$, then $e_-(0, s)(\hat{s}_+)$ has non-zero (-2)-component s. So we may assume $a \neq 0$ and $s \neq 0$. Then $e_-(a, s)(\hat{s}_+)$ has (-2)-component $s + \frac{1}{24}\psi(a, U_a(\hat{s}a))$. If this is non-zero we are done, hence assume it is zero. Note that the (-1)-component equals $b := a + \frac{1}{6}U_a(\hat{s}a)$. If this component is also 0, we would get $\frac{1}{24}\psi(a, U_a(\hat{s}a)) = \frac{1}{4}\psi(a, -a) = 0$ and hence $s = 0$, a contradiction. Now note that the inner ideal $e_-(a, s)(\mathcal{S}_+)$ contains $[e_-(a, s)(\hat{s}_+), [e_-(a, s)(\hat{s}_+), V_{tb,b}]]$, for $0 \neq t \in \mathcal{S}$. This element has (-2)-component $\psi(b, U_b(tb))$, which is non-zero. So clearly $E_-(\mathcal{A})(\mathcal{S}_+) \subseteq S$. This argument also shows that for any $a \in \mathcal{A}$ and $s \in \mathcal{S}$ with $(a, s) \neq (0, 0)$ we have $e_+(a, s)(\mathcal{S}_-) \in E_-(\mathcal{A})(\mathcal{S}_+)$. Together with $E_-(\mathcal{A})(\mathcal{S}_-) = \mathcal{S}_-$ this shows $E_-(\mathcal{A})(S) = S$.

The uniqueness claim follows from the fact that $ta = tb$ implies $a = b$ and $L_s L_t = L_{s'} L_t$ implies $s = s'$ for non-zero $t \in \mathcal{S}$, since t is conjugate invertible. □

Theorem 3.3.4. *Let $\mathcal{A}$ be a structurable division algebra with $\mathcal{S} \neq 0$. Then the set of all proper non-trivial inner ideals of $L = K(\mathcal{A})$ forms a Moufang set, with root groups*

$$\begin{aligned} U_{\mathcal{S}_+} &= E_+(\mathcal{A}); \\ U_{e_+(a,s)(\mathcal{S}_-)} &= E_-(\mathcal{A})^{e_+(a,s)}, \quad \text{for all } a \in \mathcal{A}, s \in \mathcal{S}. \end{aligned}$$

This Moufang set is isomorphic to the Moufang set $\mathbb{M}(\mathcal{A})$ as defined in [BDMS19, Theorem 5.1.6].

Proof. The proof of this claim is a *mutatis mutandis* copy of the proof of Theorem 3.2.9, replacing J with $\mathcal{A}$, $e_+(j)$ with $e_+(a, s)$ etc., using Proposition 3.3.3 in place of Proposition 3.2.8. □

SECTION 3.4

Moufang triangles

We now proceed to the next case, which will give rise to Moufang triangles, i.e., to Moufang projective planes. In fact, this case will be somewhat peculiar: the geometry we will obtain, will be the *dual double* of a projective plane (which is, in fact, a thin generalized hexagon). Likewise, the structurable algebras we will use will be the double of the division algebra coordinatizing the projective plane. (Notice, however, that they are equipped with an involution exchanging the two components; they are still central simple, as we will show below.)

Remark 3.4.1. Notice that it is impossible to find the Moufang triangles directly as the geometry of the poset of all inner ideals of the TKK Lie algebra of any central simple structurable algebra. Indeed, the inner ideals $\mathcal{S}_+$ and $\mathcal{S}_-$ would either both correspond to points or both to lines. In the former case, there cannot be a proper inner ideal containing both $\mathcal{S}_+$ and $\mathcal{S}_-$ (since such an inner ideal would have to be abelian). In the latter case, the intersection of $\mathcal{S}_+$ and $\mathcal{S}_-$ is trivial and hence there would be no point incident with both lines. (A similar argument applies if the structurable algebra is a Jordan algebra.)

Construction 3.4.2. Let F be an alternative division algebra over a field of characteristic different from 2 and 3. Set $k = Z(F)$. Then $\mathcal{A} := F \oplus F$ is a k-algebra, with multiplication

$$(a, b).(c, d) = (ac, db),$$

for all $a, b, c, d \in F$. This algebra has the involution

$$(x, y) \mapsto (y, x).$$

In particular, the subspace of skew elements is

$$\mathcal{S} = \{(x, -x) \mid x \in F\}. \tag{3.4}$$

Set $L = K(\mathcal{A})$.

Lemma 3.4.3 ([TW02, (9.15), (9.22)], [Sch66, (3.4)])**.** *For any alternative algebra F, the following identities hold:*

(i) $[e_{\sigma(1)}, e_{\sigma(2)}, e_{\sigma(3)}] = \operatorname{sgn}(\sigma)[e_1, e_2, e_3]$, *for all $e_1, e_2, e_3 \in F$ and $\sigma \in \operatorname{Sym}(3)$.*
(ii) $e^{-1}(ef) = f = (fe)e^{-1}$, *for all $e \in F^\times$ and all $f \in F$.*
(iii) $(fef)g = f(e(fg))$, *for all $e, f, g \in F$.*

Lemma 3.4.4. *$\mathcal{A}$ is a central simple structurable algebra over k.*

Proof. In principle, this follows by showing that $\mathcal{A}$ is isomorphic to one of the known central simple structurable algebras (distinguishing between whether F is associative or not and relying on the classification of alternative division algebras), but we believe that the following short and direct proof is instructive. However, see Remark 3.4.5 below.

By [All78, Theorem 13] and Lemma 3.4.3 it suffices to show $D_{a^2,a}(\mathcal{H}) = 0$ for all $a \in \mathcal{H}$. (Recall that $\mathcal{H}$ denotes the set of hermitian elements with respect to the involution $^-$ of $\mathcal{A}$.) Since F is power-associative, $[a, a^2] = 0$. Hence $D_{(e,e),(e,e)^2}(f, f) = 2[(e, e), (e^2, e^2), (f, f)]$, for all $e, f \in F$. Now $e(e^2 f) = e(e(ef)) = (ee^2)f$ by Lemma 3.4.3. Hence $D_{a^2,a}(\mathcal{H}) = 0$ and $\mathcal{A}$ is a structurable algebra.

Since $k = Z(F)$, we get $Z(\mathcal{A}) \leq k \oplus k$. The fact that $Z(\mathcal{A})$ is contained in $\mathcal{H}$ shows $Z(\mathcal{A}) = k1$. Let I be an ideal of $\mathcal{A}$. Consider $0 \neq (e, f) \in I$ arbitrary. If $e \neq 0$, then e is invertible and hence $(g, 0) = (ge^{-1}, 0)(e, f) \in I$, for all $g \in F$, by Lemma 3.4.3. Since I is closed under the involution, we get $I = \mathcal{A}$. Similarly if $f \neq 0$. $\square$

Remark 3.4.5. If F is a *composition* (division) algebra, then $\mathcal{A}$ is isomorphic to $F \otimes (k \oplus k)$, where $k \oplus k$ is the split 2-dimensional composition algebra with involution $(a, b) \mapsto (b, a)$. Indeed, mapping $(e, f) \in \mathcal{A}$ to the element $e \otimes (1, 0) + \overline{f} \otimes (0, 1)$ of $F \otimes (k \oplus k)$ yields an isomorphism of structurable algebras.

If F is an *associative* (division) algebra, then $\mathcal{A}$ is a central simple associative algebra with involution, and hence belongs to one of the (classical) classes of structurable algebras.

Lemma 3.4.6. *All elements $(e, f) \in \mathcal{A}$ with $e, f \neq 0$ are conjugate invertible, with $\widehat{(e, f)} = (f^{-1}, e^{-1})$.*

Proof. By Lemma 3.4.3(ii), we have $V_{(e,f),(f^{-1},e^{-1})} = \mathrm{id}$. $\square$

Lemma 3.4.7. *Let $0 \neq f \in F$. Then $U_{(f,0)}(\mathcal{A}) = (F, 0)$ and $U_{(0,f)}(\mathcal{A}) = (0, F)$.*

Proof. Clearly, $U_{(f,0)}(\mathcal{A}) \leq (F, 0)$. Conversely, let $g \in F$ be arbitrary. Using Lemma 3.4.3, we have

$$\begin{aligned} U_{(f,0)}(0, f^{-1}gf^{-1}) &= 2((f, 0)(f^{-1}gf^{-1}, 0))(f, 0) - ((f, 0)(0, f))(0, f^{-1}gf^{-1}) \\ &= 2(g, 0). \end{aligned}$$

The proof of the other statement is similar (or follows by applying the involution). $\square$

Lemma 3.4.8. *Let $D \in \mathrm{Der}(\mathcal{A})$. Then there is some derivation D' of F such that $D(e, f) = (D'(e), D'(f))$ for all $e, f \in F$. In particular, $D(\lambda, \mu) = 0$ for all $\lambda, \mu \in k$.*

Proof. Clearly $D(1,1) = 0$. Since $(1,1) = (1,-1)^2$, we have

$$0 = D(1,1) = D(1,-1)(1,-1) + (1,-1)D(1,-1) = 2(1,-1)D(1,-1).$$

Left multiplying by $(1,-1)$ implies $D(1,-1) = 0$. Since D is k-linear, this already implies that $D(\lambda,\mu) = 0$ for all $\lambda,\mu \in k$. In particular, $D(1,0) = D(0,1) = 0$.

For each $e \in F$, we now have $D(e,0) = D((1,0)(e,0)) = (1,0)D(e,0)$. Hence there is some derivation $D' \in \mathrm{Der}(F)$ such that $D(e,0) = (D'(e),0)$ for all $e \in F$. Similarly, there is some derivation $D'' \in \mathrm{Der}(F)$ such that $D(0,f) = (0,D''(f))$ for all $f \in F$. Hence $D(e,f) = (D'(e),D''(f))$ for all $e,f \in F$. Expressing that $D(\overline{x}) = \overline{D(x)}$ for all $x \in \mathcal{A}$ now implies $D' = D''$. □

Theorem 3.4.9. *$\mathcal{A}$ satisfies Assumption 3.1.10.*

Proof. By Lemma 3.4.4, $\mathcal{A}$ is central simple. By (3.4) and Lemma 3.4.6, Assumption 3.1.1 is satisfied.

Consider an arbitrary non-zero $V \in \mathrm{Instrl}(\mathcal{A})$ with $V^\delta(\mathcal{S}) = 0$ and $V^2 = 0$. By Lemma 1.1.51, we have $V = T_x + D$, for some $D \in \mathrm{Der}(\mathcal{A})$, with $x = (e,f)$, for some $e,f \in F$. By Lemma 3.4.8, there is a derivation D' of F such that $D(g,h) = (D'(g),D'(h))$ for all $g,h \in F$. Note that $D^\delta = D + R_{\overline{D(1)}} = D$. By assumption,

$$0 = V^\delta(1,-1) = -\psi(x,(1,-1)) + D(1,-1) = (e+f,-e-f);$$

using (1.9). Hence $f = -e$ so $x = (e,-e) \in \mathcal{S}$. Then

$$0 = V^\delta(e,-e) = -\psi((e,-e),(e,-e)) + D(e,-e) = (D'(e),-D'(e))$$

and thus $D'(e) = 0$.

Now $V(1,0) = T_x(1,0) = (3e,0)$. Since $D'(e) = 0$ and $V^2 = 0$, this implies

$$0 = V^2(1,0) = T_x(3e,0) + D(3e,0) = 3T_x(e,0) = (9e^2,0).$$

Hence $e^2 = 0$ and $e = 0$, since F is division. We conclude that $V = D$ is a derivation. In particular, $V^\epsilon = V$.

Since $V \neq 0$, there exists $f \in F$ such that $D'(f) \neq 0$. Set $y = (f,f)$ and $a = V(y) = (D'(f),D'(f))$. Then $V(a) = 0$ and hence

$$U := [V,[V,V_{y,y}]] = 2V_{a,a}.$$

Since $\overline{a} = a$, we get $V_{a,a} = L_{a^2}$ by Lemma 1.1.52. Since $\mathcal{A}$ is power-associative, this implies $V_{a,a}^2(1) = a^4$. If $U^2 = 0$ we get $4a^4 = 0$ and thus $a = 0$, a contradiction. □

Lemma 3.4.10. *The only proper inner ideals of L properly containing $\mathcal{S}_+$ are the two inner ideals $(F,0)_+ \oplus \mathcal{S}_+$ and $(0,F)_+ \oplus \mathcal{S}_+$.*

Proof. Let I be a proper non-trivial inner ideal properly containing $\mathcal{S}_+$. By Lemma 3.1.12 and Theorem 3.4.9, $I \leq \mathcal{A}_+ \oplus \mathcal{S}_+$, so I contains some non-zero $a_+ \in \mathcal{A}_+$. If a were conjugate invertible, then the operator U_a would be invertible, but then the fact that $-U_a(b)_+ = [a_+, [a_+, b_-]] \in I$ for all $b \in \mathcal{A}$ would imply that $\mathcal{A}_+ \leq I$. Since I is abelian, this is in contradiction with Lemma 1.1.53. Hence a is not conjugate invertible, so by Lemma 3.4.6, a is contained in $(F, 0)$ or $(0, F)$. Assume without loss of generality that $a \in (F, 0)$. By Lemma 3.4.7, we get $(F, 0)_+ \leq I$. If moreover $(0, f)_+ \in I$ for some some non-zero $f \in F$, then $[(0, f)_+, (f, 0)_+] = \psi((0, f), (f, 0))_+ = (-f^2, f^2)_+ \neq 0$ yields a contradiction with the fact that I is abelian. Hence $I = (F, 0)_+ \oplus \mathcal{S}_+$.

It only remains to show that $J := (F, 0)_+ \oplus \mathcal{S}_+$ is an inner ideal, i.e., that

$$[(F, 0)_+ \oplus \mathcal{S}_+, [(F, 0)_+ \oplus \mathcal{S}_+, L]] \leq (F, 0)_+ \oplus \mathcal{S}_+ = (F, 0)_+ \oplus L_2.$$

Using the 5-grading of L, this means that we only have to verify the following inclusions:

$$[(F, 0)_+, [(F, 0)_+, \mathcal{A}_-]] \leq (F, 0)_+, \tag{3.5}$$

$$[(F, 0)_+, [(F, 0)_+, \mathcal{S}_-]] = 0, \tag{3.6}$$

$$[(F, 0)_+, [\mathcal{S}_+, \mathcal{S}_-]] \leq (F, 0)_+, \tag{3.7}$$

$$[\mathcal{S}_+, [(F, 0)_+, \mathcal{S}_-]] \leq (F, 0)_+. \tag{3.8}$$

Let $e, f, x, y \in F$ be arbitrary and let $s = (x, -x)$ and $t = (y, -y)$. To show (3.5), we compute that

$$[(e, 0)_+, [(f, 0)_+, (x, y)_-]] = -V_{(f,0),(x,y)}(e, 0)_+ = (-fy.e - ey.f, 0)_+ \subseteq (F, 0)_+.$$

To show (3.6), we compute that

$$[(e, 0)_+, [(f, 0)_+, s_-]] = -V_{(e,0),(xf,0)} = 0.$$

To show (3.7), we compute that

$$[(e, 0)_+, [s_+, t_-]] = [(e, 0)_+, L_s L_t] = (-x.ye, 0)_+ \in (F, 0)_+.$$

Finally, (3.8) follows from (3.7) by the Jacobi identity because $[\mathcal{S}_+, (F, 0)_+] = 0$. $\square$

We are now ready to construct a point-line geometry from the proper non-trivial inner ideals of $K(\mathcal{A})$.

Definition 3.4.11. We define the point-line geometry $\Gamma = (\mathcal{M}, \mathcal{F})$ with as points

$$\mathcal{M} := \mathcal{M}(L) := \{I \mid I \text{ is a minimal proper non-trivial inner ideal of } L\}$$

and as lines

$$\mathcal{F} := \mathcal{F}(L) := \{I \mid I \text{ is a non-minimal proper non-trivial inner ideal of } L\},$$

with inclusion as incidence.

Lemma 3.4.12. *Let $J \in \mathcal{F}$. Then J is the union of the elements of $\mathcal{M}$ it contains. Moreover, any two distinct such elements intersect trivially and span J.*

Proof. By Theorems 3.1.14 and 3.4.9 and Lemma 3.4.10, we may assume that $J = (F,0)_+ \oplus \mathcal{S}_+$. Since $\mathcal{S}_+ \in \mathcal{M}$, we get $e_-((f,0),0)(\mathcal{S}_+) \in \mathcal{M}$, for all $f \in F$. Note that for each $(g,-g) \in \mathcal{S}$, we have

$$e_-((f,0),0)((g,-g)_+) = (g,-g)_+ - (gf,0)_+,$$

since $V_{(gf,0),(f,0)} = 0$. Moreover, by (3.5) and (3.6) and $V_{x,y}(F,0) = (F,0)$ for all $x,y \in \mathcal{A}$, also $(F,0)_+$ is an inner ideal contained in J. Since each element of $J \setminus (F,0)_+$ can be written as $(g,-g)_+ - (gf,0)_+$ for some $f,g \in F$, the first claim is proved.

The second claim is obvious since the intersection of inner ideals is again an inner ideal, the elements of $\mathcal{M}$ are minimal inner ideals and the dimension of J is twice the dimension of an element of $\mathcal{M}$. □

Lemma 3.4.13. *Two distinct points $I, J \in \mathcal{M}$ are collinear if and only if $[I,J] = 0$.*

Proof. If I and J are collinear, then they are contained in a proper inner ideal. Because this inner ideal is abelian, we get $[I,J] = 0$.

Conversely, let $I, J \in \mathcal{M}$ with $[I,J] = 0$. By Theorem 3.1.14, we may assume $I = \mathcal{S}_+$. By $[I,J] = 0$ and the fact that all non-zero elements of $\mathcal{S}$ are conjugate invertible, we get $J \leq L_0 \oplus L_1 \oplus L_2$. Suppose that there is an $x := V + a_+ + s_+ \in J$ with $s \in \mathcal{S}$, $a \in \mathcal{A}$, $V \in \mathrm{Instrl}(\mathcal{A})$ and $V \neq 0$. Since $[I,J] = 0$, we have $V^\delta(\mathcal{S}) = 0$. If $V^2 \neq 0$, then there exists some $b \in \mathcal{A}$ such that $[x,[x,b_-]] \in J$ has a non-zero (-1)-component, a contradiction. If $V^2 = 0$, then by Theorem 3.4.9, we can find $0 \neq U \in \mathrm{Instrl}(\mathcal{A})$ such that $W := [V,[V,U]]$ satisfies $W^2 \neq 0$. Hence $y := [x,[x,U]] \in J$ has 0-component W, so we can repeat the previous argument with x replaced by y and V replaced by W to get a contradiction. We conclude that $J \leq \mathcal{A}_+ \oplus \mathcal{S}_+$.

Since $J \neq I$, there exists a non-zero $a \in \mathcal{A}$ and an $s \in \mathcal{S}$ such that $y := a_+ + s_+ \in J$. For each $t \in \mathcal{S}$, the inner ideal J contains the element $[y,[y,t_-]]$, which has 0-component $-V_{a,ta}$. Since $J \leq L_1 \oplus L_2$, we must have $V_{a,ta} = 0$ for all $t \in \mathcal{S}$ and hence, by Lemma 1.1.57, we get $a \in (F,0)$ or $a \in (0,F)$. Hence J intersects one of the inner ideals $(F,0)_+ \oplus \mathcal{S}_+$ and $(0,F)_+ \oplus \mathcal{S}_+$ non-trivially. Since J is a minimal inner ideal, we conclude that either $J \subseteq (F,0)_+ \oplus \mathcal{S}_+$ or $J \subseteq (0,F)_+ \oplus \mathcal{S}_+$. Since these inner ideals both contain $I = \mathcal{S}_+$, the points I and J are indeed collinear. □

Lemma 3.4.14. *Two points $I, J \in \mathcal{M}$ that satisfy $[I,[I,J]] \neq 0$ are at distance 3 in Γ. Moreover, we then have $[i,j] \neq 0$ for all non-zero $i \in I, j \in J$.*

Proof. By Theorem 3.1.14, we may assume that $I = \mathcal{S}_+$. The assumption $[I,[I,J]] \neq 0$ then implies that J contains an element with non-zero (-2)-component. By Lemma 3.1.9 and the fact that $\mathcal{S}_+$ is fixed by $E_+(\mathcal{A})$, we may assume that $J = \mathcal{S}_-$. Since all non-zero elements of $\mathcal{S}$ are conjugate invertible, this already shows that $[i,j] \neq 0$ for all $0 \neq i \in I$ and $0 \neq j \in J$.

By Lemma 3.4.13, the points $(F,0)_+$ and $(F,0)_-$ are collinear, so we can consider the path $(\mathcal{S}_+, (F,0)_+, (F,0)_-, \mathcal{S}_-)$ to see that the points I and J are at distance at most 3. By Lemma 3.4.13, the points I and J are not collinear.

Now suppose that I and J are at distance 2 and let $K \in \mathcal{M}$ be a point collinear with both I and J. By Lemma 3.1.12 we get $K \leq (\mathcal{S}_- \oplus \mathcal{A}_-) \cap (\mathcal{A}_+ \oplus \mathcal{S}_+) = 0$, a contradiction. $\square$

The following lemma is a stronger version of Lemma 3.4.13.

Lemma 3.4.15. *Two distinct points $I, J \in \mathcal{M}$ are collinear if and only if there is some $0 \neq j \in J$ such that $[I,j] = 0$.*

Proof. Let $0 \neq j \in J$ be such that $[I,j] = 0$; we have to show that $[I,J] = 0$. By Lemma 3.4.14, we already know that $[I,[I,J]] = 0$.

By Theorem 3.1.14, we may assume that $I = \mathcal{S}_+$. Then $[I,[I,J]] = 0$ yields $J \leq L_{-1} \oplus L_0 \oplus L_1 \oplus L_2$. Hence $j = b_- + V + a_+ + s_+$ for some $a, b \in \mathcal{A}$, $V \in \mathrm{Instrl}(\mathcal{A})$ and $s \in \mathcal{S}$. Since all non-zero elements of $\mathcal{S}$ are conjugate invertible, $[I,j] = 0$ implies $b = 0$ and $V^\delta(\mathcal{S}) = 0$.

We claim that J contains an element in $L_1 \oplus L_2$ with non-zero 1-component. For each $c \in \mathcal{A}$, J contains $[j,[j,c_+]] \in L_1 \oplus L_2$, which has 1-component $V^2(c)_+$, so if $V^2 \neq 0$, then the claim holds. If $V^2 = 0$, we use Theorem 3.4.9 to find $W \in \mathrm{Instrl}(\mathcal{A})$ such that $j' := [j,[j,W]] \in L_0 \oplus L_1 \oplus L_2$ has 0-component U with $U^2 \neq 0$. Replacing j by j' then shows the claim.

Denote this element of J by $x = c_+ + t_+$ where $0 \neq c \in \mathcal{A}$ and $t \in \mathcal{S}$. Then for each $d \in \mathcal{A}$, J also contains $[x,[x,d_-]]$, which has 1-component $-U_c(d)_+$. If c were conjugate invertible, then U_c would be an invertible operator, hence the projection of J onto $\mathcal{A}_+$ would be all of $\mathcal{A}_+$. This contradicts the fact that $\dim J = \dim \mathcal{S} < \dim \mathcal{A}$. Hence c is not conjugate invertible, so $c \in (F,0)$ or $c \in (0,F)$. It follows that the minimal inner ideal J intersects one of the two non-minimal inner ideals $(F,0)_+ \oplus \mathcal{S}_+$ or $(0,F)_+ \oplus \mathcal{S}_+$ non-trivially and is hence contained in it. We conclude that indeed $[I,J] = 0$. $\square$

Lemma 3.4.16. *Let $I, J \in \mathcal{M}$ be two distinct points such that $[I,J] \neq 0$ and $[I,[I,J]] = 0$. Then there is exactly one point $K \in \mathcal{M}$ collinear with both I and J.*

Proof. By Theorem 3.1.14, we may assume $J = \mathcal{S}_+$. Since $[I,J] \neq 0$ we can find $i \in I$ and $j \in J$ such that $e := [i,j] \neq 0$. By $[I,[I,J]] = 0$ we get $I \leq L_{-1} \oplus L_0 \oplus L_1 \oplus L_2$. Hence $e \in L_1 \oplus L_2$ and $i = b_- + V + a_+ + s_+$ for some

$a, b \in \mathcal{A}$, $s \in \mathcal{S}$ and $V \in \mathrm{Instrl}(\mathcal{A})$. Note that for each $c \in \mathcal{A}$, I contains $[i, [i, c_+]]$, which has (-1)-component $-U_b(c)_-$. Since $\dim I = \dim \mathcal{S} < \dim \mathcal{A}$, Lemma 3.4.6 implies that b is contained in $(F, 0)$ or $(0, F)$. Hence e is contained in either $(F, 0)_+ \oplus \mathcal{S}_+$ or $(0, F)_+ \oplus \mathcal{S}_+$. By Lemma 3.4.12, e is contained in a unique element $E \in \mathcal{M}$. Since $J = \mathcal{S}_+$, we get $[E, J] = 0$, so E and J are collinear by Lemma 3.4.13. Since $[I, [I, J]] = 0$, we get $[I, e] = 0$, so Lemma 3.4.15 implies that I and E are collinear. Since $[I, J] \neq 0$, it already follows from Lemma 3.4.13 that I and J are at distance 2 in the collinearity graph of Γ.

Suppose now that $K \in \mathcal{M}$ is another point collinear with both I and J. By Lemma 3.4.10, E and K are contained in either $(F, 0)_+ \oplus \mathcal{S}_+$ or $(0, F)_+ \oplus \mathcal{S}_+$. Assume first that both $E, K \leq (F, 0)_+ \oplus \mathcal{S}_+$. Then Lemma 3.4.12 implies that $E \oplus K = (F, 0)_+ \oplus \mathcal{S}_+$. Since $[E, I] = 0 = [K, I]$, this implies that also $[J, I] = [\mathcal{S}_+, I] = 0$, a contradiction.

Hence we may assume that $E \leq (F, 0)_+ \oplus \mathcal{S}_+$ and $K \leq (0, F)_+ \oplus \mathcal{S}_+$. Since $E \cap \mathcal{S}_+ = 0$ by Lemma 3.4.12, the projection of E onto $\mathcal{A}_+$ is $(F, 0)_+$. Similarly, the projection of K onto $\mathcal{A}_+$ is $(0, F)_+$. Hence the projection of $E \oplus K \leq \mathcal{A}_+ \oplus \mathcal{S}_+$ onto $\mathcal{A}_+$ is all of $\mathcal{A}_+$. Since $[E \oplus K, I] = 0$, this implies that for each $c \in \mathcal{A}$, we can find some $t \in \mathcal{S}$ such that $[c_+ + t_+, I] = 0$. Now let $i = b_- + V + a_+ + s_+ \in I$ be arbitrary. Then $[c_+ + t_+, i]$ has 0-component $V_{c,b}$, hence $V_{c,b} = 0$ for all $c \in \mathcal{A}$ and in particular $U_c(b) = V_{c,b}(c) = 0$ for all $c \in \mathcal{A}$. By choosing c conjugate invertible, we get $b = 0$. This implies that $[c_+ + t_+, i]$ has 1-component $V(c)$, hence $V(c) = 0$ for all $c \in \mathcal{A}$, so also $V = 0$. So $I \leq \mathcal{A}_+ \oplus \mathcal{S}_+$, but then $[I, J] = 0$, which is again a contradiction. □

Corollary 3.4.17. *Let $I, J \in \mathcal{M}$ be two distinct points at distance d in Γ. Then*

$$\begin{aligned} d = 1 &\iff [I, J] = 0, \\ d = 2 &\iff [I, J] \neq 0 \text{ and } [I, [I, J]] = 0, \\ d = 3 &\iff [I, [I, J]] \neq 0. \end{aligned}$$

Lemma 3.4.18. *The geometry Γ does not contain triangles, quadrangles or pentagons.*

Proof. Let $I \in \mathcal{M}$ be a point and suppose that I would be contained in a triangle (I, J, K) in Γ. By Theorems 3.1.14 and 3.4.9, we may assume that $I = \mathcal{S}_+$. By Lemma 3.4.10, there are only two lines through I, namely $(F, 0)_+ \oplus \mathcal{S}_+$ and $(0, F)_+ \oplus \mathcal{S}_+$, so we must have $J \subseteq (F, 0)_+ \oplus \mathcal{S}_+$ and $K \subseteq (0, F)_+ \oplus \mathcal{S}_+$ (or conversely). Since $J \neq I \neq K$, there is some $(e, 0)_+ + s_+ \in J$ and $(0, f)_+ + t_+ \in K$ with $e, f \neq 0$. Since J and K are collinear, we have $[J, K] = 0$ by Lemma 3.4.13, so in particular $0 = [(e, 0)_+ + s_+, (0, f)_+ + t_+] = \psi((e, 0), (0, f))_+$, which is a contradiction. Hence Γ does not contain triangles.

By Lemma 3.4.16 and Corollary 3.4.17, Γ does not contain quadrangles.

Suppose now that Γ would contain a pentagon (I, J, M, N, O) for certain points $I, J, M, N, O \in \mathcal{M}$. By Theorem 3.4.9, we may again assume that $I = \mathcal{S}_+$.

By Lemma 3.4.10, we may then assume that J contains some $j = a_+ + s_+$ with $a = (f, 0)$ for some non-zero $f \in F$. Write $s = (g, -g)$ with $g \in F$. Then

$$e_+((0, f^{-1}g), 0)(a_+ + s_+) = a_+.$$

Since $e_+((0, f^{-1}g), 0)$ fixes $I = \mathcal{S}_+$, we may assume that $J = (F, 0)_+$ as well. Notice that $\langle (F, 0)_+, (F, 0)_- \rangle$ is a line through I and that any point lies on exactly two lines by Lemma 3.4.10; hence $M \leq \langle (F, 0)_+, (F, 0)_- \rangle$. So M contains some $y := (f, 0)_+ + (g, 0)_-$ with $f, g \in F$. Note that $g \neq 0$ since $J \cap M = 0$. We now observe that

$$e_+\big(0, (-fg^{-1}, fg^{-1})\big)(y) = (g, 0)_-$$

and that $e_+(0, (-fg^{-1}, fg^{-1}))$ preserves I and J. We may thus assume that $M = (F, 0)_-$.

Since O is collinear with I and not on the line IJ, Lemma 3.4.10 shows that O contains some element $s_+ + (0, f)_+$ with $s \in \mathcal{S}$ and $0 \neq f \in F$. Since O and M are at distance at most 2, we get $[O, [O, M]] = 0$. Hence

$$[s_+ + (0, f)_+, [s_+ + (0, f)_+, (g, 0)_-]] = 0$$

for all $g \in F$. In particular, the 1-component $-U_{(0,f)}(g, 0)$ must be 0 for all $g \in F$. Since $U_{(0,f)}(g, 0) = 2(0, fgf)$, we get a contradiction. We conclude that Γ does not contain pentagons. $\square$

Theorem 3.4.19. *Let $\mathcal{A}$ be the structurable algebra from Construction 3.4.2 and set $L = K(\mathcal{A})$. Consider the graph $\Omega = (V, E)$, with*

$$\begin{aligned} V &= \{I \mid I \text{ is a proper non-minimal non-trivial inner ideal of } L\}, \\ E &= \{\{I, J\} \mid I \cap J \neq 0\}. \end{aligned}$$

Then Ω is the incidence graph of the Moufang triangle equivalent with the projective plane defined over F. Its root groups can be identified with

$$U_1 = e_-((F, 0), 0), U_2 = e_-(0, \mathcal{S}), U_3 = e_-((0, F), 0),$$

and the commutator relations are as in $E_-(\mathcal{A})$.

Moreover, the geometry Γ (as introduced in Definition 3.4.11) is a thin generalized hexagon, which is the dual double[2] of this Moufang triangle.

Proof. By Lemma 3.4.13, the cycle of points

$$\big(\mathcal{S}_+,\ (F, 0)_+,\ (F, 0)_-,\ \mathcal{S}_-,\ (0, F)_-,\ (0, F)_+,\ \mathcal{S}_+\big)$$

forms a hexagon in Γ. Then Theorem 3.4.9, Lemmas 3.4.10 and 3.4.18, and Corollary 3.4.17 show that Γ is a thin generalized hexagon.

[2]The *dual double* of a point-line geometry Δ is the geometry with as point set the *flags* of Δ, i.e., the incident point-line pairs, and as line set the union of the point set and the line set of Δ.

Note that the intersection of two non-minimal proper inner ideals is a minimal inner ideal if this intersection is non-empty. This intersection is a point of the thin generalized hexagon Γ. Therefore, Ω is the incidence graph of a generalized triangle and Γ is the dual double of this triangle.

Consider the following 6-cycle in Ω:

$$(x_0, \ldots, x_5, x_0) = \big((0,F)_+ \oplus \mathcal{S}_+,\ \mathcal{S}_+ \oplus (F,0)_+,\ (F,0)_+ \oplus (F,0)_-,\ (F,0)_- \oplus \mathcal{S}_-,\\ \mathcal{S}_- \oplus (0,F)_-,\ (0,F)_- \oplus (0,F)_+,\ (0,F)_+ \oplus \mathcal{S}_+\big).$$

We now show that $e_-((F,0),0)$ fixes all neighbors of x_2 and x_3 and acts transitively on the set of all neighbors of x_1 distinct from x_2. Note that if I is a proper non-minimal inner ideal, we can identify a neighbor J of I with the minimal inner ideal $I \cap J$, since Γ is a thin generalized hexagon. Moreover, any automorphism stabilizing I and $I \cap J$ stabilizes J. So in order to check that $e_-((F,0),0)$ fixes all neighbors of x_i it suffices to check that it fixes x_i itself and all the minimal inner ideals it contains, for $i = 2, 3$, and this is clear since it fixes the inner ideals x_2 and x_3 elementwise. Similarly, in order to check that that $e_-((F,0),0)$ acts transitively on the set of all neighbors of x_1 distinct from x_2, it suffices to show that it acts transitively on the set of all minimal inner ideals in x_1 distinct from $x_1 \cap x_2 = (F,0)_+$. This follows from the fact that $[(f,0)_-, (g,-g)_+] = (-fg,0)_+$ for all $f, g \in F$ and a short computation.

By Lemma 1.2.11, it now follows that the root group U_1 coincides with $e_-((F,0),0)$; similarly, the root group U_3 coincides with $e_-((0,F),0)$. In order to determine the root group U_2, note that $e_-(0,\mathcal{S})$ fixes all neighbors of x_3 and x_4 since $[\mathcal{S}_-, x_i] = 0$, for $i = 3, 4$. Using the fact that $[(f,-f)_-, (g,0)_+] = (fg,0)_-$, we deduce that it acts transitively on the set of all neighbors of x_2 different from x_3.

In order to show that Ω is the incidence graph of a Moufang triangle it suffices to show that $E(\mathcal{A})$ acts transitively on the set of all cycles of length 6. Consider any 6-cycle $(y_0, \ldots, y_5, y_0)$. By Theorem 3.1.14, the minimal inner ideal $y_0 \cap y_1$ can be mapped onto $\mathcal{S}_+$. Consider the minimal inner ideal $I = y_3 \cap y_4$. By Corollary 3.4.17, we get $[\mathcal{S}_+, [\mathcal{S}_+, I]] \neq 0$. Then by Lemma 3.1.9, there exists an automorphism fixing $\mathcal{S}_+$ and mapping I onto $\mathcal{S}_-$. Since there are precisely two proper inner ideals through a minimal inner ideal and $[(0,F)_+, \mathcal{S}_-] \neq 0$, there exists an automorphism mapping our cycle $(y_0, \ldots, y_5, y_0)$ onto the cycle $(x_0, \ldots, x_5, x_0)$.

In order to show that Ω is the incidence graph of the Moufang triangle equivalent with the projective plane defined over F, it suffices to set $x_1(f) = e_-((f,0),0)$, $x_2(f) = e_-(0,(f,-f))$, $x_3(f) = e_-((0,f),0)$ for all $f \in F$ and compare Theorem 1.2.21 with (1.14). $\square$

Remark 3.4.20. Another way to obtain the Moufang triangles associated with an *octonion* division algebra $\mathbb{O}$, is to consider the proper non-trivial inner ideals of the exceptional Jordan algebra $H_3(\mathbb{O})$ (not of its TKK Lie algebra!). These inner ideals are either 1- or 10-dimensional; the former can be identified with the points of the projective plane and the latter with the lines of the projective plane. This is, of course, related to the fact that these Moufang triangles arise as rank 2 forms of buildings of type E_6. See [Fau73, 2(B)] and [McC04, p. 34-35].

Remark 3.4.21. As we have mentioned in the introduction of this thesis, a Lie algebra can often be obtained in more than one way as the TKK Lie algebra of a structurable algebra; see [Sta20, Theorem 5.9]. Moreover, by [LGLN07, 5.2], *any* inner ideal is the "end" of a $\mathbb{Z}$-grading, at least if $\operatorname{char} k \neq 2, 3, 5$. For some choices of inner ideals, this $\mathbb{Z}$-grading will be different from the standard 5-grading obtained via the TKK construction.

In the case that we are currently considering, the inner ideal $\mathcal{S}_+$ is, of course, the end of a $\mathbb{Z}$-grading on L. On the other hand, consider an inner ideal contained in $\mathcal{F}$ (i.e., a non-minimal proper inner ideal); then this inner ideal will also be the end of a $\mathbb{Z}$-grading on L. Indeed, note that $\operatorname{ad}_{T_{(1,-1)}}$ is a grading derivation with components

$$\begin{aligned} L_{-1} &= (0, F)_- \oplus (0, F)_+; \\ L_0 &= \mathcal{S}_- \oplus \operatorname{Instrl}(\mathcal{A}) \oplus \mathcal{S}_+; \\ L_1 &= (F, 0)_- \oplus (F, 0)_+. \end{aligned}$$

Then (L_{-1}, L_1) is a *Jordan pair* (as defined in [Loo75, 1.2]), a special case of a Kantor pair. More precisely, one checks that (L_{-1}, L_1) is isomorphic to the Jordan pair $(M_{1,2}(F), M_{1,2}(F^{\mathrm{op}}))$ via the isomorphism

$$\begin{aligned} (0, f)_- + (0, g)_+ &\mapsto 2(\overline{f}, \overline{g}); \\ (f, 0)_- + (g, 0)_+ &\mapsto (g, f), \end{aligned}$$

using [Loo75, 8.15].

SECTION 3.5

Moufang hexagons

We now come to the case of the Moufang hexagons. It is known from the classification of Moufang hexagons by Jacques Tits and Richard Weiss [TW02] that Moufang hexagons are parametrized by anisotropic cubic norm structures, or equivalently, by cubic Jordan division algebras, see Theorem 1.2.21. The structurable algebra we will use will be a matrix structurable algebra constructed from this Jordan algebra.

Notation 3.5.1. Let J be a cubic Jordan division algebra over a field of characteristic different from 2 and 3, with non-degenerate admissible form N, trace form T and Freudenthal cross product $\times$. Let $\mathcal{A}$ be the structurable algebra $M(J,1)$ of skew-dimension one. (Recall Definition 1.1.60.) Let $s := \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \in \mathcal{S}$.

Set $L = K(\mathcal{A})$. The aim of this section is to prove that the geometry of the proper non-trivial inner ideals of L, with inclusion as incidence, is a generalized hexagon.

We begin with a description of the *derivations* of $\mathcal{A}$; this will be used in the proof of Theorem 3.5.3 below.

Lemma 3.5.2. *Let $D \in \operatorname{Der}(\mathcal{A})$. Then there exist $m, n \in \operatorname{End}_k(J)$ such that*

$$D \begin{pmatrix} \alpha & l \\ j & \beta \end{pmatrix} = \begin{pmatrix} 0 & m(l) \\ n(j) & 0 \end{pmatrix}$$

for all $\alpha, \beta \in k$ and $l, j \in J$. If $D \neq 0$, then $m \neq 0 \neq n$.

Proof. The fact that $D(1) = 0$ is evident. Since $D(\mathcal{S}) \subseteq \mathcal{S}$, we have $D(s) = \lambda s$ for some $\lambda \in k$. Hence $0 = D(1) = D(s^2) = D(s)s + sD(s) = 2\lambda s^2 = 2\lambda$ and thus $D(s) = 0$. This already shows that $D\left(\begin{smallmatrix} \alpha & 0 \\ 0 & \beta \end{smallmatrix}\right)$ for all $\alpha, \beta \in k$.

We now consider an element of the form

$$x = \begin{pmatrix} 0 & l \\ 0 & 0 \end{pmatrix}.$$

Then $sx = x$ and $xs = -x$. Hence $D(x) = D(s)x + sD(x) = sD(x)$ and $-D(x) = D(x)s + xD(s) = D(x)s$. This implies that

$$D(x) = \begin{pmatrix} 0 & l' \\ 0 & 0 \end{pmatrix}$$

for some $l' \in J$. Similarly, for each $j \in J$, there is some $j' \in J$ such that $D\left(\begin{smallmatrix} 0 & 0 \\ j & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 0 \\ j' & 0 \end{smallmatrix}\right)$. Since D is k-linear, we conclude that there exist $m, n \in \operatorname{End}_k(J)$ such that

$$D \begin{pmatrix} \alpha & l \\ j & \beta \end{pmatrix} = \begin{pmatrix} 0 & m(l) \\ n(j) & 0 \end{pmatrix}$$

for all $\alpha, \beta \in k$ and $l, j \in J$.

Assume now that $m = 0$; we will show that this implies $D = 0$. We have $x^2 = \left(\begin{smallmatrix} 0 & 0 \\ l \times l & 0 \end{smallmatrix}\right)$ and $l \times l = 2l^\sharp$. Since $m = 0$, we have $D(x) = 0$. Hence also $D(x^2) = D(x)x + xD(x) = 0$, which implies $n(l^\sharp) = 0$. Since $l \in J$ was arbitrary and J is division, this implies that $n = 0$. □

Theorem 3.5.3. *Let $\mathcal{A}$ be as in Notation 3.5.1. Then $\mathcal{A}$ satisfies Assumption 3.1.10.*

Proof. Notice that $\mathcal{A}$ is central simple by [AF84, §4, Lemma 2.1]. Assumption 3.1.1 is now obviously satisfied.

Consider $V \in \mathrm{Instrl}(\mathcal{A})$ such that $V^2 = 0$, $V^\delta(s) = 0$ and $V \neq 0$. By Lemma 1.1.51, we can write $V = D + T_x$ for some $D \in \mathrm{Der}(\mathcal{A})$ and $x \in \mathcal{A}$. Write

$$x = \begin{pmatrix} \alpha & l \\ j & \beta \end{pmatrix}$$

for some $\alpha, \beta \in k$ and $j, l \in J$. Let $m, n \in \mathrm{End}_k(J)$ be as in Lemma 3.5.2. Then $D(1) = 0 = D(s)$, and thus $D^\delta(s) = 0$ and $D^\epsilon = D$. On the other hand, $T_x^\delta(s) = -\psi(x, s) = xs + s\overline{x} = \left(\begin{smallmatrix} \alpha+\beta & 0 \\ 0 & -\alpha-\beta \end{smallmatrix}\right)$. Hence the condition $V^\delta(s) = 0$ implies $\beta = -\alpha$. In particular, $x - \overline{x} = 2\alpha s$.

Now let $y = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$. Then

$$T_x(y) = xy + y(x - \overline{x}) = xy + 2\alpha ys = \begin{pmatrix} 3\alpha & 0 \\ j & 0 \end{pmatrix}$$

and hence

$$\begin{aligned} T_x^2(y) = xT_x(y) + T_x(y)(x - \overline{x}) &= \begin{pmatrix} 3\alpha^2 + T(l, j) + 6\alpha^2 & j \times j \\ 2\alpha j + 2\alpha j & 0 \end{pmatrix} \\ &= \begin{pmatrix} 9\alpha^2 + T(l, j) & 2j^\sharp \\ 4\alpha j & 0 \end{pmatrix}. \end{aligned}$$

Using Lemma 3.5.2 and the fact that $D(y) = 0$, we get

$$0 = V^2(y) = D(T_x(y)) + T_x^2(y) = \begin{pmatrix} 9\alpha^2 + T(l, j) & 2j^\sharp \\ 4\alpha j + n(j) & 0 \end{pmatrix}.$$

In particular, $j^\sharp = 0$; since J is division, this implies $j = 0$. Hence $9\alpha^2 = 0$, i.e., $\alpha = 0$. By considering $\left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ instead of y, we obtain in a similar fashion that $l = 0$. We conclude that $x = 0$.

Hence $V = D$ for some $D \in \mathrm{Der}(\mathcal{A})$. Since $V \neq 0$, it follows from Lemma 3.5.2 that we can find some $j \in J$ with $l := m(j) \neq 0$. Consider $a = \left(\begin{smallmatrix} 0 & j \\ 0 & 0 \end{smallmatrix}\right)$ and notice that $D(a) = \left(\begin{smallmatrix} 0 & l \\ 0 & 0 \end{smallmatrix}\right) \in \mathcal{H}$, $D(a)^2 = \left(\begin{smallmatrix} 0 & 0 \\ 2l^\sharp & 0 \end{smallmatrix}\right)$ and $(D(a)^2)^2 = \left(\begin{smallmatrix} 0 & 8l^{\sharp\sharp} \\ 0 & 0 \end{smallmatrix}\right)$. Since $V^2 = 0$ and $V^\epsilon = V$, we have

$$U := [V, [V, V_{a,a}]] = 2V_{D(a),D(a)} = 2L_{D(a)^2}$$

by Lemma 1.1.52. If $U^2 = 0$, then $L_{D(a)^2}^2 = 0$, which can be applied on 1 to get $(D(a)^2)^2 = 0$, hence $8l^{\sharp\sharp} = 8N(l)l = 0$. Since J is division, this implies $l = 0$ and we get a contradiction. Hence $U^2 \neq 0$. $\square$

We now consider the extremal geometry associated to L. Recall from Definition 2.3.12 that $\mathcal{E}(L)$ is the collection of one-dimensional inner ideals of L and that the non-zero elements of these inner ideals are precisely the extremal elements of L. The lines are two-dimensional subspaces of L such that every one-dimensional subspace of it is an element of $\mathcal{E}(L)$.

Proposition 3.5.4. *Let a be a non-zero element of $\mathcal{A}$. The following are equivalent:*

(a) *a_+ is extremal;*
(b) *$U_a(\mathcal{A}) \leq \langle a \rangle$;*
(c) *$U_{sa}(\mathcal{A}) \leq \langle sa \rangle$;*
(d) *$V_{a,sa} = 0$;*
(e) *$\langle s_+, a_+ \rangle$ is a line of $\Gamma(L)$.*

Proof. First note that $s_+ \in \mathcal{S}_+$ is extremal since $\langle s_+ \rangle = \mathcal{S}_+$ is the 2-component of the 5-grading on L. Recall that $s^2 = 1$.

(a)⇒(b) If a_+ is extremal, then for each $b \in \mathcal{A}$, the element $[a_+, [a_+, b_-]] = -U_a(b)_+$ must be a multiple of a_+.

(b)⇒(c) This follows immediately from the identity $U_{sa} = -L_s U_a L_s$ (see [AH81, Proposition 11.3]).

(c)⇒(d) Let $x := e_-(-sa, 0)(s_+)$. Since s_+ is extremal, so is x. Using Lemma 1.1.76 together with the fact that $s(sa) = (s^2)a = a$, we get

$$x = \tfrac{1}{24}\psi(sa, U_{sa}(a))_- - \tfrac{1}{6}U_{sa}(a)_- + \tfrac{1}{2}V_{a,sa} + a_+ + s_+.$$

Now $U_{sa}(a) \in \langle sa \rangle$ by assumption. Since $\psi(sa, sa) = 0$, it follows that

$$x = (\lambda sa)_- + \frac{1}{2}V_{a,sa} + a_+ + s_+$$

for some $\lambda \in k$.
Assume first that $\lambda \neq 0$. Since $[x, [x, s_+]] \leq L_0 \oplus L_1 \oplus L_2$, the fact that x is extremal but has a non-zero (-1)-component implies that $[x, [x, s_+]] = 0$. In particular, the 0-component $\lambda^2 V_{a,sa}$ equals 0 and hence $V_{a,sa} = 0$.
Assume next that $\lambda = 0$. Let $b \in \mathcal{A}$ be arbitrary. Then $[x, [x, b_-]]$ must be a multiple of $\frac{1}{2}V_{a,sa} + a_+ + s_+$, but this element has (-1)-component $\frac{1}{4}V_{sa,a}^2(b)_-$. Since b was arbitrary, this implies $V_{sa,a}^2 = 0$. Moreover, $V_{a,sa}^\delta(s) = -\psi(a, a) = 0$. By Lemma 1.1.56, we also have $V_{a,sa}^2 = 0$. Suppose now that $V_{a,sa} \neq 0$. By Theorem 3.5.3, there exists $W \in \mathrm{Instrl}(\mathcal{A})$ such that $[V_{a,sa}, [V_{a,sa}, W]] \notin \langle V_{a,sa} \rangle$. Since x is extremal and $[V_{a,sa}, [V_{a,sa}, W]]$ is 4 times the 0-component of $[x, [x, W]]$, this is a contradiction. Hence $V_{a,sa} = 0$ also in this case.

(d)⇒(e) Let $\lambda \in k$ be arbitrary and let $x := e_-(-\lambda sa, 0)(s_+)$. Since $V_{a,sa} = 0$ by assumption, we have

$$\mathrm{ad}(-\lambda(sa)_-)^2(s_+) = \lambda^2 V_{a,sa} = 0$$

and hence $\mathrm{ad}(-\lambda(sa)_-)^j(s_+) = 0$ for all $j \geq 2$. It follows that $x = \lambda a_+ + s_+$, and since s_+ is extremal, we already obtain that $\lambda a_+ + s_+$ is also extremal for all $\lambda \in k$. In particular, the element $a_+ + s_+$ is extremal.

It remains to show that a_+ is extremal; the result will then follow because $[s_+, a_+] = 0$. Note that for any $i \in [-2, 2]$, we have

$$[a_+ + s_+, [a_+ + s_+, L_i]] \leq L_{i+2} \oplus L_{i+3} \oplus L_{i+4}$$

and the projection of $[a_+ + s_+, [a_+ + s_+, L_i]]$ onto L_{i+2} equals $[a_+, [a_+, L_i]]$. Together with $[a_+ + s_+, [a_+ + s_+, L_i]] \leq \langle a_+ + s_+\rangle$ this implies that $[a_+, [a_+, L_i]] \leq \langle a_+\rangle$. We conclude that a_+ is indeed extremal.

(e)⇒(a) This is obvious. □

Definition 3.5.5 ([Gar01, Definition 6.1])**.** Let I be a subspace of a structurable algebra $\mathcal{A}$ of skew-dimension one. Then I is called an *inner ideal* of $\mathcal{A}$ if $U_i(\mathcal{A}) \leq I$, for all $i \in I$.

Corollary 3.5.6. *All proper non-trivial inner ideals of $\mathcal{A}$ are* 1*-dimensional.*

Proof. Let I be a proper non-trivial inner ideal of $\mathcal{A}$. Consider $a \in I$. Let $x := e_-(-sa, 0)(s_+)$. Since s_+ is extremal, so is x. Using Lemma 1.1.76 we get

$$x = \tfrac{1}{24}\psi(sa, U_{sa}(a))_- - \tfrac{1}{6}U_{sa}(a)_- + \tfrac{1}{2}V_{a,sa} + a_+ + s_+.$$

Now since a is not conjugate invertible (otherwise $I = \mathcal{A}$), we get $\psi(sa, U_{sa}(a)) = 0$ by [AF84, Theorem 2.11]. Hence

$$x = (sa')_- + \frac{1}{2}V_{a,sa} + a_+ + s_+$$

for some $a' \in I$. For any $b \in \mathcal{A}$ we get that $[x, [x, b_+]]$ has (-1)-component $-U_{sa'}(b)$. Since x is extremal we get $U_{sa'}(\mathcal{A}) \leq \langle sa'\rangle$. By Proposition 3.5.4 we get that $\langle a'\rangle$ is a 1-dimensional inner ideal contained in I. Then [Gar01, Theorem 6.12] concludes this proof. □

Remark 3.5.7. By Proposition 3.5.4, an element a of the structurable algebra $\mathcal{A}$ is *extremal*[3], i.e. $U_a(\mathcal{A}) \leq \langle a\rangle$, if and only if a_+ is extremal (in the Lie algebra). Note that the proof uses the fact that we are working with a structurable algebra of skew-dimension one and the implication from (c) to (d) relies on the assumption that J is division (via Theorem 3.5.3). One could expect that there is a more direct proof of the equivalence of (b) and (d), without going to the Lie algebra, which may hold in a more general setting.

Corollary 3.5.8. *The set of extremal elements of L contained in $\mathcal{A}_+$ equals*

$$B := \left\{ \lambda \begin{pmatrix} N(x) & x \\ x^\sharp & 1 \end{pmatrix}_+ \,\middle|\, x \in J, \lambda \in k^\times \right\} \cup \left\{ \lambda \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}_+ \,\middle|\, \lambda \in k^\times \right\}.$$

Moreover, if $\psi(a, b) = 0$ for $a, b \in B$, then a and b are linearly dependent.

[3]This notion coincides with the notion of a *singular element* in [Gar01, Definition 5.1] and a *strictly regular element* in [AF84, p. 196].

Proof. By Proposition 3.5.4, a_+ is extremal if and only if $U_a(\mathcal{A}) \le \langle a \rangle$. Recall that J is a division algebra, so in particular, if $j \in J$, then $(j^\sharp)^\sharp = N(j)j$ and if $j^\sharp = 0$, then $j = 0$. The first statement now follows from [Gar01, Lemma 5.7].

A straightforward calculation shows that for any $x, y \in J$,

$$\psi\left(\begin{pmatrix} N(x) & x \\ x^\sharp & 1 \end{pmatrix}_+, \begin{pmatrix} N(y) & y \\ y^\sharp & 1 \end{pmatrix}_+\right) = \lambda s_+$$

with $\lambda = N(x) - N(y) + T(x, y^\sharp) - T(y, x^\sharp) = N(x-y)$. Since J is division, $\lambda = 0$ if and only if $x = y$. Finally observe that

$$\psi\left(\begin{pmatrix} N(x) & x \\ x^\sharp & 1 \end{pmatrix}_+, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}_+\right) = -s_+.$$

The second statement is now clear. □

Remark 3.5.9. By Corollary 3.5.6 the only proper non-trivial inner ideals of the structurable algebra $\mathcal{A}$ are 1-dimensional, and form the Moufang set corresponding to J (see [DM20]). Proposition 3.5.4 shows that if x is an extremal element of $\mathcal{A}$, then x_+ is an extremal element of $L = K(\mathcal{A})$. Hence, we get an embedding of the Moufang set into the geometry of inner ideals of L, which will turn out to be the Moufang hexagon associated to J; see Theorem 3.5.11 below.

Lemma 3.5.10. *Any proper inner ideal I containing $\mathcal{S}_+$ is either $\mathcal{S}_+$ itself or equals $\langle a_+, s_+ \rangle$ for some extremal element $a_+ \in \mathcal{A}_+$. Moreover, any non-zero element in I is extremal.*

Proof. We may assume $I \neq \mathcal{S}_+$. By Theorem 3.5.3 and Lemma 3.1.12, we get $\mathcal{S}_+ \lneqq I \le \mathcal{A}_+ \oplus \mathcal{S}_+$. Hence $a_+ \in I$ for some non-zero $a \in \mathcal{A}$. Then I also contains $[a_+, [a_+, s_-]] = -V_{a,sa} \in L_0$, hence $V_{a,sa} = 0$. By Proposition 3.5.4, a_+ is then extremal.

Suppose that I is at least 3-dimensional. By the previous paragraph, any element of I is of the form $\mu a_+ + \lambda s_+$, for some $\lambda, \mu \in k$ and some extremal element a_+. Since $\dim(I) \ge 3$, we can find two linearly independent extremal elements a_+ and b_+ in $I \cap \mathcal{A}_+$. Since I is abelian, $\psi(a_+, b_+) = [a_+, b_+] = 0$, but this contradicts Corollary 3.5.8. Hence I must be 2-dimensional and hence equal to $\langle a_+, s_+ \rangle$ for some extremal element $a_+ \in \mathcal{A}_+$. The last claim follows from Proposition 3.5.4(e). □

Theorem 3.5.11. *Let $\mathcal{A}$ be the structurable algebra $M(J, 1)$ (of skew-dimension one) over a field of characteristic different from 2 and 3, where J is a cubic Jordan division algebra. Set $L = K(\mathcal{A})$. Consider the graph $\Omega = (V, E)$, with*

$$V = \{I \mid I \text{ is a proper non-trivial inner ideal of } L\},$$
$$E = \{\{I, K\} \mid I \lneqq K\}.$$

Then Ω is the incidence graph of a generalized hexagon.

Proof. Let I be an arbitrary proper non-trivial inner ideal. By Theorem 3.5.3 and Theorem 3.1.14, there exists an element of $E(\mathcal{A})$ mapping I to an inner ideal containing $\mathcal{S}_+$. By Lemma 3.5.10, the only proper non-trivial inner ideals containing $\mathcal{S}_+$ are $\mathcal{S}_+$ itself and the inner ideals $\langle a_+, s_+\rangle$ for an extremal element $a_+ \in \mathcal{A}_+$. Moreover, all non-zero elements of such an inner ideal are extremal, i.e., these inner ideals are singular. This implies that any line in the extremal geometry $(\mathcal{E}(L), \mathcal{F}(L))$ is a maximal singular subspace. Theorem 2.3.17 now implies that $(\mathcal{E}(L), \mathcal{F}(L))$ is a generalized hexagon (notice that the conditions of this theorem are satisfied by Corollary 1.1.72 and Corollary 3.5.8). Since all proper non-trivial inner ideals are either points or lines of the extremal geometry, we conclude that Ω is the incidence graph of a generalized hexagon. $\square$

Remark 3.5.12. In [Fau77, Chapter 11], John Faulkner defines a Lie algebra starting from a Jordan cubic division algebra J, which we will denote by $F(J)$. There is a lot of evidence that this Lie algebra is isomorphic to $K(M(J, 1))$, but we have not pursued this in detail. Indeed, in [Fau77, Chapter 12], John Faulkner proves that the geometry with as points the 1-dimensional inner ideals of $F(J)$ and as lines the 2-dimensional inner ideals of $F(J)$ containing at least two 1-dimensional inner ideals, with inclusion as incidence, form a generalized hexagon. If it is indeed true that $K(M(J, 1)) \cong F(J)$, then Theorem 3.5.11 is a generalization of Faulkners' result, in the sense that we are considering all inner ideals (rather than only the 1-dimensional ones and the 2-dimensional ones containing at least two 1-dimensional ones). Moreover, our approach also allows to identify the Moufang sets associated to J in the Moufang hexagon associated to J (see Remark 3.5.9).

Remark 3.5.13. Let us again (as in Remark 3.4.21 for the triangle case) try to obtain each inner ideal as the end of a $\mathbb{Z}$-grading. Of course, the inner ideal $\mathcal{S}_+$ is the end of a $\mathbb{Z}$-grading on L.

Let us now consider a 2-dimensional inner ideal; then this inner ideal will also arise as the end of a $\mathbb{Z}$-grading on L. Indeed, note that $\mathrm{ad}_{T_{s_0}}$ is a grading derivation with components

$$\begin{aligned}
L_{-3} &= \left(\begin{smallmatrix} 0 & 0 \\ 0 & k \end{smallmatrix}\right)_- \oplus \left(\begin{smallmatrix} 0 & 0 \\ 0 & k \end{smallmatrix}\right)_+ \\
L_{-2} &= T_{\left(\begin{smallmatrix} 0 & 0 \\ J & 0 \end{smallmatrix}\right)} \\
L_{-1} &= \left(\begin{smallmatrix} 0 & J \\ 0 & 0 \end{smallmatrix}\right)_- \oplus \left(\begin{smallmatrix} 0 & J \\ 0 & 0 \end{smallmatrix}\right)_+ \\
L_0 &= \langle T_{\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)}\rangle \oplus \langle T_{\left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)}\rangle \oplus \mathrm{Inder}(\mathcal{A}) \oplus \mathcal{S}_- \oplus \mathcal{S}_+ \\
L_1 &= \left(\begin{smallmatrix} 0 & 0 \\ J & 0 \end{smallmatrix}\right)_- \oplus \left(\begin{smallmatrix} 0 & 0 \\ J & 0 \end{smallmatrix}\right)_+ \\
L_2 &= T_{\left(\begin{smallmatrix} 0 & J \\ 0 & 0 \end{smallmatrix}\right)} \\
L_3 &= \left(\begin{smallmatrix} k & 0 \\ 0 & 0 \end{smallmatrix}\right)_- \oplus \left(\begin{smallmatrix} k & 0 \\ 0 & 0 \end{smallmatrix}\right)_+,
\end{aligned}$$

using $V_{\left(\begin{smallmatrix} 0 & J \\ 0 & 0 \end{smallmatrix}\right),\left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)} = V_{\left(\begin{smallmatrix} 0 & 0 \\ J & 0 \end{smallmatrix}\right),\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)} = V_{\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right),\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)} = V_{\left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right),\left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)} = 0$ and Lemma 1.1.51. So in this case, as opposed to the triangle case, we do *not* get a Jordan pair, and not even a Kantor pair.

Remark 3.5.14. We have chosen to use the matrix structurable algebra $M(J,1)$, but we could also have chosen $M(J,\eta)$ for any parameter $\eta \in k^\times$. Indeed, the structurable algebras $M(J,1)$ and $M(J,\eta)$ are isotopic (see, e.g., [Gar01, Proposition 4.11 and Lemma 4.13], where the result is stated for Albert algebras J but holds in general). Since isotopic structurable algebras give rise to graded-isomorphic Lie algebras under the TKK-construction by [AH81, Proposition 12.3], this does not affect the resulting geometry.

Now we show that the generalized hexagon is, in fact, a Moufang hexagon. Since we generalized the method used in Section 7 of [DMM20] in Section 4.4 to determine the root groups even if the characteristic equals 2 or 3, we will make use of the results of that section. For a self-contained proof, see [DMM20].

Theorem 3.5.15. *Let Ω be the incidence graph of a generalized hexagon from Theorem 3.5.11. Then Ω is the incidence graph of the Moufang hexagon associated to the cubic Jordan division algebra J.*

Proof. By Proposition 4.2.2 the Lie algebra $K(\mathcal{A})$ is a simple non-degenerate Lie algebra generated by its extremal elements. By (the proof of) Theorem 3.5.11 the incidence graph of the extremal geometry coincides with Ω and the extremal geometry is a generalized hexagon. In Section 4.4 we recover a cubic norm structure on a subspace of L_{-1}, see Notation 4.4.3 and Construction 4.4.11, which we now denote by J_1. Choose s_-, s_+, $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)_-$ and $\left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)_-$ as x, y, e_1' and e_2' in Notation 4.4.1. Then $e_1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)_+$ and $e_2 = -\left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)_+$ in the same notation. Then by Notation 4.4.27, J_1 is the (-1)-eigenspace of $\mathcal{A}_-$ with respect to $\mathrm{ad}_{[e_1',e_2]}$. Using $[e_1', e_2] = V_{\left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right),\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)}$ we see that J_1 coincides with $\{\left(\begin{smallmatrix} 0 & 0 \\ j & 0 \end{smallmatrix}\right)_- \mid j \in J\}$. So $\sigma : j \mapsto \left(\begin{smallmatrix} 0 & 0 \\ -j & 0 \end{smallmatrix}\right)_-$ is a bijection between the vector space J and the vector space J_1. Now in Construction 4.4.11 we define the corresponding cross product which we now denote by $\sharp_1$ as follows (recall Theorem 4.3.11(vi) and $\mathrm{char}(k) \neq 2$):

$$\begin{aligned} \left(\begin{smallmatrix} 0 & 0 \\ j & 0 \end{smallmatrix}\right)_-^{\sharp_1} &= \frac{1}{2}[\left(\begin{smallmatrix} 0 & 0 \\ j & 0 \end{smallmatrix}\right)_-, [\left(\begin{smallmatrix} 0 & 0 \\ j & 0 \end{smallmatrix}\right)_-, e_2]] \\ &= \frac{1}{2} U_{\left(\begin{smallmatrix} 0 & 0 \\ j & 0 \end{smallmatrix}\right)} \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)_- = \frac{1}{2}\left(\begin{smallmatrix} 0 & j \times j \\ 0 & 0 \end{smallmatrix}\right)_- = \begin{pmatrix} 0 & j^\sharp \\ 0 & 0 \end{pmatrix}_-, \end{aligned} \tag{3.9}$$

for any $j \in J$. Note

$$\begin{aligned} e_-(-\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)_-)(e_2) &= -\frac{1}{3}[\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)_-, \left(\begin{smallmatrix} 0 & 1^\sharp \\ 0 & 0 \end{smallmatrix}\right)_-] + \left(\begin{smallmatrix} 0 & 1^\sharp \\ 0 & 0 \end{smallmatrix}\right)_- + V_{e_2,\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)} + e_2 \\ &= s_- + \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)_- + V_{e_2,\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)} + e_2, \end{aligned}$$

hence $N_1(-\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)_-) = 1$, with N_1 the norm as is Construction 4.4.11. So we have a basepoint for N_1, and find a bijection σ_2 between $J_1' = \{\left(\begin{smallmatrix} 0 & j \\ 0 & 0 \end{smallmatrix}\right)_- \mid j \in J\}$ and J_1

as in Definition 4.4.16 and a straightforward calculation shows that this bijection is the obvious one, namely sending $\left(\begin{smallmatrix} 0 & j \\ 0 & 0 \end{smallmatrix}\right)_-$ to $\left(\begin{smallmatrix} 0 & 0 \\ -j & 0 \end{smallmatrix}\right)_-$. Then Construction 4.4.19 and (3.9) show that $\left(\begin{smallmatrix} 0 & 0 \\ j & 0 \end{smallmatrix}\right)_-^{\sharp_{J_1}} = \begin{pmatrix} 0 & 0 \\ -j^\sharp & 0 \end{pmatrix}_-$. Hence the bijection σ is an isomorphism between the cubic norm structure $(J, k, \sharp)$ and the cubic norm structure $(J_1, k, \sharp_{J_1})$. Now Theorem 4.4.38 concludes this proof. $\square$

Remark 3.5.16. The explicit description of the root groups, as described in Section 4.4.2, is related to Peirce subspaces. More precisely, if one considers the idempotent $e = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, and sets

$$\mathcal{A}_{ij} = \{x \in \mathcal{A} \mid ex = ix, xe = jx\},$$

for $i, j = 0, 1$, we get $U_2 = e_-(\mathcal{A}_{11}, 0)$, $U_3 = e_-(\mathcal{A}_{01}, 0)$, $U_5 = e_-(\mathcal{A}_{10}, 0)$ and finally $U_6 = e_-(\mathcal{A}_{00}, 0)$, where $\mathcal{A} = M(J)$, with J a cubic Jordan division algebra.

A similar remark applies to the triangle case. In this case, consider $\mathcal{A} = F \oplus F$ as in Construction 3.4.2, with F an alternative division algebra. Then $e = (1, 0)$ is an idempotent and Theorem 3.4.19 yields $U_1 = e_-(\mathcal{A}_{11}, 0)$ and $U_3 = e_-(\mathcal{A}_{00}, 0)$.

CHAPTER 4

From extremal geometries to algebraic structures

We start this chapter in Section 4.1 by associating a 5-grading to certain pairs of extremal elements and then we deduce certain properties of these gradings.

Section 4.2 is based on the last section of [CM21]. We show that any finite-dimensional simple Lie algebra over a field of characteristic different from 2, 3 is generated by its extremal elements if, and only if, L is isomorphic to the symplectic Lie algebra $\mathfrak{sp}(V, f)$ for some non-degenerate symplectic space (V, f), or L is obtained by applying the Tits-Kantor-Koecher construction to a (finite-dimensional) simple structurable algebra $\mathcal{A}$ of skew-dimension 1. This result is obtained without using any classification results on spherical buildings, (classical) Lie algebras or structurable algebras.

In Section 4.3 we again consider arbitrary simple Lie algebras generated by pure extremal elements and show that there exist certain automorphisms which depend on elements in the (-1)- and 1-part of the 5-grading induced by two hyperbolic extremal elements. Essentially, we extend the notion of algebraicity, as defined in Definition 1.1.68, to these specific Lie algebras in characteristic 2 and 3 as well, and prove that these Lie algebras satisfy this extended notion.

We start Section 4.4 by showing that we can associate a cubic norm structure to the Lie algebra if its extremal geometry contains lines and is not of a specific type, which extends a result of Section 4.2 to characteristic 2 and 3. Then we use this correspondence in the specific case of an anisotropic cubic norm structure and show that the extremal geometry is then the Moufang hexagon associated with this anisotropic cubic norm structure. We can use this result to show that a simple Lie algebra generated by its pure extremal elements is characterized by its extremal geometry if this geometry has lines. In the final subsection of Section 4.4 we have a look at the case when there are lines in the extremal geometry but there is no associated cubic norm structure.

In Section 4.5 we look at the case in which there are symplectic pairs in the extremal geometry, and also assume that after a Galois extension of degree at most 2 there are lines in the extremal geometry. In particular, we allow the line set of the extremal geometry to be the empty set. We show that, if the characteristic of the field is not 2, we can associate a quadrangular algebra to the Lie algebra. Then we use this correspondence in the specific case an anisotropic quadrangular algebra and show that the inner ideal geometry is the Moufang quadrangle associated with this anisotropic quadrangular algebra.

In the final section we consider the remaining case of an extremal geometry with no lines, nor symplectic pairs and show that the extremal points, together with the appropriate root groups, form a Moufang set.

It is advised to read Chapter 2, and especially Sections 2.1 and 2.3, before reading this chapter.

SECTION 4.1

Gradings from extremal elements

In this section we describe the 5-grading which we can associate with any two hyperbolic extremal elements, and prove several results on this grading which we will need in the rest of this chapter. In this section we do not make any assumptions on the field k over which the Lie algebra L is defined.

We start by recalling the $\mathbb{Z}$-grading from Lemma 2.3.20. This grading plays a central role in this section.

Lemma 4.1.1 ([CI06, Proposition 22]). *Suppose that there exist extremal elements $x, y \in L$ such that $g_x(y) = 1$. Then L has a $\mathbb{Z}$-grading*

$$L = L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus L_2,$$

with $L_{-2} = \langle x \rangle$, $L_2 = \langle y \rangle$, $L_0 = N_L(x) \cap N_L(y)$, $L_1 = [y, U]$ and $L_{-1} = [x, U]$, where

$$U = \{u \in L \mid g_x(u) = g_y(u) = g_x([y, u]) = 0\}.$$

Moreover, L_i is contained in the i-eigenspace of $\mathrm{ad}_{[x,y]}$, and ad_x defines a linear isomorphism from L_1 tot L_{-1} with inverse $-\mathrm{ad}_y$.

Recall that the i-component of $x \in L$ is the image of the projection of x onto L_i. In the next sections we will often consider different 5-gradings, but *the i-*

component will always refer to the component according to the grading denoted by $\oplus_{i=-2}^{2} L_i$.

Using the following lemma we can describe g_x and g_y more explicitly.

Lemma 4.1.2 ([CI06, Lemma 18, 19]). *Consider $a, b \in E$ and $l \in L$. Then*

$$\begin{aligned} g_a(b) &= g_b(a), \\ g_a([b,l]) &= -g_b([a,l]). \end{aligned}$$

Also, if $a \in E$ and $b \in L$ satisfy $[a,b] = 0$, then $g_a(b) = 0$ and $g_a([b,l]) = 0$ for all $l \in L$.

Lemma 4.1.3. *Let x, y and L_i be as in Lemma 4.1.1. Then $g_x(L_{\leq 1}) = 0 = g_y(L_{\geq -1})$.*

Proof. By the last part of Lemma 4.1.2, $g_x(L_{\leq -1}) = 0$. Lemma 21 of [CI06] together with $L_0 \leq N_L(x)$ implies $g_x(L_0) = 0$. Now by definition of U, we have $g_x(L_1) = g_x([y,U]) = 0$. The argument for $g_y(L_{\geq 0}) = 0$ is completely similar. Finally $g_y(L_{-1}) = g_y([x,U]) = g_x([y,U]) = 0$, by Lemma 4.1.2. □

The following lemma describes an automorphism which switches x and y, and also switches L_{-1} and L_1, while stabilizing L_0. In the lemma thereafter we describe certain automorphisms which stabilize the components L_i.

Lemma 4.1.4. *Let x, y and L_i be as in Lemma 4.1.1. Consider the automorphism $\varphi = \exp(y)\exp(x)\exp(y)$. Then*

$$\varphi(\lambda x + l_{-1} + l_0 + l_1 + \mu y) = \mu x + [x, l_1] + ([x,[y,l_0]] + l_0) + [y, l_{-1}] + \lambda y,$$

for any $\lambda, \mu \in k$ and $l_i \in L_i$, $i = -1, 0, 1$.

Proof. By $g_x(y) = g_y(x) = 1$, we get $\varphi(x) = y$ and $\varphi(y) = x$. For any $l_1 \in L_1$ we get $\varphi(l_1) = \exp(y)([x,l_1] + l_1) = [x,l_1] + (l_1 + [y,[x,l_1]]) = [x,l_1]$, using $g_x(L_1) = 0 = g_y(L_{-1})$, see Lemma 4.1.3. Similarly $\varphi(l_{-1}) = [y, l_{-1}]$ for all $l_{-1} \in L_{-1}$. Let $l_0 \in L_0$ be arbitrary. Let $\lambda \in k$ be such that $[x, l_0] = \lambda x$. Using Lemma 4.1.2, we get

$$[x,l_0] + g_x([y,l_0])x = \lambda x - g_y([x,l_0])x = \lambda x - \lambda g_y(x)x = 0.$$

So, using Lemma 4.1.3 multiple times and using $[y, l_0] \in L_2$ in the last line, we get

$$\begin{aligned} \varphi(l_0) &= \exp(y)(\exp(x)(l_0 + [y,l_0])) \\ &= \exp(y)(([x,l_0] + g_x([y,l_0])x) + ([x,[y,l_0]] + l_0) + [y,l_0]) \\ &= \exp(y)(([x,[y,l_0]] + l_0) + [y,l_0]) \\ &= ([x,[y,l_0]] + l_0) + ([y,l_0] + [y,[x,[y,l_0]]] + [y,l_0]) \\ &= ([x,[y,l_0]] + l_0) + (2[y,l_0] + [[y,x],[y,l_0]]]) \\ &= [x,[y,l_0]] + l_0. \end{aligned}$$

□

Lemma 4.1.5. *Let x, y and L_i be as in Lemma 4.1.1. Let $0 \neq \lambda \in k$ be arbitrary. Consider the map $\varphi_\lambda : L \to L$ defined by $\varphi_\lambda(l_i) = \lambda^i l_i$ for all $l_i \in L_i$, with $i \in \{-2, -1, 0, 1, 2\}$. Then $\varphi_\lambda \in \mathrm{Aut}(L)$.*

Proof. Clearly φ_λ is bijective. Consider $l_i \in L_i$ and $l_j \in L_j$ arbitrary, with $i, j \in \{-2, -1, 0, 1, 2\}$, then

$$\varphi_\lambda([l_i, l_j]) = \lambda^{i+j}[l_i, l_j] = [\lambda^i l_i, \lambda^j l_j] = [\varphi_\lambda(l_i), \varphi_\lambda(l_j)]. \qquad \square$$

The following lemma will ensure the non-degeneracy of certain maps later on.

Lemma 4.1.6. *Let x, y and L_i be as in Lemma 4.1.1. Assume moreover that L is simple. If $a \in L_{\sigma 1}$ satisfies $[a, b] = 0$ for all $b \in L_{\sigma 1}$, then $a = 0$, with $\sigma \in \{-, +\}$.*

Proof. Let Z_{-1} be the subspace of L consisting of all $a \in L_{-1}$ satisfying $[a, b] = 0$ for all $b \in L_{-1}$. Similarly one defines Z_1. For $z \in Z_{-1}$ and $a \in L_{-1}$ we get, using the Premet identities, Lemma 4.1.3 and $z, a \in L_{-1}$,

$$[[y, a], [y, z]] = g_y([a, z]) + g_y(z)[y, a] - g_y(a)[y, z] = 0.$$

Hence $[y, Z_{-1}] \leq Z_1$ and, since $-\operatorname{ad}_x \operatorname{ad}_y$ fixes all elements of L_{-1}, we get $[y, Z_{-1}] = Z_1$. Also note $[[L_0, Z_{-1}], L_{-1}] = [L_0, [Z_{-1}, L_{-1}]] + [Z_{-1}, [L_0, L_{-1}]] = 0$ and thus $[L_0, Z_{-1}] \leq Z_{-1}$ and similarly $[L_0, Z_1] \leq Z_1$. Moreover, by

$$[y, [Z_{-1}, L_1]] = [[y, Z_{-1}], L_1] = [Z_1, L_1] = 0$$

and Lemmas 4.1.1 and 4.1.4 we get $[Z_{-1}, L_1] = [[y, Z_{-1}], [x, L_1]] = [Z_1, L_{-1}]$. Now one verifies that

$$Z_{-1} \oplus [Z_{-1}, L_1] \oplus Z_1$$

is an ideal of L. Hence $Z_{-1} = 0$ and $Z_1 = 0$. $\square$

Lemma 4.1.7. *Consider $x, y, a, b \in E$ such that $g_x(y) = 1 = g_a(b)$. Let $L = L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus L_2$ and $L = L'_{-2} \oplus L'_{-1} \oplus L'_0 \oplus L'_1 \oplus L'_2$ be the 5-gradings corresponding to the hyperbolic pairs (x, y) and (a, b), respectively, as in Lemma 4.1.1, so with $x \in L_{-2}$, $a \in L'_{-2}$, $y \in L_2$, $b \in L'_2$.*

If $\varphi \in \mathrm{Aut}(L)$ maps L_{-2} and L_2 onto L'_{-2} and L'_2, respectively, then $\varphi(L_i) = L'_i$ for all $i \in \{-2, \dots, 2\}$.

Proof. Consider $0 \neq \lambda, \mu \in k$ such that $\varphi(x) = \lambda a$ and $\varphi(y) = \mu b$. We get $L_0 = N_L(x) \cap N_L(y)$ and $L'_0 = N_L(a) \cap N_L(b)$ by Lemma 4.1.1, so L_0 is mapped onto L'_0 by φ. Also, $L_1 = [y, U]$ and $L'_1 = [b, U']$, with U as in Lemma 4.1.1 and $U' = \{u \in L \mid g_a(u) = g_b(u) = g_a([b, u]) = 0\}$. Then

$$\begin{aligned}\varphi(U) &\leq \{u \in L \mid g_{\varphi(x)}(u) = g_{\varphi(y)}(u) = g_{\varphi(x)}([\varphi(y), u]) = 0\} \\ &= \{u \in L \mid \lambda g_a(u) = \mu g_b(u) = \lambda\mu g_a([b, u]) = 0\} = U',\end{aligned}$$

using $\lambda, \mu \neq 0$. Hence $\varphi(L_1) = \varphi([y, U]) \leq [b, U'] = L'_1$, similarly $\varphi(L_{-1}) \leq L'_{-1}$. Since φ is an automorphism, these containments are actually equalities. $\square$

The next lemma connects the grading to extremal elements in a certain relation with x.

Lemma 4.1.8. *Let x, y and L_i be as in Lemma 4.1.1. Assume moreover that L is generated by E. For all $i \in \{-1, 0, 1, 2\}$, we have*

$$E_i(x) = (E \cap L_{\leq i}) \backslash (E \cap L_{\leq i-1}).$$

Proof. This claim is shown in the proof [CI06, Theorem 28]. □

In the next three lemmas we prove some facts on the grading if the extremal geometry has lines. Recall that if L is generated by its set of extremal elements E, then there exists a bilinear symmetric form g on L which associates with the Lie bracket such that $g(x, y) = g_x(y)$ for all $x, y \in E$, see Proposition 2.3.6.

Lemma 4.1.9. *Let L be a Lie algebra generated by extremal elements. Let x, y and L_i be as in Lemma 4.1.1. If the extremal geometry of L contains lines, then for each element $a \in E_{-1}(x) \cap E_1(y)$ we have $a \in L_{-1}$.*

Proof. By Lemma 4.1.8 and a similar property for $E_i(y)$, which we can obtain by Lemma 4.1.8 and Lemma 4.1.4. □

Lemma 4.1.10. *Let x, y and L_i be as in Lemma 4.1.1 and $\sigma \in \{-, +\}$. Assume that L is generated by E. Consider $e \in E \cap L_{\sigma 1}$. Then $g_e(x) = 0$ for all $x \in L_i$ with $i \neq -\sigma 1$.*

Proof. We will assume $\sigma = +$. By $g_x(l) = 0$ for all $l \in L_{\leq 1}$ and $g_y(l) = 0$ for all $l \in L_{\geq -1}$, we get $g_e(x) = 0 = g_e(y)$. Moreover, for any $l \in L_1$, there exists $l' \in L_{-1}$ such that $l = [y, l']$ by Lemma 4.1.1. By Lemma 4.1.2 we have $g_e(l) = g_e([y, l']) = g([e, y], l') = 0$. Similarly, there exist $e' \in L_{-1}$ such that $e = [y, e']$. Consider $l \in L_0$ arbitrary, by $[y, l] \leq \langle y \rangle$ there exists a $\lambda \in k$ such that $[y, l] = \lambda y$. Then $g_e(l) = g([y, e'], l) = -g(e', [y, l]) = \lambda g_y(e') = 0$, using $e' \in L_{-1}$. □

Lemma 4.1.11. *Let x, y and L_i be as in Lemma 4.1.1. Assume moreover that L is generated by E. If the extremal geometry of L contains lines, then there exist $a_0, b_0 \in E \cap L_{-1}$ such that $[a_0, b_0] = x$. Moreover,*

$$[a_0 + b_0, -[a_0, y] + [b_0, y]] = [x, y].$$

Proof. Fix an extremal point a in $\mathcal{E}_{-1}(\langle x \rangle) \cap \mathcal{E}_1(\langle y \rangle)$. By Lemma 4.1.9 we find $a \leq L_{-1}$.

Now, by Lemma 2.1.9(f), we can find a point b collinear with $\langle x \rangle$ and special with a. Since $\mathcal{E}_{\leq 0}(a)$ and $\mathcal{E}_{\leq 1}(a)$ are subspaces of the extremal geometry, any point of the line through $\langle x \rangle$ and b, with the exception of $\langle x \rangle$, is special with a. Consequently, we may assume that b is special with y as well and is thus contained in L_{-1}, again using Lemma 4.1.9. Since a and b are special and $\langle x \rangle$ is a common neighbor, we get $[a, b] = \langle x \rangle$. Since a and $\langle y \rangle$ are special, $[a, \langle y \rangle]$ is the unique point collinear with both a and $\langle y \rangle$. Clearly, $\langle x \rangle$ is special with $[a, \langle y \rangle]$ and thus $[a, \langle y \rangle] \leq L_1$ by Lemma 4.1.9. Similarly for b and $\langle y \rangle$. Note that we can find $a_0 \in a$ and $b_0 \in b$ such that $[a_0, b_0] = x$. Now note

$$\begin{aligned}[a_0 + b_0, -[a_0, y] + [b_0, y]] &= -2g_{a_0}(y)a_0 + 2g_{b_0}(y)b_0 + [a_0, [b_0, y]] - [b_0, [a_0, y]] \\ &= [a_0, [b_0, y]] + [b_0, [y, a_0]] = -[y, [a_0, b_0]] \\ &= -[y, x] = [x, y],\end{aligned}$$

where we used $(a, \langle y \rangle), (b, \langle y \rangle) \in \mathcal{E}_1$ and the Jacobi identity. □

SECTION 4.2

Recovering a structurable algebra

In this section we describe how one can associate a (skew-dimension one) structurable algebra to a simple Lie algebra generated by its extremal elements.

Assumption 4.2.1. In this section we assume the characteristic of the field to be distinct from 2 and 3. We also assume all algebras to be finite-dimensional in this section.

Proposition 4.2.2. *If $\mathcal{A}$ is a simple skew-dimension one structurable algebra, then $K(\mathcal{A})$ is a simple non-degenerate Lie algebra generated by its extremal elements.*

Proof. By [All79, Corollary 6], $K(\mathcal{A})$ is a simple Lie algebra. Clearly s_+ is an extremal element which is not a sandwich by Proposition 1.1.61, with $0 \neq s \in \mathcal{S}$ arbitrary. Hence [CIR08, Theorem 1.1] and Corollary 2.3.11 imply that $K(\mathcal{A})$ is generated by its extremal elements and is non-degenerate, unless $K(\mathcal{A})$ is isomorphic to $W_{1,1}(5)$ and the characteristic of the field equals 5. The Witt algebra $W_{1,1}(5)$ is a 5-dimensional Lie algebra. The dimension of $K(\mathcal{A})$ equals $2\dim(\mathcal{A}) + 2\dim(\mathcal{S}) + \dim(\mathrm{Instrl}(\mathcal{A}))$. Since $1 = \dim(\mathcal{S}) < \dim(\mathcal{A})$, the Lie algebra $K(\mathcal{A})$ can never have dimension equal to 5. □

We now focus on the converse of the above proposition. We will show in a series of steps that any simple non-symplectic Lie algebra which is generated by its extremal elements can be obtained by the above construction.

Recall that by the definition of the Lie bracket on $K(\mathcal{A})$ we clearly see that this Lie algebra has a 5-grading given by $K(\mathcal{A})_j = 0$ for all $|j| > 2$ and

$$K(\mathcal{A})_{-2} = \mathcal{S}_-,\ K(\mathcal{A})_{-1} = \mathcal{A}_-,\ K(\mathcal{A})_0 = \mathrm{Instrl}(\mathcal{A}),$$
$$K(\mathcal{A})_1 = \mathcal{A}_+,\ K(\mathcal{A})_2 = \mathcal{S}_+\,.$$

Even more generally, for any Kantor pair (P^-, P^+) there exists a 5-graded Lie algebra denoted by $\mathrm{TKK}(P^-, P^+)$ and with 1-component isomorphic with P^+ and (-1)-component isomorphic with P^-. Since the construction itself is not relevant here, we refer to [AF99, §4] for more details.

Lemma 4.2.3. *Let $L = L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus L_2$ be a 5-grading of a simple Lie algebra L. Then (L_{-1}, L_1) is a Kantor pair with products defined by*

$$\{x, y, z\}^\sigma = [[x, y], z],$$

with $x, z \in L_{\sigma 1}$, $y \in L_{-\sigma 1}$ and $\sigma = \pm$. If $(L_{-1}, L_1) \neq 0$, L is graded-isomorphic to $\mathrm{TKK}(L_{-1}, L_1)$ *and $L_0 = [L_{-1}, L_1]$.*

Proof. The first statement is [AFS17, 4.2, page 728]. The second statement follows from Lemma 4.4, Definition 4.5 and Corollary 4.17 combined with Proposition 4.19 of *loc. cit.* □

Lemma 4.2.4 ([Sta20, Lemma 4.13])**.** *Let $L = \mathrm{TKK}(L_{-1}, L_1)$ be the 5-graded Lie algebra associated with the Kantor pair (L_{-1}, L_1). Then (L_{-1}, L_1) is the Kantor pair associated with a structurable algebra $\mathcal{A}$ if and only if there exist $u \in L_{-1}$ and $v \in L_1$ such that $[u, v]$ is the grading derivation of L. In that case L is graded-isomorphic to $K(\mathcal{A})$.*

Now we are able to apply these results to our setting of extremal geometries.

Lemma 4.2.5. *Let L be a simple Lie algebra generated by pure extremal elements. Let x, y and L_i be as in Lemma 4.1.1. If L is not a symplectic Lie algebra, then there exist $e \in L_{-1}$ and $f \in L_1$ such that $[x, y] = [e, f]$.*

Proof. Assume that the extremal geometry of L contains lines. Let a_0 and b_0 be as in Lemma 4.1.11. Then consider $e := a_0 + b_0$ and $f := -[a_0, y] + [b_0, y]$ and note $e \in L_{-1}, f \in L_1$. By Lemma 4.1.11 we conclude $[x, y] = [e, f]$.

Assume now that the extremal geometry does not contain any lines and that L is not symplectic. Then by Theorem 2.4.2 we find three extremal elements x, y, z where x and y generate an $\mathfrak{sl}_2$ not containing z, and $g_z(u) \neq 0$ for all extremal elements u in this $\mathfrak{sl}_2$. Denote the subalgebra of L generated by x, y and z by H. From the proof of Lemma 2.4.6 we see that $g(x, [y, z])^2 - 4g(x, y)g(x, z)g(y, z) \neq 0$. As described in [CSUW01, p. 130], we can find $z' \in E \cap H$ such that $g(x, [y, z']) = 0$, $g(x, z') = g(x, z) \neq 0$ and

$$g(y, z') = g(y, z) - \frac{g(x, [y, z])^2}{4g(x, z)g(x, y)} \neq 0,$$

by $g(x,[y,z])^2 - 4g(x,y)g(x,z)g(y,z) \neq 0$. (Recall that the extremal form f from *loc. cit.* satisfies $f = 2g$.) Then Theorem 5.2 of *loc. cit.* implies that there exists a Galois extension k'/k of degree at most 2 such that $H \otimes k'$ is a quotient of $\mathfrak{sl}_3(k')$. Since the characteristic is not equal to 3 we get that $H \otimes k'$ is isomorphic to $\mathfrak{sl}_3$. The extremal geometry of this Lie algebra is a root shadow space of type $A_{2,\{1,2\}}$. If $k' = k$, we can apply the first paragraph to find $e \in H_{-1}$ and $f \in H_1$ such that $[x,y] = [e,f]$. Then $H_{-1} \leq L_{-1}$ and $H_1 \leq L_1$ conclude the proof in this case. Assume k'/k is a degree 2 extension and let σ be the non-trivial element of its Galois group, we also denote the Lie algebra automorphism mapping $h \otimes \lambda$ onto $h \otimes \lambda^\sigma$ by σ. Since the extremal geometry of $H \otimes k'$ contains lines, we can find $a \in E(H \otimes k') \cap (H \otimes k')_{-1}$. If a^σ is a multiple of a, then a is a multiple of $a_1 \otimes 1$ for $a_1 \in H_{-1}$, and then there would be lines in the extremal geometry of H and we can apply the first paragraph. Note that a^σ is contained in both $E(H \otimes k')$ and $(H \otimes k')_{-1}$. Since the extremal geometry of $H \otimes k'$ is of type $A_{2,\{1,2\}}$, which is a generalized hexagon, $a^\sigma \notin \langle a \rangle$ implies $(a, a^\sigma) \in E_1$. Hence $[a, a^\sigma] = \lambda x$ for certain non-zero $\lambda \in k'$. Note $\lambda^\sigma x = (\lambda x)^\sigma = [a, a^\sigma]^\sigma = -[a, a^\sigma] = -\lambda x$ and thus $\lambda^\sigma = -\lambda$. Set $e = a + a^\sigma$, and $f = \lambda^{-1}[y,a] + (\lambda^{-1}[y,a])^\sigma = \lambda^{-1}[y,a] - \lambda^{-1}[y,a^\sigma]$. Since e and f are fixed by σ, they are contained in $H_{-1} \leq L_{-1}$ and $H_1 \leq L_1$, respectively. By Lemmas 4.1.2 and 4.1.3, we have $g_a(y) = g_y(a) = 0$ and $g_{a^\sigma}(y) = g_y(a^\sigma) = 0$. Hence,

$$\begin{aligned}[e,f] &= -2\lambda^{-1} g_a(y) a + 2 g_{a^\sigma}(y) a^\sigma - \lambda^{-1}[a,[y,a^\sigma]] + \lambda^{-1}[a^\sigma,[y,a]] \\ &= \lambda^{-1}[y,[a^\sigma,a]] = \lambda^{-1}\lambda[x,y] = [x,y]. \qquad \square\end{aligned}$$

Remark 4.2.6. The proof of Lemma 4.2.5 also holds for infinite-dimensional simple Lie algebras of characteristic not 2 or 3 generated by their pure extremal elements.

Theorem 4.2.7. *Let L be a simple finite-dimensional Lie algebra over a field of characteristic different from 2, 3 generated by extremal elements. Unless $L = \mathfrak{sp}(V,f)$ for some non-degenerate symplectic space (V,f), $L = K(\mathcal{A})$ for some simple skew-dimension one structurable algebra $\mathcal{A}$.*

Proof. Consider the 5-grading on L from Lemma 4.1.1. (The existence of x and y follows from Proposition 2.3.9 and Corollary 2.3.11.) Then $[x,y]$ is the grading derivation of this 5-grading.

Assume first $L_{-1} \neq \{0\} \neq L_1$. By Lemma 4.2.3, $L \cong \mathrm{TKK}(L_{-1}, L_1)$. By Lemma 4.2.5, the grading derivation equals $[u,v]$, for some $u \in L_{-1}$ and $v \in L_1$. Hence Lemma 4.2.4 implies that L is graded-isomorphic to $K(\mathcal{A})$, with $\mathcal{A}$ a structurable algebra. Recall that $\mathcal{S}$ denotes the set of skew elements of $\mathcal{A}$. By construction of $K(\mathcal{A})$, its (-2)- and 2-component are isomorphic with $\mathcal{S}$. Since L_{-2} and L_2 are 1-dimensional (by construction), we get $\dim(\mathcal{S}) = 1$.

Now assume $L_{-1} = \{0\} = L_1$. Then $L = L_{-2} \oplus L_0 \oplus L_2$ is a 3-graded Lie algebra. Hence it is a simple 3-graded Lie algebra with (-1)-component L_{-2} and 1-component L_2. Since L is simple we get $L_0 = [L_{-2}, L_2] = \langle [x,y] \rangle$ and

$L = \langle x\rangle \oplus \langle [x,y]\rangle \oplus \langle y\rangle$. Hence $L \simeq \mathfrak{sl}_2$, which is a symplectic Lie algebra. A contradiction. $\square$

It remains to consider the symplectic Lie algebras generated by extremal elements.

Example 4.2.8. Let A be an associative algebra with involution σ. Then $H(A,\sigma) := \{a \in A \mid a^\sigma = a\}$ is a Jordan algebra, with multiplication given by $a \circ b = \frac{ab+ba}{2}$. The transpose yields an involution of the associative algebra $\mathrm{Mat}_n(k)$. We denote this involution by $\top$.

Lemma 4.2.9. *The symplectic Lie algebra $\mathfrak{sp}(V,f)$, with (V,f) non-degenerate symplectic space of dimension $2n$, is isomorphic to $K(H(\mathrm{Mat}_n(k),\top))$.*

Proof. By [All79, p. 1868], the Lie algebra $K(H(\mathrm{Mat}_n(k),\top))$ is isomorphic to the Lie algebra

$$\left\{ \begin{pmatrix} x & y \\ z & -x^\top \end{pmatrix} \mid x \in \mathrm{Mat}_n(k), y,z \in H(\mathrm{Mat}_n(k),\top) \right\},$$

modulo its center, which is isomorphic to $\mathfrak{sp}(V,f)$. $\square$

Corollary 4.2.10. *Let L be a finite-dimensional simple Lie algebra over a field of characteristic different from 2, 3 generated by its extremal elements. Then $L = K(\mathcal{A})$, for some simple structurable algebra $\mathcal{A}$.*

Proof. By Theorem 4.2.7 and Lemma 4.2.9. $\square$

In the next few theorems we determine when the extremal geometry contains lines in terms of the associated structurable algebra.

Theorem 4.2.11. *Let $\mathcal{A}$ be a simple skew-dimension one structurable algebra. The extremal geometry in $K(\mathcal{A})$ contains lines if and only if $\mathcal{A}_-$ contains an extremal element.*

Proof. If the extremal geometry contains lines, then clearly $\mathcal{A}_-$ contains an extremal element, by Lemma 4.1.9. Conversely, assume $\mathcal{A}_-$ contains an extremal element a_-. Then by Lemma 4.1.8 we get that $(\mathcal{S}_-, \langle a_-\rangle) \in \mathcal{E}_{-1}$. $\square$

Corollary 4.2.12. *The extremal geometry of $K(M(J,\eta))$ contains lines, with $M(J,\eta)$ as in Definition 1.1.60.*

Proof. Consider $a = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Note that $s := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is an element of $\mathcal{S}$. Then $sa = a$. We have

$$V_{a,a}(x) = (a\bar{a})x + (x\bar{a})a - (x\bar{a})a = (a\bar{a})x,$$

for any $x \in M(J, \eta)$. Then $a\overline{a} = 0$ implies $V_{sa,a} = 0$. Hence $[a_-, [a_-, \mathcal{S}_+]] = 0$. Now we still need $[a_-, [a_-, \mathcal{A}_+]] \leq \langle a_- \rangle$ and $[a_-, [a_-, \mathrm{Instrl}(\mathcal{A})]] = 0$ in order to obtain that a_- is extremal. The former is equivalent to showing that $U_a(\mathcal{A}) \leq \langle a \rangle$. The latter boils down to $\psi(a, V_{x,y}(a)) = 0$ for all $x, y \in \mathcal{A}$. Both claims are straightforward to check. Hence Theorem 4.2.11 concludes this proof. □

Corollary 4.2.13. *A symplectic Lie algebra is not isomorphic to $K(\mathcal{A})$, with $\mathcal{A}$ a simple skew-dimension one structurable algebra.*

Proof. Let L be a symplectic Lie algebra. Assume $L = K(\mathcal{A})$, for some simple skew-dimension one structurable algebra $\mathcal{A}$. By Corollary 1.1.62 there exists a field extension $\hat{k}$ of k of degree at most 2 such that $\mathcal{A} \otimes_k \hat{k}$ is isomorphic to a structurable matrix algebra $M(J, \eta)$. But $K(\mathcal{A} \otimes_k \hat{k}) \cong K(\mathcal{A}) \otimes_k \hat{k}$. By Corollary 4.2.12 there are lines in the extremal geometry of this Lie algebra. On the other hand, $K(\mathcal{A}) \otimes_k \hat{k}$ is still a symplectic Lie algebra, which does not contain lines in its extremal geometry. We get a contradiction. □

Corollary 4.2.14. *Let $\mathcal{A}$ be a simple skew-dimension one structurable algebra. The extremal geometry in $K(\mathcal{A})$ contains lines if and only if $\mathcal{A}$ is isotopic to $M(J, 1)$, with J as in Definition 1.1.60.*

Proof. By Corollary 4.2.12 and [AH81, Proposition 12.3] one direction is obvious.

Assume now that the extremal geometry in $K(\mathcal{A})$ contains lines. By Theorem 4.2.11 we find $a \in \mathcal{A}$ non-zero such that a_+ is extremal. In particular $-U_a(b)_+ = [a_+, [a_+, b_-]] \in \langle a_+ \rangle$ for any $b \in \mathcal{A}$. Hence $U_a(\mathcal{A}) \leq \langle a \rangle$. Then [AF84, Theorem 4.6] and [Gar01, Lemma 4.13] conclude this proof. □

Remark 4.2.15. Combining the above Corollary 4.2.14 with Corollary 1.1.62, and Theorem 4.2.7 we obtain an algebraic proof of Theorem 2.4.7, in case the characteristic is not $2, 3$. Indeed, if L is a simple Lie algebra generated by its extremal elements, then Theorem 4.2.7 implies that, if L is not symplectic, there is a skew-dimension one structurable algebra $\mathcal{A}$ such that L is isomorphic to $K(\mathcal{A})$. Extending the field quadratically, if needed, we can, by Corollary 1.1.62 assume that $\mathcal{A}$ is a structurable matrix algebra and find the extremal geometry of L to contain lines by Corollary 4.2.14.

We need the automorphisms $e_\sigma(a, s)$ as defined in Definition 1.1.75, in order to prove that the inner ideal geometry is a Moufang set if there are no inner line ideals. Note that the proof that these are indeed automorphisms, see Theorem 1.1.71, depends on the classification of simple Lie algebras over algebraically closed fields of characteristic larger or equal than 5. However, if the characteristic of the field is strictly larger than 5 one does not need this classification, see Lemma 1.1.69.

Let $a \in \mathcal{A}$. For later use, we deduce the image of $\mathcal{S}_-$ under the automorphism $\exp(a_+)$. For any $t \in \mathcal{S}$ we have, as a special case of Lemma 1.1.76

$$\exp(a_+)(t_-) = t_- - ta_- - \tfrac{1}{2}V_{a,ta} + \tfrac{1}{6}U_a(ta)_+ + \tfrac{1}{24}\psi(a, U_a(ta))_+. \tag{4.1}$$

In the next theorem we prove that the automorphism group acts transitively on hyperbolic pairs of extremal points. As noted before, we have to rely on the classification of simple Lie algebras in the characteristic 5 case.

Theorem 4.2.16. *Let $\mathcal{A}$ be a simple skew-dimension one structurable algebra. Then the automorphism group of $K(\mathcal{A})$ acts transitively on the pairs $(x, y) \in \mathcal{E}_2$.*

Proof. Since $L = K(\mathcal{A})$ is simple and generated by its extremal elements, Theorem 2.3.19 shows that it suffices to show that the stabilizer of $\mathcal{S}_- \leq K(\mathcal{A})$ is transitive on the points in $\mathcal{E}_2(\mathcal{S}_-)$. By Lemma 3.1.9 any element in $\mathcal{E}_2(\mathcal{S}_-)$ equals $\langle e_-(a,t)(s_+)\rangle$, for some $a \in \mathcal{A}$ and $s, t \in \mathcal{S}$. Now note that $e_-(a,t)$ is an automorphism and $e_-(a,t)(\mathcal{S}_-) = \mathcal{S}_-$. □

Theorem 4.2.17. *Assume that the characteristic of k is not 2 or 3. Then the map*

$$\mathcal{A} \mapsto K(\mathcal{A})$$

induces a one-to-one correspondence between simple skew-dimension one structurable algebras over k (up to isotopy) and simple finite-dimensional Lie algebras over k generated by extremal elements which are not symplectic (up to isomorphism).

Proof. Let $\mathcal{A}$ be a simple skew-dimension one structurable algebra. By Proposition 4.2.2 we find that $K(\mathcal{A})$ is a simple Lie algebra generated by its extremal elements, which by Corollary 4.2.13 is not symplectic. By [AH81, Proposition 12.3], two isotopic structurable algebras yield isomorphic Lie algebras.

Consider two simple skew-dimension one structurable algebras $\mathcal{A}$ and $\mathcal{A}'$ such that $K(\mathcal{A})$ and $K(\mathcal{A}')$ are isomorphic. Since $(\mathcal{S}_-, \mathcal{S}_+)$ and $(\mathcal{S}'_-, \mathcal{S}'_+)$ are both hyperbolic pairs of extremal points, Theorem 4.2.16 implies that $K(\mathcal{A})$ and $K(\mathcal{A}')$ are graded-isomorphic and hence $\mathcal{A}$ and $\mathcal{A}'$ are isotopic by [AH81, Proposition 12.3]. Now Theorem 4.2.7 concludes this proof. □

Suppose that the simple Lie algebra L is generated by its set of extremal elements and no two linearly independent extremal elements commute. The inner ideal geometry of L only contains points, by Theorem 2.5.11. Moreover, by Lemma 2.5.8, L is not a symplectic Lie algebra, unless $L \simeq \mathfrak{sl}_2$. Hence $L = K(\mathcal{A})$ for a unique – up to isotopy – simple skew-dimension one structurable algebra $\mathcal{A}$, unless $L \simeq \mathfrak{sl}_2$. In the latter case one easily sees that all extremal points form a Moufang set. In the former case $\mathcal{S}_-$ and $\exp(a_+)(\mathcal{S}_-)$ are hyperbolic for any $0 \neq a \in \mathcal{A}$. Equation (4.1) implies $\psi(a, U_a(sa)) \neq 0$, for $0 \neq s \in \mathcal{S}$. Then [AF84, Proposition 2.11] shows that $\mathcal{A}$ is a structurable division algebra. By

(1.14) and Theorem 1.1.71 we get that $E_-(\mathcal{A}) := \{\exp(a_- + s_-) \mid a \in \mathcal{A}, s \in \mathcal{S}\}$ is a subgroup of the automorphism group of $L = K(\mathcal{A})$. Similarly for $E_+(\mathcal{A}) := \{\exp(a_+ + s_+) \mid a \in \mathcal{A}, s \in \mathcal{S}\}$. Note that Lemma 3.1.9 implies that $E_-(\mathcal{A})$ is transitive on $\mathcal{E}_2(\mathcal{S}_-)$. It is not hard to see that it is actually sharply transitive. Using [BDMS19, Theorem 5.1.1] we see that $\mathcal{E}$ is indeed a Moufang set, with root groups as below. The little projective group is $E(\mathcal{A}) = \langle E_-(\mathcal{A}), E_+(\mathcal{A})\rangle$. The multiplication in this group is defined as $fg = f \circ g$ for any f, $g \in E(\mathcal{A})$. Conversely, if $\mathcal{A}$ is a skew-dimension one structurable division algebra, it is shown in Theorem 3.3.4 that the only non-trivial inner ideals of $K(\mathcal{A})$ are 1-dimensional and form a Moufang set.

So we obtained:

Theorem 4.2.18. *Suppose L is a finite-dimensional simple Lie algebra over a field k of characteristic different from $2, 3$ generated by its set of extremal elements. If no two linearly independent extremal elements commute and $L \neq \mathfrak{sl}_2$, then $L = K(\mathcal{A})$ with $\mathcal{A}$ a skew-dimension one structurable division algebra. The set of proper non-trivial inner ideals of L equals $\mathcal{E} = \{\mathcal{S}_-\} \cup E_-(\mathcal{A})(\mathcal{S}_+)$ and is a Moufang set with root groups*

$$\begin{aligned} U_{\mathcal{S}_-} &= E_-(\mathcal{A}); \\ U_{e_-(a,s)(\mathcal{S}_+)} &= E_+(\mathcal{A})^{e_-(a,s)}, \forall a \in \mathcal{A}, s \in \mathcal{S}\,. \end{aligned}$$

Remark 4.2.19. We will show in Section 4.6 that Theorem 4.2.18 holds in characteristic 2 and 3 as well. Of course, the connection with structurable (division) algebras is lost, since these algebras are not defined in those characteristics, but the extremal points still form a Moufang set. Actually, my initial motivation to develop the whole theory and machinery in Section 4.3 was to eliminate this dependence on Anastasia Stavrova's result in Theorems 4.2.17 and 4.2.18. As mentioned before, this result of Anastasia Stavrova on algebraicity depends on the classification of simple Lie algebras if the characteristic equals 5, and this seemed to me as quite a strong result to use to prove the above theorems. That is why I started thinking about proving this algebraicity property independently in this case, using the extremal elements and the extremal geometry. Eventually, it turned out that the proof did not only work for the original aim (characteristic 5), but also for characteristics 2 and 3. In these characteristic one cannot apply results on structurable algebras as above (since these do not exist) to obtain a Jordan algebra if there are lines in the extremal geometry. So we have to show this correspondence more or less from scratch on, and that is what we do in Section 4.4. Then, in Section 4.5, we can recover a quadrangular algebra, if there are symplectic pairs in the extremal geometry, $\mathrm{char}(k) \neq 2$ and after a Galois extension of degree at most 2 there are lines in the extremal geometry.

The following two examples give examples of Lie algebras which are generated by extremal elements and which are *not* symplectic. So by Theorem 4.2.17 there should exist a corresponding structurable algebra of skew-dimension one, but it is not immediately clear how to describe this algebra. However, we will see in Section 4.5 that this algebra is coordinatized by a quadrangular algebra.

Example 4.2.20. Consider the tensor product $C_1 \otimes C_2$, where C_1 is an octonion algebra and C_2 is a composition algebra of dimension 2, 4 or 8 over a field k. As noted before, this is a structurable algebra and its set of skew elements is equal to $\{s_1 \otimes 1 + 1 \otimes s_2 \mid s_1 \in S_1, s_2 \in S_2\}$, with S_i the set of skew elements in C_i. Hence $\dim(\mathcal{S}) = \dim(C_1) + \dim(C_2) - 2$, so 8, 10 or 14 respectively. We can define the following quadratic form on $\mathcal{S}$:

$$q_A : \mathcal{S} \to k : s_1 \otimes 1 + 1 \otimes s_2 \mapsto q_1(s_1) - q_2(s_2),$$

which is called the *Albert form*. (With q_i the quadratic form associated with the composition algebra C_i.) Denote the associated bilinear form by q_A as well. We also define $(s_1 \otimes 1 + 1 \otimes s_2)^\natural = s_1 \otimes 1 - 1 \otimes s_2$. We now assume that q_A has Witt index 1 and that k has characteristic 0. If $s \in \mathcal{S}$ is not an isotropic vector, then s is conjugate invertible (see [All88, Corollary 3.13]). If $s \in \mathcal{S}$ is an isotropic vector, then for any $t \in \mathcal{S}$ we have, due to [All88, Proposition 3.3], $sts = q_A(s)t^\natural - q_A(s, t^\natural)s = -q_A(s, t^\natural)s$. Hence $[s_+, [s_+, t_-]] = [s_+, L_s L_t] = -2sts_+ \in \langle s_+ \rangle$. So since $\mathcal{S}_+$ is the end of a 5-grading, we obtain that s_+ is an extremal element of $L := K(C_1 \otimes C_2)$. Consider $s, t \in \mathcal{S}$ linearly independent and satisfying $q_A(s) = 0 = q_A(t)$. Now assume that the extremal geometry of L contains lines. If $(s_+, t_+) \in E_{-1}$, then $\mathcal{S}_+$ contains an extremal line, namely $\langle s_+, t_+ \rangle$. Otherwise, due to $[s_+, t_+] = 0$, we get $(s_+, t_+) \in E_0$. So by Lemma 2.5.5 and the fact that $\mathcal{S}_+$ is an inner ideal we get that $\mathcal{S}_+$ contains all points (and lines) of a symplecton of the extremal geometry. So in any case, $\mathcal{S}_+$ would contain extremal lines. By the previous considerations this implies that q_A has Witt index at least 2, a contradiction. Hence the extremal geometry of L does not contain lines. So, in this case, due to Theorem 2.5.11, the inner ideal geometry is a polar space. Moreover, its rank equals 2 if there does not exist an inner ideal I with $[I, I] = 0$ properly containing $\mathcal{S}_+$, by Corollary 2.5.6. So assume that I is an inner ideal containing $\mathcal{S}_+$ properly with $[I, I] = 0$. Since there exist conjugate invertible elements in $\mathcal{S}$ the fact that I is abelian implies $I \leq \mathrm{Instrl}(\mathcal{A}) \oplus \mathcal{A}_+ \oplus \mathcal{S}_+$. Consider $0 \neq V + a_+ \in I$ and $s \in \mathcal{S}$ conjugate invertible. Then $V(sb)_+ + \psi(a, sb)_+ = [V + a_+, [s_+, b_-]] \in I$ for any $b \in \mathcal{A}$. Together with $\mathcal{S}_+ \leq I$ this implies that there exists $0 \neq a_+ \in \mathcal{A}_+ \cap I$. Then $[\mathcal{S}_+, [a_+, \mathcal{S}_-]] \leq I$ implies that $L_\mathcal{S} L_\mathcal{S}(\mathcal{A}_+ \cap I) \leq \mathcal{A}_+ \cap I$. By [All88, Theorem 4.5] we get $\mathcal{A}_+ \leq I$. Now $[s_+, 1_+] = \psi(s, 1)_+ = 2s_+ \neq 0$ for any $s \in \mathcal{S}$ contradicts $[I, I] = 0$. Hence the inner ideal geometry of L is a generalized quadrangle.

Note that by Corollary 4.2.14 the Lie algebra L is also isomorphic to the TKK-construction of a form of a structurable matrix algebra.

Example 4.2.21. Let $Q : M \to k$ be a non-degenerate quadratic form on a k-vector space M. We assume that there exist a $c \in M$ such that $Q(c) = 1$. Let T be the associated bilinear form and set $\overline{x} = 2T(x, c)c - x$. Recall, Example 1.1.8, that M together with the product

$$x.y = T(x,c)y + T(y,c)x - Q(x,y)c.$$

is a Jordan algebra, which we denote by $J(Q,c)$. This Jordan algebra is central simple unless Q is isotropic and $\dim(M) = 2$ (see [McC04, II.3.3], where the bilinear form Q is slightly different). From now on we will assume $\dim(M) > 2$ and that Q has Witt index 1, so $J := J(Q,c)$ is central simple. By Example 1.1.8 we get $U_{x,x}(y) = 2Q(x,\overline{y})x - Q(x)\overline{y}$. Now $x_+ \in K(J)$ is extremal if, and only if, $U_x(J) \leq \langle x \rangle$, so if, and only if, $Q(x) = 0$. As in the previous example, the existence of lines in the extremal geometry implies the existence of lines in J_+. But then there would be a subspace V of J of dimension 2 such that $Q(v) = 0$, for all $v \in V$, contradicting that Q has Witt index 1. Hence the inner ideal geometry of $K(J)$ is a polar space of rank at least 2. Now note that J_+ is the smallest inner ideal containing two distinct extremal points $\langle x_+ \rangle$ and $\langle y_+ \rangle$ in J_+, since all elements $z \in J$ with $Q(z) \neq 0$ are invertible, i.e. U_z is invertible. Now, if the rank of the polar space were strictly larger than 2, there would be a proper abelian inner ideal I properly containing J_+. But then $0 \neq a_- + V \in I$, for $a \in J$, $V \in \mathrm{Instrl}(J)$. Since J is abelian, $[V, j_+] = V(j)_+ = 0$ for all $j \in J$ implying $V = 0$. Hence $0 = [a_-, j_+] = -V_{j,a}$ for all $j \in J$. But $0 = V_{j,a}(j) = U_j(a)$ implies $a = 0$ if j is invertible. Hence the inner ideal geometry is a polar space of rank 2, i.e. a generalized quadrangle.

Remark 4.2.22. Note that the main theorem of [CIR08] states that if a simple Lie algebra L over a field of characteristic not 2 or 3 contains one pure extremal element, then L is either generated by its pure extremal elements or L is isomorphic to the 5-dimensional Witt algebra $W_{1,1}(5)$ and the characteristic of the field equals 5.

SECTION 4.3

Deducing algebraicity

In this section we consider a simple Lie algebra over the field $k \neq \mathbb{F}_2$ which is generated by its pure extremal elements. We show that if its extremal geometry contains lines, then the 1-component (and (-1)-component) of the 5-grading coming from two hyperbolic extremal elements as in Lemma 4.1.1 is spanned by the extremal elements contained in this component. Using this property we can show that certain automorphisms which behave nicely with respect to the considered 5-grading exist. We deduce a more precise description of certain extremal elements to show uniqueness statements about these type of automorphisms. Then we translate these results to Lie algebras whose extremal geometry does not contain lines, and finally we determine how these automorphisms commute.

We will use the existence and uniqueness of these automorphisms often in the remaining sections of this chapter.

Assumption 4.3.1. We assume in this section that L is a simple Lie algebra generated by its set E of pure extremal elements. As always L is defined over the field k, which we assume to be different from $\mathbb{F}_2$.

Notation 4.3.2. By Proposition 2.3.9 we can consider $x, y \in E$ with $g_x(y) = 1$. Let $L = L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus L_2$ be the corresponding 5-grading as in Lemma 4.1.1, with $x \in L_{-2}$ and $y \in L_2$. Denote the subspace of L_{-1} spanned by $E \cap L_{-1}$ by I_{-1}. Denote the subspace of L_1 spanned by $E \cap L_1$ by I_1.

Lemma 4.3.3. *If $I_{-1} \neq \emptyset$, then L is generated by y and I_{-1}.*

Proof. Denote the subalgebra generated by y and I_{-1} by I. The condition $I_{-1} \neq \emptyset$ implies that there are lines in the extremal geometry. Indeed, consider $e \in I_{-1}$, then $g_y(e) = 0$ but $[y, e] \neq 0$ by Lemma 4.1.1, hence $(y, e) \in E_1$ and $\langle y \rangle$ and $\langle e \rangle$ have a unique common neighbor, namely $\langle [y, e] \rangle$. In particular the extremal geometry contains lines. By Lemma 4.1.11 there are two neighbors of $\langle x \rangle$ which are special to each other, hence $[I_{-1}, I_{-1}] = \langle x \rangle$ and thus $x \in I$. By Lemma 4.1.9 and the fact that $\mathcal{E}_{\leq 1}(\langle y \rangle)$ is a hyperplane (by the definition of a root filtration space), all elements in $E_{-1}(x)$ are contained in I.

Consider $z \in E_2(x)$ and assume that $\langle y \rangle$ and $\langle z \rangle$ are collinear. The line $\langle y, z \rangle$ then contains a point $\langle u \rangle \in \mathcal{E}_1(\langle x \rangle)$. Then $[x, u] \in E_{-1}(x) \cap E_1(y)$ and hence it is contained in L_{-1} by Lemma 4.1.9. Since $\mathrm{Exp}(\langle [x, u] \rangle)$ acts sharply transitively on all points of $\langle y, z \rangle$ without $\langle u \rangle$ and $\exp([x, u])(y) = y + [[x, u], y]$, we get $z \in I$. Note that since $\mathcal{E}_{\leq 1}(\langle x \rangle)$ is a hyperplane of the extremal geometry, this implies $E_{-1}(y) \subseteq I$. The hyperbolic pair (x, z) yields another 5-grading with 1-dimensional (-2)- and 2-parts. The extremal elements in the (-1)-part of this grading are again contained in I since these are contained in $E_{-1}(x)$ by Lemma 4.1.8. By applying the same argument as before, every $z' \in E_2(x) \cap E_{-1}(z)$ is contained in I.

Now assume that $\mathcal{E}_0 \neq \emptyset$, i.e. the extremal geometry is not a generalized hexagon, see Remark 2.3.18. Then Lemma 2.1.16 yields $E_2(x) \subset I$. Since every element of $\mathcal{E}_1(x)$ lies on a path of length 3 from $\langle x \rangle$ to an element of $\mathcal{E}_2(x)$ by Lemma 2.1.9(b) and every neighbor of $\langle x \rangle$ is contained in I, we get $E_1(x) \subseteq I$. By a similar argument one shows $I_1 \leq I$ and then reversing the roles of x and y one sees $E_{\geq 1}(x) \cup E_{\geq 1}(y) \cup E_{\leq -1}(x) \cup E_{\leq -1}(y) \subseteq I$. So consider $z \in E_0(x) \cap E_0(y)$. Let S and S' be the symplecta containing x and z, and y and z, respectively. Then $S \cap S' = \langle z \rangle$ since otherwise x would be collinear with a point symplectic with y, which contradicts property (D) of a root filtration space since $y \in E_2(x)$. Hence in S there exists a line containing z and at least 2 points contained in I and thus $z \in I$. Since L is generated by E we can conclude $I = L$ in this case.

Assume from now on $\mathcal{E}_0 = \emptyset$, i.e. the extremal geometry is a generalized hexagon. Consider $z \in E_2(x)$ such that $\langle z \rangle$ and $\langle y \rangle$ are at distance 2, and thus

$(y, z) \in E_1$. Recall that the common neighbor of $\langle y\rangle$ and $\langle z\rangle$ is $\langle [y, z]\rangle$. We want to show $z \in I$, so by the second paragraph of this proof we are done if $[y, z] \in E_2(x)$. Since x and y are hyperbolic we may now assume $[y, z] \in E_1(x)$. By Lemma 2.1.9(b) we can find $u \in E$ such that $u \in E_{-1}(z) \cap E_2(y)$. Using that $\mathcal{E}_{\leq 1}(x)$ is a hyperplane and replacing u with another element on the line $\langle z, u\rangle$, we may assume $u \in E_1(x)$. Since we are considering a generalized hexagon,

$$\langle x\rangle, \langle [x, [y, z]]\rangle, \langle [y, z]\rangle, \langle z\rangle, \langle u\rangle, \langle [x, u]\rangle, \langle x\rangle$$

forms an ordinary hexagon. Hence $[y, z] \in E_2([x, u])$ and since $\mathcal{E}_{\leq 1}(\langle [x, u]\rangle)$ is a hyperplane, we can replace y by another element of $\langle y, [y, z]\rangle$ in order to assume $y \in E_2([x, u])$. Using that $\mathcal{E}_{\leq 1}(\langle y\rangle)$ is a hyperplane there is a unique point $\langle b\rangle$ of the line $\langle [x, u], u\rangle$ special with $\langle y\rangle$. By construction $[x, u] \notin \langle b\rangle$. Let $\langle a\rangle$ be the common neighbor of $\langle b\rangle$ and $\langle y\rangle$. We previously showed that all neighbors of $\langle y\rangle$ and $\langle x\rangle$ are contained in I, and hence a, $[x, u]$ and thus $[a, [x, u]] \in \langle b\rangle$ are contained in I. Since two different points of the line $\langle [x, u], u\rangle$, namely $\langle [x, u]\rangle$ and $\langle b\rangle$, are contained in I, u is contained in I as well. Since $\langle u\rangle$ and $\langle [y, z]\rangle$ are special points with common neighbor $\langle z\rangle$, z is is multiple of $[u, [y, z]]$. Recall that all points collinear to y are contained in I, so we get $[y, z] \in I$ and together with $u \in I$ this yields $z \in I$.

Now consider $z \in E_2(x) \cap E_2(y)$. Let $\langle y\rangle, \langle a\rangle, \langle b\rangle, \langle z\rangle$ be a path of length 3. If $\langle a, b\rangle$ is not completely contained in $E_{\leq 1}(x)$ we can apply the previous paragraph twice to obtain $z \in I$. We can assume $a, b \in E_1(x)$. Note that $\langle [a, x]\rangle$ and $\langle [b, x]\rangle$ are extremal points collinear with x, and a and b, respectively. If $[x, a], [x, b] \notin \langle a, b\rangle$, then there exists an ordinary pentagon (or an ordinary triangle, if $[a, x] = [b, x]$), which contradicts the definition of a generalized hexagon. So the line $\langle a, b\rangle$ contains a point $\langle c\rangle$ collinear with $\langle x\rangle$. Since $\operatorname{Exp}(\langle x\rangle)$ acts sharply transitively on the points of the line $\langle a, b\rangle$ without $\langle c\rangle$, we find that $\exp(\lambda x)(a) \in \langle b\rangle$ for a certain $\lambda \in k$. Note that $z' = \exp(\lambda x)(y) = y + \lambda[x, y] + \lambda^2 g_x(y)x \in I$. By $\exp(\lambda x)(x) = x$ we get $z' \in E_2(x)$. Note that $\langle b\rangle = \exp(\lambda x)(\langle a\rangle)$ is a neighbor of $\langle z\rangle$ and of $\langle z'\rangle$, since $\langle a\rangle$ and $\langle y\rangle$ are neighbors. Hence $z' \in E_{\leq 1}(z)$. Since by construction $z' \in I$, the previous paragraph then implies $z \in I$. As in the case $\mathcal{E}_0 \neq \emptyset$ this allows us to conclude $E_{\geq 1}(x) \cup E_{\geq 1}(y) \cup E_{\leq -1}(x) \cup E_{\leq -1}(y) \subseteq I$. In this case this is the same as $E \subseteq I$ and thus $I = L$. □

Corollary 4.3.4. *Suppose $I_{-1} \neq \emptyset$. Then $I \leq L$ is an ideal if $[I, y] \leq I$ and $[I, I_{-1}] \leq I$.*

Proof. By the Jacobi identity and Lemma 4.3.3. □

To prove that L_1 and L_{-1} are generated by its extremal elements, i.e. $I_{-1} = L_{-1}$, we use the following property of polar spaces.

Lemma 4.3.5. *Let S be a non-degenerate polar space. Let x and y be non-collinear points of S. Then S is generated by x, y and $x^\perp \cap y^\perp$.*

Proof. Let X be the subspace generated by x, y and $x^\perp \cap y^\perp$. Since every line through x contains a point collinear to y, we have $x^\perp \subseteq X$, and similarly $y^\perp \subseteq X$. Consider $z \in X$ arbitrary. Let K be any line through x, if $x^\perp \cap K \neq y^\perp \cap K$, then $K \subseteq X$. So we may assume that K contains a unique point $a \in x^\perp \cap y^\perp$. By the non-degeneracy, we find $b \in x^\perp \cap y^\perp$ such that a and b are not collinear. But then K and by are opposite lines, and hence there exists a point c on K distinct from a which is collinear to a point of by distinct from y and b. Hence c lies on a line M such that $M \cap y^\perp \in by \backslash \{b, y\}$ and thus $M \cap y^\perp \neq M \cap x^\perp$. As before, $M \subseteq X$. In particular $c \in X$, so together with $a \in X$ we obtain $z \in X$. □

This lemma gives a more precise description of the subspace L_S of L spanned by the elements in S.

Lemma 4.3.6. *Let S be a symplecton containing x, then*

$$L_S = \langle x \rangle \oplus \langle \{ \langle z \rangle \in S \mid z \in L_{-1} \} \rangle \oplus \langle \{ \langle z \rangle \in S \mid z \in L_0 \} \rangle.$$

Moreover $\langle \{ \langle z \rangle \in S \mid z \in L_0 \} \rangle$ is 1-dimensional.

Proof. By Lemma 2.1.12 we get that $S \cap \mathcal{E}_0(y)$ is a point $\langle z \rangle$. By Lemma 4.1.1 $[z, y] = 0 = [z, x]$ implies $z \in L_0$. By Lemma 4.3.5 we get that L_S is spanned by x, z and all points $\langle a \rangle$ collinear to both $\langle x \rangle$ and $\langle z \rangle$. Since such a point $\langle a \rangle$ is collinear to $\langle x \rangle$, we get $a \in L_{\leq -1}$ by Lemma 4.1.8. Since $\langle a \rangle$ is collinear to $\langle z \rangle$, we get, using Lemma 2.1.12, $a \in E_1(y)$ and hence $a \in L_{\geq -1}$ by Lemma 4.1.8. Hence $a \in L_{-1}$. □

We are now ready to prove the first theorem of this section.

Theorem 4.3.7. *Assume that L is a simple Lie algebra over $k \neq \mathbb{F}_2$ generated by its pure extremal elements, and $\mathcal{F} \neq \emptyset$. Consider $x, y \in E$ with $g_x(y) = 1$ and let $L = L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus L_2$ be the associated 5-grading. Then both L_{-1} and L_1 are linearly spanned by the extremal elements contained in it.*

Proof. As before, let I_{-1} and I_1 be the subspaces of L linearly spanned by the extremal elements contained in L_{-1} and L_1, respectively. We will show that

$$I = \langle x \rangle \oplus I_{-1} \oplus [I_{-1}, I_1] \oplus I_1 \oplus \langle y \rangle$$

is an ideal of L using Corollary 4.3.4. Since L is simple the conclusion of this theorem then follows.

By Lemma 4.1.11 we see that $[x, y] \in [I_{-1}, I_1]$. Also by $a \in E \cap L_{-1}$ we get $a \in E_1(y)$ and hence $[a, y] \in I_1$. So $[I, y] \leq I$ is clear. Now we only need to show $[I, I_{-1}] \leq I$. Obviously $[\langle x \rangle \oplus I_{-1} \oplus I_1, I_{-1}] \leq I$. Similarly as before $[\langle y \rangle, I_{-1}] \leq I_1 \leq I$, so the only case left to prove is $[[I_{-1}, I_1], I_{-1}] \leq I_{-1}$. Consider arbitrary $a, b \in E \cap L_{-1}$ and $c \in E \cap L_1$. We show $[[a, c], b] \in I_{-1}$.

If $a = b$ this is obvious. If $a \in E_{-1}(b)$, then $\langle a, b\rangle$ is an inner ideal by Lemma 2.5.3 and hence $[b,[a,c]] \in \langle a,b\rangle \leq I_{-1}$. If $a \in E_0(b)$, then a and b are contained in a symplecton S. The subspace L_S spanned by all elements of S is an inner ideal by Lemma 2.5.5 and hence $[b,[a,c]] \in L_S$. By Lemma 4.3.6 we obtain that L_S is spanned by x, $\{\langle z\rangle \in S \mid z \in L_0\}$ and $\{\langle z\rangle \in S \mid z \in L_{-1}\}$. Since $[b,[a,c]] \in L_{-1}$, it is a linear combination of elements in this last set, so it is contained in I_{-1}.

Note that by the Jacobi identity and the fact that $[x,c]$ is contained in $E \cap L_{-1}$, the containment of $[b,[a,c]]$ in I_{-1} is equivalent with the containment of $[a,[b,c]]$ in I_{-1}. Note that $c \in E_{\leq 0}(a)$ and $c \in E_{\leq 0}(b)$ imply $[c,a] = 0$ and $[c,b] = 0$ respectively. These two observations allow us to assume $c \in E_{\geq 1}(a) \cap E_{\geq 1}(b)$. By the previous paragraph we can assume $a \in E_{\geq 1}(b)$, and since $x \in E_{-1}(a) \cap E_{-1}(b)$ we obtain $a \in E_1(b)$. If $c \in E_1(a)$, then $[a,c] \in E$ and since the extremal form g is associative, $g_b([a,c]) = g(b,[a,c]) = g([b,a],c) = g_{[b,a]}(c) = 0$, using $c \in E_1(x)$. Hence $[a,c] \in E_{\leq 1}(b)$, so either $[b,[a,c]] \in E$ or $[b,[a,c]] = 0$. In any case we obtain $[b,[a,c]] \in I_{-1}$. If $c \in E_1(b)$ then $[a,[b,c]] \in I_{-1}$ and thus $[b,[a,c]] \in I_{-1}$.

The only case left to consider is $a \in E_1(b)$ and $c \in E_2(a) \cap E_2(b)$. We may assume $g_c(a) = 1 = g_c(b)$. Consider $\lambda, \mu \in k$ arbitrary. Then

$$\varphi = \exp(\mu c)\exp(\lambda b)\exp(c)(a)$$

is an extremal element. Using Lemma 4.1.10, $g_c(a) = 1 = g_c(b)$ and $g_a(b) = 0$, one deduces the components of φ explicitly. Its (-2)-component equals $\lambda[b,a]$, its (-1)-component

$$a + \lambda^2 b + \lambda[b,[c,a]] + \lambda\mu[c,[b,a]] \tag{4.2}$$

and its 2-component 0. Now we will choose non-zero λ and μ such that the 0- and 1-component of φ equal 0. The 0-component of φ equals

$$(\mu+1)[c,a] + (-\lambda + \lambda^2\mu)[c,b] + \lambda\mu[c,[b,[c,a]]].$$

Now, since c is extremal the Premet identities imply $[c,[b,[c,a]]] = g_c([b,a]) - g_c(a)[c,b] - g_c(b)[c,a] = -[c,b] - [c,a]$. So for the 0-component to be zero it is sufficient that $\mu + 1 - \lambda\mu = 0$ and $-\lambda + \lambda^2\mu - \lambda\mu = 0$. Actually since the second equation follows from the first, we need non-zero λ and μ such that $\mu + 1 = \lambda\mu$. Since $|k| \geq 3$ we can always find such scalars. Now, using the associativity of g we have $g_c([b,[c,a]]) = -g_{[c,[c,b]]}(a) = -2g_c(b)g_c(a) = -2$, and we see that the 1-component of φ equals

$$(2\mu - 2\lambda\mu - 2\lambda\mu^2 + \lambda^2\mu^2 + \mu^2 + 1)c,$$

and after using $\mu + 1 = \lambda\mu$ to express this completely in terms of μ, we see that this component equals zero. We conclude that φ is contained in $E_{\leq -1}(x)$. If φ is a multiple of x, then (4.2) equals 0 and $[b,[c,a]]$ can be written as the sum of 3 extremal elements contained in L_{-1}. Indeed, obviously $a, b \in E$, but moreover

$[c, [b, a]] \in E$ since $\langle c \rangle$ and $\langle x \rangle = \langle [b, a] \rangle$ are special. If $\varphi \notin \langle x \rangle$, then $\langle \varphi, x \rangle$ is a line of the extremal geometry. Since the (-1)-component of φ is contained in $\langle \varphi, x \rangle$, it is contained in E. So by (4.2) the element $[b, [c, a]]$ can be written as the sum of 4 extremal elements contained in L_{-1}. This concludes the proof of the claim $[[I_{-1}, I_1], I_{-1}] \leq I_{-1}$. □

Now we will work towards showing the so-called *algebraicity* of the Lie algebra, using the previous theorem. We will define this property later on, but loosely speaking it means that for any element in L_{-1} there exist automorphisms depending on this element which behave nicely with respect to the 5-grading. Actually, if the characteristic is not 2, 3, or 5, this property is easily shown, see Lemma 1.1.69, so it is not surprising that we sometimes have to handle the low characteristic cases a bit more carefully.

In the next two lemmas we show that as soon as an extremal element has a certain form, it is contained in the image of $\mathrm{Exp}(\langle x \rangle)$, which we will use to show the uniqueness of certain automorphisms. Recall that $\exp(e)$ is an automorphism if e is a pure extremal element, see Proposition 2.3.5.

Lemma 4.3.8. *Assume* $\mathrm{char}(k) \neq 3$. *Consider* $l = l_{-2} + l_{-1} + e \in E$, *with* $e \in E \cap L_1$, $l_{-1} \in L_{-1}$ *and* $l_{-2} \in L_{-2}$. *Then* $l = \exp(\lambda x)(e)$, *for certain* $\lambda \in k$.

Proof. Note

$$[l, [l, [x, y]]] = [l, 2l_{-2} + l_{-1} - e] = -3[l_{-2}, e] - 2[l_{-1}, e].$$

Since l is extremal this implies $[l_{-2}, e] = 0$ and hence $l_{-2} = 0$ by Lemma 4.1.1. If $l_{-1} = 0$ there is nothing to show, so assume $l_{-1} \neq 0$. By Lemma 4.1.8 we have $l \in E_1(y)$ and thus $[l_{-1}, y] = [l, y] \in E$. By the same lemma $[l_{-1}, y] \in E_1(x)$ and thus $l_{-1} = [x, [l_{-1}, y]] \in E$. Since l_{-1}, e and $l = l_{-1} + e$ are contained in E, Lemma 2.3.15 implies $(l_{-1}, e) \in L_{-1}$. Hence $\langle l_{-1} \rangle$ is the common neighbor of the special pair $\langle x \rangle$, $\langle e \rangle$ and thus $l_{-1} \in \langle [x, e] \rangle$. □

Lemma 4.3.9. *Consider* $l = l_{-2} + l_{-1} + l_0 + y \in E$, *with* $l_{-2} \in L_{-2}$, $l_{-1} \in L_{-1}$ *and* $l_0 \in L_0$. *Then* $l = \exp(\lambda x)(y)$, *for certain* $\lambda \in k$.

Proof. By assumption l has 0 as 1-component. Let λ and μ be such that $l_{-2} = \lambda x$ and $[l_0, x] = \mu x$. Then

$$\begin{aligned}[l, [l, x]] &= [\lambda x + l_{-1} + l_0 + y, \mu x + [y, x]] \\ &= (\mu^2 x + \mu[y, x]) + (-2\lambda x - l_{-1} + 2y).\end{aligned}$$

Since l is extremal and $g_l(x) = g_x(l) = g_x(y) = 1$ by Lemma 4.1.3, we get $-l_{-1} = 2l_{-1}$, $\mu[y, x] = 2l_0$ and $4\lambda = \mu^2$. If $\mathrm{char}(k)$ is not 2 or 3, then clearly $l = \exp(-\frac{1}{2}\mu x)(y)$.

Assume now $\mathrm{char}(k) = 3$, then $\psi = \exp(\frac{1}{2}\mu x)(l)$ is an extremal element such that $\psi = l_{-1} + y$, with $l_{-1} \in L_{-1}$ as before. Assume $l_{-1} \neq 0$. By Lemma 4.1.8

$\psi \in E_1(y)$ and thus $[y, \psi] = [y, l_{-1}] \in E$. By the same lemma $[y, l_{-1}] \in E_1(x)$ and thus $l_{-1} = [[y, l_{-1}], x] \in E$. Since l_{-1}, y and $\psi = l_{-1} + y$ are contained in E, Lemma 2.3.15 implies $l_{-1} \in E_{-1}(y)$, contradicting Lemma 4.1.8. Hence $l_{-1} = 0$ and $l = \exp(-\frac{1}{2}\mu x)(y)$.

Assume now $\mathrm{char}(k) = 2$, then by the first paragraph $l_{-1} = 0$. There exist $e_1, e_2, \in E \cap L_{-1}$ such that $[e_1, e_2] = x$ by Lemma 4.1.11. By Lemma 4.1.8 we can assume that e_1 and e_2 are contained in L_{-1}. By Lemma 4.1.10 we get for any $e \in E \cap L_{-1}$ that $g_e(l) = 0$ and since $[e, y] \neq 0$, this implies that $\langle e \rangle$ and $\langle l \rangle$ are special and hence $[l, e] = [l_0, e] + [y, e]$ is extremal. Now note that by the Premet identities

$$[[l, e_1], [l, e_2]] = g_l([e_1, e_2])l + g_l(e_2)[l, e_1] - g_l(e_1)[l, e_2] = l.$$

Now, by Lemma 4.3.8 and $[l, e_1] \in E$ there exists an automorphism α in $\mathrm{Exp}(\langle x \rangle)$ such that $\alpha([l, e_1]) = [y, e_1]$. Hence the Premet identities imply

$$\begin{aligned}\alpha(l) &= [\alpha([l, e_1]), \alpha([l, e_2])] = [[y, e_1], l'_{-1} + [y, e_2]] \\ &= [[y, e_1], l'_{-1}] + g_y([e_1, e_2])y = [[y, e_1], l'_{-1}] + y,\end{aligned}$$

for a certain $l'_{-1} \in L_{-1}$. If $l'_{-1} = 0$ we are done, so assume $l'_{-1} \neq 0$. Now note that $[[[y, e_1], l'_{-1}], y] = [[y, l'_{-1}], [y, e_1]] \neq 0$ since $0 \neq [y, l'_{-1}] \in \langle [y, e_2] \rangle$ by Lemma 4.3.8. Hence $[[[y, e_1], l'_{-1}], y]$ would equal λy with $0 \neq \lambda \in k$. Then

$$0 = 2g_{\alpha(l)}(y)\alpha(l) = [\alpha(l), [\alpha(l), y]] = \lambda^2 y$$

yields a contradiction. □

Notation 4.3.10. Assume that the extremal geometry of L has lines. We set

$$E_\sigma(x, y) = \langle \{\exp(e) \mid e \in E, e \in L_{\sigma 2} \oplus L_{\sigma 1}\} \rangle \leq \mathrm{Aut}(L),$$

where $\sigma \in \{-, +\}$.

The following theorem, which we will often use in the rest of this chapter, shows that $E_\sigma(x, y)$ is a solvable subgroup of $\mathrm{Aut}(L)$ of length 2 whose commutator subgroup equals $\exp(L_{2\sigma})$. For every $l \in L_{\sigma 1}$, we show that $E_\sigma(x, y)$ contains an automorphism depending on l, more precisely it satisfies (4.3) and (4.4). Automorphisms satisfying these two properties are *not* uniquely defined, but they can only differ by an element of $\exp(L_{\sigma 2})$, see part (v). In part (iv) we show that for any scalar $\lambda \in k$ and any automorphism of $E_\sigma(x, y)$ depending on $l \in L_{\sigma 1}$ as in part (iii), we can construct an automorphism in $E_\sigma(x, y)$ depending on λl in a straightforward fashion.

Theorem 4.3.11. *Assume that L is a simple Lie algebra over $k \neq \mathbb{F}_2$ generated by its pure extremal elements, and $\mathcal{F} \neq \emptyset$. Consider $x, y \in E$ with $g_x(y) = 1$ and let $L = L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus L_2$ be the associated 5-grading. Let $\sigma \in \{-, +\}$. Then:*

(i) $[E_\sigma(x,y), \exp(L_{\sigma 2})] = 1$.

(ii) $[E_\sigma(x,y), E_\sigma(x,y)] = \exp(L_{\sigma 2})$.

(iii) *For every $l \in L_{\sigma 1}$, there exists an automorphism $\alpha_l \in E_\sigma(x,y)$ and maps $q_{\alpha_l}, n_{\alpha_l}, v_{\alpha_l}$ from L to itself such that for every $m \in L$ we have*

$$\alpha_l(m) = m + [l,m] + q_{\alpha_l}(m) + n_{\alpha_l}(m) + v_{\alpha_l}(m), \tag{4.3}$$

with

$$q_{\alpha_l}(L_i) \subseteq L_{i+\sigma 2}, n_{\alpha_l}(L_i) \subseteq L_{i+\sigma 3} \text{ and } v_{\alpha_l}(L_i) \subseteq L_{i+\sigma 4}, \tag{4.4}$$

for every $i \in \{-2,-1,0,1,2\}$.

(iv) *For any $\lambda \in k$ and any $\alpha_l \in Aut(L)$ satisfying (4.3) and (4.4), the map $\alpha_{\lambda l}$ defined by*

$$\alpha_{\lambda l}(m) = m + \lambda[l,m] + \lambda^2 q_{\alpha_l}(m) + \lambda^3 n_{\alpha_l}(m) + \lambda^4 v_{\alpha_l}(m),$$

for any $m \in L$, is also an element of $E_\sigma(x,y)$.

(v) *If $\beta_l \in \mathrm{Aut}(L)$ satisfies the same conditions (4.3) and (4.4) as α_l (with possibly different maps $q_{\beta_l}, n_{\beta_l}, v_{\beta_l}$), then $\alpha_l = \beta_l \exp(z)$ for unique $z \in L_{2\sigma}$. In particular $\beta_l \in E_\sigma(x,y)$. Moreover, we have the following alternative description of $E_\sigma(x,y)$*

$$\begin{aligned} E_\sigma(x,y) = \{\varphi \in \mathrm{Aut}(L) \mid & \textit{ There exist } l \in L_{\sigma 1} \textit{ and maps } q_{\varphi_l}, n_{\varphi_l}, v_{\varphi_l} \\ & \textit{from } L \textit{ to itself such that (4.3) and (4.4) are satisfied.}\}. \end{aligned}$$

(vi) *If $\mathrm{char}(k) \neq 2$, then there is a unique α_l as in (iii) such that $q_{\alpha_l}(m) = \frac{1}{2}[l,[l,m]]$. We denote this α_l by $e_\sigma(l)$. Its inverse is $e_\sigma(-l) = \alpha_{-l}$.*

Proof. We will assume $\sigma = +$.

(i) Consider $a, b \in E$ with $g_a(b) = 0$ and $a, b \in L_{\geq 1}$. Then for any $l \in L$ we have:

$$\begin{aligned} \exp(a)\exp(b)(l) =& \exp(a)(g_b(l)b + [b,l] + l) \\ =& (g_a([b,l])a + g_b(l)[a,b]) + (g_a(l)a + g_b(l)b + [a,[b,l]]) \\ & + [a+b,l] + l. \end{aligned} \tag{4.5}$$

Using the Premet identities and $g_a(b) = 0$ we have

$$\begin{aligned} [[a,b],[a+b,l]] =& [[a,b],[a,l]] - [[b,a],[b,l]] \\ =& g_a([b,l])a + g_a(l)[a,b] - g_a(b)[a,l] \\ & - g_b([a,l])b - g_b(l)[b,a] + g_b(a)[b,l] \\ =& g_a([b,l])a - g_b([a,l])b + (g_a(l) + g_b(l))[a,b]. \end{aligned} \tag{4.6}$$

Consider $l \in L_{\geq -1}$ arbitrary. Since $[a,b] \in L_2$ we get $[[a,b], g_b([a,l])b + g_a(l)[b,a]] = 0$. By $[[a,b],[b,[a,l]]] \in L_{\geq 3} = 0$ we get $[[a,b], g_a(l)a + g_b(l)b +$

$[b,[a,l]]] = 0$. Hence, by (4.5) with the roles of a and b reversed, (4.6) and the Jacobi identity, we see that

$$\exp(a)\exp(b)(l) = \exp([a,b])\exp(b)\exp(a)(l).$$

Since L is generated by $L_{\geq -1}$, this implies

$$\exp(a)\exp(b) = \exp([a,b])\exp(b)\exp(a).$$

If $a \in L_2$, then $[a,b] = 0$ and $g_a(b) = 0$. Since $E_+(x,y)$ is generated by $\exp(b)$ with $b \in (L_1 \oplus L_2) \cap E$, we get $[\exp(y), E_+(x,y)] = 1$.

(ii) Considering $a, b \in E \cap L_1$, then since $\langle a\rangle$ and $\langle b\rangle$ have a common neighbor, namely $\langle y\rangle$, we have $g_a(b) = 0$. So by the identities obtained in the proof of (i), we get $[\exp(a), \exp(b)] \in \operatorname{Exp}(\langle y\rangle)$. Using the commutator identities

$$[g_1, g_2g_3] = [g_1, g_3][g_1, g_2]^{g_3}, [g_1g_2, g_3] = [g_1, g_3]^{g_2}[g_2, g_3],$$

with g_1, g_2 and g_3 elements of a group G, we see that

$$[E_+(x,y), E_+(x,y)] \leq \operatorname{Exp}(\langle y\rangle)$$

follows from the fact that the commutators of the generators of $E_+(x,y)$ lie in $\exp(y)$, and that $\exp(y)$ commutes with all elements in $E_+(x,y)$ by part (i). Now in the extremal geometry there exist two neighbors of $\langle y\rangle$ which are special, see Lemma 4.1.11, so there exist $a, b \in E \cap L_1$ such that $[a,b] \neq 0$ and hence $[E_+(x,y), E_+(x,y)] = \operatorname{Exp}(\langle y\rangle)$.

(iii) Consider $l \in L_1$ arbitrary. By Theorem 4.3.7 we can write l as the sum of a finite number of extremal elements in L_1. Let n_l be the smallest natural number such that l can be written as the sum of n_l extremal elements contained in L_1. We will prove that there exists an automorphism α_l satisfying (4.3) and (4.4) by induction on n_l.
If $n_l = 1$, then l is extremal. If we set $\alpha_l = \exp(l)$, $q_{\alpha_l}(m) = g_l(m)l$ for all $m \in L$, and let n_{α_l} and v_{α_l} be the 0-map. By Proposition 2.3.5 this is an automorphism. We see that (4.3) is satisfied. By Lemma 4.1.10 and the fact that the image of q_{α_l} is $\langle l\rangle$, (4.4) holds as well.
Now assume $n_l > 1$. Then $l = e_1 + \cdots + e_{n_l}$, with $e_i \in E \cap L_1$. Set $l' = e_2 + \cdots + e_{l_n}$, then by induction there exist an automorphism $\alpha_{l'}$ and maps $q_{\alpha_{l'}}, n_{\alpha_{l'}}$ and $v_{\alpha_{l'}}$ satisfying (4.3) and (4.4). We set $\alpha_l = \exp(e_1)\alpha_{l'}$, which is an element of $E_+(x,y)$. If we set

$$q_{\alpha_l}(m) = q_{\alpha_{l'}}(m) + g_{e_1}(m)e_1 + [e_1,[l',m]]; \tag{4.7}$$
$$n_{\alpha_l}(m) = n_{\alpha_{l'}}(m) + g_{e_1}([l',m])e_1 + [e_1, q_{\alpha_{l'}}(m)]; \tag{4.8}$$
$$v_{\alpha_l}(m) = v_{\alpha_{l'}}(m) + [e_1, n_{\alpha_{l'}}(m)]; \tag{4.9}$$

for all $m \in L$, then these maps satisfy (4.3), using

$$[e_1, v_{\alpha_{l'}}(m)] \in L_{\geq(-2+1+4)} = L_{\geq 3} = 0,$$

and $g_{e_1}(q_{\alpha_{l'}}(m)) = g_{e_1}(n_{\alpha_{l'}}(m)) = g_{e_1}(v_{\alpha_{l'}}(m)) = 0$. This last identity holds by Lemma 4.1.10 and (4.4) for $q_{\alpha_{l'}}$, $n_{\alpha_{l'}}$ and $v_{\alpha_{l'}}$.
By Lemma 4.1.10, the containment of e_1 and l' in L_1, and the fact that (4.4) holds for $q_{\alpha_{l'}}$, $n_{\alpha_{l'}}$, and $v_{\alpha_{l'}}$, (4.4) holds for q_{α_l}, n_{α_l} and v_{α_l} as well.

(iv)+(v) If z is extremal and $\lambda \in k$, then

$$\exp(\lambda z) : m \mapsto m + \lambda[z, m] + \lambda^2 g_z(m)z$$

is also an automorphism by Proposition 2.3.5, and moreover an element of $E_+(x, y)$ since $\lambda z \in E$. Then by the same induction as in part (iii) and the formulas (4.7) to (4.9) we see that $\alpha_{\lambda l}$ is also an element of $E_+(x, y)$ if α_l is constructed as in (iii).
Let $\beta_l \in \mathrm{Aut}(L)$ be any automorphism satisfying (4.3) and (4.4) and let α_l be as constructed in (iii). Set $\varphi = \alpha_{-l}\alpha_l$, note that α_{-l} is well-defined by the previous paragraph. Then φ fixes L_1 and sends the extremal element x to an extremal element with zero (-1)-component. By Lemmas 4.1.4 and 4.3.9 and since L is generated by L_1 and x, we get $\varphi = \exp(\lambda y)$ for unique $\lambda \in k$. Hence $\alpha_l^{-1}\beta_l$ fixes L_1 and sends x to an extremal element with 0 as (-1)-component, so as before $\alpha_l^{-1}\beta_l = \exp(\mu y)$ for unique $\mu \in k$. This shows the first part of (v). In particular we get that $E_+(x, y)$ contains all automorphisms which satisfy (4.3) and (4.4) for certain $l \in L_1$. Now we show the converse. Consider $\beta_l = \exp(\mu y)\alpha_l$, with $\mu \in k$ arbitrary. Set $q_{\beta_l} = q_{\alpha_l} + \mathrm{ad}_{\mu y}$, $n_{\beta_l} = n_{\alpha_l} + \mathrm{ad}_{\mu y}\,\mathrm{ad}_l$ and $v_{\beta_l}(m) = v_{\alpha_l}(m) + [\mu y, q_{\alpha_l}(m)] + g_y(m)y$ for all $m \in L$. Then these maps satisfy (4.3) and (4.4), since α_l, q_{α_l}, n_{α_l} and v_{α_l} satisfy (4.4) and since $y \in L_2$. Together with (the argument in the paragraph surrounding) (4.7) to (4.9), this shows that the set of all automorphisms which satisfy (4.3) and (4.4) for certain $l \in L_1$ is closed under left multiplication by elements in $E_+(x, y)$. This concludes the proof of (v).
We now know that any other β_l satisfying (4.3) and (4.4) equals $\exp(\mu y)\alpha_l$ for unique $\mu \in k$. Then $\beta_{\lambda l} = \exp(\lambda^2\mu y)\alpha_{\lambda l} \in E_+(x, y)$, showing (iv).

(vi) Assume $\mathrm{char}(k) \neq 2$. We first show the existence of such an automorphism. Let n_l, l', $q_{\alpha_{l'}}$ and e_1 be as in the proof of part (iii). If l is extremal, then $q_{\alpha_l}(m) = g_l(m)l = \frac{1}{2}[l, [l, m]]$ so the case $n_l = 1$ is clear. Now assume $n_l > 1$ and $q_{\alpha_l'}(m) = \frac{1}{2}[l', [l', m]]$ by induction, then by (4.7) we have

$$q_{\alpha_l}(m) = \frac{1}{2}[l', [l', m]] + \frac{1}{2}[e_1, [e_1, m]] + [e_1, [l', m]].$$

Consider $\beta_l = \exp(-\frac{1}{2}[e_1, l'])\alpha_l$, then

$$\begin{aligned} q_{\beta_l}(m) &= \frac{1}{2}[l', [l', m]] + \frac{1}{2}[e_1, [e_1, m]] + \frac{1}{2}[e_1, [l', m]] \\ &\quad + \frac{1}{2}([e_1, [l', m]] + [m, [e_1, l']]) \\ &= \frac{1}{2}[l' + e_1, [l' + e_1, m]] = \frac{1}{2}[l, [l, m]]. \end{aligned}$$

The uniqueness follows by part (v) and the fact that $[x, y] \neq 0$, so the images of x under $e_+(l)$ and $\exp(\lambda y)e_+(l)$ only coincide if $\lambda = 0$. Now $e_+(-l)e_+(l)$ fixes L_1 and maps x onto an extremal element with zero as (-1)- and 0-component and hence Lemmas 4.1.4 and 4.3.9 imply $e_+(-l)e_+(l) = \mathrm{id}$. Since $q_{\beta_{-l}}(m) = \frac{1}{2}(-1)^2[l,[l,m]] = \frac{1}{2}[-l,[-l,m]]$, the automorphism β_{-l} coincides with $e_+(-l)$, by the uniqueness of $e_+(-l)$. $\square$

We now turn our attention to the case where the extremal geometry does not necessarily contain lines.

Definition 4.3.12. Let L be a Lie algebra over k with extremal elements x and y with $g_x(y) = 1$ such that we have a 5-grading as in Lemma 4.1.1. For $\sigma \in \{-,+\}$ we set

$$E_\sigma(x,y) = \{\varphi \in \mathrm{Aut}(L) \mid \text{There exist } l \in L_{\sigma 1} \text{ and maps } q_{\varphi_l}, n_{\varphi_l}, v_{\varphi_l} \\ \text{from } L \text{ to itself such that (4.3) and (4.4) are satisfied.}\}.$$

Note that this definition is consistent with our earlier definition for when the extremal geometry has lines by Theorem 4.3.11(v). We call L *algebraic* if the conclusions of Theorem 4.3.11 are satisfied, with the above definition of $E_\sigma(x,y)$.

Now we can extend the results of Theorem 4.3.11 to a larger class of Lie algebras, using a Galois descent argument.

Theorem 4.3.13. *Assume L is a Lie algebra over k with extremal elements x and y with $g_x(y) = 1$ and let k'/k be a Galois extension such that $L \otimes k'$ is a simple Lie algebra generated by its pure extremal elements, with $\mathcal{F}(L \otimes k') \neq \emptyset$ and $|k'| \geq 3$. Then $L = L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus L_2$, the 5-grading associated with x and y, as in Lemma 4.1.1, is algebraic.*

Proof. Consider $l \in L_1$ arbitrary. To ease notation, we set $l = l \otimes 1$ in $L \otimes k'$.

By assumption, we find an automorphism α_l of $L \otimes k'$ satisfying (4.3) and (4.4). Since $[x,y] \neq 0$, we can find a basis $\mathcal{B}$ of L_0, and thus of $L_0 \otimes k'$, such that $\mathcal{B}$ contains $[x,y]$. Now, by (4.4), $q_{\alpha_l}(x) \in L_0$. Hence we can find unique $b_1, \dots, b_n \in \mathcal{B} \backslash \{[x,y]\}$ and $\lambda, \lambda_1, \dots, \lambda_n \in k'$ such that $q_{\alpha_l}(x) = [x,y] \otimes \lambda + b_1 \otimes \lambda_1 + \cdots + b_n \otimes \lambda_n$. Now consider $\beta_l = \exp(\lambda y)\alpha_l$, then there exist maps q_{β_l}, n_{β_l} and v_{β_l} such that (4.3) and (4.4) are satisfied. More precisely for q_{β_l}, we have $q_{\beta_l} = q_{\alpha_l} + \mathrm{ad}_{\lambda y}$. Hence $q_{\beta_l}(x) = ([x,y]\otimes\lambda + b_1\otimes\lambda_1 + \cdots + b_n\otimes\lambda_n) + [y,x]\otimes\lambda = b_1\otimes\lambda_1 + \cdots + b_n\otimes\lambda_n$.

Consider $g \in \mathrm{Gal}(k'/k)$. Then φ_g, which sends $m \otimes \lambda$ to $m \otimes g(\lambda)$ for any $m \in L$ and $\lambda \in k'$, is an automorphism of $L \otimes k'$. Now for any $m' \in L \otimes k'$ we have, using $\varphi_g(l) = l$,

$$(\varphi_g \circ \beta_l \circ \varphi_g^{-1})(m') = m' + [l, m'] + \varphi_g(q_{\beta_l}(\varphi_g^{-1}(m'))) \\ + \varphi_g(n_{\beta_l}(\varphi_g^{-1}(m'))) + \varphi_g(v_{\beta_l}(\varphi_g^{-1}(m'))).$$

Set $\gamma_l = \varphi_g \circ \beta_l \circ \varphi_g^{-1}$, $q_{\gamma_l} = \varphi_g \circ q_{\beta_l} \circ \varphi_g^{-1}$, $n_{\gamma_l} = \varphi_g \circ n_{\beta_l} \circ \varphi_g^{-1}$ and $v_{\gamma_l} = \varphi_g \circ v_{\beta_l} \circ \varphi_g^{-1}$. Then these maps satisfy (4.3) and (4.4) since q_{β_l}, n_{β_l} and v_{β_l} satisfy (4.4) and since φ_g stabilizes the components $L_i \otimes k'$. Hence Theorem 4.3.11(v) implies that there exists $\mu \in k'$ such that $\gamma_l = \exp(\mu y)\beta_l$. In particular

$$\begin{aligned} b_1 \otimes g(\lambda_1) + \cdots + b_n \otimes g(\lambda_n) = q_{\gamma_l}(x) = q_{\exp(\mu y)\beta_l}(x) = [y,x] \otimes \mu + q_{\beta_l}(x) \\ = [y,x] \otimes \mu + b_1 \otimes \lambda_1 + \cdots + b_m \otimes \lambda_n. \end{aligned}$$

Since $b_1, \ldots, b_n$ and $[x,y]$ are linearly independent by construction, this implies $\mu = 0$. Hence $\beta_l = \varphi_g \circ \beta_l \circ \varphi_g^{-1}$, and moreover $q_{\beta_l} = \varphi_g \circ q_{\beta_l} \circ \varphi_g^{-1}$, $n_{\beta_l} = \varphi_g \circ n_{\beta_l} \circ \varphi_g^{-1}$ and $v_{\beta_l} = \varphi_g \circ v_{\beta_l} \circ \varphi_g^{-1}$. Since $g \in \mathrm{Gal}(k'/k)$ was arbitrary, we get $\varphi_g(q_{\beta_l}(m \otimes 1)) = q_{\beta_l}(m \otimes 1)$ for all $m \in L$ and $g \in \mathrm{Gal}(k'/k)$. Hence $q_{\beta_l}(L \otimes 1) \leq L \otimes 1$, since k'/k is a Galois extension and thus $\mathrm{Fix}(\mathrm{Gal}(k'/k)) = k$. Similarly, $n_{\beta_l}(L \otimes 1) \leq L \otimes 1$, $v_{\beta_l}(L \otimes 1) \leq L \otimes 1$ and thus $\beta_l(L \otimes 1) \leq L \otimes 1$. If $\beta_l(L \otimes 1) = M \otimes 1$, with M a proper subspace of L, then $\beta_l(L \otimes k') = M \otimes k'$, contradicting $\beta_l \in \mathrm{Aut}(L \otimes k')$. So $\beta_l(L \otimes 1) = L \otimes 1$ and hence we find an automorphism of L satisfying (4.3) and (4.4).

So we showed that conclusion (iii) of Theorem 4.3.11 holds. The fact that conclusion (i), (v) and (vi) hold is clear, since they hold over a field extension. Conclusion (iv) also holds since we can extend $\alpha_l \in \mathrm{Aut}(L)$ to an automorphism $\mathrm{Aut}(L \otimes k')$ such that the maps q_{α_l}, n_{α_l} and v_{α_l} fix $L \otimes 1$, and hence for $\lambda \in k$ the map $\alpha_{\lambda l}$ is an automorphism of $L \otimes k'$ stabilizing $L \otimes 1$.

Now we only have to show that conclusion (ii) holds. Since $[L_1 \otimes k', L_1 \otimes k'] \neq 0$ by Lemma 4.1.6, we have $[L_1, L_1] = L_2$. Consider $a, b \in L_1$ such that $[a,b] \neq 0$, and let α_a and α_b be automorphisms as in conclusion (iii). By (4.3) and (4.4) and conclusion (v) we get $\alpha_a \alpha_b = \exp([a,b])\alpha_b \alpha_a$. □

In the next two lemmas we determine how the obtained automorphisms commute.

In these lemmas and the corollary thereafter we use (only) the results of Theorem 4.3.11, so we can loosen our assumptions on the Lie algebra L to the assumptions made in Theorem 4.3.13 in the next two lemmas and corollary.

Lemma 4.3.14. *Consider $a, b \in L_{\sigma 1}$. Let α_a and α_b be as in Theorem 4.3.11(iii). Then $\alpha_a \alpha_b = \exp([a,b])\alpha_b \alpha_a$.*

Proof. Assume $\sigma = -$ this time. Let q_{α_a} and q_{α_b} be as in Theorem 4.3.11(iii). If $l \in L_i$, then the $(i-1)$-components of $\alpha_a \alpha_b(l)$ and $\exp([a,b])\alpha_b \alpha_a(l)$ both equal $[a+b, l]$. Moreover, the $(i-2)$-component of $\alpha_a \alpha_b(l)$ equals $q_{\alpha_a}(l) + q_{\alpha_b}(l) + [a,[b,l]]$ and the $(i-2)$-component of $\exp([a,b])\alpha_b \alpha_a$ equals $q_{\alpha_a}(l) + q_{\alpha_b}(l) + [[a,b],l] + [b,[a,l]]$. Using the Jacobi identity, these components coincide. By applying both $\alpha_a \alpha_b$ and $\exp([a,b])\alpha_b \alpha_a$ onto L_2 and noting that $[x,y] \neq 0$ we can use Theorem 4.3.11(v) to get $\alpha_a \alpha_b = \exp([a,b])\alpha_b \alpha_a$. □

If $\mathrm{char}(k) \neq 2$ we can be more precise.

Lemma 4.3.15. *Assume* $\mathrm{char}(k) \neq 2$. *Then*

$$e_\sigma(a)e_\sigma(b) = \exp(\frac{1}{2}[a,b])e_\sigma(a+b)$$

for all $a, b \in L_{\sigma 1}$ *and* $\sigma \in \{-,+\}$.

Proof. Assume $\sigma = -$. Note that both $e_-(a)e_-(b)$ and $\exp(\frac{1}{2}[a,b])e_-(a+b)$ are contained in $E_-(x,y)$. For all $l \in L_i$, the $(i-1)$-component of both $e_-(a)e_-(b)(l)$ and $\exp(\frac{1}{2}[a,b])e_-(a+b)(l)$ equals $[a+b,l]$. By Theorem 4.3.11(v) there is a unique $\lambda \in k$ such that $\exp(\lambda x)e_-(a)e_-(b) = \exp(\frac{1}{2}[a,b])e_-(a+b)$. Applying both sides to y and looking at the 0-component yields

$$\frac{1}{2}[a,[a,y]] + [a,[b,y]] + \frac{1}{2}[b,[b,y]] + \lambda[x,y] = \frac{1}{2}([[a,b],y] + [a+b,[a+b,y]])$$

and hence, using the Jacobi identity, $\lambda[x,y] = 0$ and thus $\lambda = 0$. $\square$

Now an explicit description of the maps $n_{e_\sigma(l)}$ and $v_{e_\sigma(l)}$ as in part (iii) of Theorem 4.3.11 follows if $\mathrm{char}(k) \neq 2, 3$.

Corollary 4.3.16. *If* $\mathrm{char}(k) \neq 2, 3$, *then for all* $m \in L$ *we have* $n_{e_\sigma(l)}(m) = \frac{1}{6}[l,[l,[l,m]]]$ *and* $v_{e_\sigma(l)}(m) = \frac{1}{24}[l,[l,[l,[l,m]]]]$.

Proof. Denote $e_-(l)$ by α_l as in Theorem 4.3.11(iii). Let α_{2l} be as in Theorem 4.3.11(iv). Since $q_{\alpha_{2l}} = q_{e_-(2l)}$, the uniqueness property in part (vi) of Theorem 4.3.11 yields $\alpha_{2l} = e_-(2l)$. By Lemma 4.3.15 we get $\alpha_{2l} = e_-(2l) = e_-(l)e_-(l)$. Hence $8n_{e_-(l)}(m) = n_{e_l}(m) + \frac{1}{2}[l,[l,[l,m]]] + \frac{1}{2}[l,[l,[l,m]]] + n_{e_l}(m)$ for all $m \in L$, which implies $n_{e_\sigma(l)}(m) = \frac{1}{6}[l,[l,[l,m]]]$. Similarly $v_{e_-(l)}(m) = \frac{1}{24}[l,[l,[l,[l,m]]]]$ for all $m \in L$. $\square$

The following theorem shows that $E_-(x,y)$ acts sharply transitively on the set of all extremal points which contain elements with non-zero 2-component. We will use this in Section 4.6 to show the first axiom of a Moufang set. Note that if the characteristic is not 2 or 3, this next theorem is a special case of Lemma 3.1.9.

Theorem 4.3.17. *Assume* L *is a Lie algebra over* k *with extremal elements* x *and* y *with* $g_x(y) = 1$ *and let* k'/k *be a Galois extension such that* $L \otimes k'$ *is a simple Lie algebra generated by its pure extremal elements, with* $\mathcal{F}(L \otimes k') \neq \emptyset$ *and* $|k'| \geq 3$. *Let* $L = L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus L_2$ *be the 5-grading associated with* x *and* y, *as in Lemma 4.1.1. Then*

- *every extremal element with 2-component equal to* y *equals* $\varphi(y)$, *for unique* $\varphi \in E_-(x,y)$.
- *every extremal element with* (-2)*-component equal to* x *equals* $\varphi(x)$, *for unique* $\varphi \in E_+(x,y)$.

Proof. Let e be an extremal element with 2-component equal to y. By $L_1 = [L_{-1}, y]$ and $[[l_{-1}, y], x] = -l_{-1}$ for all $l_{-1} \in L_{-1}$, there exist a unique $a \in L_{-1}$ such that $[a, y]$ is equal to the 1-component of e. By Theorem 4.3.11(iii) there exists $\alpha \in E_-(x, y)$ such that $\alpha(e) = l_{-2} + l_{-1} + l_0 + y$, with $l_i \in L_i$. By Lemma 4.3.9, there exists $\lambda \in k$ such that $\exp(\lambda x)(\alpha(e)) = y$. Note that $\exp(\lambda x)\alpha$ is contained in $E_-(x, y)$. Hence $\varphi := (\exp(\lambda x)\alpha)^{-1}$ is also contained in this subgroup $E_-(x, y)$ of $\mathrm{Aut}(L)$, and clearly $\varphi(y) = e$.

Assume $\varphi' \in E_-(x, y)$ satisfies $\varphi'(y) = e = \varphi(y)$. The definition of $E_-(x, y)$ implies $\varphi(l) = l + [a, l] = \varphi'(l)$ for all $l \in L_{-1}$. Hence the action of φ and φ' coincide on L_2 and L_{-1} and since these two subspaces generate L, the automorphisms φ and φ' coincide.

The second claim is shown completely similarly. □

Remark 4.3.18. Assume for this remark $\mathrm{char}(k) \neq 2, 3$. Recall that we already introduced a notion of algebraicity of 5-graded Lie algebras if this condition on the characteristic holds, see Definition 1.1.68. Assume L is a Lie algebra over k with extremal elements x and y with $g_x(y) = 1$ and let $L = L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus L_2$ be the associated 5-grading. Note that Corollary 4.3.16 implies that if L is algebraic according to Definition 4.3.12, it is algebraic according to Definition 1.1.68. Assume now that L is algebraic according to Definition 1.1.68. For ease of notation (we do not want to introduce notation for Kantor pairs), assume that the 5-grading on L is coming from a skew-dimension one structurable algebra $\mathcal{A}$. Then the arguments in the proof of Theorem 4.3.11(v) and the first paragraph of the proof of Lemma 4.3.9 can be used to show that $E_\sigma(\mathcal{A})$, as defined in Definition 1.1.75, coincides with $E_\sigma(x, y)$. By [BDMS19, Lemma 3.1.3], we get that L is algebraic according to Definition 4.3.12.

Remark 4.3.19. Note that we did not make any assumptions on the dimension of the Lie algebra L in this section. In particular, the results also hold for infinite-dimensional Lie algebras. In the next sections of this chapter, we will always explicitly mention when we assume L to be finite-dimensional.

SECTION 4.4

Extremal geometry with lines - recovering a cubic norm structure

In this section we prove that if L is a simple Lie algebra over a field k, with $|k| > 3$, which is generated by its pure extremal elements and if its extremal geometry contains lines, then L_1 can be decomposed as $k \oplus J \oplus J \oplus k$, for a certain cubic norm structure J and L can be defined using the maps from this cubic norm structure. (Unless the norm, which we will define later, is the 0-map.)

We briefly outline the proof strategy. By considering a pair of special extremal elements in $E \cap L_{-1}$ we can decompose L_{-1} in 4 parts. Then using the algebraicity proven in the previous section, see Theorem 4.3.11, we exploit the existence of certain extremal elements to define a cubic form and a trace on one of these four components which we denote by J. Using some uniqueness properties of certain extremal elements we can show that J has the structure of a cubic norm structure, if the cubic form is not the 0-map.

Note that if the characteristic is different from 2 or 3, we can obtain this existence of J in a more straightforward fashion, relying on the theory of (skew-dimension one) structurable algebras. See Corollary 4.2.14 and Theorem 4.2.17.

This cubic norm structure is not necessarily anisotropic, but if it is anisotropic the associated extremal geometry is the Moufang hexagon corresponding to J as in Theorem 1.2.21. This is the main theorem of Section 4.4.2. In Section 4.4.3 we look at the other extreme: the case that this cubic form is the 0-map. This turns out to be equivalent to the extremal geometry being of type $A_{n,\{1,n\}}$ (if the Lie algebra is finite-dimensional).

Notation 4.4.1. We fix some notation for this section.

The Lie algebra L is simple and generated by its pure extremal elements E and there are lines in its extremal geometry. This Lie algebra is defined over a field k and we assume $|k| > 3$.

Fix $x, y \in E$ such that $g_x(y) = 1$, the existence of these elements follows by Proposition 2.3.9. By Lemma 4.1.1 there exists a 5-grading $L = L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus L_2$ with $L_{-2} = \langle x \rangle$ and $L_2 = \langle y \rangle$.

Since the extremal geometry contains lines, Lemma 4.1.11 implies that there exist extremal elements e_1' and e_2' contained in L_{-1} such that $[e_1', e_2'] = x$. Set $e_1 = [y, e_1']$ and $e_2 = [y, e_2']$. Again by Lemma 4.1.11 we see that $[x, y] = [e_1' + e_2', e_1 - e_2]$.

4.4.1 Constructing the cubic norm structure

In this section we will construct all the maps involved in a cubic norm structure and verify that these satisfy all axioms of a cubic norm structure.

We start with the following observation, needed to obtain a second grading on the Lie algebra which we will use often in this section.

Lemma 4.4.2. *We have $g_{e_2}(e_1') = -1$.*

Proof. Using the associativity and symmetry of the extremal form g, see Proposition 2.3.6,

$$g_{e_2}(e_1') = g([y, e_2'], e_1') = -g(y, [e_1', e_2']) = -g(y, x) = -g_x(y) = -1. \qquad \square$$

We are now ready to define a subspace J which will turn out to have the structure of a cubic norm structure. Using the grading related to $(e_1', -e_2) \in E_2$ we are able to describe a decomposition of L_{-1} into 4 parts.

Notation 4.4.3. We denote the 5-grading of L obtained by considering the pair $(e_1', -e_2) \in E_2$ in Lemma 4.1.1 by

$$L = L'_{-2} \oplus L'_{-1} \oplus L'_0 \oplus L'_1 \oplus L'_2, \tag{4.10}$$

where $L'_{-2} = \langle e_1' \rangle$ and $L'_2 = \langle e_2 \rangle$. Recall from Lemma 4.1.1 that L'_i is contained in the i-eigenspace of $\operatorname{ad}_{[e_1', -e_2]} = -\operatorname{ad}_{[e_1', e_2]}$. By $[e_1', L'_{-1}] = 0 = [e_2, L'_1]$ this implies

$$[e_1', [e_2, l'_{-1}]] = l'_{-1}, \;\; [e_2, [e_1', l'_1]] = l'_1, \tag{4.11}$$

for any $l'_{-1} \in L'_{-1}$ and $l'_1 \in L'_1$.

We set

$$J = L_{-1} \cap L'_{-1}, \;\; J' = L_{-1} \cap L'_0.$$

Note that $\langle e_1' \rangle$ and $\langle e_2' \rangle$ are both collinear with $\langle x \rangle$ by Lemma 4.1.8. Since $\langle e_1' \rangle$ and $\langle y \rangle$ are special, $\langle e_1 \rangle = \langle [y, e_1'] \rangle$ is collinear with both $\langle e_1' \rangle$ and $\langle y \rangle$. Similarly for $\langle e_2 \rangle$ and hence $\langle x \rangle, \langle e_1' \rangle, \langle e_1 \rangle, \langle y \rangle, \langle e_2 \rangle, \langle e_2' \rangle, \langle x \rangle$ forms an ordinary hexagon in the extremal geometry. Then by Lemma 4.1.9 we have $x \in L'_{-1}$ and $e_2' \in L'_1$. Similarly $y \in L'_1$ and $e_1 \in L'_{-1}$.

Now, as in Lemma 4.4.2, one obtains $g_{e_1}(e_2') = 1$. We denote the 5-grading of L obtained by considering $(e_2', e_1) \in E_2$ in Lemma 4.1.1 by

$$L = L''_{-2} \oplus L''_{-1} \oplus L''_0 \oplus L''_1 \oplus L''_2, \tag{4.12}$$

where $L''_{-2} = \langle e_2' \rangle$ and $L''_2 = \langle e_1 \rangle$. Recall from Lemma 4.1.1 that L''_i is contained in the i-eigenspace of $\operatorname{ad}_{[e_2', e_1]}$. By $[e_2', L''_{-1}] = 0 = [e_1, L''_1]$ this implies

$$[e_2', [e_1, l''_{-1}]] = -l''_{-1}, \;\; [e_1, [e_2', l''_1]] = -l''_1, \tag{4.13}$$

for any $l''_{-1} \in L''_{-1}$ and $l''_1 \in L''_1$. Similarly as before, we have $y, e_1' \in L''_1$ and $x, e_2 \in L''_{-1}$.

In the second lemma of this subsection we obtain a decomposition of L_{-1} into 4 parts.

Lemma 4.4.4. *We have the following decomposition*

$$L_{-1} = \langle e_1' \rangle \oplus J \oplus J' \oplus \langle e_2' \rangle,$$

and moreover $L_{-1} \cap L'_1 = \langle e_2' \rangle$.

Proof. By Lemma 4.1.10 we get $g_{e'_1}(L_{-1}) = 0$. Together with $e'_1 \in L'_{-2}$ and Lemma 4.1.8 this yields $L_{-1} \leq L'_{-2} \oplus L'_{-1} \oplus L'_0 \oplus L'_1$. So if we consider $l \in L_{-1}$ arbitrary, then there exist unique $l'_i \in L'_i$, $i = -2, -1, 0, 1$, such that $l = l'_{-2} + l'_{-1} + l'_0 + l'_1$. By $L'_{-2} = \langle e'_1 \rangle \leq L_{-1}$ we get $l'_{-1} + l'_0 + l'_1 \in L_{-1}$. From now on we may assume $l'_{-2} = 0$. Using $[e'_1, e_2] \in L_0$, we get $l'_{-1} - l'_1 = [[e'_1, e_2], l] \in L_{-1}$.

Let $\lambda \in k$ be any non-zero scalar. Consider the map φ_λ defined by $\varphi_\lambda(l_i) = \lambda^i l_i$ for all $l_i \in L'_i$. By Lemma 4.1.5, applied to the 5-grading (4.10), φ_λ is an automorphism of L.

By $x \in L'_{-1}$ and $y \in L'_1$, we get $\varphi_\lambda(x) = \lambda^{-1}x$ and $\varphi_\lambda(y) = \lambda y$. Then Lemma 4.1.7 implies $\varphi_\lambda(L_{-1}) = L_{-1}$. Hence $\varphi_\lambda(l) = \lambda^{-1}l'_{-1} + l'_0 + \lambda l'_1$ is contained in L_{-1}. Since λ was arbitrary, $l \in L_{-1}$ implies $(\lambda^{-1}-1)l'_{-1} + (\lambda-1)l'_1 \in L_{-1}$ for all non-zero $\lambda \in k$. We showed before $l'_{-1} - l'_1 \in L_{-1}$. Hence $(\lambda^{-1}-1+\lambda-1)l'_{-1} \in L_{-1}$ for any non-zero $\lambda \in k$. If $\operatorname{char}(k) \neq 2$, consider $\lambda = -1$ to get $l'_{-1} \in L_{-1}$. If $\operatorname{char}(k) = 2$, then $|k| > 2$ implies that we can find λ such that $\lambda + \lambda^{-1} \neq 0$. We can conclude $l'_{-1} \in L_{-1}$. By $l'_{-1} - l'_1 \in L_{-1}$ and $l \in L_{-1}$, we get that both l'_1 and l'_0 are also contained in L_{-1}. Hence

$$\begin{aligned} L_{-1} &= (L'_{-2} \cap L_{-1}) \oplus (L'_{-1} \cap L_{-1}) \oplus (L'_0 \cap L_{-1}) \oplus (L'_1 \cap L_{-1}) \\ &= \langle e'_1 \rangle \oplus J \oplus J' \oplus L'_1 \cap L_{-1}. \end{aligned}$$

Consider $l'_1 \in L'_1 \cap L_{-1}$ arbitrary. By (4.11) we get $[e_2, [e'_1, l'_1]] = l'_1$. Since $[e'_1, l'_1] \in L_{-2} = \langle x \rangle$ this implies that l'_1 is a multiple of $[e_2, x] = e'_2$. We showed $L'_1 \cap L_{-1} = \langle e'_2 \rangle$ and obtain the claimed decomposition. □

Using this decomposition we obtain some more information on the Lie bracket.

Corollary 4.4.5. *We obtain the following decomposition of* L_1

$$L_1 = \langle e_1 \rangle \oplus [y, J] \oplus [y, J'] \oplus \langle e_2 \rangle,$$

and the following identities

$$\begin{aligned}
&\langle e_1 \rangle = L_1 \cap L'_{-1}, && [y, J] = L_1 \cap L'_0, && [y, J'] = L_1 \cap L'_1; && (4.14)\\
&[J, J] = 0, && [J, e'_1] = [J, e'_2] = 0, && [J, e_1] = 0; && (4.15)\\
&[J', J'] = 0, && [J', e'_1] = [J', e'_2] = 0, && [J', e_2] = 0; && (4.16)\\
&[J, [J, e_2]] \leq J', && [J', [J', e_1]] \leq J. && && (4.17)
\end{aligned}$$

Proof. By $[y, L_{-1}] = L_1$, $[x, [y, l_{-1}]] = -l_{-1}$ for all $l_{-1} \in L_{-1}$ and Lemma 4.4.4, the decomposition is clear. The identities in (4.14) follow from this new decomposition and $y \in L'_1$.

Since $[J, J] \leq L_{-2} \cap L'_{-2}$ and $x \in L'_{-1}$ spans L_{-2}, the first identity of (4.15) follows. The second identity of (4.15) follows completely similarly. Since $J \leq L'_{-1}$ and $e_1 \in L'_{-1}$, we have $[J, e_1] \leq L'_{-2} = \langle e'_1 \rangle \leq L_{-1}$. On the other hand, looking at the standard grading, $[J, e_1] \leq L_0$, which yields $[J, e_1] = 0$.

The identities of (4.16) are shown completely similarly as these of (4.15).

By $[J,[J,e_2]] \leq L_{-1}$ and $[J,[J,e_2]] \leq [L'_{-1},[L'_{-1},L'_2]] \leq L'_0$ the first identity of (4.17) is clear, similarly for the second identity. □

Lemma 4.4.6. *We can alternatively describe J and J' as follows*

$$J = \{l \in L_{-1} \mid [e'_1, l] = 0 = [e'_2, l] = [e_1, l]\}, \tag{4.18}$$
$$J' = \{l \in L_{-1} \mid [e'_1, l] = 0 = [e'_2, l] = [e_2, l]\}, \tag{4.19}$$

Proof. By (4.15) it is clear that J is contained in the set on the right hand side of (4.18).

Conversely, by $[e'_1, e'_2] \neq 0$ and Lemma 4.4.4 it is clear that the set on the right hand side of equation (4.18) is contained in $J \oplus J'$. Consider $j' \in J'$ arbitrary. By Lemma 4.1.11, $j' \in L'_0$ and $[e_1, e'_1] = [e_2, e'_2] = 0$, we get

$$\begin{aligned} -j' &= [[x,y],j'] = [[e'_1 + e'_2, e_1 - e_2], j'] \\ &= -[[e'_1, e_2], j'] + [[e'_2, e_1], j'] = [e'_2, [e_1, j']]. \end{aligned}$$

Hence $[j', e_1] = 0$ implies $j' = 0$ and (4.18) follows by (4.15). Similarly for (4.19). □

The subspace L_0 also has a decomposition, but this time it is a decomposition into 3 parts.

Lemma 4.4.7. *We have the following decomposition of L_0*

$$L_0 = (L_0 \cap L'_{-1}) \oplus (L_0 \cap L'_0) \oplus (L_0 \cap L'_1), \tag{4.20}$$

and moreover

$$L_0 \cap L'_1 = [J, e_2] \tag{4.21}$$
$$L_0 \cap L'_{-1} = [e'_1, [y, J']] \tag{4.22}$$

Proof. By Lemma 4.1.10 applied to $e'_1 \in L_{-1}$ we get $g_{e'_1}(L_0) = 0$ and hence Lemma 4.1.8 yields $L_0 \leq L'_{-2} \oplus L'_{-1} \oplus L'_0 \oplus L'_1$. By considering the extremal element $e_2 \in L_1$ instead of e'_1, we get $L_0 \leq L'_{-1} \oplus L'_0 \oplus L'_1 \oplus L'_2$. Hence $L_0 \leq L'_{-1} \oplus L'_0 \oplus L'_1$. Now by the last sentence of the first paragraph and the second and third paragraph of the proof of Lemma 4.4.4 we obtain decomposition (4.20).

Consider $l_0 \in L_0 \cap L'_1$ arbitrary. By Lemma 4.1.1 there exists a (unique) $l \in L'_{-1}$ such that $l_0 = [l, e_2]$. Since $e_2 \in L_1$ and $l_0 \in L_0$, we get, using Lemma 4.4.4, Corollary 4.4.5, and (4.20), $l \in L_{-1}$. Hence $l \in L_{-1} \cap L'_{-1} = J$. Clearly $[J, e_2] \leq L_0 \cap L'_1$. This shows equation (4.21). Equation (4.22) is shown completely similarly. □

In the next lemma we deduce some connections between grading (4.10) and grading (4.12).

Lemma 4.4.8. *We have*

$$L_{-1} \cap L''_{-1} = J', \qquad L_{-1} \cap L''_0 = J, \qquad L_{-1} \cap L''_1 = \langle e'_1 \rangle, \tag{4.23}$$

$$L_0 \cap L'_1 = L_0 \cap L''_{-1}, \quad L_0 \cap L'_0 = L_0 \cap L''_0, \quad L_0 \cap L'_{-1} = L_0 \cap L''_1. \tag{4.24}$$

Proof. As in Lemma 4.4.4 we get that

$$L_{-1} = \langle e'_2 \rangle \oplus (L_{-1} \cap L''_{-1}) \oplus (L_{-1} \cap L''_0) \oplus \langle e'_1 \rangle,$$

with $\langle e'_1 \rangle = L_{-1} \cap L''_1$. Then as in (4.18) and (4.19) we can describe these two intersections without using the 5-gradings, but now the roles of e_1 and e_2 are reversed. Hence (4.23) follows

By the same arguments used to prove (4.21), we get $L_0 \cap L''_1 = [J', e_1]$, and similarly $L_0 \cap L''_{-1} = [e'_2, [y, J]]$. Consider the automorphism φ from Lemma 4.1.4. Now note that $[y, L_0 \cap L''_{-1}] \leq L_2 \cap L''_{1-1} = 0$, since $y \in L''_1$. Together with Lemma 4.1.4 this yields

$$L_0 \cap L''_{-1} = \varphi(L_0 \cap L''_{-1}) = \varphi([e'_2, [y, J]]) = [e_2, J] = L_0 \cap L'_1,$$

and similarly $L_0 \cap L''_1 = L_0 \cap L'_{-1}$. Now consider $l \in L_0 \cap L'_0$. By Lemma 4.1.11 and $[e'_1, e_1] = 0 = [e'_2, e_2]$, we get

$$0 = [[x, y], l] = [[e'_1 + e'_2, e_1 - e_2], l] = -[[e'_1, e_2], l] + [[e'_2, e_1], l] = [[e'_1, e_2], l].$$

Hence l is contained in the 0-eigenspace of $\mathrm{ad}_{[e'_1, e_2]}$. By the decomposition of L_0 in (4.20), we get $l \in L'_0$. Hence $L_0 \cap L''_0 \leq L_0 \cap L'_0$, and $L_0 \cap L'_0 \leq L_0 \cap L''_0$ follows similarly. □

The next lemma on the uniqueness of certain extremal elements will be used to define the maps involved in the cubic norm structure.

Lemma 4.4.9. *For every $a \in J$, there is a unique extremal element of the following form*

$$a_{-2} + a_{-1} + [a, e_2] + e_2,$$

with $a_{-1} \in J'$ and $a_{-2} \in L_{-2}$. Moreover it is equal to $\alpha_a(e_2)$, for unique $\alpha_a \in E_-(x, y)$ satisfying (4.3) and (4.4).

Proof. By Theorem 4.3.11(iii) there exists an automorphism α_a satisfying (4.3) and (4.4) and hence an extremal element

$$\alpha_a(e_2) = a_{-2} + a_{-1} + [a, e_2] + e_2,$$

with $a_{-1} \in L_{-1}$ and $a_{-2} \in L_{-2}$. Note that $\langle e_2 \rangle$ and $\langle e'_2 \rangle$ are collinear extremal points. Applying the automorphism α_a yields that $\langle \alpha_a(e_2) \rangle$ and $\langle e'_2 \rangle = \langle [a, e'_2] + e'_2 \rangle = \langle \alpha_a(e'_2) \rangle$ are collinear extremal points, using $[J, e'_2] = 0$. Consider the 5-grading from (4.12), associated with the pair $(e'_2, e_1) \in E_2$. Since $\langle e'_2 \rangle = L''_{-2}$

and $\alpha_a(\langle e_2\rangle)$ are collinear, Lemma 4.1.8 implies $\alpha_a(e_2) \in L''_{-2} \oplus L''_{-1}$. Hence $a_{-1} \in \langle e'_2\rangle \oplus J'$ by (4.23) and the similar decomposition of L_{-1} into 4 parts as in Lemma 4.4.4. Now since $[x, e_2] = [x, [y, e'_2]] = [[x, y], e'_2] = -e'_2$, there exists $\lambda \in k$ such that $\exp(\lambda x)(\alpha_a(e_2))$ has (-1)-component contained in J'. This shows that there exists an extremal element of the claimed form. Assume from now on $a_{-1} \in J'$ and replace α_a by $\exp(\lambda x)\alpha_a$. Now we show the uniqueness claims.

Suppose $\varphi = a'_{-2}+a'_{-1}+[a, e_2]+e_2$ is an extremal element such that $a'_{-2} \in L_{-2}$ and $a'_{-1} \in J'$. Then $\alpha_{-a}(\varphi)$ has 0-component 0.[1] Moreover, its (-1)-component equals $a'_{-1}+a_{-1}-[a,[a,e_2]]$. By assumption $a'_{-1} \in J'$, by the previous paragraph $a_{-1} \in J'$, and by (4.17) we get $[a,[a,e_2]] \in J$. Hence the (-1)-component of $\alpha_{-a}(\varphi)$ is contained in $J' \leq L'_0$. Note that the (-2)-component of $\alpha_{-a}(e_2)$ is contained in L'_{-1} and its only other non-zero component equals $e_2 \in L'_2$. By Lemma 4.3.9, applied to the grading (4.10) associated with $(e'_1, -e_2)$, we get $\alpha_{-a}(\varphi) = \exp(\lambda e'_1)(e_2)$ for certain $\lambda \in k$. Considering the 0-components of these extremal elements yields $0 = \lambda[e_1, e'_2]$ and thus $\lambda = 0$ and $e_2 = \alpha_{-a}(\varphi)$. Since $\alpha_{-a}^{-1} = \exp(\lambda x)\alpha_a$, for certain $\lambda \in k$, we get $\varphi = \exp(\lambda x)\alpha_a(e_2)$. By $-[x, e_2] = e'_2 \notin J'$, and the fact that the (-1)-components of φ and $\alpha_a(e_2)$ are contained in J', we get $\lambda = 0$. This shows the uniqueness of this extremal element.

Note that $[j, e_2] = 0$ for $j \in J$ implies $0 = [e'_1, [j, e_2]] = [j, [e'_1, e_2]] = -j$ using $j \in L'_{-1}$ and $[j, e'_1] = 0$. Consider $\psi \in E_-(x, y)$ such that $\psi(e_2) = \alpha_a(e_2)$. Comparing 0-components of both elements and using that $[j, e_2] = 0$ implies $j = 0$, we get $\psi = \exp(\lambda x)\alpha_a$ for certain $\lambda \in k$ by Theorem 4.3.11(v). But again, as in the end of the previous paragraph, the fact that (-1)-component of the extremal element has to lie in J' implies $\lambda = 0$, showing the uniqueness of the automorphism. □

Lemma 4.4.10. *If $e \in E$ satisfies $(e'_2, e) \in E_{-1}$ and $(e, x) \in E_1$, then there exist unique $a \in J$, $a_{-1} \in J'$ and $a_{-2} \in L_{-2}$ such that*

$$a_{-2} + a_{-1} + [a, e_2] + e_2 \in \langle e'_2, e\rangle.$$

Proof. By $(x, e) \in E_1$, Lemma 4.1.3 yields that $e = l_{-2} + l_{-1} + l_0 + l_1$ for unique $l_i \in L_i$, $i = -2, -1, 0, 1$. Consider the 5-grading (4.12) associated with the pair (e'_2, e_1). Recall $\langle e_2\rangle = L''_{-2}$. Hence Lemma 4.1.8 together with $(e'_2, e) \in E_{-1}$ implies $e \in L''_{-2} \oplus L''_{-1}$. So Lemma 4.4.4 and (4.20), (4.23) and (4.24) imply $l_{-1} \in \langle e'_2\rangle \oplus J'$ and $l_0 = [a, e_2]$ for certain $a \in J$. Note that without loss of generality we may change e by adding a multiple of e'_2 to e. We can thus assume $l_{-1} \in J'$.

Since $(x, e'_2) \in E_{-1}$, $\langle e'_2\rangle$ is the unique common neighbor of the special pair of extremal points $\langle x\rangle$ and $\langle e\rangle$. Hence $0 \neq [x, e] \in \langle e'_2\rangle$. After rescaling of e, we

[1] The automorphism α_{-a} is obtained from the automorphism α_a of the previous paragraph as in Theorem 4.3.11(iv).

can assume $[x, e] = -e_2'$. The (-1)-component of $[x, e]$ equals $[x, l_1]$ and hence $-l_1 = [y, [x, l_1]] = [y, [x, e]] = -[y, e_2'] = -e_2$. So the 1-component of e equals e_2.

The uniqueness claim follows from the uniqueness properties in Lemma 4.4.9 and $e_2' \notin J$. □

We can now define some maps.

Construction 4.4.11. We define maps $N : J \to k$, $\sharp : J \to J'$, $T : J \times J' \to k$, $\times : J \times J \to J'$ as follows:

$$\begin{aligned} N(a)x &= a_{-2}, \text{ for any } a \in J \text{ with } a_{-2} \text{ as in Lemma 4.4.9;} \\ a^\sharp &= a_{-1}, \text{ for any } a \in J \text{ with } a_{-1} \text{ as in Lemma 4.4.9;} \\ T(a,b)x &= [a,b], \text{ for any } a \in J, b \in J'; \\ a \times b &= [a, [b, e_2]], \text{ for any } a, b \in J. \end{aligned}$$

Note that by $[J, J] = 0$, $\times$ is a symmetric map. Completely similarly, one defines maps $N' : J' \to k$, $\sharp' : J' \to J$, $\times' : J' \times J' \to J$.

Using Lemma 4.4.9 we can deduce some identities.

Lemma 4.4.12. *For all $a, b \in J$ and $\lambda \in k$, we have*

$$(\lambda a)^\sharp = \lambda^2 a^\sharp; \tag{4.25}$$

$$N(\lambda a) = \lambda^3 N(a); \tag{4.26}$$

$$(a+b)^\sharp = a^\sharp + a \times b + b^\sharp; \tag{4.27}$$

$$N(a+b) = N(a) + T(b, a^\sharp) + T(a, b^\sharp) + N(b); \tag{4.28}$$

$$(a^\sharp)^{\sharp'} = N(a)a. \tag{4.29}$$

Proof. Equations (4.25) and (4.26) follow immediately from Lemma 4.4.9 and Theorem 4.3.11(iv).

There exist automorphisms α_a and α_b such that $\alpha_a(e_2) = N(a)x + a^\sharp + [a, e_2] + e_2$ and $\alpha_b(e_2) = N(b)x + b^\sharp + [b, e_2] + e_2$ by Lemma 4.4.9. Hence

$$\begin{aligned} \alpha_b(\alpha_a(e_2)) =& (N(a)x + T(b, a^\sharp)x + q_{\alpha_b}([a, e_2]) + N(b)x) \\ &+ (a^\sharp + a \times b + b^\sharp) + [a + b, e_2] + e_2 \end{aligned}$$

is also an extremal element. On the other hand

$$\begin{aligned} \alpha_a(\alpha_b(e_2)) =& (N(b)x + T(a, b^\sharp)x + q_{\alpha_a}([b, e_2]) + N(a)x) \\ &+ (b^\sharp + b \times a + a^\sharp) + [b + a, e_2] + e_2 \end{aligned}$$

is also an extremal element. Since $b^\sharp + b \times a + a^\sharp \in J'$, we can use Lemma 4.4.9 to obtain (4.27) and

$$\begin{aligned} N(a+b)x &= N(a)x + T(b, a^\sharp)x + q_{\alpha_b}([a, e_2]) + N(b)x \\ &= N(a)x + T(a, b^\sharp)x + q_{\alpha_a}([b, e_2]) + N(b)x. \end{aligned}$$

In particular, we have:

$$T(a, b^\sharp)x - q_{\alpha_b}([a, e_2]) = T(b, a^\sharp)x - q_{\alpha_a}([b, e_2]). \tag{4.30}$$

Since $b \in J$ is arbitrary, we can replace it by λb for arbitrary $\lambda \in k$. By Theorem 4.3.11(iv), (4.25), and the linearity of T, the left hand side of (4.30) is quadratic in λ, while the right hand side is linear in λ. Then $|k| \geq 3$ implies that both the left and the right hand side of (4.30) equal 0. So we obtain (4.28).

Consider $a \in J$ arbitrary. Consider the extremal element $\varphi = N(a)x + a^\sharp + [a, e_2] + e_2$. Since $a^\sharp \in J'$, we get

$$g_{e_1}(a^\sharp) = g([y, e_1'], a^\sharp) = g(y, [e_1', a^\sharp]) = 0.$$

First assume $N(a) \neq 0$. Then Lemma 4.1.10, $g_{e_1}(a^\sharp) = 0$ and $[e_2, e_1] \neq 0$ imply $(\varphi, e_1) \in E_1$ and hence $-N(a)e_1' + [a^\sharp, e_1] + [[a, e_2], e_1] - y = [\varphi, e_1] \in E$. Using the automorphism from Lemma 4.1.4 we get $-x + [x, [[a, e_2], e_1]] + [[y, a^\sharp], [x, e_1]] - N(a)e_1 \in E$. Now using $[a, e_1] = 0$, we have $[x, [[a, e_2], e_1]] = -[x, [a, [e_1, e_2]] = -a$. On the other hand $[[y, a^\sharp], [x, e_1]] = [x, [[y, a^\sharp], e_1]] - [[x, [y, a^\sharp]], e_1] = [x, [y, [a^\sharp, e_1]]] + [a^\sharp, e_1] = [[x, y], [a^\sharp, e_1]] + [a^\sharp, e_1] = [a^\sharp, e_1]$. Multiplying the obtained extremal element by $-N(a)^{-1}$, we get $N(a)^{-1}x + N(a)^{-1}a + [-N(a)^{-1}a^\sharp, e_1] + e_1$. So by definition $(-N(a)^{-1}a^\sharp)^{\sharp'} = N(a)^{-1}a$. Using (4.25), which also holds for $\sharp'$ by a similar argument, we get $(a^\sharp)^{\sharp'} = N(a)a$, so we obtain (4.29) if $N(a) \neq 0$.

Now assume $N(a) = 0$ and $a^\sharp \neq 0$. By Lemma 4.1.8 we get $(\varphi, y) \in E_1$ and hence $[a^\sharp, y] = [\varphi, y] \in E$. Similarly $-a^\sharp = [x, [y, a^\sharp]] \in E$. Hence $\exp(a^\sharp)(e_1) = [a^\sharp, e_1] + e_1$, using $g_{e_1}(a^\sharp) = 0$, and thus $(a^\sharp)^{\sharp'} = 0 = N(a)a$. Finally, if $a^\sharp = 0$ and $N(a) = 0$, then (4.29) is trivially satisfied. □

Lemma 4.4.13. *For all $a, b, c \in J$ we have*

$$T(a, a^\sharp) = 3N(a); \tag{4.31}$$

$$a \times a = 2a^\sharp; \tag{4.32}$$

$$T(c, a \times b) = T(a, b \times c); \tag{4.33}$$

$$a^\sharp \times' (a \times b) = N(a)b + T(b, a^\sharp)a; \tag{4.34}$$

$$a^\sharp \times' b^\sharp = -(a \times b)^{\sharp'} + T(b, a^\sharp)b + T(a, b^\sharp)a. \tag{4.35}$$

Proof. By [TW02, (15.16), (15.18)] and (4.25) to (4.29) . (It is stated for a slightly different situation, but it continues to hold in this case. Also note that we assumed $|k| > 3$.) □

In the following lemma we deduce some Lie brackets in terms of T, $\times$ and $\times'$.

Lemma 4.4.14. *Let $i, j, l \in J$ and $i', j', l' \in J'$ be arbitrary, then*

$$\begin{aligned} [j, [i, [y, l]]] &= T(j, i \times l)e'_1; && (4.36) \\ [j, [i, [y, l']]] &= -T(j, l')i - T(i, l')j + (i \times j) \times' l'; && (4.37) \\ [j, [i', [y, l]]] &= (j \times l) \times' i' - T(j, i')l; && (4.38) \\ [j, [i', [y, l']]] &= -(i' \times' l') \times j; && (4.39) \\ [j', [i', [y, l']]] &= T(i' \times' l', j')e'_2; && (4.40) \end{aligned}$$

Proof. By Lemma 4.4.4 and $[j, [i, [y, l]]] \in L_1 \cap L'_{-2}$, we get $[j, [i, [y, l]]] = \lambda e'_1$ for certain $\lambda \in k$. Using $[e'_1, e'_2] = x$ and (4.15), we get

$$-\lambda x = [e'_2, [j, [i, [y, l]]]] = -[j, [i, [l, e_2]]] = -[j, i \times l] = -T(j, i \times l)x,$$

and hence (4.36) holds. Similarly, (4.40) holds.

By applying the Jacobi identity multiple times while using $[j, [i, l']] \in L_{-3} = 0$, we have

$$\begin{aligned} [j, [i, [y, l']]] &= [y, [j, [i, l']]] - [[y, j], [i, l']] - [j, [[y, i], l']] \\ &= 0 - [[y, j], T(i, l')x] - [[y, i], [j, l']] + [l', [j, [y, i]]] \\ &= -T(i, l')j - T(j, l')i + [l', [j, [y, i]]]. \end{aligned}$$

By $y = [e_1, e_2]$ and (4.15), we get

$$[l', [j, [y, i]] = [l', [j, [[e_1, e_2], i]]] = [l', [[j, [i, e_2]], e_1]] = (i \times j) \times' l'. \qquad (4.41)$$

So we recover (4.37).

By the Jacobi identity and $j \in L_{-1}$,

$$[j, [i', [y, l]]] = [j, [y, [i', l]]] + [j, [l, [y, i']]] = T(l, i')j + [j, [l, [y, i']]],$$

so, by (4.37), (4.38) holds.

Equation (4.39) follows by $y = [e_1, e_2]$ and a completely similar calculation as (4.41). □

Now note that by the simplicity of L, (4.15) and (4.16) and Lemma 4.1.11, we have

$$L_0 = \langle [x, y] \rangle + [J + J', [y, J + J']] + [J, e_1] + [J', e_2] + \langle [e'_1, e_2] \rangle.$$

By Lemma 4.4.14, the definition of T, $\sharp$ and $\sharp'$, (4.15) and (4.16) and the Jacobi identity, we can determine all Lie brackets of these elements in L_0 with an element of L_{-1} in terms of T, $\sharp$ and $\sharp'$. By Lemma 4.1.4, the same holds for the Lie brackets with elements in L_1. Since $L_2 = [L_1, L_1]$, $L_{-2} = [L_{-1}, l_{-1}]$ and $L_0 = [L_{-1}, L_1] + [L_{-2}, L_2]$, the Lie bracket of elements in L_0 with any element in L can be expressed in terms of T, $\times$ and $\times'$. Also, clearly the Lie bracket of any two elements in $L_{\leq -1}$ can be expressed in terms of T, similarly for $L_{\geq 1}$.

Lemma 4.4.15. *If $N \neq 0$, then we can re-choose e_1' and e_2' in Notation 4.4.1 such that there exist $c \in J$ with $N(c) = 1$.*

Proof. Consider $a \in J$ such that $N(a) \neq 0$. Note that $[N(a)e_1', N(a)^{-1}e_2'] = [e_1', e_2'] = x$. Hence if we replace e_1' by $N(a)e_1'$ and e_2' by $N(a)^{-1}e_2'$, then the conditions in Notation 4.4.1 are satisfied. Also note that the two subspaces defined in Notation 4.4.3 with respect to these new extremal elements are precisely J and J' by (4.18) and (4.19). Denote the norm obtained in Construction 4.4.11 by N_1. Since $N(a)x + a^\sharp + [a, e_2] + e_2 \in E$, we get $x + N(a)^{-1}a^\sharp + [a, N(a)^{-1}e_2] + N(a)^{-1}e_2 \in E$. Hence $N_1(a) = 1$. □

Using this c, which we call a *basepoint*, we can define isomorphisms from J to J' and from J' to J. For the rest of this section assume $N \neq 0$.

Definition 4.4.16. Define the map $\sigma_1 : J \to J'$ by

$$\sigma_1(j) = T(j, c^\sharp)c^\sharp - c \times j,$$

for all $j \in J$ and the map $\sigma_2 : J' \to J$ by

$$\sigma_2(j') = T(c, j')c - j' \times' c^\sharp,$$

for all $j' \in J'$.

We now show some properties of these two maps.

Lemma 4.4.17. *We have $\sigma_2\sigma_1 = \mathrm{id}_J$ and $\sigma_1\sigma_2 = \mathrm{id}_{J'}$.*

Proof. Consider $j \in J$ arbitrary. Then

$$\begin{aligned}\sigma_2(\sigma_1(j)) &= T(c, T(j, c^\sharp)c^\sharp)c - T(c, c \times j)c - T(j, c^\sharp)c^\sharp \times' c^\sharp + c^\sharp \times' (c \times j)\\ &= T(j, c^\sharp)T(c, c^\sharp)c - T(j, c \times c)c - 2T(j, c^\sharp)(c^\sharp)^{\sharp'} + N(c)j + T(j, c^\sharp)c\\ &= 3T(j, c^\sharp)c - 2T(j, c^\sharp)c - 2T(j, c^\sharp)c + j + T(j, c^\sharp)c\\ &= j,\end{aligned}$$

using (4.29) and (4.31) to (4.34) and $N(c) = 1$. Similarly for the second identity. □

Lemma 4.4.18. *We have $\sigma_2 \circ \sharp = \sharp' \circ \sigma_1$.*

Proof. Consider $j \in J$ arbitrary, then

$$\begin{aligned}\sigma_2(j^\sharp) &= T(c, j^\sharp)c - j^\sharp \times' c^\sharp\\ &= T(c, j^\sharp)c + (j \times c)^{\sharp'} - T(c, j^\sharp)c - T(j, c^\sharp)j\\ &= (j \times c)^{\sharp'} - T(j, c^\sharp)j,\end{aligned}$$

using (4.35). On the other hand

$$\begin{aligned}\sigma_1(j)^{\sharp'} &= (T(j,c^\sharp)c^\sharp - c\times j)^{\sharp'}\\ &= T(j,c^\sharp)^2(c^\sharp)^{\sharp'} - T(j,c^\sharp)c^\sharp \times' (c\times j) + (c\times j)^{\sharp'}\\ &= T(j,c^\sharp)^2 c - T(j,c^\sharp)j - T(j,c^\sharp)^2 c + (c\times j)^{\sharp'}\\ &= (j\times c)^{\sharp'} - T(j,c^\sharp)j,\end{aligned}$$

using $N(c)=1$, (4.25), (4.27), (4.29) and (4.34). □

We are now ready to define the cubic norm structure and prove that it indeed satisfies all properties of a cubic norm structure.

Construction 4.4.19. Consider $a,b\in J$ arbitrary, then define the maps $T_J : J\times J\to k$, $\times_J : J\times J\to J$ and $\sharp_J : J\to J$ as follows:

$$T_J(a,b) = T(a,\sigma_1(b)); \tag{4.42}$$

$$a\times_J b = \sigma_2(a\times b); \tag{4.43}$$

$$a^{\sharp_J} = \sigma_2(a^\sharp). \tag{4.44}$$

Proposition 4.4.20. *Let L be as in Notation 4.4.1 and J be as in Notation 4.4.3. Assume N, as defined in Construction 4.4.11, to be non-zero. The data $(J,k,N,\sharp_J,T_J,\times_J,c)$ is a non-degenerate cubic norm structure.*

Proof. Recall that by Lemma 1.1.19 we only have to show that conditions (i), (ii), (iv), (v), (vii), (x) and (xi) of Definition 1.1.15 are satisfied.

First of all note that $\times_J$ is symmetric and bilinear by construction. For T_J, note that

$$T_J(a,b) = T(a,T(b,c^\sharp)c^\sharp) - T(a,c\times b) = T(a,c^\sharp)T(b,c^\sharp) - T(c,a\times b),$$

by (4.33), and hence T_J is symmetric and bilinear.

Condition (i) follows from the linearity of σ_2 and (4.25).

Condition (ii) is precisely (4.26).

Condition (iv) follows from the linearity of σ_2 and (4.27).

Condition (v) follows from $\sigma_1\sigma_2 = \mathrm{id}_{J'}$ and (4.28).

Condition (vii) follows from $\sigma_1\sigma_2 = \mathrm{id}_{J'}$, Lemma 4.4.18 and (4.29).

Condition (x) follows by $\sigma_2(c^\sharp) = T(c,c^\sharp)c - c^\sharp\times' c^\sharp = 3N(c)c - 2N(c)c = c$, using (4.29), (4.31) and (4.32) and $N(c)=1$.

Condition (xi) follows from

$$\begin{aligned}c\times_J j = \sigma_2(c\times j) &= T(c,c\times j)c - c^\sharp\times'(c\times j)\\ &= T(j,c\times c)c - N(c)j - T(j,c^\sharp)c\\ &= T(j,c^\sharp)c - j = T_J(j,c) - j,\end{aligned}$$

using (4.32) to (4.34), $\sigma_1(c) = c^\sharp$ and $N(c) = 1$, with $j \in J$ arbitrary.

The cubic norm structure is non-degenerate by (4.15), the definition of T_J and Lemma 4.1.6. $\square$

Similarly as in Construction 4.4.19 and Proposition 4.4.20 we can define maps for J' such that $(J', k, N', \sharp_{J'}, T_{J'}, \times_{J'}, c^\sharp)$ is a cubic norm structure.

Lemma 4.4.21. *The map σ_1 is an isomorphism between the cubic norm structures $(J, k, N, \sharp_J, T_J, \times_J, c)$ and $(J', k, N', \sharp_{J'}, T_{J'}, \times_{J'}, c^\sharp)$.*

Proof. Clearly $\sigma_1(c) = c^\sharp$. For any $j \in J$ we have $\sigma_1(j^{\sharp_J}) = \sigma_1(\sigma_2(j^\sharp)) = j^\sharp = \sigma_2(\sigma_1(j^\sharp))) = \sigma_1(\sigma_1(j)^{\sharp'}) = \sigma_1(j)^{\sharp_{J'}}$, using Lemmas 4.4.17 and 4.4.18. $\square$

4.4.2 The anisotropic case: Moufang hexagons

We now look more closely at the case corresponding to a generalized hexagon.

Theorem 4.4.22. *We assume that L is as in Notation 4.4.1. The extremal geometry of L is a generalized hexagon (i.e., is of type $G_{2,2}$) if, and only if, $N \neq 0$ and the cubic norm structure is anisotropic or $J = 0$.*

Proof. First assume that there exist a non-zero $j \in J$ such that $N(j) = 0$. We will show that the extremal geometry is not a generalized hexagon. By definition of N, $\sharp$, (4.25) and (4.26), we get that

$$e_\lambda = \lambda^2 j^\sharp + \lambda[j, e_2] + e_2$$

is an extremal element, for every $\lambda \in k$.

By property (vii) of a cubic norm structure we may assume $j^\sharp = 0$. Since then $e_2 \in E$ and $\lambda[j, e_2] + e_2$ for all non-zero $\lambda \in k$, Lemma 2.3.15 implies that $[j, e_2] \in E$. By Lemma 4.1.8, we have $([y, e_2], y) \in E_0$. Now note that a generalized hexagon has no symplectic pairs, so the extremal geometry is not a generalized hexagon.

Now assume that the extremal geometry is not a generalized hexagon. By Remark 2.3.18 there exist symplectic pairs. By Corollary 2.1.14 the line $\langle x, e_1' \rangle$ is contained in a symplecton S. Then by Proposition 2.1.6(d), Lemma 4.3.6 and $(e_1', e_2') \in E_1$, there exists $z \in E \cap L_{-1}$ such that $(z, e_1') \in E_0$ and $(z, e_2') \in E_{-1}$. In particular $[z, e_2'] = 0$ and $z \in L'_{-1} \oplus L'_{-2}$ by Lemma 4.1.8. By (4.15) we get $z \in L'_{-1} \cap L_{-1} = J$. Now $\exp(z)$ is an automorphism of L and hence $g_z(e_2)z + [z, e_2] + e_2$ is an extremal element. Note $g_z(e_2) = g(z, e_2) = -g(z, [e_2', y]) = -g([z, e_2'], y) = 0$. So $[z, e_2] + e_2$ is an extremal element. Since $z \in J$, Lemma 4.4.9 and the definition of N imply $N(z) = 0$. $\square$

We first handle the case $J = 0$.

Lemma 4.4.23. *The Lie algebra L is isomorphic to $\mathfrak{sl}_3/Z(\mathfrak{sl}_3)$ if and only if $J = 0$ if and only if the extremal geometry is isomorphic to $\Gamma(V, V^*)$, with V a 3-dimensional vector space (which is of type $A_{2,\{1,2\}}$).*

Proof. Note that $\mathfrak{sl}_3$, the Lie algebra of traceless (3×3)-matrices over k, is simple only if $\mathrm{char}(k) \neq 3$. If $\mathrm{char}(k) = 3$ then its center $Z = Z(\mathfrak{sl}_3)$ is 1-dimensional and consists of the multiples of the identity matrix. After modding out this center the Lie algebra is simple. Let $E_{i,j}$ be the (3×3)-matrix over k with all entries equal to 0, except for the (i, j)-entry, which equals 1. Set $\overline{E_{i,j}} = E_{i,j} + Z$.

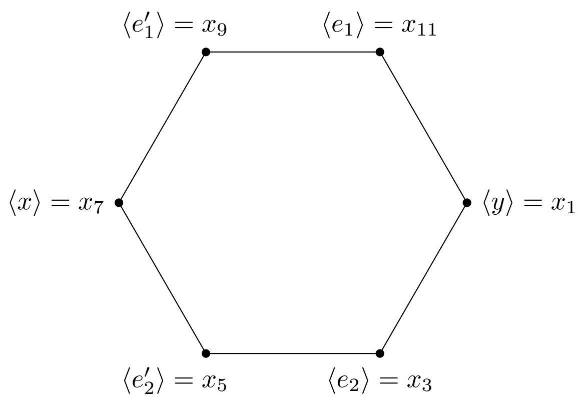
Assume first $J = 0$. Then mapping x, y, e'_1, e'_2, e_1, e_2, $[e'_1, e_2]$ and $[e'_2, e_1]$ to $\overline{E_{3,1}}$, $-\overline{E_{1,3}}$, $\overline{E_{3,2}}$, $\overline{E_{2,1}}$, $-\overline{E_{1,2}}$, $\overline{E_{2,3}}$, $-\overline{E_{2,2}} + \overline{E_{3,3}}$ and $\overline{E_{1,1}} - \overline{E_{2,2}}$, respectively, defines an isomorphism between L and $\mathfrak{sl}_3/Z$. (Note that L_0 is spanned by $[e'_1, e_2]$ and $[e'_2, e_1]$ by $[e'_1, e_1] = [e'_2, e_2] = 0$ and Lemma 4.1.11.)

Note that a geometry of type $A_{2,\{1,2\}}$ is a generalized hexagon with only two lines through every point. Hence if the extremal geometry is of type $A_{2,\{1,2\}}$, Theorem 4.4.22 and Lemma 4.4.29 imply $J = 0$.[2]

If $J = 0$, then $L \cong \mathfrak{sl}_3/Z$ and it is straightforward to check that the extremal geometry is isomorphic to a geometry of type $A_{2,\{1,2\}}$, more precisely, it is isomorphic to $\Gamma(V, V^*)$ as in Example 2.1.3, with V a 3-dimensional vector space over k. $\square$

Assumption 4.4.24. Assume for the rest of this subsection that $N \neq 0$ and that the associated cubic norm structure is anisotropic, so the extremal geometry is a generalized hexagon.

Now we proceed to show that the generalized hexagon is, in fact, a Moufang hexagon and we determine the root groups. In the rest of this subsection, we fix a cycle $(x_0, x_1, \ldots, x_{11}, x_0)$ of length 12 in $\Omega := \mathcal{E} \cup \mathcal{F}$, the incidence graph of the extremal geometry, corresponding to the following cycle of length 6 in the extremal geometry (which is a generalized hexagon):



[2]The proof of Lemma 4.4.29 does not depend on this lemma.

So explicitly, we have $x_0 = \langle e_1, y\rangle$, $x_1 = \langle y\rangle$, etc.

Let $U_1, \dots, U_6$ be the root groups, as defined in Notation 1.2.9. So U_1 is the subgroup of $\mathrm{Aut}(\Omega)$ which fixes all neighbors of x_2, x_3, x_4, x_5 and x_6 (in Ω), and similarly for the other root groups. It turns out that, except for U_1, all these root groups are subgroups of $E_-(x, y)$. In order to determine the root groups explicitly, we can use Lemma 1.2.11 to see that it suffices to show that the claimed subgroup (of $E_-(x, y)$) fixes all the neighbors of a set of 5 distinguished vertices and acts transitively on the set of neighbors of another distinguished vertex, with one neighbor excluded. We first determine U_2, U_4 and U_6 using the following lemma. In order to do this, we make the following easy observation.

Lemma 4.4.25. *Consider extremal elements e, e' such that $(e, e') \in E_{-1}$. Then $\exp(e)$ fixes all lines (in the extremal geometry) through $\langle e'\rangle$. In particular, $\exp(e)$ fixes $\langle e'\rangle$.*

Proof. Since $(e, e') \in E_{-1}$, we get $g_e(e') = 0$ and $[e, e'] = 0$. Hence $\exp(e)$ fixes e'. Consider $z \in E$ such that $(e', z) \in E_{-1}$. Then $(e, z) \in E_{\leq 0}$ or $(e, z) \in E_1$. In the former case $[e, z] = 0$ and $g_e(z) = 0$, so $\exp(e)$ fixes the line $\langle e', z\rangle$. In the latter case $\langle [e, z]\rangle$ is equal to the common neighbor of $\langle x\rangle$ and $\langle e\rangle$, i.e. it is equal to $\langle e'\rangle$. Also note that $z \in E_1(e)$ implies $g_e(z) = 0$ and thus $\exp(e)(z) = [e, z]+z \in \langle e', z\rangle$, so $\exp(e)$ fixes the line $\langle e', z\rangle$. □

Corollary 4.4.26. *For the root groups U_2, U_4 and U_6, we have*

$$U_2 = \{\exp(\lambda e'_2) \mid \lambda \in k\},\ U_4 = \{\exp(\lambda x) \mid \lambda \in k\},\ U_6 = \{\exp(\lambda e'_1) \mid \lambda \in k\}.$$

Moreover, U_i acts transitively on the set of all neighbors of x_i distinct from x_{i+1}, for $i = 2, 4, 6$.

Proof. By Lemma 4.4.25, $\mathrm{Exp}(\langle e'_2\rangle) = \mathrm{Exp}(x_5)$ fixes all neighbors of x_3, x_5 and x_7. The neighbors of $x_4 = \langle e'_2, e_2\rangle$ are all the 1-dimensional subspaces contained in this 2-dimensional space. Since $\exp(e'_2)$ fixes e'_2 and e_2 it fixes all neighbors of x_4. A similar argument holds for x_6. Now we still need to check that the subgroup $U_2 = \{\exp(\lambda e'_2) \mid \lambda \in k\} = \mathrm{Exp}(\langle e'_2\rangle)$ acts transitively on the set of all neighbors of $x_2 = \langle y, e_2\rangle$ distinct from $x_3 = \langle e_2\rangle$. Since $(e'_2, y) \in E_1$ implies $\exp(e'_2)(y) = [e'_2, y] + y = -e_2 + y$, this is obvious. This shows the claim for U_2. The proof for U_4 and U_6 is similar. □

We introduce the following notation in order to obtain the other type of root groups.

Notation 4.4.27. For any $a \in J$, let $e_-(a) \in E_-(x, y)$ be the unique automorphism from Lemma 4.4.9 and set

$$E_-(J) = \{e_-(a) \mid a \in J\}.$$

Similarly for any $a' \in J'$, let $e_-(a') \in E_-(x,y)$ be the unique automorphism such that

$$e_-(a')(e_1) = N'(a')x + a'^{\sharp'} + [a', e_1] + e_1,$$

and define $E_-(J')$ accordingly.

Lemma 4.4.28. *For every $a \in J$*

$$e_-(a)^{-1} = e_-(-a),$$

and moreover $E_-(J)$ is a subgroup of $\mathrm{Aut}(L)$. The same holds for $E_-(J')$.

Proof. Consider $a \in J$ arbitrary, then, by definition, $e_-(-a)(e_-(a)(e_2))$ has (-1)-component equal to $a^\sharp - [a, [a, e_2]] + (-a)^\sharp$ which is contained in J by (4.17). Moreover its 0-component equals $[a, e_2] + [-a, e_2] = 0$, hence Lemma 4.4.9 implies $e_-(-a)(e_-(a)(e_2)) = e_2$. By the uniqueness claim about the automorphism in Lemma 4.4.9 we get $e_-(-a)e_-(a) = \mathrm{id} = e_-(0)$.

Clearly $E_-(J)$ contains the identity of $\mathrm{Aut}(L)$. Consider $a, b \in J$ arbitrary, then $e_-(b)^{-1}(e_-(a)(e_2)) = e_-(-b)(e_-(a)(e_2))$ has 0-component $[a-b, e_2]$ and (-1)-component $a^\sharp - [b, [a, e_2]] + (-b)^\sharp$, which is contained in J. So, by definition of $E_-(J)$, we get $e_-(b)^{-1}e_-(a) = e_-(-b+a)$ and hence $E_-(J)$ is a subgroup of $\mathrm{Aut}(L)$. ☐

In order to determine the other root groups it is necessary to describe all lines through x_5 and x_9. One can easily modify the argument to get a suitable description for all lines through x_3 and x_{11} as well. (For example by Lemma 4.1.4.)

Lemma 4.4.29. *The only lines of the extremal geometry containing x_5 different from x_6 are*

$$x_5 \oplus e_-(a)(e_2), \ \textit{with } a \in J.$$

The only lines of the extremal geometry containing x_9 different from x_8 are

$$x_9 \oplus e_-(a')(e_1), \ \textit{with } a' \in J'.$$

Proof. Recall that we assume J to be anisotropic, so our extremal geometry is a generalized hexagon by Theorem 4.4.22. Hence any two distinct extremal points are either collinear, special, or hyperbolic (at distance 3).

Consider any line l of the extremal geometry through $\langle e_2' \rangle = x_5$ different from x_6. Consider any point p distinct from $\langle e_2' \rangle$ on this line. If p is collinear with $\langle x \rangle$, then the extremal geometry would contain an ordinary 3-gon, namely $\langle x \rangle$, $\langle e_2' \rangle$, p, a contradiction. Since $\langle e_2' \rangle$ is a common neighbor of both $\langle x \rangle$ and p, p and $\langle x \rangle$ are special. Hence Lemma 4.4.10 implies that this line l through $\langle e_2' \rangle$ contains the extremal point $\langle e_-(a)(e_2) \rangle$. Completely similarly for the lines through x_9. ☐

Lemma 4.4.30. *For any $a \in J$*

$$e_-(a)(e_1) = e_1.$$

Proof. Since $[a, e_1'] = 0$, the extremal elements $e_-(a)(e_1)$ and $e_-(a)(e_1') = e_1'$ are collinear. Hence Lemma 4.4.29 together with $[a, e_1] = 0$ imply $e_-(a)(e_1) = \lambda e_1' + e_1$ for certain $\lambda \in k$. By $(e_1, e_2) \in E_1$, also

$$(\lambda e_1' + e_1, N(a)x + a^\sharp + [a, e_2] + e_2) = (e_-(a)(e_1), e_-(a)(e_2)) \in E_1. \tag{4.45}$$

Now using the associativity of g and (4.16), we get

$$g(e_1, a^\sharp) = g([y, e_1'], a^\sharp) = g(y, [e_1', a^\sharp]) = 0.$$

Together with Lemmas 4.1.10 and 4.4.2 this yields $g(\lambda e_1' + e_1, N(a)x + a^\sharp + [a, e_2] + e_2) = \lambda g(e_1', e_2) = -\lambda$. Now (4.45) implies $\lambda = 0$. □

Lemma 4.4.31. *For the root groups U_3 and U_5, we have*

$$U_3 = E_-(J'), \ U_5 = E_-(J).$$

Moreover, U_i acts transitively on all neighbors of x_i distinct from x_{i+1}, for $i = 3, 5$.

Proof. We show the claim for U_5. Let $a \in J$ be arbitrary. Observe that $e_-(a)$ fixes all neighbors of x_6, x_8 and x_{10} by $[a, x] = 0 = [a, e_1'] = [a, e_2'] = [a, e_1]$ and Lemma 4.4.30. By Lemma 4.1.8 we see that any neighbor of $x_7 = \langle x \rangle$ is of the form $\langle x \rangle \oplus \langle l_{-1} \rangle$, for some extremal element $l_{-1} \in L_{-1}$. Clearly $e_-(a)(\langle x, l_{-1} \rangle) = \langle x, l_{-1} \rangle$. Hence $e_-(a)$ also fixes all neighbors of x_7, i.e., all lines of the extremal geometry through $\langle x \rangle$.

We now need to check that $e_-(a)$ fixes all lines through $x_9 = \langle e_1' \rangle$. By Lemma 4.4.29 every line through x_9 different from x_8 contains $e_-(a')(e_1)$ for certain $a' \in J'$. Using Lemmas 4.3.14 and 4.4.30 and $[x, [a', e_1]] = [a', [x, e_2]] = -[a', e_2'] = 0$, we get

$$\begin{aligned} e_-(a)(e_-(a')(e_1)) &= \exp([a, a'])(e_-(a')(e_-(a)(e_1))) = \exp([a, a'])(e_-(a')(e_1)) \\ &= e_-(a')(e_1) + \lambda e_1' \in \langle e_-(a')(e_1), e_1' \rangle, \end{aligned}$$

where $\lambda \in k$ is such that $\lambda x = [a, a']$. So $e_-(a)$ fixes all neighbors of x_6, x_7, x_8, x_9 and x_{10}. By Lemma 4.4.29, we see that $E_-(J)$ acts transitively on the set of all neighbors of x_5 different from x_6. □

We now get to the final root group, which is the hardest to determine.

Consider the automorphism $\varphi = \exp(-e_2)\exp(e'_1)\exp(-e_2)$. Note that by Lemma 4.1.4 applied to grading obtained from the hyperbolic pair $(e'_1, -e_2)$, i.e. $L = L'_{-2} \oplus L'_{-1} \oplus L'_0 \oplus L'_1 \oplus L'_2$, we obtain

$$\varphi(e'_1) = -e_2,\ \varphi(e_2) = -e'_1, \tag{4.46}$$

$$\varphi(l'_{-1}) = [-e_2, l'_{-1}],\ \varphi(l'_0) = l'_0 - [e'_1, [e_2, l_0]],\ \varphi(l'_1) = [e'_1, l'_1], \tag{4.47}$$

for all $l'_i \in L'_i$, $i = -1, 0, 1$.

Lemma 4.4.32. *The root group U_1 is equal to*

$$U_1 = \varphi E_-(J)\varphi^{-1},$$

and acts transitively on the set of all neighbors of x_1 distinct from x_2.

Proof. Using $x \in L'_{-1}$ and (4.47), we get $\varphi(x) = -[e_2, x] = -[[y, e'_2], x] = -e'_2$. Similarly, $\varphi(e'_2) = x$, $\varphi(y) = -e_1$ and $\varphi(e_1) = y$. Together with (4.46) this implies that φ maps $(x_0, x_1, x_2, \ldots, x_{10}, x_{11}, x_0)$ to $(x_0, x_{11}, x_{10}, \ldots, x_2, x_1, x_0)$. Hence $\varphi U_5 \varphi^{-1}$ fixes all neighbors of $\varphi(x_6) = x_6$, $\varphi(x_7) = x_5$, $\varphi(x_8) = x_4$, $\varphi(x_9) = x_3$, and $\varphi(x_{10}) = x_2$. Moreover, by Lemma 1.2.12 the root group U_5 acts transitively on the neighbors of x_{11} different from x_{10} as well. Hence $\varphi U_5 \varphi^{-1}$ acts transitively on all neighbors of $\varphi(x_{11}) = x_1$ distinct from $\varphi(x_{10}) = x_2$ and thus $\varphi U_5 \varphi^{-1} = U_1$.

By Lemma 4.4.31 we get $\varphi U_5 \varphi^{-1} = \varphi E_-(J)\varphi^{-1}$. □

Notation 4.4.33. Note that since $J \leq L'_{-1}$, we get by (4.47) that $\varphi(J) = [J, e_2]$. So we will denote $\varphi e_-(j)\varphi^{-1}$ by $e_-([j, e_2])$, and denote the subgroup of $\mathrm{Aut}(L)$ consisting of these automorphisms by $E_-([J, e_2])$. Then $U_1 = E_-([J, e_2])$.

Corollary 4.4.34. *For any $a \in J$*

$$e_-([a, e_2])(e'_1) = e'_1 + a - a^\sharp + N(a)e'_2.$$

The non-zero multiples of these extremal elements, together with the non-zero multiples of e'_2, are all extremal elements contained in L_{-1}.

Proof. Consider $a \in J$ arbitrary. By (4.47) together with $x \in L'_{-1}$, $[a, e_2] \in L'_{-1+2} = L'_1$, $a^\sharp \in L'_0$ and $[a^\sharp, e_2] = 0$, see (4.16), we get $\varphi(x) = -[e_2, x] = -e'_2$, $\varphi([a, e_2]) = [e'_1, [a, e_2]] = [a, [e'_1, e_2]] = -a$ and $\varphi(a^\sharp) = a^\sharp$, respectively. Hence

$$\begin{aligned} e_-([a, e_2])(e'_1) &= -\varphi(e_-(a)(e_2)) = -\varphi(N(a)x + a^\sharp + [a, e_2] + e_2) \\ &= N(a)e'_2 - a^\sharp + a + e'_1. \end{aligned}$$

Now, since every line through $\langle x \rangle = x_7$ contains a unique extremal point contained in L_{-1} (see Lemma 4.1.8) and U_1 acts transitively on all lines through $\langle x \rangle$ distinct from $\langle x, e'_2 \rangle$ by Lemma 1.2.12, the last claim is clear as well. □

Lemma 4.4.35. *Every cycle $(y_0, \dots, y_{11}, y_0)$ of length 12 in Ω can be mapped onto the cycle $(x_0, \dots, x_{11}, x_0)$ by an element of $\mathrm{Aut}(L)$.*

Proof. We may assume that y_1 is 1-dimensional. By Theorem 2.3.19, the extremal point y_1 can be mapped onto $\langle y \rangle$. So we may assume $y_1 = \langle y \rangle = x_1$. Now y_7 is at distance 3 from y_1 in the extremal geometry. This is equivalent with y_7 having a non-zero (-2)-component by Lemma 4.1.8. By Theorem 4.3.17, there exists an element of $E_+(x, y)$ mapping y_7 onto $\langle x \rangle$ while fixing $\langle y \rangle$. So we may assume $y_7 = \langle x \rangle = x_7$. Since y_5 is collinear with y_7 and at distance 2 from y_1, Lemma 4.1.9 implies that y_5 is a 1-dimensional subspace of L_{-1}. By Corollary 4.4.34 and Lemma 4.4.32 we may then assume $y_5 = x_5$. Similarly, we may assume that $y_9 = x_9$. Since y_3 is the unique neighbor of $y_1 = x_1$ and $y_5 = x_5$, we get $y_3 = x_3$, and similarly $y_{11} = x_{11}$. □

Proposition 4.4.36. *Ω is the incidence graph of a Moufang hexagon.*

Proof. This follows from Lemmas 4.4.31, 4.4.32 and 4.4.35 and Corollary 4.4.26. □

We now determine the commutator relations.

Lemma 4.4.37. *The commutator relations between U_2, U_3, U_4, U_5 and U_6 are as in $E_-(x, y)$, see Lemma 4.3.14. The commutator relations with U_1 are trivial except for:*

$$\begin{aligned}
[e_-([a, e_2]), \exp(\lambda e_1')] &= \exp(\lambda N(a) e_2') e_-(\lambda a^\sharp) \exp(\lambda^2 N(a) x) e_-(\lambda a), \\
[e_-([a, e_2]), e_-(b)] &= \exp(-T(a^\sharp, b) e_2') e_-(-a \times b) \exp(T(a, b^\sharp) x), \\
[e_-([a, e_2]), e_-(b')] &= \exp(-T(a, b') e_2'),
\end{aligned}$$

for all $a, b \in J$ and $b' \in J'$ and $\lambda \in k$.

Proof. We now deduce the commutator relation between U_1 and U_6, all other commutator relations can be deduced in a similar fashion, but this one is the most difficult to obtain. Also note that one can use Lemma 4.3.15 to determine the commutator relations between elements of U_i and U_j, with $i, j \in \{2, 3, 4\}$.

Consider $a \in J$ and $\lambda \in k$ arbitrary. Set

$$\begin{aligned}
\psi &= [e_-([a, e_2]), \exp(\lambda e_1')] \\
&= e_-(-[a, e_2]) \exp(-\lambda e_1') e_-([a, e_2]) \exp(\lambda e_1'),
\end{aligned}$$

where we used Lemma 4.4.28. By Theorem 1.2.13 there exist $b \in J$, $b' \in J'$ and $\gamma, \mu \in k$ such that

$$\psi = \exp(\gamma e_2') e_-(b') \exp(\mu x) e_-(b).$$

We get $\psi\exp(-\lambda e_1') = \exp(\gamma e_2')e_-(b')\exp(\mu x)e_-(b)\exp(-\lambda e_1')$. Now we determine the image of y under both sides of the previous equality to deduce b, b', γ and μ. Note that by Lemma 4.4.30

$$\begin{aligned} e_-(b)(y) &= e_-(b)([e_1,e_2]) = [e_-(b)(e_1), e_-(b)(e_2)] \\ &= [e_1, N(b)x + b^\sharp + [b,e_2] + e_2] \\ &= N(b)[[y,e_1'],x] + [e_1,b^\sharp] + [b,[e_1,e_2]] + y \\ &= N(b)e_1' + [e_1,b^\sharp] + [b,y] + y, \end{aligned}$$

and similarly $e_-(b')(y) = -N'(b')e_2' + [b'^{\sharp'}, e_2] + [b', y] + y$. Now a straightforward calculation shows that the 1-component and 0-component of $\exp(\gamma e_2')e_-(b')\exp(\mu x)e_-(b)\exp(-\lambda e_1')(y)$ are $\lambda e_1 + [b,y] + [b',y] - \gamma e_2$ and

$$[e_1,b^\sharp] + \mu[x,y] + \lambda[b',e_1] + [b',[b,y]] + [b'^{\sharp'}, e_2] + \lambda\gamma[e_2',e_1] + \gamma[e_2',[b,y]], \tag{4.48}$$

respectively.

By $e_-(-a)(e_1) = e_1$, see (4.15), and $\varphi^{-1}(y) = e_1$, we get $e_-(-[a,e_2])(y) = \varphi e_-(-a)\varphi^{-1}(y) = y$. So we deduce by $e_1 = [y,e_1']$ and Corollary 4.4.34 that $e_-(-[a,e_2])(e_1)$ equals $e_1 - [y,a] - [y,a^\sharp] - N(a)e_2$. Hence the 0-component of $\psi\exp(-\lambda e_1')(y) = e_-(-[a,e_2])(\lambda e_1 + y)$ equals 0 and its 1-component equals $\lambda e_1 + \lambda[a,y] + \lambda[a^\sharp,y] - \lambda N(a)e_2$. So we can already conclude $b = \lambda a$, $b' = \lambda a^\sharp$ and $\gamma = \lambda N(a)$.

Now we deduce μ using that (4.48) equals zero. By (4.20) and

$$\begin{aligned} \mu[x,y] + [b',[b,y]] + \lambda\gamma[e_2',e_1] = \mu[x,y] + \lambda^2[a^\sharp,[a,y]] + \lambda^2 N(a)[e_2',e_1] &\in L_0', \\ [e_1,b^\sharp] + \lambda[b',e_1] &\in L_{-1}', \\ [b'^{\sharp'}, e_2] + \gamma[e_2',[b,y]] &\in L_1', \end{aligned}$$

all these components have to equal zero.

In particular we get by $e_2 \in L_1$, $[a^\sharp, e_2] = 0 = [y,e_2] = [e_2',e_2]$, and $[[a,e_2],y] \in L_2 \cap L_2' = 0$,

$$\begin{aligned} 0 &= [\mu[x,y] + \lambda^2[a^\sharp,[a,y]] + \lambda^2 N(a)[e_2',e_1], e_2] \\ &= \mu[[x,y],e_2] + \lambda^2[[a^\sharp,[a,y]],e_2] + \lambda^2 N(a)[[e_2',e_1],e_2] \\ &= \mu e_2 + \lambda^2[a^\sharp,[[a,e_2],y]] + \lambda^2 N(a)[e_2',y] = (\mu - \lambda^2 N(a))e_2. \end{aligned}$$

We obtain $\mu = \lambda^2 N(a)$. □

Theorem 4.4.38. *Let L be as in Notation 4.4.1 and J as in Notation 4.4.3. Assume J to be anisotropic. (Or, equivalently, the extremal geometry to be a generalized hexagon and $J \neq 0$, by Theorem 4.4.22.) The incidence graph $\Omega = \mathcal{E} \cup \mathcal{F}$ of the extremal geometry of L is the incidence graph of the Moufang hexagon associated to the anisotropic cubic norm structure $(J, \sharp_J)$.*

Proof. Consider the following parametrization:

$$
\begin{aligned}
x_1(a) &= e_-([a, e_2]), & x_4(t) &= \exp(tx),\\
x_2(t) &= \exp(te_2'), & x_5(a) &= e_-(a),\\
x_3(a) &= e_-(-\sigma_1(a)), & x_6(t) &= \exp(-te_1'),
\end{aligned}
$$

for all $a \in J$ and $t \in k$. Using Lemma 4.4.37, we see that the commutator relations are the same as in Theorem 1.2.21. □

Theorem 4.4.39. *Let L be a simple Lie algebra over k generated by its pure extremal elements such that its extremal geometry Γ is of type $G_{2,2}$. Assume $|k| > 3$. If L' is a simple Lie algebra generated by its pure extremal elements such that its extremal geometry is isomorphic to Γ, then $L \cong L'$.*

Proof. Denote the extremal geometry of L' by Γ'.

If Γ is of type $A_{2,\{1,2\}}$, this is handled in Lemma 4.4.23.

By Theorem 4.4.38, there exist cubic norm structures $(J_1, \sharp_{J_1})$ and $(J_2, \sharp_{J_2})$ such that Γ is the Moufang hexagon associated with $(J_1, \sharp_{J_1})$ and Γ' is the Moufang hexagon associated with $(J_2, \sharp_{J_2})$. By Theorem 1.2.23 the cubic norm structures $(J_1, \sharp_{J_1})$ and $(J_2, \sharp_{J_2})$ are isotopic. Hence there exists $d \in J_1 \backslash \{0\}$ such that $(J_2, \sharp_{J_2})$ is isomorphic to $(J_1, (\sharp_{J_1})_d)$.

First suppose $d = 1$, i.e. $(J_1, \sharp_{J_1})$ and $(J_2, \sharp_{J_2})$ are isomorphic. By using the bijection σ_1, we can identify J_1' with J_1. Then Lemma 4.4.4, the definition of T, T_{J_1}, $\sharp$ and $\sharp_{J_1}$, and Lemma 4.4.14 imply that we can define L purely in terms of the cubic norm structure. Obviously, an isomorphism of cubic norm structures then yields a 5-graded isomorphism.

Now suppose $d \in J_1^\times$ arbitrary. Consider the maps

$$
\begin{aligned}
\sigma_{1,d} &: J_1 \to J_1' : j \mapsto N(d)(T(j, d^\sharp)d^\sharp - d \times j);\\
\sigma_{2,d} &: J_1' \to J_1 : j' \mapsto N(d)^{-1}(T(d, j')d - j' \times' d^\sharp).
\end{aligned}
$$

These maps satisfy $\sigma_{2,d}\sigma_{1,d} = \mathrm{id}_J$, $\sigma_{1,d}\sigma_{2,d} = \mathrm{id}_J'$, and $\sigma_{2,d} \circ \sharp = \sharp' \circ \sigma_{1,d}$, by exactly the same arguments as in the proofs of Lemmas 4.4.17 and 4.4.18. Similarly, one can define $T_{J_1,d}$, $\sharp_{J_1,d}$ and $\times_{J_1,d}$ as in Construction 4.4.19 and obtain a cubic norm structure as in Proposition 4.4.20. Now note that, by using the definition of T_{J_1}, $\sharp_{J_1}$ and $\times_{J_1}$ and the identities $\sigma_1\sigma_2 = \mathrm{id}_{J_1'}$, $\sigma_2\sigma_1 = \mathrm{id}_{J_1}$, $\sigma_2 \circ \sharp = \sharp' \circ \sigma_1$, we have

$$
\begin{aligned}
N(d)j^{\sharp_{J_1,d}} &= N(d)\sigma_{2,d}(j^\sharp) = T(d, j^\sharp)d - j^\sharp \times' d^\sharp\\
&= T_{J_1}(d, \sigma_2(j^\sharp))d - \sigma_1(j^{\sharp_{J_1}}) \times' \sigma_1(d^{\sharp_{J_1}})\\
&= T_{J_1}(d, j^{\sharp_{J_1}})d - \sigma_2(j^{\sharp_{J_1}} \times d^{\sharp_{J_1}}) = T_{J_1}(d, j^{\sharp_{J_1}})d - (j^{\sharp_{J_1}} \times_{J_1} d^{\sharp_{J_1}}),
\end{aligned}
$$

for all $j \in J$. So by using different bijections in Construction 4.4.19 we see that we may assume $d = 1$, which concludes this proof. (Recall (1.3).) □

Remark 4.4.40. Note that in [CRS15, CF18], see Theorem 2.3.16, it is shown that the extremal geometry determines the Lie algebra if the extremal geometry contains lines and is not of type $G_{2,2}$. Hence the preceding theorem settles this last case.

Corollary 4.4.41. *Let L be a simple finite-dimensional Lie algebra over k generated by its pure extremal elements such that its extremal geometry Γ contains lines. Assume $|k| > 3$. If L' is a simple Lie algebra generated by its pure extremal elements such that its extremal geometry is isomorphic to Γ, then $L \cong L'$.*

Proof. By Theorems 2.3.16 and 4.4.39 and Remark 2.3.18. □

Remark 4.4.42. To show that the extremal geometry is Moufang as soon as it is a generalized hexagon, one can use the theory of abstract root subgroups by Timmesfeld [Tim01], see [CM21, Theorem 7.2]. Using the classification of Moufang hexagons one could then also obtain an anisotropic cubic norm structure. Possibly, but this will certainly require some effort, one could deduce the description of the Lie algebra in terms of a cubic norm structure as described in this section, but we have not pursued this. Moreover, since one has to rely on the classification of Moufang hexagons in this argument, one would not get an elementary proof for Theorem 4.4.39. Now we do get an elementary proof for this theorem.

4.4.3 The case $N = 0$: Type $A_{n,\{1,n\}}$

Now we have a more thorough look at the case $N = 0$. We start by showing that most other maps are also the zero map then.

Lemma 4.4.43. *If $N = 0$, then $\sharp = 0$, $\times = 0$, $N = 0$, $\sharp' = 0$ and $\times' = 0$.*

Proof. By (4.25) and (4.28), we have $\lambda^2 T(b, a^\sharp) + \lambda T(a, b^\sharp) = 0$, for any $a, b \in J$ and $\lambda \in k$. Since $|k| > 2$, we get $T(b, a^\sharp) = 0 = T(a, b^\sharp)$. Since a and b are arbitrary, Lemma 4.1.6 implies $a^\sharp = 0 = b^\sharp$. Equations (4.27) and (4.29) imply all other identities. □

In the following theorem we describe which extremal geometries correspond to the case $N = 0$. First, we need two lemmas.

Lemma 4.4.44. *Consider the root filtration space $\Gamma(V, W^*)$ for a certain vector space V and subspace W^* of V^* which separates the points of the projective space of V. Then, for any pair of special points p and p', every point collinear with the unique common neighbor of p and p' is collinear with p or p'.*

Proof. Recall from Example 2.1.3 that the points of this root filtration space are $(\langle v\rangle, \langle\phi\rangle) \in \mathbb{P}(V) \times \mathbb{P}(W^*)$ such that $\phi(v) = 0$. Consider two points $(\langle v\rangle, \langle\phi\rangle)$ and

$(\langle w\rangle, \langle\psi\rangle)$ (hence $\phi(v) = 0 = \psi(w)$). From Example 2.1.3 we see that these are special points if either $\psi(v) = 0$ or $\phi(w) = 0$, but not both. Assume without loss of generality the latter. The point collinear with both $(\langle v\rangle, \langle\phi\rangle)$ and $(\langle w\rangle, \langle\psi\rangle)$ is $(\langle w\rangle, \langle\phi\rangle)$. Now note that the only points collinear with $(\langle w\rangle, \langle\phi\rangle)$ are $(\langle w\rangle, \langle\varphi\rangle)$, with $\varphi \in W^*$ such that $\varphi(w) = 0$, and $(\langle u\rangle, \langle\phi\rangle)$, with $u \in V$ such that $\phi(u) = 0$. The former type of points is collinear with $(\langle w\rangle, \langle\psi\rangle)$ and the latter type of points is collinear with $(\langle v\rangle, \langle\phi\rangle)$. Hence Γ has the claimed property. □

Lemma 4.4.45. *Consider a root filtration space Γ such that there exists a special pair of points p and p' such that every point special with p and collinear with the unique common neighbor of p and p' is collinear with p' Then either Γ isomorphic to $\Gamma(V, W^*)$, for a certain vector space V and subspace W^* of V^* which separates the points of the projective space of V or Γ is a generalized hexagon such that every point is contained in exactly two lines.*

Proof. By [CI07, Theorem 13, Theorem 15] the root filtration space Γ either only has symplecta of rank at least 3, or is a generalized hexagon, or is as in Example 2.1.4 with the polar space having rank 3, or is isomorphic to $\Gamma(V, W^*)$ for a certain vector space V and subspace W^* of V^* which separates the points of the projective space of V.

First assume that Γ only has symplecta of rank at least 3. Let p and p' be any pair of special points, let q be its common neighbor. By Corollary 2.1.14 there is a symplecton through every line. In particular there exists a symplecton S containing the line qp'. By Corollary 2.1.13 we get that $\mathcal{E}_{-1}(p) \cap S$ is a line L containing q. Since the rank of the symplecton S is at least 3, there exists a line through q not collinear with all points of qp' nor L. Let u be any point on this line distinct from q, then Corollary 2.1.13 implies $u \in \mathcal{E}_1(p)$ and since u and p' are not collinear but are contained in a symplecton we get $u \in \mathcal{E}_0(p')$.

Assume Γ is a generalized hexagon. Then there do not exist symplectic pairs of points. Let p and p' be a special pair with common neighbor q. We can assume that there exists a neighbor t of q not on pq nor on $p'q$, since otherwise we have nothing to show. If t is collinear with p, then p, q, t would be an ordinary 3-gon in the generalized hexagon. Hence t is special with p. Similarly t is special with p'.

Now assume that the root filtration space has as points the lines of a polar space of rank 3, see Example 2.1.4. Note that the maximal singular subspaces of this root filtration space consists of all lines in a plane of the polar space. If there is a line in the polar space which is contained in exactly two planes of the polar space, then we can apply [CI07, Theorem 35] to obtain that Γ is isomorphic to $\Gamma(V, W^*)$, with V and W^* as before, and then there is nothing to prove. So we can assume that in the polar space any line is contained in at least 3 planes of the polar space. Two lines L and M of this polar space are special if each of them contains exactly one point collinear with all points of the other line, denote these points by p and q, respectively. The unique neighbor of L and M is the line pq.

Since there are 3 distinct planes through pq, we can find a plane π through the line pq different form the plane on pq on L and different from the plane on pq and M. Since the rank of the polar space is 3, this line L is not collinear with all points of π and similarly M is not collinear with all points of π. Consider a line N in this plane through a point of pq distinct from p and q. Then pq and N are collinear, but since by construction $L \cap N = \emptyset = M \cap N$, N is not collinear with L nor M. Note that since not every point of N is collinear with every point of L, N and L are special.

Hence we can conclude that if there exists a special pair of points p and p' such that every point special with p and collinear with the unique common neighbor of p and p' is collinear with p', then Γ is isomorphic to $\Gamma(V, W^*)$, with V and W^* as before, or Γ is a generalized hexagon with every point on precisely two lines. □

Theorem 4.4.46. *Let L be as in Notation 4.4.1 and N as in Construction 4.4.11. The extremal geometry of L is isomorphic to $\Gamma(V, W^*)$ for a certain vector space V and subspace W^* of V^* which separates the points of the projective space of V if and only if $N = 0$.*

Proof. Assume $N = 0$, then $\sharp = 0$ by Lemma 4.4.43. By definition of N, $\sharp$ and Lemma 4.4.9 we get $[a, e_2] + e_2 \in E$ for all $a \in J$. Then obviously $\lambda[a, e_2] + e_2 = [\lambda a, e_2] + e_2$ is also contained in E, for any $\lambda \in k$. Now Lemma 2.3.15 implies that $\langle [a, e_2], e_2 \rangle$ is a line of the extremal geometry and in particular $([a, e_2] + e_2, e_2) \in E_{-1}$ for all $0 \neq a \in J$. Note that $\langle x \rangle$ and $\langle e_2 \rangle$ are a pair of special extremal points, with common neighbor $\langle e_2' \rangle = \langle [x, e_2] \rangle$. By Lemma 4.4.10 any line R through $\langle e_2' \rangle$ containing an extremal element special with x contains

$$N(a)x + a^\sharp + [a, e_2] + e_2 = [a, e_2] + e_2$$

for some $a \in J$. As noted before, this extremal element is contained in $E_{-1}(e_2)$ if $a \neq 0$. Since $\langle e_2' \rangle$ and $\langle e_2 \rangle$ are collinear, all extremal points of this line R are collinear with $\langle e_2 \rangle$. By Lemma 4.4.45, we obtain that the extremal geometry is isomorphic to $\Gamma(V, W^*)$ or $J = 0$. But if $J = 0$, we can apply Lemma 4.4.23.

Conversely, assume that the extremal geometry is isomorphic to $\Gamma(V, W^*)$. Then by Lemma 4.4.44 any neighbor of $\langle e_2' \rangle$ which is special with $\langle x \rangle$ has to be collinear with $\langle e_2 \rangle$. This, by the same arguments as in the previous paragraph, is equivalent with

$$N(a)x + a^\sharp + [a, e_2] + e_2$$

being contained in $E_{-1}(e_2)$ for all $a \in J$. In particular $0 = [N(a)x + a^\sharp + [a, e_2] + e_2, e_2] = N(a)[x, e_2] + [a^\sharp, e_2] + [[a, e_2], e_2]$ for all $a \in J$. Since $[x, e_2] = -e_2' \neq 0$, we get $N(a) = 0$ for all $a \in J$. □

In particular, if the Lie algebra is finite-dimensional we have the following corollary.

Corollary 4.4.47. *Let L be a simple Lie algebra over k generated by its pure extremal elements such that its extremal geometry Γ is of type $A_{n,\{1,n\}}$, for a certain $n \in \mathbb{N}$, and $|k| > 3$. If L' is a simple Lie algebra generated by its pure extremal elements such that its extremal geometry is isomorphic to Γ, then $L \cong L'$.*

Proof. Let x, y, $L = \oplus_{i=-2}^{2} L_i$, e_1', e_2', J, and N be as in Section 4.4.1. By Theorem 4.4.46, we get $N = 0$. First we show that $\dim(J) = n-2$, and then we show that we can find bases for J and J' such that T has a canonical form, only depending on n. By Lemma 4.4.4, the definition of T, $\sharp$, and Lemma 4.4.14, the Lie bracket on L can be determined completely in terms of T. Since we can apply the same argument for L', we see that L and L' are then isomorphic.

Let V be a k-vector space of dimension $n+1$. The extremal geometry of type $A_{n,\{1,n\}}$ is isomorphic with $\Gamma(V, V^*)$. Since $\langle x\rangle$ is a point of the extremal geometry Γ of type $A_{n,\{1,n\}}$, there exist $v \in V$ and $\phi \in V^*$ with $\phi(v) = 0$ such that $\langle x\rangle$ can be identified with $(\langle v\rangle, \langle \phi\rangle)$. From the definition of lines in $\Gamma(V, V^*)$ we obtain that there are precisely two maximal singular subspaces containing $(\langle v\rangle, \langle \phi\rangle)$, namely $\{(\langle w\rangle, \langle \phi\rangle) \mid w \in V$ such that $\phi(w) = 0\}$ and $\{(\langle v\rangle, \langle \psi\rangle) \mid \psi \in V^*$ such that $\psi(v) = 0\}$. Both these singular subspaces have the same rank, namely n. Consider $a \in J$ arbitrary. As shown in the proof of Theorem 4.4.46, we get $[a, e_2] \in E$. By Lemma 4.1.10 and $[a, e_2] \in L_0$, we get $([a, e_2], e_1') \in E_{\leq 1}$. Now note that $[a, e_1'] = 0$ and $a \in L_{-1}'$ imply $[e_1', [a, e_2]] = [a, [e_1', e_2]] = -a$. Hence $([a, e_2], e_1') \in E_1$ and $a \in E$. So $J\backslash\{0\} \subseteq E$. By Lemma 2.3.15, $(a, b) \in E_{-1}$ for all linearly independent $a, b \in J$. By Lemma 4.1.8, $(a, x) \in E_{-1}$ for all $a \in J$. By Lemma 4.1.8, but now applied to the grading (4.10) associated with e_1' and e_2, we get $(a, e_1') \in E_{-1}$ for all $a \in J$. Hence $\{\langle \lambda x + j + \mu e_1'\rangle \mid j \in J, \lambda, \mu \in k, (\lambda, j, \mu) \neq (0,0,0)\}$ is a singular subspace of the extremal geometry of L containing $\langle x\rangle$. By $[e_1', e_2'] = x$, (4.14) and Lemmas 4.1.6 and 4.1.8, we get that it is a maximal singular subspace. Its rank is $\dim(\langle x, J, e_1'\rangle) = \dim(J) + 2$. Since this rank equals n, $\dim(J)$ equals $n-2$.

By (4.15) and Lemma 4.1.6, $T : J \times J' \to k$ is a non-degenerate bilinear form. It is well-known that we can find bases for J and J' such that T has a canonical form, but for the sake of completeness we indicate a proof. For $j \in J$, set $j^\perp = \{j' \in J' \mid T(j, j') = 0\}$. Consider two linearly independent elements b_1 and b_2 of J. Then by the non-degeneracy of T, $b_1^\perp$ and $b_2^\perp$ are two distinct hyperplanes of J'. So we can find b_1' and b_2' in J' such that $T(b_1, b_1') = 0 = T(b_2, b_2')$, while $T(b_1, b_2') = 1 = T(b_2, b_1')$. By repeating this argument, we can find linearly independent subsets $\{b_1, \ldots, b_{2m}\}$ and $\{b_1', \ldots, b_{2m}'\}$ of J and J', where $n-2 = 2m$ or $n-2 = 2m+1$, such that $T(b_i, b_j') = 0$ if (i,j) does not equal $(2l+1, 2l+2)$ or $(2l+2, 2l+1)$ for certain $l \in \{0, \ldots, m-1\}$ and $T(b_{2l+1}, b_{2l+2}') = 1 = T(b_{2l+2}, b_{2l+1}')$ for all $l \in \{0, \ldots, m-1\}$. If $n-2 = 2m$ we obtain a canonical form for T. If $n-2 = 2m+1$, then the non-degeneracy of T implies that $b_1^\perp \cap \cdots \cap b_{2m}^\perp$ has dimension 1 and is, by construction, not a subspace of $\langle b_1', \ldots, b_{2m}'\rangle$. Hence, we can find $b_{n-2}' = b_{2m+1}' \in J'$ such that $T(b_i, b_{n-2}') = 0$

if $i \neq n-2$ and $T(b_{n-2}, b'_{n-2}) = 1$. Hence we obtain a canonical form for T in this case as well. □

Remark 4.4.48. Note that Corollary 4.4.47 is certainly not a new result, see Theorem 2.3.16. It appeared for the first time in (the proof of) the main theorem of the doctoral thesis of Kieran Roberts, see Theorem 2 of [Rob12] and the remarks before that theorem. But it is a nice application of the machinery that we developed in this chapter. The above argument can probably be modified to deal with the extremal geometries isomorphic to $\Gamma(V, W^*)$, with V infinite dimensional and W^* a subspace of V^* separating the points of V. This problem has recently been studied in [CM21].

SECTION 4.5

Extremal geometry with symplectic pairs - recovering a quadrangular algebra

In this section L is a simple Lie algebra generated by its pure extremal elements which has an extremal geometry with symplectic pairs of extremal points. Moreover, we assume that after a Galois extension of degree at most 2, the extremal geometry has lines. Let x and y be extremal elements such that $g_x(y) = 1$ and let $L = L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus L_2$ be the corresponding 5-grading, as in Lemma 4.1.1.

We start Section 4.5.1 by showing that L_{-1} can be decomposed as $M \oplus X \oplus M'$ into 3 parts, using a second 5-grading, as in Section 4.4.1. Then we proceed to show that we can define certain maps on M and X such that, if $\mathrm{char}(k) \neq 2$, M and X together with these maps form a quadrangular algebra. This quadrangular algebra might or might not be isotropic.

In Section 4.5.2 we first show that the inner ideal geometry, as defined in Section 2.5, is a generalized quadrangle if and only if the associated quadrangular algebra is anisotropic. Then we continue this subsection with determining the root groups of this generalized quadrangle. We end this subsection by showing that the generalized quadrangle is the Moufang quadrangle corresponding to the quadrangular algebra associated with (M, X), as in Theorem 1.2.20.

In Section 4.5.3 we determine another 5-grading on L, whose ends are *not* one-dimensional subspaces.

Assumption 4.5.1. In this section we assume that L is a simple Lie algebra, defined over a field k with $|k| > 2$, which is generated by its set of pure extremal elements such that

(i) there exists a Galois extension k' of k of degree at most 2 such that the extremal geometry of $L \otimes k'$ contains lines;

(ii) there exist symplectic pairs of extremal elements.

Note that by Theorem 4.3.13 the 5-gradings on L as in Lemma 4.1.1 are algebraic. (In Theorem 2.4.7 we show that $L \otimes k'$ is simple.) Also note that if $\mathrm{char}(k) \neq 2$ then by Theorem 2.4.7 Assumption (i) is satisfied if L is not a symplectic Lie algebra.

4.5.1 Constructing the quadrangular algebra

By assumption there exist symplectic pairs of extremal points, i.e. $\mathcal{E}_0 \neq \emptyset$. In the next lemma we show that this implies the existence of certain extremal elements, which we can then use to define another grading on L and obtain a decomposition of L_{-1} into 3 parts.

Lemma 4.5.2. *There exist extremal elements x, y, a and b such that $g_x(y) = g_a(b) = 1$ and $(x, a), (x, b), (y, a), (y, b) \in E_0$.*

Proof. If the extremal geometry of L does not contain lines, then this follows from Theorem 2.4.7.

Assume that the extremal geometry of L contains lines. By assumption (ii) of 4.5.1 there exist $x \in E$ and a symplecton S of the extremal geometry containing $\langle x \rangle$. By Proposition 2.3.9 there exist $y \in E$ such that $g_x(y) = 1$. Using Lemma 2.1.12 there exists $a \in E$ such that $\langle a \rangle \in S$ and $(a, y) \in E_0$. Now $a \in S$ and property (D) of a root filtration space imply $(x, a) \in E_0$. By Lemma 2.1.9(c) there exists a symplecton T containing x and containing an extremal point hyperbolic with $\langle a \rangle$. In fact, by Lemma 2.1.11, all extremal points in T not collinear with or equal to $\langle x \rangle$ are hyperbolic with $\langle a \rangle$. Then, using Lemma 2.1.12 again, we find $b \in E$ such that $\langle b \rangle \in T$ is symplectic with $\langle y \rangle$. Since $(x, y) \in E_2$, Lemma 2.1.11 implies that $(b, x) \in E_0$. As noted before, this implies $(a, b) \in E_2$. By rescaling a we obtain $g_a(b) = 1$. $\square$

Notation 4.5.3. Let x, y, a, b be as in Lemma 4.5.2.

Denote the 5-grading on L associated with the hyperbolic pair (x, y) as in Lemma 4.1.1 by

$$L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus L_2, \tag{4.49}$$

so with $L_{-2} = \langle x \rangle$.

Denote the 5-grading on L associated with the hyperbolic pair (a, b) as in Lemma 4.1.1 by

$$L'_{-2} \oplus L'_{-1} \oplus L'_0 \oplus L'_1 \oplus L'_2, \tag{4.50}$$

so with $L'_{-2} = \langle a \rangle$. Now set

$$M = L_{-1} \cap L'_{-1}, \; X = L_{-1} \cap L'_0, \; M' = L_{-1} \cap L'_1.$$

Note that by Lemma 4.1.8 and $(x,a),(y,a) \in E_0$ we get $a \in L_0$ and similarly $b \in L_0$. Also recall, see Lemma 4.1.1, that L'_i is contained in the i-eigenspace of $\mathrm{ad}_{[a,b]}$. Similarly, $x, y \in L'_0$.

Lemma 4.5.4. *We have $g_a(L_{-2}\oplus L_{-1}\oplus L_1\oplus L_2) = 0 = g_b(L_{-2}\oplus L_{-1}\oplus L_1\oplus L_2)$.*

Proof. Since $(x,a) \in E_0$, we get $g_a(x) = 0$. Similarly one obtains $g_a(L_2) = 0 = g_b(L_{-2} \oplus L_2)$. Consider $l \in L_{-1}$, then $[x,[y,l]] = -l$. Using the associativity of g, see Proposition 2.3.6,

$$-g_a(l) = g(a,[x,[y,l]]) = g([a,x],[y,l]) = g(0,[y,l]) = 0.$$

Similarly one obtains $g_a(L_1) = 0 = g_b(L_{-1} \oplus L_1)$. $\square$

Lemma 4.5.5. *We have the following decompositions*

$$L_{-1} = M \oplus X \oplus M',$$

$$L_1 = [y,M] \oplus [y,X] \oplus [y,M'].$$

Proof. By Lemma 4.5.4 we get $g_b(L_{-1}) = 0$ and hence we obtain $L_{-1} \leq L'_{-1} \oplus L'_0 \oplus L'_1 \oplus L'_2$ by Lemma 4.1.3 applied to the grading (4.50). Similarly, $L_{-1} \leq L'_{-2} \oplus L'_{-1} \oplus L'_0 \oplus L'_1$. Hence $L_1 \leq L'_{-1} \oplus L'_0 \oplus L'_1$.

Consider $l \in L_1$ arbitrary, then there exist unique $l'_{-1} \in L'_{-1}$, $l'_0 \in L'_0$ and $l'_1 \in L'_1$ such that $l = l'_{-1} + l'_0 + l'_1$. Using $[a,b] \in L_0$, we get $-l'_{-1} + l'_1 = [[a,b],l] \in L_{-1}$.

Now, consider a non-zero scalar $\lambda \in k$. Consider the map φ_λ defined by $\varphi_\lambda(l) = \lambda^i l_i$ for all $l_i \in L'_i$. By Lemma 4.1.5, applied to the 5-grading (4.50), φ_λ is an automorphism of L.

Note that $x, y \in L'_0$ and hence $\varphi_\lambda(x) = x$ and $\varphi_\lambda(y) = y$. By Lemma 4.1.7 we obtain $\varphi_\lambda(L_{-1}) = L_{-1}$. Hence $\varphi_\lambda(l) = \lambda^{-1}l'_{-1} + l'_0 + \lambda l'_1$ is contained in L_{-1}. Since λ was arbitrary, $l \in L_{-1}$ implies $(\lambda^{-1}-1)l'_{-1} + (\lambda-1)l'_1 \in L_{-1}$ for all non-zero $\lambda \in k$. We showed before $-l'_{-1} + l'_1 \in L_{-1}$. Hence $(\lambda - 1 + \lambda^{-1} - 1)l'_1 \in L_{-1}$ for any non-zero $\lambda \in k$. If $\mathrm{char}(k) \neq 2$, consider $\lambda = -1$ to get $l'_1 \in L_1$. If $\mathrm{char}(k) = 2$, then $|k| > 2$ implies that we can find λ such that $\lambda + \lambda^{-1} \neq 0$. We can conclude $l'_1 \in L_{-1}$, so by $-l'_{-1} + l'_1 \in L_{-1}$ and $l \in L_{-1}$, we see $l'_{-1}, l'_0 \in L_{-1}$. This shows the first decomposition, the second one is similar. (Or follows by Lemma 4.1.4.) $\square$

By combining both gradings we obtain some more information on the Lie bracket.

Corollary 4.5.6. *The following identities and inclusions hold*

$$[b, M] = M', \ [a, M] = M'; \tag{4.51}$$
$$[M, M] = [M', M'] = [M, X] = [M', X] = 0; \tag{4.52}$$
$$[X, [X, [y, M]] \leq M, \ [X, [X, [y, M']] \leq M'; \tag{4.53}$$
$$[X, [M, [y, M']]] \leq X, \ [X, [M', [y, M]]] \leq X; \tag{4.54}$$
$$[a, M] = 0 = [b, M']; \tag{4.55}$$
$$[a, X] = 0 = [b, X]. \tag{4.56}$$

Proof. Consider $m \in M$ arbitrary. Since $m \in L'_{-1}$, and $b \in L'_2$, we get $[b, m] \in L_{-1} \cap L'_1 = M'$. By Lemma 4.1.1 we get $[a, [b, m]] = -m$, and similarly we get $[b, [a, m']] = -m'$ for all $m' \in M'$. Hence (4.51).

By $x \in L'_0$ we get $L_2 \cap L'_i = 0$ for $i = -2, -1, 1, 2$. This implies (4.52).

By $y \in L'_0$, $M \leq L'_{-1}$ implies $[y, M] \leq L'_{-1}$. Hence $[X, [X, [y, M]] \leq L_{-1} \cap L'_{-1} = M$ and similarly $[X, [X, [y, M']] \leq M'$, so (4.53) holds.

By $[X, [M, [y, M']]] \leq L'_0$ and $[X, [M', [y, M]]] \leq L'_0$, (4.54) holds.

By $M \leq L'_{-1}$ and $a \in L'_{-2}$, $[a, M] \leq L'_3 = 0$ and similarly $[b, M'] = 0$. Hence (4.55).

Consider $z \in X$ arbitrary. Then $z \in L'_0$ and $\langle a \rangle = L'_{-2}$ implies that $[z, a] = \lambda a$ for certain $\lambda \in k$. Since $[z, a] \in L_{-1}$ and $a \in L_0$, we get $\lambda = 0$. Hence $[X, a] = 0$ and similarly $[b, X] = 0$. Hence (4.56). □

We will now exploit the uniqueness of certain extremal elements to define a quadratic form on M.

Lemma 4.5.7. *For every $m \in M$ there is a unique $\lambda \in k$ such that*

$$l_m := \lambda x + [m, b] + b \tag{4.57}$$

is an extremal element. Moreover, $l_m = l_{m'}$ if and only if $m = m'$, where $m, m' \in M$.

Proof. First note that there is at least one $\lambda \in k$ such that (4.57) holds, by applying any automorphism obtained in Theorem 4.3.11(iii) to $b \in L_0$.

Assume that that there are two distinct scalars such that (4.57) holds. Then the 2-dimensional subspace $\langle x, [m, b] + b \rangle$ contains three distinct extremal elements, so Lemma 2.3.15 implies that this is a line of the extremal geometry. This contradicts $E_{\leq -1}(x) \leq L_{-2} \oplus L_{-1}$, see Lemma 4.1.8.

By the previous paragraph, $l_m = l_{m'}$ if, and only if, $[m, b] = [m', b]$. Now by $m, m' \in L'_{-1}$, we get $[a, [m, b]] = m$ and $[a, [m', b]] = m'$. Hence $[m, b] = [m', b]$ if and only if $m = m'$. □

Notation 4.5.8. Define the map $Q : M \to k$ by letting $Q(m)$ be the unique $\lambda \in k$ from Lemma 4.5.7. Define the bilinear form $T : M \times M \to k$ by

$$T(m, m')x = [m, [m', b]],$$

with $m, m' \in M$ arbitrary.

Lemma 4.5.9. *The map Q is a quadratic form on M, with corresponding bilinear form T.*

Proof. Consider $\lambda \in k$ and $m, m' \in M$ arbitrary. Let α_m be as in Theorem 4.3.11(iii), then $q_{\alpha_m}(b) + [m, b] + b \in E$. By Theorem 4.3.11(iv), $\lambda^2 q_{\alpha_m}(b) + \lambda[m, b] + b \in E$. Hence $Q(\lambda m) = \lambda^2 Q(m)$.

Let $\alpha_{m'}$ be as in Theorem 4.3.11(iii). Then

$$\begin{aligned}(Q(m')x + Q(m)x + [m, [m', b]]) + [m' + m, b] + b &= \alpha_m(Q(m')x + [m', b] + b) \\ &= \alpha_m(\alpha_{m'}(b)) \in E.\end{aligned}$$

By Lemma 4.5.7 this yields $Q(m + m') - Q(m) - Q(m') = T(m, m')$. The map T is bilinear by construction and hence Q is a quadratic form on M. □

In the next few lemmas we show the existence of certain extremal elements depending on Q and show that Q is not the zero-map.

Lemma 4.5.10. *For any $m \in M$*

$$Q(m)x + m + a \in E.$$

Proof. By definition, $Q(m)x + [m, b] + b \in E$. Using Lemma 4.1.4, applied to the grading (4.50), we find an automorphism φ flipping the components of that grading. Using $[a, m] = 0$ (see (4.55)), $[a, x] = 0 = [b, x]$, and $m \in L'_{-1}$ in the last equation, we get

$$\begin{aligned}\varphi(Q(m)x + [m, b] + b) &= Q(m)x + [a, [m, b]] + a \\ &= Q(m)x + [m, [a, b]] + a = Q(m)x + m + a.\end{aligned}$$ □

Lemma 4.5.11. *If $Q(m) = 0$ for $0 \neq m \in M$, then m and $[b, m]$ are contained in E.*

Proof. By definition of Q and Lemma 4.5.9 we get $\lambda[m, b] + b \in E$ for all $\lambda \in k$. Note $[m, b] \neq 0$ by Lemma 4.5.7 (for $m' = 0$ in that lemma). By Lemma 2.3.15, we get $[m, b] \in E$. Since $[m, b] \in L'_1$ and $a \in L'_{-2}$, this implies $(a, [b, m]) \in E_{\leq 1}$ and thus $0 \neq -m = [a, [b, m]] \in E$. □

Corollary 4.5.12. *For any $m \in M$*

$$x + [m, b] + Q(m)b \in E.$$

Proof. Assume first that $m \in M$ satisfies $Q(m) \neq 0$. By Lemma 4.5.9

$$Q(m)^{-1}x + Q(m)^{-1}[m, b] + b = Q(Q(m)^{-1}m)x + [Q(m)^{-1}m, b] + b \in E.$$

Hence $x + [m, b] + Q(m)b \in E$.

Assume now that $m \in M$ satisfies $Q(m) = 0$. By Lemma 4.5.11 we have $[m, b] \in E \cap L_{-1}$. Hence $(x, [m, b]) \in E_{-1}$ by Lemma 4.1.8 and in particular $x + [m, b] \in E$. □

Lemma 4.5.13. *There exists $m \in M$ such that $Q(m) \neq 0$.*

Proof. First we show $M \neq 0$. Suppose on the contrary $M = 0$. By (4.51), then also $M' = 0$. Hence Lemma 4.5.5 and (4.56) imply $[[a, b], L_{-1}] = 0$. Using $[a, y] = [b, y] = 0 = [x, a] = [y, a]$, we get $[[a, b], L_{-2}] = 0 = [[a, b], L_2]$. Hence $[[a, b], L_1] = [[a, b], [y, L_{-1}]] = 0$ and thus $[[a, b], L_0] = [[a, b], [L_{-1}, L_1] + [L_{-2}, L_2]] = 0$. We conclude $[[a, b], L] = 0$, so the simplicity of L implies $[a, b] = 0$, a contradiction.

Consider $0 \neq m \in M$ arbitrary. If $Q(m) \neq 0$, we are done. We may thus assume $Q(m) = 0$. By Lemma 4.5.11 the component L_{-1} then contains extremal elements, so by Lemma 4.1.8 the extremal geometry contains lines.

Consider $z \in E_{-1}(b) \cap E_{-1}(x)$ arbitrary. Then by Lemma 4.1.8 we get $z \in (L_{-2} \oplus L_{-1}) \cap (L'_2 \oplus L'_1)$. Together with $x \in L'_0$ and the decomposition from Lemma 4.5.5 this implies $z \in L_{-1} \cap L'_1 = M'$. Now $\langle x \rangle$ and $\langle b \rangle$ are contained in a symplecton in the extremal geometry, and in such a symplecton we can find two points both collinear to $\langle x \rangle$ and $\langle b \rangle$ but not to each other. I.e., there exist $z_1, z_2 \in E_{-1}(b) \cap E_{-1}(x)$ such that $(z_1, z_2) \in E_0$. If $z_1 + z_2 \in E$, then $\langle z_1, z_2 \rangle$ is a line in the extremal geometry by Lemma 2.3.15, contradicting $(z_1, z_2) \in E_0$. So $z_1 + z_2 \notin E$, and by (4.51) there exist $m \in M$ such that $[m, b] = z_1 + z_2$. If $Q(m) = 0$, then Lemma 4.5.11 implies $[m, b] \in E$, contradicting $z_1 + z_2 \notin E$. □

In the next convention we ensure that the image of Q contains 1.

Convention 4.5.14. Consider $m \in M$ such that $Q(m) \neq 0$. Set $x' := Q(m)x$ and $y' := Q(m)^{-1}y$. Then x', y', a and b still satisfy the conclusions from Lemma 4.5.2. Note that the two gradings (4.49) and (4.50) from Notation 4.5.3 remain the same with respect to these new extremal elements and hence the decomposition from Lemma 4.5.5 is still the same. Now $x' + [m, b] + b = Q(m)x + [m, b] + b \in E$ implies $Q'(m) = 1$, with Q' as defined in Notation 4.5.8, with respect to these new extremal elements.

Assume from now on that we started with these extremal elements x', y', a and b in Notation 4.5.3, so we get $Q'(m) = 1$ for certain $m \in M$. To ease notation, we will again write x, y and Q instead of x', y' and Q'. We will also choose a fixed element $m \in M$ such that $Q(m) = 1$ and denote it by 1.

Note that by Lemma 4.5.10 and Lemma 4.1.4, applied to the grading (4.50), also $Q(-m)x + [a, [b, -m]] + a = Q(m)x + m + a$ is contained in E for all $m \in$

M, where we used $[a,m]=0$ (see (4.55)). As in Notation 4.5.8 we can define $Q': M' \mapsto k$ by letting $Q'(m') \in k$ be such that $Q(m')x + [m',a] + a \in E$ for $m' \in M'$. Since $Q(m)x + [[b,m],a] + a \in E$, we get $Q'([b,m]) = Q(m)$. In particular $Q'([b,1]) = 1$. We set $1' = [b,1]$.

Lemma 4.5.15. *For any $m \in M$*

$$Q(m)[a,b] - Q(m)[x,y] = [[b,m],[y,m]]; \tag{4.58}$$

$$Q(m)[a,b] + Q(m)[x,y] = [m,[y,[b,m]]]. \tag{4.59}$$

Proof. Consider $m \in M$ arbitrary. By Corollary 4.5.12 we have $x + [m,b] + Q(m)b \in E$. Using Theorem 4.3.17, we find an automorphism $\varphi \in E_+(x,y)$ such that $\varphi(x) = x + [m,b] + Q(m)b$. Note $[m,b] = [[y,[m,b]],x]$. So $\varphi = \alpha_{[y,[m,b]]}$, with $\alpha_{[y,[m,b]]}$ as in Theorem 4.3.11(iii). Hence, the $(i+1)$-component of $\varphi(l_i)$ equals $[[y,[m,b]],l_i]$ for all $l_i \in L_i$. In particular $\varphi(a) = a + [[y,[m,b]],a] + \lambda y$ for certain $\lambda \in k$. By $[y,a] = 0 = [a,m]$ and $m \in L'_{-1}$, we get $[[y,[m,b]],a] = [y,[[m,b],a]] = [y,[[a,b],m]]] = -[y,m]$. So $\varphi(a) = a - [y,m] + \lambda y$, this is an extremal element. By Lemma 4.5.7 and twice applying Lemma 4.1.4, once to the grading (4.49) and once to the grading (4.50), this scalar λ is unique. On the other hand, by Lemma 4.5.10 $Q(m)x - m + a = Q(-m)x - m + a \in E$, and after applying the automorphism from Lemma 4.1.4 we get $a - [y,m] + Q(m)y \in E$. Hence $\lambda = Q(m)$.

By construction $[x,a] = 0$, hence $[\varphi(x), \varphi(a)] = \varphi([x,a]) = 0$. Now note that the 0-component of $[\varphi(x), \varphi(a)]$ equals

$$Q(m)[x,y] - Q(m)[a,b] - [[m,b],[y,m]].$$

Since this is the 0-component of 0, we obtain (4.58).

Note $[[b,m],m] = T(m,m)x = 2Q(m)x$ by Lemma 4.5.9. Hence

$$[[b,m],[y,m]] = -[m,[[b,m],y]] + 2Q(m)[y,x] = [m,[y,[b,m]]] - 2Q(m)[x,y],$$

and, together with (4.58), this yields (4.59). □

Notation 4.5.16. Define the maps $h : X \times X \to M$ and $\cdot : X \times M \to X : (z,m) \mapsto z \cdot m$ as follows

$$h(x_1,x_2) = [x_2,[x_1,[y,1]]]; \tag{4.60}$$

$$z \cdot m = [z,[m,[y,1']]]. \tag{4.61}$$

Note that the image of these maps is indeed contained in M and X, respectively, by (4.53) and (4.54).

In the next two lemmas we show that these maps satisfy certain properties of a quadrangular algebra. Note that we do *not* assume the characteristic to be different from 2 in these lemmas.

Recall the involution σ of M associated with Q from Definition 1.1.1.

Lemma 4.5.17. *For all $z \in X$ and $m \in M$*

$$z \cdot 1 = z; \tag{4.62}$$
$$(z \cdot m) \cdot m^\sigma = Q(m)z. \tag{4.63}$$

Proof. Consider $z \in X$ and $v \in M$ arbitrary. Using (4.59) for $m = 1$ we get

$$z \cdot 1 = [z, [1, [y, 1']]] = [z, [a, b] + [x, y]].$$

Using $z \in X = L_{-1} \cap L'_0$, we get $z \cdot 1 = z$.

By (4.59) applied for $m = 1$, $z \cdot v \in L'_0 \cap L_{-1}$ and $[v, 1'] = [v, [b, 1]] = -T(v, 1)x$, we get

$$\begin{aligned}
(z \cdot v) \cdot v^\sigma &= [z \cdot v, [v^\sigma, [y, 1']]] = -[z \cdot v, [v, [y, 1']]] + T(1, v)[z \cdot v, [1, [y, 1']]] \\
&= -[[z, [v, [y, 1']]], [v, [y, 1']]] + T(1, v)[z \cdot v, [a, b] + [x, y]] \\
&= T(v, 1)[[z, [y, x]], [v, [y, 1']]] + [[z, [1', [v, y]]], [v, [y, 1']]] + T(1, v)z \cdot v \\
&= -T(v, 1)[z, [v, [y, 1']]] + [[z, [1', [v, y]]], [v, [y, 1']]] + T(1, v)z \cdot v \\
&= -T(v, 1)z \cdot v + [[z, [1', [v, y]]], [v, [y, 1']]] + T(1, v)z \cdot v \\
&= [[z, [1', [v, y]]], [v, [y, 1']]].
\end{aligned}$$

Now $1' = [b, 1]$, $[y, b] = 0$ and the Jacobi identity yield $[v, [y, 1']] = [[v, b], [y, 1]] + [b, [v, [y, 1]]]$. Note $[v, [y, 1]] \in L'_{-2}$, so $[b, [v, [y, 1]]]$ is a multiple of $[a, b]$. Together with $[z, [1', [v, y]]] \in L'_0$ we get

$$[[z, [1', [v, y]]], [b, [v, [y, 1]]]] = 0.$$

Hence

$$\begin{aligned}
(z \cdot v) \cdot v^\sigma &= [[z, [1', [v, y]]], [v, [y, 1']]] \\
&= [[z, [1', [v, y]]], [[v, b], [y, 1]]] \\
&= [[1', [z, [v, y]]], [[v, b], [y, 1]]] \\
&= -[[v, b], [[y, 1], [1', [z, [v, y]]]]] + [[y, 1], [[v, b], [1', [z, [v, y]]]]]
\end{aligned} \tag{4.64}$$

using $[1', z] = 0$ in the third equality. Now we will analyze the two terms in the last line more carefully.

By $y, z \in L'_0$ and $v, 1 \in L'_{-1}$, we get $[[y, 1], [z, [v, y]]] \in L_1 \cap L'_{-2} = 0$ and hence

$$\begin{aligned}
[[y, 1], [1', [z, [v, y]]]] &= [[[y, 1], 1'], [z, [v, y]]] + [1', [[y, 1], [z, [v, y]]]] \\
&= [-[a, b] + [x, y], [z, [v, y]]] = [z, [v, y]],
\end{aligned} \tag{4.65}$$

using (4.58) and $[z, [v, y]] \in L_0 \cap L'_{-1}$.

By $y, z \in L'_0$, $v \in L'_{-1}$, $1' \in L'_1$ and $b \in L'_2$

$$[[v, b], [1', [z, [v, y]]]] \in L'_1 \cap L_{-2} = 0. \tag{4.66}$$

Now using (4.64) to (4.66), $[[v, b], z] = 0$ (by (4.52)) and (4.58), we get

$$
\begin{aligned}
(z \cdot v) \cdot v^\sigma &= -[[v, b], [z, [v, y]]] = -[z, [[v, b], [v, y]]] \\
&= -Q(v)[z, [a, b] - [x, y]] = Q(v)z.
\end{aligned}
$$

We obtain (4.63). □

Lemma 4.5.18. *For all $x_1, x_2 \in X$ and $m \in M$*

$$
\begin{aligned}
T(h(x_1, x_2), 1)x &= [x_1, x_2]; && (4.67) \\
T(h(x_1 \cdot m, x_2), 1) &= T(h(x_1, x_2), m); && (4.68) \\
h(x_1, x_2 \cdot m) &= h(x_2, x_1 \cdot m) + T(h(x_1, x_2), 1)m. && (4.69)
\end{aligned}
$$

Proof. Using $x_1 \in X = L_{-1} \cap L_0'$ and (4.58) for $m = 1$, we get

$$[x_1, x_2] = -[x_2, [x_1, [x, y]]] = [x_2, [x_1, [1', [y, 1]]]].$$

Now using the Jacobi identity and $[1', x_1] = 0 = [1', x_2]$, we get

$$
\begin{aligned}
[x_1, x_2] &= [1', [x_2, [x_1, [y, 1]]]] = -[h(x_1, x_2), 1'] \\
&= -[h(x_1, x_2), [b, 1]] = T(h(x_1, x_2), 1)x,
\end{aligned}
$$

i.e., we obtain (4.67). Now using (4.67), $[x_1, m] = 0 = [x_2, m]$ and the Jacobi identity we get

$$
\begin{aligned}
T(h(x_1 \cdot m, x_2), 1)x = [x_1 \cdot m, x_2] &= [[x_1, [m, [y, 1']]], x_2] \\
&= -[m, [x_2, [x_1, [y, 1']]]].
\end{aligned}
$$

By $[b, y] = [b, x_1] = [b, x_2] = 0$ and the Jacobi identity applied multiple times

$$[x_2, [x_1, [y, 1']]] = [b, [x_2, [x_1, [y, 1]]]] = [b, h(x_1, x_2)].$$

Hence

$$
\begin{aligned}
T(h(x_1 \cdot m, x_2), 1)x &= -[m, [b, h(x_1, x_2)]] \\
&= T(h(x_1, x_2), m)x,
\end{aligned}
$$

so (4.68) holds.

We now show that (4.69) holds and start by considering the case $m = 1$. Using (4.62) it suffices to show the following identity

$$
\begin{aligned}
h(x_1, x_2) = [x_2, [x_1, [y, 1]]] &= -[[x_1, x_2], [y, 1]] + [x_1, [x_2, [y, 1]]] \\
&= -T(h(x_1, x_2), 1)[x, [y, 1]] + h(x_2, x_1) \qquad (4.70) \\
&= T(h(x_1, x_2), 1)1 + h(x_2, x_1),
\end{aligned}
$$

where we used $1 \in L_{-1}$ and (4.67).

Now we show the more general case, so let $m \in M$ be arbitrary. Using the Jacobi identity, (4.67) and (4.68)

$$\begin{aligned} h(x_1, x_2 \cdot m) &= [x_2 \cdot m, [x_1, [y, 1]]] = [x_1, [x_2 \cdot m, [y, 1]]] + [[x_2 \cdot m, x_1], [y, 1]] \\ &= [x_1, [x_2 \cdot m, [y, 1]]] + T(h(x_2 \cdot m, x_1), 1)[x, [y, 1]] \\ &= [x_1, [x_2 \cdot m, [y, 1]]] - T(h(x_2, x_1), m)1. \end{aligned}$$

By $[m, 1'] = [m, [b, 1]] = -T(m, 1)x$, $[1', x_2] = 0$ (see (4.52)) and $x_2 \in X \leq L_{-1}$, we have

$$\begin{aligned} [x_1, [x_2 \cdot m, [y, 1]]] &= [x_1, [[x_2, [m, [y, 1']]], [y, 1]]] \\ &= -T(m, 1)[x_1, [[x_2, [y, x]], [y, 1]]] - [x_1, [[x_2, [1', [m, y]]], [y, 1]]] \\ &= T(m, 1)[x_1, [x_2, [y, 1]]] - [x_1, [[1', [x_2, [m, y]]], [y, 1]]] \\ &= T(m, 1)h(x_2, x_1) - [x_1, [1', [[x_2, [m, y]], [y, 1]]]] \\ &\quad + [x_1, [[x_2, [m, y]], [1', [y, 1]]]]. \end{aligned}$$

Note $[[x_2, [m, y]], [y, 1]] \in L_1 \cap L'_{-2} = 0$. By $[x_2, [m, y]] \in L_0 \cap L'_{-1}$ and (a special case of) (4.58) we get

$$[x_1, [[x_2, [m, y]], [1', [y, 1]]]] = [x_1, [[x_2, [m, y]], [a, b] - [x, y]]] = [x_1, [x_2, [m, y]]].$$

Hence

$$h(x_1, x_2 \cdot m) = T(m, 1)h(x_2, x_1) - T(h(x_2, x_1), m)1 + [x_1, [x_2, [m, y]]]. \tag{4.71}$$

By (4.67) and $m \in L_{-1}$

$$\begin{aligned} [x_1, [x_2, [m, y]]] &= [[x_1, x_2], [m, y]] + [x_2, [x_1, [m, y]]] \\ &= T(h(x_1, x_2), 1)[x, [m, y]] + [x_2, [x_1, [m, y]]] \\ &= T(h(x_1, x_2), 1)m + [x_2, [x_1, [m, y]]]. \end{aligned} \tag{4.72}$$

We can interchange the roles of x_1 and x_2 in (4.71) to obtain

$$[x_2, [x_1, [m, y]]] = h(x_2, x_1 \cdot m) - T(m, 1)h(x_1, x_2) + T(h(x_1, x_2), m)1. \tag{4.73}$$

Then (4.71) to (4.73) combine to

$$\begin{aligned} h(x_1, x_2 \cdot m) &= T(m, 1)h(x_2, x_1) - T(h(x_2, x_1), m)1 + T(h(x_1, x_2), 1)m \\ &\quad + h(x_2, x_1 \cdot m) - T(m, 1)h(x_1, x_2) + T(h(x_1, x_2), m)1 \\ &= h(x_2, x_1 \cdot m) + T(h(x_1, x_2), 1)m + (h(x_2, x_1) - h(x_1, x_2))T(m, 1) \\ &\quad - T(h(x_2, x_1) - h(x_1, x_2), m)1 \\ &= h(x_2, x_1 \cdot m) + T(h(x_1, x_2), 1)m - T(h(x_1, x_2), 1)T(m, 1)1 \\ &\quad + T(T(h(x_1, x_2), 1)1, m)1 \\ &= h(x_2, x_1 \cdot m) + T(h(x_1, x_2), 1)m, \end{aligned}$$

where we used (4.70) in the second last equality. □

Assume for the rest of this subsection that $\mathrm{char}(k) \neq 2$, then we can define the following maps.

Notation 4.5.19. Set $\theta : X \times M \to M : (x_1, v) \mapsto \frac{1}{2}h(x_1, x_1 \cdot v)$ and $\pi : X \to M : x_1 \mapsto \theta(x_1, 1)$. (Consistent with Lemma 1.1.31 and Notation 1.1.28.)

Remark 4.5.20. If $\mathrm{char}(k) = 2$, then the above definition of θ does not work. It is quite likely, using well-chosen α_{x_1} as in Theorem 4.3.11 instead of $e_-(x_1)$, with $x_1 \in X$, that one can construct a map θ such that properties (vi) and (vii) are satisfied. It seems that it is more difficult to obtain the other properties.

We now determine the image of y under the automorphism $e_-(x_1)$, with $x_1 \in X$. Recall that since $\mathrm{char}(k) \neq 2$, the automorphism $e_-(x_1)$ is well-defined, see Theorem 4.3.11(vi).

Lemma 4.5.21. *For all $x_1 \in X$ and $m \in M$*

$$e_-(x_1)([y, m]) = \frac{1}{2}[x_1, [x_1, [y, m]]] + [x_1, [y, m]] + [y, m]; \tag{4.74}$$

$$e_-(x_1)(y) = Q(\pi(x_1))x - x_1\pi(x_1) + \frac{1}{2}[x_1, [x_1, y]] + [x_1, y] + y. \tag{4.75}$$

Proof. To ease notation, we set $\varphi = e_-(x_1)$. By $b \in L_0$ and $[b, x_1] = 0$, see (4.56), we get $\varphi(b) = b$. Similarly $\varphi(a) = a$. Since $a \in L'_{-2}$ and $b \in L'_2$, Lemma 4.1.7 applied to the grading (4.50) implies $\varphi(L'_{-1}) \leq L'_{-1}$. Since $[y, m] \in L'_{-1}$ and $x \in L'_0$, $\varphi([y, m])$ has (-2)-component equal to 0. Hence (4.74).

We now show (4.75). By $\varphi(b) = b$, $[x_1, b] = 0 = [y, b]$, (4.74) and $[b, 1] = 1'$, we get

$$\begin{aligned} \varphi([[y, 1], b]) &= [\varphi([y, 1]), b] = [\pi(x_1) + [x_1, [y, 1]] + [y, 1], b] \\ &= [\pi(x_1), b] - [x_1, [y, 1']] - [y, 1']. \end{aligned} \tag{4.76}$$

By Lemma 4.1.5 we have $[[y, 1], [y, [1, b]]] = \lambda y$ with $\lambda \in k$ such that $\lambda x = [1, [1, b]]$. Now using the definition of T, $[1, [1, b]] = T(1, 1)x = 2Q(1)x = 2x$. Together with (4.74) and (4.76) this yields

$$\begin{aligned} 2\varphi(y) &= [\varphi([y, 1]), \varphi([[y, 1], b])] \\ &= [\pi(x_1) + [x_1, [y, 1]] + [y, 1], [\pi(x_1), b] - [x_1, [y, 1']] - [y, 1']] \\ &= [\pi(x_1), [\pi(x_1), b]] + (-[\pi(x_1), [x_1, [y, 1']]] + [[x_1, [y, 1]], [\pi(x_1), b]]) \\ &\quad + [x_1, [x_1, y]] + 2[x_1, y] + 2y, \end{aligned} \tag{4.77}$$

where we used Theorem 4.3.11(vi) in the last line. Clearly

$$[\pi(x_1), [\pi(x_1), b]] = T(\pi(x_1), \pi(x_1))x = 2Q(\pi(x_1))x. \tag{4.78}$$

By $[\pi(x_1), [x_1, [y, 1]]] \in L_{-1} \cap L'_{-2} = 0$, and $[b, y] = 0 = [b, x_1]$ we get

$$\begin{aligned} -[\pi(x_1), [x_1, [y, 1']]] + [[x_1, [y, 1]], [\pi(x_1), b]] &= -x_1\pi(x_1) + [\pi(x_1), [x_1, [y, [1, b]]]] \\ &= -2x_1\pi(x_1). \end{aligned} \tag{4.79}$$

Now (4.77) to (4.79) imply (4.75). □

We now deduce the last property we need in order to prove that (X, M) has the structure of a quadrangular algebra.

Lemma 4.5.22. *For all $x_1 \in X$ and $v \in M$, we have*

$$(x_1 \cdot \pi(x_1)) \cdot v = x_1 \cdot \theta(x_1, v). \tag{4.80}$$

Proof. First note that by (4.62) and the definition of π,

$$(x_1 \cdot \pi(x_1)) \cdot 1 = x_1 \cdot \pi(x_1) = x_1 \cdot \theta(x_1, 1).$$

Since (4.80) is linear in v, we can now assume $T(v, 1) = 0$. In particular $[v, 1'] = -[v, [1, b]] = -T(v, 1)x = 0$. We also get, using $g_y(v) = 0 = g_y(1')$ and $0 = g_y(0) = g_y([v, 1'])$, that

$$\begin{aligned}[][[v, [y, 1']], y] &= -[[y, v], [y, 1']] + 0 \\ &= -g_y([v, 1']) - [g_y(1')y, v] + [g_y(v)y, 1'] = 0.\end{aligned}$$

Hence $0 = e_-(x_1)([[v, [y, 1']], y]) = [e_-(x_1)([v, [y, 1']]), e_-(x_1)(y)]$ and we will now exploit that the (-1)-component of this element equals zero to obtain the claimed equality. By definition of $\cdot$ and (4.67),

$$\begin{aligned} e_-(x_1)([v, [y, 1']]) &= \frac{1}{2}[x_1, x_1 \cdot v] + x_1 \cdot v + [v, [y, 1']] \\ &= \frac{1}{2}T(h(x_1, x_1 \cdot v), 1)x + x_1 \cdot v + [v, [y, 1']]. \end{aligned}$$

Together with (4.75) this yields that the (-1)-component of $[e_-(x_1)([v, [y, 1']]), e_-(x_1)(y)]$ equals

$$\begin{aligned} &\frac{1}{2}T(h(x_1, x_1 \cdot v), 1)[x, [x_1, y]] + \frac{1}{2}[x_1 \cdot v, [x_1, [x_1, y]]] + [x_1 \cdot \pi(x_1), [v, [y, 1']]] \\ &\qquad = \frac{1}{2}T(h(x_1, x_1 \cdot v), 1)x_1 + \frac{1}{2}[x_1 \cdot v, [x_1, [x_1, y]]] + (x_1 \cdot \pi(x_1)) \cdot v, \end{aligned} \tag{4.81}$$

which thus has to equal 0. Now $[[y, 1], [y, 1']] = -2y$ yields

$$\begin{aligned} -2[x_1, [x_1, y]] &= [x_1, [x_1, [[y, 1], [y, 1']]]] \\ &= [[x_1, [x_1, [y, 1]]], [y, 1']] + 2[[x_1, [y, 1]], [x_1, [y, 1']]] \\ &\quad + [[y, 1], [x_1, [x_1, [y, 1']]]]. \end{aligned} \tag{4.82}$$

By definition of h and $\cdot$,

$$[x_1 \cdot v, [[x_1, [x_1, [y, 1]]], [y, 1']]] = [x_1 \cdot v, [h(x_1, x_1), [y, 1']]] = (x_1 \cdot v) \cdot h(x_1, x_1). \tag{4.83}$$

Since $(x_1 \cdot v) \cdot h(x_1, x_1), x_1 \cdot v, x_1, y \in L'_0$, $1 \in L'_{-1}$, $1' \in L'_1$ and $[b, (x_1 \cdot v) \cdot h(x_1, x_1)] = [b, x_1] = [b, y] = [b, x_1 \cdot v] = 0$, Lemma 4.1.4 applied to the grading (4.50) associated with (a, b), yields

$$\begin{aligned}[x_1 \cdot v, [[y, 1], [x_1, [x_1, [y, 1']]]]] &= [x_1 \cdot v, [[y, [b, 1]], [x_1, [x_1, [y, [a, 1']]]]]] \\ &= -[x_1 \cdot v, [[y, 1'], [x_1, [x_1, [y, 1]]]]] \\ &= (x_1 \cdot v) \cdot h(x_1, x_1). \end{aligned} \tag{4.84}$$

We also have

$$\begin{aligned}[x_1 \cdot v, [[x_1, [y, 1]], [x_1, [y, 1']]]] &= [[x_1 \cdot v, [x_1, [y, 1]]], [x_1, [y, 1']]] \\ &\quad + [[x_1, [y, 1]], [x_1 \cdot v, [x_1, [y, 1']]]] \\ &= x_1 \cdot h(x_1, x_1 \cdot v) + [[x_1, [y, 1]], [x_1 \cdot v, [x_1, [y, 1']]]] \\ &= 2x_1 \cdot h(x_1, x_1 \cdot v), \end{aligned} \tag{4.85}$$

where the last equality is obtained in a similar fashion as (4.84). Now the fact that (4.81) equals 0, together with (4.82) to (4.85), implies

$$0 = \frac{1}{2} T(h(x_1, x_1 \cdot v), 1)x_1 - (x_1 \cdot v) \cdot \pi(x_1) - 2x_1 \cdot \theta(x_1, v) + (x_1 \cdot \pi(x_1)) \cdot v. \tag{4.86}$$

Now note that by $[x_1, x_1] = 0$ and (4.67), $T(\pi(x_1), 1) = 0$. Together with $T(v, 1) = 0$, (1.4) yields

$$-(x_1 \cdot v) \cdot \pi(x_1) = T(\pi(x_1), v)x_1 + (x_1 \cdot \pi(x_1)) \cdot v. \tag{4.87}$$

By (4.67) and (4.68)

$$\begin{aligned} 2T(\pi(x_1), v) = T(h(x_1 \cdot v, x_1), 1) &= [x_1 \cdot v, x_1] \\ &= -[x_1, x_1 \cdot v] = -T(h(x_1, x_1 \cdot v), 1). \end{aligned}$$

Together with (4.86) and (4.87) this yields (4.80). □

Corollary 4.5.23. *If* $\mathrm{char}(k) \neq 2$*, then* $(k, M, Q, 1, X, \cdot, h, \theta)$ *is a quadrangular algebra, with* M*,* Q*,* X*,* 1*,* $\cdot$*,* h *and* θ *as in Notation 4.5.3, Notation 4.5.8, Notation 4.5.16 and Notation 4.5.19.*

Proof. By their definition and the fact that the Lie bracket is a bilinear map, h and $\cdot$ are bilinear maps. Note that if $0 \neq m \in M$ satisfies $T(m, n) = 0$ for all $n \in M$, then $[m, M'] = [m, [M, b]] = 0$ by (4.51). But then $[m, L_{-1}] = 0$ by (4.52), contradicting Lemma 4.1.6. Hence Q is a non-degenerate quadratic form by Lemma 4.5.9. Then Lemma 1.1.31 and (4.62), (4.63), (4.68), (4.69) and (4.80) conclude this proof. □

4.5.2 Moufang quadrangles

Assume in this subsection $\mathrm{char}(k) \neq 2$. We will show that the inner ideal geometry, as defined in Section 2.5, is a generalized quadrangle if and only if the associated quadrangular algebra is anisotropic. Then, we will proceed to characterize this quadrangle more precisely, we will show that this quadrangle is the Moufang quadrangle associated with the quadrangular algebra. Let M, Q, X, 1, $\cdot$, h and θ as in Notation 4.5.3, Notation 4.5.8, Notation 4.5.16 and Notation 4.5.19.

Lemma 4.5.24. *Assume that the extremal geometry does not contain lines. If $e \in E$ satisifes $[a, e] = 0 = [b, e]$, then e is*

- *contained in L_0 if $[e, x] = 0 = [e, y]$;*
- *a multiple of $\exp(\lambda x)e_-(x_1)(y)$, with $\lambda \in k$ and $x_1 \in X$, if $[e, x] \neq 0$;*
- *a multiple of $\exp(\lambda y)e_+([x_1, y])(x)$, with $\lambda \in k$ and $x_1 \in X$, if $[e, y] \neq 0$.*

Proof. By $0 \neq [a, l'_1] \in L'_{-1}$ for all $l'_1 \in L'_1$ and $0 \neq [a, b] \in L'_0$, $[e, a] = 0$ implies $e \in L'_{-2} \oplus L'_{-1} \oplus L'_0$. Similarly, $[e, b] = 0$ implies $e \in L'_0 \oplus L'_1 \oplus L'_2$. Hence $e \in L'_0$. By Lemma 4.5.5 this implies $e = l_{-2} + l_{-1} + l_0 + l_1 + l_2$ with $l_i \in L_i \cap L'_0$, $i \in \{-2, -1, 0, 1, 2\}$. If $l_2 \neq 0$, we may assume $l_2 = y$. Note $l_1 \in L_1 \cap L'_0 = [X, y]$. So by Theorem 4.3.17 and Theorem 4.3.11(v) there exists unique $x_1 \in X$ and $\lambda \in k$ such that $\exp(\lambda x)e_-(x_1)(y) = e$. Completely similarly, if the (-2)-component of e is non-zero, it is a multiple of $\exp(\lambda' y)e_+([x'_1, y])(x)$ for unique $x'_1 \in X$ and $\lambda' \in k$.

Now assume that e has 2- and (-2)-component equal to 0. If its (-1)-component is non-zero, then $[e, y] = [l_{-1}, y] + [l_0, y]$ is an extremal element. By Lemma 4.1.8, $\langle [l_{-1}, y] + [l_0, y], y \rangle$ is a line of the extremal geomtry, a contradiction. Similarly, the 1-component of e equals 0. Hence $e \in L_0$, as claimed. Now, by the standard 5-grading, $[e, x] = \lambda x$, for certain $\lambda \in k$, and hence $[e, [e, x]] = \lambda[e, x] = \lambda^2 x$. But $e \in E$ implies $[e, [e, x]] \in \langle e \rangle$, a contradiction. Hence $[x, e] = 0$ and similarly $[y, e] = 0$.

Consider $e \in E$ such that $[a, e] = [b, e] = 0 = [x, e] = [y, e]$. Since e having a non-zero 2-component (respectively, (-2)-component) implies $[x, e] \neq 0$ (respectively $[y, e] \neq 0$), the previous paragraph shows $e \in L_0$.

Consider $e \in E$ such that $[a, e] = [b, e] = 0 \neq [y, e]$. If both its (-2)- and 2-component equal 0, we get $e \in L_0$ and $[y, e] = 0$. If its (-2)-component is non-zero there is nothing to prove anymore. If its (-2)-component equals 0, its 2-component is non-zero and hence $e = l_0 + l_1 + l_2$, for certain $l_i \in L_i$, by the second sentence in the second paragraph of this proof. By the 5-grading there exists $\lambda \in k$ such that $[l_0, y] = \lambda y$, as before $[e, [e, y]] = [e, \lambda y] = \lambda^2 y$ is contained in $\langle e \rangle$ and hence $\lambda = 0$, implying $[y, e] = 0$, a contradiction. □

Now we can characterize when the inner ideal geometry is a generalized quadrangle, but we first introduce some notation for the lines of this inner ideal geometry, see Section 2.5.

Notation 4.5.25. Let $\mathcal{I}$ be the set of inner ideals I containing at least two extremal points and such that the only proper non-trivial inner ideals of I are extremal points.

Theorem 4.5.26. *The inner ideal geometry of L, as defined in Section 2.5, with $\mathcal{E}$ as point set, and $\mathcal{I}$ as line set, is a generalized quadrangle if and only if the quadrangular algebra from Corollary 4.5.23 is anisotropic.*

Proof. To ease notation we will denote the quadrangular algebra from Corollary 4.5.23 by (X, M).

Assume that the inner ideal geometry forms a generalized quadrangle.

If the extremal geometry contains lines then this inner ideal geometry coincides with the extremal geometry, see Theorem 2.5.11. In the extremal geometry there exist hyperbolic points and these points lie at distance 3. Since in a generalized quadrangle (and more generally, a polar space) points lie at distance at most 2, we obtain a contradiction.

So there are no lines in the extremal geometry. If Q is isotropic, Lemma 4.5.11 implies that there exist extremal elements contained in L_{-1} and hence lines in the extremal geometry by Lemma 4.1.8. We can now assume Q to be anisotropic.

Assume that there exists a non-zero $x_1 \in X$ such that $\pi(x_1) = 0$. Then $e_-(x_1)(y) = \frac{1}{2}[x_1, [x_1, y]] + [x_1, y] + y$, by (4.75). By $[a, x_1] = 0 = [a, y]$, see (4.56), we get $[e_-(x_1)(y), a] = 0$. On the other hand,

$$[y, [x_1, [x_1, y]]] = [[y, x_1], [x_1, y]] - [x_1, [y, [y, x_1]] = 0, \tag{4.88}$$

using $[y, [y, x_1]] \in L_3 = 0$, and thus $[y, e_-(x_1)(y)] = 0$. Hence $e_-(x_1)(y)$ is an extremal element symplectic to both a and y, since $\mathcal{E}_{-1} = \emptyset$ because there are no lines in the extremal geometry. By Lemma 2.5.5, a and y generate an inner ideal I which contains only extremal points as non-trivial proper inner ideals. These types of inner ideals are the lines of our inner ideal geometry, so in order to obtain a generalized quadrangle, every point symplectic with two distinct extremal points contained in I has to be contained in I. Now by Theorem 4.5.43 and Lemma 4.1.4, $\langle a \rangle \oplus [y, M] \oplus \langle y \rangle$ is an inner ideal containing a and y, so it has to contain I. (It is actually equal to I, but this is irrelevant for now.) As noted before, $e_-(x_1)(y)$ is symplectic with both a and y. But the 1-component of $e_-(x_1)(y)$ equals $[x_1, y]$, which is a non-zero element of $[y, X]$. Since $X \cap M = 0$, the extremal element $e_-(x_1)(y)$ is not contained in I, a contradiction with the assumption that the inner ideal geometry is a generalized quadrangle. We can conclude $\pi(x_1) \neq 0$ for all $0 \neq x_1 \in X$. Together with the fact that Q is anisotropic, we get that the quadrangular algebra is anisotropic.

Conversely, asumme (X, M) is anisotropic.

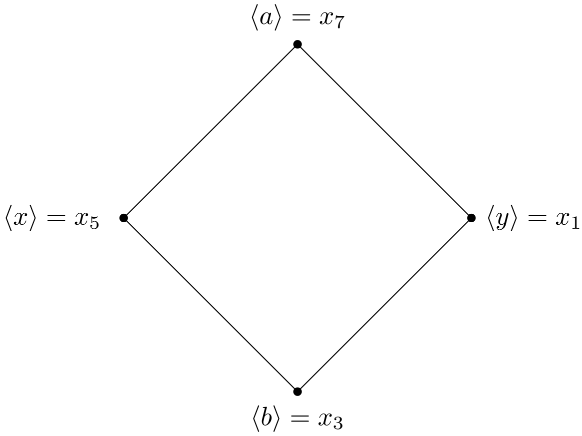
If there are lines in the extremal geometry, there is $e \in L$ such that (x, e) and (b, e) are contained in E_{-1}. Hence $e \in L_{-2} \oplus L_{-1}$ and $e \in L'_2 \oplus L'_1$ by Lemma 4.1.8. By $x \in L'_0$ and Lemma 4.5.5, $e \in L_{-1} \cap L'_1 = M'$. Since $a \in L'_{-2}$, we get $(e, a) \in$

E_1 and $[e,a] \in E$. Hence $\exp([e,a])(b) = g_{[e,a]}(b)[e,a] + [[e,a],b] + b = e + b$ using $e \in L'_{-1}$ and $g([a,e],b) = g(a,[e,b]) = 0$, by the associativity of g and $[e,b] = 0$. Now Lemma 4.5.7 and the definition of Q imply $Q([e,a]) = 0$, contradicting the assumption that the quadrangular algebra is anisotropic.

Now, since the extremal geometry has no lines but it does have symplectic pairs, the inner ideal geometry forms a polar space by Theorem 2.5.11. We now show that its rank equals 2, i.e., it is a generalized quadrangle. If the rank is at least 3, then for every line of the inner ideal geometry, there exists an extremal point not on this line which is symplectic with two (and thus all) points on this line. As before, consider the inner ideal I generated by the symplectic pair of extremal elements y and a. This inner ideal is contained in $\langle a\rangle \oplus [M,y] \oplus \langle y\rangle$. Assume that there exists an extremal point e not on the line I but symplectic with both $\langle y\rangle$ and $\langle a\rangle$. Then e and the line I span a plane in the inner ideal geometry. Since this inner ideal geometry is a polar space and $[x,y] \neq 0$, there exists an extremal point e' in this plane which is symplectic with $\langle b\rangle$ but hyperbolic with $\langle x\rangle$. Note that since $\langle y\rangle$ is symplectic with both $\langle a\rangle$ and e, it is symplectic with e', i.e. $[e',y] = 0$. In particular $e' \leq L_{\geq 0}$. So Lemma 4.5.24 together with $[a,e'] = 0 = [b,e']$ implies that there exist $\lambda \in k$ and $x_1 \in X$ such that e' is a multiple of $\exp(\lambda x)e_-(x_1)(y)$. By (4.75) we get that the 0-component of e' equals $\frac{1}{2}[x_1,[x_1,y]] + \lambda[x,y]$. Now note $[\frac{1}{2}[x_1,[x_1,y]] + \lambda[x,y], y] = 2\lambda y$, using (4.88). Hence $[e',y] = 0$ implies $\lambda = 0$. Now since $e' \leq L_{\geq 0}$, (4.75) yields $\pi(x_1) = 0$, contradicting the assumption that the quadrangular algebra is anisotropic. So the inner ideal geometry is indeed a generalized quadrangle. □

Assumption 4.5.27. From now on we assume that the quadrangular algebra from Corollary 4.5.23 is anisotropic. Hence the inner ideal geometry is a generalized quadrangle, and in particular $\mathcal{E}_{-1} = \mathcal{E}_1 = \emptyset$.

Now we proceed to show that the generalized quadrangle is, in fact, a Moufang quadrangle and we determine the root groups. In the rest of this subsection, we fix a cycle $(x_0, x_1, \ldots, x_7, x_0)$ of length 8 in $\Omega := \mathcal{E} \cup \mathcal{I}$, the incidence graph of the inner ideal geometry of L, corresponding to the following cycle of length 4 in the generalized quadrangle:



We determine the points x_0, x_2, x_4 and x_6 of the incidence graph Ω more precisely.

Lemma 4.5.28. *We have*

$$x_4 = \langle x\rangle \oplus M' \oplus \langle b\rangle, \qquad x_6 = \langle x\rangle \oplus M \oplus \langle a\rangle;$$
$$x_0 = \langle a\rangle \oplus [M, y] \oplus \langle y\rangle, \qquad x_2 = \langle b\rangle \oplus [M', y] \oplus \langle y\rangle.$$

Proof. By definition, x_6 is the minimal inner ideal containing both a and x. Now $M = [a, M'] = [a, [x, [y, M']]] \leq I$, using (4.51). Hence $\langle x\rangle \oplus M \oplus \langle a\rangle$ is contained in I. Since it is also an inner ideal by Theorem 4.5.43, it coincides with I. Completely similarly for the other identities. □

Let U_1, U_2, U_3 and U_4 be the root groups, as defined in Notation 1.2.9. So U_1 is the subgroup of $\mathrm{Aut}(\Omega)$ which fixes all neighbors of x_2, x_3, x_4, and similarly for the other root groups. It turns out that, except for U_1, all these root groups are subgroups of $E_-(x, y)$. In order to determine the root groups explicitly, we can use Lemma 1.2.11 to see that it suffices to show that the claimed subgroup (of $E_-(x, y)$) fixes all the neighbors of a set of 3 distinguished vertices and acts transitively on the set of neighbors of another distinguished vertex, with one neighbor excluded. We first determine U_2 and U_4.

Lemma 4.5.29. *Any extremal element in x_4 is a multiple of x or a multiple of $e_-(m)(b)$, with $m \in M$. Any extremal element in x_6 is a multiple of x or a multiple of $e_-(m')(a)$, with $m' \in M'$.*

Proof. Consider $e \in x_4$ with $e \in E$ arbitrary. If the 0-component of e is non-zero, then by (4.51) and Lemmas 4.5.7 and 4.5.28 it is a multiple of $e_-(m)(b)$, with $m \in M$. If the 0-component of e equals 0, then by Lemma 4.1.8, there are lines in the extremal geometry, unless e is a multiple of x. Now note that by assumption the extremal geometry does not contain lines. Similarly for x_6. □

Notation 4.5.30. Set $E_-(M) = \{e_-(m) \mid m \in M\}$ and similarly for $E_-(M')$. Set $E_-(X, x) = \{\exp(\lambda x)e_-(x_1) \mid \lambda \in k, x_1 \in X\}$. Note that by (4.52), Lemma 4.3.15 and Theorem 4.3.11(i) these are subgroups of $E_-(x, y)$.

Lemma 4.5.31. *All automorphisms of $E_-(x, y)$ fix all elements of $\mathcal{I}$ containing x.*

Proof. Consider $\varphi \in E_-(x, y)$ arbitrary. By Theorem 4.3.11(v) there exist $\lambda \in k$ and $l_{-1} \in L_{-1}$ such that $\varphi = \exp(\lambda x)e_-(l_{-1})$. Let $I \in \mathcal{I}$ be such that $x \in I$ and $e \in I$, with $e \in E$, $e \notin \langle x\rangle$. By Assumption 4.5.1(i) the extremal geometry of $L \otimes k'$ contains lines. By Corollary 2.5.10, $I \otimes k' = (L \otimes k')_S$, where S is the symplecton in the extremal geometry of $L \otimes k'$ containing $\langle e\rangle$ and $\langle x\rangle$. By Lemma 4.3.6, $I = \langle x\rangle \oplus M'' \oplus \langle e'\rangle$, for a certain subspace M'' of L_{-1} and $e' \in L_0 \cap E$.

By the 5-grading on L, $\varphi(l) \in L_{-2} + l$ for all $l \in L_{-1}$. Hence it suffices to show $\varphi(e') \in I$. Now note $\varphi(e') = \mu x + [l_{-1}, e'] + e'$ for certain $\mu \in k$. Since I is an inner ideal, $[l_{-1}, e'] = [e', [x, [y, l_{-1}]]] \in I$. Together with $x, e' \in I$, we get $\varphi(e') \in I$. $\square$

Lemma 4.5.32. *For the root groups U_2 and U_4, we have*

$$U_2 = E_-(M'),\ U_4 = E_-(M).$$

Moreover, U_i acts transitively on all neighbors of x_i distinct from x_{i+1}, for $i = 2, 4$.

Proof. Consider $m \in M$ arbitrary. By $[m, a] = 0$, we get obtain $e_-(m)(a) = \frac{1}{2}[m, [m, a]] + [m, a] + a = a$. By (4.52), $e_-(m)(n) = n$ for all $n \in M$. Together with $e_-(m)(x) = x$ this implies that $e_-(m)$ fixes all neighbors of x_6, i.e. all extremal points contained in $x_6 = \langle x\rangle \oplus M \oplus \langle a\rangle$. Since $m \in M = L_{-1} \cap L'_{-1}$, $e_-(m)$ is contained in both $E_-(x, y)$ and $E_-(a, b)$. Hence Lemma 4.5.31 implies that $e_-(m)$ fixes all neighbors of $\langle x\rangle = x_5$ and $\langle a\rangle = x_7$ (in Ω). Now note that by Lemma 4.5.29, the group $E_-(M)$ acts transitively on all neighbors of x_4 different from x_3. So we showed $U_4 = E_-(M)$. Completely similarly for U_2. $\square$

Lemma 4.5.33. *For the root group U_3, we have*

$$U_3 = e_-(X, x).$$

Moreover, U_3 acts transitively on all neighbors of x_3 distinct from x_4.

Proof. Consider $x_1 \in X$ arbitrary. Since $[x_1, M] = 0 = [x_1, a]$, see (4.52) and (4.56), $e_-(x_1)$ fixes all elements of $x_6 = \langle x\rangle \oplus M \oplus \langle a\rangle$, in particular it fixes all extremal points contained in x_6, i.e. the neighbors of x_6 in Ω. Similarly, $e_-(x_1)$ fixes all neighbors of x_4. By Lemma 4.5.31, $e_-(x_1)$ fixes all neighbors of $\langle x\rangle = x_5$. Completely similarly $\mathrm{Exp}(\langle x\rangle)$, and thus $E_-(X, \langle x\rangle)$, fixes all neighbors of x_4, x_5 and x_6.

We now show that $E_-(X,x)$ acts transitively on the neighbors of x_3 different from x_4, then $U_3 = E_-(X,x)$ follows.

Consider a neighbor $I \in \mathcal{I}$ of $x_3 = \langle b \rangle$, i.e. $I \in \mathcal{I}$ such that $b \in I$. By Theorem 4.5.26 the inner ideal geometry is a generalized quadrangle. Since $[a,b] \neq 0$, this implies that there exists $e \in E \cap I$ such that $[e,a] = 0$. Moreover, since $[b,y] = 0$, $\langle b \rangle$ is the only extremal point of I symplectic with $\langle y \rangle$. Hence $[e,y] \neq 0$ and similarly $[e,x] \neq 0$. Now Lemma 4.5.24 implies that there exist $\lambda \in k$ and $x_1 \in X$ such that $e = \exp(\lambda x)e_-(x_1)(y)$. □

For the final root group we need three lemmas. (These lemmas do *not* rely on the condition that the quadrangular algebra is anisotropic.)

Lemma 4.5.34. *For all $m \in M$ and $l \in L$, $e_-(m)(l) = \frac{1}{2}[m,[m,l]] + [m,l] + l$. We also get $e_-(m)(y) = Q(m)a + [m,y] + y$.*

Proof. For all $l \in L_{-2} \oplus L_{-1} \oplus L_0$, this is true by definition. By Lemma 4.1.4 applied twice to Corollary 4.5.12, $Q(m)a + [m,y] + y \in E$. On the other hand $[m,[m,y]]$ is contained in the inner ideal x_6, and since it is also contained in L_0, it is a multiple of a. By $[x,y] \notin \langle a \rangle = L'_{-2}$, since $[x,y] \in L'_0$, Theorem 4.3.17 implies $e_-(m)(y) = Q(m)a + [m,y] + y$. Consider $l_{-1} \in L_{-1}$ arbitrary. Then $e_-(m)([y,l_{-1}]) = [Q(m)a+[m,y]+y, [m,l_{-1}]+l_{-1}]$ has 2-component $[Q(m)a,[m,l_{-1}]] = 0$ by $[a,x] = 0$. Hence $e_-(m)(l_1) = \frac{1}{2}[m,[m,l_1]] + [m,l_1] + l_1$ for all $l_1 \in L_1$, which concludes this proof. □

Lemma 4.5.35. *Set $\varphi = e_-(1)e_+([y,1'])e_-(1)$. Then $\varphi(b) = x$, $\varphi(x) = b$, $\varphi(y) = a$, $\varphi(a) = y$ and $\varphi(x_1) = [[y,1'],x_1]$ for all $x_1 \in X$.*

Proof. Note $e_-(1)(b) = Q(1)x + [1,b] + b = x + [1,b] + b$. Note that $[[y,[b,1]],[1,b]]$ and $[[y,[b,1]],[[y,[b,1]],x]]$ are contained in $L'_2 \cap L_0 = \langle b \rangle$ and $[[y,[b,1]],b] \in L'_3 = 0$. So by the obvious generalization of Lemma 4.5.34 to $[y,M']$, we get

$$e_+([y,[b,1]])(e_-(1)(b)) = x + (-[x,[y,[b,1]]] + [1,b]) + \mu b = x + \mu b$$

for certain $\mu \in k$. If $\mu \neq 0$, then Lemma 4.5.7 (applied for $m = 0$) yields a contradiction with $b \in E$. Together with $e_-(1)(x) = x$, this yields $\varphi(b) = x$. Completely similarly, $\varphi(x) = b$, $\varphi(y) = a$, $\varphi(a) = y$.

Consider $x_1 \in X$ arbitrary. Using (4.52), $e_-(1)(x_1) = x_1$. Now, a generalization of Lemma 4.5.34 to $[y,M']$, and $[[y,1'],[[y,1'],x_1]] \in L_1 \cap L'_2 = 0$, yields

$$e_+([y,1'])(x_1) = x_1 + [[y,1'],x_1].$$

Together with $x_1 \in L'_0 \cap L_{-1}$ and (4.59) for $m = 1$ we get

$$\begin{aligned}\varphi(x_1) &= e_-(1)e_+([y,1'])(x_1) = e_-(1)(x_1 + [[y,1'],x_1]) \\ &= x_1 + (-\frac{1}{2}[1,x_1] - x_1 + [[y,1'],x_1]) = [[y,1'],x_1].\end{aligned}$$ □

Lemma 4.5.36. *The subalgebra L_0 decomposes as*

$$L_0 = (L'_{-2} \cap L_0) \oplus (L'_{-1} \cap L_0) \oplus (L'_0 \cap L_0) \oplus (L'_1 \cap L_0) \oplus (L'_2 \cap L_0),$$

with moreover

$$L'_{-1} \cap L_0 = [X, [y, 1]] \text{ and } L'_1 \cap L_0 = [X, [y, 1']]. \tag{4.89}$$

Proof. Let φ be as in Lemma 4.5.35. Lemmas 4.1.7 and 4.5.35 imply $\varphi(L_i) = L'_{-i}$ and $\varphi(L'_i) = L_{-i}$ for all $i \in \{-2, -1, 0, 1, 2\}$. In particular $\varphi(X) = \varphi(L_{-1} \cap L'_0) = L'_1 \cap L_0$. By Lemma 4.5.35, $\varphi(X) = [X, [y, 1']]$. Similarly, one deduces $L_0 \cap L'_{-1} = [X, [y, 1]]$.

Consider $l \in L_0$ arbitrary, then there exist (unique) $l'_i \in L'_i$ such that $l = l'_{-2} + l'_{-1} + l'_0 + l'_1 + l'_2$. Since L'_{-2} and L'_2 are contained in L_0, $l_{-1} + l_0 + l_1 \in L_0$. Now by completely the same argument as in the proof of Lemma 4.5.5 we get $l_{-1}, l_0, l_1 \in L_0$. □

Lemma 4.5.37. *For the root group U_1, we have*

$$U_1 = \{\exp(\lambda b)e_-([x_1, [y, 1']]) \mid \lambda \in k, x_1 \in X\}.$$

Moreover, U_1 acts transitively on all neighbors of x_1 distinct from x_2.

Proof. By interchanging the roles of (x, y) and (b, a) in this whole subsection, and then applying Lemma 4.5.33 while using (4.89). □

Lemma 4.5.38. *Every cycle $(y_0, \dots, y_7, y_0)$ of length 8 in Ω can be mapped onto the cycle $(x_0, \dots, x_7, x_0)$ by an element of $\mathrm{Aut}(L)$.*

Proof. We may assume that y_1 is 1-dimensional. By Theorem 2.3.19, the extremal point y_1 can be mapped onto $\langle y \rangle$. So we may assume $y_1 = \langle y \rangle = x_1$. Now y_5 is at distance 2 from y_1 in the inner ideal geometry. This is equivalent with $[y_5, y] \neq 0$. If y_5 has 0 as (-2)-component, it would commute with y by Lemma 4.1.8 and the fact that there are no special pairs of extremal elements. (Since there would otherwise be collinear extremal points by Proposition 2.3.7(d)) Hence y_5 has a non-zero (-2)-component, so by Theorem 4.3.17 there exists an element of $E_+(x, y)$ mapping y_5 onto $\langle x \rangle$ and fixing $\langle y \rangle$. So we may assume $y_5 = \langle x \rangle = x_5$. By Lemma 4.5.37 there exists an automorphism fixing x and y, while sending y_0 onto x_0. So we may assume $y_0 = x_0$. Since $x_5 = y_5$, x_6, x_7, $x_0 = y_0$ is the unique shortest path between y_5 and y_0, we get $y_6 = x_6$ and $y_7 = x_7$. Similarly $y_2 = x_2$, $y_3 = x_3$, $y_4 = x_4$. □

Proposition 4.5.39. *The inner ideal geometry $(\mathcal{E}, \mathcal{I})$ is a Moufang quadrangle.*

Proof. This follows from Lemmas 4.5.32, 4.5.33, 4.5.37 and 4.5.38. □

Before determining the commutator relations, we need a lemma on commuting automorphisms.

Lemma 4.5.40. *Consider $e \in L_0 \cap E$ and $l \in L_{-1}$ such that $[l, e] = 0$, then $e_-(l)$ and $\exp(e)$ commute.*

Proof. Note that $e \in L_0 \leq N_L(x)$ implies $[e, [e, x]] \in \langle e \rangle \cap \langle x \rangle = 0$. Hence $[e, x] = 0$ and $\exp(e)(x) = e$. Similarly, $\exp(e)(y) = y$. For all $l' \in L_1$

$$\begin{aligned} \exp(e)(e_-(l)(l')) &= \exp(e)([l, l'] + l') = [l, l'] + (l' + [e, l']) \\ e_-(l)(\exp(e)(l')) &= e_-(l)(l' + [e, l']) = [l, l'] + (l' + [e, l']), \end{aligned}$$

where we used $[e, [e, l']] \in L_{-1} \cap \langle e \rangle = 0$, $[l, e] = 0$, $[e, [l, l']] \leq [e, \langle x \rangle] = 0$ and

$$[l, [e, l']] = [[l, e], l'] + [e, [l, l']] = 0.$$

So both automorphisms coincide on L_{-1}. Since L is generated by y and L_{-1}, it now suffices to show that they coincide on y. By Theorem 4.3.17 it suffices to show that the images of y under these automorphisms have the same 0- and 1-components. One easily checks, using $[e, l] = 0$ and $[e, y] = 0$, that these are indeed equal, and more precisely, they are equal to $\frac{1}{2}[l, [l, y]]$ and $[l, y]$, respectively. □

Lemma 4.5.41. *The commutator relations between U_1, U_2, U_3, and U_4 are trivial except for*

$$[\exp(\lambda b)e_-([x_1, [y, 1']]), \exp(\mu x)e_-(x_2)] = e_-([h(x_1, x_2), b]), \tag{4.90}$$

$$\begin{aligned} [\exp(\lambda b)e_-([x_1, [y, 1']]), e_-(m)] &= e_-([\theta(x_1, m) - \lambda m, b])e_-(-x_1 \cdot m) \\ &\quad \exp(\lambda Q(m)x), \end{aligned} \tag{4.91}$$

$$[e_-(m'), e_-(m)] = \exp([m', m]), \tag{4.92}$$

for all $m \in M$, $m' \in M'$, $x_1, x_2 \in X$ and $\lambda, \mu \in k$.

Proof. Consider $x_1, x_2 \in X$ and $\lambda, \mu \in k$. By Lemma 4.5.40 and (4.56), $\exp(\lambda b)$ commutes with $e_-(x_2)$. By Theorem 4.3.11(i) applied to the 5-grading associated with (a, b), $\exp(\lambda b)$ also commutes with $e_-([x_1, [y, 1']])$. Using $x \in L'_0$ and $[x_1, [y, 1']] \in L'_1$, Lemma 4.5.40 applied to the 5-grading associated with (a, b) shows that $\exp(\mu x)$ commutes with $e_-([x_1, [y, 1']])$. So in order to show (4.90) we may without loss of generality assume $\lambda = \mu = 0$.

Set

$$\begin{aligned} \varphi &= [e_-([x_1, [y, 1']]), e_-(x_2)] \\ &= e_-(-[x_1, [y, 1']])e_-(-x_2)e_-([x_1, [y, 1']])e_-(x_2), \end{aligned}$$

where we used Lemma 4.3.15 to deduce the inverses of these automorphisms. By Theorem 1.2.13 there exists $m' \in M'$ such that $\varphi = e_-(m')$. We get

$\varphi \exp(-x_2) = e_-(m')e_-(-x_2)$. Now we determine the image of y under both sides of the previous equality to deduce m'. In fact, we only need to deduce the 1-component of these images.

By $[[x_1,[y,1']],y] \in L_2 \cap L_1' = 0$ and $y \in L_0'$, we get $e_-([x_1,[y,1']])(y) = y$. By (4.75), $e_-(-x_2)(y) = Q(\pi(x_2))x + x_2\pi(x_2) + \frac{1}{2}[x_2,[x_2,y]] - [x_2,y] + y$. Note that all components of $e_-(-x_2)(y)$ are contained in L_0', denote these components by l_i. Then, by $l_i \in L_0'$ and $[x_1,[y,1']] \in L_1'$,

$$e_-(-[x_1,[y,1']])(l_i) = \frac{1}{2}[[x_1,[y,1']],[[x_1,[y,1']],l_i]] - [[x_1,[y,1']],l_i] + l_i.$$

In particular, by $[x_1,[y,1']] \in L_0$, $e_-(-[x_1,[y,1']])(l_i)$ is also contained in L_i. Hence the 1-component of $\varphi e_-(-x_2)(y)$ equals

$$\begin{aligned} e_-(-[x_1,[y,1']])(-[x_2,y]) &= [[x_1,[y,1']],[x_2,y]] - [x_2,y] \\ &= -[[b,h(x_1,x_2)],y] + [x_2,[y,[[y,1'],x_1]]] - [x_2,y] \\ &= [[h(x_1,x_2),b] - x_2,y], \end{aligned}$$

using $[[x_1,[y,1']],[[x_1,[y,1']],[x_2,y]]] \in L_1 \cap L_2' = 0$, $[y,[[y,1'],x_1]] \in L_2 \cap L_1' = 0$ and $[b,y] = [b,x_1] = [b,x_2] = 0$.

On the other hand the 1-component of $e_-(m')e_-(-x_2)(y)$ equals $[m' - x_2, y]$. We conclude $m' = [h(x_1,x_2),b]$, showing (4.90).

Equation (4.92) follows by Lemma 4.3.15.

Now we deduce the most difficult commutator relation, (4.91). Consider $\lambda \in k$, $x_1 \in X$ and $m \in M$ arbitrary. Set $\varphi = [\exp(\lambda b)e_-([x_1,[y,1']]), e_-(m)]$. Again, by Theorem 1.2.13 there exist $m' \in M$, $x_2 \in X$ and $\mu \in k$ such that $\varphi = e_-(m')\exp(\mu x)e_-(x_2)$. We now determine the 0- and 1-component of $\varphi e_-(-m)$ in two different ways in order to deduce m', λ and x_2. As before, $e_-([x_1,[y,1']])(y) = y$, and by $[b,y] = 0$ we get $\exp(\lambda b)(y) = y$. By Lemma 4.5.34, we get $e_-(-m)(y) = Q(m)a - [m,y] + y$. Now we determine the image of m under $e_-(-[x_1,[y,1']])$. By a generalization of (4.74), $[x_1,b] = [y,b] = [x_1 \cdot m, b] = 0$ and the definition of h, we get

$$\begin{aligned} e_-(-[x_1,[y,1']])(m) &= \frac{1}{2}[[x_1,[y,1']],[[x_1,[y,1']],m]] - [[x_1,[y,1']],m] + m \\ &= -\frac{1}{2}[[x_1,[y,1']],x_1 \cdot m] + x_1 \cdot m + m \\ &= \frac{1}{2}[b,h(x_1,x_1 \cdot m)] + x_1 \cdot m + m \\ &= [b,\theta(x_1,m)] + x_1 \cdot m + m, \end{aligned}$$

and hence, using $[b,X] = 0$,

$$\exp(-\lambda b)(e_-(-[x_1,[y,1']])(-[m,y])) = [[\theta(x_1,m) - \lambda m, b] - x_1 \cdot m - m, y],$$

which is the 1-component of $\varphi e_-(-m)(y)$ since the components L_i of our standard 5-grading are fixed by both $\exp(-\lambda b)$ and $e_-(-[x_1,[y,1']])$. On the other hand, the 1-component of

$$e_-(m')\exp(\mu x)e_-(x_2)e_-(-m)(y)$$

equals $[m'+x_2-m,y]$. We obtain $m'=[\theta(x_1,m)-\lambda m,b]$ and $x_2=-x_1\cdot m$.

Now we only need to determine μ. Note that the 0-component l of

$$e_-(m')\exp(\mu x)e_-(x_2)e_-(-m)(y)$$

can be decomposed as in Lemma 4.5.36. The component of l contained in L_0' equals $\mu[x,y]+\frac{1}{2}[x_2,[x_2,y]]-[m',[m,y]]$. By the previous calculations, the 0-component of $\varphi e_-(-m)(y)$ equals $\exp(-\lambda b)e_-(-[x_1,[y,1']])(Q(m)a)$. By definition of $e_-(-[x_1,[y,1']])$, $a\in L_{-2}'$, $[x_1,[y,1']]\in L_1'$ and $b\in L_2'$, the component of $\exp(\lambda b)e_-(-[x_1,[y,1']])(Q(m)a)$ contained in L_0' equals

$$\frac{1}{2}Q(m)[[x_1,[y,1]],[[x_1,[y,1]],a]-\lambda Q(m)[b,a],$$

which thus has to equal $\mu[x,y]+\frac{1}{2}[x_2,[x_2,y]]-[m',[m,y]]$. Now taking the Lie bracket with y on both sides of this equality yields $2\mu y-[[m',[m,y]],y]=0$. Together with $[y,[y,m]]=0$ and Lemma 4.1.4 this yields

$$\begin{aligned}2\mu x&=-[m,m']=-[m,[\theta(x_1,m)-\lambda m,b]]\\&=-T(m,\theta(x_1,m)-\lambda m)x=(-T(m,\theta(x_1,m))+2\lambda Q(m))x.\end{aligned}$$

Note $2T(m,\theta(x_1,m))=T(h(x_1,x_1\cdot m),m)=T(h(x_1\cdot m,x_1\cdot m),1)=[x_1\cdot m,x_1\cdot m]=0$ by (4.67) and (4.68). Now (4.91) follows since $\exp(x)$ commutes with all elements of $E_-(x,y)$, see Theorem 4.3.11(i). □

Theorem 4.5.42. *Let L be as in Assumption 4.5.1 and $\operatorname{char}(k)\neq 2$. Assume that the quadrangular algebra on (X,M), as constructed in Section 4.5.1, is anisotropic. Consider $\Omega=\mathcal{E}\cup\mathcal{I}$, the incidence graph of the inner ideal geometry. (See Notation 4.5.25 for the definition of $\mathcal{I}$.) Then Ω is the incidence graph of the Moufang quadrangle associated to the quadrangular algebra on (X,M).*

Proof. Recall that, by Proposition 4.5.39, Ω is indeed the incidence graph of a Moufang quadrangle. Consider the following parametrization of the root groups:

$$\begin{aligned}x_1(x_1,\lambda)&=\exp(-\lambda b)e_-([x_1,[y,1']]), & x_3(x_1,\lambda)&=\exp(-\lambda x)e_-(x_1),\\ x_2(m)&=e_-([b,m]), & x_4(m)&=e_-(m),\end{aligned}$$

for all $m\in M$, $x_1\in X$ and $\lambda\in k$. Now note $[m_1,[m_2,b]]=T(m_1,m_2)x$ for all $m_1,m_2\in M$. Hence by Lemma 4.5.41 we see that the commutator relations are the same as in Theorem 1.2.20. □

4.5.3 Other grading

In this short subsection we construct another 5-grading on L, see also Figure 4.1, and make a few remarks.

Theorem 4.5.43. *Consider*

$$L''_i = \{l \in L_{i_1} \cap L'_{i_2} \mid i_1, i_2 \in \mathbb{Z}, i_1 + i_2 = i\},$$

for $i \in \mathbb{Z}$. Then

$$L = L''_{-2} \oplus L''_{-1} \oplus L''_0 \oplus L''_1 \oplus L''_2,$$

and moreover this is a $\mathbb{Z}$-grading. If $\mathrm{char}(k) \neq 2$*, we have more precisely*

$$L''_{-2} = \langle x\rangle \oplus M \oplus \langle a\rangle,\ \ L''_2 = \langle b\rangle \oplus [y, M'] \oplus \langle y\rangle$$
$$L''_{-1} = X \oplus [X, [y, 1]],\ \ L''_1 = [X, y] \oplus [X, [y, 1']]$$

Proof. By $x, y \in L'_0$ and Lemma 4.5.5 it is clear that $L''_i = 0$ if $|i| > 2$. By Lemmas 4.5.5 and 4.5.36, $L = \oplus_{i=-2}^{2} L''_i$ follows. This is a $\mathbb{Z}$-grading because both $\oplus_{i=-2}^{2} L_i$ and $\oplus_{i=-2}^{2} L'_i$ are $\mathbb{Z}$-gradings.

The more precise descriptions of L''_{-2}, L''_{-1}, L''_1 and L''_2 follow by Lemmas 4.5.5 and 4.5.36. □

Remark 4.5.44. Assume $\mathrm{char}(k) \neq 2$, and assume that the quadrangular algebra constructed in Section 4.5.1 is anisotropic. Let $x_1, x_2, x_3 \in X$ be arbitrary. By Lemma 4.1.4 applied to the grading (4.50), together with $[X, b] = 0 = [y, b]$ and $[[x_1, [y, 1]], [x_2, [x_3, [y, 1']]]] \in X$, we get

$$\begin{aligned}[][[x_1, [y, 1]], [x_2, [x_3, [y, 1']]]] &= [[x_1, [y, 1']], [x_2, [x_3, [y, [a, 1']]]]] \\ &= -[[x_1, [y, 1']], [x_2, [x_3, [y, 1]]]]. \end{aligned} \tag{4.93}$$

Then $-2y = [[y, 1], [y, 1']]$, the Jacobi identity, (4.93), $[X, M] = 0$, and the definition of h, imply

$$\begin{aligned} -2[x_1, [x_2, [x_3, y]]] &= [x_1, [x_2, [x_3, [[y, 1], [y, 1']]]]] \\ &= \sum_{i<j, j\neq k, i\neq k} 2[[x_i, [x_j, [y, 1]]], [x_k, [y, 1']]] \\ &= 2 \sum_{i<j, j\neq k, i\neq k} x_k \cdot h(x_j, x_i) \\ &= 2(x_1 \cdot h(x_3, x_2) + x_2 \cdot h(x_3, x_1) + x_3 \cdot h(x_2, x_1)). \end{aligned}$$

Also note, by (4.62), $[x_1, x_2] = T(h(x_1, x_2), 1) = 2g(x_1, x_2)$, with g as in the proof of Lemma 1.1.31. In [BDM13] they give X a multiplication and involution such that the obtained algebra is a structurable algebra of skew-dimension one, if $\mathrm{char}(k) \neq 2, 3$. Comparing the above identities with the ones in Theorem 5.4 of *loc. cit.* shows that the subalgebra

$$\langle x \rangle \oplus X \oplus ([X, [X, y]] + [x, y]) \oplus [X, y] \oplus \langle y \rangle$$

of L (and L_0') is isomorphic to the TKK-Lie algebra of the skew-dimension one structurable algebra on X as in Theorem 5.4 of *loc. cit.*

Remark 4.5.45. In [BDM15] they construct a quadrangular algebra out of a so-called special J-ternary algebra, where $J = J(Q', c')$ is a Jordan algebra, with Q' a quadratic form of Witt index 1 and basepoint c'. If we assume that $\mathrm{char}(k) \neq 2$ and that the quadrangular algebra from Section 4.5.1 is anisotropic, then it easy to see that $L''_{-2} \oplus [L''_{-2}, L''_2] \oplus L''_2$ is the TKK-Lie algebra of the Jordan algebra $J = J(Q', c')$, with Q' a quadratic form of Witt index 1 and $Q'(c') = 1$. It is likely that we can give L''_{-1} the structure of a J-ternary algebra.

Remark 4.5.46. Assume that Q is anisotropic. (By Lemmas 4.1.8 and 4.5.11 this condition is automatically satisfied if the extremal geometry has no lines.) Assume moreover $\mathrm{char}(k) \neq 2$. Consider $m \in M$ such that $Q(m) \neq 0$ and $T(m, 1) = 0$. (By Assumption 4.5.1(i) and Lemma 4.3.6, M has dimension at least 2.) By Lemma 4.5.34 we get $[1, [m, y]] = [m, [1, y]] = \frac{1}{2}T(1, m)a = 0$ and by using the automorphism of Lemma 4.1.4 applied to the grading (4.50) this implies $[[b, m], [1', y]] = 0$. Together with (4.58) and (4.59) this implies

$$\begin{aligned}
[1 + [b, m], -Q(m)^{-1}[y, m] + [y, 1']] &= [1, [y, 1']] - Q(m)^{-1}[[b, m], [y, m]] \\
&= ([a, b] + [x, y]) - ([a, b] - [x, y]) = 2[x, y].
\end{aligned}$$

If $\mathrm{char}(k) \neq 2, 3$, then Lemma 4.2.4 shows that we can give $L_{-1} = M \oplus X \oplus M'$ the structure of a skew-dimension one structurable algebra. In fact, we proved the stronger statement in Theorem 4.2.17 that the TKK-construction yields a one-to-one correspondence between skew-dimension one structurable algebras (up to isotopy) and finite-dimensional simple Lie algebras generated by their extremal elements which are not symplectic (up to isomorphism). Note however that, by the above, for this specific associated structurable algebra $1 + [b, m]$ will play the role of the unit. Then, using the Lie bracket one can, in principle, deduce the involution and multiplication on $M \oplus X \oplus M'$ in terms of h, $\cdot$ and Q.

Consider a skew-dimension one structurable algebra $\mathcal{A}$ which is not isotopic to a matrix structurable algebra. By Corollary 4.2.14 the extremal geometry of $K(\mathcal{A})$ contains no lines. By Theorems 2.4.7 and 4.2.17 Assumption 4.5.1(i) is satisfied. So if there are symplectic pairs in the extremal geometry, then $\mathcal{A}$ is isotopic to the structurable algebra on $M \oplus X \oplus M'$. If there are no symplectic pairs, then any two distinct points of the extremal geometry are hyperbolic (since the existence of special pairs implies the existence of collinear pairs of extremal points). But then $\mathcal{A}$ is a structurable division algebra by Theorem 4.2.18. An idea for further research is thus describing a direct construction of a skew-dimension one structurable algebra out of a quadrangular algebra. Moreover, it would be desirable to have a direct proof that any non-division skew-dimension one structurable algebra is either isotopic to a matrix structurable algebra or to the one obtained from a quadrangular algebra.

In Chapter 1 we mentioned a procedure due to Bruce Allison and John Faulkner to construct skew-dimension one structurable algebras which are not isotopic to a matrix structurable algebra, namely a generalized Cayley-Dickson construction, see [AF84]. It seems highly likely that one can also obtain the structurable algebra on $M \oplus X \oplus M'$ by this Cayley-Dickson process.

Remark 4.5.47. Note that if there are lines in the extremal geometry *and* there are symplectic pairs of extremal elements, then the results of this section and the previous section show that L_{-1} has two different decompositions, one related to a cubic norm structure (unless the norm is the 0-map) and one to a quadrangular algebra. The existence of lines and symplectic pairs implies that this cubic norm structure is isotropic and that the quadratic form associated with the quadrangular algebra has to be isotropic. It would be interesting to investigate whether there is a more direct connection between this cubic norm structure and this quadrangular algebra, preferably a construction with as input an isotropic cubic norm structure and as output a quadrangular algebra whose quadratic form is isotropic (and conversely).

Example 4.5.48. If $X = 0$, $\mathrm{char}(k) \neq 2$ and Q is anisotropic, then $\oplus_{i=-2}^{2} L''_i = L''_{-2} \oplus L''_0 \oplus L''_2$ is precisely the Lie algebra of Example 4.2.21. (The quadratic form in that example is precisely the quadratic form Q' of Remark 4.5.45.)

SECTION 4.6

Moufang sets

In a relatively easy manner we will show that if all pairs of distinct extremal points are hyperbolic and the Lie algebra is not symplectic, then the set of extremal points $\mathcal{E}$ forms a Moufang set. (In characteristic 2, there is an additional assumption.)

Assumption 4.6.1. In this section we assume that L is a simple Lie algebra generated by its set of pure extremal elements such that

(i) there exists a Galois extension k' of k such that the extremal geometry of the simple Lie algebra $L \otimes k$ contains lines;
(ii) $\mathcal{E}_{-1} = \mathcal{E}_0 = \mathcal{E}_1 = \emptyset$.

Remark 4.6.2. By Theorem 4.3.13 the conclusions of Theorem 4.3.11 hold and hence all lemmas and theorems from Section 4.3 that we apply in this section can

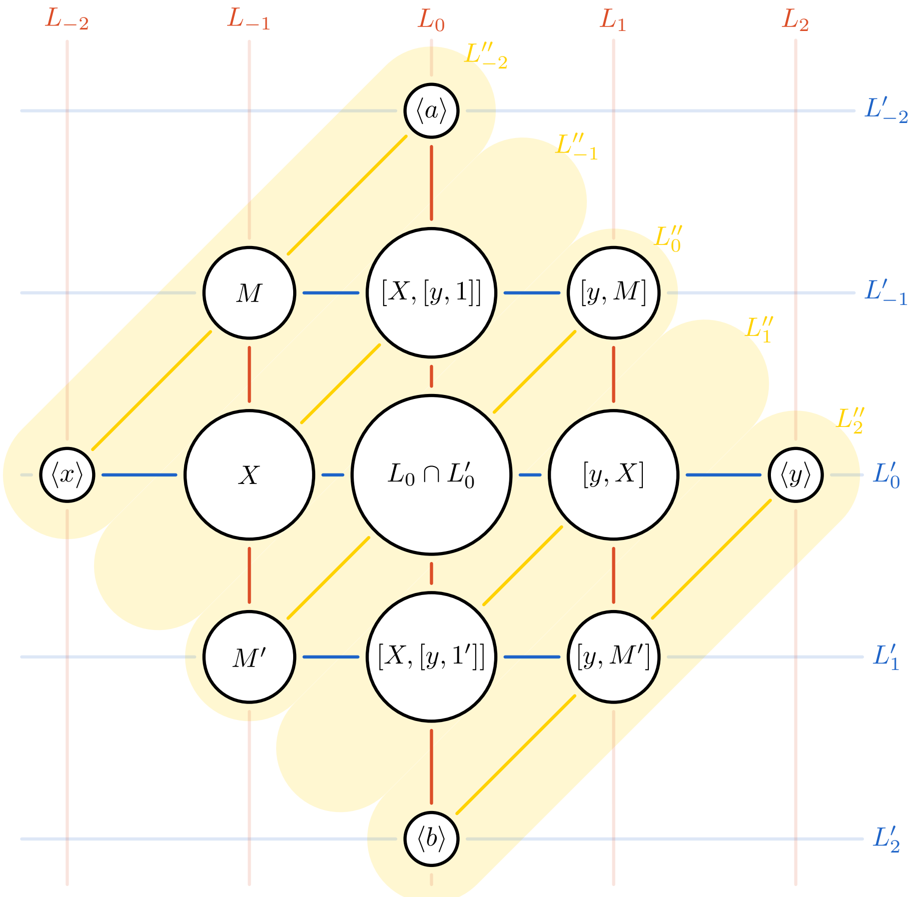


Figure 4.1: Three gradings on L

actually be applied. Also note that $|k'| > 3$, since k'/k is a proper field extension. Indeed, note that by assumption there are no lines in the extremal geometry of L, but there are lines in the extremal geometry of $L \otimes k'$.

Remark 4.6.3. Note that, by Theorem 2.4.7, Assumption 4.6.1(i) follows from Assumption 4.6.1(ii) if L is not a symplectic Lie algebra and $\operatorname{char}(k) \neq 2$. In fact if L is a symplectic Lie algebra such that Assumption 4.6.1(ii) is satisfied, then L is isomorphic to $\mathfrak{sl}_2$, and the extremal points of that Lie algebra also form a Moufang set.

Consider $x, y \in E$ such that $g_x(y) = 1$.

Notation 4.6.4. In this section we set $\varphi = \exp(y)\exp(x)\exp(y)$, the automorphism from Lemma 4.1.4. Recall that this automorphism interchanges x and y, and sends the component L_i onto the component L_{-i}.

Lemma 4.6.5. *We have $\varphi E_-(x, y)\varphi^{-1} = E_+(x, y)$.*

Proof. Consider $\alpha_l \in E_-(x, y)$, with $l \in L_{-1}$, as in Theorem 4.3.11. So there exist maps q_{α_l}, n_{α_l} and v_{α_l} from L to itself such that (4.3) and (4.4) hold. By (4.3)

$$(\varphi\alpha_l\varphi^{-1})(m) = m + [\varphi(l), m] + (\varphi q_{\alpha_l}\varphi^{-1})(m) + (\varphi n_{\alpha_l}\varphi^{-1})(m) + (\varphi v_{\alpha_l}\varphi^{-1})(m)$$

for all $m \in L$. Since φ and φ^{-1} map L_i onto L_{-i}, (4.4) implies

$$(\varphi q_{\alpha_l}\varphi^{-1})(L_i) \subseteq L_{i+2},\ (\varphi n_{\alpha_l}\varphi^{-1})(L_i) \subseteq L_{i+3},\ (\varphi v_{\alpha_l}\varphi^{-1})(L_i) \subseteq L_{i+4}.$$

We obtain $\varphi\alpha_l\varphi^{-1} \in E_+(x, y)$ by Theorem 4.3.11(v). Hence $\varphi E_-(x, y)\varphi^{-1} \subseteq E_+(x, y)$ and similarly $\varphi^{-1}E_+(x, y)\varphi \subseteq E_-(x, y)$. So

$$E_-(x, y) = \varphi^{-1}(\varphi E_-(x, y)\varphi^{-1})\varphi \subseteq \varphi^{-1}E_+(x, y)\varphi \subseteq E_-(x, y),$$

and hence all containments are actually equalities. □

Lemma 4.6.6. *Any $e \in \mathcal{E}$ not coinciding with $\langle x\rangle$ or $\langle y\rangle$ equals $\alpha(\langle x\rangle)$ and $\beta(\langle y\rangle)$ for unique $\alpha \in E_+(x, y)$ and $\beta \in E_-(x, y)$.*

Proof. Since $g_x(L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_1) = 0$ by Lemma 4.1.3, Assumption 4.6.1(ii) and Lemma 4.1.8 imply that e contains an element with 2-component equal to y. Then Theorem 4.3.17 implies that there exists a unique $\beta \in E_-(x, y)$ such that $\beta(\langle y\rangle) = e$. Using Lemmas 4.1.4 and 4.6.5 the other claim follows as well. □

Notation 4.6.7. For all $e \in \mathcal{E}$ not equal to $\langle y\rangle$ or $\langle x\rangle$ we set $U_e = \alpha E_-(x, y)\alpha^{-1}$, with $\alpha \in E_+(x, y)$ as in Lemma 4.6.6. For $\langle x\rangle$ we set $U_{\langle x\rangle} = E_-(x, y)$ and for $\langle y\rangle$ we set $U_{\langle y\rangle} = E_+(x, y)$.

Theorem 4.6.8. *The data $(\mathcal{E}, (U_e)_{e\in\mathcal{E}})$ is a Moufang set.*

Proof. By $\mathrm{id} \in E_-(x,y)$, $\mathrm{id}(\langle y\rangle) = \langle y\rangle$ and Lemma 4.6.6, we get that $E_-(x,y) = U_{\langle x\rangle}$ acts sharply transitively on $\mathcal{E}\setminus\{\langle x\rangle\}$ while obviously fixing $\langle x\rangle$. For all $e \in \mathcal{E}\setminus\{\langle y\rangle\}$ this implies by definition of U_e that it acts sharply transitively on $\mathcal{E}\setminus\{e\}$ while fixing e. Again by $\mathrm{id} \in E_+(x,y)$ and Lemma 4.6.6 the group $U_{\langle y\rangle}$ acts sharply transitively on $\mathcal{E}\setminus\{\langle y\rangle\}$ while fixing $\langle y\rangle$. This shows the first axiom of a Moufang set.

By definition of the root groups, we have $G^+ = \langle E_-(x,y), E_+(x,y)\rangle$. So in order to prove $U_e^g = U_{g(e)}$ for all $g \in G^+$ and $e \in \mathcal{E}$ it suffices to show this for all $g \in E_-(x,y)$ and $e \in \mathcal{E}$, since it is clear for all $g \in E_+(x,y)$ by construction.

Consider $e \in \mathcal{E}\setminus\{\langle x\rangle, \langle y\rangle\}$ arbitrary and let α and β be as in Lemma 4.6.6. Since $\beta \neq \mathrm{id}$, $\beta^{-1} \neq \mathrm{id}$ and hence by Lemma 4.6.6 there exists a unique $\gamma \in E_+(x,y)$ such that $\gamma(\beta^{-1}(y)) \in \langle x\rangle$. By definition $U_e = \alpha E_-(x,y)\alpha^{-1}$. By Lemma 4.6.5

$$\gamma\beta^{-1}U_e\beta\gamma^{-1} = \gamma\beta^{-1}\alpha\varphi^{-1}E_+(x,y)\varphi\alpha^{-1}\beta\gamma^{-1} = (U_{\langle y\rangle})^{\gamma\beta^{-1}\alpha\varphi^{-1}}. \tag{4.94}$$

Now $\gamma\beta^{-1}\alpha\varphi^{-1}(y) = \gamma(\beta^{-1}\alpha)(x)) \in \gamma(\langle y\rangle) = \langle y\rangle$. Similarly $\gamma\beta^{-1}\alpha\varphi^{-1}(x) = (\gamma\beta^{-1})(\alpha(y)) = (\gamma\beta^{-1})(y) \in \langle x\rangle$, by definition of γ. Set $\delta = \gamma\beta^{-1}\alpha\varphi^{-1}$. By Lemma 4.1.7 we get $\delta(L_i) = L_i$.

Now consider $\alpha_l \in E_+(x,y)$, with $l \in L_1$, as in Theorem 4.3.11(iii), so with associated maps q_{α_l}, n_{α_l}, v_{α_l}. Then for all $m \in L$

$$\begin{aligned}(\delta\alpha_l\delta^{-1})(m) = m + [\delta(l), m] + \delta(q_{\alpha_l}(\delta^{-1}(m))) \\ + \delta(n_{\alpha_l}(\delta^{-1}(m))) + \delta(v_{\alpha_l}(\delta^{-1}(m))).\end{aligned}$$

Now since δ fixes the components of the 5-grading, setting $\beta_l = \delta\alpha_l\delta^{-1}$, $q_{\beta_l} = \delta q_{\alpha_l}\delta^{-1}$, $n_{\beta_l} = \delta n_{\alpha_l}\delta^{-1}$ and $v_{\beta_l} = \delta v_{\alpha_l}\delta^{-1}$, we see that these maps satisfy (4.3) and (4.4). Hence $\beta_l \in E_+(x,y)$ and thus $U_{\langle y\rangle}^{\delta} = E_+(x,y)^{\delta} = E_+(x,y) = U_{\langle y\rangle}$. Together with (4.94) this implies

$$U_{\alpha(\langle x\rangle)} = U_{\beta(\langle y\rangle)} = U_e = \beta\gamma^{-1}U_{\langle y\rangle}\gamma\beta^{-1} = \beta U_{\langle y\rangle}\beta^{-1} = U_{\langle y\rangle}^{\beta},$$

using $\gamma \in E_+(x,y)$. Now $U_e^g = U_{g(e)}$ for all $g \in E_-(x,y)$ and $e \in \mathcal{E}$ follows. $\square$

APPENDIX A

Nederlandstalige samenvatting

Historische context

Structureerbare algebra's

In 1978 introduceerde Bruce Allison een klasse van niet-associatieve algebra's, structureerbare algebra's genaamd, die de klasse van de Jordan algebra's omvat. Elke structureerbare algebra $\mathcal{A}$ heeft een involutie en dus een deelruimte $\mathcal{S}$ van scheve elementen ten opzichte van deze involutie. De Jordan algebra's zijn precies de structureerbare algebra's met een triviale involutie. Een ander voorbeeld van structureerbare algebra's zijn associatieve algebra's met involutie. Structureerbare algebra's zijn geclassificeerd door Bruce Allison in [All78] als de karakteristiek gelijk is aan 0, en door Oleg Smirnov in [Smi92] als de karakteristiek minstens 7 is (en hij ontdekte ook een structureerbare algebra van dimensie 35 die in de oorspronkelijke classificatie ontbrak, zie [Smi90]).

In [All79] beschrijft Bruce Allison hoe men vertrekkende van een structureerbare algebra een 5-gegradeerde Lie algebra kan construeren. De uiteinden van deze gradering zijn isomorf met $\mathcal{S}$. Als $\mathcal{S} = 0$, i.e. we beschouwen een Jordan algebra, krijgen we eigenlijk een 3-gegradeerde Lie algebra. Deze constructie van een 3-gegradeerde Lie algebra uitgaande van een Jordan algebra is te danken aan Jacques Tits, Max Koecher en Issai Kantor, zie [Tit62, Koe67, Kan64]. We noemen deze constructie van gegradeerde Lie algebra's uit structureerbare algebra's de Tits–Kantor–Koecher-constructie, of, kortweg, de TKK-constructie. In [All79] wordt aangetoond dat alle isotrope Lie algebra's in karakteristiek 0 verkregen worden door de TKK-constructie toe te passen op een structureerbare algebra, in het bijzonder verkrijgt men de exceptionele Lie algebra's. Bijvoorbeeld, de TKK-constructie toegepast op een *Brown algebra*, een 56-dimensionale structureerbare algebra van scheef-dimensie één, geeft een Lie algebra van type E_8.

Elke lineaire algebraïsche groep heeft een geassocieerde Lie algebra. Als de algebraïsche groep isotroop is en het onderliggende veld heeft karakteristiek verschillend van 2 en 3, dan ontstaat de Lie algebra (of preciezer, zijn afgeleide algebra) ook via de TKK-constructie vertrekkende van een *structureerbare algebra*, vaak op meer dan één manier [Sta20, Stelling 5.9].

Inwendige idealen

Een belangrijk begrip in deze thesis is dat van een inwendig ideaal van een Lie algebra. Een deelruimte I van een Lie algebra L wordt een inwendig ideaal genoemd indien $[I, [I, L]] \leq I$. Inwendige idealen in Lie algebra's werden geïntroduceerd door John Faulkner in [Fau73] en verder onderzocht door Georgia Benkart in haar doctoraatsthesis [Ben74]; zie ook [Ben77, Ben76]. Als de Lie algebra enkelvoudig is en over een algebraïsch gesloten veld van karakteristiek 0 gedefinieerd is, dan zijn de inwendige idealen in detail bestudeerd in [DFLGGL12]. Bovendien heeft John Faulkner onder dezelfde voorwaarden deze inwendige idealen in verband gebracht met meetkundes; zie [Fau73]. In het recente boek [FL19] spelen inwendige idealen in Lie algebra's ook een cruciale rol.

Arjeh Cohen en Gabor Ivanyos hebben in [CI06] het concept van een *extremale meetkunde* in een Lie algebra geïntroduceerd. Een element van een Lie algebra heet *extremaal* als het een één-dimensionaal inwendig ideaal opspant (als de karakteristiek gelijk is aan 2, moet aan enkele bijkomende voorwaarden voldaan zijn), en de overeenkomstige extremale meetkunde heeft als puntenverzameling de verzameling van al die één-dimensionale inwendige idealen. Onder bepaalde voorwaarden hebben deze extremale meetkunden de structuur van zogenaamde wortel schaduwruimten van sferische gebouwen, zie [CI06, CI07]. Meer recent hebben Hans Cuypers, Yael Fleischmann, Kieran Roberts en Sergey Shpectorov [CRS15, CF17, CF18] onderzocht hoe deze enkelvoudige door extremale elementen gegenereerde Lie algebra's worden gekarakteriseerd door hun extremale meetkunde.

Het concept van een inwendig ideaal bestaat ook in de theorie van Jordan algebra's. (In feite werd het concept in Jordan algebra's geïntroduceerd vóórdat het in Lie algebra's werd geïntroduceerd). De inwendige idealen van Jordan algebra's zijn bestudeerd (en in veel gevallen geclassificeerd) in [McC71] en ze kunnen ook gebruikt worden om (exceptionele) meetkundes te beschrijven, zie [Fau70]. In [Gar01] toont Skip Garibaldi aan dat sommige van de inwendige idealen van een (gespleten) Brown-algebra gerelateerd zijn aan een gebouw van type E_7.

Sferische gebouwen

Sferische gebouwen zijn door Jacques Tits geïntroduceerd als een hulpmiddel om isotrope enkelvoudige lineaire algebraïsche groepen over willekeurige velden te bestuderen. Deze sferische gebouwen horende bij algebraïsche groepen voldoen altijd aan de zogenaamde *Moufang* eigenschap, die zegt dat de automorfismegroep van zo'n gebouw zeer transitief is (op een zeer precieze manier). Als de rang van het gebouw (die samenvalt met de relatieve rang van de algebraïsche groep) 1 is, dan noemen we het gebouw een *Moufang verzameling*; is deze 2, dan noemen we het gebouw een *Moufang veelhoek*. Afhankelijk van het relatief type, is de Moufang veelhoek een *Moufang driehoek* (relatief type A_2), *Moufang vierhoek* (relatief type B_2 of BC_2) of een *Moufang zeshoek* (relatief type G_2). De Moufang veelhoeken zijn in detail geclassificeerd en onderzocht in [TW02].

Alle gekende voorbeelden van zogenaamde echte Moufang verzamelingen met abelse wortelgroepen komen voort uit (kwadratische) Jordan delingsalgebra's, zie [DMW06, DMS08, Grü15]. Meer algemeen komen alle gekende voorbeelden van echte Moufang verzamelingen met (abelse of niet-abelse) wortelgroepen zonder elementen van orde 2 of 3 voort uit structureerbare delingsalgebra's, zie [BDMS19].

De algebraïsche structuren die Moufang driehoeken coördinatiseren (i.e., Moufang vlakken), zijn de alternatieve delingsalgebra's. De algebraïsche structuren die de Moufang zeshoeken coördinatiseren zijn de anisotrope kubische normstructuren (i.e., kubische Jordan delingsalgebra's).

De classificatie van de Moufang vierhoeken in [TW02] verdeelt ze onder in verschillende klassen, en elke klasse heeft zijn eigen overeenkomstige algebraïsche structuur. Om de meeste van deze vierhoeken, en in het bijzonder de exceptionele vierhoeken, op een uniforme manier te kunnen behandelen, is in [Wei06] het begrip van een *vierhoekige algebra* geïntroduceerd. Recent, in [MW19], werd het begrip van een vierhoekige algebra uitgebreid om isotrope vierhoekige algebra's toe te laten. In [BDM13, BDM15], hebben Lien Boelaert en Tom De Medts twee verschillende verbanden gelegd tussen structureerbare algebra's en vierhoekige algebra's (van exceptioneel type). In het eerste artikel hebben de geassocieerde structureerbare algebra's scheef-dimensie één, terwijl in het laatste artikel de geassocieerde structureerbare algebra's tensorproducten van compositie-algebra's zijn. Wanneer de artikels gepubliceerden werden was het onduidelijk hoe deze twee verschillende constructies met elkaar te verbinden.

Overzicht

In Hoofdstuk 1 introduceren we de algebraïsche structuren die nodig zijn in dit proefschrift en bespreken we overeenkomstige meetkundige structuren, namelijk Moufang veelhoeken en Moufang verzamelingen.

We beginnen Hoofdstuk 2 met de introductie van een zeer specifiek soort meetkunde, namelijk de *wortel filtratie ruimten*. In Sectie 2.2 beschouwen we deelmeetkunden van deze wortel filtratie ruimten die gefixeerd worden door een involutie en tonen we aan dat, onder enkele bijkomende voorwaarden, deze deelmeetkunde een *polaire ruimte* vormt. In de volgende sectie lichten we de constructie van een wortel filtratie ruimte in bepaalde Lie algebra's L gegenereerd door zogenaamde zuivere extremale elementen toe. Meer precies construeren we een extremale meetkunde $\Gamma = \Gamma(L)$, waarvan we de punten- en lijnenverzamelingen noteren met respectievelijk $\mathcal{E} = \mathcal{E}(L)$ en $\mathcal{F} = \mathcal{F}(L)$. De verzameling $\mathcal{E}$ is precies de verzameling van 1-dimensionale ruimten opgespannen door de zuivere extremale elementen van L, en de lijnverzameling $\mathcal{F}$ bestaat uit bepaalde 2-dimensionale deelruimten van L, incidentie is gewoon inclusie. Als $\mathcal{F} \neq \emptyset$, dan heeft de extremale meetkunde de structuur van een wortel filtratie ruimte. Deze constructie werd voor het eerst geïntroduceerd door Arjeh Cohen en Gabor Ivanyos [CI06]. In [CI07] hebben zij deze wortel filtratie ruimten geclassificeerd. In Sectie 2.4 beschouwen we het geval $\mathcal{F} = \emptyset$. Als we aannemen dat er zogenaamde *symplectische* paren van extremale elementen bestaan en de Lie algebra niet symplectisch is, dan kunnen we de extremale punten van L identificeren met de punten van deze polaire ruimte gefixeerd door een involutie uit Sectie 2.2. In de laatste sectie breiden we het begrip van een extremale meetkunde uit tot dat van een inwendige ideaal meetkunde en tonen we aan dat deze meetkunde ofwel een wortel filtratie ruimte is (en samenvalt met de extremale meetkunde), ofwel een polaire ruimte, ofwel gewoon een verzameling zonder lijnen is.

In Hoofdstuk 3 construeren we Moufang verzamelingen, Moufang driehoeken en Moufang zeshoeken met behulp van inwendige idealen van Lie algebra's bekomen door de TKK-constructie toe te passen op bepaalde structureerbare algebra's. De drie verschillende soorten structureerbare algebra's die we gebruiken zijn respectievelijk:

- structureerbare delingsalgebra's; zie Sectie's 3.2, 3.3;
- algebra's $D \oplus D^{\mathrm{op}}$, waarbij D een alternatieve delingsalgebra is, voorzien van de uitwisselings involutie; zie Sectie 3.4;
- matrix structureerbare algebra's $M(J, 1)$, waarbij J een kubische Jordan delingsalgebra is; zie Sectie 3.5.

In elk geval bepalen we ook de wortelgroepen rechtstreeks in termen van de structureerbare algebra.

In Hoofdstuk 4 vinden we bepaalde algebraïsche structuren terug, namelijk structureerbare algebra's, kubische normstructuren en vierhoekige algebra's, als we geschikte veronderstellingen maken over de extremale meetkunde. De eerste sectie van dit hoofdstuk verzamelt enkele eigenschappen van de 5-graderingen geassocieerd met bepaalde paren van extremale elementen. De uiteinden van deze 5-gradering, d.w.z. het (-2)- en 2-deel, zijn één-dimensionaal. Met behulp van deze 5-gradering tonen we in Sectie 4.2 aan dat als de karakteristiek niet gelijk is aan 2 of 3 dat elke niet-symplectische enkelvoudige Lie algebra gegenereerd door zijn extremale elementen verkregen wordt door de TKK-constructie toe te passen op een structureerbare algebra van scheef-dimensie één. In het bijzonder, als de extremale meetkunde lijnen bevat, is deze structureerbare algebra isotoop met een zogenaamde matrix structureerbare algebra. We eindigen deze sectie door aan te tonen dat als de eerder vermelde inwendige ideaal meetkunde geen lijnen bevat, de extremale punten, samen met bepaalde wortelgroepen, een Moufang verzameling vormen.

Daarna breiden we sommige resultaten van Sectie 4.2 uit tot velden van karakteristiek 2 en 3. Meer precies tonen we in Sectie 4.3 aan dat als L een enkelvoudige Lie algebra is, gegenereerd door haar extremale elementen zodanig dat $\mathcal{F}(L) \neq \emptyset$, dat dan de 1-component van de eerder genoemde 5-gradering lineair voortgebracht is door haar extremale elementen. We kunnen dit dan gebruiken om het bestaan aan te tonen van automorfismen die compatibel zijn met deze gradering. Met behulp van een afdalingsargument verkrijgen we enkele uitspraken in grotere algemeenheid. In Sectie 4.4 gebruiken we deze automorfismen en twee verschillende 5-gradering van L om een kubische normstructuur te vinden, tenminste als de extremale meetkunde lijnen bevat en niet van zogenaamd type $A_{n,\{1,n\}}$ is. We tonen ook aan dat de extremale meetkunde een Moufang zeshoek is als en slechts als J een anisotrope kubische normstructuur is, en we bepalen de wortelgroepen expliciet. Hiermee kunnen we aantonen dat een eindig-dimensionale enkelvoudige Lie algebra L gegenereerd door zijn zuivere extremale elementen met $\mathcal{F}(L) \neq \emptyset$ bepaald is door zijn extremale meetkunde. In de laatste subsectie van Sectie 4.4 beschouwen we het geval dat de extremale meetkunde van het type $A_{n,\{1,n\}}$ is.

In Sectie 4.5 beschouwen we enkelvoudige Lie algebra's L over k gegenereerd door hun zuivere extremale elementen die symplectische paren bevatten en zodanig dat er een Galois-extensie k'/k van graad hoogstens 2 bestaat zodat de extremale meetkunde van $L \otimes_k k'$ lijnen bevat. We veronderstellen niet (per se) $\mathcal{F}(L) = \emptyset$. Met behulp van twee verschillende 5-graderingen en automorfismen verkregen uit Sectie 4.3, kunnen we een vierhoekige algebra terugvinden als de karakteristiek van het veld niet 2 is. Indien $\mathcal{F}(L) = \emptyset$, dan volgt uit Hoofdstuk 2 dat de inwendige ideaal meetkunde een polaire ruimte vormt. We tonen aan dat de inwendige ideaal meetkunde een polaire ruimte van rang 2 is, i.e. een veralgemeende vierhoek, als en slechts als de corresponderende vierhoekige algebra anisotroop is. In dat geval tonen we aan dat deze vierhoek precies de Moufang vierhoek is geassocieerd met deze anisotrope vierhoekige algebra.

In het laatste deel van dit hoofdstuk beschouwen we het geval dat L een enkelvoudige Lie algebra is, gegenereerd door zuivere extremale elementen met $\mathcal{F}(L) = \emptyset$ en dat er geen symplectische paren zijn. Dan, onder enkele bijkomende veronderstellingen indien de karakteristiek 2 is, vormt de verzameling $\mathcal{E}(L)$ samen met geschikte wortelgroepen, die we bekomen door Sectie 4.3, een Moufang verzameling.

Hoofdstuk 2 is gebaseerd op de eerste 7 secties van [CM21]. Hoofdstuk 3 is gebaseerd op [DMM20]. Sectie 4.2 is gebaseerd op de laatste sectie van [CM21].

De andere secties van Hoofdstuk 4 zijn (nog) niet gepubliceerd.

Bibliography

[AF84] Bruce Allison and John Faulkner. A Cayley-Dickson process for a class of structurable algebras. *Transactions of the American Mathematical Society*, 283(1):185–210, 1984.

[AF99] Bruce Allison and John Faulkner. Elementary groups and invertibility for Kantor pairs. *Communications in Algebra*, 27(2):519–556, 1999.

[AFS17] Bruce Allison, John Faulkner, and Oleg Smirnov. Weyl images of Kantor pairs. *Canadian Journal of Mathematics. Journal Canadien de Mathématiques*, 69(4):721–766, 2017.

[AH81] Bruce Allison and Wolfgang Hein. Isotopes of some nonassociative algebras with involution. *Journal of Algebra*, 69(1):120–142, 1981.

[All78] Bruce Allison. A class of nonassociative algebras with involution containing the class of Jordan algebras. *Mathematische Annalen*, 237(2):133–156, 1978.

[All79] Bruce Allison. Models of isotropic simple Lie algebras. *Communications in Algebra*, 7(17):1835–1875, 1979.

[All88] Bruce Allison. Tensor products of composition algebras, Albert forms and some exceptional simple Lie algebras. *Transactions of the American Mathematical Society*, 306(2):667–695, 1988.

[BC13] Francis Buekenhout and Arjeh M. Cohen. *Diagram geometry*, volume 57 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer, Heidelberg, 2013.

[BDM13] Lien Boelaert and Tom De Medts. Exceptional Moufang quadrangles and structurable algebras. *Proceedings of the London Mathematical Society. Third Series*, 107(3):590–626, 2013.

[BDM15] Lien Boelaert and Tom De Medts. A new construction of Moufang quadrangles of type E_6, E_7 and E_8. *Transactions of the American Mathematical Society*, 367(5):3447–3480, 2015.

[BDMS19] Lien Boelaert, Tom De Medts, and Anastasia Stavrova. Moufang sets and structurable division algebras. *Mem. Amer. Math. Soc.*, 259(1245):v+90, 2019.

[Ben74] Georgia Benkart. *Inner ideals and the structure of Lie algebras.* ProQuest LLC, Ann Arbor, MI, 1974. Thesis (Ph.D.)–Yale University.

[Ben76] Georgia Benkart. The Lie inner ideal structure of associative rings. *Journal of Algebra*, 43(2):561–584, 1976.

[Ben77] Georgia Benkart. On inner ideals and ad-nilpotent elements of Lie algebras. *Transactions of the American Mathematical Society*, 232:61–81, 1977.

[BK51] R. H. Bruck and Erwin Kleinfeld. The structure of alternative division rings. *Proceedings of the American Mathematical Society*, 2:878–890, 1951.

[Bou68] N. Bourbaki. *Éléments de mathématique. Fasc. XXXIV. Groupes et algèbres de Lie. Chapitre IV: Groupes de Coxeter et systèmes de Tits. Chapitre V: Groupes engendrés par des réflexions. Chapitre VI: systèmes de racines.* Actualités Scientifiques et Industrielles, No. 1337. Hermann, Paris, 1968.

[CF17] Hans Cuypers and Yael Fleischmann. A geometric characterization of the symplectic Lie algebra. *arXiv e-prints*, page arXiv:1707.02095, July 2017.

[CF18] Hans Cuypers and Yael Fleischmann. A geometric characterization of the classical Lie algebras. *Journal of Algebra*, 502:1–23, 2018.

[CI06] Arjeh M. Cohen and Gábor Ivanyos. Root filtration spaces from Lie algebras and abstract root groups. *Journal of Algebra*, 300(2):433–454, 2006.

[CI07] Arjeh M. Cohen and Gábor Ivanyos. Root shadow spaces. *European Journal of Combinatorics*, 28(5):1419–1441, 2007.

[CIR08] Arjeh M. Cohen, Gábor Ivanyos, and Dan Roozemond. Simple Lie algebras having extremal elements. *Koninklijke Nederlandse Akademie van Wetenschappen. Indagationes Mathematicae. New Series*, 19(2):177–188, 2008.

[CM21] Hans Cuypers and Jeroen Meulewaeter. Extremal elements in Lie algebras, buildings and structurable algebras. *Journal of Algebra*, 580:1–42, 2021.

[CO19] Hans Cuypers and Marc Oostendorp. A geometric characterization of the special linear and unitary Lie algebras, preprint. 2019.

[CRS15] Hans Cuypers, Kieran Roberts, and Sergey Shpectorov. Recovering the Lie algebra from its extremal geometry. *Journal of Algebra*, 441:196–215, 2015.

[CSUW01] Arjeh M. Cohen, Anja Steinbach, Rosane Ushirobira, and David Wales. Lie algebras generated by extremal elements. *Journal of Algebra*, 236(1):122–154, 2001.

[DFLGGL08] Cristina Draper, Antonio Fernández López, Esther García, and Miguel Gómez Lozano. The socle of a nondegenerate Lie algebra. *Journal of Algebra*, 319(6):2372–2394, 2008.

[DFLGGL12] Cristina Draper, Antonio Fernández López, Esther García, and Miguel Gómez Lozano. The inner ideals of the simple finite dimensional Lie algebras. *Journal of Lie Theory*, 22(4):907–929, 2012.

[DM19] Tom De Medts. Structurable algebras of skew-dimension one and hermitian cubic norm structures. *Communications in Algebra*, 47(1):154–172, 2019.

[DM20] Tom De Medts. Known examples of Moufang sets. Available online at https://algebra.ugent.be/~tdemedts/research-publications.php, July 2020.

[DMM20] Tom De Medts and Jeroen Meulewaeter. *Inner ideals and structurable algebras: Moufang sets, triangles and hexagons*. *arXiv e-prints*, August 2020. page arXiv:2008.02700.

[DMS08] Tom De Medts and Yoav Segev. Identities in Moufang sets. *Trans. Amer. Math. Soc.*, 360(11):5831–5852, 2008.

[DMS09] Tom De Medts and Yoav Segev. A course on Moufang sets. *Innov. Incidence Geom.*, 9:79–122, 2009.

[DMW06] Tom De Medts and Richard M. Weiss. Moufang sets and Jordan division algebras. *Math. Ann.*, 335(2):415–433, 2006.

[Fau70] John R. Faulkner. *Octonion planes defined by quadratic Jordan algebras*. Memoirs of the American Mathematical Society, No. 104. American Mathematical Society, Providence, R.I., 1970.

[Fau73] John R. Faulkner. On the geometry of inner ideals. *Journal of Algebra*, 26:1–9, 1973.

[Fau77] John R. Faulkner. Groups with Steinberg relations and coordinatization of polygonal geometries. *Memoirs of the American Mathematical Society*, 10(185):iii+135, 1977.

[FL19] Antonio Fernández López. *Jordan structures in Lie algebras*, volume 240 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2019.

[Fle15] Yael Fleischmann. *A Geometric Approach to Classical Lie Algebras*. PhD thesis, TU Eindhoven, 2015.

[Gar01] R. Skip Garibaldi. Structurable algebras and groups of type E_6 and E_7. *Journal of Algebra*, 236(2):651–691, 2001.

[GGLN11] Esther García, Miguel Gómez Lozano, and Erhard Neher. Nondegeneracy for Lie triple systems and Kantor pairs. *Canadian Mathematical Bulletin. Bulletin Canadien de Mathématiques*, 54(3):442–455, 2011.

[Grü15] Matthias Grüninger. On the identities defining a quadratic Jordan division algebra. *Arch. Math. (Basel)*, 104(1):11–24, 2015.

[Jac58] Nathan Jacobson. Composition algebras and their automorphisms. *Rendiconti del Circolo Matematico di Palermo. Serie II*, 7:55–80, 1958.

[Kan64] I. L. Kantor. Classification of irreducible transitive differential groups. *Doklady Akademii Nauk SSSR*, 158:1271–1274, 1964.

[Kle51] Erwin Kleinfeld. Alternative division rings of characteristic 2. *Proceedings of the National Academy of Sciences of the United States of America*, 37:818–820, 1951.

[Koe67] Max Koecher. Imbedding of Jordan algebras into Lie algebras. I. *American Journal of Mathematics*, 89:787–816, 1967.

[LGLN07] Antonio Fernández López, Esther García, Miguel Gómez Lozano, and Erhard Neher. A construction of gradings of Lie algebras. *International Mathematics Research Notices. IMRN*, (16):Art. ID rnm051, 34, 2007.

[Loo75] Ottmar Loos. *Jordan pairs*. Lecture Notes in Mathematics, Vol. 460. Springer-Verlag, Berlin-New York, 1975.

[McC69] Kevin McCrimmon. The Freudenthal-Springer-Tits constructions of exceptional Jordan algebras. *Transactions of the American Mathematical Society*, 139:495–510, 1969.

[McC70] Kevin McCrimmon. The Freudenthal-Springer-Tits constructions revisited. *Transactions of the American Mathematical Society*, 148:293–314, 1970.

[McC71] Kevin McCrimmon. Inner ideals in quadratic Jordan algebras. *Transactions of the American Mathematical Society*, 159:445–468, 1971.

[McC04] Kevin McCrimmon. *A taste of Jordan algebras*. Universitext. Springer-Verlag, New York, 2004.

[MW19] Bernhard Mühlherr and Richard M. Weiss. Isotropic quadrangular algebras. *Journal of the Mathematical Society of Japan*, 71(4):1321–1380, 2019.

[PR86] Holger P. Petersson and Michel L. Racine. Jordan algebras of degree 3 and the Tits process. *Journal of Algebra*, 98(1):211–243, 1986.

[Rac72] M. L. Racine. A note on quadratic Jordan algebras of degree 3. *Transactions of the American Mathematical Society*, 164:93–103, 1972.

[Rob12] Kieran Roberts. *Lie Algebras and Incidence Geometries*. PhD thesis, University of Birmingham, 2012.

[Sch66] Richard D. Schafer. *An introduction to nonassociative algebras*. Pure and Applied Mathematics, Vol. 22. Academic Press, New York-London, 1966.

[Shu11] Ernest E. Shult. *Points and lines*. Universitext. Springer, Heidelberg, 2011.

[Smi90] O. N. Smirnov. An example of a simple structurable algebra. *Akademiya Nauk SSSR. Sibirskoe Otdelenie. Institut Matematiki. Algebra i Logika*, 29(4):491–499, 504, 1990.

[Smi92] O. N. Smirnov. Simple and semisimple structurable algebras. In *Proceedings of the International Conference on Algebra, Part 2 (Novosibirsk, 1989)*, volume 131 of *Contemp. Math.*, pages 685–694. Amer. Math. Soc., Providence, RI, 1992.

[Sta20] Anastasia Stavrova. *5-Graded simple Lie algebras, structurable algebras, and Kantor pairs*. *arXiv e-prints*, December September 2020. page arXiv: 1712.05288.

[Tim01] Franz Georg Timmesfeld. *Abstract root subgroups and simple groups of Lie type*, volume 95 of *Monographs in Mathematics*. Birkhäuser Verlag, Basel, 2001.

[Tit62] J. Tits. Une classe d'algèbres de Lie en relation avec les algèbres de Jordan. *Nederl. Akad. Wetensch. Proc. Ser. A 65 = Indag. Math.*, 24:530–535, 1962.

[Tit66] J. Tits. Classification of algebraic semisimple groups. In *Algebraic Groups and Discontinuous Subgroups (Proc. Sympos. Pure Math., Boulder, Colo., 1965)*, pages 33–62. Amer. Math. Soc., Providence, R.I., 1966.

[Tit74] Jacques Tits. *Buildings of spherical type and finite BN-pairs.* Lecture Notes in Mathematics, Vol. 386. Springer-Verlag, Berlin-New York, 1974.

[Tit76] J. Tits. Non-existence de certains polygones généralisés. I. *Inventiones Mathematicae*, 36:275–284, 1976.

[Tit79] J. Tits. Non-existence de certains polygones généralisés. II. *Inventiones Mathematicae*, 51(3):267–269, 1979.

[TW02] Jacques Tits and Richard M. Weiss. *Moufang polygons.* Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2002.

[vdBS59] F. van der Blij and T. A. Springer. The arithmetics of octaves and of the group G_2. *Nederl. Akad. Wetensch. Proc. Ser. A 62 = Indag. Math.*, 21:406–418, 1959.

[Wei79] Richard Weiss. The nonexistence of certain Moufang polygons. *Inventiones Mathematicae*, 51(3):261–266, 1979.

[Wei06] Richard M. Weiss. *Quadrangular algebras*, volume 46 of *Mathematical Notes.* Princeton University Press, Princeton, NJ, 2006.

[Zel80] E. I. Zelmanov. Absolute zero-divisors in Jordan pairs and Lie algebras. *Matematicheskiĭ Sbornik. Novaya Seriya*, 112(154)(4(8)):611–629, 1980.

[ZK90] E. I. Zelmanov and A. I. Kostrikin. A theorem on sandwich algebras. volume 183, pages 106–111, 225. 1990. Translated in Proc. Steklov Inst. Math. **1**991, no. 4, 121–126, Galois theory, rings, algebraic groups and their applications (Russian).