# EXTREMAL ELEMENTS IN LIE ALGEBRAS, BUILDINGS AND STRUCTURABLE ALGEBRAS

HANS CUYPERS AND JEROEN MEULEWAETER

Abstract. An extremal element in a Lie algebra $\mathfrak{g}$ over a field of characteristic not 2 is an element $x \in \mathfrak{g}$ such that $[x,[x,\mathfrak{g}]]$ is contained in the linear span of $x$. The linear span of an extremal element, called an extremal point, is an inner ideal of $\mathfrak{g}$, i.e. a subspace $I$ satisfying $[I,[I,\mathfrak{g}]] \leq I$. We show that in characteristic different from $2,3$ the geometry with point set the set of extremal points and as lines the minimal inner ideals containing at least two extremal points is a Moufang spherical building, or in case there are no lines a Moufang set.

This last result on the Moufang sets is obtained by connecting Lie algebras to structurable algebras, a class of non-associative algebras with involution generalizing Jordan algebras. It is shown that in characteristic different from $2,3$ each finite-dimensional simple Lie algebra generated by extremal elements is either a symplectic Lie algebra or can be obtained by applying the Tits-Kantor-Koecher construction to a skew-dimension one structurable algebra. Various relations between the Lie algebra $\mathfrak{g}$ and its extremal geometry on the one hand and the associated structurable algebra on the other hand are investigated.

## 1. Introduction

There exist well known close connections between simple linear algebraic groups, (classical) Lie algebras, and (algebraic) spherical buildings. Tits associates spherical buildings to simple linear algebraic groups in [Tit74]. Moreover, every linear algebraic group has an associated Lie algebra. We will focus on direct connections between Lie algebras and buildings.

A main tool in this paper is the notion of an inner ideal of a Lie algebra, introduced by Faulkner in [Fau73]. An inner ideal of a Lie algebra $\mathfrak{g}$ with Lie product $[\cdot,\cdot]$ is a linear subspace $I$ of $\mathfrak{g}$ satisfying $[I,[I,\mathfrak{g}]] \leq I$. The recent monograph [FL19] has inner ideals as one of its main subjects.

Faulkner [Fau73] showed that the set of all inner ideals of a split simple Lie algebra over a field of characteristic 0 carries the structure of a spherical building.

More recently, Cohen et al. [CSUW01, CI06, CIR08, Coh12] studied simple Lie algebras generated by extremal elements, i.e. elements whose linear span is a 1-dimensional inner ideal. Their main results provide a direct connection between simple Lie algebras generated by extremal elements and so-called root shadow spaces of spherical buildings.

We define the inner ideal geometry of a Lie algebra to be the point-line geometry with the 1-dimensional inner ideals as points and the minimal proper inner ideals containing at least two points as lines. Building upon the work of Cohen et al. we are able to prove the following results.

**Theorem 1.1.** *Suppose $\mathfrak{g}$ is a finite-dimensional simple Lie algebra generated by its extremal elements over a field $\mathbb{F}$ of characteristic not 2. If $\mathfrak{g}$ contains two commuting and linearly independent extremal elements, then the inner ideal geometry is a root shadow space of a spherical Moufang building of rank at least 2.*

*If $\mathfrak{g}$ does not contain two commuting and linearly independent extremal elements, then the inner ideal geometry contains no lines. If moreover the characteristic of $\mathbb{F}$ is not 2 or 3 then the set of extremal points of $\mathfrak{g}$ forms a Moufang set.*

See Theorem 6.11, Theorem 7.2 and Theorem 8.38 for a proof of this theorem.

*Date*: *Submitted*: October 6, 2020; *Revised:* March 19, 2021.

2010 *Mathematics Subject Classification.* Primary 51E24, 16W10, 17B60, 51A50; Secondary 51A45.

*Key words and phrases.* Building, Lie algebra, structurable algebra.

The first author is the corresponding author.

In most cases, in particular if the rank of the associated building is at least 3, either the inner ideal geometry determines the Lie algebra $\mathfrak{g}$, up to isomorphism, or the inner ideal geometry of the Lie algebra $\hat{\mathfrak{g}}$ obtained by extending the field $\mathbb{F}$ quadratically determines $\hat{\mathfrak{g}}$, up to isomorphism. See [CF17] and Theorems 4.16 and 5.7.

Structurable algebras have been introduced by Bruce Allison in [All78] in order to construct isotropic simple Lie algebras using a generalization of the Tits-Kantor-Koecher construction. Structurable algebras are a generalization of Jordan algebras. A structurable algebra $\mathcal{A}$ is an algebra with an involution $^-$ such that $\overline{xy} = \overline{y}\,\overline{x}$ for all $x, y \in \mathcal{A}$ and satisfying some additional conditions; see Section 8 for details. These algebras are defined over fields of characteristic not 2 or 3. This implies that we can write $\mathcal{A}$ as

$$\mathcal{A} = \mathcal{H} \oplus \mathcal{S}$$

where $\mathcal{H} = \{x \in \mathcal{A} \mid \overline{x} = x\}$ and $\mathcal{S} = \{x \in \mathcal{A} \mid \overline{x} = -x\}$.

The dimension of $\mathcal{S}$ is called the *skew-dimension* of $\mathcal{A}$. We obtain the following characterization of simple Lie algebras generated by extremal elements.

**Theorem 1.2.** *Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra over a field $\mathbb{F}$ of characteristic different from 2, 3. Then $\mathfrak{g}$ is generated by its extremal elements if and only if $\mathfrak{g}$ is isomorphic to the symplectic Lie algebra $\mathfrak{sp}(V, f)$ for some non-degenerate symplectic space $(V, f)$, or $\mathfrak{g}$ is obtained by applying the Tits-Kantor-Koecher construction to a finite-dimensional simple structurable algebra $\mathcal{A}$ of skew-dimension 1.*

The above results are obtained without using any classification results on spherical buildings, (classical) Lie algebras or structurable algebras. (Except for the claim about the Moufang set if the characteristic equals 5, then one does need classification results.) They provide not only both a geometric and algebraic framework to study Lie algebras, but also offer new ways to explore Moufang buildings and in particular Moufang sets, quadrangles or hexagons, see for example [BDM13, BDMS19, DMM20].

The paper is organised as follows. In Section 2 we discuss root filtration spaces. Most of the results presented there are based on the work of Cohen and Ivanyos [CI07]. Section 3 is devoted to subgeometries of root filtration spaces fixed by some involutive automorphism. In particular we show that such subgeometries carry the structure of a polar space. Lie algebras generated by extremal elements will be considered in the Sections 4, 5 and 6. In these sections we study the extremal geometry of a Lie algebra generated by extremal elements. Building upon the work of Cohen et al. [CI06] we obtain a proof that the inner ideal geometry of such a Lie algebra containing two linearly independent but commuting elements is a root shadow space of a spherical building of rank at least 2. Such buildings are shown to be Moufang in Section 7.

In the last section, Section 8, we study the connection between Lie algebras generated by extremal elements, their extremal geometry, and structurable algebras. We provide a proof of Theorem 1.2. Using the connection between Lie algebras generated by extremal elements and structurable algebras, we are also able to use the results of [DMM20] to show that the extremal points in a simple Lie algebra $\mathfrak{g}$ form a Moufang set, under the conditions stated in the second part of Theorem 1.1.

**Acknowledgments:** The second author would like to thank the first author for his hospitality during his research visit at Eindhoven University of Technology in November 2018. The second author is a PhD Fellow of the Research Foundation Flanders (Belgium) (F.W.O.-Vlaanderen), grant number 166032/1128720N.

## 2. Root filtration spaces

We first recall some definitions from incidence geometry.

**Definition 2.1.** A *partial linear space* $(\mathcal{E}, \mathcal{F})$ is a pair consisting of a set $\mathcal{E}$ of *points* and a set $\mathcal{F}$ of *lines*, which are subsets of $\mathcal{E}$ of size at least 2, such that each pair of points from $\mathcal{E}$ is in at most one line in $\mathcal{F}$. We call a partial linear space *thick* if every line contains at least 3 points.

**Definition 2.2.** A *subspace* of a partial linear space $(\mathcal{E}, \mathcal{F})$ is a subset of the point set with the property that all points of each line meeting it in two points are contained in the subset. A subspace is called *singular* if any two points in it are *collinear*, i.e. are contained in a common line. Subspaces are often identified with the partial linear space induced on them by the lines they contain.

**Definition 2.3.** The *rank* of a singular subspace is the length of a maximal chain of proper non-trivial subspaces. The *rank* of a partial linear space is the supremum of all ranks of maximal singular subspaces.

**Definition 2.4.** A *geometric hyperplane*, or just hyperplane, is a subspace meeting each line non-trivially.

**Definition 2.5.** As the intersection of subspaces is again a subspace, we can define the subspace *generated* by a subset $X$ of the point set to be the intersection of all subspaces containing $X$.

In this section we consider root filtration spaces.

**Definition 2.6.** A thick partial linear space $\Gamma = (\mathcal{E}, \mathcal{F})$ is a *root filtration space with filtration* $\mathcal{E}_i$, $-2 \leq i \leq 2$, if the sets $\mathcal{E}_i$, with $-2 \leq i \leq 2$, provide a partition of $\mathcal{E} \times \mathcal{E}$ into five symmetric relations satisfying the following for all $x, y, z \in \mathcal{E}$:

(A) The relation $\mathcal{E}_{-2}$ is equality.
(B) The relation $\mathcal{E}_{-1}$ is collinearity of distinct points.
(C) For each $(x, y) \in \mathcal{E}_1$, there is a unique point, denoted by $[x, y]$, such that if $z \in \mathcal{E}_i(x) \cap \mathcal{E}_j(y)$, then $[x, y] \in \mathcal{E}_{\leq i+j}(z)$.
(D) If $(x, y) \in \mathcal{E}_2$, then $\mathcal{E}_{\leq 0}(x) \cap \mathcal{E}_{\leq -1}(y) = \emptyset$.
(E) The subsets $\mathcal{E}_{\leq i}(x)$ are subspaces of $\Gamma$.
(F) The subset $\mathcal{E}_{\leq 1}(x)$ is a geometric hyperplane.
(G) $\mathcal{E}_2(x)$ is non-empty.
(H) $\Gamma$ is connected.

Here $\mathcal{E}_{\leq i}$ is the union of all $\mathcal{E}_j$ with $-2 \leq j \leq i$. Moreover, for $x \in \mathcal{E}$ the set $\mathcal{E}_i(x)$ is the set of all $y \in \mathcal{E}$ with $(x, y) \in \mathcal{E}_i$, and, similarly, $\mathcal{E}_{\leq i}(x)$ is the set of all $y \in \mathcal{E}$ with $(x, y) \in \mathcal{E}_{\leq i}$.

A point pair $x, y$ is called *collinear* if $(x, y) \in \mathcal{E}_{-1}$, *polar* if $(x, y) \in \mathcal{E}_0$, *special* if $(x, y) \in \mathcal{E}_1$, and finally *hyperbolic* if $(x, y) \in \mathcal{E}_2$.

**Remark 2.7.** Note that by conditions (G) and (H) the set $\mathcal{E}_{-1}(x)$ is non-empty for each point $x \in \mathcal{E}$, and, moreover, by (D) and (F) even $\mathcal{E}_1(x)$ is non-empty. The fact that these sets are non-empty is sometimes referred to as the root filtration space $\Gamma$ being *non-degenerate*.

Root filtration spaces have been studied by Cohen and Ivanyos [CI07]. The main result is that a root filtration space $\Gamma = (\mathcal{E}, \mathcal{F})$ in which singular subspaces have finite rank are so-called *root shadow spaces* of spherical buildings. This means that the building is of type $X_n$ for some spherical Dynkin diagram and the points in $\mathcal{E}$ are flags whose types are the nodes from the set $J$ of nodes adjacent to the node extending the Dynkin diagram to an affine diagram. We call such a root shadow space of type $X_{n,J}$ or just $X_{n,j}$ if $J = \{j\}$. For a detailed discussion of root shadow spaces, the reader is referred to [BC13, Section 11.6] or [Shu11, Chapter 17].

**Example 2.8.** Let $V$ be a vector space over a skew field of dimension $n+1$ at least 3 and $\mathbb{P}(V)$ be the corresponding projective geometry. Take for $\mathcal{E}$ the set of incident point-hyperplane pairs of $\mathbb{P}(V)$. Lines in $\mathcal{F}$ consist of all point-hyperplane pairs $(p, H)$, where either $H$ is running through the set of all hyperplanes containing a codimension 2 subspace $K$ on $p$, or $p$ is running through a 2-dimensional subspace $L$ inside $H$. Then $\Gamma$ is a root shadow space of type $\mathrm{A}_{n,\{1,n\}}$, where $n+1$ is the dimension of $V$.

The space $\Gamma$ admits the following filtration. Let $x = (p, H)$ and $y = (q, K)$ be incident point-hyperplane pairs in $\mathcal{E}$. Then

- $(x, y) \in \mathcal{E}_{-2} \Leftrightarrow x = y$;
- $(x, y) \in \mathcal{E}_{-1} \Leftrightarrow x \neq y$ and $p = q$ or $H = K$;

- $(x, y) \in \mathcal{E}_0 \quad \Leftrightarrow p \neq q,\ H \neq K,\ p \in K$ and $q \in H$;
- $(x, y) \in \mathcal{E}_1 \quad \Leftrightarrow p \in K$ but $q \notin H$, or $p \notin K$ but $q \in H$;
- $(x, y) \in \mathcal{E}_2 \quad \Leftrightarrow p \notin K$ and $q \notin H$.

Suppose $W^*$ is a subspace of $V^*$ with the property that for any two linearly independent $v, v' \in V$ there is a $\phi \in W^*$ with $\phi(v) \neq 0$ but $\phi(v') = 0$. We say $W^*$ *separates* points of the projective space of $V$. If we allow the dimension of $V$ to be infinite, then we can construct more examples by choosing our hyperplanes as kernels of elements inside a subspace $W^*$ of $V^*$ separating the points of the projective space of $V$. This geometry is then denoted by $\Gamma(V, W^*)$.

**Definition 2.9.** A *polar space* is a partial linear space satisfying the *one or all* axiom:

a point is collinear with one or all points of a line.

Note that a point $p$ is collinear with itself, if there is a line containing $p$.

A polar space is called *degenerate* if it contains a point $p$ collinear with all other points, and *non-degenerate* otherwise.

In a non-degenerate polar space, singular subspaces are projective spaces.

**Example 2.10.** Examples of polar spaces are obtained from polarities. Let $\perp$ be a polarity of a space $\mathbb{P}$, then let $\mathcal{E}$ be the set of *absolute points*, i.e. the points $p$ with $p \in p^\perp$ and as lines the *absolute lines*, i.e, the lines $l$ where for each point $p \in l$ we have $l \subseteq p^\perp$. This polar space is non-degenerate if $p^\perp$ is never the full projective space.

A non-degenerate polar space of rank $n$ with all lines having at least 3 points is a root shadow space of type $\mathrm{BC}_{n,1}$.

Let $\mathcal{E}$ be the set of lines of a non-degenerate polar space of rank at least 3. Take for $\mathcal{F}$ the sets of the form $\{\ell \in \mathcal{E} \mid p \in \ell, \ell \subset \pi\}$, where $(p, \pi)$ runs over the set of point-singular plane pairs with $p \in \pi$. Then $(\mathcal{E}, \mathcal{F})$ is a root shadow space of type $C_{n,2}$ (or $\mathrm{B}_{n,2}$ or $\mathrm{D}_{n,2}$, depending on the polar space). It is a root filtration space with filtration defined as follows:

- $(l, m) \in \mathcal{E}_{-2} \Leftrightarrow l = m$;
- $(l, m) \in \mathcal{E}_{-1} \Leftrightarrow l, m$ are contained in a singular plane;
- $(l, m) \in \mathcal{E}_0 \quad \Leftrightarrow l, m$ do intersect but are not contained in some singular subspace, or $l, m$ do not intersect but are contained in singular subspace;
- $(l, m) \in \mathcal{E}_1 \quad \Leftrightarrow l$ contains a unique point collinear with all points of $m$;
- $(l, m) \in \mathcal{E}_2 \quad \Leftrightarrow$ each point on $l$ is collinear with a unique point of $m$.

The main result of Cohen and Ivanyos [CI07] reads as follows.

**Theorem 2.11** ([CI07, Theorem 1])**.** *Suppose that $\Gamma$ is a non-degenerate root filtration space of finite rank. Then $\Gamma$ is isomorphic to a root shadow space of type $\mathrm{A}_{n,\{1,n\}}$ $(n \geq 2)$, $\mathrm{B}_{n,2}$, $\mathrm{C}_{n,2}$ $(n \geq 3)$, $\mathrm{D}_{n,2}$ $(n \geq 4)$, $\mathrm{E}_{6,2}$, $\mathrm{E}_{7,1}$, $\mathrm{E}_{8,8}$, $\mathrm{F}_{4,1}$ or $\mathrm{G}_{2,2}$.*

The key to this result is the construction of so-called symplecta. Cohen and Ivanyos showed:

**Proposition 2.12** ([CI07])**.** *Let $\Gamma$ be a non-degenerate root filtration space. Suppose the relation $\mathcal{E}_0$ is non-empty. Then $\Gamma$ contains a collection $\mathcal{S}$ of subspaces such that every pair of points $x, y$ with $(x, y) \in \mathcal{E}_0$ is contained in a unique element $S \in \mathcal{S}$.*

*Moreover, for each $S \in \mathcal{S}$ we have:*

(a) *$S$ is a non-degenerate polar space of rank at least 2.*
(b) *If $x$ is collinear to two non-collinear points of $S$, then it is contained in $S$.*
(c) *For all points $x, y \in S$ we have $(x, y) \in \mathcal{E}_{\leq 0}$.*
(d) *For each point $x$ the set of points in $\mathcal{E}_{\leq -1}(x) \cap S$ is either empty or contains a line.*

*Proof.* By Theorem 13 of [CI07] we find that $\Gamma$ contains a collection of symplecta satisfying (a)–(d), or every line is contained in a unique maximal singular subspace.

In the latter case, Theorem 29, Proposition 23 and Theorem 35 of [CI07] imply that $\Gamma$ is either isomorphic to $\Gamma(V, W^*)$ for some vector space $V$ and subspace $W^*$ of its dual, see Example 2.8, or to the root filtration space of lines in a polar space of rank 3 as described in Example 2.10. Also in these cases we find a collection of symplecta if the dimension of $V$ is at least 4, see Example 2.13 and Example 2.14 below. One easily checks properties (a)–(d). $\square$

The elements of $\mathcal{S}$ are called *symplecta*.

**Example 2.13.** In the root filtration space $\Gamma = (\mathcal{E}, \mathcal{F})$ of Example 2.8, two point-hyperplane pairs $(p, H)$ and $(q, K)$ of $\mathcal{E}$ are in relation $\mathcal{E}_0$ if and only if $p \neq q$ and $H \neq K$, but $p, q \in H \cap K$. The unique symplecton containing $(p, H)$ and $(q, K)$ is the set of point-hyperplane pairs $(r, L)$ where $r$ is on the line through $p$ and $q$ and $L$ contains $H \cap K$. This symplecton carries the structure of a generalized quadrangle, i.e. a rank 2 polar space.

**Example 2.14.** The symplecta of the root filtration space $(\mathcal{E}, \mathcal{F})$ of lines in a polar space $(P, L)$ as in Example 2.10 come in two types.

The first are the symplecta determined by two intersecting lines of $L$ which are not in a singular subspace. It consists of all the lines on the intersection point. Suppose $S$ is a symplecton of this type and consists of all lines in $L$ on a fixed point $p \in P$. Then any line $\ell \in L \setminus S$ is in relation $\mathcal{E}_{-1}$ with none of the lines of $S$, or the line is in a singular plane with $p$ and $\ell$ is in relation $\mathcal{E}_{-1}$ with all lines on $S$ which are inside this plane. These lines form an element of $\mathcal{F}$, showing (d) of Proposition 2.12 holds true in this case.

A symplecton of the second type is the set of all lines in a singular subspace which is a projective 3-space. (Note that this is a polar space.) In case the polar space has rank $\leq 3$, this class of symplecta is empty.

We provide some extra information on root filtration spaces that will be used in the next section.

The following two lemmas can also be found in [CI06] and [CI07] and are concerned with a non-degenerate root filtration space $\Gamma = (\mathcal{E}, \mathcal{F})$.

**Lemma 2.15.** *Let $x, y \in \mathcal{E}$.*

(a) *If $(x, y) \in \mathcal{E}_1$, then there is a unique point $z$ in $\mathcal{E}_{-1}(x) \cap \mathcal{E}_{-1}(y)$.*
(b) *If $(x, y) \in \mathcal{E}_1$, then there is a point $z$ in $\mathcal{E}_2(x) \cap \mathcal{E}_{-1}(y)$.*
(c) *If $(x, y) \in \mathcal{E}_0$, then there is a point $z$ in $\mathcal{E}_0(x) \cap \mathcal{E}_2(y)$.*
(d) *If $(x, y) \in \mathcal{E}_0$, then there is a point $z$ in $\mathcal{E}_{-1}(x) \cap \mathcal{E}_1(y)$.*
(e) *If $(x, y) \in \mathcal{E}_2$, then there is a point $z$ in $\mathcal{E}_{-1}(x) \cap \mathcal{E}_1(y)$.*

*Proof.* Statement (a) is [CI06, Lemma 1(ii)]. Statement (b) follows from [CI06, Lemma 1(v), Lemma 4] and (c) is [CI07, Lemma 8(ii)]. Statement (d) follows from (c). Indeed, let $v$ be a point in $\mathcal{E}_0(x) \cap \mathcal{E}_2(y)$, then a common neighbour of $z$ of $x$ and $v$ is, by property (D), in $\mathcal{E}_1(y)$. The final statement follows from Condition (F) of Definition 2.6. $\square$

**Lemma 2.16** ([CI06, Lemma 1(vi)], Pentagon Property)**.** *Let $x_1, \ldots, x_5$ be five points forming a pentagon, i.e., $x_i$ and $x_j$ are collinear if and only if $i - j \equiv 1 \pmod 5$. Then $(x_i, x_{i+2}) \in \mathcal{E}_0$ for $i \in \{1, 2, 3\}$.*

We analyse the relation between a point and a symplecton.

**Lemma 2.17.** *Consider a point $x$ and a symplecton $S$ of $\Gamma$ with $\mathcal{E}_{\leqslant -1}(x) \cap S = \emptyset$. Then either $S \subseteq \mathcal{E}_{\leqslant 0}(x)$ or $A := \mathcal{E}_{\leqslant 0}(x) \cap S$ is a non-empty singular subspace of $S$. In the latter case, any point of $S$ not in $A$ but collinear with a point of $A$ is in $\mathcal{E}_1(x)$ and all other points of $S$ not in $A$ are in $\mathcal{E}_2(x)$.*

*Proof.* Fix a point $x$ and symplecton $S$ such that $x$ is not collinear to any point of $S$. Either $S$ is contained in $\mathcal{E}_{\leqslant 0}(x)$ or we can find $y \in S$ which is contained in $\mathcal{E}_1(x) \cup \mathcal{E}_2(x)$. Suppose we are in the latter case. If $y \in \mathcal{E}_2(x)$, then, as $S$ is of rank at least 2, we can find a line on $y$ inside $S$. This line meets the geometric hyperplane $\mathcal{E}_{\leq 1}(x)$ in a point which, by condition (D), is in $\mathcal{E}_1(x)$. So we may assume $y \in S$ to be inside $\mathcal{E}_1(x)$. Let $z = [x, y]$ be the unique common neighbour of $x$ and $y$. By Proposition 2.12(d), the subspace $\mathcal{E}_{-1}(z) \cap S$ of $S$ is singular and contains a line $L$.

Now by Lemma 2.15 there is a point $v$ collinear with $x$ and in $\mathcal{E}_2(y)$. The line $L$ meets the geometric hyperplane $\mathcal{E}_{\leq 1}(v)$ in a point $u$. If the points $v, x, z, u, [v, u]$ form a pentagon, Lemma 2.16 implies that $(x, u) \in \mathcal{E}_0$. Assume they do not form a pentagon. Note that $v \in \mathcal{E}_1(z)$ since $v \in \mathcal{E}_2(y)$ and hence $z$ cannot be collinear to $v$ or $[v, u]$, otherwise $(v, z) \in \mathcal{E}_0$ in the latter case. So $x$ and $[v, u]$ are collinear, which also implies that $(x, u) \in \mathcal{E}_0$.

This shows that each point of $S \cap \mathcal{E}_1(x)$ is on a line of $S$ meeting $\mathcal{E}_0(x) \cap S$ in a point. In particular $A := \mathcal{E}_{\leqslant 0}(x) \cap S$ is a non-empty subspace of $S$.

Now let $d \in A$ be arbitrary and let $e \in S \backslash A$ be any point collinear with $d$. If $T$ is the unique symplecton containing $x$ and $d$, then $e$ is collinear with a point, and hence with a line of $T$. As $T$ is a polar space, we can find a point $p$ on this line collinear with $x$, and $x$ and $e$ are at distance 2. As $e \notin A$, we have $(x, e) \in \mathcal{E}_1$. Moreover, $p = [x, e]$ is collinear with both $d$ and $e$ and hence with all the points of the line through $d$ and $e$.

It remains to show that $A$ is singular. Assume there are two non-collinear points $i, j$ in $A$. Then let $k$ be a point collinear with both $i$ and $j$. Assume $k \notin A$. In the previous paragraph we showed that there exists a neighbour $p$ of $x$ collinear with the line $ik$, and similarly a neighbour $q$ collinear to all points of the line $jk$. Since these two neighbours of $x$ can not coincide (using that $\mathcal{E}_{-1}(h) \cap S$ is always a singular subspace for any point $h$ not in $S$), we get $(x, k) \in \mathcal{E}_0$. Hence $k \in A$, a contradiction with the assumption $k \notin A$. So, all common neighbours in $S$ of $i$ and $j$ are contained in $A$. This shows that $A$ is a convex non-singular subspace of $S$. Hence $A = S$, which contradicts our initial assumption. □

We analyse this somewhat further.

**Lemma 2.18.** *Suppose $S$ is a symplecton and $x$ a point such that $\mathcal{E}_2(x) \cap S$ is non-empty. Then there is a unique point $y \in S \cap \mathcal{E}_0(x)$ and on every line on $y$ inside $S$ there is a singular plane containing a unique point collinear with $x$.*

*Proof.* By Lemma 2.17, $S \cap \mathcal{E}_0(x)$ is a non-empty singular subspace. If $y$ is any element of this intersection, all lines on this point are contained in $\mathcal{E}_{\leq 1}(x)$ by this lemma. But then the fact that $S$ is a non-degenerate polar space implies that $\mathcal{E}_0(x) \cap S$ consists of precisely one point, which we call $y$.

Now assume $T$ is the symplecton on $x$ and $y$. Then every point $z \in S$ collinear with $y$ is collinear to all the points of a line $\ell$ on $y$ in $T$. The line $\ell$, and hence the plane on $z$ and $\ell$ contains a unique point collinear to $x$. □

If $X$ is a subset of $\mathcal{E}$, then we call $X$ connected, if the graph on $X$ induced by collinearity (i.e., the relation $\mathcal{E}_{-1}$) is connected.

We conclude this section with two lemmas on the connectedness of some complements of subspaces. The following lemma is well known.

**Lemma 2.19.** *Let $S$ be a thick non-degenerate polar space of rank at least 2. If $H$ is a proper geometric hyperplane of $S$, then $S \setminus H$ is connected.*

*Proof.* Let $x, y$ be non-collinear points of $S$ not in $H$. We will show that there is a path of collinear points from $x$ to $y$ outside $H$.

If $\ell$ is a line on $x$, then $\ell$ contains a unique point $z$ collinear to $y$. Clearly we can assume that $z$ is in $H$. Now pick a point $x'$ on $\ell$ distinct from $x$ and $z$. Moreover let $y'$ be a point on the line through $y$ and $z$ different from $y$ and $z$. Let $m$ be a line on $x'$ different from $\ell$ and not in a singular subspace with $\ell$. Then both $y$ and $y'$ are collinear with points $u, v \in m$ different from $x'$. If $u = v$, then $u = v$ is also collinear to $z$. Since $u = v$ is collinear to $x'$, this contradicts $\ell$ and $m$ not to be in a singular subspace. As at least one of $u$ or $v$ is not in $H$, we find that at least one of $x, x', u, y$ or $x, x', v, y', y$ is a path from $x$ to $y$ outside $H$. □

**Lemma 2.20.** *Suppose $\mathcal{E}_0$ is non-empty. Let $x \in \mathcal{E}$. The complement of the geometric hyperplane $\mathcal{E}_{\leqslant 1}(x)$ in $\mathcal{E}$ is connected.*

*Proof.* Let $\mathcal{S}$ be as in Proposition 2.12. Let $x \in \mathcal{E}$ and set $H = \mathcal{E}_{\leq 1}(x)$. Suppose $y, z \in \mathcal{E} \setminus H$. We will show that $y$ and $z$ are connected by a path in the collinearity graph, which does not contain points from $H$. Clearly we can assume that $y$ and $z$ are non-collinear. If $z \in \mathcal{E}_0(y)$, then $y$ and $z$ are contained in a symplecton $S$ and we can apply Lemma 2.19 to see that $y$ and $z$ are connected by a path outside $H$.

Assume now that $z$ is a point not contained in $H$ but inside $\mathcal{E}_1(y)$. Clearly, we can assume that the unique neighbour $c$ of $y$ and $z$ is contained in $H$. Consider a symplecton $S$ containing the line $cz$. Then there exists a line $\ell$ in $S$ such that $\ell \subseteq \mathcal{E}_{\leq -1}(y)$. If $\ell$ meets $H$ in a single point, then, again using Lemma 2.19, we find a path from $y$ via a point of $\ell$ to $z$ without using points from $H$.

Thus we can assume that $\ell$ is in $H$. Now $A := \mathcal{E}_0(x) \cap S$ is a non-empty singular subspace contained in $H$. By Lemma 2.18, we find that $A$ consists of a single point, call it $a$. Note $a \notin \mathcal{E}_{-1}(y)$ and hence $a \notin l$. Since $\ell \leq H \cap S$, Lemma 2.17 implies that $a$ and $\ell$ are contained in a singular plane $\pi$ of $S$. In particular, $S$ has rank at least 3 and we can find a point $d \in S$ which is collinear to all points of $\ell$ but not to $a$. But then $d \in \mathcal{E}_{\leq 0}(y)$ and $d \notin H$. By the above, we can find paths from $y$ to $d$ and from $d$ to $z$ not containing any point from $H$.

Finally assume now that $y$ is at distance 3 with $z$. Let $(z, a, b, y)$ be any path of length 3 in $\Gamma$ from $z$ to $y$. If $a$ or $b$ are not in $H$, we can apply the above to the pairs $a, y$ or $b, z$, respectively and find a path from $y$ to $z$. So assume $a, b \in H$.

By Lemma 2.15, we can find a neighbour $c$ of $z$ at distance 3 with $b$. Since $H$ is a geometric hyperplane the line $cz$ intersects $H$ in a unique point. Since $\mathcal{E}_{\leqslant 1}(b)$ is a geometric hyperplane as well and $b$ and $z$ are at distance 2, we may assume $c \notin H$. Since $\mathcal{E}_{\leqslant 1}(c)$ is a geometric hyperplane, the line $by$ contains a point $d$ at distance (at most) 2 from $c$. If $d = b$, then $c$ and $b$ are at distance (at most) 2, a contradiction. Since $b \in H$ and $y \notin H$, $d \notin H$. Note that $d$ is at distance 2 from $c$. By the above we find a path outside $H$ from $y$ to $z$ via $d$ and $c$. $\square$

**Corollary 2.21.** *Suppose $\mathcal{E}_0$ is non-empty. Let $x \in \mathcal{E}$. Then $\Gamma$ is generated by $\mathcal{E}_{\leq 1}(x) \cup \{z\}$, for each point $z \in \mathcal{E}_2(x)$.*

## 3. Subgeometries of root filtration spaces fixed by involutions

We proceed with the notation of the previous section. So, suppose $\Gamma = (\mathcal{E}, \mathcal{F})$ is a (non-degenerate) root filtration space with filtration $\mathcal{E}_i, -2 \leq i \leq 2$.

**Definition 3.1.** Let $\sigma$ be an involution in the automorphism group of $\Gamma$. We define the geometry $\Gamma_\sigma = (\mathcal{E}_\sigma, \mathcal{S}_\sigma)$ to be the point-line geometry with as point set the set $\mathcal{E}_\sigma$ of points of $\Gamma$ fixed by $\sigma$, and as line set $\mathcal{S}_\sigma$, the set of symplecta of $\Gamma$ fixed (as a set of points) by $\sigma$ and containing at least two non-collinear points of $\mathcal{E}_\sigma$. Incidence is defined by inclusion.

Notice that a line $S \in \mathcal{S}_\sigma$ can be identified with the subsets $\mathcal{E}_\sigma \cap S$, as any two of its non-collinear points determine the symplecton $S$ uniquely. This also implies that we can consider $\Gamma_\sigma$ to be a partial linear space.

**Example 3.2.** Let $V$ be a vector space over a skew field of dimension $n+1$ at least 3. Let $\Gamma$ be the root filtration space of type $\mathrm{A}_{n,\{1,n\}}$ described in Example 2.8. Let $\sigma$ be a non-degenerate polarity on $\mathbb{P}(V)$. Then $\sigma$ induces an automorphism of order 2 on $\Gamma$, which we also denote by $\sigma$.

The points fixed by $\sigma$ are the point-hyperplane pairs $(p, p^\sigma)$ with $p \in p^\sigma$. So, the points in $\mathcal{E}_\sigma$ can be identified with the points of $\mathbb{P}(V)$ that are absolute with respect to $\sigma$, i.e. points $p$ with $p \in p^\sigma$.

As $\sigma$ maps a line of $\Gamma$ consisting of point-hyperplane pairs with a fixed point to a line consisting of point-hyperplane pairs in which the hyperplane is fixed, we find that $\sigma$ does not fix any line of $\Gamma$.

A symplecton $S$ of $\Gamma$ fixed set-wise by $\sigma$ meets $\mathcal{E}_\sigma$ in the point-hyperplane pairs $(p, p^\sigma)$ where $p$ is absolute and running over an absolute line of $\mathbb{P}(V)$, which is a line which is contained in $p^\sigma$ for any point $p$ on it.

The geometry $\Gamma_\sigma$ is isomorphic to the polar space defined by $\sigma$ on $\mathbb{P}(V)$, i.e. the point-line geometry of absolute points and lines with respect to $\sigma$.

This example generalizes to the following:

**Proposition 3.3.** *Let $\sigma$ be an involution in the automorphism group of $\Gamma$. Suppose that $\sigma$ does not fix any line of $\Gamma$. Then $\Gamma_\sigma$ satisfies the one-or-all-axiom: if $x \in \mathcal{E}_\sigma$ and $S \in \mathcal{S}_\sigma$, then $x$ is collinear in $\Gamma_\sigma$ with one or all points $y \in \mathcal{E}_\sigma$ that are contained in $S$.*

*Proof.* Suppose that $\sigma$ fixes a singular subspace $V$ of $\Gamma$. For any $x \in V$, we have $x^\sigma \in V$. Hence, if $x^\sigma \neq x$, then the line $xx^\sigma$ is fixed by $\sigma$. On the other hand, if $x^\sigma = x$ for all $x \in V$, then any line in $V$ is (point-wise) fixed by $\sigma$. So we obtain that the only singular subspaces of $\Gamma$ fixed by $\sigma$ are points. I.e., the points of $\Gamma_\sigma$.

Consider an arbitrary point $x$ of $\Gamma_\sigma$ and an arbitrary symplecton $S$ of $\Gamma_\sigma$ not containing $x$.

Suppose first that $x$ is collinear with a point of $S$. Then, by Proposition 2.12, $\mathcal{E}_{-1}(x) \cap S$ is a singular subspace $V$ of $S$ containing a line. Now $(\mathcal{E}_{-1}(x) \cap S)^\sigma = \mathcal{E}_{-1}(x^\sigma) \cap S^\sigma = \mathcal{E}_{-1}(x) \cap S$. Hence $\mathcal{E}_{-1}(x) \cap S$ should be a point by the previous paragraph, a contradiction.

This implies that any point of $S$ has distance at least 2 with $x$. By Lemma 2.17 either all points of $S$ are in $\mathcal{E}_0(x)$ or $A := \mathcal{E}_{\leqslant 0}(x) \cap S$ is a singular subspace. Assume the former. Then any point $y$ of $S$ fixed by $\sigma$ is contained in a unique symplecton with $x$. Since $x$ and $y$ are fixed by $\sigma$ this symplecton is also fixed. I.e., $x$ and $y$ lie on a line in $\Gamma_\sigma$. Since the points in $S$ fixed by $\sigma$ are precisely the points of the line $S$ in $\Gamma_\sigma$, we find $x$ to be collinear with all points of the line $S$ in $\Gamma_\sigma$.

Assume now that $A$ is a singular subspace. Since $x$ and $S$ are fixed by $\sigma$, the subspace $A$ is fixed by $\sigma$ as well. Hence, as follows from the first part of this proof, $A$ consists of a unique point $y$. Moreover, $y$ is fixed by $\sigma$. In particular, in $\Gamma_\sigma$ the point $x$ is collinear with the unique point $y$ of $S$. □

The above proposition indeed implies that the fixed point geometry $\Gamma_\sigma$ is a polar space. However, in general this polar space may be thin (with just two points on a line) or degenerate (i.e., there is a point collinear to all other points). Moreover, it can also happen that $\Gamma_\sigma$ contains no lines, or even no points.

## 4. Lie algebras generated by extremal elements

Now we turn our attention to Lie algebras. In particular, Lie algebras generated by extremal elements.

In this section we provide some definitions and collect some results on extremal elements, mainly from [CI06].

**Definition 4.1.** Let $\mathfrak{g}$ be a Lie algebra over the field $\mathbb{F}$. A non-zero element $x \in \mathfrak{g}$ is called *extremal* if there is a map $g_x : \mathfrak{g} \to \mathbb{F}$, called the *extremal form* on $x$, such that

$$[x,[x,y]] = 2g_x(y)x, \tag{1}$$

and moreover

$$\big[[x,y],[x,z]\big] = g_x\big([y,z]\big)x + g_x(z)[x,y] - g_x(y)[x,z], \tag{2}$$

and

$$\big[x,[y,[x,z]]\big] = g_x\big([y,z]\big)x - g_x(z)[x,y] - g_x(y)[x,z], \tag{3}$$

for every $y, z \in \mathfrak{g}$.

The last two identities are called the *Premet identities*. If the characteristic of $\mathbb{F}$ is not 2, then the Premet identities follow from equation (1), see [CI06, Definition 14].

As a consequence, $\big[x,[x,\mathfrak{g}]\big] \subseteq \mathbb{F}x$ for any extremal element $x \in \mathfrak{g}$. We call $x \in \mathfrak{g}$ a *sandwich* if $[x,[x,y]] = 0$ and $[x,[y,[x,z]]] = 0$ for every $y, z \in \mathfrak{g}$. So, a sandwich is an element $x$ for which the extremal form $g_x$ can be chosen to be identically zero. We introduce the convention that *$g_x$ is identically zero whenever $x$ is a sandwich in $\mathfrak{g}$.* An extremal element is called *pure* if it is not a sandwich.

We denote the set of extremal elements of a Lie algebra $\mathfrak{g}$ by $E(\mathfrak{g})$ or, if $\mathfrak{g}$ is clear from the context, by $E$. Accordingly, we denote the set $\{\mathbb{F}x | x \in E(\mathfrak{g})\}$ of *extremal points* in the projective space on $\mathfrak{g}$ by $\mathcal{E}(\mathfrak{g})$ or $\mathcal{E}$.

We continue with some examples.

**Example 4.2.** Let $V$ be a vector space over a field $\mathbb{F}$ with dual space $V^*$. Suppose $W^*$ is a subspace of $V^*$ separating the points of the projective space of $V$.

On $V \otimes W^*$ we can define a Lie bracket by linear extension of the following product for pure tensors $v \otimes \phi$ and $w \otimes \psi$:

$$[v \otimes \phi, w \otimes \psi] = (v \otimes \psi)\phi(w) - (w \otimes \phi)\psi(v).$$

The Lie algebra thus obtained will be denoted by $\mathfrak{g}(V \otimes W^*)$.

A pure tensor $v \otimes \phi$ is called singular if $\phi(v) = 0$. Let $g$ be the $\mathbb{F}$-bilinear form on $V \otimes W^*$ defined by

$$g(v \otimes \phi, w \otimes \psi) = -\psi(v)\phi(w)$$

for $v \otimes \phi, w \otimes \psi \in V \otimes W^*$. Then for all singular pure tensors $v \otimes \phi$ and tensors $w \otimes \psi$ we have

$$\begin{aligned}\big[v \otimes \phi, [v \otimes \phi, w \otimes \psi]\big] &= \big[v \otimes \phi, \phi(w)v \otimes \psi - \psi(v)w \otimes \phi\big] \\ &= -\psi(v)\phi(w)v \otimes \phi - \psi(v)\phi(w)v \otimes \phi \\ &= -2\psi(v)\phi(w)v \otimes \phi \\ &= 2g(v \otimes \phi, w \otimes \psi)v \otimes \phi.\end{aligned}$$

In characteristic $\neq 2$ this implies that the singular pure tensors are extremal elements in $\mathfrak{g}(V \otimes W^*)$ with extremal form at the singular tensor $v \otimes \phi$ given by $g(v \otimes \phi, \cdot)$. This also holds true in characteristic 2. (It is straightforward, but somewhat tedious, to check that the Premet identities also hold.) The subalgebra of $\mathfrak{g}(V \otimes W^*)$ generated by the singular tensors is denoted by $\mathfrak{s}(V \otimes W^*)$.

An element $v \otimes \phi \in V \otimes V^*$ acts linearly on $V$ by

$$(v \otimes \phi)(w) = \phi(w)v$$

for all $w \in V$. This provides an isomorphism between $\mathfrak{g}(V \otimes V^*)$ and the finitary general linear Lie algebra $\mathfrak{fgl}(V)$. (A linear map is finitary if its kernel has finite codimension.) Under this isomorphism the subalgebra $\mathfrak{s}(V \otimes V^*)$ is mapped isomorphically to $\mathfrak{fsl}(V)$, the finitary special Lie algebra, or $\mathfrak{sl}(V)$ in case $V$ is finite-dimensional.

In the sequel of this paper we often identify the Lie algebra $\mathfrak{g}(V \otimes V^*)$ as well as its subalgebras with the finitary general linear Lie algebra $\mathfrak{fgl}(V)$, or the corresponding subalgebras.

As was noticed by the anonymous referee, $V \otimes W^*$ carries the structure of an associative algebra whose product is defined as the linear expansion of the product

$$(v \otimes \phi)(w \otimes \psi) = (v \otimes \psi)\phi(w).$$

The associated Lie algebra is then $\mathfrak{g}(V \otimes W^*)$. A pure tensor $v \otimes \phi$ being singular is equivalent to its square being zero in the associative algebra. So, if $a$ is such a singular pure tensor, then $[a,[a,b]] = -2aba$ which is a scalar multiple of $a$, for all $b \in \mathfrak{g}(V \otimes W^*)$.

**Example 4.3.** Let $\sigma$ be a field automorphism of order $\leq 2$ of the field $\mathbb{F}$, and $V$ be a vector space over $\mathbb{F}$. Assume that the characteristic of $\mathbb{F}$ is different from 2. Moreover, suppose $f$ is an antisymmetric $\sigma$-sesquilinear form on $V$, linear in the second coordinate. So, for all $u, v, w \in V$ and $\lambda, \mu \in \mathbb{F}$ we have:

$$\begin{aligned} f(v,w) &= -f(w,v)^\sigma, \\ f(v, \lambda w + \mu u) &= f(v,w)\lambda + f(v,u)\mu. \end{aligned}$$

Then for each vector $v \in V$ the map $f_v : V \to \mathbb{F}$, with $f_v(w) = f(v,w)$ for all $w \in V$ is an element of $V^*$. By $S_f(V \otimes V^*)$ or just $S_f$ we denote the subspace of $V \otimes V^*$ spanned by the pure tensors $v \otimes f_v$, with $v \in V$. (We say it is spanned by the $f$-symmetric elements.)

The space $S_f(V \otimes V^*)$ is closed under the Lie bracket:

$$\begin{aligned}[v \otimes f_v, w \otimes f_w] &= (v \otimes f_w)f(v,w) - (w \otimes f_v)f(w,v) \\ &= f(v,w)v \otimes f_w + w \otimes f_{f(v,w)v} \\ &= (f(v,w)v + w) \otimes f_{f(v,w)v+w} - f(v,w)v \otimes f_{f(v,w)v} - w \otimes f_w.\end{aligned}$$

The corresponding Lie subalgebra of $\mathfrak{g}(V \otimes V^*)$ will be denoted by $\mathfrak{s}_f(V \otimes V^*)$ or, for short, $\mathfrak{s}_f$.

If $f$ is a non-degenerate symplectic form, then $\mathfrak{s}_f$ is simple and can be identified with the finitary Lie algebra $\mathfrak{fsp}(V,f)$, i.e., the Lie subalgebra of $\mathfrak{fgl}(V)$ of finitary linear transformations $t : V \to V$ satisfying $f(t(v),w) = -f(v,t(w))$ for all $v,w \in V$. Moreover, its set of extremal elements is the set of all elements $v \otimes f_v$, where $0 \neq v \in V$. See [CF17, Theorem 3.1, Propositions 3.5 and 3.6].

If $\sigma$ is non-trivial and $f$ a non-degenerate skew-Hermitian form with positive Witt index, then the elements $v \otimes f_v$, where $0 \neq v \in V$ with $f(v,v) = 0$ generate a subalgebra of $\mathfrak{s}(V \otimes V^*)$, which can be identified with the finitary special unitary Lie algebra $\mathfrak{fsu}(V,f)$. The extremal elements in this subalgebra are the elements $v \otimes f_v$, where $0 \neq v \in V$ and $f(v,v) = 0$. See for example [CO19].

In view of the last remark in the previous example, we can also view $\mathfrak{s}_f(V \otimes V^*)$ as the Lie algebra of skew elements in $\mathfrak{g}(V \otimes V^*)$ with respect to the involution on the associative algebra $V \otimes V^*$ induced by $v \otimes f_w \mapsto -w \otimes f_v$.

We will now revise some general theory on extremal elements. For $x \in E$ and $\lambda \in \mathbb{F}$ we define the map $\exp(x,\lambda) : \mathfrak{g} \to \mathfrak{g}$ by

$$\exp(x,\lambda)(y) = y + \lambda[x,y] + \lambda^2 g_x(y)x,$$

for all $y \in \mathfrak{g}$.

**Proposition 4.4** ([CI06, Lemma 15]). *Let $x \in E$ be pure and $\lambda \in \mathbb{F}$. Then $\exp(x,\lambda)$ is an automorphism of $\mathfrak{g}$.*

Let $x \in E$ be pure. By $\mathrm{Exp}(x)$ we denote the set $\{\exp(x,\lambda) \mid \lambda \in \mathbb{F}\}$. Since, for $\lambda,\mu \in \mathbb{F}$, we have $\exp(x,\lambda)\exp(x,\mu) = \exp(x,\lambda+\mu)$, we find that $\mathrm{Exp}(x)$ is a subgroup of $\mathrm{Aut}(\mathfrak{g})$ isomorphic to the additive group of $\mathbb{F}$.

Clearly, $\mathrm{Exp}(x) = \mathrm{Exp}(\lambda x)$ for $\lambda \in \mathbb{F}^*$. Therefore we can define $\mathrm{Exp}(\langle x \rangle)$ to be equal to $\mathrm{Exp}(x)$.

**Proposition 4.5** ([CI06, Proposition 20]). *Suppose that $\mathfrak{g}$ is generated by its extremal elements (as a Lie algebra). The extremal elements span the vector space $\mathfrak{g}$. Moreover, there is a bilinear form $g : \mathfrak{g} \times \mathfrak{g} \to \mathbb{F}$, such that for all $x,y \in E$ we have $g(x,y) = g_x(y)$. The form $g$ is symmetric and associates with the Lie product $[\cdot,\cdot]$ on $\mathfrak{g}$.*

The form $g$ is called the *extremal form* on $\mathfrak{g}$. As the form $g$ is associative, its radical $\mathrm{Rad}(g) := \{u \in \mathfrak{g} \,|\, g_u(z) = 0, \forall\ z \in \mathfrak{g}\}$ is an ideal in $\mathfrak{g}$. Notice that the extremal form $f$ from [CSUW01] satisfies $f = 2g$.

**Proposition 4.6** ([CI06, Lemma 21, 24, 25 and 27]). *Suppose that $\mathfrak{g}$ is generated by its extremal elements and $x,y \in E$ are pure. Then we have one of the following:*

(a) *$\mathbb{F}x = \mathbb{F}y$;*
(b) *$[x,y] = 0$ and $\lambda x + \mu y \in E \cup \{0\}$ for all $\lambda,\mu \in \mathbb{F}$;*
(c) *$[x,y] = 0$ and $\lambda x + \mu y \in E$ only if $\lambda = 0$ or $\mu = 0$;*
(d) *$z := [x,y] \in E$, and $x,z$ and $y,z$ are as in case* (b)*;*
(e) *$\langle x,y \rangle$ is isomorphic to $\mathfrak{sl}_2(\mathbb{F})$.*

*Moreover, $g(x,y) \neq 0$ if and only if $\langle x,y \rangle$ is isomorphic to $\mathfrak{sl}_2(\mathbb{F})$.*

Based on the previous proposition, we define the following relations on $E \times E$.

**Definition 4.7.** For $x, y \in E$ extremal elements we define

$$(x,y) \in \begin{cases} E_{-2}, & \iff \mathbb{F}x = \mathbb{F}y, \\ E_{-1}, & \iff [x,y] = 0, (x,y) \notin E_{-2} \text{ and } \mathbb{F}x + \mathbb{F}y \subseteq E \cup \{0\}\ , \\ E_0, & \iff [x,y] = 0 \text{ and } (x,y) \notin E_{-2} \cup E_{-1}, \\ E_1, & \iff [x,y] \neq 0 \text{ and } g(x,y) = 0, \\ E_2, & \iff g(x,y) \neq 0. \end{cases}$$

For the corresponding extremal points $\langle x\rangle, \langle y\rangle$, we define

$$(\langle x\rangle, \langle y\rangle) \in \mathcal{E}_i \iff (x,y) \in E_i.$$

Let $x \in E$. Then $y \in E_i(x)$ denotes that $(x,y) \in E_i$. By $E_{\leq i}(x)$ we denote the set $\bigcup_{-2\leq j\leq i} E_j(x)$. Similarly, if $x \in \mathcal{E}$, then $\mathcal{E}_i(x)$ consists of all $y$ with $(x,y) \in \mathcal{E}_i$, and $\mathcal{E}_{\leq i}(x)$ denotes $\bigcup_{-2\leq j\leq i} \mathcal{E}_j(x)$.

**Proposition 4.8.** *Suppose $\mathfrak{g}$ is a Lie algebra defined over a field $\mathbb{F}$ of characteristic different from 2, generated by its set of extremal elements $E$. Suppose moreover that $\mathfrak{g}$ contains a pure extremal element. Then the extremal form $g$ is non-degenerate and $(\mathcal{E}, \mathcal{E}_2)$ is connected if and only if $\mathfrak{g}$ is simple.*

*Proof.* First assume that $\mathfrak{g}$ is simple. As $\mathfrak{g}$ contains a pure extremal element, the form $g$ is non-trivial and its radical $\mathrm{Rad}(g)$ is not $\mathfrak{g}$. This radical is an ideal of $\mathfrak{g}$, so $\mathfrak{g}$ being simple implies this radical to be trivial and $g$ to be non-degenerate.

Since sandwiches are contained in the radical of $g$, all elements of $E$ are pure. Now by Theorem 28 of [CI06] we find $(\mathcal{E}, \mathcal{E}_2)$ to be connected.

Now assume $g$ to be non-degenerate and $(\mathcal{E}, \mathcal{E}_2)$ to be connected. Let $\mathfrak{i}$ be a non-zero ideal of $\mathfrak{g}$. Then we can find an extremal element $x$ and $i \in \mathfrak{i}$ such that $g(x,i) \neq 0$. But then $[x,[x,i]] = g(x,i)x \in \mathfrak{i}$, which implies that $\langle x\rangle$ is in $\mathfrak{i}$.

Applying the same argument to $(x,y) \in \mathcal{E}_2$ instead of $(i,x)$ we find that all neighbours of $\langle x\rangle$ in the graph $(\mathcal{E}, \mathcal{E}_2)$ are in $\mathfrak{i}$. By connectedness of the latter graph we even find all elements of $\mathcal{E}$ in $\mathfrak{i}$, implying $\mathfrak{i} = \mathfrak{g}$, and proving $\mathfrak{g}$ to be simple. □

The case that $g$ is trivial and all extremal elements are sandwich elements is covered by the next result:

**Proposition 4.9** ([ZK90]). *If $\mathfrak{g}$ is a finite-dimensional Lie algebra generated by its sandwich elements, then $\mathfrak{g}$ is nilpotent.*

**Corollary 4.10.** *If $\mathfrak{g}$ is a simple finite-dimensional Lie algebra defined over a field of characteristic different from 2 generated by its extremal elements, then $\mathfrak{g}$ does not contain a sandwich element.*

*Proof.* If $\mathfrak{g}$ does contain a pure element, then Proposition 4.8 shows that $\mathfrak{g}$ does not contain sandwich elements. If $\mathfrak{g}$ does not contain pure elements, then $\mathfrak{g}$ is generated by its sandwich elements and, by Proposition 4.9, we find $\mathfrak{g}$ to be nilpotent. □

**Remark 4.11.** Originally we cited [Zel80], in which Proposition 4.9 has been proven under the extra condition that the characteristic of $\mathbb{F}$ is different from 2 and 3. As was pointed out to us by the anonymous referee, this actually holds more generally by [ZK90].

**Definition 4.12.** Let $\mathcal{E}$ be the set of extremal points of the Lie algebra $\mathfrak{g}$ and let $\mathcal{F}$ be the set of projective lines $\mathbb{F}x + \mathbb{F}y$ for $(x,y) \in \mathcal{E}_{-1}$. Hereby, we identify a 2-space with the set of 1-spaces it contains. Then the point-line space $(\mathcal{E}, \mathcal{F})$ is called the *extremal geometry* of $\mathfrak{g}$. We denote it by $\Gamma(\mathfrak{g})$, or in case it is clear what $\mathfrak{g}$ is, by $\Gamma$.

The *rank* of $\Gamma(\mathfrak{g})$ is the maximal dimension (as a linear subspace of $\mathfrak{g}$) of a subspace $X$ of $\Gamma(\mathfrak{g})$ in which any two points are collinear.

If $x$ and $y$ are two extremal points with $(x,y) \in \mathcal{E}_2$, then they generate an $\mathfrak{sl}_2$-subalgebra. This subalgebra is generated by any two of its extremal points. The intersection of $\mathcal{E}$ with this subalgebra is called the $\mathfrak{sl}_2$-*line* on $x$ and $y$.

**Example 4.13.** Let $V$ be a vector space over a field $\mathbb{F}$ of dimension at least 3 and $W^*$ a subspace of $V^*$ separating the points of the projective space of $V$. Let $\mathcal{E}$ be the set of extremal points in $\mathfrak{s}(V \otimes W^*)$, which is, as follows from [Fle15, Corollary 3.4.11], the set of elements $\langle v \otimes \phi \rangle$ where $v \in V$ and $\phi \in W^*$ with $\phi(v) = 0$. Then each extremal point $\langle v \otimes \phi \rangle$ corresponds to an incident point-hyperplane pair $(\langle v \rangle, \ker(\phi))$ of $\mathbb{P}(V)$. The extremal geometry with point set $\mathcal{E}$ is isomorphic to the geometry $\Gamma(V, W^*)$ described in Example 2.8. In particular, if $V$ has dimension $n+1 < \infty$, then the extremal geometry is a root shadow space of type $\mathrm{A}_{n,\{1,n\}}$.

**Theorem 4.14** ([CI06, Theorem 28])**.** *Suppose $\mathfrak{g}$ is a simple Lie algebra generated by its set of pure extremal elements $E$. If the set $\mathcal{E}_{-1}$ is not empty, then $(\mathcal{E}, \mathcal{F})$ is a root filtration space.*

The following lemma characterizes collinearity.

**Lemma 4.15** ([CI07, Lemma 27])**.** *Suppose $\mathfrak{g}$ is a generated by its set of pure extremal elements $E$. Let $x, y \in E$ be linearly independent and pure. Then $(x, y) \in E_{-1} \Leftrightarrow$ there are $\lambda, \mu \in \mathbb{F}^*$ with $\lambda x + \mu y \in E$.*

Combining Theorem 2.11 and the main result of Cuypers-Roberts-Shpectorov [CRS15, Theorem 1.1] and Cuypers-Fleischmann [CF18, Theorem 1.1] we obtain:

**Theorem 4.16.** *Suppose that the extremal geometry $\Gamma$ of a simple Lie algebra $\mathfrak{g}$, generated by its set of pure extremal elements, has finite rank.*

*If $\mathcal{E}_{-1} \neq \emptyset$, we find $\Gamma$ to be isomorphic to a root shadow space of type $\mathrm{A}_{n,\{1,n\}}$ $(n \geq 2)$, $\mathrm{BC}_{n,2}$ $(n \geq 3)$, $\mathrm{D}_{n,2}$ $(n \geq 4)$, $\mathrm{E}_{6,2}$, $\mathrm{E}_{7,1}$, $\mathrm{E}_{8,8}$, $\mathrm{F}_{4,1}$ or $\mathrm{G}_{2,2}$.*

*Furthermore, if both $\mathcal{E}_{-1}$ and $\mathcal{E}_0$ are non-empty, then $\mathfrak{g}$ is determined, up to isomorphism, by its extremal geometry $\Gamma$.*

The only case in which $\mathcal{E}_{-1} \neq \emptyset$, but $\mathcal{E}_0 = \emptyset$, is when the extremal geometry is a root shadow space of type $\mathrm{G}_{2,2}$.

Note that the labeling of the Coxeter diagrams follows [Bou68].

## 5. Lie algebras generated by extremal elements with no lines

Theorem 4.16 excludes Lie algebras generated by extremal elements in which $\mathcal{E}_{-1} = \emptyset$ (and hence also $\mathcal{E}_1$ is empty, by property (G)). In this section we will study the Lie algebras generated by extremal elements with $\mathcal{E}_{-1} = \emptyset$ somewhat closer.

From now on we assume $\mathfrak{g}$ to be a simple Lie algebra over a field $\mathbb{F}$ generated by its set $E$ of pure extremal elements such that $\mathcal{E} \times \mathcal{E} = \mathcal{E}_{-2} \cup \mathcal{E}_0 \cup \mathcal{E}_2$. Moreover, we assume the field $\mathbb{F}$ to be of characteristic different from 2. By Proposition 4.6 we get:

**Lemma 5.1.** *Any two extremal elements in $\mathcal{E}$ either commute or generate an $\mathfrak{sl}_2(\mathbb{F})$.*

Examples of Lie algebras generated by extremal elements that either commute or generate an $\mathfrak{sl}_2$ are the finitary symplectic and special unitary ones described in Example 4.3.

In [CF17, Theorem 1.1] we find the following characterization of the symplectic Lie algebras.

**Theorem 5.2.** *Let $\mathfrak{g}$ be a simple Lie algebra over the field $\mathbb{F}$ of characteristic $\neq 2$ and generated by its set of pure extremal elements. Assume the following:*

(a) *any two extremal elements $x$ and $y$ in $\mathfrak{g}$ either commute or generate an $\mathfrak{sl}_2(\mathbb{F})$;*
(b) *for any three extremal elements $x, y, z$ in $\mathfrak{g}$ with $[x, y] \neq 0$, there is an extremal element $u$ in the subalgebra $\langle x, y \rangle$ commuting with $z$.*

*Then $\mathfrak{g}$ is isomorphic to $\mathfrak{fsp}(V, f)$ for some non-degenerate symplectic space $(V, f)$.*

*Moreover, under this isomorphism the extremal elements in $\mathfrak{g}$ are mapped to rank 1 elements in $\mathfrak{fsp}(V, f)$.*

In the proof of the above theorem, as provided in [CF17], particular Lie subalgebras generated by three extremal elements play an important role. These subalgebras are also of importance in our setting. They are described in the following example.

**Example 5.3.** Let $(V, f)$ be a non-degenerate symplectic space containing three linearly independent $u, v, w$ with $f(u, v) = 1, f(v, w) = 1$ and $f(u, w) = 0$.

Then the subalgebra of $\mathfrak{s}_f$ generated by the three extremal elements $u \otimes f_u$, $v \otimes f_v$, and $w \otimes f_w$ is 6-dimensional and we denote it by $\mathfrak{sp}_3(\mathbb{F})$. It contains a 1-dimensional center spanned by $(u + w) \otimes f_{u+w}$. Notice that this center is non-trivial, as there is a vector $x \in V$ with $f(u + w, x) = 1$. Modulo this center, $\mathfrak{sp}_3(\mathbb{F})$ is isomorphic to an extension of $\mathfrak{sl}_2$ by its natural 2-dimensional module, which we denote by $\mathfrak{psp}_3(\mathbb{F})$.

The group $\langle \mathrm{Exp}(\langle u \otimes f_u\rangle), \mathrm{Exp}(\langle v \otimes f_v\rangle), \mathrm{Exp}(\langle w \otimes f_w\rangle)\rangle$ acts transitively on the non-central extremal points of $(\mathfrak{p})\mathfrak{sp}_3(\mathbb{F})$. Moreover, for any two non-central commuting extremal points $x$ and $y$ of $(\mathfrak{p})\mathfrak{sp}_3(\mathbb{F})$ we find that every extremal point $z$ commuting with $x$ also commutes with $y$. A maximal set of pairwise commuting non-central extremal points is called a *transversal* of $(\mathfrak{p})\mathfrak{sp}_3(\mathbb{F})$. The transversal on $\langle u \otimes f_u\rangle$ consists of all extremal points $\langle x \otimes f_x\rangle$, where $x \in \langle u, w\rangle$ but not in $\langle u - w\rangle$. In particular, it contains $|\mathbb{F}|$ extremal points. Notice that this transversal is contained in the linear span of $u \otimes f_u$, $w \otimes f_w$ and $[u \otimes f_u, [w \otimes f_w, z]]$ for any non-central $z$ of $(\mathfrak{p})\mathfrak{sp}_3(\mathbb{F})$ not in the transversal. It intersects each $\mathfrak{sl}_2$-line spanned by two non-central points of $(\mathfrak{p})\mathfrak{sp}_3(\mathbb{F})$ in exactly one extremal point.

We note that the (non-central) extremal points and $\mathfrak{sl}_2$-lines in $(\mathfrak{p})\mathfrak{sp}_3(\mathbb{F})$ form a dual affine plane, i.e. a projective plane from which a point and all the lines on this point are removed. If $\infty$ denotes the removed point, then the union of $\{\infty\}$ and any transversal is a removed line.

All these properties can be checked easily. For more details, see [CF17, Example 3.8].

Any triple $(x, y, z)$ of distinct extremal elements from $E$ with $[x, z] = 0$, and $[x, y] \neq 0 \neq [y, z]$ is called a *symplectic triple*. A symplectic triple of extremal points is a triple $(x, y, z)$ such that there exist $x_1 \in x$, $y_1 \in y$ and $z_1 \in z$ such that $(x_1, y_1, z_1)$ is a symplectic triple of extremal elements.

**Proposition 5.4** ([CF17, Proposition 4.2])**.** *A symplectic triple $(x, y, z)$ of extremal elements of the Lie algebra $\mathfrak{g}$ generates either a subalgebra isomorphic to $\mathfrak{sp}_3(\mathbb{F})$, in which case it is of dimension 6, or to $\mathfrak{psp}_3(\mathbb{F})$ of dimension 5.*

*Under this isomorphism $x$, $y$ and $z$ are mapped onto scalar multiples of pure tensors of $\mathfrak{sp}_3(\mathbb{F})$ or $\mathfrak{psp}_3(\mathbb{F})$, respectively.*

For each $x \in \mathcal{E}$ we denote by $x^\perp$ the set $\mathcal{E}_{\leq 0}(x)$.

**Lemma 5.5.** *Consider a point $z$ on the transversal on $x$ and $y$ in the subalgebra generated by a symplectic triple $(x, u, y)$. Then $x^\perp \cap y^\perp \subseteq z^\perp$.*

*Proof.* By Example 5.3, $x, y, z$ are contained in the linear span of $x, y$ and $[x, [y, u]]$. Now if $v \in x^\perp \cap y^\perp$, then by associativity of $g$ we have $g(v, [x, [y, u]]) = g([v, x], [y, u]) = g(0, [y, u]) = 0$. So $g(v, z) = 0$ and $v \in z^\perp$. □

**Lemma 5.6.** *Let $x, y, z \in E$ be linearly independent and such that $g(x, y)$, $g(x, z)$ and $g(y, z)$ are all non-zero. If there is no extremal element $u \in \langle x, y\rangle$ commuting with $z$, then there exists a quadratic extension $\hat{\mathbb{F}}$ of $\mathbb{F}$ such that $\mathfrak{g} \otimes_{\mathbb{F}} \hat{\mathbb{F}}$ contains extremal lines.*

*Proof.* Suppose $x, y$ and $z \in E$ such that $g(x, y)$, $g(x, z)$ and $g(y, z)$ are all non-zero. Moreover, assume that there is no extremal element $u \in \langle x, y\rangle$ commuting with $z$. Note that the elements $u_\lambda = g(x, y)x + \lambda^2 y + \lambda[x, y] = \exp(x, 1/\lambda)(\lambda^2 y)$, where $\lambda \in \mathbb{F}$, are extremal. Now

$$g(z, u_\lambda) = g(x, y)g(z, x) + \lambda^2 g(z, y) + \lambda g(z, [x, y])$$

either takes the value 0 for some value $\lambda = \lambda_1 \in \mathbb{F}$ and we find that $u_{\lambda_1}$ does not commute with $z$ but $g(z, u_{\lambda_1}) = 0$ (case (d) of Proposition 4.6), which implies $\mathcal{E}_{-1} \neq \emptyset$, a contradiction, or, as the characteristic of $\mathbb{F}$ is different from 2, we find two distinct elements $\lambda_1$ and $\lambda_2$ in a quadratic extension $\hat{\mathbb{F}}$ of $\mathbb{F}$ with $g(z, u_{\lambda_1}) = g(z, u_{\lambda_2}) = 0$.

Suppose we are in the latter case. Then inside $\mathfrak{g} \otimes_{\mathbb{F}} \hat{\mathbb{F}}$ we find the following. As $\langle u_{\lambda_1}, u_{\lambda_2} \rangle$ contains $\langle x, y \rangle$, the element $z$ cannot commute with both $u_{\lambda_1}$ and $u_{\lambda_2}$, which then implies that $z$ and at least one of $u_{\lambda_1}$ and $u_{\lambda_2}$ are in relation (d) of Proposition 4.6. But then $\mathfrak{g} \otimes_{\mathbb{F}} \hat{\mathbb{F}}$ contains extremal lines. □

Now we are in a position to prove the main result of this section.

**Theorem 5.7.** *Let $\mathfrak{g}$ be a simple Lie algebra over the field $\mathbb{F}$ of characteristic $\neq 2$ and generated by its set of pure extremal elements such that any two extremal elements $x$ and $y$ in $\mathfrak{g}$ either commute or generate an $\mathfrak{sl}_2(\mathbb{F})$.*

*Then either $\mathfrak{g}$ is isomorphic to the finitary symplectic Lie algebra $\mathfrak{fsp}(V, f)$ for some non-degenerate symplectic space $(V, f)$, or there is a quadratic extension $\hat{\mathbb{F}}$ of $\mathbb{F}$ such that the Lie algebra $\hat{\mathfrak{g}} := \mathfrak{g} \otimes_{\mathbb{F}} \hat{\mathbb{F}}$ is a simple Lie algebra generated by its extremal elements and with its extremal geometry $\Gamma := \Gamma(\hat{\mathfrak{g}})$ being a root filtration space.*

*In the latter case, the extremal points of the Lie algebra $\mathfrak{g}$ form the point set of the geometry $\Gamma_\sigma$, which is a non-degenerate thick polar space. Here $\sigma$ denotes the automorphism of $\Gamma$ induced by the unique field automorphism of order 2 of the extension $\hat{\mathbb{F}}$ of $\mathbb{F}$.*

*Proof.* By Theorem 5.2 we either have that $\mathfrak{g}$ is a finitary symplectic Lie algebra as in the theorem, or by Lemma 5.6 there is a quadratic field extension $\hat{\mathbb{F}}$ of $\mathbb{F}$ such that the Lie algebra $\hat{\mathfrak{g}} := \mathfrak{g} \otimes_{\mathbb{F}} \hat{\mathbb{F}}$ is generated by its extremal elements and its extremal geometry contains lines.

Let $\sigma$ denote the automorphism of $\Gamma$ induced by the unique field automorphism of order 2 of the extension $\hat{\mathbb{F}}$ of $\mathbb{F}$. Then $\sigma$ induces an automorphism of $\hat{\mathfrak{g}}$, also denoted by $\sigma$, that acts on $x \otimes \lambda$ by $(x \otimes \lambda)^\sigma = x \otimes \lambda^\sigma$. Note that $\sigma$ does not fix a line in $\hat{\mathfrak{g}}$, otherwise the extremal geometry in $\mathfrak{g}$ would contain lines (using Lemma 6.9). We identify $\mathfrak{g}$ and its elements with the subalgebra and elements of $\hat{\mathfrak{g}}$ fixed by $\sigma$.

We claim that $\hat{\mathfrak{g}}$ is simple. Let $\mathfrak{i}$ be a non-trivial ideal of $\hat{\mathfrak{g}}$, and suppose $0 \neq i \in \mathfrak{i}$. Then, as the extremal elements in $E$ linearly span $\mathfrak{g}$, we find that over $\hat{\mathbb{F}}$ they also span $\hat{\mathfrak{g}}$. So, we can express $i$ as $x_1 \otimes \lambda_1 + \cdots + x_k \otimes \lambda_k$ where $x_j \in E$ are linearly independent and $0 \neq \lambda_j \in \hat{\mathbb{F}}$. After replacing $i$ with a scalar multiple, we can assume $\lambda_1 = 1$. Then $i + i^\sigma$ is a non-trivial element in $\mathfrak{i} + \mathfrak{i}^\sigma$ fixed by $\sigma$. But then the subspace spanned by the elements in $\mathfrak{i} + \mathfrak{i}^\sigma$ fixed by $\sigma$ forms a non-trivial ideal of $\mathfrak{g}$, which by simplicity of $\mathfrak{g}$ equals $\mathfrak{g}$. This implies that $\mathfrak{i} + \mathfrak{i}^\sigma = \hat{\mathfrak{g}}$. Let $\hat{g}$ be the extremal form on $\hat{\mathfrak{g}}$. For $x \in E$ we can find (up to switching $\mathfrak{i}$ and $\mathfrak{i}^\sigma$) an element $y \in \mathfrak{i}$ with $\hat{g}(x, y) \neq 0$. As in the proof of Proposition 4.8 we then find $x$ and also $E$ to be in $\mathfrak{i}$. As $E$ generates $\hat{\mathfrak{g}}$ we find $\mathfrak{i} = \hat{\mathfrak{g}}$, proving simplicity of $\hat{\mathfrak{g}}$.

Clearly $\sigma$ also induces an automorphism (again denoted by $\sigma$) on the extremal geometry $\Gamma$ of $\hat{\mathfrak{g}}$. The set $\mathcal{E}$ is the point set of $\Gamma_\sigma$. The relation $\perp$ denotes the relation of being equal or collinear in $\Gamma_\sigma$. By Proposition 3.3 we know that $\Gamma_\sigma$ is a polar space. Moreover, as for each point $x \in \mathcal{E}$, the set $\mathcal{E}_2(x)$ is non-empty, no point is collinear to all and hence $\Gamma_\sigma$ is non-degenerate. But in a non-degenerate polar space a line on two collinear points $x, y$ equals $(x^\perp \cap y^\perp)^\perp$. Now as follows from Lemma 5.5, any transversal on $x, y$ is contained in the line on $x$ and $y$. As transversals contain $|\mathbb{F}| \geq 3$ points, the line on $x$ and $y$ is thick, provided there is at least one transversal on $x$ and $y$. This follows from the following observations.

Let $x = x_1, \ldots, x_n = y$ be a shortest path from $x$ to $y$ in the $\mathcal{E}_2$-graph. Such path exists by Proposition 4.8. If $n > 3$, then, according to Example 5.3 and Proposition 5.4, $x_1^\perp$ meets the $\mathfrak{sl}_2$-line on $x_2$ and $x_3$ in $x_3$, while $x_4^\perp$ meets this line in $x_2$. As such $\mathfrak{sl}_2$-line contains at least 3 points, we find a point on the line which is in relation $\mathcal{E}_2$ with both $x_1$ and $x_4$, contradicting that we have a shortest path from $x$ to $y$. In particular we find that there is a symplectic triple on $x$ and $y$ and thus a transversal. □

**Example 5.8.** Suppose $\mathfrak{g}$ is a Lie algebra as in the hypothesis of Theorem 5.7, but not isomorphic to a finitary symplectic Lie algebra. If moreover, the extremal geometry $\Gamma$ of $\hat{\mathfrak{g}} := \mathfrak{g} \otimes_{\mathbb{F}} \hat{\mathbb{F}}$ is isomorphic to $\Gamma(V, W^*)$ for some $\hat{\mathbb{F}}$-vector space $V$ and subspace $W^*$ of $V^*$ separating the points of the projective space of $V$, then it can be shown that the involution $\sigma$ induces a Hermitian polarity on $\mathbb{P}(V)$ and that $\Gamma_\sigma$ is isomorphic to the polar space of absolute points with respect to this polarity. See also Example 3.2. In this particular case we can identify the Lie algebra $\hat{\mathfrak{g}}$, up to a center, with the Lie algebra $\mathfrak{s}(V \otimes W^*)$ and $\mathfrak{g}$, up to a center, with the corresponding finitary unitary Lie algebra $\mathfrak{fsu}(V, h)$, where $h$ is a Hermitian form with associated involution $\sigma$. This has been worked out by Marc Oostendorp and the first author in [CO19].

## 6. The geometry of inner ideals

In this section we assume that $\mathfrak{g}$ is a simple Lie algebra over a field $\mathbb{F}$ generated by its set of pure extremal elements $E$. Its extremal geometry is denoted by $\Gamma$.

In the first few lemmas we allow the field $\mathbb{F}$ to be of characteristic 2.

**Definition 6.1.** An *inner ideal* of $\mathfrak{g}$ is a linear subspace $\mathfrak{i}$ such that

$$[\mathfrak{i}, [\mathfrak{i}, \mathfrak{g}]] \subseteq \mathfrak{i}.$$

Extremal points are inner ideals.

An *inner line ideal* is an inner ideal $\mathfrak{i}$, properly contained in $\mathfrak{g}$, and containing at least two extremal points, which is minimal with respect to this property.

**Definition 6.2.** The *inner ideal geometry* of a Lie algebra $\mathfrak{g}$ is the point-line geometry with $\mathcal{E}(\mathfrak{g})$ as point set, and as lines the intersections of $\mathcal{E}(\mathfrak{g})$ with the inner line ideals of $\mathfrak{g}$.

Notice that by the minimality of inner line ideals, this geometry is a partial linear space.

If $X \subset \mathcal{E}$ then we denote by $\mathfrak{g}_X$ the Lie subalgebra generated by $X$.

**Lemma 6.3.** *Suppose $\ell$ is a line of $\Gamma$. Then the 2-dimensional subalgebra $\mathfrak{g}_\ell$ is an inner line ideal of $\mathfrak{g}$.*

*Proof.* Let $x, y \in E$ be two linearly independent elements from $E$ such that $\langle x\rangle$ and $\langle y\rangle$ are points of $\ell$. Then $[x, y] = 0$, and $x + y \in E$. So, for all $z \in \mathfrak{g}$ we have

$$[x + y, [x + y, z]] = [x, [x, z]] + 2[x, [y, z]] + [y, [y, z]]$$

and thus, if the characteristic is not 2,

$$[x, [y, z]] \in \mathfrak{g}_\ell\,.$$

This shows $\mathfrak{g}_\ell$ to be an inner ideal, which clearly is an inner line ideal. Now assume that the characteristic is 2.[1] If $z \in \mathcal{E}_{\leq 0}(\langle x\rangle)$, then $[x, [y, z]] = [y, [x, z]] = 0$, by the Jacobi identity and $[x, y] = 0 = [x, z]$. If $z \in \mathcal{E}_1(\langle x\rangle)$, then $[x, [y, z]] = [y, [x, z]]$ as before. Since $[x, z] \in \mathcal{E}$ is a neighbour of $\langle x\rangle$, $\langle y\rangle$ and $[x, z]$ are at distance at most 2. If $[x, z] \in \mathcal{E}_{\leq 0}(\langle y\rangle)$ we get $[y, [x, z]] = 0$. If $\langle y\rangle$ and $[x, z]$ are special, then clearly $\langle x\rangle = [y, [x, z]]$. We conclude $[x, [y, z]] \leq \langle x\rangle$ for all $z \in \mathcal{E}_{\leq 1}(\langle x\rangle)$. Similarly, $[x, [y, z]] = [y, [x, z]]] \leq \langle y\rangle$ for all $z \in \mathcal{E}_{\leq 1}(\langle y\rangle)$. By Corollary 2.21 and [CI06, Lemma 4] we get that $[x, [y, \mathfrak{g}]] \leq \langle x, y\rangle = \mathfrak{g}_\ell$, which concludes the proof. □

**Lemma 6.4.** *Suppose $\Gamma$ contains lines. Then every inner line ideal equals $\mathfrak{g}_\ell$ for some line $\ell$ of $\Gamma$.*

*Proof.* Suppose $\mathfrak{i}$ is an inner line ideal containing two points $x, y \in \mathcal{E}$. If $(x, y) \in \mathcal{E}_{-1}$, then by minimality $\mathfrak{i} = \mathfrak{g}_\ell$, where $\ell$ is the line on $x$ and $y$.

If $(x, y) \in \mathcal{E}_1$, then let $z$ be an element in $\mathcal{E}_2(x)$ collinear with $y$. Then $[x, y] = [x, [[x, y], z]] = [x, [y, [z, x]]]$ is in $\mathfrak{i}$, using the Jacobi identity and $[y, z] = 0$. The inner line ideal $\mathfrak{g}_\ell$, where $\ell$ is the line through $x$ and $[x, y]$, is thus contained in $\mathfrak{i}$. This contradicts the minimality of $\mathfrak{i}$.

If $(x, y) \in \mathcal{E}_0$, we can find $z \in \mathcal{E}_1(x) \cap \mathcal{E}_{-1}(y)$ by Lemma 2.15. Let $S$ be the symplecton through $x$ and $y$. Then $\mathcal{E}_{-1}(z) \cap S$ is a line $\ell$ through $y$. Hence $[x, z]$ is the unique point of $\ell$ collinear with $x$. By [CI06, Lemma 4] there exists an extremal point $u$ such that $[u, y] = z$. Hence $\mathfrak{i}$ contains $[x, [y, u]] = [x, z] \in \ell$ and thus $\mathfrak{g}_l \leq \mathfrak{i}$, again contradicting the minimality.

Finally assume that $(x, y) \in \mathcal{E}_2$. Then there is a path $x, u, z, y$ from $x$ to $y$ and $[u, y] = z$ and $[x, z] = u$. So $u = [x, [y, u]]$ is in $\mathfrak{i}$, which again leads to a contradiction. □

[1] The proof is actually valid for all characteristics but the argument in the previous paragraph is more elegant if the characteristic is not 2.

**Lemma 6.5.** *Suppose $S$ is a symplecton of $\Gamma$. Then $\mathfrak{g}_S$ is an inner ideal. Moreover, if $\mathfrak{i}$ is an inner ideal containing two non-collinear points of $S$ then $\mathfrak{g}_S \subseteq \mathfrak{i}$.*

*Proof.* Suppose $x, y \in \mathcal{E}$ are points of $S$. If $x, y$ are collinear we find for each point $z \in \mathcal{E}$ that $[x, [y, z]] \subseteq \mathfrak{g}_S$. If $(x, y) \in \mathcal{E}_0$, then for each point $z \in \mathcal{E}$ we have, as $[x, y] = 0$,

$$[x, [y, z]] = -[y, [z, x]] = [y, [x, z]].$$

So, for all $z \in \mathcal{E}_{\leq 0}(x) \cup \mathcal{E}_{\leq 0}(y)$ we find

$$[x, [y, z]] = 0.$$

If $z \in \mathcal{E}_1(x)$, then $[x, z]$ is a point at distance 1 from $x$ and $\mathcal{E}_{-1}([x, z]) \cap S$ contains a line $\ell$. Then either $[y, [x, z]] = 0$, or $[x, z] \in \mathcal{E}_1(y)$ and $[y, [x, z]]$ is a point on $\ell$ and thus in $S$. By symmetry of the argument we now have for all $z \in \mathcal{E}_{\leq 1}(x) \cup \mathcal{E}_{\leq 1}(y)$ that

$$[x, [y, z]] \subseteq \mathfrak{g}_S\,.$$

As $\mathcal{E}_0(y)$ contains a point $z$ in $\mathcal{E}_2(x)$, by Lemma 2.15, and $\Gamma$ and hence also $\mathfrak{g}$ is generated by $\mathcal{E}_{\leq 1}(x) \cup \{z\}$, see Corollary 2.21, we find that

$$[x, [y, \mathfrak{g}]] \subseteq \mathfrak{g}_S\,.$$

Suppose $\mathfrak{i}$ is an inner ideal containing the two non-collinear points $x$ and $y$ of $S$. By Lemma 2.15 there is a point $u \in \mathcal{E}_0(x) \cap \mathcal{E}_2(y)$.

Then Lemma 2.18 implies that each point $z \in S$ collinear with $x$ and $y$ has a common neighbour $v$ with $x$ and $u$. But then $v \in \mathcal{E}_1(y)$. Note that by [CI06, Lemma 4] there exists $u \in \mathcal{E}$ collinear with $v$ and special with $x$, hence $[x, u] = v$. Together with $[y, v] = z$ this yields $z = [y, v] = [y, [x, u]] \in \mathfrak{i}$. So any common neighbour of $x$ and $y$ is in $\mathfrak{i}$. Repeating this argument, we find that all points of $S$ are in $\mathfrak{i}$ and $\mathfrak{i}$ contains $\mathfrak{g}_S$. □

**Corollary 6.6.** *Let $P$ be a convex subspace of $\Gamma$ such that for all $(x, y) \in P \times P$ we have $(x, y) \in \mathcal{E}_{\leq 0}$. Then $\mathfrak{g}_P$ is an inner ideal.*

From now on we assume the characteristic of $\mathbb{F}$ to be different from 2.

**Lemma 6.7.** *If $\mathfrak{i}$ is an inner ideal containing two points $x$ and $y$ with $(x, y) \in \mathcal{E}_2$, then $\mathfrak{i} = \mathfrak{g}$.*

*Proof.* Suppose $x, y$ are in $\mathfrak{i}$ with $(x, y) \in E_2$. We may assume $g_x(y) = 1$. Then $[x, [y, [y, x]] = 2[x, y]$ is contained in $\mathfrak{i}$. Now consider the 5-grading on $\mathfrak{g}$ as in Lemma 8.18. Then $\mathfrak{i}$ contains $[[x, y], [[x, y], \mathfrak{g}_1]] = \mathfrak{g}_1$ and similarly $\mathfrak{g}_{-1}$. Now note $[u, [v, [x, y]]] = -[u, v]$ for all $u \in \mathfrak{g}_{-1}$ and $v \in \mathfrak{g}_1$. Note that $\mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus ([\mathfrak{g}_{-1}, \mathfrak{g}_1] \oplus [\mathfrak{g}_{-2}, \mathfrak{g}_2]) \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$ is an ideal of $\mathfrak{g}$ and hence it equals $\mathfrak{g}$. We get $\mathfrak{i} = \mathfrak{g}$. □

In the remaining lemmas we consider the case that there are no lines in $\Gamma$. As we have seen in the previous section, we either find $\mathfrak{g}$ to be isomorphic to a finitary symplectic Lie algebra, or there is a quadratic field extension $\hat{\mathbb{F}}$ of $\mathbb{F}$ such that $\hat{\mathfrak{g}} := \mathfrak{g} \otimes \hat{\mathbb{F}}$ is a simple Lie algebra generated by its set of extremal elements $\hat{E}$ and the extremal geometry $\hat{\Gamma}$ contains lines.

We first handle the symplectic case.

**Lemma 6.8.** *Let $\mathfrak{g}$ be $\mathfrak{s}_f(V \otimes V^*)$, where $(V, f)$ is a non-degenerate symplectic space. If $\mathfrak{i}$ is an inner line ideal, then there is a singular 2-dimensional subspace $U$ of $V$ such that $\mathfrak{i}$ is the subspace spanned by the elements $v \otimes f_v$, with $0 \neq v \in U$.*

*Proof.* Suppose $\mathfrak{i}$ contains $v \otimes f_v$ and $w \otimes f_w$ with $v, w$ linearly independent. If $f(v, w) \neq 0$, then Lemma 6.7 applies. So assume that $f(v, w) = 0$. Consider an extremal element $u \otimes f_u$. Then

$$\begin{aligned}
[v \otimes f_v, [w \otimes f_w, u \otimes f_u]] &= f(w, u)[v \otimes f_v, w \otimes f_u + u \otimes f_w] \\
&= f(w, u) f(v, u)\big(w \otimes f_v + v \otimes f_w\big) \\
&= f(v, u) f(w, u)\big((v + w) \otimes f_{v+w} - v \otimes f_v - w \otimes f_w\big).
\end{aligned}$$

As the latter element is in $\langle t \otimes f_t \mid t \in \langle v, w \rangle \rangle$, the proposition follows immediately. (Note that there exist $u \in V$ such that $f(u, v) \neq 0 \neq f(u, w)$.) □

Denote by $\sigma$ the field automorphism of order 2 of $\hat{\mathbb{F}}$ fixing $\mathbb{F}$ as well as the induced automorphisms of $\hat{\mathfrak{g}}$ and its extremal geometry $\hat{\Gamma}$, such that $\Gamma = \hat{\Gamma}_\sigma$.

**Lemma 6.9.** *For any subspace $I$ of $\hat{\mathfrak{g}}$ fixed by $\sigma$ there exists a subspace $J$ of $\mathfrak{g}$ such that $J \otimes \hat{\mathbb{F}} = I$.*

*Proof.* Let $\mathcal{B}$ be a basis of $\mathfrak{g}$. Consider $0 \neq a \in I$ such that $a^\sigma$ is an $\hat{\mathbb{F}}$-multiple of $a$. We can write $a$ uniquely as $b_1 \otimes \lambda_1 + \cdots + b_n \otimes \lambda_n$, for some $\lambda_1, \ldots, \lambda_n \in \hat{\mathbb{F}}$ and $b_1, \ldots, b_n \in \mathcal{B}$, with all $b_i$ different from each other. We may assume $\lambda_1 \neq 0$. By considering $a' = \lambda_1^{-1} a$ instead of $a$ and using that fact that $a'^\sigma$ should be a multiple of $a'$, we get $a'^\sigma = a'$ and $a' = x \otimes 1$, for some $x \in \mathfrak{g}$.

Consider $a \in I$ such that $a^\sigma$ is not a multiple of $a$. Then $v = a + a^\sigma$ is non-zero and fixed by $\sigma$. Now consider an element $\lambda \in \hat{\mathbb{F}}$ not fixed by $\sigma$. Set $w = \lambda a + \lambda^\sigma a^\sigma$, then $w^\sigma = w$. Moreover $w$ is not a multiple of $v$ by construction. Hence the subspace $\langle a, a^\sigma \rangle$ contains linearly independent elements $x_1 \otimes 1$ and $x_2 \otimes 1$, for some $x_1, x_2 \in \mathfrak{g}$.

Let $\mathcal{B}_I$ be a basis of $I$. If for $b \in \mathcal{B}_I$ we find $b^\sigma$ to be a multiple of itself, we can replace it by a $b' \in \langle b \rangle$ which is fixed by $\sigma$ by the first paragraph. If it is not a multiple of $b$ then we can find an element of $b'$ in $\langle b, b^\sigma \rangle$ which is fixed by $\sigma$ and is linearly independent of the other basis elements by the second paragraph. Applying this procedure for any basis element, we see that we can assume that any $b$ is fixed by $\sigma$, and the $\mathbb{F}$-subspace $J = \langle b \mid b \in \mathcal{B}_I \rangle$ satisfies $I = J \otimes \hat{\mathbb{F}}$. □

**Corollary 6.10.** *If $\mathfrak{i}$ is an inner line ideal of $\mathfrak{g}$, then $\mathfrak{i}$ meets $\mathcal{E}$ in the set of all the points of a line of $\hat{\Gamma}_\sigma$.*

*Proof.* Let $x$ and $y$ be two points in $\mathfrak{i}$. If $(x, y) \in \mathcal{E}_0$, then the minimal inner ideal of $\hat{\mathfrak{g}}$ containing $x, y$ is the subspace $\mathfrak{j}$ of $\hat{\mathfrak{g}}$ spanned by all the extremal points of $\hat{\mathfrak{g}}$ inside the symplecton on $x$ and $y$, by Lemma 6.5. Note that $\mathfrak{j}$ is fixed by $\sigma$. But then Lemma 6.9 implies that $\mathfrak{i}$ meets $\mathcal{E}$ in the points of $S$ fixed by $\sigma$.

If $(x, y) \in \mathcal{E}_2$, then by Lemma 6.7 we find that the minimal inner ideal of $\hat{\mathfrak{g}}$ containing $x$ and $y$ is $\hat{\mathfrak{g}}$, which implies $\mathfrak{i} = \mathfrak{g}$, contradicting that $\mathfrak{i}$ is a proper inner ideal of $\mathfrak{g}$. □

Combining all results from the above we find the following.

**Theorem 6.11.** *Suppose $\mathfrak{g}$ is a simple Lie algebra generated by its pure extremal elements over a field of characteristic not 2.*

*Then we have one of the following:*

(a) *The extremal geometry $\Gamma$ of $\mathfrak{g}$ contains lines and equals the inner ideal geometry, which is then a root filtration space.*
(b) *The extremal geometry $\Gamma$ of $\mathfrak{g}$ contains no lines, but $\mathfrak{g}$ contains two commuting, but linearly independent extremal elements; the inner ideal geometry is a non-degenerate polar space of rank at least 2.*
(c) *The Lie algebra $\mathfrak{g}$ does not contain commuting, but linearly independent extremal elements, and the inner ideal geometry has no lines.*

*Proof.* If the extremal geometry $\Gamma$ of $\mathfrak{g}$ does contain lines, then Lemmas 6.3 and 6.4 and Theorem 4.14 imply that its inner ideal geometry equals $\Gamma$, which is then a root filtration space.

So, we can assume that $\Gamma$ contains no lines. Then, by Theorem 5.7, either $\mathfrak{g}$ is isomorphic to a symplectic Lie algebra or there is a quadratic field extension $\hat{\mathbb{F}}$ of $\mathbb{F}$ such that the extremal geometry $\hat{\Gamma}$ of $\hat{\mathfrak{g}} := \mathfrak{g} \otimes_{\mathbb{F}} \hat{\mathbb{F}}$ does have lines.

If $\mathfrak{g}$ is isomorphic to $\mathfrak{fsp}(V, f)$ for some non-degenerate symplectic space $(V, f)$ of dimension at least 2, then Lemma 6.8 shows that the inner ideal geometry is isomorphic to the non-degenerate polar space defined by $(V, f)$. If the dimension is 2, then $\mathfrak{g} \simeq \mathfrak{sl}_2$ has no inner line ideals. If the dimension is at least 3 (and hence at least 4), then the inner ideal geometry is isomorphic to the symplectic polar space.

If $\mathfrak{g}$ is not a symplectic Lie algebra, then the point sets of $\Gamma$ and $\hat{\Gamma}_\sigma$ can be identified, where $\sigma$ is the automorphism induced on $\hat{\Gamma}$ by the unique field automorphism of $\hat{\mathbb{F}}$ of order 2 fixing $\mathbb{F}$. Applying Corollary 6.10 we find that the inner ideal geometry is isomorphic with $\hat{\Gamma}_\sigma$, which is a non-degenerate polar space by Theorem 5.7 containing lines if and only if $\mathfrak{g}$ contains two distinct but commuting extremal points. □

## 7. Transitivity and Moufang property

In this section we show that the automorphism group works transitive on specific pairs of extremal elements and that the inner ideal geometries with lines are root shadow spaces of spherical Moufang buildings.

**Proposition 7.1.** *Let $\mathfrak{g}$ be simple Lie algebra generated by its set $E$ of pure extremal elements and defined over a field of characteristic not 2.*

*Suppose $\mathcal{E}_{-1} \cup \mathcal{E}_0 \cup \mathcal{E}_1 \neq \emptyset$. Then $\mathrm{Aut}(\mathfrak{g})$ is transitive on the pairs $(x, y) \in \mathcal{E}_2$.*

*Proof.* We first prove the automorphism group of $\mathfrak{g}$ to be transitive on $\mathcal{E}$.

Let $x, y \in E$ with $g(x, y) = g(y, x) \neq 0$. Then, after replacing $y$ by a scalar multiple, we can assume $g(x, y) = g(y, x) = 1$. But then $\exp(x, 1)y = y + [x, y] + x = \exp(y, -1)x$. So, $\exp(y, 1)\exp(x, 1)$ maps $y$ to $x$. This implies that for each $(x, y) \in \mathcal{E}_2$ there is an automorphism of $\mathfrak{g}$ that maps $x$ to $y$.

As $\mathfrak{g}$ is simple, we find the graph $(\mathcal{E}, \mathcal{E}_2)$ to be connected by Proposition 4.8, and hence by the above the automorphism group of $\mathfrak{g}$ to be transitive on $\mathcal{E}$.

Next we will show that the automorphism group of $\mathfrak{g}$ is transitive on the pairs $(x, y) \in \mathcal{E}_2$. By the above we can assume that $x$ is fixed and we have to show that the stabilizer of $x$ is transitive on the points in $\mathcal{E}_2(x)$.

First we assume that $\Gamma$ contains lines. Let $x \in \mathcal{E}$ and $y, z \in \mathcal{E}_2(x)$. Assume that $y$ and $z$ are collinear. Then we can find a point $u \in \mathcal{E}_1(x)$ on the line through $y$ and $z$. The group $\mathrm{Exp}([x, u])$ stabilizes $x$ and is transitive on the points on $yz$ different from $u$. In particular, there is an element $g \in \mathrm{Exp}([x, u])$ fixing $x$ and mapping $y$ to $z$. If $\Gamma$ contains symplecta, then by connectedness of $\mathcal{E}_2(x)$, see Lemma 2.20, we find that the stabilizer of $x$ in the group $\mathrm{Aut}(\mathfrak{g})$ is transitive on $\mathcal{E}_2(x)$.

If $\Gamma$ contains lines but no symplecta, then it is a generalized hexagon. Now assume $(y, z) \in \mathcal{E}_1$ (and still $y, z \in \mathcal{E}_2(x)$), then either $[y, z] \in \mathcal{E}_2(x)$ or $[y, z] \in \mathcal{E}_1(x)$. By the above there are elements in the stabilizer of $x$ mapping $y$ to $[y, z]$ and $[y, z]$ to $z$ in the former case. But then there is an automorphism of $\mathfrak{g}$ fixing $x$ and mapping $y$ to $z$. So, assume $[y, z] \in \mathcal{E}_1(x)$. Let $u$ be a second point collinear with $y$ and also in $\mathcal{E}_1(x)$. Then the point $[x, u]$ is in $\mathcal{E}_2([y, z])$, and there is a point $v$ on $z[y, z]$ in $\mathcal{E}_1([x, u])$. Let $w = [v, [x, u]]$. In $\mathrm{Exp}(u)$ we find an element $g_1$ that maps a point $y_1$ on the line $y[y, z]$ but different from $[y, z]$ to $[y, z]$. This element maps $x$ to an element $x_1$ on $x[x, u]$ different from $[x, u]$. In $\mathrm{Exp}(w)$ there is an element $g_2$ that maps $x_1$ back to $x$. This element maps $[y, z]$ to an element $z_1$ on the line $z[y, z]$ different from $[y, z]$. So, $g_1 g_2$ stabilizes $x$ and maps $y_1$ to $z_1$. And by the above $y_1$ is in the orbit of $y$ under the stabilizer of $x$, while $z$ and $z_1$ are also in a single orbit. But then $y$ and $z$ are in a single orbit under this stabilizer.

If $z \in \mathcal{E}_2(y)$, then let $y, u, v, z$ be a path of collinear points from $y$ to $z$. Then by the above we can assume that $u, v \in \mathcal{E}_1(x)$. Using Lemma 2.16, this implies that there is a point $w$ on the line $uv$ collinear with $x$. But then we find that there is an element $g$ in $\mathrm{Exp}(x)$ mapping $v$ to $u$. Hence $z^g$ is in $\mathcal{E}_{\leq 1}(y)$ and by the above we find $y$ and $z$ to be in a single orbit under the stabilizer of $x$. This finishes the case that $\Gamma$ contains lines.

So, now assume that $\Gamma$ contains no lines. Then, as we saw in Theorem 6.11 the minimal inner ideals induce a non-degenerate polar space on $\Gamma$. Let $y, z \in \mathcal{E}_2(x)$. If $z \in \mathcal{E}_0(y)$, then the subalgebra of $\mathfrak{g}$ generated by the symplectic triple $x, y, z$ is isomorphic to $(\mathfrak{p})\mathfrak{sp}_3(\mathbb{F})$ by Proposition 5.4. We can find an element in the group generated by $\mathrm{Exp}(x), \mathrm{Exp}(y)$ and $\mathrm{Exp}(z)$ fixing $x$ and mapping $y$ to $z$. Indeed, let $g$ be a non-trivial element of $\mathrm{Exp}(x)$ mapping $y$ to $y' \neq y$. Then the subalgebra generated by $y'$ and $z$ is isomorphic to $(\mathfrak{p})\mathfrak{sl}_2$, since if $[y', z] = 0$ we find $[y', y] = 0$, by Example 5.3, which yields a contradiction. In this subalgebra we find an extremal point $v$ different from but commuting with $x$. This point is in the intersection of the transversal on $x$ with the $\mathfrak{sl}_2$-line on $z$ and $y'$. In $\mathrm{Exp}(v)$ there is an element $g_1$ mapping $y'$ to $z$. So, $gg_1$ fixes $x$ but maps $y$ to $z$.

Now using the connectedness of the commuting graph of $\mathcal{E}_2(x)$, which is a hyperplane complement in the polar space induced by the minimal inner ideals, see Lemma 2.19, we find the stabilizer of $x$ to be transitive on $\mathcal{E}_2(x)$. ☐

**Theorem 7.2.** *Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra over a field $\mathbb{F}$ of characteristic not 2, generated by its set of extremal elements.*

*If its inner ideal geometry $\Gamma$ contains lines, then it is a root shadow space of a spherical Moufang building.*

*Proof.* By Theorem 6.11 we find $\Gamma(\mathfrak{g})$ to be a root shadow space of a building of rank at least 2, provided that $\Gamma(\mathfrak{g})$ contains lines. By Tits [Tit74] spherical buildings of rank at least 3 are Moufang. So we only have to be concerned with the case that the extremal geometry $\Gamma(\mathfrak{g})$ is a generalized quadrangle or generalized hexagon.

The subgroups $\mathrm{Exp}(x)$, with $x \in \mathcal{E}$ form a class of abstract transvection groups in the subgroups of $\mathrm{Aut}(\mathfrak{g})$ that they generate. See [CI06, Tim01]. Now Timmesfeld proves that in both cases under consideration this subgroup acts highly transitive on $\Gamma(\mathfrak{g})$. In particular, the arguments of [Tim01, pp. 198–199] apply to show that $\Gamma(\mathfrak{g})$ is Moufang if it is a generalized hexagon, and [Tim01, pp. 155] if $\Gamma(\mathfrak{g})$ is a generalized quadrangle. ☐

## 8. Extremal geometries and skew-dimension one structurable algebras

In this section we describe how one can associate a (skew-dimension one) structurable algebra to a simple Lie algebra generated by its extremal elements. We start with some background material on structurable algebras. (This material is based on [BDMS19, Chapter 2] and we refer to this article for a more detailed exposition.) In this section we assume the characteristic of the field to be distinct from 2 and 3. All algebras are assumed to be finite-dimensional.

**Definition 8.1.** A *structurable algebra* over a field $\mathbb{F}$ of characteristic not 2 or 3 is a finite-dimensional, unital $\mathbb{F}$-algebra $\mathcal{A}$ with involution $^-$, i.e. $\overline{x\,y} = \overline{y}\,\overline{x}$, such that

$$[V_{x,y}, V_{z,w}] = V_{V_{x,y}(z),w} - V_{z,V_{y,x}(w)},$$

for all $x, y, z, w \in \mathcal{A}$, where the left hand side denotes the Lie bracket of the two operators and with $V_{x,y}$ for $x, y \in \mathcal{A}$ the linear operator defined by

$$V_{x,y}(z) := (x\overline{y})z + (z\overline{y})x - (z\overline{x})y$$

for all $z \in \mathcal{A}$. We define the linear operator $U_a$ by $U_a(b) = V_{a,b}(a)$, for all $a, b \in \mathcal{A}$.

**Definition 8.2.** Let $\mathcal{A}$ be a structurable algebra; then $\mathcal{A} = \mathcal{H} \oplus \mathcal{S}$, with

$$\mathcal{H} = \{h \in \mathcal{A} \mid \overline{h} = h\} \text{ and } \mathcal{S} = \{s \in \mathcal{A} \mid \overline{s} = -s\}.$$

The elements of $\mathcal{H}$ are called *Hermitian elements*, the elements of $\mathcal{S}$ are called *skew elements*. The dimension of $\mathcal{S}$ is called the *skew-dimension* of $\mathcal{A}$.

The centre and ideals of a structurable algebra are defined as in [BDMS19, 2.1.3].

**Definition 8.3.** Let $\mathcal{A}$ be a structurable algebra. An element $u \in \mathcal{A}$ is said to be *conjugate invertible* if there exists an element $\hat{u} \in \mathcal{A}$ such that $V_{u,\hat{u}} = \mathrm{id}$. If $u$ is conjugate invertible, then the element $\hat{u}$ is uniquely defined. If each element in $\mathcal{A} \setminus \{0\}$ is conjugate invertible, $\mathcal{A}$ is called a *structurable division algebra.*

**Example 8.4.** Some examples of structurable algebras:

*(i)* The Jordan algebras are precisely the structurable algebras with the identity as involution.
*(ii)* If $C_i$ is a composition algebra over $\mathbb{F}$ with involution $\sigma_i$, for $i = 1, 2$, then the $\mathbb{F}$-algebra $C_1 \otimes_{\mathbb{F}} C_2$, together with the involution

$$- = \sigma := \sigma_1 \otimes \sigma_2$$

is a structurable algebra.

A *skew-dimension one structurable algebra* is a structurable algebra with $\dim(\mathcal{S}) = 1$. Since the class of skew-dimension one structurable algebras plays a vital role in this section, we discuss this class in some more detail.

**Lemma 8.5** ([AF84, Lemma 2.1])**.** *Let $\mathcal{A}$ be a simple skew-dimension one structurable algebra. Fix a non-zero element $s_0 \in \mathcal{S}$, so $\mathcal{S} = \langle s_0 \rangle$. Then $s_0^2 = \mu 1$ for some $\mu \in \mathbb{F}^*$, $s_0(s_0 x) = (x s_0) s_0 = \mu x$ for all $x \in \mathcal{A}$ and $\mathcal{A}$ is central simple.*

We will now discuss an important class of structurable algebras of skew-dimension one, which are called structurable matrix algebras.

**Definition 8.6.** Let $J$ be a Jordan algebra over a field $\mathbb{F}$, let $T : J \times J \to \mathbb{F}$ be a symmetric bilinear form, let $\times : J \times J \to J$ be a symmetric bilinear map, and let $N : J \to \mathbb{F}$ be a cubic form such that one of the following holds:

- $J$ is a cubic Jordan algebra with a non-degenerate admissible form $N$, with basepoint 1, trace form $T$, and Freudenthal cross product $\times$; see, for instance, [KMRT98, §38].
- $J$ is a Jordan algebra of a non-degenerate quadratic form $q$ with basepoint 1, and $T$ is the linearization of $q$. In this case, $N$ and $\times$ are the zero maps.
- $J = 0$, and the maps $N$, $T$ and $\times$ are the zero maps. In this case, $J$ is not unital.

Fix a constant $\eta \in \mathbb{F}^*$. We now define the *structurable matrix algebra* $M(J, \eta)$ as follows. Let

$$\mathcal{A} = \left\{ \begin{pmatrix} k_1 & j_1 \\ j_2 & k_2 \end{pmatrix} \mid k_1, k_2 \in \mathbb{F}, j_1, j_2 \in J \right\},$$

and define the multiplication and the involution by the formulae

$$\begin{pmatrix} k_1 & j_1 \\ j_2 & k_2 \end{pmatrix} \begin{pmatrix} k_1' & j_1' \\ j_2' & k_2' \end{pmatrix} = \begin{pmatrix} k_1 k_1' + \eta T(j_1, j_2') & k_1 j_1' + k_2' j_1 + \eta (j_2 \times j_2') \\ k_1' j_2 + k_2 j_2' + j_1 \times j_1' & k_2 k_2' + \eta T(j_2, j_1') \end{pmatrix},$$

$$\overline{\begin{pmatrix} k_1 & j_1 \\ j_2 & k_2 \end{pmatrix}} = \begin{pmatrix} k_2 & j_1 \\ j_2 & k_1 \end{pmatrix},$$

for all $k_1, k_2, k_1', k_2' \in \mathbb{F}$ and $j_1, j_2, j_1', j_2' \in J$. It is shown in [All78, Section 8.v] and [AF84, Section 4] that $M(J, \eta)$ is a central simple structurable algebra.

The following proposition relates all simple skew-dimension one structurable algebras to these structurable matrix algebras.

**Proposition 8.7** ([All90, Theorem 1.13])**.** *Let $\mathcal{A}$ be a simple structurable algebra of skew-dimension one with $s_0^2 = \mu 1$. Then $\mathcal{A}$ is isomorphic to a structurable matrix algebra $M(J, \eta)$ if and only if $\mu$ is a square in $\mathbb{F}^*$.*

**Corollary 8.8.** *Let $\mathcal{A}$ be a simple structurable algebra of skew-dimension one. Then there exists a field extension $\hat{\mathbb{F}}/\mathbb{F}$ of degree at most 2 such that $\mathcal{A} \otimes_{\mathbb{F}} \hat{\mathbb{F}}$ is isomorphic to a structurable matrix algebra over $\hat{\mathbb{F}}$.*

**Definition 8.9.** A *Kantor pair* is a pair of $\mathbb{F}$-vector spaces $P = (P^-, P^+)$ together with two trilinear maps

$$\{\cdot, \cdot, \cdot\}^\sigma : P^\sigma \times P^{-\sigma} \times P^\sigma \to P^\sigma, \ \sigma \in \{+, -\},$$

such that

$$[V_{x,y}, V_{a,b}] = V_{V_{x,y}a,b} - V_{a,V_{y,x}b},$$
$$K_{x,a} V_{y,z} + V_{z,y} K_{x,a} = K_{K_{x,a}y,z},$$

with $x, a, z \in P^\sigma$ and $y, b \in P^{-\sigma}$. The involved operators are defined as follows: $V_{x,y} z = \{x, y, z\}^\sigma$ and $K_{a,b} c = \{a, c, b\}^\sigma - \{b, c, a\}^\sigma$ for all $x, z, a, b \in P^\sigma$ and $y, c \in P^{-\sigma}$.

Kantor pairs are a generalization of structurable algebras: if $\mathcal{A}_+$ and $\mathcal{A}_-$ are two isomorphic copies of a structurable algebra $\mathcal{A}$, then $(\mathcal{A}_-, \mathcal{A}_+)$ is a Kantor pair, with $\{x, y, z\} = 2V_{x,y} z$, for $x, z \in \mathcal{A}_\sigma$ and $y \in \mathcal{A}_{-\sigma}$.

In order to define the Lie algebra associated to a structurable algebra $\mathcal{A}$ we need some more concepts related to structurable algebras.

**Definition 8.10.** It follows from the definition of a structurable algebra that the linear span $\mathrm{Span}\{V_{x,y} \mid x, y \in \mathcal{A}\}$ is a Lie subalgebra of $\mathrm{End}(\mathcal{A})$. We denote that Lie algebra by $\mathrm{Instr}(\mathcal{A})$.

**Definition 8.11.** The following map is called *the skewer* :

$$\psi : \mathcal{A} \times \mathcal{A} \to \mathcal{S} : (x, y) \mapsto x\overline{y} - y\overline{x}.$$

**Definition 8.12.** For each $A \in \mathrm{End}(\mathcal{A})$, we define new $\mathbb{F}$-linear maps

$$\begin{aligned} A^{\epsilon} &= A - L_{A(1)+\overline{A(1)}}, \\ A^{\delta} &= A + R_{\overline{A(1)}}, \end{aligned}$$

where $L_x$ and $R_x$ denote the left and right multiplication by an element $x \in \mathcal{A}$, respectively.

**Definition 8.13.** Consider two copies $\mathcal{A}_+$ and $\mathcal{A}_-$ of $\mathcal{A}$ with corresponding isomorphisms $\mathcal{A} \to \mathcal{A}_+ : a \mapsto a_+$ and $\mathcal{A} \to \mathcal{A}_- : a \mapsto a_-$, and let $\mathcal{S}_+ \subset A_+$ and $\mathcal{S}_- \subset A_-$ be the corresponding subspaces of skew-elements. Define the vector space

$$K(\mathcal{A}) = \mathcal{S}_- \oplus \mathcal{A}_- \oplus \mathrm{Instr}(\mathcal{A}) \oplus \mathcal{A}_+ \oplus \mathcal{S}_+.$$

As in [All79, §3], we define a Lie algebra on $K(\mathcal{A})$ as the unique extension of the Lie algebra on $\mathrm{Instr}(\mathcal{A})$ satisfying:

$$\begin{aligned} &[V, a_+] := (Va)_+ \in \mathcal{A}_+, && [V, a_-] := (V^{\epsilon}a)_- \in \mathcal{A}_- \\ &[V, s_+] := (V^{\delta}s)_+ \in \mathcal{S}_+ && [V, s_-] := (V^{\epsilon\delta}s)_- \in \mathcal{S}_- \\[1ex] &[s_+, a_+] := 0, && [s_-, a_-] := 0, \\ &[s_+, a_-] := (sa)_+ \in \mathcal{A}_+, && [s_-, a_+] := (sa)_- \in \mathcal{A}_-, \\[1ex] &[a_+, b_-] := V_{a,b} \in \mathrm{Instr}(\mathcal{A}), && \\ &[a_+, b_+] := \psi(a, b)_+ \in \mathcal{S}_+, && [a_-, b_-] := \psi(a, b)_- \in \mathcal{S}_- \\[1ex] &[s_+, t_+] := 0, && [s_-, t_-] := 0, \\ &[s_+, t_-] := L_s L_t \in \mathrm{Instr}(\mathcal{A}). && \end{aligned}$$

for all $a, b \in \mathcal{A}$, $s, t \in \mathcal{S}$, $V \in \mathrm{Instr}(\mathcal{A})$.

**Proposition 8.14.** *If $\mathcal{A}$ is a simple skew-dimension one structurable algebra, then $K(\mathcal{A})$ is a simple non-degenerate Lie algebra generated by its extremal elements.*

*Proof.* By [All79, Corollary 6], $K(\mathcal{A})$ is a simple Lie algebra. Clearly $s_+$ is an extremal element which is not a sandwich by Lemma 8.5, with $0 \neq s \in \mathcal{S}$ arbitrary. Hence [CIR08, Theorem 1.1] and Corollary 4.10 imply that $K(\mathcal{A})$ is generated by its extremal elements and is non-degenerate, unless $K(\mathcal{A})$ is isomorphic to $W_{1,1}(5)$ and the characteristic of the field equals 5. The Witt algebra $W_{1,1}(5)$ is a 5-dimensional Lie algebra. The dimension of $K(\mathcal{A})$ equals $2\dim(\mathcal{A}) + 2\dim(\mathcal{S}) + \dim(\mathrm{Instr}(\mathcal{A}))$. Since $1 = \dim(\mathcal{S}) < \dim(\mathcal{A})$, the Lie algebra $K(\mathcal{A})$ can never have dimension equal to 5. □

We now focus on the converse of the above proposition. We will show in a series of steps that any simple non-symplectic Lie algebra which is generated by its extremal elements can be obtained by the above construction.

From the definition of the Lie bracket we clearly see that the Lie algebra $K(\mathcal{A})$ has a 5-grading given by $K(\mathcal{A})_j = 0$ for all $|j| > 2$ and

$$K(\mathcal{A})_{-2} = \mathcal{S}_-, \quad K(\mathcal{A})_{-1} = \mathcal{A}_-, \quad K(\mathcal{A})_0 = \mathrm{Instr}(\mathcal{A}), \quad K(\mathcal{A})_1 = \mathcal{A}_+, \quad K(\mathcal{A})_2 = \mathcal{S}_+.$$

Even more generally, for any Kantor pair $(P^-, P^+)$ there exists a 5-graded Lie algebra denoted by $\mathrm{TKK}(P^-, P^+)$ and with 1-component isomorphic with $P^+$ and $(-1)$-component isomorphic with $P^-$. Since the construction itself is not relevant here, we refer to [AF99, §4] for more details.

**Definition 8.15.** If $\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$ is a 5-grading of a Lie algebra $\mathfrak{g}$, the grading derivation is the derivation $\zeta$ of $\mathfrak{g}$ given by

$$\zeta(x) = i \cdot x,$$

for all $x \in \mathfrak{g}_i$, with $i$ between $-2$ and 2. If $\mathfrak{g}$ contains an element $\zeta$ such that $\mathrm{ad}_\zeta$ is the grading derivation we call $\zeta$ the grading derivation of $\mathfrak{g}$ by abuse of language.

**Lemma 8.16.** *Let $\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$ be a 5-grading of a simple Lie algebra $\mathfrak{g}$. Then $(\mathfrak{g}_{-1}, \mathfrak{g}_1)$ is a Kantor pair with products defined by*

$$\{x, y, z\}^\sigma = [[x, y], z],$$

*with $x, z \in \mathfrak{g}_{\sigma 1}$, $y \in \mathfrak{g}_{-\sigma 1}$ and $\sigma = \pm$. If $(\mathfrak{g}_{-1}, \mathfrak{g}_1) \neq 0$, $\mathfrak{g}$ is graded-isomorphic to $\mathrm{TKK}(\mathfrak{g}_{-1}, \mathfrak{g}_1)$ and $\mathfrak{g}_0 = [\mathfrak{g}_{-1}, \mathfrak{g}_1]$.*

*Proof.* The first statement is [AFS17, 4.2, page 728]. The second statement follows from Lemma 4.4, Definition 4.5 and Corollary 4.17 combined with Proposition 4.19 of *loc. cit.*. □

**Lemma 8.17** ([Sta20, Lemma 4.13])**.** *Let $\mathfrak{g} = \mathrm{TKK}(\mathfrak{g}_{-1}, \mathfrak{g}_1)$ be the 5-graded Lie algebra associated with the Kantor pair $(\mathfrak{g}_{-1}, \mathfrak{g}_1)$. Then $(\mathfrak{g}_{-1}, \mathfrak{g}_1)$ is the Kantor pair associated with a structurable algebra $\mathcal{A}$ if and only if there exist $u \in \mathfrak{g}_{-1}$ and $v \in \mathfrak{g}_1$ such that $[u, v]$ is the grading derivation of $\mathfrak{g}$. In that case $\mathfrak{g}$ is graded-isomorphic to $K(\mathcal{A})$.*

Now we are able to apply these results to our setting of extremal geometries.

**Lemma 8.18** ([CI06, Proposition 22])**.** *Let $\mathfrak{g}$ be a Lie algebra generated by extremal elements. Suppose that there exist extremal elements $x, y \in \mathfrak{g}$ such that $g_x(y) = 1$. Then $\mathfrak{g}$ has a 5-grading*

$$\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2,$$

*with $\mathfrak{g}_{-2} = \langle x \rangle$ and $\mathfrak{g}_2 = \langle y \rangle$. Moreover, $\mathfrak{g}_i$ is contained in the $i$-eigenspace of $\mathrm{ad}_{[x,y]}$ and $\mathrm{ad}_x$ defines a linear isomorphism from $\mathfrak{g}_1$ tot $\mathfrak{g}_{-1}$ with inverse $-\mathrm{ad}_y$.*

**Lemma 8.19.** *Let $\mathfrak{g}$ be a simple Lie algebra generated by extremal elements. Let $x, y \in E(\mathfrak{g})$ such that $g_x(y) = 1$. Consider the 5-grading*

$$\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2,$$

*with $\mathfrak{g}_{-2} = \langle x \rangle$ and $\mathfrak{g}_2 = \langle y \rangle$ as in the above Lemma 8.18. If the extremal geometry of $\mathfrak{g}$ contains lines, then for each element $a \in \mathcal{E}_{-1}(\langle x \rangle) \cap \mathcal{E}_1(\langle y \rangle)$ we have $a \leq \mathfrak{g}_{-1}$.*

*Proof.* By $g_x(y) \neq 0$, the extremal points $\langle x \rangle$ and $\langle y \rangle$ are at distance 3 in the extremal geometry. We can find an element $a \in \mathcal{E}_{-1}(\langle x \rangle) \cap \mathcal{E}_1(\langle y \rangle)$. Consider $0 \neq a' \in a$. Then $[[x, y], a'] = [x, [y, a']] = \lambda a'$, for some non-zero $\lambda \in \mathbb{F}$, by the Jacobi identity, $[x, a'] = 0$ and the fact that $\langle x \rangle$ and $[y, a]$ are special and have $a$ as common neighbour. By Lemma 8.18 we conclude $a \leq \mathfrak{g}_i$. But then $a \in \mathcal{E}_{-1}(\langle x \rangle) \cap \mathcal{E}_1(\langle y \rangle)$ implies $i = -1$. □

**Lemma 8.20.** *Let $\mathfrak{g}$ be a simple Lie algebra generated by extremal elements. Let $x, y$ and $\mathfrak{g}_i$ be as in Lemma 8.18. If $\mathfrak{g}$ is not a symplectic Lie algebra, then there exist $e \in \mathfrak{g}_{-1}$ and $f \in \mathfrak{g}_1$ such that $[x, y] = [e, f]$.*

*Proof.* Assume that the extremal geometry of $\mathfrak{g}$ contains lines. Then fix an extremal point $a$ in $\mathcal{E}_{-1}(\langle x \rangle) \cap \mathcal{E}_1(\langle y \rangle)$. By the previous Lemma 8.19 we find $a \leq \mathfrak{g}_{-1}$.

Now, by [CI06, Lemma 4], we can find a point $b$ collinear with $\langle x \rangle$ and special with $a$. Since $\mathcal{E}_{\leq 0}(a)$ and $\mathcal{E}_{\leq 1}(a)$ are subspaces of the extremal geometry, any point of the line through $\langle x \rangle$ and $b$, with the exception of $\langle x \rangle$, is special with $a$. Consequently, we may assume that $b$ is special with $y$ as well and is thus contained in $\mathfrak{g}_{-1}$, again using Lemma 8.19. Since $a$ and $b$ are special and $\langle x \rangle$ is a common neighbour, we get $[a, b] = \langle x \rangle$. Since $a$ and $\langle y \rangle$ are special, $[a, \langle y \rangle]$ is the unique point collinear with both $a$ and $\langle y \rangle$. Clearly, $\langle x \rangle$ is special with $[a, \langle y \rangle]$ and thus $[a, \langle y \rangle] \leq \mathfrak{g}_1$ by

Lemma 8.19. Similarly for $b$ and $\langle y\rangle$. Note that we can find $a_0 \in a$ and $b_0 \in b$ such that $[a_0, b_0] = x$. Now consider $e := a_0 + b_0$ and $f := -[a_0, y] + [b_0, y]$. Note $e \in \mathfrak{g}_{-1}, f \in \mathfrak{g}_1$, and moreover

$$\begin{aligned}[e, f] &= [a_0 + b_0, -[a_0, y] + [b_0, y]] = -g_{a_0}(y)a_0 + g_{b_0}(y)b_0 + [a_0, [b_0, y]] - [b_0, [a_0, y]] \\ &= [a_0, [b_0, y]] + [b_0, [y, a_0]] = -[y, [a_0, b_0]] = -[y, x] = [x, y],\end{aligned}$$

where we used $(a, \langle y\rangle), (b, \langle y\rangle) \in \mathcal{E}_1$ and the Jacobi identity.

Assume now that the extremal geometry does not contain any lines and that $\mathfrak{g}$ is not symplectic. Then by Theorem 5.2 we find three extremal elements $x, y, z$ where $x$ and $y$ generate an $\mathfrak{sl}_2$ not containing $z$, and $g_z(u) \neq 0$ for all extremal elements $u$ in this $\mathfrak{sl}_2$. Denote the subalgebra of $\mathfrak{g}$ generated by $x, y$ and $z$ by $\mathfrak{h}$. From the proof of Lemma 5.6 we see that $g(x, [y, z])^2 - 4g(x, y)g(x, z)g(y, z) \neq 0$. As described in [CSUW01, p. 130], we can find $z' \in E \cap \mathfrak{h}$ such that $g(x, [y, z']) = 0$, $g(x, z') = g(x, z) \neq 0$ and

$$g(y, z') = g(y, z) - \frac{g(x, [y, z])^2}{4g(x, z)g(x, y)} \neq 0,$$

by $g(x, [y, z])^2 - 4g(x, y)g(x, z)g(y, z) \neq 0$. (Recall that the extremal form $f$ from *loc. cit.* satisfies $f = 2g$.) Then Theorem 5.2 of *loc. cit.* implies that $\mathfrak{h}$ is a quotient of $\mathfrak{sl}_3(\mathbb{F})$. Since the characteristic is not equal to 3 we get that $\mathfrak{h}$ is isomorphic to $\mathfrak{sl}_3$. The extremal geometry of this Lie algebra is a root shadow space of type $A_{2,\{1,2\}}$. Hence we can apply the previous paragraphs to find $e \in \mathfrak{h}_{-1}$ and $f \in \mathfrak{h}_1$ such that $[x, y] = [e, f]$. Then $\mathfrak{h}_{-1} \leq \mathfrak{g}_{-1}$ and $\mathfrak{h}_1 \leq \mathfrak{g}_1$ conclude this proof. □

**Theorem 8.21.** *Let $\mathfrak{g}$ be a simple finite-dimensional Lie algebra over a field of characteristic different from 2, 3 generated by extremal elements. Unless $\mathfrak{g} = \mathfrak{fsp}(V, f)$ for some non-degenerate symplectic space $(V, f)$, we find $\mathfrak{g} = K(\mathcal{A})$ for some simple skew-dimension one structurable algebra $\mathcal{A}$.*

*Proof.* Consider the 5-grading on $\mathfrak{g}$ from Lemma 8.18. (The existence of $x$ and $y$ follows from Proposition 4.8.) Then $[x, y]$ is the grading derivation of this 5-grading.

Assume first $\mathfrak{g}_{-1} \neq \{0\} \neq \mathfrak{g}_1$. By Lemma 8.16, $\mathfrak{g} \cong \mathrm{TKK}(\mathfrak{g}_{-1}, \mathfrak{g}_1)$. By Lemma 8.20, the grading derivation equals $[u, v]$, for some $u \in \mathfrak{g}_{-1}$ and $v \in \mathfrak{g}_1$. Hence Lemma 8.17 implies that $\mathfrak{g}$ is graded-isomorphic to $K(\mathcal{A})$, with $\mathcal{A}$ a structurable algebra. Recall that $\mathcal{S}$ denotes the set of skew elements of $\mathcal{A}$. By construction of $K(\mathcal{A})$, its $(-2)$- and 2-component are isomorphic with $\mathcal{S}$. Since $\mathfrak{g}_{-2}$ and $\mathfrak{g}_2$ are 1-dimensional (by construction), we get $\dim(\mathcal{S}) = 1$.

Now assume $\mathfrak{g}_{-1} = \{0\} = \mathfrak{g}_1$. Then $\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_2$ is a 3-graded Lie algebra. Hence it is a simple 5-graded Lie algebra with $(-1)$-component $\mathfrak{g}_{-2}$ and 1-component $\mathfrak{g}_2$. By Lemma 8.16 we get $\mathfrak{g}_0 = [\mathfrak{g}_{-2}, \mathfrak{g}_2] = \langle [x, y]\rangle$ and $\mathfrak{g} = \langle x\rangle \oplus \langle [x, y]\rangle \oplus \langle y\rangle$. Hence $\mathfrak{g} \simeq \mathfrak{sl}_2$, which is a symplectic Lie algebra. A contradiction. □

It remains to consider the symplectic Lie algebras generated by extremal elements.

**Example 8.22.** Let $A$ be an associative algebra with involution $\sigma$. Then $H(A, \sigma) := \{a \in A \mid a^\sigma = a\}$ is a Jordan algebra, with multiplication given by $a \circ b = \frac{ab+ba}{2}$. The transpose yields an involution of the associative algebra $\mathrm{Mat}_n(\mathbb{F})$. We denote this involution by $\top$.

**Lemma 8.23.** *The symplectic Lie algebra $\mathfrak{sp}(V, f)$, with $(V, f)$ non-degenerate symplectic space of dimension $2n$, is isomorphic to $K(H(\mathrm{Mat}_n(\mathbb{F}), \top))$.*

*Proof.* By [All79, p. 1868], the Lie algebra $K(H(\mathrm{Mat}_n(\mathbb{F}), \top))$ is isomorphic to the Lie algebra

$$\left\{ \begin{pmatrix} x & y \\ z & -x^\top \end{pmatrix} \mid x \in \mathrm{Mat}_n(\mathbb{F}), y, z \in H(\mathrm{Mat}_n(\mathbb{F}), \top) \right\},$$

modulo its center, which is isomorphic to $\mathfrak{sp}(V, f)$. □

**Corollary 8.24.** *Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra over a field of characteristic different from 2, 3 generated by its extremal elements. Then $\mathfrak{g} = K(\mathcal{A})$, for some simple structurable algebra $\mathcal{A}$.*

*Proof.* By Theorem 8.21 and Lemma 8.23. □

In the next few theorems we determine when the extremal geometry contains lines in terms of the associated structurable algebra.

**Theorem 8.25.** *Let $\mathcal{A}$ be a simple skew-dimension one structurable algebra. The extremal geometry in $K(\mathcal{A})$ contains lines if and only if $\mathcal{A}_-$ contains an extremal element.*

*Proof.* If the extremal geometry contains lines, then clearly $\mathcal{A}_-$ contains an extremal element, by Lemma 8.19. Conversely, assume $\mathcal{A}_-$ contains an extremal element $a_-$. Then by the claim in the seventh line of the proof of [CI06, Theorem 28] we get that $(\mathcal{S}_-, \langle a_- \rangle) \in \mathcal{E}_{-1}$. $\square$

**Corollary 8.26.** *The extremal geometry of $K(M(J,\eta))$ contains lines, with $M(J,\eta)$ as in Definition 8.6.*

*Proof.* Consider $a = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Note that $s := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is an element of $\mathcal{S}$. Then $sa = a$. We have

$$V_{a,a}(x) = (a\overline{a})x + (x\overline{a})a - (x\overline{a})a = (a\overline{a})x,$$

for any $x \in M(J,\eta)$. Then $a\overline{a} = 0$ implies $V_{sa,a} = 0$. Hence $[a_-, [a_-, \mathcal{S}_+]] = 0$. Now we still need $[a_-, [a_-, \mathcal{A}_+]] \leq \langle a_- \rangle$ and $[a_-, [a_-, \mathrm{Instr}(\mathcal{A})]] = 0$ in order to obtain that $a_-$ is extremal. The former is equivalent to showing that $U_a(\mathcal{A}) \leq \langle a \rangle$. The latter boils down to $\psi(a, V_{x,y}(a)) = 0$ for all $x, y \in \mathcal{A}$. Both claims are straightforward to check. Hence Theorem 8.25 concludes this proof. $\square$

**Corollary 8.27.** *A symplectic Lie algebra is not isomorphic to $K(\mathcal{A})$, with $\mathcal{A}$ a simple skew-dimension one structurable algebra.*

*Proof.* Let $\mathfrak{g}$ be a symplectic Lie algebra. Assume $\mathfrak{g} = K(\mathcal{A})$, for some simple skew-dimension one structurable algebra $\mathcal{A}$. By Corollary 8.8 there exists a field extension $\hat{\mathbb{F}}$ of $\mathbb{F}$ of degree at most 2 such that $\mathcal{A} \otimes_{\mathbb{F}} \hat{\mathbb{F}}$ is isomorphic to a structurable matrix algebra $M(J,\eta)$. But $K(\mathcal{A} \otimes_{\mathbb{F}} \hat{\mathbb{F}}) \cong K(\mathcal{A}) \otimes_{\mathbb{F}} \hat{\mathbb{F}}$. By Corollary 8.26 there are lines in the extremal geometry of this Lie algebra. On the other hand, $K(\mathcal{A}) \otimes_{\mathbb{F}} \hat{\mathbb{F}}$ is still a symplectic Lie algebra, which does not contain lines in its extremal geometry. We get a contradiction. $\square$

**Definition 8.28.** Two structurable algebras $\mathcal{A}$ and $\mathcal{A}'$ are isotopic if there exists a vector space isomorphism $\psi : \mathcal{A} \to \mathcal{A}'$ such that there exists $\chi \in \mathrm{Hom}_k(\mathcal{A}, \mathcal{A}')$ such that

$$\psi(V_{x,y}z) = V_{\psi(x),\chi(y)}\psi(z), \forall x, y, z \in \mathcal{A}\,.$$

**Corollary 8.29.** *Let $\mathcal{A}$ be a simple skew-dimension one structurable algebra. The extremal geometry in $K(\mathcal{A})$ contains lines if and only if $\mathcal{A}$ is isotopic to $M(J,1)$, with $J$ as in Definition 8.6.*

*Proof.* By Corollary 8.26 and [AH81, Proposition 12.3] one direction is obvious.

Assume now that the extremal geometry in $K(\mathcal{A})$ contains lines. By Theorem 8.25 we find $a \in \mathcal{A}$ non-zero such that $a_+$ is extremal. In particular $-U_a(b)_+ = [a_+, [a_+, b_-]] \in \langle a_+ \rangle$ for any $b \in \mathcal{A}$. Hence $U_a(\mathcal{A}) \leq \langle a \rangle$. Then [AF84, Theorem 4.6] and [Gar01, Lemma 4.13] conclude this proof. $\square$

**Remark 8.30.** Combining the above Corollary 8.29 with Corollary 8.8, and Theorem 8.21 we obtain an algebraic proof of Theorem 5.7, in case the characteristic is not $2, 3$. Indeed, if $\mathfrak{g}$ is a simple Lie algebra generated by its extremal elements, then Theorem 8.21 implies that, if $\mathfrak{g}$ is not symplectic, there is a skew-dimension one structurable algebra $\mathcal{A}$ such that $\mathfrak{g}$ is isomorphic to $K(\mathcal{A})$. Extending the field quadratically, if needed, we can, by Corollary 8.8 assume that $\mathcal{A}$ is a structurable matrix algebra and find the extremal geometry of $\mathfrak{g}$ to contain lines by Corollary 8.29.

We first define automorphisms of the Lie algebra $K(\mathcal{A})$, where $\mathcal{A}$ is a (skew-dimension one) structurable algebra, in order to prove that the inner ideal geometry is a Moufang set if there are no inner line ideals. Note that Theorem 8.32 depends on the classification of simple Lie algebras over algebraically closed fields of characteristic larger or equal than 5. However, if the characteristic of the field is strictly larger than 5 one does not need this classification, see for example [BDMS19, Lemma 3.1.7].

**Definition 8.31.** Let $\mathfrak{g}$ be a finite-dimensional 5-graded Lie algebra over a field $\mathbb{F}, \operatorname{char} \mathbb{F} \neq 2, 3$. We say that $\mathfrak{g}$ is *algebraic* if for any $(x, s) \in \mathfrak{g}_{\sigma 1} \oplus \mathfrak{g}_{\sigma 2}$, where $\sigma = \pm$, the endomorphism

$$\exp(x + s) := \sum_{i=0}^{4} \frac{1}{i!} \operatorname{ad}(x + s)^i$$

of $\mathfrak{g}$ is a Lie algebra automorphism. We say that a structurable algebra $\mathcal{A}$ over $\mathbb{F}$ is algebraic, if $K(\mathcal{A})$ is algebraic in the above sense.

**Theorem 8.32** ([Sta20, Theorem 2.10, Theorem 3.4])**.** *Any central simple structurable algebra over a field of characteristic different from 2 and 3 is algebraic.*

Let $a \in \mathcal{A}$. For later use, we deduce the image of $\mathcal{S}_-$ under the automorphism $\exp(a_+)$. For any $t \in \mathcal{S}$ we have:

$$\exp(a_+)(t_-) = t_- - ta_- - \tfrac{1}{2} V_{a,ta} + \tfrac{1}{6} U_a(ta)_+ + \tfrac{1}{24} \psi(a, U_a(ta))_+. \tag{4}$$

The next (technical) lemma appears in [DMM20]. Actually, we only need a weaker version of it, described in the following corollary.

**Lemma 8.33.** *Let $\mathcal{A}$ be a central simple structurable algebra with $\mathcal{S} \neq 0$ and every non-zero element of $\mathcal{S}$ conjugate invertible. Any minimal inner ideal in the Lie algebra $K(\mathcal{A})$ containing an element with non-zero 2-component equals $\exp(a_- + t_-)(\mathcal{S}_+)$, for some $a \in \mathcal{A}$ and $t \in \mathcal{S}$.*

**Corollary 8.34.** *Let $\mathcal{A}$ be a simple skew-dimension one structurable algebra. Any extremal element $x$ in the Lie algebra $K(\mathcal{A})$ generating an extremal point contained in $\mathcal{E}_2(\mathcal{S}_-)$ equals $\exp(a_- + t_-)(s_+)$, for some $a \in \mathcal{A}$ and $t \in \mathcal{S}$ and $0 \neq s \in \mathcal{S}$.*

Now we prove a theorem similar to Proposition 7.1. Note that we now allow $\mathcal{E}_{-1} \cup \mathcal{E}_0 \cup \mathcal{E}_1 = \emptyset$, but still require that the characteristic is not equal to 2 or 3. The techniques in this proof are algebraic as opposed to geometric arguments in the proof of Proposition 7.1. Moreover, as mentioned before, here we have to rely on the classification of simple Lie algebras in the characteristic 5 case.

**Theorem 8.35.** *Let $\mathcal{A}$ be a simple skew-dimension one structurable algebra. Then the automorphism group of $K(\mathcal{A})$ acts transitively on the pairs $(x, y) \in \mathcal{E}_2$.*

*Proof.* Since $\mathfrak{g} = K(\mathcal{A})$ is simple, the first paragraph in the proof of Proposition 7.1 shows that it suffices to show that the stabilizer of $\mathcal{S}_- \leq K(\mathcal{A})$ is transitive on the points in $\mathcal{E}_2(\mathcal{S}_-)$. By Corollary 8.34 any element in $\mathcal{E}_2(\mathcal{S}_-)$ equals $\langle \exp(a_- + t_-)(s_+) \rangle$, for some $a \in \mathcal{A}$ and $s, t \in \mathcal{S}$. Now note that $\exp(a_- + t_-)$ is an automorphism and $\exp(a_- + t_-)(\mathcal{S}_-) = \mathcal{S}_-$. □

**Theorem 8.36.** *Assume that the characteristic of $\mathbb{F}$ is not 2 or 3. Then the map*

$$\mathcal{A} \mapsto K(\mathcal{A})$$

*induces a one-to-one correspondence between simple skew-dimension one structurable algebras over $\mathbb{F}$ (up to isotopy) and simple Lie algebras over $\mathbb{F}$ generated by extremal elements which are not symplectic (up to isomorphism).*

*Proof.* Let $\mathcal{A}$ be a simple skew-dimension one structurable algebra. By Proposition 8.14 we find that $K(\mathcal{A})$ is a simple Lie algebra generated by its extremal elements, which by Corollary 8.27 is not symplectic. By [AH81, Proposition 12.3], two isotopic structurable algebras yield isomorphic Lie algebras.

Consider two simple skew-dimension one structurable algebras $\mathcal{A}$ and $\mathcal{A}'$ such that $K(\mathcal{A})$ and $K(\mathcal{A}')$ are isomorphic. Since $(\mathcal{S}_-, \mathcal{S}_+)$ and $(\mathcal{S}'_-, \mathcal{S}'_+)$ are both hyperbolic pairs of extremal points, Theorem 8.35 implies that $K(\mathcal{A})$ and $K(\mathcal{A}')$ are graded-isomorphic and hence $\mathcal{A}$ and $\mathcal{A}'$ are isotopic by [AH81, Proposition 12.3].

Now Theorem 8.21 concludes this proof. □

We briefly recall the definition of a Moufang set.

**Definition 8.37.** Let $X$ be a set (with $|X| \geq 3$) and $\{U_x \mid x \in X\}$ be a collection of subgroups of $\mathrm{Sym}(X)$. The data $(X, \{U_x\}_{x\in X})$ is a *Moufang set* if the following two properties are satisfied:

- For each $x \in X$, $U_x$ fixes $x$ and acts sharply transitively on $X\backslash\{x\}$.
- For each $g \in G^+ := \langle U_x \mid x \in X\rangle \leq \mathrm{Sym}(X)$ and each $y \in X$ we have $U_y^g = U_{y.g}$.[2]

The group $G^+$ is called the *little projective group* of the Moufang set, and the groups $U_x$ are called the *root groups*.

Suppose that the simple Lie algebra $\mathfrak{g}$ is generated by its set of extremal elements and no two linearly independent extremal elements commute. The inner line ideal geometry of $\mathfrak{g}$ only contains points, by Theorem 6.11. Moreover, by Lemma 6.8, $\mathfrak{g}$ is not a symplectic Lie algebra, unless $\mathfrak{g} \simeq \mathfrak{sl}_2$. Hence $\mathfrak{g} = K(\mathcal{A})$ for a unique – up to isotopy – simple skew-dimension one structurable algebra $\mathcal{A}$, unless $\mathfrak{g} \simeq \mathfrak{sl}_2$. In the latter case one easily sees that all extremal points form a Moufang set. In the former case $\mathcal{S}_-$ and $\exp(a_+)(\mathcal{S}_-)$ are hyperbolic for any $0 \neq a \in \mathcal{A}$. (4) implies $\psi(a, U_a(sa)) \neq 0$, for $0 \neq s \in \mathcal{S}$. Then [AF84, Proposition 2.11] shows that $\mathcal{A}$ is a structurable division algebra. By [BDMS19, Lemma 3.1.3] and Theorem 8.32 we get that $E_-(\mathcal{A}) := \{\exp(a_- + s_-) \mid a \in \mathcal{A}, s \in \mathcal{S}\}$ is a subgroup of the automorphism group of $\mathfrak{g} = K(\mathcal{A})$. Similarly for $E_+(\mathcal{A}) := \{\exp(a_+ + s_+) \mid a \in \mathcal{A}, s \in \mathcal{S}\}$. Note that Corollary 8.34 implies that $E_-(\mathcal{A})$ is transitive on $\mathcal{E}_2(\mathcal{S}_-)$. It is not hard to see that it is actually sharply transitive. Using [BDMS19, Theorem 5.1.1] we see that $\mathcal{E}$ is indeed a Moufang set, with root groups as below. The little projective group is $E(\mathcal{A}) = \langle E_-(\mathcal{A}), E_+(\mathcal{A})\rangle$. The multiplication in this group is defined as $fg = g \circ f$ for any $f, g \in E(\mathcal{A})$, in order to be compatible with the convention on the group action mentioned in Definition 8.37. Conversely, if $\mathcal{A}$ is a skew-dimension one structurable division algebra, it is shown in [DMM20] that the only non-trivial inner ideals of $K(\mathcal{A})$ are 1-dimensional and form a Moufang set.

So we obtained:

**Theorem 8.38.** *Suppose $\mathfrak{g}$ is a finite-dimensional simple Lie algebra over a field $\mathbb{F}$ of characteristic different from $2, 3$ generated by its set of extremal elements. If no two linearly independent extremal elements commute and $\mathfrak{g} \neq \mathfrak{sl}_2$, then $\mathfrak{g} = K(\mathcal{A})$ with $\mathcal{A}$ a skew-dimension one structurable division algebra. The set of proper non-trivial inner ideals of $\mathfrak{g}$ equals $\mathcal{E} = \{\mathcal{S}_-\} \cup E_-(\mathcal{A})(\mathcal{S}_+)$ and is a Moufang set with root groups*

$$U_{\mathcal{S}_-} = E_-(\mathcal{A});$$
$$U_{e_-(a,s)(\mathcal{S}_+)} = E_+(\mathcal{A})^{e_-(a,s)}, \forall a \in \mathcal{A}, s \in \mathcal{S}.$$

Notice that Theorem 6.11, Theorem 7.2 and the above result prove Theorem 1.1.

We end this paper with some examples in which we explore the relation between various types of structurable algebras, the corresponding Lie algebras and their extremal geometries.

**Example 8.39.** Let $J$ be a cubic Jordan division algebra. In [DMM20] it is shown that the non-trivial inner ideals of $\mathfrak{g} := K(M(J, 1))$ form a generalized hexagon. This geometry coincides with the extremal geometry associated to $\mathfrak{g}$.

Conversely, assume that $\mathfrak{g}$ is a simple Lie algebra over a field of characteristic not 2 or 3 generated by its extremal elements such that its extremal geometry is a thick generalized hexagon. Since this extremal geometry has lines, Corollary 8.29 implies that $\mathfrak{g} \cong K(M(J, 1))$, with $J$ as in Definition 8.6. Assume that there exists $0 \neq j \in J$ such that $j^\sharp = 0$. Then, by $j \times j = 2j^\sharp = 0$, we get $V_{a,sa} = V_{a,a} = 0$, with $a = \begin{pmatrix} 0 & j \\ 0 & 0 \end{pmatrix}$. Set $b = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. By (the proof of) Corollary 8.26, $\langle b_+\rangle$ is an extremal point. Similarly one can check that $\langle a_+\rangle$ is an extremal point. One sees $[a_+, b_+] = \psi(a, b)_+ = 0$. So either $(\langle a_+\rangle, \langle b_+\rangle)$ is in $\mathcal{E}_{-1}$ or in $\mathcal{E}_0$. In the former case, the extremal geometry contains a singular plane spanned by $s_+, a_+$, and $b_+$, a contradiction with the fact that this geometry is a generalized hexagon. In the latter case, we also get a contradiction, since then there are ordinary quadrangles in the geometry. Hence $j^\sharp = 0$ implies $j = 0$, for $j \in J$. If we look back at Definition 8.6, this implies that $J$ is either a cubic Jordan algebra or $J = 0$. In the

[2] As is common in the theory of Moufang sets, we denote the group action on the right.

former case, the identity $(j^\sharp)^\sharp = N(j)j$ shows that $N(j) = 0$ implies $j = 0$. I.e., $J$ is division. In the latter case, it is straightforward to check that one obtains the Lie algebra $\mathfrak{sl}_3$ and hence its extremal geometry is a thin generalized hexagon (namely a geometry of type $\mathrm{A}_{2,\{1,2\}}$).

**Example 8.40.** Assume $\mathbb{F}$ is an algebraically closed field of characteristic 0. Consider a cubic Jordan algebra $J$ with a non-degenerate admissible form $N$, with basepoint 1, trace form $T$, and Freudenthal cross product $\times$. If $J$ has dimension 3, 6, 9, 15 or 27, then $K(M(J,1))$ is of type $\mathrm{D}_4$, $\mathrm{F}_4$, $\mathrm{E}_6$, $\mathrm{E}_7$ and $\mathrm{E}_8$ respectively by [All79, p. 1871] (note that the relative and absolute type coincide in this case). Then the extremal geometry in $K(M(J,1))$ is of type $\mathrm{D}_{4,2}$, $\mathrm{F}_{4,1}$, $\mathrm{E}_{6,2}$, $\mathrm{E}_{7,1}$ and $\mathrm{E}_{8,8}$ respectively, by Theorem 4.16.

**Example 8.41.** Consider the tensor product $C_1 \otimes C_2$, where $C_1$ is an octonion algebra and $C_2$ is a composition algebra of dimension 2, 4 or 8 over a field $\mathbb{F}$. As noted before, this is a structurable algebra and its set of skew elements is equal to $\{s_1 \otimes 1 + 1 \otimes s_2 \mid s_1 \in S_1, s_2 \in S_2\}$, with $S_i$ the set of skew elements in $C_i$. Hence $\dim(\mathcal{S}) = \dim(C_1) + \dim(C_2) - 2$, so 8, 10 or 14 respectively. We can define the following quadratic form on $\mathcal{S}$:

$$q_A : \mathcal{S} \to \mathbb{F} : s_1 \otimes 1 + 1 \otimes s_2 \mapsto q_1(s_1) - q_2(s_2),$$

which is called the *Albert form*. (With $q_i$ the quadratic form associated with the composition algebra $C_i$.) Denote the associated bilinear form by $q_A$ as well. We also define $(s_1 \otimes 1 + 1 \otimes s_2)^\natural = s_1 \otimes 1 - 1 \otimes s_2$. We now assume that $q_A$ has Witt index 1 and that $\mathbb{F}$ has characteristic 0. If $s \in \mathcal{S}$ is not an isotropic vector, then $s$ is conjugate invertible (see [All88, Corollary 3.13]). If $s \in \mathcal{S}$ is an isotropic vector, then for any $t \in \mathcal{S}$ we have, due to [All88, Proposition 3.3], $sts = q_A(s)t^\natural - q_A(s,t^\natural)s = -q_A(s,t^\natural)s$. Hence $[s_+,[s_+,t_-]] = [s_+, L_s L_t] = -2sts_+ \in \langle s_+\rangle$. So since $\mathcal{S}_+$ is the end of a 5-grading, we obtain that $s_+$ is an extremal element of $\mathfrak{g} := K(C_1 \otimes C_2)$. Consider $s,t \in \mathcal{S}$ linearly independent and satisfying $q_A(s) = 0 = q_A(t)$. Now assume that the extremal geometry of $\mathfrak{g}$ contains lines. If $(s_+,t_+) \in E_{-1}$, then $\mathcal{S}_+$ contains an extremal line, namely $\langle s_+,t_+\rangle$. Otherwise, due to $[s_+,t_+] = 0$, we get $(s_+,t_+) \in E_0$. So by Lemma 6.5 and the fact that $\mathcal{S}_+$ is an inner ideal we get that $\mathcal{S}_+$ contains all points (and lines) of a symplecton of the extremal geometry. So in any case, $\mathcal{S}_+$ would contain extremal lines. By the previous considerations this implies that $q_A$ has Witt index at least 2, a contradiction. Hence the extremal geometry of $\mathfrak{g}$ does not contain lines. So, in this case, due to Theorem 6.11, the inner ideal geometry is a polar space. Moreover, its rank equals 2 if there does not exist an inner ideal $I$ with $[I,I] = 0$ properly containing $\mathcal{S}_+$, by Corollary 6.6. So assume that $I$ is an inner ideal containing $\mathcal{S}_+$ properly with $[I,I] = 0$. Since there exist conjugate invertible elements in $\mathcal{S}$ the fact that $I$ is abelian implies $I \leq \mathrm{Instr}(\mathcal{A}) \oplus \mathcal{A}_+ \oplus \mathcal{S}_+$. Consider $0 \neq V + a_+ \in I$ and $s \in \mathcal{S}$ conjugate invertible. Then $V(sb)_+ + \psi(a,sb)_+ = [V + a_+, [s_+, b_-]] \in I$ for any $b \in \mathcal{A}$. Together with $\mathcal{S}_+ \leq I$ this implies that there exists $0 \neq a_+ \in \mathcal{A}_+ \cap I$. Then $[\mathcal{S}_+, [a_+, \mathcal{S}_-]] \leq I$ implies that $L_{\mathcal{S}} L_{\mathcal{S}}(\mathcal{A}_+ \cap I) \leq \mathcal{A}_+ \cap I$. By [All88, Theorem 4.5] we get $\mathcal{A}_+ \leq I$. Now $[s_+, 1_+] = \psi(s,1)_+ = 2s_+ \neq 0$ for any $s \in \mathcal{S}$ contradicts $[I,I] = 0$. Hence the inner ideal geometry of $\mathfrak{g}$ is a Moufang quadrangle.

Note that by Corollary 8.29 the Lie algebra $\mathfrak{g}$ is also isomorphic to the TKK-construction of a form of a structurable matrix algebra.

**Example 8.42.** Let $Q : M \to \mathbb{F}$ be a non-degenerate quadratic form on a $\mathbb{F}$-vector space $M$. We assume that there exist a $c \in M$ such that $Q(c) = 1$. Let

$$Q(x,y) = \frac{1}{2}(Q(x+y) - Q(x) - Q(y))$$

be the associated bilinear form and set $T(x) = Q(x,c)$ and $\overline{x} = 2T(x)c - x$. Define the following product on $M$:

$$x.y = T(x)y + T(y)x - Q(x,y)c.$$

With this product, $M$ is a Jordan algebra, which we denote by $\mathrm{Jord}(Q,c)$. This Jordan algebra is central simple unless $Q$ is isotropic and $\dim(M) = 2$ (see [McC04, II.3.3], where the bilinear form $Q$ is slightly different). From now on we will assume $\dim(M) > 2$ and that $Q$ has Witt index 1, so $J := \mathrm{Jord}(Q,c)$ is central simple. By [McC04, II.3.3.1] we get $U_{x,x}(y) = 2Q(x,\overline{y})x - Q(x)\overline{y}$. Now $x_+ \in K(J)$ is extremal if and only if $U_x(J) \leq \langle x\rangle$, so if and only if $Q(x) = 0$. As in the previous example, the existence of lines in the extremal geometry implies the existence of lines in $J_+$. But then there would be a subspace $V$ of $J$ of dimension 2 such that $Q(v) = 0$, for all $v \in V$, contradicting that $Q$ has Witt index 1. Hence the inner ideal geometry of $K(J)$ is a polar space of rank at least 2. Now note that $J_+$ is the smallest inner ideal containing two distinct extremal points $\langle x_+\rangle$ and $\langle y_+\rangle$ in $J_+$, since all elements $z \in J$ with $Q(z) \neq 0$ are invertible, i.e. $U_z$ is invertible. Now, if the rank of the polar space would be strictly larger than 2, there would be a proper abelian inner ideal $I$ properly containing $J_+$. But then $0 \neq a_- + V \in I$, for $a \in J$, $V \in \mathrm{Instr}(J)$. Since $J$ is abelian, $[V, j_+] = V(j)_+ = 0$ for all $j \in J$ implying $V = 0$. Hence $0 = [a_-, j_+] = -V_{j,a}$ for all $j \in J$. But $0 = V_{j,a}(j) = U_j(a)$ implies $a = 0$ if $j$ is invertible. Hence the extremal geometry is a polar space of rank 2 which is Moufang, i.e. a Moufang quadrangle.

**Example 8.43.** Quadrangular algebras have been introduced by Richard Weiss [Wei06] in order to understand the exceptional Moufang quadrangles of type $E_6$, $E_7$, $E_8$ and $F_4$. These algebras are generalizations of pseudo-quadratic forms. In [BDM13], it has been show that, in characteristic different from $2, 3$, each quadrangular algebra gives rise to a skew-dimension one structurable algebra. This structurable algebra is division and hence the inner ideal geometry of the associated Lie algebra forms a Moufang set. So one does not obtain the Moufang quadrangle associated to the quadrangular algebra, but a Moufang set, most likely coming from a residue of the Moufang quadrangle.

## References


[AF84] B. N. Allison and J. R. Faulkner. A Cayley-Dickson process for a class of structurable algebras. *Transactions of the American Mathematical Society*, 283(1):185–210, 1984.

[AF99] Bruce N. Allison and John R. Faulkner. Elementary groups and invertibility for Kantor pairs. *Communications in Algebra*, 27(2):519–556, 1999.

[AFS17] Bruce Allison, John Faulkner, and Oleg Smirnov. Weyl images of Kantor pairs. *Canadian Journal of Mathematics. Journal Canadien de Mathématiques*, 69(4):721–766, 2017.

[AH81] B. N. Allison and W. Hein. Isotopes of some nonassociative algebras with involution. *Journal of Algebra*, 69(1):120–142, 1981.

[All78] B. N. Allison. A class of nonassociative algebras with involution containing the class of Jordan algebras. *Mathematische Annalen*, 237(2):133–156, 1978.

[All79] B. N. Allison. Models of isotropic simple Lie algebras. *Communications in Algebra*, 7(17):1835–1875, 1979.

[All88] B. N. Allison. Tensor products of composition algebras, Albert forms and some exceptional simple Lie algebras. *Transactions of the American Mathematical Society*, 306(2):667–695, 1988.

[All90] B. N. Allison. Simple structurable algebras of skew-dimension one. *Communications in Algebra*, 18(4):1245–1279, 1990.

[BC13] Francis Buekenhout and Arjeh M. Cohen. *Diagram geometry*, volume 57 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer, Heidelberg, 2013.

[BDM13] Lien Boelaert and Tom De Medts. Exceptional Moufang quadrangles and structurable algebras. *Proceedings of the London Mathematical Society. Third Series*, 107(3):590–626, 2013.

[BDMS19] Lien Boelaert, Tom De Medts, and Anastasia Stavrova. Moufang sets and structurable division algebras. *Mem. Amer. Math. Soc.*, 259(1245):v+90, 2019.

[Bou68] N. Bourbaki. *Éléments de mathématique. Fasc. XXXIV. Groupes et algèbres de Lie. Chapitre IV: Groupes de Coxeter et systèmes de Tits. Chapitre V: Groupes engendrés par des réflexions. Chapitre VI: systèmes de racines*. Actualités Scientifiques et Industrielles, No. 1337. Hermann, Paris, 1968.

[CF17] Hans Cuypers and Yael Fleischmann. A geometric characterization of the symplectic Lie algebra. *arXiv e-prints*, page arXiv:1707.02095, July 2017.

[CF18] Hans Cuypers and Yael Fleischmann. A geometric characterization of the classical Lie algebras. *Journal of Algebra*, 502:1–23, 2018.

[CI06] Arjeh M. Cohen and Gábor Ivanyos. Root filtration spaces from Lie algebras and abstract root groups. *Journal of Algebra*, 300(2):433–454, 2006.

[CI07] Arjeh M. Cohen and Gábor Ivanyos. Root shadow spaces. *European Journal of Combinatorics*, 28(5):1419–1441, 2007.

[CIR08] Arjeh M. Cohen, Gábor Ivanyos, and Dan Roozemond. Simple Lie algebras having extremal elements. *Koninklijke Nederlandse Akademie van Wetenschappen. Indagationes Mathematicae. New Series*, 19(2):177–188, 2008.

[CO19] Hans Cuypers and Marc Oostendorp. A geometric characterization of the special linear and unitary Lie algebras, preprint. 2019.

[Coh12] Arjeh M. Cohen. The geometry of extremal elements in a Lie algebra. In *Buildings, finite geometries and groups*, volume 10 of *Springer Proc. Math.*, pages 15–35. Springer, New York, 2012.

[CRS15] Hans Cuypers, Kieran Roberts, and Sergey Shpectorov. Recovering the Lie algebra from its extremal geometry. *Journal of Algebra*, 441:196–215, 2015.

[CSUW01] Arjeh M. Cohen, Anja Steinbach, Rosane Ushirobira, and David Wales. Lie algebras generated by extremal elements. *Journal of Algebra*, 236(1):122–154, 2001.

[DMM20] Tom De Medts and Jeroen Meulewaeter. *Inner ideals and structurable algebras: Moufang sets, triangles and hexagons. arXiv e-prints*, 2020. page arXiv:2008.02700.

[Fau73] John R. Faulkner. On the geometry of inner ideals. *Journal of Algebra*, 26:1–9, 1973.

[FL19] Antonio Fernández López. *Jordan structures in Lie algebras*, volume 240 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2019.

[Fle15] Yael Fleischmann. *A Geometric Approach to Classical Lie Algebras*. PhD thesis, TU Eindhoven, 2015.

[Gar01] R. Skip Garibaldi. Structurable algebras and groups of type $E_6$ and $E_7$. *Journal of Algebra*, 236(2):651–691, 2001.

[KMRT98] Max-Albert Knus, Alexander Merkurjev, Markus Rost, and Jean-Pierre Tignol. *The book of involutions*, volume 44 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 1998. With a preface in French by J. Tits.

[McC04] Kevin McCrimmon. *A taste of Jordan algebras*. Universitext. Springer-Verlag, New York, 2004.

[Shu11] Ernest E. Shult. *Points and lines*. Universitext. Springer, Heidelberg, 2011. Characterizing the classical geometries.

[Sta20] Anastasia Stavrova. *5-Graded simple Lie algebras, structurable algebras, and Kantor pairs. arXiv e-prints*, September 2020. page arXiv: 1712.05288.

[Tim01] Franz Georg Timmesfeld. *Abstract root subgroups and simple groups of Lie type*, volume 95 of *Monographs in Mathematics*. Birkhäuser Verlag, Basel, 2001.

[Tit74] Jacques Tits. *Buildings of spherical type and finite BN-pairs*. Lecture Notes in Mathematics, Vol. 386. Springer-Verlag, Berlin-New York, 1974.

[Wei06] Richard M. Weiss. *Quadrangular algebras*, volume 46 of *Mathematical Notes*. Princeton University Press, Princeton, NJ, 2006.

[Zel80] E. I. Zelmanov. Absolute zero-divisors in Jordan pairs and Lie algebras. *Matematicheskiĭ Sbornik. Novaya Seriya*, 112(154)(4(8)):611–629, 1980.

[ZK90] E. I. Zelmanov and A. I. Kostrikin. A theorem on sandwich algebras. volume 183, pages 106–111, 225. 1990. Translated in Proc. Steklov Inst. Math. **1**991, no. 4, 121–126, Galois theory, rings, algebraic groups and their applications (Russian).

*Current address*, Hans Cuypers: Department of Mathematics and Computer Science, Eindhoven University of Technology, Den Dolech 2, P.O. Box 513, 5600 MB, Eindhoven, The Netherlands,

*Email address*, Hans Cuypers: `f.g.m.t.cuypers@tue.nl`

*Current address*, Jeroen Meulewaeter: Department of Mathematics: Algebra and Geometry, Ghent University, Krijgslaan 281, 9000 Ghent, Belgium

*Email address*, Jeroen Meulewaeter: `Jeroen.Meulewaeter@UGent.be`