Journal of Scientific Computing manuscript No.
(will be inserted by the editor)

On a reconstruction of a solely time-dependent source in a time-fractional diffusion equation with non-smooth solutions

A. S. Hendy · **K. Van Bockstal**

Received: date / Accepted: date

Abstract An inverse source problem for a non-automonous time fractional diffusion equation of order $(0 < \beta < 1)$ is considered in a bounded Lipschitz domain in $\mathbb{R}^d$. The missing solely time-dependent source is recovered from an additional integral measurement. The existence, uniqueness and regularity of a weak solution is studied. We design two numerical algorithms based on Rothe's method over uniform and graded grids, derive a priori estimates and prove convergence of iterates towards the exact solution. An essential feature of the fractional subdiffusion problem is that the solution lacks the smoothness near the initial time, although it would be smooth away from $t = 0$. Rothe's method on a uniform grid addresses the existence of a such a solution (non-smooth with t^γ term where $1 > \gamma > \beta$) under low regularity assumptions, whilst Rothe's method over graded grids has the advantage to cope better with the behaviour at $t = 0$ (also here t^β is included in the class of admissible solutions) for the considered problems. The theoretical obtained results are supported by numerical experiments and stay valid in case of smooth solutions to the problem.

Keywords Inverse source problem · Reconstruction · Fractional diffusion · Uniform and nonuniform (graded) meshes · Prior estimates · Convergence.

Mathematics Subject Classification (2010) 35A15 · 35R11 · 47G20 · 65M12

A. S. Hendy
Department of Mathematics, Faculty of Science
Benha University, Benha 13511, Egypt
Present address:
Department of Computational Mathematics and Computer Science
Institute of Natural Sciences and Mathematics
Ural Federal University, 19 Mira St., Yekaterinburg 620002, Russia
E-mail: ahmed.hendy@fsc.bu.edu.eg

K. Van Bockstal
Research Group NaM^2, Department of Electronics and Information systems,
Ghent University, Krijgslaan 281, 9000 Ghent, Belgium
E-mail: karel.vanbockstal@ugent.be, `http://users.ugent.be/~kavbocks/`

1 Introduction

Let $\Omega \subset \mathbb{R}^d$ ($d \in \mathbb{N}$) be a bounded domain with a Lipschitz continuous boundary $\partial\Omega$. The outer normal vector on $\partial\Omega$ is denoted by ν. The final time is denoted by T, $Q_T := \Omega \times (0,T]$ and $\Sigma_T := \partial\Omega \times (0,T]$. Consider a general second-order linear differential operator given by

$$L(\mathbf{x},t)u(\mathbf{x},t) = -\nabla \cdot (A(\mathbf{x},t)\nabla u(\mathbf{x},t)) + c(t)u(\mathbf{x},t), \tag{1}$$

where ($(\mathbf{x},t) \in Q_T$)

$$A(\mathbf{x},t) = (a_{i,j}(\mathbf{x},t))_{i,j=1,\ldots,d}\,.$$

The goal of this contribution is to determine the couple $\{h(t),u\}$ such that

$$\begin{cases} \left(\partial_t^\beta u\right)(\mathbf{x},t) + (Lu)(\mathbf{x},t) & \\ \qquad = h(t)f(\mathbf{x}) + \displaystyle\int_0^t F(\mathbf{x},s,u(\mathbf{x},s))\ \mathrm{d}s & (\mathbf{x},t) \in Q_T, \\ (-A(\mathbf{x},t)\nabla u(\mathbf{x},t)) \cdot \nu(\mathbf{x}) = g(\mathbf{x},t) & (\mathbf{x},t) \in \Sigma_T, \\ u(\mathbf{x},0) = \tilde{u}_0(\mathbf{x}) & \mathbf{x} \in \Omega, \end{cases} \tag{2}$$

and the following integral measurement are satisfied

$$\int_\Omega u(\mathbf{x},t)\ \mathrm{d}\mathbf{x} = m(t). \tag{3}$$

Here, $\partial_t^\beta u$ denotes the Caputo derivative of order $\beta \in (0,1)$ defined by

$$\left(\partial_t^\beta u\right)(\mathbf{x},t) = \partial_t \int_0^t \frac{(t-s)^{-\beta}}{\Gamma(1-\beta)} [u(\mathbf{x},s) - \tilde{u}_0(\mathbf{x})]\ \mathrm{d}s, \qquad (\mathbf{x},t) \in Q_T, \tag{4}$$

where Γ denotes the Gamma function. Unlike to the classical diffusion equation, the model (2) contains a time fractional-order derivative. This is supported by a macroscopic level observation, the diffusion process comes from the random motion of individual particles, and the use of the first-order time derivative in the canonical model rests on a Brownian motion assumption. It is shown experimentally that anomalous diffusion in which the mean-square variance grows slower (subdiffusion) than in a Gaussian process can give a satisfactory fitting to some experimental data [29]. The model (2) as in [6] can be used to handle a setup in a microwave heating process used in various applications in industry, e.g. in ceramics and in food processing. A supply of external energy is targeted at a controlled level by the microwave generating equipment. However, a spatial and temporal variation for the dielectric constant of the target material is leading to spatially heterogeneous conversion of electromagnetic energy to heat. This can be due to a source term $h(t)f(x)$ in (2), where $h(t)$ is proportional to the external energy source power and $f(x)$ is the microwave energy local conversion rate. It is noticed that this spatial variation of absorbing material has no high effect on the thermal diffusivity, this is reasoned to another material at higher concentration. There is a high demand to mention that the temperature is not so high that temperature dependence of the dielectric constant is important, as in thermal runaway studies [35]. If $u(\mathbf{x},t)$ denotes the concentration of the absorbed energy in this model, then its integral over all volume of the material determines the time dependence absorbed energy. The above inverse problem mentioned here for such a model gives an idea of how total energy content might have an external control. The integral condition eq. (3) arises naturally and can be used as supplementary information in the determination of the source term. Such type of condition can model various physical phenomena in the context of many fields [5].

Before starting with discussing the most relevant articles concerning this article, we note some interesting studies about the recognition of the space-dependent part of the source for fractional equations, e.g. [17,6,53,50,37,52,49,41]. Identification of the time-dependent part of the source term in a fractional diffusion equation assuming various boundary conditions and additional measurements has been studied in [36,51,10,38]. The well-posedness of the problem in a one-dimensional setting with nonlocal boundary conditions, $F = 0$ and a measurement in the form of integral over the domain is studied in [6]. In [17] the problem is addressed assuming the unit square domain and a nonlocal boundary condition. In [53] the authors studied the recognition of the space-dependent source in a 1D case from final data; a numerical scheme based on a local discontinuous Galerkin method is investigated. In [50,37] the problem is assumed for the fractional diffusion equation with only space-dependent coefficients, uniqueness is proven and a regularization method is applied. In [52] the authors proposed the Landweber regularization technique for the backward problem in a fractional diffusion equation. The Tikhonov regularization method is applied on the problem considering a simple fractional diffusion equation in the one-dimensional case in [49]. In [38] an ISP for a semilinear time-fractional diffusion equation with a solely time-dependent unknown source was studied. The existence and uniqueness of a weak solution for the ISP were proved. The missing source term was recovered from an integral-type measurement over the domain. A numerical algorithm based on Rothe's method was established and the convergence of approximations towards the exact solution was demonstrated. However, the analysis in this paper gives that the solution is continuously differentiable on the closed time interval. An essential feature of the time Caputo fractional subdiffusion problem is that the solution lacks the smoothness near the initial time although it would be smooth away from $t = 0$. The related discussions are referred to [4,8,28,36,27]. Stynes et al [43] showed that under proper regularity and compatibility assumptions, the one-dimensional subdiffusion problem $\partial_t^\beta u - pu_{xx} + c(x)u = f(x,t)$ with homogeneous Dirichlet boundary condition has a unique classical solution, and there exists a constant C_u such that

$$|u''(t)| \leqslant C_u(1 + t^{\beta-2}), \quad 0 < t \leqslant T. \tag{5}$$

In [18, Section 6.1], this result is generalized to the case $\Omega \subset \mathbb{R}^d$ for $d \in \{2,3\}$. A rigorous numerical analysis of time fractional nonlinear parabolic partial differential equations, with fractional derivative of order $\alpha \in (0,1)$ in time is presented in [9]. A complete solution theory is done for the time fractional nonlinear parabolic partial differential equations with a Lipschitz nonlinear source term. Nonetheless, as pointed out by [4,8,30,31], most of existing works on $L1$ or high order $L1$-type approximations of the Caputo fractional derivative assumed that the solution is smooth at the initial time $t = 0$. The corresponding error estimates are always restrictive since often the solution does not have the required regularity. The Rothe's method in [38] was based on a compatibility condition between the initial data, sources and boundary condition, and so the corresponding error estimates are also restrictive. More recently, the authors in [8,43] analyzed the $L1$-formula on graded time grids by taking into account the initial singularity in the Caputo's fractional subdiffusion problems. They obtained the error estimate under the regularity assumption (5) by using the discrete maximum principle and direct analysis of the local truncation error with the basic assumption

$$\tau_{i-1} \leqslant \tau_i, \quad 2 \leqslant i \leqslant n,$$

where n is the number of time discretization intervals. This assumption has a reasonableness because graded meshes that concentrate the grid points near $t=0$ would be necessary to recompense the lack of smoothness there. With the aid of the discrete fractional Grönwall inequality and the global consistency error analysis, a sharp error estimate reflecting the regularity of solution is obtained for a simple $L1$-scheme [22].

The added value of this paper focuses on the global (in time) solvability of the ISP (2)-(3) under low regularity assumptions and in the proposition of two approximation schemes based on different approximations of the Caputo fractional derivative. We formulate again the ISP for both schemes into an appropriate direct (nonlocal) formulation following the approach in [38] for one of them and graded $L1$-approximation in the other scheme. We propose an interesting variational technique based on elimination of $h(t)$ from (2) by (3). The first proposed numerical scheme is based on a semi-discretization in time by Rothe's method (cf. [11,34]) by following the considered uniform mesh approximation in [38]. We show the existence of approximations at each time step of the time partitioning and we derive suitable stability results. Using a recent result of Van Bockstal [45,46] for the forward problem, we show how to establish the existence of a unique solution under lower regularity assumptions than in [38]. We show that the solution is continuous in the time frame, which takes also into account non-smooth solutions of the form t^γ with $\gamma \in (\beta,1)$. Afterwards by following the benefits of graded times grids, we also propose a semi discretization in time by Rothe's method based on the graded $L1$-formula. In terms of the graded Rothe scheme, we derive also the stability results and show the convergence of approximations in suitable function spaces assuming that the solution satisfies the bounds (5). This approach has the advantage to cope better from numerical viewpoint with the behaviour at $t=0$ in case of non-smooth solutions for the considered problems ($\partial_t u$ blows up as $t \to 0^+$). Finally, we present several numerical examples supporting the obtained convergence results in both uniform and graded mesh schemes.

Remark 1 (Additional notations) We denote by $(\cdot,\cdot)$ the standard inner product in $\mathrm{L}^2(\Omega)$ and by $\|\cdot\|$ its induced norm. The norm $|\cdot|_\mathrm{e}$ denotes the Euclidian norm. The space $\mathrm{L}^1_{\mathrm{loc}}$ consists of all functions that are integrable on any compact subset of their domain of definition. Consider an abstract Banach space X with norm $\|\cdot\|_X$. Its dual space is denoted by X^*. Let $p \geqslant 1$. The space $\mathrm{L}^p((0,T),X)$ consists of measurable functions $u : (0,T) \to X$ such that

$$\|u\|_{\mathrm{L}^p((0,T),X)} = \left(\int_0^T \|u(t)\|_X^p \,\mathrm{d}t \right)^{1/p} < \infty.$$

The space $\mathrm{C}\left([0,T],X\right)$ consists of continuous functions $u : [0,T] \to X$ satisfying

$$\|u\|_{\mathrm{C}([0,T],X)} = \max_{t\in[0,T]} \|u(t)\|_X < \infty.$$

The space $\mathrm{L}^\infty\left((0,T),X\right)$ consists of all measurable functions $u : (0,T) \to X$ that are essentially bounded. The values C, ε and C_ε are considered to be generic and positive constants (independent of the discretization parameter) throughout the paper, where ε is arbitrarily small and C_ε arbitrarily large, i.e. $C_\varepsilon = C\left(1+\varepsilon+\frac{1}{\varepsilon}\right)$. The same notation for different constants is used, but the meaning should be clear from the context.

Remark 2 (The regularity of m determined in terms of u can be of the assumed regularity (5)*)* In comparion with [43], we consider an additional term $\int_0^t F(u(s))\ \mathrm{d}s$ in the problem (2). We show that the solution u to the corresponding forward problem has the assumed regularity (5) in the special case that F is linear in u. We take $\Omega = (0,l)$ with $l > 0$ and $F(u) = -u$, i.e.

$$\begin{cases} \left(\partial_t^\beta u\right)(x,t) + (Lu)(x,t) = -\int_0^t u(x,s)\ \mathrm{d}s & (x,t) \in Q_T, \\ u'(0) = u'(l) = 0 & (x,t) \in \Sigma_T, \\ u(x,0) = \tilde{u}_0(x) & x \in \Omega, \end{cases} \tag{6}$$

where

$$Lu = -pu_{xx} + cu, \quad p,c > 0.$$

We use the Fourier method (separation of variables) to construct the solution to (6), cfr. [27]. Prescribing a particular solution of the form $u(x,t) = X(x)T(t)$ leads to the following fractional differential equation

$$\left(\partial_t^\beta T\right)(t) + \lambda T(t) + \int_0^t T(s)\ \mathrm{d}s = 0, \quad t \in (0,T), \tag{7}$$

and the eigenvalue problem

$$\begin{cases} (LX)(x) = \lambda X(x) & x \in (0,l) \\ X'(0) = X'(l) = 0, \end{cases} \tag{8}$$

where λ is the separation constant. Since the operator L is symmetric and elliptic, it follows from the Sturm-Liouville theory (see e.g. [47]) that there exists a sequence of eigenvalues $\{\lambda_i\}$ satisfying $0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots$ with corresponding orthonormal eigenfunctions $\{X_i\}$ such that the couple $\{\lambda_i, X_i\}$ is solving problem (8) for any $i \in \mathbb{N}$. Next, we solve problem (7) for $\lambda = \lambda_i$ by using the Laplace transform method. We obtain that

$$\mathscr{L}\left[T_i(t)\right](z) = \frac{z^\beta}{z^{\beta+1} + \lambda_i z + 1} T_i(0), \quad i \in \mathbb{N}.$$

From [16, Lemma 5], it follows for $i \in \mathbb{N}$ that

$$T_i(t) = c_i \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \frac{(-\lambda_i)^{k_2} \binom{k_1+k_2}{k_2}}{\Gamma(k_2\beta + k_1(\beta+1)+1)} t^{k_2\beta + k_1(\beta+1)}$$

$$= c_i S^{1;0;0}_{1;0;0}\left(\begin{matrix}[1,1,1]\,;\,; \\ \\ [1,\beta+1,\beta]\,;\,;\end{matrix}\quad t^{\beta+1}, -\lambda_i t^{\beta}\right),$$

where S represents a generalization of the Kampé De Fériet series [42]

$$S^{A;B;B'}_{C;D;D'}\left(\begin{matrix}[(a),(\theta),(\varphi)]\,;[(b),(\psi)]\,;[(b'),(\psi')] \\ \\ [(c),(\delta),(\varepsilon)]\,;[(d),(\eta)]\,;[(d'),(\eta')]\end{matrix}\quad z_1, z_2\right)$$
$$= \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} \frac{\prod_{j=1}^{A}\Gamma(a_j+k_1\theta_j+k_2\varphi_j)\prod_{j=1}^{B}\Gamma(b_j+k_1\psi_j)\prod_{j=1}^{B'}\Gamma(b'_j+k_2\psi'_j)}{\prod_{j=1}^{C}\Gamma(c_j+k_1\delta_j+k_2\varepsilon_j)\prod_{j=1}^{D}\Gamma(d_j+k_1\eta_j)\prod_{j=1}^{D'}\Gamma(d'_j+k_2\eta'_j)}\frac{z_1^{k_1}z_2^{k_2}}{k_1!k_2!}.$$

Hence, the solution to problem (6) is given by

$$u(x,t) = \sum_{i=1}^{\infty}(\tilde{u}_0, X_i)X_i(x) S^{1;0;0}_{1;0;0}\left(\begin{matrix}[1,1,1]\,;\,; \\ \\ [1,\beta+1,\beta]\,;\,;\end{matrix}\quad t^{\beta+1}, -\lambda_i t^{\beta}\right)$$

and it is absolutely convergent as [42, Eq. (3.10)] is satisfied, i.e.

$$|u(x,t)| \leqslant C, \quad (x,t) \in \overline{Q_T}.$$

Moreover, we have that

$$\partial_t u(x,t) = -\sum_{i=1}^{\infty}(\tilde{u}_0, X_i)X_i(x)\lambda_i t^{\beta-1}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\frac{(-\lambda_i)^{k_2}\binom{k_1+k_2+1}{k_2+1}}{\Gamma(k_2\beta+k_1(\beta+1)+\beta)}t^{k_2\beta+k_1(\beta+1)}$$
$$= -t^{\beta-1}\sum_{i=1}^{\infty}(\tilde{u}_0, X_i)X_i(x)\lambda_i S^{1;0;1}_{1;0;1}\left(\begin{matrix}[2,1,1]\,;\,;[1,1] \\ \\ [\beta,\beta+1,\beta]\,;\,;[2,1]\end{matrix}\quad t^{\beta+1}, -\lambda_i t^{\beta}\right),$$

and therefore

$$|\partial_t u(x,t)| \leqslant C t^{\beta-1}, \quad (x,t) \in Q_T.$$

Repeating this procedure, we conclude that the regularity (5) on the solution to problem (2) is a reasonable assumption.

2 Weak formulation

First, we state the assumptions on the data functions in the differential operator eq. (1). The matrix $A = (a_{ij}(\mathbf{x},t))$ is a $d \times d$ matrix-valued function such that $A \in \mathbf{L}^{\infty}(\overline{Q_T}) := \left(\mathrm{L}^{\infty}(\overline{Q_T})\right)^{d\times d}$ is uniformly elliptic, i.e. there exists a constant $\alpha > 0$ such that

$$\sum_{i,j=1}^{d} a_{ij}(\mathbf{x},t)\xi_i\xi_j \geqslant \alpha\,|\xi|^2, \quad \text{for a.a. } (\mathbf{x},t) \in \overline{Q_T} \text{ and for all } \xi \in \mathbb{R}^d. \tag{9}$$

Moreover, we suppose that $c \in \mathrm{L}^\infty([0,T])$ such that

$$c(t) \geqslant 0, \qquad t \in [0,T].$$

We associate a bilinear form $\mathscr{L}$ with the differential operator L defined in eq. (1) as follows

$$\mathscr{L}(t)\,(u(t),\varphi) := (Lu,\varphi) = (A(t)\nabla u(t),\nabla\varphi) + c(t)\,(u(t),\varphi),$$

with $u(t),\varphi \in \mathrm{H}^1(\Omega)$. Using the properties above, there exists a positive constant C such that

$$\mathscr{L}(t)\,(u,\varphi) \leqslant C\,\|u\|_{\mathrm{H}^1(\Omega)}\,\|\varphi\|_{\mathrm{H}^1(\Omega)}, \quad \forall u,\varphi \in \mathrm{H}^1(\Omega), \tag{10}$$

and

$$\mathscr{L}(t)\,(\varphi,\varphi) \geqslant \alpha\,\|\nabla\varphi\|^2, \quad \forall\varphi \in \mathrm{H}^1(\Omega). \tag{11}$$

Further, we rewrite the Caputo fractional derivative defined in eq. (4) as follows

$$\left(\partial_t^\beta u\right)(\mathbf{x},t) = \partial_t\left(k*(u-\tilde{u}_0)\right)(\mathbf{x},t), \qquad (\mathbf{x},t) \in Q_T,$$

with

$$k(t) = \frac{t^{-\beta}}{\Gamma(1-\beta)}, \qquad t > 0.$$

The symbol '$*$' stands for the convolution product defined by $(k*z)(t) = \int_0^t k(t-s)z(s)\ \mathrm{d}s$. The singular kernel $k \in \mathrm{L}^1(0,T)$ satisfies $k(t) \geqslant 0$ for $t > 0$. Moreover, $\partial_t k \in \mathrm{L}^1_{\mathrm{loc}}(0,T)$ with $\partial_t k(t) \leqslant 0$ for all $t > 0$. Finally, we note that k is strongly positive definite as also $\partial_{tt}k(t) \geqslant 0$ for all $t > 0$ [33, Corollary 2.2].

Now, we want to obtain an expression for the unknown source function h in terms of the unknown function u and the data. Using the measurement (3), we integrate the partial differential equation (PDE) in (2) over the domain Ω and we apply the Divergence Theorem to obtain that

$$h(t) = \frac{(k*m')(t) + \int_\Gamma g(\mathbf{x},t)\ \mathrm{d}\mathbf{x} + c(t)m(t) - \displaystyle\int_\Omega \left(\int_0^t F(\mathbf{x},s,u(\mathbf{x},s))\ \mathrm{d}s\right)\ \mathrm{d}\mathbf{x}}{\int_\Omega f(\mathbf{x})\ \mathrm{d}\mathbf{x}}, \tag{12}$$

where we have assumed that

$$\int_\Omega f(\mathbf{x})\ \mathrm{d}\mathbf{x} \neq 0$$

and that the measurement m is absolutely continuous in the time-variable. Note that at this position we need the Neumann boundary condition and the assumption that the coefficient c in eq. (1) is solely time-dependent as we are not able to obtain the uniqueness of a solution to the ISP when u is not integrated over time in the expression for h. The variational formulation of problem (2)-(3) can now be defined as follows:

$$\begin{gathered}\text{search } \{h,u\} \in \mathrm{L}^2(0,T) \times \mathrm{L}^2\left((0,T),\mathrm{H}^1(\Omega)\right) \text{ with}\\ \partial_t(k*(u-\tilde{u}_0)) \in \mathrm{L}^2\left((0,T),\mathrm{H}^1_0(\Omega)^*\right)\\ \text{such that for a.a. } t\in(0,T) \text{ it holds that}\end{gathered}$$

$$\begin{aligned}&\langle \partial_t(k*(u-\tilde{u}_0))(t),\varphi\rangle_{\mathrm{H}^1(\Omega)^*\times\mathrm{H}^1(\Omega)} + \mathscr{L}(t)\,(u(t),\varphi)\\ &\quad = h(t)\,(f,\varphi) + \left(\int_0^t F(s,u(s))\,\mathrm{d}s,\varphi\right) - (g(t),\varphi)_\Gamma, \quad \forall\varphi\in\mathrm{H}^1(\Omega), \quad (13)\end{aligned}$$

$$\text{with } h(t) \text{ given by (12).}$$

3 Existence of a unique solution

The existence of a solution is shown by the aid of Rothe's time-discretization method. The time interval $[0,T]$ is divided into $n\in\mathbb{N}$ equidistant subintervals $[t_{i-1},t_i]$ with length $\tau=\frac{T}{n}<1$. The approximation of a function z at time $t=t_i$, $0\leqslant i\leqslant n$, is denoted by z_i. The same notation is also used for any given function. Moreover, we approximate the time derivative at time t_i by the backward Euler difference, i.e.

$$\partial_t z(t_i) \approx \delta z_i = \frac{z_i - z_{i-1}}{\tau}, \qquad 1\leqslant i\leqslant n.$$

Finally, the time discrete convolution is defined as follows

$$(k*z)(t_i) \approx (k*z)_i := \sum_{l=1}^{i} k_{i+1-l} z_l \tau. \qquad (14)$$

We define

$$(k*z)_0 := 0. \qquad (15)$$

From [40, Lemma 3.2], it follows that for a sequence $(v_i)_{i\in\mathbb{N}}$ in $\mathrm{H}^1(\Omega)$ it holds that

$$\begin{aligned}\sum_{i=1}^{j}\langle\delta(k*v)_i,v_i\rangle_{\mathrm{H}^1(\Omega)^*\times\mathrm{H}^1(\Omega)}\tau &= \sum_{i=1}^{j}\left(\delta(k*v)_i,v_i\right)\tau\\ &\geqslant \frac{1}{2}\left(k*\|v\|^2\right)_j + \frac{1}{2}\sum_{i=1}^{j}k_i\|v_i\|^2\tau, \quad j\in\mathbb{N}, \quad (16)\end{aligned}$$

with

$$\left(k*\|v\|^2\right)_j := \sum_{l=1}^{j} k_{j+1-l}\|v_l\|^2\tau.$$

The following crucial relation holds true

$$\delta(k*u)_i = k_i\tilde{u}_0 + (k*\delta u)_i, \quad i\geqslant 1. \qquad (17)$$

We approximate problem (12)-(13) at time $t = t_i$ as follows: Find $u_i \in \mathrm{H}^1(\Omega)$ and $h_i \in \mathbb{R}$ such that for all $\varphi \in \mathrm{H}^1(\Omega)$ it holds that

$$\begin{aligned}\langle (k*\delta u)_i, \varphi\rangle_{\mathrm{H}^1(\Omega)^*\times \mathrm{H}^1(\Omega)} &+ \mathscr{L}_i(u_i,\varphi) \\ &= h_i\,(f,\varphi) + \left(\sum_{l=1}^{i} F(t_l,u_{l-1})\tau, \varphi\right) - (g_i,\varphi)_\Gamma, \quad (18)\end{aligned}$$

with

$$h_i = \frac{(k*m')_i + \int_\Gamma g_i(\mathbf{x})\ \mathrm{d}\mathbf{x} + c_i m_i - \int_\Omega \left(\sum_{l=1}^{i} F(\mathbf{x},t_l,u_{l-1}(\mathbf{x}))\tau\right)\ \mathrm{d}\mathbf{x}}{\int_\Omega f(\mathbf{x})\ \mathrm{d}\mathbf{x}}. \quad (19)$$

Note that $(k*m')_i$ can also be approximated for $i \geqslant 1$ as

$$(k*m')_i \approx (k*\delta m)_i = \sum_{l=1}^{i} k_{i+1-l}\delta m_l \tau = k_1 m_i - k_i m_0 + \sum_{l=1}^{i-1} \delta k_{i-l+1} m_l \tau. \quad (20)$$

The system (18)-(19) is decoupled. First, for given $i = 1,\dots,n$, we determine h_i from (19) and afterwards we solve the elliptic problem (18). Using the time discrete convolution (14), the discrete problem (18) can be equivalently written as

$$a_i(u_i,\varphi) = \langle \tilde{F}_i, \varphi\rangle, \quad \forall \varphi \in \mathrm{H}^1(\Omega), \quad (21)$$

with

$$a_i(u_i,\varphi) = k(\tau)\,(u_i,\varphi) + \mathscr{L}_i(u_i,\varphi)$$

and

$$\begin{aligned}\langle \tilde{F}_i, \varphi\rangle = h_i\,(f,\varphi) &+ \left(\sum_{l=1}^{i} F(t_l,u_{l-1})\tau, \varphi\right) - (g_i,\varphi)_\Gamma \\ &+ k(\tau)\,(u_{i-1},\varphi) - \sum_{l=1}^{i-1} k_{i+1-l}\,(u_l - u_{l-1},\varphi). \quad (22)\end{aligned}$$

The uniform boundedness of h_i and $\|u_i\|_{\mathrm{H}^1(\Omega)}$ is crucial in the existence proof later. These results are established in the next lemmas. We need to assume that eq. (12) is well-defined at $t = 0$ and we consider

$$h_0 = \frac{(k*m')(0) + \int_\Gamma g_0(\mathbf{x})\ \mathrm{d}\mathbf{x} + c_0 m_0}{\int_\Omega f(\mathbf{x})\ \mathrm{d}\mathbf{x}} \in \mathbb{R}. \quad (23)$$

Note that $(k*m')(0)$ is not necessarily equal to zero or well-defined for $m \in \mathrm{C}^1((0,T])$ as the following example shows: when $m(t) = t^\gamma$ with $\gamma \in (0,\infty)$, we get that

$$(k*m')(t) = \begin{cases} \frac{t^{-\beta+\gamma}\Gamma(\gamma+1)}{\Gamma(-\beta+\gamma+1)} & \gamma \neq \beta, \\ \Gamma(\beta+1) & \gamma = \beta, \end{cases}$$

and

$$(k*m')(0)=\lim_{t\searrow 0}(k*m')(t)=\begin{cases}0 & \gamma>\beta,\\ \Gamma(\beta+1) & \gamma=\beta,\\ \infty & \gamma<\beta.\end{cases}$$

In the next lemmas, we put conditions on m such that $(k*m')_i$ is uniformly bounded and $(k*m')(0)=0$, respectively. We note that the existence of a unique solution on every time step follows from the Lax-Milgram lemma and it is stated in the following lemma.

Lemma 1 *Assume that* $\tilde{u}_0\in \mathrm{L}^2(\Omega)$*,* $f\in \mathrm{L}^2(\Omega)$ *with* $\int_\Omega f(\mathbf{x})\ \mathrm{d}\mathbf{x}\neq 0$*,* F *be globally Lipschitz continuous in all variables and* $g\in \mathrm{L}^2\left((0,T),\mathrm{L}^2(\Gamma)\right)$*. Moreover, assume* $m\in \mathrm{C}^1((0,T])$ *satisfying*

$$\left|\frac{\partial^l m}{\partial t^l}(t)\right|\leqslant C(1+t^{\gamma-l})\quad \textit{for } l=0,1,\quad \textit{and for } 0<t\leqslant T;$$

with $\gamma\in(\beta,1)$ *fixed. Then, for any* $i=1,2,\dots,n$*, there exists a unique couple* $\{u_i,h_i\}\in \mathrm{H}^1(\Omega)\times\mathbb{R}$ *solving* (18)-(19).

Remark 3 Note that under the conditions on m in Lemma 1, we have that $(k*m')_i$ is uniformly bounded as

$$\begin{aligned}\left|(k*m')_i\right| &\leqslant C\sum_{l=1}^{i}k_{i+1-l}(1+t_l^{\gamma-1})\tau\\ &\leqslant C\int_0^{t_i}(t_i-s)^{-\beta}\ \mathrm{d}s+C\int_0^{t_i}(t_i-s)^{-\beta}s^{\gamma-1}\ \mathrm{d}s\\ &\leqslant C\left(T^{1-\beta}+T^{\gamma-\beta}\right).\end{aligned}$$

Moreover, we have that

$$\lim_{t\searrow 0}\left|(k*m')(t)\right|\leqslant C\lim_{t\searrow 0}\int_0^t(t-s)^{-\beta}(1+s^{\gamma-1})\ \mathrm{d}s\leqslant C\lim_{t\searrow 0}\left(t^{1-\beta}+t^{\gamma-\beta}\right)=0.$$

The following lemmas are required to ensure the existence of a solution to (12)-(13) and to prove the convergence of approximations towards this solution. The first lemma follows the lines of [38, Lemma 3.3]. Therefore, we omit its proof.

Lemma 2 *Let the assumptions of Lemma 1 be fulfilled. Then, positive constants* C *and* τ_0 *exist such that for and* $\tau<\tau_0$ *and every* $j=1,2,\dots,n$*, the following relations hold*

$$\left(k*\|u\|^2\right)_j+\sum_{i=1}^{j}k_i\|u_i\|^2\tau+\sum_{i=1}^{j}\|u_i\|^2_{\mathrm{H}^1(\Omega)}\tau\leqslant C,\qquad \left|h_j\right|\leqslant C.$$

Lemma 3 *Let the assumptions of Lemma 1 be fulfilled. Assume that* $\tilde{u}_0 \in \mathrm{H}^1(\Omega)$*,* $A^\top = A$*,* $\partial_t A \in \left(\mathrm{L}^\infty(Q_T)\right)^{d\times d}$*,* $c(t) \geqslant \tilde{c}_0 > 0$ *for* $t \in [0,T]$*, and* $\partial_t c \in \mathrm{L}^\infty(0,T)$*. Moreover, assume that eq.* (23) *is satisfied with* $(k * m')(0) = 0$*, i.e.* $m \in \mathrm{C}^1((0,T])$ *and* $g \in \mathrm{C}^1\left((0,T], \mathrm{L}^2(\Gamma)\right)$ *satisfying*

$$\left|\frac{\partial^l m}{\partial t^l}(t)\right| \leqslant C(1+t^{\gamma-l}) \qquad \text{for } l = 0,1 \quad \text{and for } 0 < t \leqslant T; \tag{24}$$

$$\left|(k*m')'(t)\right| \leqslant C(1+t^{\gamma-\beta-1}) \qquad \text{for } 0 < t \leqslant T; \tag{25}$$

$$\left\|\frac{\partial^l g}{\partial t^l}(t)\right\|_{\mathrm{L}^2(\Gamma)} \leqslant C(1+t^{\gamma-l}) \qquad \text{for } l = 0,1 \quad \text{and for } 0 < t \leqslant T; \tag{26}$$

with $\gamma \in (\beta,1)$ *fixed. Then, there exist positive constants C and* τ_0 *such that for every* $j = 1,2,\ldots,n$ *and* $\tau < \tau_0$*, the followings relations hold true*

$$\left\|u_j\right\|^2_{\mathrm{H}^1(\Omega)} + \sum_{i=1}^j \left\|u_i - u_{i-1}\right\|^2_{\mathrm{H}^1(\Omega)} \leqslant C \quad \text{and} \quad \sum_{i=1}^j \left|\delta h_i\right| \tau \leqslant C.$$

Proof We set $\varphi = \delta u_i \tau$ in eq. (18) and sum the result up for $i = 1,\ldots,j$ with $1 \leqslant j \leqslant n$. We obtain that

$$\begin{aligned}\sum_{i=1}^j \langle (k*\delta u)_i, \delta u_i\rangle_{\mathrm{H}^1(\Omega)^*\times \mathrm{H}^1(\Omega)} \tau + \sum_{i=1}^j \mathscr{L}_i(u_i,\delta u_i)\tau \\ = \sum_{i=1}^j h_i\,(f,\delta u_i)\,\tau + \sum_{i=1}^j \left(\sum_{l=1}^i F(t_l,u_{l-1})\tau, \delta u_i\right)\tau - \sum_{i=1}^j (g_i,\delta u_i)_\Gamma\,\tau.\end{aligned} \tag{27}$$

The positivity of the first term on the left-hand side (LHS) follows from [39, Eq. 3.2]. Taking into account the symmetry of A, we have that

$$\begin{aligned}\sum_{i=1}^j (A_i \nabla u_i, \nabla \delta u_i)\,\tau = \frac{1}{2}(A_j \nabla u_j, \nabla u_j) - \frac{1}{2}(A_0 \nabla \tilde{u}_0, \nabla \tilde{u}_0) \\ - \frac{1}{2}\sum_{i=1}^j (\delta A_i \nabla u_{i-1}, \nabla u_{i-1})\,\tau \\ + \frac{1}{2}\sum_{i=1}^j (A_i(\nabla u_i - \nabla u_{i-1}), \nabla u_i - \nabla u_{i-1}).\end{aligned}$$

It follows from Lemma 2 that

$$\sum_{i=1}^j (A_i \nabla u_i, \nabla \delta u_i)\,\tau \geqslant \frac{\alpha}{2}\left\|\nabla u_j\right\|^2 - C + \frac{\alpha}{2}\sum_{i=1}^j \left\|\nabla u_i - \nabla u_{i-1}\right\|^2.$$

Analogously, we get that

$$\sum_{i=1}^j c_i\,(u_i,\delta u_i)\,\tau \geqslant \frac{\tilde{c}_0}{2}\left\|u_j\right\|^2 - C + \frac{\tilde{c}_0}{2}\sum_{i=1}^j \left\|u_i - u_{i-1}\right\|^2.$$

Note that we don't obtain information about δu_i from the terms on the LHS of eq. (27). The terms on the right-hand side (RHS) of eq. (27) can be estimated by using the following per partes formula: for any sequences of real numbers $\{z_i\}_{i=0}^j$ and $\{w_i\}_{i=0}^j$, it holds that

$$\sum_{i=1}^j z_i(w_i - w_{i-1}) = z_j w_j - z_0 w_0 - \sum_{i=1}^j w_{i-1}(z_i - z_{i-1}),$$

Thus for the first term on the RHS of eq. (27), we have that

$$\sum_{i=1}^j h_i\left(f, \delta u_i\right)\tau = h_j\left(f, u_j\right) - h_0\left(f, \tilde{u}_0\right) - \sum_{i=1}^j \delta h_i\left(f, u_{i-1}\right)\tau. \tag{28}$$

First, we derive a bound on δh_i by applying the δ-operation on eq. (19). We obtain that

$$\delta h_i = \frac{\delta(k*m')_i + \int_\Gamma \delta g_i(\mathbf{x})\ \mathrm{d}\mathbf{x} + \delta(c_i m_i) - \int_\Omega f(\mathbf{x}, t_i, u_{i-1}(\mathbf{x}))\ \mathrm{d}\mathbf{x}}{\int_\Omega f(\mathbf{x})\ \mathrm{d}\mathbf{x}}, \quad i \geqslant 1.$$

We use the mean value theorem and (24-26) to obtain for $i \geqslant 2$ that

$$\begin{aligned} \left|\delta(k*m')_i\right| &\leqslant C(1 + t_{i-1}^{\gamma-\beta-1}), \\ \|\delta g_i\|_{\mathrm{L}^2(\Gamma)} &\leqslant C\left(1 + t_{i-1}^{\gamma-1}\right), \\ |\delta(c_i m_i)| = |m_i \delta c_i + c_{i-1}\delta m_i| &\leqslant C\left(1 + t_{i-1}^{\gamma-1}\right). \end{aligned}$$

Moreover, we have as $\tau < 1$ and $\gamma \in (\beta, 1)$ that

$$\begin{aligned} |\delta(c_1 m_1)| &= \frac{|c_1 m_1 - c_0 m_0|}{\tau} \leqslant C\left(\tau^{-1} + \tau^{\gamma-1}\right) \leqslant C\tau^{-1}, \\ \left|\delta(k*m')_1\right| &= \left|k_1 m_1'\right| \leqslant C\left(\tau^{-\beta} + \tau^{\gamma-\beta-1}\right) \leqslant C\tau^{-1}, \\ \|\delta g_1\|_{\mathrm{L}^2(\Gamma)} &= \frac{|g_1 - g_0|}{\tau} \leqslant C\left(\tau^{-1} + \tau^{\gamma-1}\right) \leqslant C\tau^{-1}. \end{aligned}$$

Therefore, we have that

$$\begin{aligned} |\delta h_1| &\leqslant C + C\|\tilde{u}_0\| + C\tau^{-1}, \\ |\delta h_i| &\leqslant C + C\|u_{i-1}\| + Ct_{i-1}^{\gamma-1} + Ct_{i-1}^{\gamma-\beta-1}, \quad i \geqslant 2. \end{aligned} \tag{29}$$

Therefore, using that $t^{\gamma-1}, t^{\gamma-\beta-1} \in \mathrm{L}^1(0,T)$ and Lemma 2, we see that

$$\begin{aligned} \left|\sum_{i=1}^j \delta h_i\left(f, u_{i-1}\right)\tau\right| &= \left|\tau\delta h_1\left(f, \tilde{u}_0\right) + \sum_{i=2}^j \delta h_i\left(f, u_{i-1}\right)\tau\right| \\ &\leqslant C + C\sum_{i=2}^j \left(1 + \|u_{i-1}\| + t_{i-1}^{\gamma-1} + t_{i-1}^{\gamma-\beta-1}\right)\|u_{i-1}\|\,\tau \end{aligned}$$

$$\leqslant C+C\sum_{i=1}^{j-1}\left(t_i^{\gamma-1}+t_i^{\gamma-\beta-1}\right)\|u_i\|^2\,\tau.$$

Hence, from eq. (28), using Lemma 2, we obtain that

$$\left|\sum_{i=1}^{j} h_i\left(f,\delta u_i\right)\tau\right| \leqslant C_{\varepsilon_1}+\varepsilon_1\left\|u_j\right\|^2+C\sum_{i=1}^{j-1}\left(t_i^{\gamma-1}+t_i^{\gamma-\beta-1}\right)\|u_i\|^2\,\tau.$$

For the second term on the RHS of eq. (27), we deduce by the Lipschitz continuity of F and Lemma 2 that

$$\begin{aligned}\left|\sum_{i=1}^{j}\left(\sum_{l=1}^{i}F(t_l,u_{l-1})\tau,\delta u_i\right)\tau\right| &= \left|\left(\sum_{l=1}^{j}F(t_l,u_{l-1})\tau,u_j\right)-\sum_{i=1}^{j}\left(F(t_i,u_{i-1}),u_{i-1}\right)\tau\right|\\ &\leqslant C_{\varepsilon_2}+\varepsilon_2\left\|u_j\right\|^2.\end{aligned}$$

Using the conditions (26) on g and the mean value theorem, we have that

$$\begin{aligned}\left|\sum_{i=1}^{j}(g_i,\delta u_i)_\Gamma\,\tau\right| &= \left|(g_j,u_j)_\Gamma-(g_0,\tilde{u}_0)_\Gamma-\sum_{i=1}^{j}(\delta g_i,u_{i-1})_\Gamma\,\tau\right|\\ &\leqslant C_{\varepsilon_3}+\varepsilon_3\left\|u_j\right\|^2_{\mathrm{H}^1(\Omega)}+\|\delta g_1\|\,\|u_0\|\,\tau+\sum_{i=2}^{j}\|\delta g_i\|_{\mathrm{L}^2(\Gamma)}\,\|u_{i-1}\|_{\mathrm{L}^2(\Gamma)}\,\tau\\ &\leqslant C_{\varepsilon_3}+\varepsilon_3\left\|u_j\right\|^2_{\mathrm{H}^1(\Omega)}+C\sum_{i=2}^{j}(1+t_{i-1}^{\gamma-1})\,\|u_{i-1}\|_{\mathrm{L}^2(\Gamma)}\,\tau\\ &\leqslant C_{\varepsilon_3}+\varepsilon_3\left\|u_j\right\|^2_{\mathrm{H}^1(\Omega)}+C\sum_{i=1}^{j-1}t_i^{\gamma-1}\,\|u_i\|^2_{\mathrm{H}^1(\Omega)}\,\tau.\end{aligned}$$

Collecting all the estimates above, we arrive at

$$\begin{aligned}&\left(\frac{\tilde{c}_0}{2}-\varepsilon_1-\varepsilon_2-\varepsilon_3\right)\left\|u_j\right\|^2+\left(\frac{\alpha}{2}-\varepsilon_3\right)\left\|\nabla u_j\right\|^2\\ &\qquad+\frac{\tilde{c}_0}{2}\sum_{i=1}^{j}\|u_i-u_{i-1}\|^2+\frac{\alpha}{2}\sum_{i=1}^{j}\|\nabla u_i-\nabla u_{i-1}\|^2\\ &\leqslant C_{\varepsilon_1,\varepsilon_2,\varepsilon_3}+C\sum_{i=1}^{j-1}\left(t_i^{\gamma-1}+t_i^{\gamma-\beta-1}\right)\|u_i\|^2\,\tau+C\sum_{i=1}^{j-1}t_i^{\gamma-1}\,\|u_i\|^2_{\mathrm{H}^1(\Omega)}\,\tau.\end{aligned}$$

First, we fix $\varepsilon_i>0, i=1,\ldots,3$ and τ sufficient small such that

$$\left\|u_j\right\|^2_{\mathrm{H}^1(\Omega)}+\sum_{i=1}^{j}\|u_i-u_{i-1}\|^2_{\mathrm{H}^1(\Omega)}\leqslant C+C\sum_{i=1}^{j-1}\left(t_i^{\gamma-1}+t_i^{\gamma-\beta-1}\right)\|u_i\|^2_{\mathrm{H}^1(\Omega)}\,\tau.$$

We apply the discrete Grönwall lemma [2, Corollary 15.5] to conclude the first estimate. The second estimate follows from eq. (29) and the first estimate.

Corollary 1 *Let the assumptions of Lemma 3 be fulfilled. Then, there exist positive constants C and τ_0 such that for every $j = 1,2,\ldots,n$ and $\tau < \tau_0$, the following relation hold*

$$\left\|(k*\delta u)_j\right\|_{\mathrm{H}^1(\Omega)^*} \leqslant C.$$

Now, we introduce the following Rothe functions

$$u_n : [0,T] \to \mathrm{L}^2(\Omega) : t \mapsto \begin{cases} \tilde{u}_0 & t = 0, \\ u_{i-1} + (t - t_{i-1})\delta u_i & t \in (t_{i-1}, t_i], \quad 1 \leqslant i \leqslant n; \end{cases}$$

$$\overline{u}_n : [0,T] \to \mathrm{L}^2(\Omega) : t \mapsto \begin{cases} \tilde{u}_0 & t = 0, \\ u_i & t \in (t_{i-1}, t_i], \quad 1 \leqslant i \leqslant n; \end{cases}$$

$$\tilde{u}_n : [0,T] \to \mathrm{L}^2(\Omega) : t \mapsto \begin{cases} \tilde{u}_0 & t \in [0,\tau], \\ \overline{u}_n(t-\tau) & t \in (t_{i-1}, t_i], \quad 2 \leqslant i \leqslant n. \end{cases}$$

Similarly, we define $\overline{k}_n$, $\overline{\mathscr{L}}_n$, $\overline{F}_n$, $\overline{g}_n$, $\overline{m}_n$, $\overline{m'}_n$ and $\overline{h}_n$. Moreover, we define

$$(k*u)_n : [0,T] \to \mathrm{L}^2(\Omega) : t \mapsto \begin{cases} 0 & t = 0 \\ (k*u)_{i-1} + (t - t_{i-1})\delta(k*u)_i & t \in (t_{i-1}, t_i] \end{cases}.$$

Using these so-called Rothe's functions and eq. (17), the variational system (18)-(19) can be rewritten on the whole time frame as

$$\begin{aligned} &\langle \partial_t (k*u)_n(t) - \overline{k}_n(t)\tilde{u}_0, \varphi \rangle_{\mathrm{H}^1(\Omega)^* \times \mathrm{H}^1(\Omega)} + \overline{\mathscr{L}}_n(t)(\overline{u}_n(t), \varphi) \\ &\quad = \overline{h}_n(t)\,(f,\varphi) + \left(\int_0^t \overline{F}_n(s, \tilde{u}_n(s))\, \mathrm{d}s, \varphi \right) - (\overline{g}_n(t), \varphi)_\Gamma, \quad \forall \varphi \in \mathrm{H}^1(\Omega), \end{aligned} \tag{30}$$

and

$$\begin{aligned} \overline{h}_n(t)\,(f,1) &= (\overline{k}_n * \overline{m'}_n)(t) + \overline{c}_n(t)\overline{m}_n(t) \\ &\qquad + (\overline{g}_n(t), 1)_\Gamma - \left(\int_0^t \overline{F}_n(s, \tilde{u}_n(s))\, \mathrm{d}s, 1 \right). \end{aligned} \tag{31}$$

Note that

$$\overline{\mathscr{L}}_n(t)\,(\overline{u}_n(t), \varphi) = \left(\overline{A}_n(t)\nabla\overline{u}_n(t), \nabla\varphi\right) + \overline{c}_n(t)\,(\overline{u}_n(t), \varphi).$$

Theorem 1 (Existence)
Suppose that the conditions of Lemma 3 are fulfilled. Then there exists a couple $\{u,h\}$ to the problem (12)-(13) *with*

$$u \in \mathrm{C}\left([0,T], \mathrm{H}^1(\Omega)^*\right) \cap \mathrm{L}^\infty\left((0,T), \mathrm{H}^1(\Omega)\right), \; h \in \mathrm{L}^\infty(0,T)$$

and

$$\partial_t (k*(u - \tilde{u}_0)) \in \mathrm{L}^\infty\left((0,T), \mathrm{H}^1(\Omega)^*\right).$$

Proof The sequence $(\overline{u}_n)_{n\in\mathbb{N}}$ is uniformly bounded and 2-mean equicontinuous in $\mathrm{L}^2(Q_T) = \mathrm{L}^2\left((0,T),\mathrm{L}^2(\Omega)\right)$ by Lemma 3. Hence, from the Riesz-Frechét-Kolmogorov [20, Theorem 2.13.1], we have the existence of an element $u \in \mathrm{L}^2\left((0,T),\mathrm{L}^2(\Omega)\right)$ and a subsequence $(\overline{u}_{n_l})_{l\in\mathbb{N}}$ of $(\overline{u}_n)_{n\in\mathbb{N}}$ such that

$$\overline{u}_{n_l} \to u \text{ in } \mathrm{L}^2\left((0,T),\mathrm{L}^2(\Omega)\right) \text{ as } l \to \infty.$$

From Lemma 2 and the reflexivity of the space $\mathrm{L}^2\left((0,T),\mathrm{H}^1(\Omega)\right)$, we have the existence of a subsequence (indexed by n_l again) such that

$$\overline{u}_{n_l} \rightharpoonup u \text{ in } \mathrm{L}^2\left((0,T),\mathrm{H}^1(\Omega)\right) \text{ as } l \to \infty. \tag{32}$$

From Lemma 3, we obtain that $u \in \mathrm{L}^\infty\left((0,T),\mathrm{H}^1(\Omega)\right)$ and that

$$\int_0^T \left\|\overline{u}_{n_l}(t) - \tilde{u}_{n_l}(t)\right\|^2 \mathrm{d}t \leqslant \tau_{n_l} \sum_{i=1}^{n_l} \|u_i - u_{i-1}\|^2 \leqslant C\tau_{n_l},$$

i.e. we have that

$$\tilde{u}_{n_l} \to u \text{ in } \mathrm{L}^2\left((0,T),\mathrm{L}^2(\Omega)\right) \text{ as } l \to \infty.$$

Moreover, Lemma 2 ensures the boundedness of $\overline{h}_{n_l}$ in the reflexive space $\mathrm{L}^2(0,T)$. Hence, there exists a function $h \in \mathrm{L}^2(0,T)$ and a subsequence of $\overline{h}_{n_l}$ (denoted by the same symbol again) such that

$$\overline{h}_{n_l} \rightharpoonup h \quad \text{in} \quad \mathrm{L}^2(0,T),$$

Therefore, for all $\eta \in (0,T)$ it holds that

$$\int_0^\eta \overline{h}_{n_l}(t)\,\mathrm{d}t \to \int_0^\eta h(t)\,\mathrm{d}t \quad \text{as} \quad l \to \infty.$$

This ends the overview of the convergence results that are necessary to prove the existence of a solution to (12)-(13).

Now, we integrate eq. (30) in time over $(0,\eta) \subset (0,T)$ for the resulting subsequence to get that

$$\begin{aligned}
\langle (k*u)_{n_l}(\eta), \varphi\rangle_{\mathrm{H}^1(\Omega)^*\times\mathrm{H}^1(\Omega)} &- \int_0^\eta \langle \tilde{u}_0\overline{k}_{n_l}(t), \varphi\rangle_{\mathrm{H}^1(\Omega)^*\times\mathrm{H}^1(\Omega)}\,\mathrm{d}t \\
&+ \int_0^\eta \overline{\mathscr{L}}_{n_l}(t)(\overline{u}_{n_l}(t), \varphi)\,\mathrm{d}t = \int_0^\eta \overline{h}_{n_l}(t)\,(f,\varphi)\,\mathrm{d}t \\
&+ \int_0^\eta \left(\int_0^t \overline{F}_{n_l}(s, \tilde{u}_{n_l}(s))\,\mathrm{d}s, \varphi\right)\mathrm{d}t - \int_0^\eta \left(\overline{g}_{n_l}(t), \varphi\right)_\Gamma \mathrm{d}t.
\end{aligned} \tag{33}$$

We have that $\left\|\overline{A}_{n_l} - A\right\|_\infty \to 0$ a.e. in $\overline{Q_T}$ as $l \to \infty$ and $\left|\overline{c}_{n_l} - c\right| \to 0$ a.e. in $[0,T]$ as $l \to \infty$. Moreover, $\left|\overline{g}_{n_l} - g\right| \to 0$ a.e. on $\overline{\Sigma_T}$, $\overline{F}_{n_l} \to F$, $\overline{k}_{n_l} \to k$, $\overline{m}_{n_l} \to m$ and $\overline{m'}_{n_l} \to m'$ in $(0,T)$ as $l \to \infty$. It holds that (see [45, Theorem 3.1] by considering the $\int_0^1 \tilde{f}(\tau_{n_l},\beta)\mu(\beta)\,\mathrm{d}\beta$ integrals in the proof as $\tilde{f}(\tau_{n_l},\beta)$)

$$\lim_{l\to\infty}\left|\int_0^T \langle (k*u)_{n_l}(t), \varphi\rangle_{\mathrm{H}^1(\Omega)^*\times\mathrm{H}^1(\Omega)}\,\mathrm{d}t - \int_0^T \langle (k*\overline{u}_{n_l})(t), \varphi\rangle_{\mathrm{H}^1(\Omega)^*\times\mathrm{H}^1(\Omega)}\,\mathrm{d}t\right| = 0. \tag{34}$$

The limit transition of the other terms in eq. (33) is standard and can be done using the considerations above. Now, we integrate the equation eq. (33) again in time over $\eta \in (0,\xi) \subset (0,T)$. Then, using eq. (32) and eq. (34), we are allowed to pass to the limit for $l \to \infty$. We get that

$$\begin{aligned}\int_0^\xi \langle (k*u)(\eta),\varphi\rangle_{\mathrm{H}^1(\Omega)^*\times\mathrm{H}^1(\Omega)}\,\mathrm{d}\eta - \int_0^\xi\int_0^\eta \langle \tilde{u}_0 k(t),\varphi\rangle_{\mathrm{H}^1(\Omega)^*\times\mathrm{H}^1(\Omega)}\,\mathrm{d}t\,\mathrm{d}\eta \\ + \int_0^\xi\int_0^\eta \mathscr{L}(t)(u(t),\varphi)\,\mathrm{d}t\,\mathrm{d}\eta = \int_0^\xi\int_0^\eta h(t)\,(f,\varphi)\,\mathrm{d}t\,\mathrm{d}\eta \\ + \int_0^\xi\int_0^\eta \left(\int_0^t F(s,u(s))\ \mathrm{d}s,\varphi\right)\mathrm{d}t\,\mathrm{d}\eta - \int_0^\xi\int_0^\eta (g(t),\varphi)_\Gamma\,\mathrm{d}t\,\mathrm{d}\eta. \quad (35)\end{aligned}$$

Differentiating this relation with respect to ξ leads to

$$\begin{aligned}\langle (k*u)(\xi),\varphi\rangle_{\mathrm{H}^1(\Omega)^*\times\mathrm{H}^1(\Omega)} - \int_0^\xi \langle \tilde{u}_0 k(t),\varphi\rangle_{\mathrm{H}^1(\Omega)^*\times\mathrm{H}^1(\Omega)}\,\mathrm{d}t \\ + \int_0^\xi \mathscr{L}(t)(u(t),\varphi)\,\mathrm{d}t = \int_0^\xi h(t)\,(f,\varphi)\,\mathrm{d}t \\ + \int_0^\xi \left(\int_0^t F(s,u(s))\ \mathrm{d}s,\varphi\right)\mathrm{d}t - \int_0^\xi (g(t),\varphi)_\Gamma\,\mathrm{d}t. \quad (36)\end{aligned}$$

Hence, as $u \in \mathrm{L}^\infty\left((0,T),\mathrm{H}^1(\Omega)\right)$, we have that

$$\lim_{\xi\searrow 0}\langle (k*u)(\xi),\varphi\rangle_{\mathrm{H}^1(\Omega)^*\times\mathrm{H}^1(\Omega)} = 0,$$

i.e. $(k*u)(0) = 0$ in $\mathrm{H}^1(\Omega)^*$. Moreover, differentiating eq. (36) with respect to ξ (and replacing ξ by t) gives that u is satisfying

$$\begin{aligned}\langle \partial_t(k*u)(t) - k(t)\tilde{u}_0,\varphi\rangle_{\mathrm{H}^1(\Omega)^*\times\mathrm{H}^1(\Omega)} + \mathscr{L}(t)\,(u(t),\varphi) \\ = h(t)\,(f,\varphi) + \left(\int_0^t F(s,u(s))\ \mathrm{d}s,\varphi\right) - (g(t),\varphi)_\Gamma, \quad \forall\varphi\in\mathrm{H}^1(\Omega), \quad (37)\end{aligned}$$

i.e. the couple $\{h,u\}$ solves eq. (13). Next, we show that u is continuous in the time variable with values in $\mathrm{H}^1(\Omega)^*$. Since $\partial_t(k*u)(t) - k(t)\tilde{u}_0 = \partial_t\,(k*(u-\tilde{u}_0))\,(t)$ and $(k*(u-\tilde{u}_0))\,(0) = 0$ in $\mathrm{H}^1(\Omega)^*$, integrating eq. (37) in time gives that $(k*(u-\tilde{u}_0))(t)$ is absolutely continuous with values in $\mathrm{H}^1(\Omega)^*$. Note that $g(t) = \frac{t^{\beta-1}}{\Gamma(\beta)}$ satisfies $g*k = 1$ and thus

$$(g*\partial_t\,(k*(u-\tilde{u}_0)))(t) = \partial_t\,(g*k*(u-\tilde{u}_0))\,(t) = u(t) - \tilde{u}_0 \quad \text{in } \mathrm{H}^1(\Omega)^*.$$

Therefore, applying this convolution operation on eq. (37) implies that u is continuous in time and thus $u \in \mathrm{C}\left([0,T],\mathrm{H}^1(\Omega)^*\right)$.

Next, we make the limit transition in the measured problem. We integrate eq. (31) in time over $(0,\eta) \subset (0,T)$ for the resulting subsequence to get that

$$
\begin{aligned}
(f,1)\int_0^\eta \overline{h}_{n_l}(t)\,\mathrm{d}t = \int_0^\eta (\overline{k}_{n_l} * \overline{m'}_{n_l})(t)\,\mathrm{d}t + \int_0^\eta \overline{c}_{n_l}(t)\overline{m}_{n_l}(t)\,\mathrm{d}t \\
+ \int_0^\eta \left(\overline{g}_{n_l}(t),1\right)_\Gamma \mathrm{d}t - \int_0^\eta \left(\int_0^t \overline{F}_{n_l}(s,\tilde{u}_{n_l}(s))\ \mathrm{d}s, 1\right)\mathrm{d}t. \qquad (38)
\end{aligned}
$$

For the first term on the RHS of eq. (38), we use Young's inequality for convolutions to obtain that

$$
\left|\int_0^\eta ([\overline{k}_{n_l} - k] * \overline{m'}_{n_l})(t)\,\mathrm{d}t\right| \leqslant \left\|\overline{k}_{n_l} - k\right\|_{\mathrm{L}^1(0,T)} \left\|\overline{m'}_{n_l}\right\|_{\mathrm{L}^1(0,T)} \leqslant C \left\|\overline{k}_{n_l} - k\right\|_{\mathrm{L}^1(0,T)} \to 0
$$

and

$$
\left|\int_0^\eta (k * \left[\overline{m'}_{n_l} - m'\right])(t)\,\mathrm{d}t\right| \leqslant C \left\|\overline{m'}_{n_l} - m'\right\|_{\mathrm{L}^1(0,T)} \to 0
$$

as $l \to \infty$. Hence, passing to the limit $l \to \infty$ in eq. (38) and differentiating the result with respect to η gives that $\{h,u\}$ solves problem eq. (12). From eq. (12) it follows that $h \in \mathrm{L}^\infty(0,T)$ and hence $\partial_t(k*(u-\tilde{u}_0)) \in \mathrm{L}^\infty\left((0,T),\mathrm{H}^1(\Omega)^*\right)$ considering eq. (13). This concludes the proof.

In the next theorem, we establish the uniqueness of a solution under an additional assumption on F.

Theorem 2 (Uniqueness)
Suppose that the conditions of Lemma 3 are fulfilled. Moreover, assume that F is linear in u, i.e. $F(\mathbf{x},t,u) = \mathscr{G}(\mathbf{x},t)u$ *with* $\partial_t\mathscr{G} \in \mathrm{L}^\infty(Q_T)$. *Then there exists a unique couple* $\{u,h\}$ *to the problem* (12)-(13) *with*

$$
u \in \mathrm{C}\left([0,T],\mathrm{H}^1(\Omega)^*\right) \cap \mathrm{L}^\infty\left((0,T),\mathrm{H}^1(\Omega)\right),\ h \in \mathrm{L}^\infty(0,T)
$$

and

$$
\partial_t(k*(u-\tilde{u}_0)) \in \mathrm{L}^\infty\left((0,T),\mathrm{H}^1(\Omega)^*\right).
$$

Proof The existence of a solution follows from Theorem 2. Now, we prove the uniqueness of a solution by contradiction. We suppose that two solutions $\{h_1,u_1\}$ and $\{h_2,u_2\}$ solve problem (12)-(13). Then, the differences $h := h_1 - h_2$ and $u := u_1 - u_2$ are satisfying

$$
\begin{aligned}
\langle \partial_t\,(k*u)\,(t),\varphi\rangle_{\mathrm{H}^1(\Omega)^*\times\mathrm{H}^1(\Omega)} + \mathscr{L}(t)\,(u(t),\varphi) \\
= h(t)\,(f,\varphi) + \left(\int_0^t \mathscr{G}(s)u(s)\ \mathrm{d}s, \varphi\right), \quad \forall\varphi \in \mathrm{H}^1(\Omega), \qquad (39)
\end{aligned}
$$

and

$$
h(t) = \frac{\left(\int_0^t \mathscr{G}(s)u(s)\ \mathrm{d}s, 1\right)}{(f,1)}. \qquad (40)
$$

Integrating eq. (39) with respect to time over $t \in (0,\eta) \subset (0,T)$, taking $\varphi = u(\eta)$ and integrating again over $\eta \in (0,\xi) \subset (0,T)$ give us

$$
\begin{aligned}
&\int_0^\xi ((k*u)(\eta),u(\eta))\,\mathrm{d}\eta+\int_0^\xi\int_0^\eta \mathscr{L}(t)(u(t),u(\eta))\,\mathrm{d}t\,\mathrm{d}\eta\\
&=\int_0^\xi\left(\int_0^\eta h(t)\,\mathrm{d}t\right)(f,u(\eta))\,\mathrm{d}\eta+\int_0^\xi\left(\int_0^\eta\int_0^t \mathscr{G}(s)u(s)\ \mathrm{d}s\,\mathrm{d}t,u(\eta)\right)\mathrm{d}\eta. \qquad (41)
\end{aligned}
$$

The first term on the LHS of eq. (41) is positive as k is a positive definite kernel. Using integration by parts and $A^{\top}=A$, we get that

$$
\begin{aligned}
\int_0^\xi\left(\int_0^\eta A(t)\nabla u(t)\,\mathrm{d}t,\nabla u(\eta)\right)\mathrm{d}\eta&=\frac{1}{2}\left(A(\xi)\left[\int_0^\xi \nabla u(s)\ \mathrm{d}s\right],\int_0^\xi \nabla u(t)\,\mathrm{d}t\right)\\
&\quad+\frac{1}{2}\int_0^\xi\left(\int_0^\eta \nabla u(t)\,\mathrm{d}t,\partial_t A(\eta)\left[\int_0^\eta \nabla u(s)\ \mathrm{d}s\right]\right)\mathrm{d}\eta\\
&\quad-\left(\int_0^\xi \partial_t A(t)\left[\int_0^t \nabla u(s)\ \mathrm{d}s\right]\mathrm{d}t,\int_0^\xi \nabla u(t)\,\mathrm{d}t\right).
\end{aligned}
$$

Hence, using eq. (11) and the ε-Young inequality, we get that

$$
\begin{aligned}
&\int_0^\xi\left(\int_0^\eta A(t)\nabla u(t)\,\mathrm{d}t,\nabla u(\eta)\right)\mathrm{d}\eta\\
&\qquad\geqslant\left(\frac{\alpha}{2}-\varepsilon_1\right)\left\|\int_0^\xi \nabla u(t)\,\mathrm{d}t\right\|^2-C_{\varepsilon_1}\int_0^\xi\left\|\int_0^\eta \nabla u(t)\,\mathrm{d}t\right\|^2\mathrm{d}\eta.
\end{aligned}
$$

Similarly, we obtain that

$$
\begin{aligned}
&\left|\int_0^\xi\left(\int_0^\eta c(t)u(t)\,\mathrm{d}t,u(\eta)\right)\mathrm{d}\eta\right|\\
&\qquad\geqslant\left(\frac{\tilde{c}_0}{2}-\varepsilon_2\right)\left\|\int_0^\xi u(t)\,\mathrm{d}t\right\|^2-C_{\varepsilon_2}\int_0^\xi\left\|\int_0^\eta u(t)\,\mathrm{d}t\right\|^2\mathrm{d}\eta.
\end{aligned}
$$

From eq. (40), using partial integration, it follows that

$$
\begin{aligned}
|h(t)|&=\left|\frac{1}{(f,1)}\left[\left(-\int_0^t \partial_s\mathscr{G}(s)\left(\int_0^s u(t)\,\mathrm{d}t\right)\ \mathrm{d}s+\mathscr{G}(t)\int_0^t u(s)\ \mathrm{d}s,1\right)\right]\right|\\
&\leqslant C\left\|\int_0^t u(s)\ \mathrm{d}s\right\|,\qquad \text{for } t\in[0,T]. \qquad (42)
\end{aligned}
$$

Hence, using the ε-Young inequality, we estimate the first term in the RHS of eq. (41) as follows

$$
\begin{aligned}
&\left|\int_0^\xi\left(\int_0^\eta h(t)\,\mathrm{d}t\right)(f,u(\eta))\,\mathrm{d}\eta\right|\\
&=\left|-\int_0^\xi h(\eta)\left(\int_0^\eta (f,u(t))\,\mathrm{d}t\right)\mathrm{d}\eta+\left(\int_0^\xi h(t)\,\mathrm{d}t\right)\left(\int_0^\xi (f,u(t))\,\mathrm{d}t\right)\right|\\
&\leqslant C_{\varepsilon_3}\int_0^\xi |h(t)|^2\,\mathrm{d}t+C\int_0^\xi\left\|\int_0^\eta u(t)\,\mathrm{d}t\right\|^2\mathrm{d}\eta+\varepsilon_3\left\|\int_0^\xi u(t)\,\mathrm{d}t\right\|^2
\end{aligned}
$$

$$\overset{(42)}{\leqslant} C_{\varepsilon_3}\int_0^\xi \left\|\int_0^t u(s)\,\mathrm{d}s\right\|^2 \mathrm{d}t + \varepsilon_3 \left\|\int_0^\xi u(t)\,\mathrm{d}t\right\|^2$$

The second term on the RHS of eq. (41) can be handled as follows

$$\begin{aligned}
&\left|\int_0^\xi \left(\int_0^\eta \int_0^t \mathscr{G}(s)u(s)\,\mathrm{d}s\,\mathrm{d}t, u(\eta)\right)\mathrm{d}\eta\right| \\
&= \left|-\int_0^\xi \left(-\int_0^\eta \partial_s\mathscr{G}(s)\left(\int_0^s u(t)\,\mathrm{d}t\right)\mathrm{d}s + \mathscr{G}(\eta)\int_0^\eta u(s)\,\mathrm{d}s, \int_0^\eta u(t)\,\mathrm{d}t\right)\mathrm{d}\eta\right. \\
&\quad \left. + \left(\int_0^\xi \left[-\int_0^t \partial_s\mathscr{G}(s)\left(\int_0^s u(t)\,\mathrm{d}t\right)\mathrm{d}s + \mathscr{G}(t)\int_0^t u(s)\,\mathrm{d}s\right]\mathrm{d}t, \int_0^\xi u(t)\,\mathrm{d}t\right)\right| \\
&\leqslant C_{\varepsilon_4}\int_0^\xi \left\|\int_0^t u(s)\,\mathrm{d}s\right\|^2 \mathrm{d}t + \varepsilon_4 \left\|\int_0^\xi u(t)\,\mathrm{d}t\right\|^2.
\end{aligned}$$

Therefore, it follows that

$$\begin{aligned}
\left(\frac{\tilde{c}_0}{2} - \varepsilon_2 - \varepsilon_3 - \varepsilon_4\right)\left\|\int_0^\xi u(t)\,\mathrm{d}t\right\|^2 + \left(\frac{\alpha}{2} - \varepsilon_1\right)\left\|\int_0^\xi \nabla u(t)\,\mathrm{d}t\right\|^2& \\
\leqslant C_{\varepsilon_{1,2,3,4}}\int_0^\xi \left\|\int_0^t u(s)\,\mathrm{d}s\right\|^2 \mathrm{d}t.&
\end{aligned}$$

Fixing $\varepsilon_i, i = 1,2,3,4$ sufficiently small such that $\varepsilon_1 < \frac{\alpha}{2}$ and $\varepsilon_2 + \varepsilon_3 + \varepsilon_4 < \frac{\tilde{c}_0}{2}$, and applying the Grönwall argument, we obtain that $\int_0^\xi u(t)\,\mathrm{d}t = 0$ a.e. in Ω for all $\xi \in (0,T)$. Differentiating with respect to ξ gives that $u = 0$ a.e. in Q_T. Moreover, from eq. (42), it follows that $h = 0$ in $(0,T)$.

Remark 4 The convergences of Rothe's functions towards the solution have been shown for a subsequence in Theorem 1. Taking into account the uniqueness of a solution, we obtain that the whole sequence of Rothe's functions converge against the solution.

Remark 5 The results of Theorem 1 and Theorem 2 stay valid if the conditions (24-26) are replaced by (with $C_1, C_2 \geqslant 0$ and at least one of these constants non-zero)

$$\begin{aligned}
\left|\frac{\partial^l m}{\partial t^l}(t)\right| &\leqslant C_1 + C_2 t^{\gamma - l} && \text{for } l = 0,1 \quad \text{and for } 0 < t \leqslant T; \\
\left|(k * m')'(t)\right| &\leqslant C_1 + C_2 t^{\gamma - \beta - 1} && \text{for } 0 < t \leqslant T; \\
\left\|\frac{\partial^l g}{\partial t^l}(t)\right\|_{\mathrm{L}^2(\Gamma)} &\leqslant C_1 + C_2 t^{\gamma - l} && \text{for } l = 0,1 \quad \text{and for } 0 < t \leqslant T.
\end{aligned}$$

Hence, the obtained results stay valid if the measurement m belongs to $\mathrm{C}^2([0,T])$, which can be possibly caused by the properties of the functional F.

In the next section, we extend the class of admissible solutions to non-smooth solutions containing a t^β-term. We here first introduce a novel approach in which we construct the Rothe discretization based on nonuniform meshes. This novel approach will be more adequate to deal with the singularity near $(t = 0)$ which is the main theme of time Caputo fractional derivative of order $0 < \beta < 1$ due to its singular kernel.

4 Rothe time discretization based on graded meshes

First, we introduce the $L1$- approximation of the Caputo fractional derivative. It is given by

$$
{}_0D^{\beta}_{t_j}u = \frac{\tau^{-\beta}}{\Gamma(2-\beta)} \sum_{k=1}^{j} a_{j-k}\left(u_k - u_{k-1}\right) + Q^j = \frac{\tau^{-\beta}}{\Gamma(2-\beta)} \sum_{k=1}^{j} b_{j-k} u_k + Q^j
$$

where

$$
a_i = (i+1)^{1-\beta} - i^{1-\beta}
$$

and

$$
b_i = a_i - a_{i-1}, \quad b_0 = a_0 \quad \text{and } b_j = -a_{j-1}.
$$

If $u \in C^2([0,T])$ then $\max_{1\leqslant j\leqslant n}\left|Q^j\right| \leqslant C\tau^{2-\beta}$, see [44, Lemma 4.1] or [23, Eq. 3.3]. Note that the smoothness of the initial condition in (2) and the forcing term F in the r.h.s. of the governing equation (2) doesn't always imply the smoothness of the exact solution. This leads to some loss of accuracy in the approximation of the Caputo fractional derivative, that is $\max_{1\leqslant j\leqslant n}\left|Q^j\right| \leqslant C\tau^{\beta}$ [43].

Now, we consider a graded time-partitioning of the time frame $[0,T]$. We set $t_j = T(j/n)^r$ for $j = 0,1,\ldots,n$, where the constant mesh grading $r \geqslant 1$ is chosen by the user. If $r = 1$, then the mesh is uniform. We put $\tau_j := t_j - t_{j-1}$ for $j = 1,\ldots,n$. If u is absolutely continuous in the time variable, then the $L1$-approximation on the graded meshes to the Caputo fractional derivative of order $\beta \in (0,1)$ at the node t_i is given by [3,32,54]

$$
\begin{aligned}
\left.\frac{\partial^{\beta} u}{\partial t^{\beta}}\right|_{t=t_i} &= \int_0^{t_i} \frac{\partial u(s)}{\partial s} \mathrm{g}_{1-\beta}(t_i - s)\ \mathrm{d}s \\
&\approx \sum_{l=1}^{i} \frac{u_l - u_{l-1}}{\tau_l} \int_{t_{l-1}}^{t_l} \mathrm{g}_{1-\beta}(t_i - s)\ \mathrm{d}s + Q^i \\
&= \sum_{l=1}^{i} \tilde{a}_{i,l}\left(u_l - u_{l-1}\right) + Q^i,
\end{aligned} \tag{43}
$$

where Q^i is the truncation error, the kernel $\mathrm{g}_{\beta}(t) = \frac{t^{\beta-1}}{\Gamma(\beta)}$, and the coefficients $\tilde{a}_{i,l}$ can be evaluated by

$$
\tilde{a}_{i,l} = \frac{\mathrm{g}_{2-\beta}(t_i - t_{l-1}) - \mathrm{g}_{2-\beta}(t_i - t_l)}{\tau_l}, \quad 1 \leqslant l \leqslant i. \tag{44}
$$

Note that $\tilde{a}_{i,i} = \frac{\tau_i^{-\beta}}{\Gamma(2-\beta)}$ for $i = 1,\ldots,n$. Further, we define

$$\frac{\partial^\beta u}{\partial t^\beta}(t_i) \approx D_n^\beta u_i := \sum_{l=1}^{i} \tilde{a}_{i,l}\,(u_l - u_{l-1}) = \sum_{l=1}^{i} \tilde{a}_{i,l}\,\delta u_l \tau_l. \tag{45}$$

These coefficients are also sometimes denoted by $\tilde{a}_{i-l}$ in literature, see e.g. [22]. The coefficients $\tilde{a}_{j,i}$ are positive for $1 \le i \le j$. The integral mean-value theorem implies that the non-uniform $L1$ coefficient (44) satisfies [22, Eq. (2.3)]

$$\tilde{a}_{i,l-1} < \mathrm{g}_{1-\beta}(t_i - t_{l-1}) < \tilde{a}_{i,l}, \quad 1 \leqslant l \leqslant i, \tag{46}$$

i.e. $\tilde{a}_{j,i}$ is increasing for $i = 1,\ldots,j$, which also implies its concavity. We can rewrite the discretization eq. (45) as

$$D_n^\beta u_i = \tilde{a}_{i,i} u_i - \tilde{a}_{i,1}\tilde{u}_0 + \sum_{l=1}^{i-1} \left(\tilde{a}_{i,l} - \tilde{a}_{i,l+1}\right) u_l. \tag{47}$$

A bound on the truncation error Q^i for the graded mesh can be found in the lemma below, see [43, Lemma 5.1].

Lemma 4 *Assume that* $u \in \mathrm{C}^2((0,T])$ *and there exists positive constants* C *such that*

$$\left|\frac{\partial^l u}{\partial t^l}(t)\right| \leqslant C(1 + t^{\beta - l}) \quad \textit{for } l = 0,1,2, \quad \textit{and for } 0 < t \leqslant T. \tag{48}$$

If the nonuniform grid fulfills

$$\tau_{j-1} \leqslant \tau_j, \quad 2 \leqslant j \leqslant n, \tag{49}$$

then the following inequality is achieved for $j \geqslant 1$,

$$\left|Q^j\right| = \left|\frac{\partial^\beta}{\partial t^\beta} u(t_j) - D_n^\beta u(t_j)\right| \leqslant C j^{-\min\{r\beta, 2-\beta\}}. \tag{50}$$

It can be marked also that the optimal graded mesh is obtained when $r_{\mathrm{opt}} := (2-\beta)/\beta$, and this gives us the most possible high rate of convergence $\mathscr{O}\left(n^{-\{2-\beta\}}\right)$. Additionally, if we choose $r > r_{\mathrm{opt}}$, this will increase the temporal mesh near $t = T$ and so the constant multiplier C will be increased [43, p. 17].

Using the graded mesh, we approximate problem (12)-(13) at time $t = t_i$ as follows: Find $u_i \in \mathrm{H}^1(\Omega)$ and $h_i \in \mathbb{R}$ such that for all $\varphi \in \mathrm{H}^1(\Omega)$ it holds that

$$\begin{aligned}\left(D_n^\beta u_i, \varphi\right)_{\mathrm{H}^1(\Omega)^* \times \mathrm{H}^1(\Omega)} &+ \mathscr{L}_i(u_i, \varphi) \\ &= h_i(f,\varphi) + \left(\sum_{l=1}^{i} F(t_l, u_{l-1})\tau_l, \varphi\right) - (g_i, \varphi)_\Gamma,\end{aligned} \tag{51}$$

and

$$h_i = \frac{D_n^\beta m_i + (g_i,1)_\Gamma + c_i m_i - \left(\sum_{l=1}^{i} F(t_l,u_{l-1})\tau_l, 1\right)}{(f,1)}, \tag{52}$$

Please note that eq. (51) and eq. (52) are linear in u_j and h_j, respectively, and both relations are decoupled. Thus for a given $i \in \{1,\cdots,n\}$, we first determine h_i from eq. (52) and then we solve (51). Afterwards we increase i to $i+1$. The discrete problem (51) can be equivalently written as

$$a_i(u_i,\varphi) = \langle \tilde{F}_i, \varphi\rangle, \quad \forall \varphi \in \mathrm{H}^1(\Omega), \tag{53}$$

with

$$a_i(u_i,\varphi) = \tilde{a}_{i,i}(u_i,\varphi) + \mathscr{L}_i(u_i,\varphi)$$

and

$$\langle \tilde{F}_i, \varphi\rangle = h_i(f,\varphi) + \left(\sum_{l=1}^{i} F(t_l,u_{l-1})\tau_l, \varphi\right) - (g_i,\varphi)_\Gamma \\ + \tilde{a}_{i,i}(u_{i-1},\varphi) - \sum_{l=1}^{i-1} \tilde{a}_{i,l}(u_l - u_{l-1},\varphi).$$

We mention a fractional Grönwall lemma [22, Lemma 2.2] related to graded meshes, which will be used when establishing a priori estimates for u_j and h_j.

Lemma 5 (Nonuniform discrete fractional Grönwall inequality) *For any finite time $t_n = T > 0$, and a given nonnegative sequence $(\lambda_l)_{l=0}^{n-1}$, assume that there exists a constant λ, independent of time-steps, such that $\lambda \geqslant \sum_{l=0}^{n-1} \lambda_l$ and let $\{u_i\}_{i=1}^n$ and $\{\zeta_i, g_i\}_{i=1}^n$ be sequences of non-negative numbers that satisfy*

$$D_n^\beta (u_i)^2 \leqslant \sum_{l=1}^{i} \lambda_{i-l}(u_l)^2 + u_i(\zeta_i + g_i), \qquad \forall\, i = 1,\ldots,n. \tag{54}$$

If the time grids satisfy (49) *with the maximum time grid $\tau_n \leqslant \tau^\triangle = \sqrt[\beta]{\frac{1}{2\Gamma(2-\beta)\lambda}}$, the following inequality holds*

$$u_i \leqslant 2\left(u_0 + \mathrm{g}_{1+\beta}(t_i) \max_{1\leqslant j\leqslant i}(\zeta_j + g_j)\right) E_\beta(2\lambda t_i^\beta), \qquad 1 \leqslant i \leqslant n, \tag{55}$$

where E_β denotes the Mittag–Leffler function with parameter β. The estimate (55) *is also valid if the condition* (54) *is replaced by*

$$D_n^\beta (u_i)^2 \leqslant \sum_{l=1}^{i} \lambda_{i-l}(u_l)^2 + \zeta_i + g_i, \qquad \forall\, i = 1,\ldots,n.$$

Lemma 6 *Assume that* $\tilde{u}_0 \in \mathrm{L}^2(\Omega)$, $f \in \mathrm{L}^2(\Omega)$ *with* $\int_\Omega f(\mathbf{x})\ \mathrm{d}\mathbf{x} \neq 0$, $c(t) \geqslant \tilde{c}_0 > 0$ *for* $t \in [0,T]$, F *be globally Lipschitz continuous in all variables and* $g \in \mathrm{L}^\infty\left((0,T), \mathrm{L}^2(\Gamma)\right)$. *Moreover, suppose that* $m \in \mathrm{C}^1((0,T])$ *is satisfying*

$$\left|\frac{\partial^l m}{\partial t^l}(t)\right| \leqslant C(1+t^{\beta-l}) \quad \text{for } l = 0,1,2, \quad \text{and for } 0 < t \leqslant T;$$

Then, for any $i = 1,2,\dots,n$, *there exists a unique couple* $\{u_i, h_i\} \in \mathrm{H}^1(\Omega) \times \mathbb{R}$ *solving* (51)-(52) *and there exist positive constants C such that*

$$\max_{1\leqslant i\leqslant n} \left\{ D_n^\beta \|u_i\|^2 + \|u_i\|^2_{\mathrm{H}^1(\Omega)} \right\} \leqslant C, \quad \max_{1\leqslant i\leqslant n} |h_i| \leqslant C.$$

Proof The existence of a unique solution on every time step t_i follows from the Lax-Milgram lemma. We set $\varphi = u_i$ in eq. (51) to get

$$\left(D_n^\beta\, u_i, u_i\right) + \mathscr{L}_i(u_i,u_i) = h_i(f,u_i) + \left(\sum_{l=1}^{i} F(t_l,u_{l-1})\tau_l, u_i\right) - (g_i,u_i)_\Gamma. \tag{56}$$

Let us start from eq. (52). From the assumptions on m and by the global Lipschitz continuity of F, we can deduce that

$$|h_i| \leqslant C + C\sum_{l=1}^{i} \|F(t_l,u_{l-1})\|\tau_l \leqslant C + C\sum_{l=0}^{i-1} \|u_l\|\tau_l. \tag{57}$$

Applying the definition (47) of $D_n^\beta u_i$ and the monotone property (46), we have that

$$\begin{aligned}
2\left(D_n^\beta u_i, u_i\right) &= 2\tilde{a}_{i,i}(u_i,u_i) + 2\sum_{j=1}^{i-1}\left(\tilde{a}_{i,j} - \tilde{a}_{i,j+1}\right)(u_j,u_i) - 2\tilde{a}_{i,1}(\tilde{u}_0,u_i) \\
&\geqslant 2\tilde{a}_{i,i}\|u_i\|^2 - \sum_{j=1}^{i-1}\left(\tilde{a}_{i,j+1} - \tilde{a}_{i,j}\right)\|u_i\|^2 - \tilde{a}_{i,1}\|u_i\|^2 \\
&\quad - \sum_{j=1}^{i-1}\left(\tilde{a}_{i,j+1} - \tilde{a}_{i,j}\right)\|u_j\|^2 - \tilde{a}_{i,1}\|\tilde{u}_0\|^2 \\
&= \tilde{a}_{i,i}\|u_i\|^2 - \sum_{j=2}^{i}\tilde{a}_{i,j}\|u_{j-1}\|^2 + \sum_{j=1}^{i-1}\tilde{a}_{i,j}\|u_j\|^2 - \tilde{a}_{i,1}\|\tilde{u}_0\|^2 \\
&= \sum_{j=1}^{i}\tilde{a}_{i,j}\left(\|u_j\|^2 - \|u_{j-1}\|^2\right).
\end{aligned} \tag{58}$$

Then, by eq. (43), we deduce that

$$\left(D_n^\beta u_i, u_i\right) \geqslant \frac{1}{2} D_n^\beta \|u_i\|^2. \tag{59}$$

From eq. (9) and $c(t) \geqslant \tilde{c}_0 > 0$ for $t \in [0,T]$, we get that

$$\mathscr{L}_i(u_i,u_i) \geqslant \min\{\alpha, \tilde{c}_0\}\, \|u_i\|^2_{\mathrm{H}^1(\Omega)}.$$

For the first term on the RHS of eq. (56), using the ε-Young inequality, we see that

$$\begin{aligned}|h_i(f,u_i)| &\leqslant C_{\varepsilon_1}|h_i|^2+\varepsilon_1|(f,u_i)|^2\\ &\overset{(57)}{\leqslant} C_{\varepsilon_1}+C_{\varepsilon_1}\sum_{l=0}^{i-1}\|u_l\|^2\tau_l+\varepsilon_1\|u_i\|^2,\end{aligned}$$

For the other terms on the RHS of eq. (56), we use the trace theorem and the property that $k\in \mathrm{L}^1(0,T)$ in order to obtain that

$$\left|\left(\sum_{l=1}^{i}F(t_l,u_{l-1})\tau_l,u_i\right)\right|\leqslant C_{\varepsilon_2}+C_{\varepsilon_2}\sum_{l=0}^{i-1}\|u_l\|^2\tau_l+\varepsilon_2\|u_i\|^2,$$

$$|(g_i,u_i)_\Gamma|\leqslant C_{\varepsilon_3}+\varepsilon_3\|u_i\|^2_{\mathrm{H}^1(\Omega)},$$

Collecting all the previous estimates give

$$\begin{aligned}&\frac{1}{2}D_n^\beta\|u_i\|^2+(\min\{\alpha,\tilde{c}_0\}-\varepsilon_1-\varepsilon_2-\varepsilon_3)\|u_i\|^2_{\mathrm{H}^1(\Omega)}\\ &\qquad\leqslant C_{\varepsilon_1,\varepsilon_2,\varepsilon_3}+(C_{\varepsilon_1}+C_{\varepsilon_2})\sum_{l=0}^{i-1}\|u_l\|^2\tau_l.\end{aligned}\tag{60}$$

Next, we fix $\varepsilon_i>0, i=1,\ldots,3$, sufficient small and we obtain that

$$D_n^\beta\|u_i\|^2\leqslant C+C\sum_{l=1}^{i-1}\|u_l\|^2\tau_l.$$

An application of the discrete fractional Grönwall's lemma 5 gives that

$$\|u_i\|\leqslant 2\left(\|\tilde{u}_0\|+C\mathrm{g}_{1+\beta}(t_i)\right)E_\beta(Ct_i^\beta),\tag{61}$$

which means that $\|u_i\|\leqslant C$, and this implies to $|h_i|\leqslant C$ by monitoring eq. (57). Moreover, from eq. (60), it follows that $D_n^\beta\|u_i\|^2+\|\nabla u_i\|^2\leqslant C$ for all $i=1,\ldots,n$.

The Rothe function $\tilde{u}_n$ is on the graded mesh defined as

$$\tilde{u}_n:[0,T]\to \mathrm{L}^2(\Omega):t\mapsto\begin{cases}\tilde{u}_0 & t\in[0,\tau_1]\\ u_{i-1} & t\in(t_{i-1},t_i],\quad 2\leqslant i\leqslant n.\end{cases}$$

The other Rothe functions are defined as before. Then, the variational system (51)-(52) can be rewritten on the whole time frame as

$$\begin{aligned}&\langle D_n^\beta\overline{u}_n(t),\varphi\rangle_{\mathrm{H}^1(\Omega)^*\times\mathrm{H}^1(\Omega)}+\overline{\mathscr{L}}_n(t)(\overline{u}_n(t),\varphi)\\ &\quad=\overline{h}_n(t)(f,\varphi)+\left(\int_0^t\overline{F}_n(s,\tilde{u}_n(s))\,\mathrm{d}s,\varphi\right)-(\overline{g}_n(t),\varphi)_\Gamma,\quad\forall\varphi\in\mathrm{H}^1(\Omega),\end{aligned}\tag{62}$$

and

$$\overline{h}_n(t)(f,1)=(D_n^\beta\overline{m}_n(t)+\overline{c}_n(t)\overline{m}_n(t)$$

$$+(\overline{g}_n(t),1)_\Gamma - \left(\int_0^t \overline{F}_n(s,\tilde{u}_n(s))\,\mathrm{d}s,1\right). \quad (63)$$

In the following theorem we show the existence of a solution for the graded scheme under appropriate assumptions on the solution. We only explain the differences in comparison with Theorem 1.

Theorem 3 (Existence and uniqueness: graded mesh)
Suppose that the conditions of Lemma 6 are fulfilled. Moreover, assume that $r\beta > 1$ *and that the solution* u *of* (12)-(13) *and the measurement* m *satisfy the bounds*

$$\left|\frac{\partial^l u}{\partial t^l}(\mathbf{x},t)\right| \leqslant C(1+t^{\beta-l}) \quad \text{for } l=0,1,2, \quad \text{and for a.a. } \mathbf{x}\in\Omega \text{ and } 0<t\leqslant T; \quad (64)$$

$$\left|\frac{\partial^l m}{\partial t^l}(t)\right| \leqslant C(1+t^{\beta-l}) \quad \text{for } l=0,1,2, \quad \text{and for } 0<t\leqslant T. \quad (65)$$

Then there exists a unique couple $\{u,h\}$ *to the problem* (12)-(13) *with*

$$u\in \mathrm{C}\left([0,T],\mathrm{L}^2(\Omega)\right)\cap \mathrm{L}^\infty\left((0,T),\mathrm{H}^1(\Omega)\right),\ h\in \mathrm{L}^\infty(0,T)$$

and

$$k*\partial_t u\in \mathrm{L}^\infty\left((0,T),\mathrm{L}^2(\Omega)\right).$$

Proof We point out the differences in the limit transitions in comparison with the proof of Theorem 1. From eq. (64) and the γ-Hölder continuity of x^γ when $\gamma\in(0,1)$, we have for $|t-s|\leqslant 1$ that

$$\|u(t)-u(s)\| = \left\|\int_s^t \partial_\xi u(\xi)\,\mathrm{d}\xi\right\| \leqslant C\,|t-s|^\beta,$$

i.e. $u\in \mathrm{C}\left([0,T],\mathrm{L}^2(\Omega)\right)$. Hence, we have that

$$\int_0^T \left\|\overline{u}_{n_l}(t)-\tilde{u}_{n_l}(t)\right\|^2 \mathrm{d}t = \sum_{i=1}^{n_l} \tau_i\,\|u_i-u_{i-1}\|^2 \leqslant C\tau_{n_l}^{2\beta}\sum_{i=1}^{n_l}\tau_i = CT\tau_{n_l}^{2\beta} \xrightarrow{l\to\infty} 0.$$

As in Theorem 1, we have that

$$\overline{u}_{n_l}\to u \text{ in } \mathrm{L}^2\left((0,T),\mathrm{L}^2(\Omega)\right) \text{ as } l\to\infty,$$

$$\overline{u}_{n_l}\rightharpoonup u \text{ in } \mathrm{L}^2\left((0,T),\mathrm{H}^1(\Omega)\right) \text{ as } l\to\infty,$$

and

$$\tilde{u}_{n_l}\to u \text{ in } \mathrm{L}^2\left((0,T),\mathrm{L}^2(\Omega)\right) \text{ as } l\to\infty.$$

Finally, we prove that

$$\left|\int_0^T \langle\partial_t^\beta u(t),\varphi\rangle_{\mathrm{H}^1(\Omega)^*\times \mathrm{H}^1(\Omega)}\,\mathrm{d}t - \int_0^T \langle D_{n_l}^\beta \overline{u}_{n_l}(t),\varphi\rangle_{\mathrm{H}^1(\Omega)^*\times \mathrm{H}^1(\Omega)}\,\mathrm{d}t\right| \xrightarrow{l\to\infty} 0$$

by showing that

$$\int_0^T \left\|\partial_t^\beta u(t) - D_{n_l}^\beta \overline{u}_{n_l}(t)\right\|\mathrm{d}t \xrightarrow{l\to\infty} 0. \quad (66)$$

We have that

$$
\begin{aligned}
&\int_0^T \left\| \partial_t^\beta u(t) - D_{n_l}^\beta \overline{u}_{n_l}(t) \right\| \mathrm{d}t \\
&= \sum_{i=1}^{n_l} \int_{t_{i-1}}^{t_i} \left\| \partial_t^\beta u(t) - D_{n_l}^\beta u(t_i) \right\| \mathrm{d}t \\
&\leqslant \sum_{i=1}^{n_l} \int_{t_{i-1}}^{t_i} \Bigg[\left\| \int_0^{t_i} k(t_i - s) \partial_t u(s) \ \mathrm{d}s - D_{n_l}^\beta u(t_i) \right\| \\
&\qquad + \left\| - \int_t^{t_i} k(t_i - s) \partial_t u(s) \ \mathrm{d}s \right\| \\
&\qquad + \left\| \int_0^t \left(k(t-s) - k(t_i - s) \right) \partial_t u(s) \ \mathrm{d}s \right\| \Bigg] \mathrm{d}t.
\end{aligned}
$$

From Lemma 4, if $r\beta > 1$, we get that

$$
\begin{aligned}
&\sum_{i=1}^{n_l} \int_{t_{i-1}}^{t_i} \left\| \int_0^{t_i} k(t_i - s) \partial_t u(s) \ \mathrm{d}s - D_{n_l}^\beta u(t_i) \right\| \\
&\qquad\qquad \leqslant C \tau_{n_l} \sum_{i=1}^{n_l} i^{-\min\{r\beta, 2-\beta\}} \leqslant C \tau_{n_l} \zeta(\min\{r\beta, 2-\beta\}) \xrightarrow{l\to\infty} 0,
\end{aligned}
$$

where ζ denotes the Euler–Riemann zeta function. Moreover, for $t \in (t_{i-1}, t_i]$, we have that

$$
\begin{aligned}
&\left\| \int_t^{t_i} k(t_i - s) \partial_t u(s) \ \mathrm{d}s \right\| \\
&\leqslant C \int_t^{t_i} k(t_i - s) \left(1 + s^{\beta - 1} \right) \ \mathrm{d}s \\
&= C \left[\frac{(t_i - t)^{1-\beta}}{\Gamma(2-\beta)} + \frac{\mathrm{B}_1(\beta, 1-\beta) - \mathrm{B}_{\frac{t}{t_i}}(\beta, 1-\beta)}{\Gamma(1-\beta)} \right] \xrightarrow{l\to\infty} 0
\end{aligned}
$$

and

$$
\begin{aligned}
&\left\| \int_0^t \left(k(t-s) - k(t_i - s) \right) \partial_t u(s) \ \mathrm{d}s \right\| \\
&\leqslant C \int_0^t \left(k(t-s) - k(t_i - s) \right) \left(1 + s^{\beta-1} \right) \ \mathrm{d}s \\
&= C \left[\frac{t^{1-\beta}}{\Gamma(2-\beta)} + \frac{\mathrm{B}_1(\beta, 1-\beta)}{\Gamma(1-\beta)} - \left(\frac{t_i^{1-\beta} - (t_i - t)^{1-\beta}}{\Gamma(2-\beta)} + \frac{\mathrm{B}_{\frac{t}{t_i}}(\beta, 1-\beta)}{\Gamma(1-\beta)} \right) \right] \\
&\leqslant C \left[\frac{\tau_i^{1-\beta}}{\Gamma(2-\beta)} + \frac{\mathrm{B}_1(\beta, 1-\beta) - \mathrm{B}_{\frac{t}{t_i}}(\beta, 1-\beta)}{\Gamma(1-\beta)} \right] \xrightarrow{l\to\infty} 0,
\end{aligned}
$$

where $\mathrm{B}_x(a,b)$ is the incomplete beta function. Therefore, eq. (66) is satisfied. Analogously, using eq. (65), we can show that

$$
\int_0^T \left| \partial_t^\beta m(t) - D_{n_l}^\beta \overline{m}_{n_l}(t) \right| \mathrm{d}t \xrightarrow{l\to\infty} 0.
$$

Now, we integrate eq. (62) and eq. (63) in time over $(0,\eta) \subset (0,T)$ for the resulting subsequence. Then, we pass to the limit $l \to \infty$ and afterwards we differentiate the result with respect to η. This gives that $\{h,u\}$ solves problem (12)-(13). Finally, we prove the uniqueness of a solution by contradiction. There is a clear difference with the proof of Theorem 2. Here, we have that

$$\|(k * \partial_t u)(t)\| \leqslant C \int_0^t k(t-s)\left(1+s^{\beta-1}\right)\, \mathrm{d}s \leqslant C(1+t^{1-\beta}),$$

i.e. $k*\partial_t u \in \mathrm{L}^\infty\left((0,T),\mathrm{L}^2(\Omega)\right)$. Similarly, from eq. (12), it follows that $h \in \mathrm{L}^\infty(0,T)$. Instead of using the positive definiteness of k as was done in Theorem 2, we can here use [19, Corollary 2]. As a consequence we will obtain the uniqueness of a solution without the additional assumption that F is linear in u as it was necessary in Theorem 2. So we suppose that two solutions $\{h_1,u_1\}$ and $\{h_2,u_2\}$ solve problem eq. (12)-eq. (13). Then, the differences $h := h_1 - h_2$ and $u := u_1 - u_2$ are satisfying

$$\begin{aligned} &((k*\partial_t u)(t),\varphi) + \mathscr{L}(t)(u(t),\varphi) \\ &\quad = h(t)(f,\varphi) + \left(\int_0^t [F(s,u_1(s)) - F(s,u_2(s))]\, \mathrm{d}s, \varphi\right), \quad \forall \varphi \in \mathrm{H}^1(\Omega), \end{aligned} \tag{67}$$

and

$$h(t) = \frac{\left(\int_0^t [F(s,u_2(s)) - F(s,u_1(s))]\, \mathrm{d}s, 1\right)}{(f,1)}. \tag{68}$$

From eq. (68), it follows that

$$|h(t)| \leqslant C\int_0^t \|u(s)\|\, \mathrm{d}s, \quad \text{for } t \in [0,T]. \tag{69}$$

Taking $\varphi = u(t)$ in eq. (39) and integrating with respect to time over $(0,\eta) \subset (0,T)$ gives

$$\begin{aligned} &\int_0^\eta (\partial_t (k*u)(t), u(t))\,\mathrm{d}t + \int_0^\eta \mathscr{L}(t)(u(t),u(t))\,\mathrm{d}t \\ &\quad = \int_0^\eta h(t)(f,u(t))\,\mathrm{d}t + \int_0^\eta \left(\int_0^t [F(s,u_1(s)) - F(s,u_2(s))]\, \mathrm{d}s, u(t)\right)\mathrm{d}t, \end{aligned} \tag{70}$$

where we used Leibniz's rule for differentiation under the integral sign, i.e.

$$\partial_t (k*u) = k*\partial_t u + k\tilde{u}_0 = k*\partial_t u, \qquad \text{a.e. in } Q_T.$$

Using the ε-Young inequality and the global Lipschitz continuity of F, we estimate the terms in the RHS of eq. (70) as follows

$$\begin{aligned} \left|\int_0^\eta h(t)(f,u(t))\,\mathrm{d}t\right| &\leqslant C_{\varepsilon_1}\int_0^\eta |h(t)|^2\,\mathrm{d}t + \varepsilon_1\int_0^\eta \|u(t)\|^2\,\mathrm{d}t \\ &\overset{\text{eq. (69)}}{\leqslant} C_{\varepsilon_1}\int_0^\eta\left(\int_0^t \|u(s)\|^2\, \mathrm{d}s\right)\mathrm{d}t + \varepsilon_1\int_0^\eta \|u(t)\|^2\,\mathrm{d}t \end{aligned}$$

and

$$\int_0^\eta \left(\int_0^t [F(s,u_1(s)) - F(s,u_2(s))] \, \mathrm{d}s, u(t) \right) \mathrm{d}t$$
$$\leqslant C_{\varepsilon_2} \int_0^\eta \left(\int_0^t \|u(s)\|^2 \, \mathrm{d}s \right) \mathrm{d}t + \varepsilon_2 \int_0^\eta \|u(t)\|^2 \, \mathrm{d}t.$$

Therefore, from eq. (11) and [19, Corollary 2], it follows that

$$\left(\frac{k(T)}{2} - \varepsilon_1 - \varepsilon_2 \right) \int_0^\eta \|u(t)\|^2 \, \mathrm{d}t + \alpha \int_0^\eta \|\nabla u(t)\|^2 \, \mathrm{d}t$$
$$\leqslant C_{\varepsilon_1,\varepsilon_2} \int_0^\eta \left(\int_0^t \|u(s)\|^2 \, \mathrm{d}s \right) \mathrm{d}t.$$

Fixing ε_1 and ε_2 sufficiently small such that $\varepsilon_1 + \varepsilon_2 < \frac{k(T)}{2}$, and applying the Grönwall argument, we obtain that $u = 0$ a.e. in Q_T. Moreover, from eq. (69), it follows that $h = 0$ a.e. in $(0,T)$.

Remark 6 The condition $r\beta > 1$ in Theorem 3 is satisfied when r equals its optimal value $r_{\mathrm{opt}} := (2-\beta)/\beta$.

Remark 7 Theorem 3 also stays valid when the conditions on u and m are replaced by (see also Remark 5)

$$\left| \frac{\partial^l u}{\partial t^l}(\cdot,t) \right| \leqslant C_1 + C_2 t^{\beta - l} \quad \text{for } l = 0,1,2, \quad \text{and for } 0 < t \leqslant T;$$
$$\left| \frac{\partial^l m}{\partial t^l}(t) \right| \leqslant C_1 + C_2 t^{\beta - l} \quad \text{for } l = 0,1,2, \quad \text{and for } 0 < t \leqslant T.$$

Following the proof of Lemma 6 and Theorem 3, we have the following result concerning the forward problem.

Theorem 4 *Consider*

$$\begin{cases} \left(\partial_t^\beta u\right)(\mathbf{x},t) - \nabla \cdot (A(\mathbf{x},t)\nabla u(\mathbf{x},t)) & \\ \qquad +c(\mathbf{x},t)u(\mathbf{x},t) = f(\mathbf{x},t) & (\mathbf{x},t) \in Q_T, \\ \nabla u(\mathbf{x},t) \cdot \nu = 0 & (\mathbf{x},t) \in \Sigma_T, \\ u(\mathbf{x},0) = \tilde{u}_0(\mathbf{x}) & \mathbf{x} \in \Omega. \end{cases}$$

Then this problem has a unique weak solution $u \in \mathrm{C}\left([0,T],\mathrm{L}^2(\Omega)\right) \cap \mathrm{L}^\infty\left((0,T),\mathrm{H}^1(\Omega)\right)$ *with* $k * \partial_t u \in \mathrm{L}^\infty\left((0,T),\mathrm{L}^2(\Omega)\right)$ *if*

- $\tilde{u}_0 \in \mathrm{L}^2(\Omega)$;
- $f \in \mathrm{L}^\infty\left((0,T),\mathrm{H}^1(\Omega)^*\right)$;
- $A \in \left(\mathrm{L}^\infty\left(\overline{Q_T}\right)\right)^{d\times d}$ *is uniformly elliptic;*
- $c \in \mathrm{L}^\infty\left(\overline{Q_T}\right)$ *such that* $c \geqslant \tilde{c}_0 > 0$;
- *u satisfies the bounds* $\left\| \frac{\partial^l u}{\partial t^l}(t) \right\| \leqslant C(1 + t^{\beta - l})$ *for* $l = 0,1,2,$ *and for* $0 < t \leqslant T$.

The space $\mathrm{H}^1(\Omega)$ *is replaced by* $\mathrm{H}^1_0(\Omega)$ *and* $c \geqslant 0$ *in the case of a homogeneous Dirichlet boundary condition.*

5 Numerical experiments

In this section, our interest is devoted to quantify how the layer in the solution at $t = 0$ and the grading of the mesh influence the convergence of the computed solution by means of the constructed (non)uniform Rothe scheme. We compare the different algorithms proposed in the previous sections on the basis of several numerical experiments. In these experiments, we consider $T = 0.5$ and $\Omega = (0,1)$. We assume that $A = I$ and $c = 1$. We take the number of time discretization intervals equal to $n = 200$ and we use $r = (2-\beta)/\beta$ for the graded grid parameter. The source F is defined by

$$F(t,u) = -tu,$$

and satisfies the condition in Theorem 2. We consider as exact solution the non-smooth function ($\partial_t u$ blows up as $t \to 0^+$) prescribed by

$$u_{\mathrm{ex}}(x,t) = \left(t^3 + t^\gamma\right)\sin(x), \qquad \gamma \in (0,1).$$

The corresponding exact source is given by

$$h_{\mathrm{ex}}(t) = \frac{\left(6t^{3-\beta} + 2\left(t^\gamma + t^3\right)\Gamma(4-\beta)\right)\Gamma(-\beta+\gamma+1) + t^{-\beta+\gamma}\Gamma(\gamma+1)\Gamma(4-\beta)}{\Gamma(-\beta+\gamma+1)\Gamma(4-\beta)} + \frac{t^5\gamma + 2t^5 + 5t^{\gamma+2}}{5\gamma+10}, \qquad f_{\mathrm{ex}}(x) = \sin(x).$$

The exact measurement is given by

$$m_{\mathrm{ex}}(t) = (1-\cos(1))\left(t^3 + t^\gamma\right), \qquad \gamma \in (0,1).$$

We see immediately that for $1 > \gamma \geqslant \beta > 0$ the conditions on m in Theorem 3 are fulfilled.

Moreover, from

$$(k * m'_{\mathrm{ex}})(t) = \begin{cases} (1-\cos(1))\,\frac{t^{\gamma-\beta}\Gamma(\gamma+1)\Gamma(4-\beta)+6t^{3-\beta}\Gamma(\gamma-\beta+1)}{\Gamma(4-\beta)\Gamma(\gamma-\beta+1)} & \gamma > 0, \gamma \neq \beta, \\ (1-\cos(1))\,\frac{\Gamma(\beta+1)\Gamma(4-\beta)+6t^{3-\beta}}{\Gamma(4-\beta)} & \gamma = \beta > 0; \end{cases}$$

it follows that the conditions on m in Lemma 3 are satisfied for $1 > \gamma > \beta$. Note that $(k * m'_{\mathrm{ex}})(0) \neq 0$ if $\gamma = \beta$.

A randomly generated uncorrelated noise is added to the additional measurement in order to simulate the inherent errors present in real measurements

$$m_\varepsilon(t) = m_{\mathrm{ex}}(t)\left(1 + \varepsilon\mathscr{R}(t)\right),$$

where ε represents the percentage of noise, e.g. $\varepsilon = 0.01$ for 1% noise, and $\mathscr{R}$ is a random number in the interval $[-1,1]$ that changes in time. Afterwards, the noisy data is regularized by using the nonlinear least-squares method to obtain a function approximating the noisy data. The approximating function has the following form

$$m_{\varepsilon,\mathrm{reg}}(t) = \boldsymbol{\alpha_5 t^{\alpha_4}} + \alpha_3 t^3 + \alpha_2 t^2 + \alpha_1 t + \alpha_0, \quad \alpha_i \in \mathbb{R}.$$

It consists out of a power function and a polynomial. It is this function that is used in the computations later.

The solution to the inverse source problem is obtained by applying the algorithms proposed in Section 3 (uniform scheme based on convolution quadrature) and Section 4 (graded $L1$-scheme). We use eq. (20) and eq. (47) for the derivation of $(k * m')_i$ and $D_n^{\beta} m_i$, respectively. The corresponding forward problems at each time step are solved numerically by the finite element method (FEM) using the first-order (P1–FEM) Lagrange polynomials for the space discretization. The number of space discretization intervals is taken to be equal to 100. The finite element library DOLFIN [25,26] from the FEniCS project [24,1] is used to solve the forward problems. In the following, the exact value for the source is compared to its corresponding numerically retrieved value (for both algorithms) in four experiments. The CPU time (in seconds, Intel® Core™ i7-1065G7 Processor) for the different experiments is similar, and the results are obtained in approximately one minute. Note that at each time step one has to use the numerical solutions at all preceding time levels, which has as a consequence that the CPU time is increasing fast when increasing the number of time discretization intervals.

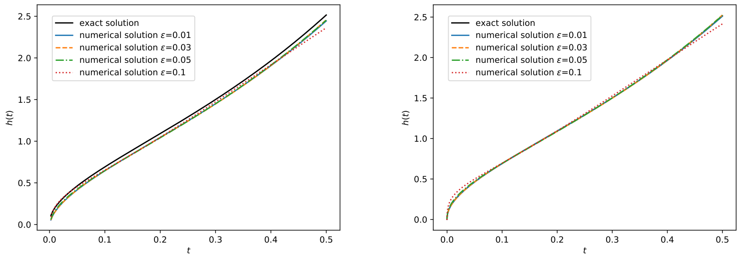
We consider the following experiments depending on the value of γ in u_{ex} and the order of the fractional derivative:

- Experiment 1: $\gamma = 0.9$ and $\beta = 0.5$;
- Experiment 2: $\gamma = \beta = 0.5$;
- Experiment 3: $\gamma = \beta = 0.2$;
- Experiment 4: $\gamma = \beta = 0.8$.

In all these experiments, the noise levels are given by $\varepsilon \in \{0.01, 0.03, 0.05, 0.1\}$. We depict the results for four different relations between γ and β in Figures 1 to 4. In Experiment 1, we consider $\gamma = 0.9$ and $\beta = 0.5$. We remember that in this situation, the conditions for the convergence of both algorithms are satisfied. This is reflected by the numerical results given in Figure 2.

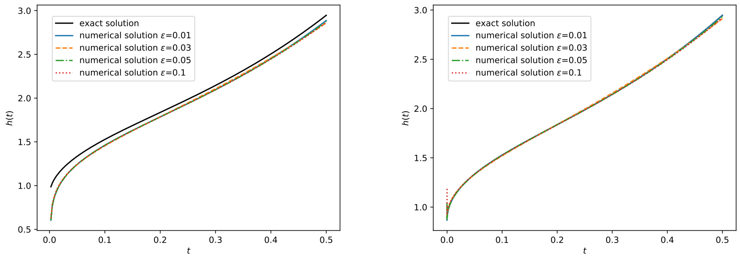
The results for the uniform mesh are depicted in Figure 1(a), whilst the result for the graded mesh are shown in Figure 1(b). We see from these pictures that both approximations are accurate but the absolute error is the smallest for the graded $L1$-scheme. This is as expected because the $L1$-approximation of the Caputo derivative (45) is more accurate than the approximation by the discrete convolution (14).

In the following three experiments ($\gamma = \beta = 0.5$, $\gamma = \beta = 0.2$, $\gamma = \beta = 0.8$, respectively), only the conditions for the convergence of the graded $L1$-scheme are satisfied. However, we show also the results obtained via the convolution quadrature. We may conclude from Figures 2 to 4 that for the graded $L1$-scheme an accurate numerical approximation for the unknown source is obtained. We note that in Experiment 4 the approximation obtained via the graded $L1$-scheme is less accurate than in the previous experiments as the graded grid parameter r becomes closer to 1 for increasing β. We see also that for small value of β a reasonable approximation via the uniform scheme can be obtained. It is clear from these experiments that Rothe's method over graded meshes copes better with the behaviour at $t = 0$ of the solution. As well-known, the graded mesh is coarser near $t = 0$, which gives an increased accuracy around $t = 0$ for all experiments in comparison with the uniform mesh. Finally, we note that performing the same experiments with $c = 0$ gives similar results, which suggests that the condition $c \geqslant c_0 > 0$ made in the analysis can be relaxed.



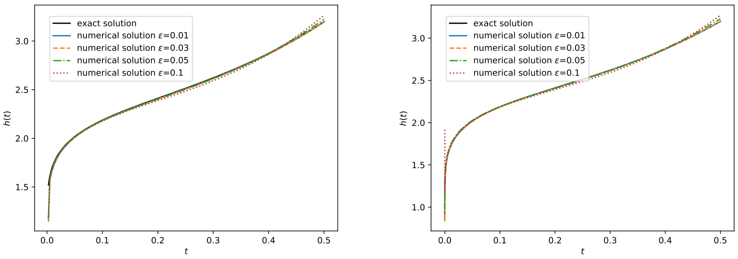
(a) (b)

Fig. 1 Experiment 1 ($\gamma = 0.9, \beta = 0.5$): The exact source and its numerical approximations using (a) uniform mesh and (b) graded mesh, obtained for various levels of noise.



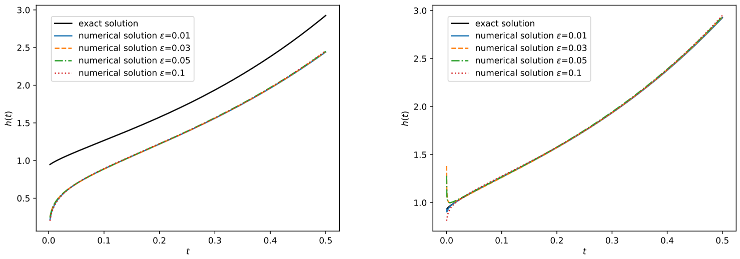
(a) (b)

Fig. 2 Experiment 2 ($\gamma = \beta = 0.5$): The exact source and its numerical approximations using (a) uniform mesh and (b) graded mesh, obtained for various levels of noise.



(a) (b)

Fig. 3 Experiment 3 ($\gamma = \beta = 0.2$): The exact source and its numerical approximations using (a) uniform mesh and (b) graded mesh, obtained for various levels of noise.



(a) (b)

Fig. 4 Experiment 4 ($\gamma = \beta = 0.8$): The exact source and its numerical approximations using (a) uniform mesh and (b) graded mesh, obtained for various levels of noise.

Remark 8 We have regularized the noisy measurement m_ε by using the nonlinear least-squares method. We need at least that the non-power function part m_p of the measurement m_ε belongs to $\mathrm{L}^2(0,T)$. Then the least square method is sufficient to determine our goals as for any $\delta > 0$ the equality $m_p = P_\delta + \phi_\delta$ holds true where P_δ is a polynomial and $\|\phi_\delta\|_{\mathrm{L}^2(0,T)} < \delta$. We suggest (if needed) to use kriging (which is an interpolation method used in geostatistics [48]) to get an idea about the measurement (and eventually the degree of the polynomial) before using the least-squares method.

Remark 9 In a future work, it would be interesting to apply other regularization methods to deal with the noisy measurement $m_\varepsilon \in \mathrm{L}^2(0,T)$, e.g. Tikhonov regularization [55,21,7]. The main idea of that approach is turning the inverse source problem to solve the variational problem

$$\min_{h \in \mathrm{L}^2(0,T)} J(h), \tag{71}$$

where

$$J(h) = \frac{1}{2} \left\| \int_\Omega u[h](\mathbf{x}, \cdot) \ \mathrm{d}\mathbf{x} - m_\varepsilon(\cdot) \right\|^2_{\mathrm{L}^2(0,T)} + \frac{\mu}{2} \|h\|^2_{\mathrm{L}^2(0,T)}, \tag{72}$$

where $\mu > 0$ is a regularization parameter and m_ε is a noisy function of m satisfying $\|m_\varepsilon - m\|_{\mathrm{L}^2(0,T)} \leqslant \varepsilon$. The first term denotes the defect between the exact data and the noisy data, and the second term is a penalty term for stabilizing the numerical solution. The conjugate gradient method can be used to find the minimizer of the functional (72). It is well known that the key work is to find the gradient of the functional (72), which can be obtained by constructing a sensitivity problem and an adjoint problem. In the same manner, a wavelet approach based on Meyer wavelet theory can be developed to regularize the source inverse problem under consideration. We refer to [12,13,15,14] for more details.

Finally, it should be noted that the main purpose of this contribution is to show theoretically (based on energy estimates) and numerically (based on a novel formulated nonuniform Rothe scheme) how to reconstruct a time-dependent source from the knowledge of an integral measurement for a non-autonomous time Caputo fractional diffusion equation of order $0 < \beta < 1$. The case of considering a noisy $m_\varepsilon \in \mathrm{L}^2(0,T)$ will be for future consideration by turning the inverse source problem to solve the variational problem and this can be done by using Tikhonov regularization or by Meyer wavelet regularization techniques.

6 Conclusion

A theoretical and numerical determination of a time-dependent source from the knowledge of an integral measurement have been studied for a non-autonomous fractional diffusion equation of order β. Two convergent and stable algorithms based on Rothe's method have been proposed for the recovery of the missing source. More specifically, in the first scheme a standard convolution quadrature for weakly singular kernels is used (on a uniform time grid), whilst in the second scheme the graded $L1$-approximation has been considered. Employing the first approximation, we have shown the existence of a unique weak solution for a big class of admissible solutions. The assumptions made on the measurement and data gave us the possibility to deal with non-smooth solutions containing a t^γ term with $\gamma \in (\beta, 1)$ fixed. The scheme on the basis of the graded $L1$-approximation has been shown to be convergent when the grading grid parameter r is strictly larger than $\frac{1}{\beta}$ and when the solution satisfies the bounds (64) (i.e. also terms t^β can be considered). The numerical experiments have been implemented using the FEM. It has been illustrated that the second approach also for the inverse problem under consideration has the advantage to cope better from numerical viewpoint with the behaviour at $t = 0$ for the considered problems. Future work can be concerned with relaxing the condition on the coefficient c to $c \geqslant 0$, and the convergence of the graded $L1$-scheme without assuming any bounds on the solution and/or $r > \frac{1}{\beta}$. Future research can also concern the derivation of error bounds between the exact and numerical solution, the further development of the numerical scheme and experiments, and the validation of the assumed regularity on the solution (and so the measurement) in the case that F is nonlinear in u.

Acknowledgements The authors are grateful to the handling editor and the anonymous referees for their constructive feedback and helpful suggestions, which highly improved the paper. The authors would also like to thank Professor Vladimir G. Pimenov of Ural Federal University and Professor Marián Slodička of Ghent University, for their generosity and guidance, which has always been so valuable to them.

This version of the article has been accepted for publication, after peer review but is not the Version of Record and does not reflect post-acceptance improvements, or any corrections. The Version of Record is available online at: `http://dx.doi.org/10.1007/s10915-021-01704-8`. Use of this Accepted Version is subject to the publisher's Accepted Manuscript terms of use `https://www.springernature.com/gp/open-research/policies/acceptedmanuscript-terms`.

Declarations

Funding

A. S. Hendy wishes to acknowledge the support of RFBR Grant 19-01-00019. K. Van Bockstal is supported by a postdoctoral fellowship of the Research Foundation - Flanders (106016/12P2919N).

Conflicts of interest/Competing interests

The authors have no conflicts of interest to declare that are relevant to the content of this article.

Availability of data and material

Data sharing is not applicable to this article as no datasets were generated or analysed during the current study.

Code availability

The authors used the open-source computing platform FEniCS for computations.

References

1. Alnæs, M.S., Blechta, J., Hake, J., Johansson, A., Kehlet, B., Logg, A., Richardson, C., Ring, J., Rognes, M.E., Wells, G.N.: The FEniCS project version 1.5. Archive of Numerical Software **3**(100), 9–23 (2015)
2. Bainov, D., Simeonov, P.: Integral inequalities and applications. Mathematics and Its Applications. East European Series. 57. Dordrecht, Kluwer Academic Publishers (1992)
3. Brunner, H.: The numerical solution of weakly singular volterra integral equations by collocation on graded meshes. Mathematics of computation **45**(172), 417–437 (1985)
4. Brunner, H., Ling, L., Yamamoto, M.: Numerical simulations of 2d fractional subdiffusion problems. Journal of Computational Physics **229**(18), 6613–6622 (2010)
5. Cannon, J.R., Esteva, S.P., Van Der Hoek, J.: A galerkin procedure for the diffusion equation subject to the specification of mass. SIAM Journal on Numerical Analysis **24**(3), 499–515 (1987)
6. Ismailov, M.I., Çiçek, M.: Inverse source problem for a time-fractional diffusion equation with nonlocal boundary conditions. Applied Mathematical Modelling **40**(7), 4891–4899 (2016)

7. Jiang, D., Liu, Y., Wang, D.: Numerical reconstruction of the spatial component in the source term of a time-fractional diffusion equation. Advances in Computational Mathematics **46**(3), 43 (2020)
8. Jin, B., Lazarov, R., Zhou, Z.: An analysis of the L1 scheme for the subdiffusion equation with nonsmooth data. IMA Journal of Numerical Analysis **36**(1), 197–221 (2016)
9. Jin, B., Li, B., Zhou, Z.: Numerical analysis of nonlinear subdiffusion equations. SIAM Journal on Numerical Analysis **56**(1), 1–23 (2018)
10. Jin, B., Rundell, W.: A tutorial on inverse problems for anomalous diffusion processes. Inverse Problems **31**(3), 035003 (2015)
11. Kačur, J.: Method of rothe in evolution equations. In: Equadiff 6, pp. 23–34. Springer (1986)
12. Karimi, M., Moradlou, F., Hajipour, M.: On regularization and error estimates for the backward heat conduction problem with time-dependent thermal diffusivity factor. Communications in Nonlinear Science and Numerical Simulation **63**, 21–37 (2018)
13. Karimi, M., Moradlou, F., Hajipour, M.: Regularization technique for an inverse space-fractional backward heat conduction problem. Journal of Scientific Computing **83**(2), 37 (2020)
14. Karimi, M., Moradlou, F., Hajipour, M.: On the ill-posed analytic continuation problem: An order optimal regularization scheme. Applied Numerical Mathematics **161**, 311–332 (2021)
15. Karimi, M., Zallani, F., Sayevand, K.: Wavelet regularization strategy for the fractional inverse diffusion problem. Numerical Algorithms (2020)
16. Kazem, S.: Exact solution of some linear fractional differential equations by Laplace transform. Int. J. Nonlinear Sci. **16**(1), 3–11 (2013)
17. Kirane, M., Malik, S.A., Al-Gwaiz, M.A.: An inverse source problem for a two dimensional time fractional diffusion equation with nonlocal boundary conditions. Mathematical Methods in the Applied Sciences **36**(9), 1056–1069 (2013)
18. Kopteva, N.: Error analysis of the $L1$ method on graded and uniform meshes for a fractional-derivative problem in two and three dimensions. Math. Comput. **88**(319), 2135–2155 (2019)
19. Kubica, A., Yamamoto, M.: Initial-boundary value problems for fractional diffusion equations with time-dependent coefficients. Fractional Calculus and Applied Analysis **21**(2), 276–311 (2018)
20. Kufner, A., John, O., Fučík, S.: Function Spaces. Monographs and textbooks on mechanics of solids and fluids. Noordhoff International Publishing, Leyden (1977)
21. Li, Y.S., Sun, L.L., Zhang, Z.Q., Wei, T.: Identification of the time-dependent source term in a multi-term time-fractional diffusion equation. Numerical Algorithms **82**(4), 1279–1301 (2019)
22. Liao, H.l., Li, D., Zhang, J.: Sharp error estimate of the nonuniform l1 formula for linear reaction-subdiffusion equations. SIAM Journal on Numerical Analysis **56**(2), 1112–1133 (2018)
23. Lin, Y., Xu, C.: Finite difference/spectral approximations for the time-fractional diffusion equation. Journal of Computational Physics **225**(2), 1533–1552 (2007)
24. Logg, A., Mardal, K.A., Wells, G.N., et al.: Automated Solution of Differential Equations by the Finite Element Method. Springer, Berlin, Heidelberg (2012)
25. Logg, A., Wells, G.N.: DOLFIN: Automated Finite Element Computing. ACM Trans. Math. Software **37**(2), 28 (2010)
26. Logg, A., Wells, G.N., Hake, J.: DOLFIN: a C++/Python Finite Element Library, chap. 10. Springer, Berlin, Heidelberg (2012)
27. Luchko, Y.: Initial-boundary-value problems for the one-dimensional time-fractional diffusion equation. Fractional Calculus and Applied Analysis **15**(1), 141–160 (2012)
28. McLean, W.: Regularity of solutions to a time-fractional diffusion equation. The ANZIAM Journal **52**(2), 123–138 (2010)
29. Metzler, R., Klafter, J.: The random walk's guide to anomalous diffusion: a fractional dynamics approach. Physics reports **339**(1), 1–77 (2000)
30. Mustapha, K.: An implicit finite-difference time-stepping method for a sub-diffusion equation, with spatial discretization by finite elements. IMA Journal of Numerical Analysis **31**(2), 719–739 (2011)
31. Mustapha, K., AlMutawa, J.: A finite difference method for an anomalous sub-diffusion equation, theory and applications. Numerical Algorithms **61**(4), 525–543 (2012)
32. Mustapha, K., McLean, W.: Superconvergence of a discontinuous galerkin method for fractional diffusion and wave equations. SIAM Journal on Numerical Analysis **51**(1), 491–515 (2013)
33. Nohel, J.A., Shea, D.F.: Frequency domain methods for Volterra equations. Advances in Mathematics **22**(3), 278–304 (1976)
34. Rektorys, K.: The method of discretization in time and partial differential equations. Equadiff 5 pp. 293–296 (1982)
35. Roussy, G., Bennani, A., Thiebaut, J.M.: Temperature runaway of microwave irradiated materials. Journal of applied physics **62**(4), 1167–1170 (1987)

36. Sakamoto, K., Yamamoto, M.: Initial value/boundary value problems for fractional diffusion-wave equations and applications to some inverse problems. Journal of Mathematical Analysis and Applications **382**(1), 426–447 (2011)
37. Sakamoto, K., Yamamoto, M.: Inverse source problem with a final overdetermination for a fractional diffusion equation. Math. Control Relat. Fields **1**(4), 509–518 (2011)
38. Slodička, M., Šišková, K.: An inverse source problem in a semilinear time-fractional diffusion equation. Computers & Mathematics with Applications **72**(6), 1655–1669 (2016)
39. Slodička, M.: Numerical solution of a parabolic equation with a weakly singular positive-type memory term. Electron. J. Differ. Equ. **1997**, paper 9, 12 (1997)
40. Slodička, M., Šišková, K.: An inverse source problem in a semilinear time-fractional diffusion equation. Computers & Mathematics with Applications **72**(6), 1655–1669 (2016)
41. Slodička, M., Šišková, K., Van Bockstal, K.: Uniqueness for an inverse source problem of determining a space dependent source in a time-fractional diffusion equation. Applied Mathematics Letters **91**, 15–21 (2019)
42. Srivastava, H.M., Daoust, M.C.: A note on the convergence of Kampé de Fériet's double hypergeometrics series. Math. Nachr. **53**, 151–159 (1972). doi: 10.1002/mana.19720530114
43. Stynes, M., O'Riordan, E., Gracia, J.L.: Error analysis of a finite difference method on graded meshes for a time-fractional diffusion equation. SIAM Journal on Numerical Analysis **55**(2), 1057–1079 (2017)
44. Sun, Z.z., Wu, X.: A fully discrete difference scheme for a diffusion-wave system. Applied Numerical Mathematics **56**(2), 193–209 (2006)
45. Van Bockstal, K.: Existence and uniqueness of a weak solution to a non-autonomous time-fractional diffusion equation (of distributed order). Applied Mathematics Letters **109**, 106540 (2020)
46. Van Bockstal, K.: Existence of a unique weak solution to a nonlinear non-autonomous time-fractional wave equation (of distributed-order). Mathematics **8**(8), 1283 (2020)
47. Vladimirov, V.S.: Equations of mathematical physics. (Uravneniya matematicheskoj fiziki.) 2. überarb. und erg. Auflage. Moskau: Verlag "Nauka", Hauptredaktion für physikalisch-mathematische Literatur. 512 S. R. 1.05 (1971). (1971)
48. Wahba, G.: Spline Models for Observational Data. Society for Industrial and Applied Mathematics (1990)
49. Wang, J.G., Zhou, Y.B., Wei, T.: Two regularization methods to identify a space-dependent source for the time-fractional diffusion equation. Applied Numerical Mathematics **68**, 39–57 (2013)
50. Wei, T., Wang, J.: A modified quasi-boundary value method for an inverse source problem of the time-fractional diffusion equation. Applied Numerical Mathematics **78**, 95–111 (2014)
51. Wei, T., Zhang, Z.: Reconstruction of a time-dependent source term in a time-fractional diffusion equation. Engineering Analysis with Boundary Elements **37**(1), 23 – 31 (2013)
52. Yang, F., Liu, X., Li, X.X.: Landweber iterative regularization method for identifying the unknown source of the modified Helmholtz equation. Boundary Value Problems **2017**(1), 91 (2017)
53. Yeganeh, S., Mokhtari, R., Hesthaven, J.S.: Space-dependent source determination in a time-fractional diffusion equation using a local discontinuous Galerkin method. BIT **57**(3), 685–707 (2017)
54. Zaky, M.A., Hendy, A.S., Macías-Díaz, J.E.: Semi-implicit galerkin–legendre spectral schemes for nonlinear time-space fractional diffusion–reaction equations with smooth and nonsmooth solutions. Journal of Scientific Computing **82**(1), 1–27 (2020)
55. Zhang, M., Liu, J.: Identification of a time-dependent source term in a distributed-order time-fractional equation from a nonlocal integral observation. Computers & Mathematics with Applications **78**(10), 3375–3389 (2019)