

A space-time discretization for an electromagnetic problem with moving non-magnetic conductor

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Abstract

This paper deals with a space-time discretization scheme for an eddy current problem in a multi-component domain with a moving non-magnetic conductor. We incorporate the Coulomb gauge to the formulation, then propose a fully-discrete finite element scheme combined with backward Euler's method to find an approximation of the solution to the variational system. The convergence of the scheme is proved, and the error estimates for the first-order Lagrangian finite elements are established. Under appropriate assumptions on the weak solution and the given initial data, we show the optimal convergence rate for the space discretization and the suboptimal rate for time discretization. Some numerical experiments are performed to support the obtained theoretical results.

Keywords: space-time discretization, finite element method, error estimates, moving non-magnetic conductor, Reynolds transport theorem
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1. Introduction

The electrodynamics of moving bodies plays a central role in many industrial and engineering processes such as induction heating, induction hardening, electromagnetic deforming and magneto-hydrodynamics. The system is originated by a source current that generates electromagnetic field in the surrounding space, which afterwards induces electric currents in the conductors. For a slowly varying system, the eddy current model is a common approach to describe the process, which simplifies Maxwell's equations by neglecting the dielectric displacement currents. In spite of many studies that deal with the electromagnetism of continuous media, e.g. [1, 2], there are just a few papers dealing with the analysis of numerical methods for the eddy current problem on a multi-media system with moving conductors.

A couple of excellent works were devoted by A. Bermúdez et al. (cf. [3, 4]), wherein the authors studied the mathematical and numerical analysis of a transient eddy current problem on a moving domain. The authors restricted their attention to a cylindrical symmetric domain that allowed them to state the eddy current equations in terms of the azimuthal component defined on a two-dimensional meridian section. Moreover, in the context of a non-magnetic system, the constant of vacuum was considered as an approximation of the magnetic permeability in the whole domain.

The article [3] provides the well-posedness of the variational system in this context. In [4], the authors proposed a finite element method in space combined with a backward Euler time scheme for the numerical analysis of the problem. In Theorem 9, optimal error estimates (for the magnetic induction) were obtained by assuming additional regularity on the solution than what had been proved in the previous work. However, from a mathematical point of view, the additional assumption about H^2 -regularity of the solution in time is a strong assumption.

In order to release the continuity assumption on the magnetic permeability, in the paper [5], we investigated an electromagnetic contact problem with moving conductor wherein the material coefficients were allowed to have jumps

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at the interfaces. A time-discrete scheme was proposed based on backward Euler's (Rothe's) method and analyzed by aid of the Reynolds transport theorem. This is up to now the first work proving the well-posedness of the variational system for the setting of a jumping permeability. However, establishing error estimates still remains an open problem. In the next work [6], we provided an enhancement of the previous paper by considering an electromagnetic multi-media problem with moving non-magnetic conductor (i.e. constant magnetic permeability). A new time-discrete scheme by means of the saddle-point formulation was proposed to solve numerically the corresponding variational system. The convergence of the numerical scheme to a unique weak solution was proved. Moreover, we derived the error estimates on the solution (associated with the time-discretization) under appropriate assumptions on the given data, but without additional assumptions on the solution.

The present paper introduces a space-time finite element scheme combined with backward Euler's method for the electromagnetic problem with a moving non-magnetic conductor considered in [6]. To the best of our knowledge, there is still a challenge to deal with finite element analysis of the saddle-point approach of the eddy current problem with moving conductor. In order to overcome the difficulties raised by a discrete saddle-point problem, we relax the divergence-free condition by incorporating the Coulomb gauge in the formulation. The time derivative acting on the moving conductors leads us to a degenerate parabolic problem, and handling the terms on moving subdomains in the sense of a space-time discretization scheme are the highlights of this contribution.

This paper is organized as follows. In the next section, we start with explaining the mathematical formulation of the model. The well-posedness of the variational system is shown in Section 3. Then in Section 4, we propose a space-time finite element method for solving numerically the variational problems. We prove the convergence of the scheme to the weak solution by deriving the error estimates of this full discretization as functions of the time step τ and the error of the orthogonal projections. Some appropriate assumptions on the given initial data and also on the weak solution are made to obtain the error estimates for the first-order Lagrangian finite element spaces, see Theorems 4.2 and 4.3. The convergence rate of the proposed scheme is optimal for the space discretization and suboptimal with respect to time. Some numerical results are presented in Section 5 to assess the performance of the proposed scheme as well as to support the theoretical analysis.

2. Mathematical model

We consider an electromagnetic process that takes place in an open, simply-connected bounded Lipschitz and convex domain $\Omega \subset \mathbb{R}^3$. This space is occupied by a static conductive coil Π , a moving workpiece Σ and the surrounding air, see Figure 1. The subdomains Π and Σ both are supposed to be open, connected Lipschitz domains, and are separate from each other during the whole time frame $[0, T]$. For the reason of convenience for the finite element analysis, the domain Ω and the fixed coil Π are also supposed to be polyhedrons. The following notations will be frequently used throughout the paper: \mathbf{n} denotes the outward unit normal vector, $\Theta := \Sigma \cup \Pi$ is the union of the conductors, $\Xi := \Omega \setminus \overline{\Theta}$ is the air subdomain and $\Gamma := \Gamma_{\text{in}} \cup \Gamma_{\text{out}}$ are the intersection of the boundaries $\partial\Omega$ and $\partial\Pi$ supposed that $|\Gamma_{\text{in}}| > 0$ and $|\Gamma_{\text{out}}| > 0$.

Let us describe the movement of the workpiece in the manner of a motion of the body $\Sigma(0)$, which is a smooth function

$$\Phi : \Sigma(0) \times [0, T] \rightarrow \mathbb{R}^3$$

with $\Phi_t := \Phi(\cdot, t)$, for each fixed $t \in [0, T]$, a bijective, smooth mapping (a deformation) of $\Sigma(0)$, i.e.

$$\mathbf{x} \mapsto \Phi(\mathbf{x}, t); \quad \widetilde{\Sigma} := \bigcup_{t \in [0, T]} \overline{\Sigma(t)} \subset \Omega; \quad \det \nabla \Phi(\mathbf{x}, t) > 0, \quad \forall (\mathbf{x}, t) \in \Sigma(0) \times [0, T]. \quad (1)$$

Here, we refer to

$$\Sigma(t) = \Phi(\Sigma(0), t)$$

as the region of space occupied by $\Sigma(0)$ at time t , or in other words, the place of the workpiece at time t . In order to work with spatial description rather than material description of fields associated to the motion, we introduce the trajectory

$$\mathbb{T} := \{(\mathbf{x}, t) : \mathbf{x} \in \Sigma(t), t \in [0, T]\},$$

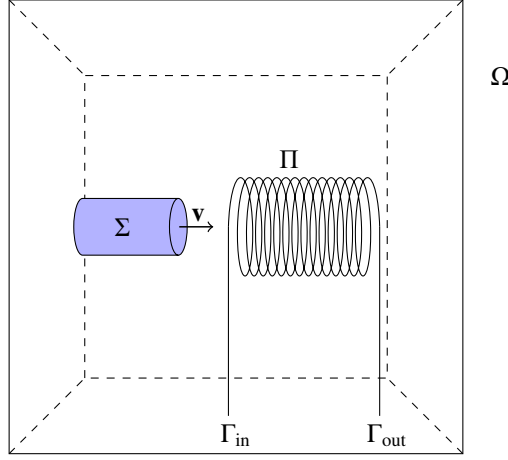


Figure 1: The domain Ω of the problem consisting of the workpiece Σ , the coil Π with the interfaces Γ_{in} and Γ_{out} , and the surrounding air.

and the reference mapping

$$\rho : \mathbb{T} \rightarrow \Sigma(0) : (x, t) \mapsto \Phi_t^{-1}(x).$$

The spatial description of the velocity of the workpiece is defined as $\mathbf{v}(x, t) := \dot{\Phi}(\rho(x, t), t)$. Since the surrounding air moves also, we can extend the velocity vector \mathbf{v} defined on the trajectory \mathbb{T} to the whole space-time domain $\bar{\Omega} \times [0, T]$ with the assumptions that $\mathbf{v} = \mathbf{0}$ on the coil Π and $\mathbf{v} \in \mathbf{C}^1(\bar{\Omega} \times [0, T])$. This means that the workpiece Σ is allowed to deform, but Σ has to stay a Lipschitz domain during the movement.

In this work, we restrict our attention to non-magnetic materials occupying the subdomains, e.g. an aluminum workpiece and a copper coil surrounded by air. In such a situation, the constant of vacuum $\mu_0 > 0$ can be considered as a good approximation of the magnetic permeability of the materials, i.e.

$$\mu = \mu_0 \quad \text{in } \Omega.$$

The electrical conductivity σ vanishes on the air, otherwise it is assumed to be a strictly positive constant on each conductor, i.e.

$$\sigma(t) = \begin{cases} \sigma_{\Pi} & \text{in } \Pi \\ \sigma_{\Sigma} & \text{in } \Sigma(t) \\ 0 & \text{in } \Omega \setminus \overline{\Theta(t)}. \end{cases}$$

The electromagnetic process can be described by the eddy current equations (the so-called magneto-quasistatic system), which originate from Maxwell's equations by neglecting the electric displacement \mathbf{D} , i.e.

$$\nabla \cdot \mathbf{B} = 0, \tag{2a}$$

$$\nabla \times \mathbf{E} = -\partial_t \mathbf{B}, \tag{2b}$$

$$\nabla \times \mathbf{H} = \mathbf{J}. \tag{2c}$$

At the interfaces of the different materials, the electromagnetic fields may behave in different ways. The following interface conditions constitute a perfect contact electromagnetic process

$$[[\mathbf{B} \cdot \mathbf{n}]]_{\partial\Theta \setminus \Gamma} = 0, \quad [[\mathbf{H} \times \mathbf{n}]]_{\partial\Theta \setminus \Gamma} = \mathbf{0},$$

where $[[\cdot]]$ denotes the jump of the field passing through the interfaces. Moreover, we impose a boundary condition on the magnetic induction \mathbf{B} : the homogeneous normal component, i.e.

$$\mathbf{B} \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega.$$

Since the convex-polyhedron domain Ω is open, simply-connected and bounded, the solenoidal magnetic induction \mathbf{B} has a unique divergence-free vector potential $\mathbf{A} \in \mathbf{H}^1(\Omega)$ satisfying $\mathbf{A} \times \mathbf{n} = \mathbf{0}$ on $\partial\Omega$, i.e. $\mathbf{B} = \nabla \times \mathbf{A}$, see [7, Theorem 3.6 on p. 48]. Substituting this statement into Faraday's law (2b) gives us a unique function $\phi \in H^1(\Omega)$ (up to a constant) such that (cf. [7, Theorem 2.9])

$$\mathbf{E} + \partial_t \mathbf{A} = -\nabla \phi.$$

Hence, we are able to investigate the electromagnetic problem in the $\mathbf{A} - \phi$ formulation. In the case of a moving conductor, Ohm's law states the relation between the current density \mathbf{J} and the electromagnetic fields as follow

$$\mathbf{J} = \sigma(\mathbf{E} + \mathbf{v} \times \mathbf{B}).$$

Therefore, the total current density \mathbf{J} can be split into two parts: the source current $\mathbf{J}_s = -\sigma \nabla \phi$ and the eddy current $\mathbf{J}_e = -\sigma \partial_t \mathbf{A} + \sigma \mathbf{v} \times (\nabla \times \mathbf{A})$. In the coil Π , the source current results from an external current density imposed on the interface Γ_{in} in the direction of the normal. We denote $j = \mathbf{J}_s \cdot \mathbf{n}$ and formulate the source current in the coil Π by the following boundary value problem

$$\begin{cases} \nabla \cdot (-\sigma \nabla \phi) = 0 & \text{in } \Pi \times (0, T), \\ -\sigma \nabla \phi \cdot \mathbf{n} = 0 & \text{on } (\partial\Pi \setminus \Gamma) \times (0, T), \\ -\sigma \nabla \phi \cdot \mathbf{n} = -j & \text{on } \Gamma_{\text{in}} \times (0, T), \\ -\sigma \nabla \phi \cdot \mathbf{n} = j & \text{on } \Gamma_{\text{out}} \times (0, T). \end{cases} \quad (3)$$

In order to guarantee the solvability of this problem, we require the following additional compatibility condition

$$\int_{\Gamma} j(s, t) \, ds = 0 \quad \forall t \in [0, T], \quad (4)$$

where

$$j = \begin{cases} -J & \text{on } \Gamma_{\text{in}}, \\ J & \text{on } \Gamma_{\text{out}}. \end{cases}$$

In what follows, we append a characteristic function χ_{Π} to the source current $-\sigma \nabla \phi$ to indicate the fact that the source current exists only in the coil Π . For the extension of ϕ on the whole domain Ω , we refer to the model presented in [5]. Ampère's law (2c) together with all considerations above lead us to the following equation

$$\sigma \partial_t \mathbf{A} + \mu_0^{-1} \nabla \times \nabla \times \mathbf{A} + \chi_{\Pi} \sigma \nabla \phi - \sigma \mathbf{v} \times (\nabla \times \mathbf{A}) = \mathbf{0}. \quad (5)$$

Now, we deal with the divergence-free condition of the vector potential \mathbf{A} . A possible approach is reformulating the variational system of \mathbf{A} in the sense of a saddle-point problem (see [6] for more details). Another common treatment is incorporating the Coulomb gauge in the formulation (cf. [8, 9]), which is done by appending the following penalty term to the left-hand side (LHS) of the relation (5)

$$-\alpha \nabla(\nabla \cdot \mathbf{A}),$$

where $\alpha > 0$ plays the role of a penalty coefficient. Please note that this technique is an approximation approach, and the value of α depends on how much stress is put on the divergence-free condition. Moreover, we impose a homogeneous Dirichlet boundary condition on $\partial\Omega$, which makes sense thanks to the consideration that the boundary $\partial\Omega$ is sufficiently far from the conductors. Finally, together with an initial condition at time $t = 0$, we derive the following initial-boundary value problem for the vector potential \mathbf{A}

$$\begin{cases} \sigma \partial_t \mathbf{A} + \mu_0^{-1} \nabla \times \nabla \times \mathbf{A} - \alpha \nabla(\nabla \cdot \mathbf{A}) + \chi_{\Pi} \sigma \nabla \phi - \sigma \mathbf{v} \times (\nabla \times \mathbf{A}) = \mathbf{0} & \text{in } \Omega \times (0, T), \\ \mathbf{A} = \mathbf{0} & \text{on } \partial\Omega \times (0, T), \\ \mathbf{A}(\cdot, 0) = \mathbf{A}_0 & \text{in } \Theta(0). \end{cases} \quad (6)$$

In the next sections, we will discuss the well-posedness and the full-discretization of the problem (3)-(4)-(6).

Remark 2.1 (Extension of \mathbf{A}_0 on the whole domain Ω). *The system (6) forms a degenerate parabolic problem because the electrical conductivity σ vanishes outside of the trajectory \mathbb{T} . Therefore, the initial guest \mathbf{A}_0 is only given on the conductors $\Theta(0)$, where σ is strictly positive. Throughout this paper, we assume that $\mathbf{A}_0 \in \mathbf{H}^1(\Theta(0))$ satisfies $\mathbf{A}_0 = \mathbf{0}$ on Γ . Given $f \in \mathbf{H}^{-1}(\Xi(0))$, by solving the following boundary value problem in the air subdomain*

$$\begin{cases} \Delta \mathbf{u} = f & \text{in } \Xi(0), \\ \mathbf{u} = \mathbf{A}_0 & \text{on } \partial\Theta(0) \setminus \Gamma, \\ \mathbf{u} = \mathbf{0} & \text{on } \partial\Omega \setminus \Gamma, \end{cases}$$

we are able to extend \mathbf{A}_0 to $\tilde{\mathbf{A}}_0$ defined on the whole domain Ω such that $\tilde{\mathbf{A}}_0 \in \mathbf{H}_0^1(\Omega)$. From now on, we identify \mathbf{A}_0 as a $\mathbf{H}_0^1(\Omega)$ -function in the sense of its extension $\tilde{\mathbf{A}}_0$.

3. Well-posedness

First of all, some standard functional settings are introduced for scalar fields. The corresponding bold symbols are defined for vector or tensor fields in the same way. The Lebesgue space $L^2(\Omega)$ of square-integrable fields is associated with the inner product $(\cdot, \cdot)_\Omega$ and its induced norm $\|\cdot\|_{L^2(\Omega)}$. The Sobolev spaces $H^1(\Omega)$, $H_0^1(\Omega)$ and $H^2(\Omega)$ are equipped with the following norms

$$\begin{aligned} \|\psi\|_{H^1(\Omega)} &= \|\psi\|_{L^2(\Omega)} + \|\nabla\psi\|_{L^2(\Omega)}, \\ \|\psi\|_{H_0^1(\Omega)} &= \|\nabla\psi\|_{L^2(\Omega)}, \\ \|\psi\|_{H^2(\Omega)} &= \sum_{|\alpha| \leq 2} \|D^\alpha \psi\|_{L^2(\Omega)}. \end{aligned}$$

The following identity which holds true for any vector field $\boldsymbol{\varphi} \in \mathbf{H}_0^1(\Omega)$ will be useful for further analysis

$$\|\nabla\boldsymbol{\varphi}\|_{L^2(\Omega)}^2 = \|\nabla \times \boldsymbol{\varphi}\|_{L^2(\Omega)}^2 + \|\nabla \cdot \boldsymbol{\varphi}\|_{L^2(\Omega)}^2. \quad (7)$$

We consider the following subspace of $H^1(\Pi)$

$$Z = \{\psi \in H^1(\Pi) : (\psi, 1)_\Pi = 0\},$$

which inherits the norm of $H^1(\Pi)$. The well-known Poincaré-Wirtinger inequality gives the existence of a constant $C > 0$ such that

$$\|\psi - c_\psi\|_{H^1(\Pi)} \leq C \|\nabla\psi\|_{L^2(\Pi)} \quad \forall \psi \in H^1(\Pi), \quad (8)$$

where

$$c_\psi = \frac{1}{|\Pi|} \int_\Pi \psi(x) \, dx. \quad (9)$$

Therefore, Z is a Hilbert space with the induced norm $\|\nabla\psi\|_{L^2(\Pi)}$.

Now, let us recall some Banach spaces for abstract functions. Given X be an arbitrary Banach space equipped with the norm $\|\cdot\|_X$. The spaces $C([0, T], X)$ and $\text{Lip}([0, T], X)$ consist of all continuous and Lipschitz continuous functions $u : [0, T] \rightarrow X$ provided the usual norm

$$\|u\|_{C([0, T], X)} = \max_{0 \leq t \leq T} \|u(t)\|_X.$$

The Bochner spaces $L^2((0, T), X)$ and $L^\infty((0, T), X)$ are the Banach spaces of all measurable abstract functions $u : (0, T) \rightarrow X$ such that

$$\|u\|_{L^2((0, T), X)} := \left(\int_0^T \|u(t)\|_X^2 \, dt \right)^{1/2} < +\infty, \quad \|u\|_{L^\infty((0, T), X)} := \text{ess sup}_{t \in (0, T)} \|u(t)\|_X < +\infty.$$

The Sobolev-Bochner space $H^1((0, T), X)$ is defined as

$$H^1((0, T), X) = \{u \in L^2((0, T), X) : \partial_t u \in L^2((0, T), X)\},$$

with the norm

$$\|u\|_{H^1((0, T), X)} = \|u\|_{L^2((0, T), X)} + \|\partial_t u\|_{L^2((0, T), X)}.$$

Please note that $H^1((0, T), X) \hookrightarrow C([0, T], X)$, see [10, Lemma 7.3].

We use $a \lesssim b$ ($a \gtrsim b$, resp.) instead of $a \leq Cb$ ($a \geq Cb$, resp.), where $C > 0$ is an arbitrary constant depending only on the given data. The positive number ε expresses an arbitrary small constant, while C_ε is an arbitrary large constant, depending on ε .

Before going to the variational formulations, we mention a useful tool for the analysis of PDEs with time-dependent domains. Let $\omega(t)$ be a Lipschitz moving domain which is associated with a velocity vector \mathbf{v} being of class \mathbf{C}^1 and f a scalar abstract function satisfying $f(t) \in \mathbf{W}^{1,1}(\omega(t))$ and $\partial_t f(t) \in L^1(\omega(t))$ for all $t \in (0, T)$. Then the Reynolds transport theorem (cf. [11, p. 78]) together with a density argument implies that

$$\frac{d}{dt} \int_{\omega(t)} f \, d\mathbf{x} = \int_{\omega(t)} \partial_t f \, d\mathbf{x} + \int_{\partial\omega(t)} f \mathbf{v} \cdot \mathbf{n} \, ds. \quad (10)$$

We present also some auxiliary results for handling time-dependent boundary terms, which will be used frequently in further analysis. Let $f, g \in H^1(\omega(t))$, then the Divergence theorem allows us to estimate that

$$\begin{aligned} \int_{\partial\omega(t)} fg(\mathbf{v} \cdot \mathbf{n}) \, ds &= \int_{\omega(t)} g \nabla f \cdot \mathbf{v} \, d\mathbf{x} + \int_{\omega(t)} f \nabla g \cdot \mathbf{v} \, d\mathbf{x} + \int_{\omega(t)} fg(\nabla \cdot \mathbf{v}) \, d\mathbf{x} \\ &\lesssim \int_{\omega(t)} |g| |\nabla f| \, d\mathbf{x} + \int_{\omega(t)} |f| |\nabla g| \, d\mathbf{x} + \int_{\omega(t)} |f| |g| \, d\mathbf{x} \end{aligned} \quad (11)$$

$$\leq \varepsilon \|\nabla f\|_{L^2(\omega(t))}^2 + \varepsilon \|\nabla g\|_{L^2(\omega(t))}^2 + C_\varepsilon \|f\|_{L^2(\omega(t))}^2 + C_\varepsilon \|g\|_{L^2(\omega(t))}^2. \quad (12)$$

When $f = g$, we get a Nečas-like inequality

$$\int_{\partial\omega(t)} f^2(\mathbf{v} \cdot \mathbf{n}) \, ds \leq \varepsilon \|\nabla f\|_{L^2(\omega(t))}^2 + C_\varepsilon \|f\|_{L^2(\omega(t))}^2. \quad (13)$$

Please note that all constants do not depend on the time variable, and the inequalities (11-13) are still valid for vector fields in $\mathbf{H}^1(\omega(t))$.

For ease of reference, we list down here all assumptions used throughout the paper.

1. (AS1): the domain Ω is an open, simply-connected bounded Lipschitz and convex polyhedron; the workpiece Σ and the polyhedral coil Π are open, connected Lipschitz subdomains and are separate from each other (see Section 2 for more details);
2. (AS2): the magnetic permeability μ is a constant in the whole domain Ω ; the electrical conductivity σ vanishes on the air, otherwise it is a positive constant on each conductor (see Section 2 for more details);
3. (AS3): $\mathbf{A}_0 \in \mathbf{H}_0^1(\Omega)$ in the sense of the extension described in Remark 2.1;
4. (AS4): $j \in \text{Lip}([0, T], H^{-1/2}(\Gamma))$ and $\mathbf{v} \in \mathbf{C}^1(\overline{\Omega} \times [0, T])$.

Now, we introduce the variational system corresponding to the problems (3), (4) and (6): find $\phi(t) \in Z$ and $\mathbf{A}(t) \in \mathbf{H}_0^1(\Omega)$ with $\partial_t \mathbf{A}(t) \in L^2(\Theta(t))$ such that the following identities are satisfied for any $\psi \in Z$ and $\boldsymbol{\varphi} \in \mathbf{H}_0^1(\Omega)$ and for a.a. $t \in (0, T)$

$$\sigma_\Pi (\nabla \phi(t), \nabla \psi)_\Pi + (j(t), \psi)_\Gamma = 0, \quad (14)$$

$$\begin{aligned}
& (\sigma(t)\partial_t \mathbf{A}(t), \boldsymbol{\varphi})_{\Theta(t)} + \mu_0^{-1} (\nabla \times \mathbf{A}(t), \nabla \times \boldsymbol{\varphi})_{\Omega} + \alpha (\nabla \cdot \mathbf{A}(t), \nabla \cdot \boldsymbol{\varphi})_{\Omega} \\
& \quad + \sigma_{\Pi} (\nabla \phi(t), \boldsymbol{\varphi})_{\Pi} - \sigma_{\Sigma} (\mathbf{v}(t) \times (\nabla \times \mathbf{A}(t)), \boldsymbol{\varphi})_{\Sigma(t)} = 0.
\end{aligned} \tag{15}$$

Here, the duality pairing $\langle j(t), \psi \rangle$ on $H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)$ can be viewed as the continuous extension of $L^2(\Gamma)$ -inner product. In the next theorem, the well-posedness of the variational system (14-15) is provided, which can be achieved by performing similar arguments presented in [5, 6].

Theorem 3.1 (Well-posedness). *Let the assumptions (AS1-AS4) be fulfilled. Then the variational system (14-15) admits exactly one solution (ϕ, \mathbf{A}) satisfying $\phi \in \text{Lip}([0, T], Z)$ and $\mathbf{A} \in L^\infty((0, T), \mathbf{H}_0^1(\Omega))$ with $\partial_t \mathbf{A} \in L^2((0, T), \mathbf{L}^2(\Omega))$ and $\|\sqrt{\sigma} \mathbf{A}\|_{L^2(\Theta)} \in C([0, T])$.*

The following higher interior regularity of the solution \mathbf{A} can be obtained by following the same lines of the proof of [12, Theorem 8.8], which involves the elliptic operator $\mu_0^{-1} \nabla \times \nabla \times \mathbf{A} - \alpha \nabla(\nabla \cdot \mathbf{A})$.

Corollary 3.1 (Interior regularity). *Let the assumptions (AS1-AS4) be fulfilled. Then $\mathbf{A} \in L^2((0, T), \mathbf{H}^2(\Sigma'))$ for any subdomain $\Sigma' \subset\subset \Omega$ (i.e. $\bar{\Sigma}' \subset \Omega$).*

By means of the saddle-point approach (cf. [13]), we are able to reformulate (14) in an equivalent formulation by which the variational system reads as: find $\phi(t) \in H^1(\Pi)$, $\beta(t) \in \mathbb{R}$ and $\mathbf{A}(t) \in \mathbf{H}_0^1(\Omega)$ with $\partial_t \mathbf{A}(t) \in \mathbf{L}^2(\Theta(t))$ such that the following identities are satisfied for any $\psi \in H^1(\Pi)$, $\lambda \in \mathbb{R}$ and $\boldsymbol{\varphi} \in \mathbf{H}_0^1(\Omega)$ and for a.a. $t \in (0, T)$

$$\sigma_{\Pi} (\nabla \phi(t), \nabla \psi)_{\Pi} + (j(t), \psi)_{\Gamma} + (\beta(t), \psi)_{\Pi} = 0, \tag{16}$$

$$(\phi(t), \lambda)_{\Pi} = 0, \tag{17}$$

$$\begin{aligned}
& (\sigma(t)\partial_t \mathbf{A}(t), \boldsymbol{\varphi})_{\Theta(t)} + \mu_0^{-1} (\nabla \times \mathbf{A}(t), \nabla \times \boldsymbol{\varphi})_{\Omega} + \alpha (\nabla \cdot \mathbf{A}(t), \nabla \cdot \boldsymbol{\varphi})_{\Omega} \\
& \quad + \sigma_{\Pi} (\nabla \phi(t), \boldsymbol{\varphi})_{\Pi} - \sigma_{\Sigma} (\mathbf{v}(t) \times (\nabla \times \mathbf{A}(t)), \boldsymbol{\varphi})_{\Sigma(t)} = 0.
\end{aligned} \tag{18}$$

The following satisfaction of the inf-sup condition guarantees the solvability of the saddle-point formulation (16-17), which implies the equivalence of the equation (14) and the mixed problem (16-17).

Lemma 3.1 (Inf-sup condition). *The operator $d : H^1(\Pi) \times \mathbb{R} \rightarrow \mathbb{R}$ defined by*

$$d(\psi, \lambda) = (\psi, \lambda)_{\Pi}$$

satisfies the Ladyzhenskaya-Babuška-Brezzi condition (or the so-called inf-sup condition), i.e. there exists a constant $C > 0$ such that

$$\sup_{\psi \in H^1(\Pi), \psi \neq 0} \frac{d(\psi, \lambda)}{\|\psi\|_{H^1(\Pi)}} \geq C |\lambda| \quad \forall \lambda \in \mathbb{R}. \tag{19}$$

Proof. Obviously, the inequality (19) holds true for $\lambda = 0$. For any $\lambda \in \mathbb{R} \setminus \{0\}$, we set $\psi = \lambda$ to have that

$$\frac{d(\lambda, \lambda)}{\|\lambda\|_{H^1(\Pi)}} = \frac{\|\lambda\|_{L^2(\Pi)}^2}{\|\lambda\|_{H^1(\Pi)}} = \sqrt{|\Pi|} |\lambda|.$$

□

Remark 3.1. *The problem (3-4) admits a unique weak solution in the space Z . An equivalent variational formulation using the Lagrangian multiplier method is represented by (16) and (17). There are also other possibilities. Let us just mention the following one. The classical variational formulation of (3-4) reads as (14), i.e.*

$$\begin{aligned}
& \sigma_{\Pi} (\nabla \phi(t), \nabla \psi)_{\Pi} = -(j(t), \psi)_{\Gamma} \quad \forall \psi \in H^1(\Pi), \\
& (1, \phi(t))_{\Pi} = 0.
\end{aligned}$$

The last formulation can be “stabilized” as follows (incorporating the term $(1, \phi(t))_{\Pi}$ into the formulation)

$$\beta(u, \psi) := \sigma_{\Pi} (\nabla u, \nabla \psi)_{\Pi} + (1, u)_{\Pi} (1, \psi)_{\Pi} = -(j(t), \psi)_{\Gamma} \quad \forall \psi \in H^1(\Pi).$$

Subtracting the last two relations from each other and setting $\psi = u - \phi(t)$, we get that

$$0 = \sigma_{\Pi} \|\nabla u - \nabla \phi(t)\|_{\mathbf{L}^2(\Pi)}^2 + ((1, u - \phi(t))_{\Pi})^2 + (1, \phi(t))_{\Pi} (1, u - \phi(t))_{\Pi} = \sigma_{\Pi} \|\nabla u - \nabla \phi(t)\|_{\mathbf{L}^2(\Pi)}^2 + ((1, u - \phi(t))_{\Pi})^2,$$

which means that $\nabla u = \nabla \phi(t)$ and $(1, u)_{\Pi} = (1, \phi(t))_{\Pi} = 0$. Invoking the Poincaré-Wirtinger inequality (8), we have that $u = \phi(t)$ in Z . Please note that the bilinear form β is invertible. In fact, the relation

$$\beta(u, u) = \sigma_{\Pi} \|\nabla u\|_{\mathbf{L}^2(\Pi)}^2 + ((1, u)_{\Pi})^2 = 0$$

implies from the first term that u is a constant and the second term gives that the constant is equal to zero.

4. Space-time discretization

In this section, we propose a fully-discrete finite element scheme combined with backward Euler's method to find an approximation of the solution to the variational system. First of all, the coil $\bar{\Pi}$ and the domain $\bar{\Omega}$ are subdivided into finite sets of distinct tetrahedrons \mathcal{T}_{Π} and \mathcal{T}_{Ω} , respectively, such that any face of a tetrahedron is either a face of another tetrahedron or a portion of the boundary, and

$$\bar{\Pi} = \bigcup_{K \in \mathcal{T}_{\Pi}} \bar{K}, \quad \bar{\Omega} = \bigcup_{K \in \mathcal{T}_{\Omega}} \bar{K}, \quad \mathcal{T}_{\Pi} \subset \mathcal{T}_{\Omega}.$$

These partitions are possible because the coil Π and the domain Ω are assumed to be Lipschitz polyhedrons (cf. [14]). From now on, we denote by \mathcal{T} the mesh of the domain Ω , which contains the triangulation of Π , and suppose that there is a regular family of meshes $\{\mathcal{T}^h : h > 0\}$, where h denotes the mesh size. We note that the mesh of the domain Ω does not necessarily fit the moving workpiece. This allows us to use a fixed mesh during the whole time range, which saves the cost of a re-meshing process. In the numerical experiments, we ignore the calculation errors caused by this ill-fitting and keep in touch with the moving workpiece by using a characteristic function, i.e.

$$\chi_{\Sigma}(\mathbf{x}) = \begin{cases} 1 & \text{if } \mathbf{x} \in \Sigma, \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore, we consider two finite element subspaces \mathbf{V}^h of $\mathbf{H}^1(\Pi)$ and \mathbf{V}_0^h of $\mathbf{H}_0^1(\Omega)$ such that

$$\lim_{h \rightarrow 0} \inf_{\psi^h \in \mathbf{V}^h} \|\psi - \psi^h\|_{\mathbf{H}^1(\Pi)} = 0, \quad (20)$$

$$\lim_{h \rightarrow 0} \inf_{\varphi^h \in \mathbf{V}_0^h} \|\varphi - \varphi^h\|_{\mathbf{H}_0^1(\Omega)} = 0, \quad (21)$$

for any $\psi \in \mathbf{H}^1(\Pi)$ and any $\varphi \in \mathbf{H}_0^1(\Omega)$. Let $\mathbf{P}^h : \mathbf{H}^1(\Pi) \rightarrow \mathbf{V}^h$ and $\mathbf{P}^h : \mathbf{H}_0^1(\Omega) \rightarrow \mathbf{V}_0^h$ be orthogonal projection operators, which are associated with bilinear bounded and $\mathbf{H}^1(\Pi)$ -elliptic and $\mathbf{H}_0^1(\Omega)$ -elliptic forms, respectively. For instance, we can introduce the orthogonal projection operators \mathbf{P}^h and \mathbf{P}^h such that, for each $\phi \in \mathbf{H}^1(\Pi)$ and $\mathbf{A} \in \mathbf{H}_0^1(\Omega)$

$$(\phi, \psi^h)_{\Pi} + (\nabla \phi, \nabla \psi^h)_{\Pi} = (\mathbf{P}^h \phi, \psi^h)_{\Pi} + (\nabla \mathbf{P}^h \phi, \nabla \psi^h)_{\Pi}, \quad (22)$$

$$\mu_0^{-1} (\nabla \times \mathbf{A}, \nabla \times \varphi^h)_{\Omega} + \alpha (\nabla \cdot \mathbf{A}, \nabla \cdot \varphi^h)_{\Omega} = \mu_0^{-1} (\nabla \times \mathbf{P}^h \mathbf{A}, \nabla \times \varphi^h)_{\Omega} + \alpha (\nabla \cdot \mathbf{P}^h \mathbf{A}, \nabla \cdot \varphi^h)_{\Omega} \quad (23)$$

are valid for any $\psi^h \in \mathbf{V}^h$ and any $\varphi^h \in \mathbf{V}_0^h$. Céa's lemma [15, Theorem 2.4.1] gives us the existence of a positive constant C such that

$$\|\psi - \mathbf{P}^h \psi\|_{\mathbf{H}^1(\Pi)} \leq C \inf_{\psi^h \in \mathbf{V}^h} \|\psi - \psi^h\|_{\mathbf{H}^1(\Pi)}, \quad (24)$$

$$\|\varphi - \mathbf{P}^h \varphi\|_{\mathbf{H}_0^1(\Omega)} \leq C \inf_{\varphi^h \in \mathbf{V}_0^h} \|\varphi - \varphi^h\|_{\mathbf{H}_0^1(\Omega)}. \quad (25)$$

The time interval $[0, T]$ is partitioned into $n \in \mathbb{N}$ equidistant subintervals with the time step $\tau = \frac{T}{n}$. We denote by $\{\phi_i^h, \beta_i^h, \mathbf{A}_i^h\}$ the fully-discrete approximation of $\{\phi, \beta, \mathbf{A}\}$ at the time-point $t_i = i\tau$, where \mathbf{A}_0^h is defined as the orthogonal projection of $\mathbf{A}_0 \in \mathbf{H}_0^1(\Omega)$ into \mathbf{V}_0^h , i.e.

$$\mathbf{A}_0^h = \mathbf{P}^h \mathbf{A}_0 \in \mathbf{V}_0^h.$$

The following notations are introduced for any function z and any time-dependent domain ω

$$z_i = z(t_i), \quad \delta z_i = \frac{z_i - z_{i-1}}{\tau}, \quad \omega_i = \omega(t_i).$$

After time and space discretization, a fully-discrete scheme is ready to be defined. For any $i = 1, 2, \dots, n$, the following problems are solved: find $\phi_i^h \in \mathbf{V}^h, \beta_i^h \in \mathbb{R}$ and $\mathbf{A}_i^h \in \mathbf{V}_0^h$ such that the identities

$$\sigma_{\Pi} \left(\nabla \phi_i^h, \nabla \psi^h \right)_{\Pi} + \left(j_i, \psi^h \right)_{\Gamma} + \left(\beta_i^h, \psi^h \right)_{\Pi} = 0, \quad (26)$$

$$\left(\phi_i^h, \lambda^h \right)_{\Pi} = 0, \quad (27)$$

$$\left(\sigma_i \delta \mathbf{A}_i^h, \boldsymbol{\varphi}^h \right)_{\Theta_i} + \mu_0^{-1} \left(\nabla \times \mathbf{A}_i^h, \nabla \times \boldsymbol{\varphi}^h \right)_{\Omega} + \alpha \left(\nabla \cdot \mathbf{A}_i^h, \nabla \cdot \boldsymbol{\varphi}^h \right)_{\Omega} + \sigma_{\Pi} \left(\nabla \phi_i^h, \boldsymbol{\varphi}^h \right)_{\Pi} - \sigma_{\Sigma} \left(\mathbf{v}_i \times (\nabla \times \mathbf{A}_i^h), \boldsymbol{\varphi}^h \right)_{\Sigma_i} = 0 \quad (28)$$

are satisfied for any $\psi^h \in \mathbf{V}^h, \lambda^h \in \mathbb{R}$ and $\boldsymbol{\varphi}^h \in \mathbf{V}_0^h$. We first solve the saddle-point problem (26-27) for any $i = 0, 1, \dots, n$, then we use their solution and the given \mathbf{A}_{i-1}^h to solve (28) for any $i = 1, 2, \dots, n$.

Remark 4.1 (Discrete inf-sup condition). *If $\mathbb{R} \subset \mathbf{V}^h$, then we can follow the proof of Lemma 3.1 to show that the discrete inf-sup condition is satisfied, i.e. there exists a positive constant C such that*

$$\sup_{\psi^h \in \mathbf{V}^h, \psi^h \neq 0} \frac{d(\psi^h, \lambda^h)}{\|\psi^h\|_{\mathbf{H}^1(\Pi)}} \geq C |\lambda^h| \quad \forall \lambda^h \in \mathbb{R}. \quad (29)$$

Moreover, we can set $\psi^h = 1 \in \mathbf{V}^h$ in (22) to get that

$$\left(\phi - \mathbf{P}^h \phi, 1 \right)_{\Pi} = 0 \quad \forall \phi \in \mathbf{Z},$$

which means that $\mathbf{P}^h \phi \in \mathbf{Z} \cap \mathbf{V}^h$ for any $\phi \in \mathbf{Z}$.

The following lemma shows the solvability of the fully-discrete variational system (26-28).

Lemma 4.1 (Solvability). *Let the assumptions (AS1-AS4) be fulfilled. Moreover, assume that $\mathbb{R} \subset \mathbf{V}^h$. Then, there exist a unique couple $(\phi_0^h, \beta_0^h) \in (\mathbf{Z} \cap \mathbf{V}^h) \times \mathbb{R}$ and a positive constant τ_0 such that for any $i = 1, 2, \dots, n$ and any $\tau < \tau_0$, the variational system (26-28) admits a unique solution triplet $(\phi_i^h, \beta_i^h, \mathbf{A}_i^h) \in (\mathbf{Z} \cap \mathbf{V}^h) \times \mathbb{R} \times \mathbf{V}_0^h$.*

Proof. Let us rewrite the variational problems (26-28) in the following forms

$$\begin{aligned} a(\phi_i^h, \psi^h) + d(\psi^h, \beta_i^h) &= -(j_i, \psi^h)_{\Gamma}, \\ d(\phi_i^h, \lambda^h) &= 0, \\ b_i(\mathbf{A}_i^h, \boldsymbol{\varphi}^h) &= \frac{1}{\tau} \left(\sigma_i \mathbf{A}_{i-1}^h, \boldsymbol{\varphi}^h \right)_{\Theta_i} - \sigma_{\Pi} \left(\nabla \phi_i^h, \boldsymbol{\varphi}^h \right)_{\Pi}, \end{aligned}$$

where d is as in Lemma 3.1, the bilinear forms $a : \mathbf{V}^h \times \mathbf{V}^h \rightarrow \mathbb{R}$ and $b_i : \mathbf{V}_0^h \times \mathbf{V}_0^h \rightarrow \mathbb{R}$ are defined by

$$\begin{aligned} a(\phi^h, \psi^h) &= \sigma_{\Pi} \left(\nabla \phi^h, \nabla \psi^h \right)_{\Pi}, \\ b_i(\mathbf{A}^h, \boldsymbol{\varphi}^h) &= \frac{1}{\tau} \left(\sigma_i \mathbf{A}^h, \boldsymbol{\varphi}^h \right)_{\Theta_i} + \mu_0^{-1} \left(\nabla \times \mathbf{A}^h, \nabla \times \boldsymbol{\varphi}^h \right)_{\Omega} + \alpha \left(\nabla \cdot \mathbf{A}^h, \nabla \cdot \boldsymbol{\varphi}^h \right)_{\Omega} - \sigma_{\Sigma} \left(\mathbf{v}_i \times (\nabla \times \mathbf{A}^h), \boldsymbol{\varphi}^h \right)_{\Sigma_i}. \end{aligned}$$

It is easy to see that the bilinear forms a and d are bounded. Moreover, the form a is also $(\mathbf{Z} \cap \mathbf{V}^h)$ -coercive thanks to the Poincaré-Wirtinger inequality (8), and d satisfies the discrete inf-sup condition (29). We follow the

Brezzi theorem [13, Corollary 1.1] to get the existence of a unique solution $(\phi_i^h, \beta_i^h) \in (\mathbb{Z} \cap \mathbb{V}^h) \times \mathbb{R}$ to (26-27) for any $i = 0, 1, \dots, n$.

Regarding the bilinear form b_i , we can easily show that the following inequalities hold true for any $\mathbf{A}^h, \boldsymbol{\varphi}^h \in \mathbb{V}_0^h$

$$\begin{aligned} |b_i(\mathbf{A}^h, \boldsymbol{\varphi}^h)| &\lesssim \left(\frac{1}{\tau} + 1\right) \|\nabla \mathbf{A}^h\|_{\mathbf{L}^2(\Omega)} \|\nabla \boldsymbol{\varphi}^h\|_{\mathbf{L}^2(\Omega)}, \\ b_i(\mathbf{A}^h, \mathbf{A}^h) &\geq \left(\frac{1}{\tau} - C_\varepsilon\right) \|\mathbf{A}^h\|_{\mathbf{L}^2(\Theta_i)}^2 + (1 - \varepsilon) \|\nabla \mathbf{A}^h\|_{\mathbf{L}^2(\Omega)}^2. \end{aligned}$$

Fixing a sufficiently small $\varepsilon > 0$, and then choosing $\tau < \tau_0$ show that b_i is a bounded and $\mathbf{H}_0^1(\Omega)$ -elliptic bilinear form. Therefore, the Lax-Milgram lemma [16, Theorem 18.E] gives us the unique solution \mathbf{A}_i^h to problem (28) for any $i = 1, 2, \dots, n$, which concludes the proof. \square

Now, we perform some basic stability estimates for fully-discrete solutions.

Lemma 4.2 (Stability estimates for ϕ_i^h and β_i^h). *Let the assumptions (AS1-AS4) be fulfilled. Moreover, assume that $\mathbb{R} \subset \mathbb{V}^h$. Then there exists a constant $C > 0$ such that*

$$(i) \quad \max_{1 \leq l \leq n} \|\nabla \delta \phi_l^h\|_{\mathbf{L}^2(\Pi)} + \max_{1 \leq l \leq n} |\delta \beta_l^h| \leq C, \quad (30)$$

$$(ii) \quad \max_{0 \leq l \leq n} \|\nabla \phi_l^h\|_{\mathbf{L}^2(\Pi)} + \max_{0 \leq l \leq n} |\beta_l^h| \leq C. \quad (31)$$

Proof. (i) Subtracting (26) and (27) for $i = i - 1$ from themselves, then setting $\psi^h = \delta \phi_i^h, \lambda^h = \delta \beta_i^h$ gives that

$$\sigma_\Pi \|\nabla \delta \phi_i^h\|_{\mathbf{L}^2(\Pi)}^2 \tau = - (j_i - j_{i-1}, \delta \phi_i^h)_\Gamma.$$

We use the Cauchy-Schwarz and ε -Young inequalities to estimate the right-hand side (RHS) as follows

$$\begin{aligned} |(j_i - j_{i-1}, \delta \phi_i^h)_\Gamma| &\stackrel{(4)}{=} |(j_i - j_{i-1}, \delta \phi_i^h + c_{\delta \phi_i^h})_\Gamma| \\ &\lesssim \|j_i - j_{i-1}\|_{\mathbf{H}^{-1/2}(\Gamma)} \|\delta \phi_i^h + c_{\delta \phi_i^h}\|_{\mathbf{H}^{1/2}(\Gamma)} \\ &\lesssim \|\delta \phi_i^h + c_{\delta \phi_i^h}\|_{\mathbf{H}^1(\Pi)} \tau \\ &\stackrel{(8)}{\lesssim} \varepsilon \|\nabla \delta \phi_i^h\|_{\mathbf{L}^2(\Pi)}^2 \tau + C_\varepsilon \tau, \end{aligned}$$

where $c_{\delta \phi_i^h}$ is defined as (9) when replacing ψ by $\delta \phi_i^h$. Fixing a sufficiently small $\varepsilon > 0$, we get that

$$\|\nabla \delta \phi_i^h\|_{\mathbf{L}^2(\Pi)} \lesssim 1.$$

In addition, by means of the discrete inf-sup condition (29), we are able to deduce that

$$|\delta \beta_i^h| \stackrel{(29)}{\lesssim} \sup_{\psi^h \in \mathbb{V}^h, \psi^h \neq 0} \frac{d(\psi^h, \delta \beta_i^h)}{\|\psi^h\|_{\mathbf{H}^1(\Pi)}} = \sup_{\psi^h \in \mathbb{V}^h, \psi^h \neq 0} \frac{-\sigma_\Pi (\nabla \delta \phi_i^h, \nabla \psi^h)_\Pi - (\delta j_i, \psi^h)_\Gamma}{\|\psi^h\|_{\mathbf{H}^1(\Pi)}} \lesssim \|\nabla \delta \phi_i^h\|_{\mathbf{L}^2(\Pi)} + \|\delta j_i\|_{\mathbf{H}^{-1/2}(\Gamma)} \lesssim 1.$$

(ii) The desired estimate can be derived as a result of the relation (30), which completes the proof. \square

The stability estimate for \mathbf{A}_i^h can be achieved by following the lines in the proof of [6, Lemma 4.3(i)].

Lemma 4.3 (Stability estimate for \mathbf{A}_i^h). *Let the assumptions (AS1-AS4) be fulfilled. Moreover, assume that $\mathbb{R} \subset \mathbb{V}^h$. Then, there exist positive constants C and τ_0 such that for any $\tau < \tau_0$, there holds*

$$\sum_{i=1}^n \|\delta \mathbf{A}_i^h\|_{\mathbf{L}^2(\Theta_i)}^2 \tau + \max_{1 \leq l \leq n} \|\nabla \mathbf{A}_l^h\|_{\mathbf{L}^2(\Omega)}^2 + \sum_{i=1}^n \|\nabla \mathbf{A}_i^h - \nabla \mathbf{A}_{i-1}^h\|_{\mathbf{L}^2(\Omega)}^2 \leq C. \quad (32)$$

Now, we are in the position to investigate the convergence and the error estimates for the proposed space-time discretization. Let us introduce the following piecewise-constant and piecewise-affine in time (Rothe's) fields and subdomains

$$\begin{aligned}\bar{\phi}_n^h(t) &= \phi_i^h, & \bar{\beta}_n^h(t) &= \beta_i^h & \text{for } t \in (t_{i-1}, t_i], \\ \bar{\mathbf{A}}_n^h(t) &= \mathbf{A}_i^h, & \underline{\mathbf{A}}_n^h(t) &= \mathbf{A}_{i-1}^h, & \mathbf{A}_n^h(t) &= \mathbf{A}_{i-1}^h + (t - t_{i-1}) \delta \mathbf{A}_i^h & \text{for } t \in (t_{i-1}, t_i], \\ \bar{j}_n(t) &= j_i, & \bar{\sigma}_n(t) &= \sigma_i, & \bar{\mathbf{v}}_n(t) &= \mathbf{v}_i, & \bar{\Sigma}_n(t) &= \Sigma_i, & \bar{\Theta}_n(t) &= \Theta_i & \text{for } t \in (t_{i-1}, t_i]\end{aligned}$$

for every $i = 1, 2, \dots, n$, with the initial data

$$\begin{aligned}\bar{\phi}_n^h(0) &= \phi_0^h, & \bar{\beta}_n^h(0) &= \beta_0^h, & \bar{\mathbf{A}}_n^h(0) &= \underline{\mathbf{A}}_n^h(0) = \mathbf{A}_n^h(0) = \mathbf{A}_0^h, \\ \bar{j}_n(0) &= j_0, & \bar{\sigma}_n(0) &= \sigma_0, & \bar{\mathbf{v}}_n(0) &= \mathbf{v}_0, & \bar{\Sigma}_n(0) &= \Sigma_0, & \bar{\Theta}_n(0) &= \Theta_0.\end{aligned}$$

The following differences between Rothe's functions are derived from the stability estimate (32)

$$\left\| \bar{\mathbf{A}}_n^h - \underline{\mathbf{A}}_n^h \right\|_{\mathbf{L}^2((0,T), \mathbf{H}_0^1(\Omega))}^2 = \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \left\| \mathbf{A}_i^h - \mathbf{A}_{i-1}^h \right\|_{\mathbf{H}_0^1(\Omega)}^2 dt = \sum_{i=1}^n \left\| \nabla \mathbf{A}_i^h - \nabla \mathbf{A}_{i-1}^h \right\|_{\mathbf{L}^2(\Omega)}^2 \tau \stackrel{(32)}{\lesssim} \tau, \quad (33)$$

$$\left\| \bar{\mathbf{A}}_n^h - \mathbf{A}_n^h \right\|_{\mathbf{L}^2((0,T), \mathbf{H}_0^1(\Omega))}^2 = \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \frac{(t_i - t)^2}{\tau^2} \left\| \mathbf{A}_i^h - \mathbf{A}_{i-1}^h \right\|_{\mathbf{H}_0^1(\Omega)}^2 dt \leq \sum_{i=1}^n \left\| \nabla \mathbf{A}_i^h - \nabla \mathbf{A}_{i-1}^h \right\|_{\mathbf{L}^2(\Omega)}^2 \tau \stackrel{(32)}{\lesssim} \tau. \quad (34)$$

Using Rothe's functions and domains, we rewrite the fully-discrete problems (26-28) in the continuous sense: for any $\psi^h \in \mathbf{V}^h$, $\lambda^h \in \mathbb{R}$ and $\boldsymbol{\varphi}^h \in \mathbf{V}_0^h$ and for all $t \in (0, T]$, it holds that

$$\sigma_{\Pi} \left(\nabla \bar{\phi}_n^h(t), \nabla \psi^h \right)_{\Pi} + \left(\bar{j}_n(t), \psi^h \right)_{\Gamma} + \left(\bar{\beta}_n^h(t), \psi^h \right)_{\Pi} = 0, \quad (35)$$

$$\left(\bar{\phi}_n^h(t), \lambda^h \right)_{\Pi} = 0, \quad (36)$$

$$\begin{aligned}\left(\bar{\sigma}_n(t) \partial_t \mathbf{A}_n^h(t), \boldsymbol{\varphi}^h \right)_{\bar{\Theta}_n(t)} + \mu_0^{-1} \left(\nabla \times \bar{\mathbf{A}}_n^h(t), \nabla \times \boldsymbol{\varphi}^h \right)_{\Omega} + \alpha \left(\nabla \cdot \bar{\mathbf{A}}_n^h(t), \nabla \cdot \boldsymbol{\varphi}^h \right)_{\Omega} \\ + \sigma_{\Pi} \left(\nabla \bar{\phi}_n^h(t), \boldsymbol{\varphi}^h \right)_{\Pi} - \sigma_{\Sigma} \left(\bar{\mathbf{v}}_n(t) \times (\nabla \times \bar{\mathbf{A}}_n^h(t)), \boldsymbol{\varphi}^h \right)_{\bar{\Sigma}_n(t)} = 0.\end{aligned} \quad (37)$$

The next two lemmas show crucial results about the stability of the proposed scheme.

Lemma 4.4. *Let the assumptions (AS1-AS4) be fulfilled. Moreover, assume that $\mathbb{R} \subset \mathbf{V}^h$. Then there exists a positive constant C such that for all $t \in [0, T]$, it holds that*

$$\left\| \nabla \bar{\phi}_n^h(t) - \nabla \phi(t) \right\|_{\mathbf{L}^2(\Pi)}^2 \leq C \left(\tau^2 + \left\| \nabla \phi(t) - \nabla \mathbf{P}^h \phi(t) \right\|_{\mathbf{L}^2(\Pi)}^2 \right). \quad (38)$$

Proof. Subtracting (14) from (35) with $\psi = \psi^h = \bar{\phi}_n^h(t) - \mathbf{P}^h \phi(t) \in Z \cap \mathbf{V}^h$ gives that

$$\sigma_{\Pi} \left(\nabla \bar{\phi}_n^h(t) - \nabla \phi(t), \nabla \bar{\phi}_n^h(t) - \nabla \mathbf{P}^h \phi(t) \right)_{\Pi} + \left(\bar{j}_n(t) - j(t), \bar{\phi}_n^h(t) - \mathbf{P}^h \phi(t) \right)_{\Gamma} = 0.$$

The first term can be rearranged and estimated as

$$\begin{aligned}\left(\nabla \bar{\phi}_n^h(t) - \nabla \phi(t), \nabla \bar{\phi}_n^h(t) - \nabla \mathbf{P}^h \phi(t) \right)_{\Pi} &= \left\| \nabla \bar{\phi}_n^h(t) - \nabla \phi(t) \right\|_{\mathbf{L}^2(\Pi)}^2 + \left(\nabla \bar{\phi}_n^h(t) - \nabla \phi(t), \nabla \phi(t) - \nabla \mathbf{P}^h \phi(t) \right)_{\Pi} \\ &\geq (1 - \varepsilon) \left\| \nabla \bar{\phi}_n^h(t) - \nabla \phi(t) \right\|_{\mathbf{L}^2(\Pi)}^2 - C_{\varepsilon} \left\| \nabla \phi(t) - \nabla \mathbf{P}^h \phi(t) \right\|_{\mathbf{L}^2(\Pi)}^2.\end{aligned}$$

Invoking the Lipschitz continuity of j , we handle the second term as follows

$$\begin{aligned}
& \left| \left(\bar{j}_n(t) - j(t), \bar{\phi}_n^h(t) - \mathbf{P}^h \phi(t) \right)_{\Gamma} \right| \stackrel{(4)}{=} \left| \left(\bar{j}_n(t) - j(t), \bar{\phi}_n^h(t) - \mathbf{P}^h \phi(t) + c_\phi \right)_{\Gamma} \right| \\
& \lesssim \|\bar{j}_n(t) - j(t)\|_{\mathbf{H}^{-1/2}(\Gamma)} \|\bar{\phi}_n^h(t) - \mathbf{P}^h \phi(t) + c_\phi\|_{\mathbf{H}^{1/2}(\Gamma)} \\
& \lesssim \|\bar{\phi}_n^h(t) - \mathbf{P}^h \phi(t) + c_\phi\|_{\mathbf{H}^1(\Pi)} \tau \\
& \stackrel{(8)}{\lesssim} \varepsilon \|\nabla \bar{\phi}_n^h(t) - \nabla \mathbf{P}^h \phi(t)\|_{\mathbf{L}^2(\Pi)}^2 + C_\varepsilon \tau^2 \\
& \leq \varepsilon \|\nabla \bar{\phi}_n^h(t) - \nabla \phi(t)\|_{\mathbf{L}^2(\Pi)}^2 + \varepsilon \|\nabla \phi(t) - \nabla \mathbf{P}^h \phi(t)\|_{\mathbf{L}^2(\Pi)}^2 + C_\varepsilon \tau^2,
\end{aligned}$$

where c_ϕ is defined as (9) when considering $\bar{\phi}_n^h(t) - \mathbf{P}^h \phi(t)$ instead of ψ . Therefore, fixing a sufficiently small $\varepsilon > 0$, we can conclude the proof. \square

Lemma 4.5. *Let the assumptions (A1-A4) be satisfied. Moreover, assume that $\mathbb{R} \subset \mathbf{V}^h$.*

(i) *There exist positive constants C and τ_0 such that for any $\tau < \tau_0$, the following relation holds true for all $\xi \in [0, T]$*

$$\begin{aligned}
& \|\mathbf{A}_n^h(\xi) - \mathbf{A}(\xi)\|_{\mathbf{L}^2(\Theta(\xi))}^2 + \int_0^\xi \|\nabla \mathbf{A}_n^h(t) - \nabla \mathbf{A}(t)\|_{\mathbf{L}^2(\Omega)}^2 dt \leq C \left(\tau + \int_0^\xi \|\nabla \phi(t) - \nabla \mathbf{P}^h \phi(t)\|_{\mathbf{L}^2(\Pi)}^2 dt \right. \\
& \quad \left. + \|\mathbf{A}_0 - \mathbf{P}^h \mathbf{A}_0\|_{\mathbf{L}^2(\Omega)}^2 + \sqrt{\int_0^\xi \|\mathbf{A}(t) - \mathbf{P}^h \mathbf{A}(t)\|_{\mathbf{L}^2(\Omega)}^2 dt} + \int_0^\xi \|\nabla \mathbf{A}(t) - \nabla \mathbf{P}^h \mathbf{A}(t)\|_{\mathbf{L}^2(\Omega)}^2 dt \right). \quad (39)
\end{aligned}$$

(ii) *If $\mathbf{A} \in \mathbf{H}^1((0, T), \mathbf{H}_0^1(\Omega))$ then there exist positive constants C and τ_0 such that for any $\tau < \tau_0$, the following relation holds true for all $\xi \in [0, T]$*

$$\begin{aligned}
& \|\mathbf{A}_n^h(\xi) - \mathbf{A}(\xi)\|_{\mathbf{L}^2(\Theta(\xi))}^2 + \int_0^\xi \|\nabla \mathbf{A}_n^h(t) - \nabla \mathbf{A}(t)\|_{\mathbf{L}^2(\Omega)}^2 dt \\
& \leq C \left(\tau + \int_0^\xi \|\nabla \phi(t) - \nabla \mathbf{P}^h \phi(t)\|_{\mathbf{L}^2(\Pi)}^2 dt + \|\mathbf{A}(\xi) - \mathbf{P}^h \mathbf{A}(\xi)\|_{\mathbf{L}^2(\Omega)}^2 + \|\mathbf{A}_0 - \mathbf{P}^h \mathbf{A}_0\|_{\mathbf{L}^2(\Omega)}^2 \right. \\
& \quad \left. + \int_0^\xi \|\nabla \mathbf{A}(t) - \nabla \mathbf{P}^h \mathbf{A}(t)\|_{\mathbf{L}^2(\Omega)}^2 dt + \int_0^\xi \|\partial_t \mathbf{A}(t) - \mathbf{P}^h \partial_t \mathbf{A}(t)\|_{\mathbf{L}^2(\Omega)}^2 dt \right). \quad (40)
\end{aligned}$$

Proof. First of all, we summarize some auxiliary results which will be useful for further calculations. We denote the lowest upper discrete time-point of t with respect to the time step τ by \bar{t}_n . It holds that

$$\begin{aligned}
& \left| \int_0^\xi \|\partial_t \mathbf{A}_n^h(t)\|_{\mathbf{L}^2(\bar{\Theta}_n(t))}^2 dt - \int_0^\xi \|\partial_t \mathbf{A}_n^h(t)\|_{\mathbf{L}^2(\Theta(t))}^2 dt \right| \stackrel{(10)}{=} \left| \int_0^\xi \int_t^{\bar{t}_n} \int_{\partial\Theta(\eta)} |\partial_t \mathbf{A}_n^h(t)|^2 (\mathbf{v} \cdot \mathbf{n})(\eta) ds d\eta dt \right| \\
& \stackrel{(11)}{\lesssim} \tau \int_0^\xi \|\partial_t \mathbf{A}_n^h(t)\|_{\mathbf{H}_0^1(\Omega)}^2 dt
\end{aligned}$$

$$\leq \sum_{i=1}^n \|\nabla \mathbf{A}_i^h - \nabla \mathbf{A}_{i-1}^h\|_{\mathbf{L}^2(\Omega)}^2 \stackrel{(32)}{\lesssim} 1,$$

which implies that

$$\int_0^\xi \|\partial_t \mathbf{A}_n^h(t)\|_{\mathbf{L}^2(\Theta(t))}^2 dt \lesssim \int_0^\xi \|\partial_t \mathbf{A}_n^h(t)\|_{\mathbf{L}^2(\bar{\Theta}_n(t))}^2 dt + 1 \stackrel{(32)}{\lesssim} 1. \quad (41)$$

In addition, for $t \in (0, T)$, we have that

$$\begin{aligned} \|\mathbf{A}_n^h(t) - \mathbf{P}^h \mathbf{A}(t)\|_{\mathbf{L}^2(\bar{\Sigma}_n(t))}^2 - \|\mathbf{A}_n^h(t) - \mathbf{P}^h \mathbf{A}(t)\|_{\mathbf{L}^2(\Sigma(t))}^2 &\stackrel{(10)}{=} \int_\tau^{\bar{\tau}_n} \int_{\partial \Sigma(\eta)} |\mathbf{A}_n^h(t) - \mathbf{P}^h \mathbf{A}(t)|^2 (\mathbf{v} \cdot \mathbf{n})(\eta) ds d\eta \\ &\stackrel{(11)}{\lesssim} \|\nabla \mathbf{A}_n^h(t) - \nabla \mathbf{P}^h \mathbf{A}(t)\|_{\mathbf{L}^2(\Omega)}^2 \tau, \end{aligned}$$

which allows us to arrive at

$$\|\mathbf{A}_n^h(t) - \mathbf{P}^h \mathbf{A}(t)\|_{\mathbf{L}^2(\bar{\Sigma}_n(t))}^2 \lesssim \|\mathbf{A}_n^h(t) - \mathbf{P}^h \mathbf{A}(t)\|_{\mathbf{L}^2(\Sigma(t))}^2 + \|\nabla \mathbf{A}_n^h(t) - \nabla \mathbf{P}^h \mathbf{A}(t)\|_{\mathbf{L}^2(\Omega)}^2 \tau. \quad (42)$$

(i) We subtract (18) for $\varphi = \varphi^h$ from (37), then we set $\varphi^h = \mathbf{A}_n^h(t) - \mathbf{P}^h \mathbf{A}(t)$ and integrate the result over the time interval $(0, \xi) \subset (0, T)$ to get that

$$\begin{aligned} &\int_0^\xi \left((\bar{\sigma}_n(t) \partial_t \mathbf{A}_n^h(t), \mathbf{A}_n^h(t) - \mathbf{P}^h \mathbf{A}(t))_{\bar{\Theta}_n(t)} - (\sigma(t) \partial_t \mathbf{A}(t), \mathbf{A}_n^h(t) - \mathbf{P}^h \mathbf{A}(t))_{\Theta(t)} \right) dt \\ &+ \mu_0^{-1} \int_0^\xi \left(\nabla \times \bar{\mathbf{A}}_n^h(t) - \nabla \times \mathbf{A}(t), \nabla \times \mathbf{A}_n^h(t) - \nabla \times \mathbf{P}^h \mathbf{A}(t) \right)_\Omega dt \\ &+ \alpha \int_0^\xi \left(\nabla \cdot \bar{\mathbf{A}}_n^h(t) - \nabla \cdot \mathbf{A}(t), \nabla \cdot \mathbf{A}_n^h(t) - \nabla \cdot \mathbf{P}^h \mathbf{A}(t) \right)_\Omega dt + \sigma_\Pi \int_0^\xi \left(\nabla \bar{\phi}_n^h(t) - \nabla \phi(t), \mathbf{A}_n^h(t) - \mathbf{P}^h \mathbf{A}(t) \right)_\Pi dt \\ &- \sigma_\Sigma \int_0^\xi \left((\bar{\mathbf{v}}_n(t) \times (\nabla \times \bar{\mathbf{A}}_n^h(t)), \mathbf{A}_n^h(t) - \mathbf{P}^h \mathbf{A}(t))_{\bar{\Sigma}_n(t)} - (\mathbf{v}(t) \times (\nabla \times \mathbf{A}(t)), \mathbf{A}_n^h(t) - \mathbf{P}^h \mathbf{A}(t))_{\Sigma(t)} \right) dt = 0. \quad (43) \end{aligned}$$

We rearrange the relation (43) in the following form

$$\begin{aligned} &\int_0^\xi (\sigma(t) \partial_t (\mathbf{A}_n^h(t) - \mathbf{A}(t)), \mathbf{A}_n^h(t) - \mathbf{A}(t))_{\Theta(t)} dt \\ &+ \mu_0^{-1} \int_0^\xi \|\nabla \times \mathbf{A}_n^h(t) - \nabla \times \mathbf{A}(t)\|_{\mathbf{L}^2(\Omega)}^2 dt + \alpha \int_0^\xi \|\nabla \cdot \mathbf{A}_n^h(t) - \nabla \cdot \mathbf{A}(t)\|_{\mathbf{L}^2(\Omega)}^2 dt \\ &= - \int_0^\xi \left((\bar{\sigma}_n(t) \partial_t \mathbf{A}_n^h(t), \mathbf{A}_n^h(t) - \mathbf{P}^h \mathbf{A}(t))_{\bar{\Theta}_n(t)} - (\sigma(t) \partial_t \mathbf{A}_n^h(t), \mathbf{A}_n^h(t) - \mathbf{P}^h \mathbf{A}(t))_{\Theta(t)} \right) dt \\ &- \int_0^\xi (\sigma(t) \partial_t (\mathbf{A}_n^h(t) - \mathbf{A}(t)), \mathbf{A}(t) - \mathbf{P}^h \mathbf{A}(t))_{\Theta(t)} dt - \sigma_\Pi \int_0^\xi \left(\nabla \bar{\phi}_n^h(t) - \nabla \phi(t), \mathbf{A}_n^h(t) - \mathbf{P}^h \mathbf{A}(t) \right)_\Pi dt \end{aligned}$$

$$\begin{aligned}
& -\mu_0^{-1} \int_0^\xi \left(\nabla \times \bar{\mathbf{A}}_n^h(t) - \nabla \times \mathbf{A}_n^h(t), \nabla \times \mathbf{A}_n^h(t) - \nabla \times \mathbf{P}^h \mathbf{A}(t) \right)_\Omega dt \\
& -\mu_0^{-1} \int_0^\xi \left(\nabla \times \mathbf{A}_n^h(t) - \nabla \times \mathbf{A}(t), \nabla \times \mathbf{A}(t) - \nabla \times \mathbf{P}^h \mathbf{A}(t) \right)_\Omega dt \\
& -\alpha \int_0^\xi \left(\nabla \cdot \bar{\mathbf{A}}_n^h(t) - \nabla \cdot \mathbf{A}_n^h(t), \nabla \cdot \mathbf{A}_n^h(t) - \nabla \cdot \mathbf{P}^h \mathbf{A}(t) \right)_\Omega dt - \alpha \int_0^\xi \left(\nabla \cdot \mathbf{A}_n^h(t) - \nabla \cdot \mathbf{A}(t), \nabla \cdot \mathbf{A}(t) - \nabla \cdot \mathbf{P}^h \mathbf{A}(t) \right)_\Omega dt \\
& + \sigma_\Sigma \int_0^\xi \left((\bar{\mathbf{v}}_n(t) - \mathbf{v}(t)) \times (\nabla \times \bar{\mathbf{A}}_n^h(t)), \mathbf{A}_n^h(t) - \mathbf{P}^h \mathbf{A}(t) \right)_{\bar{\Sigma}_n(t)} dt \\
& + \sigma_\Sigma \int_0^\xi \left((\mathbf{v}(t) \times (\nabla \times \mathbf{A}(t)), \mathbf{A}_n^h(t) - \mathbf{P}^h \mathbf{A}(t))_{\bar{\Sigma}_n(t)} - (\mathbf{v}(t) \times (\nabla \times \mathbf{A}(t)), \mathbf{A}_n^h(t) - \mathbf{P}^h \mathbf{A}(t))_{\Sigma(t)} \right) dt \\
& + \sigma_\Sigma \int_0^\xi \left(\mathbf{v}(t) \times (\nabla \times \bar{\mathbf{A}}_n^h(t) - \nabla \times \mathbf{A}(t)), \mathbf{A}_n^h(t) - \mathbf{P}^h \mathbf{A}(t) \right)_{\bar{\Sigma}_n(t)} dt =: \sum_{i=1}^{10} S_i.
\end{aligned}$$

The first term on the LHS can be estimated using the Reynolds transport theorem as follows

$$\begin{aligned}
& \int_0^\xi \left(\sigma(t) \partial_t (\mathbf{A}_n^h(t) - \mathbf{A}(t)), \mathbf{A}_n^h(t) - \mathbf{A}(t) \right)_{\Theta(t)} dt \\
& \stackrel{(10)}{=} \frac{1}{2} \int_0^\xi \frac{d}{dt} \int_{\Theta(t)} \sigma(t) |\mathbf{A}_n^h(t) - \mathbf{A}(t)|^2 d\mathbf{x} dt - \frac{1}{2} \int_0^\xi \int_{\partial\Theta(t)} \sigma(t) |\mathbf{A}_n^h(t) - \mathbf{A}(t)|^2 (\mathbf{v} \cdot \mathbf{n})(t) ds dt \\
& \stackrel{(13)}{\gtrsim} \|\mathbf{A}_n^h(\xi) - \mathbf{A}(\xi)\|_{\mathbf{L}^2(\Theta(\xi))}^2 - C \|\mathbf{A}_0 - \mathbf{P}^h \mathbf{A}_0\|_{\mathbf{L}^2(\Omega)}^2 - \varepsilon \int_0^\xi \|\nabla \mathbf{A}_n^h(t) - \nabla \mathbf{A}(t)\|_{\mathbf{L}^2(\Omega)}^2 dt - C_\varepsilon \int_0^\xi \|\mathbf{A}_n^h(t) - \mathbf{A}(t)\|_{\mathbf{L}^2(\Theta(t))}^2 dt.
\end{aligned}$$

Invoking the identity (7) gives that

$$\mu_0^{-1} \int_0^\xi \|\nabla \times \mathbf{A}_n^h(t) - \nabla \times \mathbf{A}(t)\|_{\mathbf{L}^2(\Omega)}^2 dt + \alpha \int_0^\xi \|\nabla \cdot \mathbf{A}_n^h(t) - \nabla \cdot \mathbf{A}(t)\|_{\mathbf{L}^2(\Omega)}^2 dt \geq \min(\mu_0^{-1}, \alpha) \int_0^\xi \|\nabla \mathbf{A}_n^h(t) - \nabla \mathbf{A}(t)\|_{\mathbf{L}^2(\Omega)}^2 dt.$$

We handle the term S_1 as follows

$$\begin{aligned}
|S_1| & \stackrel{(10)}{=} \left| \int_0^\xi \int_t^{\bar{t}_n} \int_{\partial\Theta(\eta)} \sigma(\eta) \partial_t \mathbf{A}_n^h(t) \cdot (\mathbf{A}_n^h(t) - \mathbf{P}^h \mathbf{A}(t)) (\mathbf{v} \cdot \mathbf{n})(\eta) ds d\eta dt \right| \\
& \stackrel{(11)}{\lesssim} \tau \int_0^\xi \|\partial_t \mathbf{A}_n^h(t)\|_{\mathbf{H}^1(\Omega)} \|\mathbf{A}_n^h(t) - \mathbf{P}^h \mathbf{A}(t)\|_{\mathbf{H}^1(\Omega)} dt \\
& \leq C_\varepsilon \int_0^\xi \|\bar{\mathbf{A}}_n^h(t) - \underline{\mathbf{A}}_n^h(t)\|_{\mathbf{H}_0^1(\Omega)}^2 dt + \varepsilon \int_0^\xi \|\mathbf{A}_n^h(t) - \mathbf{P}^h \mathbf{A}(t)\|_{\mathbf{H}_0^1(\Omega)}^2 dt
\end{aligned}$$

$$\stackrel{(33)}{\leq} C_\varepsilon \tau + \varepsilon \int_0^\xi \|\nabla \mathbf{A}_n^h(t) - \nabla \mathbf{A}(t)\|_{\mathbf{L}^2(\Omega)}^2 dt + \varepsilon \int_0^\xi \|\nabla \mathbf{A}(t) - \nabla \mathbf{P}^h \mathbf{A}(t)\|_{\mathbf{L}^2(\Omega)}^2 dt.$$

It holds that

$$|S_2| \lesssim \sqrt{\int_0^\xi \|\partial_t \mathbf{A}_n^h(t) - \partial_t \mathbf{A}(t)\|_{\mathbf{L}^2(\Theta(t))}^2 dt} \sqrt{\int_0^\xi \|\mathbf{A}(t) - \mathbf{P}^h \mathbf{A}(t)\|_{\mathbf{L}^2(\Theta(t))}^2 dt} \stackrel{(41)}{\lesssim} \sqrt{\int_0^\xi \|\mathbf{A}(t) - \mathbf{P}^h \mathbf{A}(t)\|_{\mathbf{L}^2(\Omega)}^2 dt}.$$

By adding $\pm \mathbf{A}(t)$, applying Lemma 4.4 and the Friedrichs inequality, we have the bound for the term S_3 as follows

$$\begin{aligned} |S_3| &\leq C_\varepsilon \int_0^\xi \|\nabla \bar{\phi}_n^h(t) - \nabla \phi(t)\|_{\mathbf{L}^2(\Pi)}^2 dt + \varepsilon \int_0^\xi \|\mathbf{A}_n^h(t) - \mathbf{P}^h \mathbf{A}(t)\|_{\mathbf{L}^2(\Pi)}^2 dt \\ &\stackrel{(38)}{\leq} C_\varepsilon \tau^2 + C_\varepsilon \int_0^\xi \|\nabla \phi(t) - \nabla \mathbf{P}^h \phi(t)\|_{\mathbf{L}^2(\Pi)}^2 dt + \varepsilon \int_0^\xi \|\nabla \mathbf{A}_n^h(t) - \nabla \mathbf{A}(t)\|_{\mathbf{L}^2(\Omega)}^2 dt + \varepsilon \int_0^\xi \|\nabla \mathbf{A}(t) - \nabla \mathbf{P}^h \mathbf{A}(t)\|_{\mathbf{L}^2(\Omega)}^2 dt. \end{aligned}$$

The same addition trick is applied to the terms S_4 and S_6 to obtain that

$$|S_4| + |S_6| \stackrel{(34)}{\leq} C_\varepsilon \tau + \varepsilon \int_0^\xi \|\nabla \mathbf{A}_n^h(t) - \nabla \mathbf{A}(t)\|_{\mathbf{L}^2(\Omega)}^2 dt + \varepsilon \int_0^\xi \|\nabla \mathbf{A}(t) - \nabla \mathbf{P}^h \mathbf{A}(t)\|_{\mathbf{L}^2(\Omega)}^2 dt.$$

Moreover, we get that

$$|S_5| + |S_7| \leq \varepsilon \int_0^\xi \|\nabla \mathbf{A}_n^h(t) - \nabla \mathbf{A}(t)\|_{\mathbf{L}^2(\Omega)}^2 dt + C_\varepsilon \int_0^\xi \|\nabla \mathbf{A}(t) - \nabla \mathbf{P}^h \mathbf{A}(t)\|_{\mathbf{L}^2(\Omega)}^2 dt.$$

Using the Lipschitz continuity of \mathbf{v} , the term S_8 can be estimated by the Cauchy-Schwarz and ε -Young inequalities as

$$\begin{aligned} |S_8| &\lesssim \tau \int_0^\xi \|\nabla \times \bar{\mathbf{A}}_n^h(t)\|_{\mathbf{L}^2(\Omega)} \|\mathbf{A}_n^h(t) - \mathbf{P}^h \mathbf{A}(t)\|_{\mathbf{L}^2(\Omega)} dt \\ &\leq C_\varepsilon \tau^2 \int_0^\xi \|\nabla \bar{\mathbf{A}}_n^h(t)\|_{\mathbf{L}^2(\Omega)}^2 dt + \varepsilon \int_0^\xi \|\nabla \mathbf{A}_n^h(t) - \nabla \mathbf{P}^h \mathbf{A}(t)\|_{\mathbf{L}^2(\Omega)}^2 dt \\ &\stackrel{(32)}{\leq} C_\varepsilon \tau^2 + \varepsilon \int_0^\xi \|\nabla \mathbf{A}_n^h(t) - \nabla \mathbf{A}(t)\|_{\mathbf{L}^2(\Omega)}^2 dt + \varepsilon \int_0^\xi \|\nabla \mathbf{A}(t) - \nabla \mathbf{P}^h \mathbf{A}(t)\|_{\mathbf{L}^2(\Omega)}^2 dt. \end{aligned}$$

The Reynolds transport theorem is used for the term S_9 to get that

$$\begin{aligned} |S_9| &= \sigma_\Sigma \left| \int_0^\xi \int_t^{\bar{t}_n} \frac{d}{d\eta} \int_{\Sigma(\eta)} (\mathbf{v}(t) \times (\nabla \times \mathbf{A}(t))) \cdot (\mathbf{A}_n^h(t) - \mathbf{P}^h \mathbf{A}(t)) dx d\eta dt \right| \\ &\stackrel{(10)}{=} \sigma_\Sigma \left| \int_0^\xi \int_t^{\bar{t}_n} \int_{\partial \Sigma(\eta)} (\mathbf{v}(t) \times (\nabla \times \mathbf{A}(t))) \cdot (\mathbf{A}_n^h(t) - \mathbf{P}^h \mathbf{A}(t)) (\mathbf{v} \cdot \mathbf{n})(\eta) ds d\eta dt \right| \end{aligned}$$

$$\begin{aligned}
& \stackrel{(11)}{\lesssim} \int_0^\xi \int_t^{\bar{t}_n} \|\nabla \times \mathbf{A}(t)\|_{\mathbf{H}^1(\Sigma(\eta))} \|\mathbf{A}_n^h(t) - \mathbf{P}^h \mathbf{A}(t)\|_{\mathbf{H}_0^1(\Omega)} d\eta dt \\
& \lesssim C_\varepsilon \int_0^\xi \left(\int_t^{\bar{t}_n} \|\nabla \times \mathbf{A}(t)\|_{\mathbf{H}^1(\Sigma)} dt \right)^2 + \varepsilon \int_0^\xi \|\nabla \mathbf{A}_n^h(t) - \nabla \mathbf{P}^h \mathbf{A}(t)\|_{\mathbf{L}^2(\Omega)}^2 dt \\
& \leq C_\varepsilon \tau^2 + \varepsilon \int_0^\xi \|\nabla \mathbf{A}_n^h(t) - \nabla \mathbf{A}(t)\|_{\mathbf{L}^2(\Omega)}^2 dt + \varepsilon \int_0^\xi \|\nabla \mathbf{A}(t) - \nabla \mathbf{P}^h \mathbf{A}(t)\|_{\mathbf{L}^2(\Omega)}^2 dt.
\end{aligned}$$

We recall that the subdomain $\widetilde{\Sigma}$ is defined in (1). Moreover, note that in the last step we used Corollary 3.1 and that $\|\nabla \times \mathbf{A}(t)\|_{\mathbf{H}^1(\widetilde{\Sigma})}$ is independent of η in order to obtain $C_\varepsilon \tau^2$. For S_{10} , we have the following estimate

$$\begin{aligned}
|S_{10}| & \leq \varepsilon \int_0^\xi \|\nabla \overline{\mathbf{A}}_n^h(t) - \nabla \mathbf{A}(t)\|_{\mathbf{L}^2(\Omega)}^2 dt + C_\varepsilon \int_0^\xi \|\mathbf{A}_n^h(t) - \mathbf{P}^h \mathbf{A}(t)\|_{\mathbf{L}^2(\widetilde{\Sigma}_n(t))}^2 dt \\
& \stackrel{(42)}{\lesssim} \varepsilon \int_0^\xi \|\nabla \overline{\mathbf{A}}_n^h(t) - \nabla \mathbf{A}(t)\|_{\mathbf{L}^2(\Omega)}^2 dt + C_\varepsilon \int_0^\xi \|\mathbf{A}_n^h(t) - \mathbf{P}^h \mathbf{A}(t)\|_{\mathbf{L}^2(\Sigma(t))}^2 dt + C_\varepsilon \tau \int_0^\xi \|\nabla \mathbf{A}_n^h(t) - \nabla \mathbf{P}^h \mathbf{A}(t)\|_{\mathbf{L}^2(\Omega)}^2 dt \\
& \stackrel{(34)}{\lesssim} \varepsilon \tau + (\varepsilon + C_\varepsilon \tau) \int_0^\xi \|\nabla \mathbf{A}_n^h(t) - \nabla \mathbf{A}(t)\|_{\mathbf{L}^2(\Omega)}^2 dt \\
& \quad + C_\varepsilon \int_0^\xi \|\mathbf{A}_n^h(t) - \mathbf{A}(t)\|_{\mathbf{L}^2(\Theta(t))}^2 dt + C_\varepsilon (\tau + 1) \int_0^\xi \|\nabla \mathbf{A}(t) - \nabla \mathbf{P}^h \mathbf{A}(t)\|_{\mathbf{L}^2(\Omega)}^2 dt.
\end{aligned}$$

Fixing a sufficiently small positive ε , then choosing a sufficient small τ , and applying a Grönwall argument, we obtain the desired result.

(ii) The difference in comparison to (i) lies in a different handling of the term S_2 , which is allowed due to the additional assumption that $\partial_t \mathbf{A} \in \mathbf{L}^2((0, T), \mathbf{H}_0^1(\Omega))$. First, we rewrite this term by using the Reynolds transport theorem as follows

$$\begin{aligned}
|S_2| & = \left| \int_0^\xi \int_{\Theta(t)} \partial_t (\sigma(t)(\mathbf{A}_n^h(t) - \mathbf{A}(t)) \cdot (\mathbf{A}(t) - \mathbf{P}^h \mathbf{A}(t))) dx dt - \int_0^\xi \int_{\Theta(t)} \sigma(t)(\mathbf{A}_n^h(t) - \mathbf{A}(t)) \cdot \partial_t (\mathbf{A}(t) - \mathbf{P}^h \mathbf{A}(t)) dx dt \right| \\
& \stackrel{(10)}{=} \left| \int_0^\xi \frac{d}{dt} \int_{\Theta(t)} \sigma(t)(\mathbf{A}_n^h(t) - \mathbf{A}(t)) \cdot (\mathbf{A}(t) - \mathbf{P}^h \mathbf{A}(t)) dx dt \right. \\
& \quad \left. - \int_0^\xi \int_{\partial\Theta(t)} \sigma(t)(\mathbf{A}_n^h(t) - \mathbf{A}(t)) \cdot (\mathbf{A}(t) - \mathbf{P}^h \mathbf{A}(t)) (\mathbf{v} \cdot \mathbf{n})(t) ds dt \right. \\
& \quad \left. - \int_0^\xi \int_{\Theta(t)} \sigma(t)(\mathbf{A}_n^h(t) - \mathbf{A}(t)) \cdot \partial_t (\mathbf{A}(t) - \mathbf{P}^h \mathbf{A}(t)) dx dt \right|.
\end{aligned}$$

Then, we can get the following estimate

$$|S_2| \stackrel{(12)}{\lesssim} \varepsilon \|\mathbf{A}_n^h(\xi) - \mathbf{A}(\xi)\|_{\mathbf{L}^2(\Theta(\xi))}^2 + C_\varepsilon \|\mathbf{A}(\xi) - \mathbf{P}^h \mathbf{A}(\xi)\|_{\mathbf{L}^2(\Omega)}^2 + \|\mathbf{A}_0 - \mathbf{P}^h \mathbf{A}_0\|_{\mathbf{L}^2(\Omega)}^2 + \varepsilon \int_0^\xi \|\nabla \mathbf{A}_n^h(t) - \nabla \mathbf{A}(t)\|_{\mathbf{L}^2(\Omega)}^2 dt$$

$$+ \int_0^\xi \|A_n^h(t) - A(t)\|_{\mathbf{L}^2(\Theta(t))}^2 dt + C_\varepsilon \int_0^\xi \|\nabla A(t) - \nabla \mathbf{P}^h A(t)\|_{\mathbf{L}^2(\Omega)}^2 dt + \int_0^\xi \|\partial_t A(t) - \mathbf{P}^h \partial_t A(t)\|_{\mathbf{L}^2(\Omega)}^2 dt.$$

The rest of proof follows the same lines as in part (i). \square

The next theorem shows the convergence of the space-time discretization scheme. It is an immediate consequence of the limit transitions (20-21), Céa's lemma (24-25), Lemma 4.4 and Lemma 4.5(i).

Theorem 4.1 (Convergence). *Let the assumptions (AS1-AS4) be fulfilled. Moreover, assume that $\mathbb{R} \subset \mathbf{V}^h$. Then the following convergences hold true*

$$\begin{aligned} \overline{\phi}_n^h &\rightarrow \phi && \text{in } L^2((0, T), Z), \\ A_n^h &\rightarrow A, \quad \overline{A}_n^h \rightarrow A && \text{in } L^2((0, T), \mathbf{H}_0^1(\Omega)). \end{aligned}$$

For an instance, now we consider the lowest order of Lagrangian family finite elements to show explicitly the convergence rate of our proposed scheme. The finite element spaces are now given by

$$\begin{aligned} \mathbf{V}^h &= \{\psi^h \in C(\overline{\Pi}) : \psi^h|_K \in P_1 \quad \text{for all } K \in \mathcal{T}_\Pi^h\}, \\ \mathbf{V}_0^h &= \{\varphi^h \in C(\overline{\Omega}) : \varphi^h|_{\partial\Omega} = \mathbf{0}, \varphi^h|_K \in (P_1)^3 \quad \text{for all } K \in \mathcal{T}^h\}, \end{aligned}$$

where P_1 is the space of all first-order polynomials. Please note that $\mathbb{R} \subset \mathbf{V}^h$. Let v_i ($1 \leq i \leq 4$) be the vertices of a tetrahedron K and $\psi \in C(\overline{\Pi}), \varphi \in C(\overline{\Omega})$, then the vertex degrees of freedom on K are respectively determined by

$$M_v(\psi) = \{\psi(v_i), \quad 1 \leq i \leq 4\}, \quad M_v(\varphi) = \{\varphi_j(v_i), \quad 1 \leq i \leq 4, 1 \leq j \leq 3\}.$$

One can easily see that those finite element spaces are unisolvent, $H^1(\Pi)$ and $\mathbf{H}_0^1(\Omega)$ conforming, respectively. On the other hand, the famous Sobolev embedding theorem in \mathbb{R}^3 (cf. [17, Theorem 3.5]) states that $H^{s+1}(\Omega) \hookrightarrow C(\overline{\Omega})$ if $s > \frac{1}{2}$. Therefore, for any $\psi \in H^{s+1}(\Pi)$ and $\varphi \in \mathbf{H}^{p+1}(\Omega)$ ($s, p > \frac{1}{2}$), we are able to define interpolation operators π_h and π_n by requiring that

$$M_v(\psi - \pi_h \psi) = M_v(\varphi - \pi_h \varphi) = \{0\}.$$

Then, [17, Theorem 5.48] gives the existence of constants $C > 0$ independent of h, ψ and φ such that

$$\|\psi - \pi_h \psi\|_{H^1(\Pi)} \leq Ch^s \|\psi\|_{H^{s+1}(\Pi)}, \quad (44)$$

$$\|\varphi - \pi_h \varphi\|_{\mathbf{H}_0^1(\Omega)} \leq Ch^p \|\varphi\|_{\mathbf{H}^{p+1}(\Omega)}, \quad (45)$$

for any $\psi \in H^{s+1}(\Pi), \varphi \in \mathbf{H}^{p+1}(\Omega)$ with $\frac{1}{2} < s, p \leq 1$. Combining Céa's lemma (24-25), Lemmas 4.4 and 4.5, and the relations (44-45) together, we obtain the following error estimates for the first-order Lagrangian finite elements.

Theorem 4.2 (Error estimate on ϕ). *Let $s \in (\frac{1}{2}, 1]$ and the assumptions (AS1-AS4) be satisfied. In addition, we assume that $\phi \in \text{Lip}([0, T], Z) \cap C([0, T], H^{s+1}(\Pi))$. Then there exists a constant $C > 0$ such that for any $t \in [0, T]$, the following error estimate holds true*

$$\left\| \nabla \overline{\phi}_n^h(t) - \nabla \phi(t) \right\|_{\mathbf{L}^2(\Pi)}^2 \leq C(\tau^2 + h^{2s}). \quad (46)$$

Theorem 4.3 (Error estimates on A). *Let $p \in (\frac{1}{2}, 1], q \in (\frac{1}{2}, 1]$ and the assumptions of Theorem 4.2 be fulfilled.*

(i) *Suppose that $A_0 \in \mathbf{H}_0^1(\Omega) \cap \mathbf{H}^{q+1}(\Omega)$ and the weak solution $A \in L^2((0, T), \mathbf{H}_0^1(\Omega) \cap \mathbf{H}^{p+1}(\Omega))$, then there exist positive constants C and τ_0 such that for any $\tau < \tau_0$ and for all $\xi \in [0, T]$, it holds that*

$$\|A_n^h(\xi) - A(\xi)\|_{\mathbf{L}^2(\Theta(\xi))}^2 + \int_0^\xi \|\nabla A_n^h(t) - \nabla A(t)\|_{\mathbf{L}^2(\Omega)}^2 dt \leq C(\tau + h^p). \quad (47)$$

(ii) Suppose that $\mathbf{A}_0 \in \mathbf{H}_0^1(\Omega) \cap \mathbf{H}^{q+1}(\Omega)$ and the weak solution $\mathbf{A} \in \mathbf{H}^1((0, T), \mathbf{H}_0^1(\Omega) \cap \mathbf{H}^{p+1}(\Omega))$, then there exist positive constants C and τ_0 such that for any $\tau < \tau_0$ and for all $\xi \in [0, T]$, it holds that

$$\|\mathbf{A}_n^h(\xi) - \mathbf{A}(\xi)\|_{\mathbf{L}^2(\Theta(\xi))}^2 + \int_0^\xi \|\nabla \mathbf{A}_n^h(t) - \nabla \mathbf{A}(t)\|_{\mathbf{L}^2(\Omega)}^2 dt \leq C(\tau + h^{\min(2s, 2p, 2q)}). \quad (48)$$

Please note that in the special case that $p = q = s = 1$, we get the convergence rate $O(\sqrt{\tau} + h)$ of the proposed space-time discretization scheme.

5. Numerical results

In this section, we perform some numerical experiments with the penalty coefficient $\alpha = \mu_0^{-1}$ to evaluate the error of the proposed space-time discretization scheme when h decreases. In all experiments, the domain Ω is considered as a unit cube, which contains a cylindrical aluminum workpiece Σ with radius $r = 0.08$ and height $h = 0.3$. In Section 5.1, we neglect the coil Π and the source current j , and we consider two experiments with an exact solution \mathbf{A} . Afterwards, in Section 5.2, we perform a numerical simulation, where a copper coil Π is included in the setting. The constant permeability of vacuum is $\mu_0 = 4\pi\text{E-}07$, whilst the electrical conductivity $\sigma_\Sigma = 3.5\text{E}7$ (see [18]) and $\sigma_\Pi = 5.96\text{E}7$ (see [19]). In addition, the following assumptions are made: the final time $T = 0.5$, and the movement of the workpiece is defined by the velocity vector $\mathbf{v} = [0, 0, 1]^T$ along the z -axis. The numerical scheme is implemented in the software package FreeFEM [20] using the first-order Lagrangian finite elements.

5.1. Experiments with exact solution

In order to focus on the error of the moving-related quantity \mathbf{A} , we simplify our investigated model by neglecting the coil Π and the source current j . The exact solutions for the vector potential \mathbf{A} are given by

$$\mathbf{A}_{ex1} = (1 + t^2) \begin{pmatrix} y^2 - z^2 \\ z^2 - x^2 \\ x^2 - y^2 \end{pmatrix}, \quad \mathbf{A}_{ex2} = \exp(2t) \begin{pmatrix} \exp(y) \sin(z) \\ \exp(z) \sin(x) \\ \exp(x) \sin(y) \end{pmatrix}.$$

An appropriate function f , an initial guess \mathbf{A}_0 and a non-homogeneous Dirichlet boundary condition (derived from the exact solution) are added to the initial-boundary value problem (6). We use the norm in the space $\mathbf{L}^2((0, T), \mathbf{H}_0^1(\Omega))$ to calculate the error between the fully-discrete solution $\overline{\mathbf{A}}_n^h$ and the exact solution \mathbf{A} , i.e.

$$E_A = \int_0^T \|\nabla \overline{\mathbf{A}}_n^h(t) - \nabla \mathbf{A}(t)\|_{\mathbf{L}^2(\Omega)}^2 dt.$$

In order to validate the convergence rate of the numerical scheme, we fix a time step $\tau = 2^{-8}$ to calculate the error with respect to the mesh size h . The errors shown in Figure 2 are corresponding to different uniform meshes when dividing each edge of the domain Ω into N equidistant intervals, where $N = 5, 10, 15, 20, 25, 30, 35$ and 40 .

We note that two exact solutions are smooth in the time and space variables. Therefore, a convergence rate $E_A = O(h^2)$ should be obtained as a consequence of Theorem 4.3 and the relation (34). Indeed, the obtained convergence rates (the slope of the corresponding linear regression lines) are in accordance with the predicted order of convergence.

5.2. Numerical simulation

Now, we simulate an industrial electromagnetic process based on numerical simulations performed in [21, 6]. A demonstration of the mesh of the domain is depicted in Figure 3, where an external current density $j = 5\text{E-}6$ is pushed into the coil Π through the interface Γ_{in} . At time $t = 0$, we initiate the value of the vector potential $\mathbf{A}_0 = \mathbf{0}$.

We calculate a reference solution in order to evaluate the convergence rate of the proposed scheme without knowing the exact solution. The domain Ω is partitioned into 649286 tetrahedrons by choosing $N = 20$ equidistant intervals on each edge and the time step is $\tau = 2^{-8}$. The reference current density $\nabla \phi_{\text{ref}}$ and the reference magnetic induction

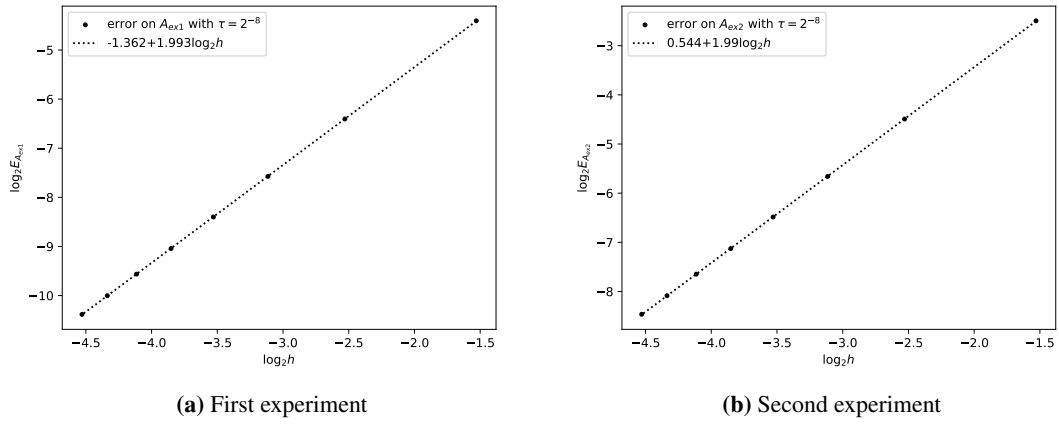


Figure 2: Convergence rate with respect to h for the experiments on a logarithmic scale.

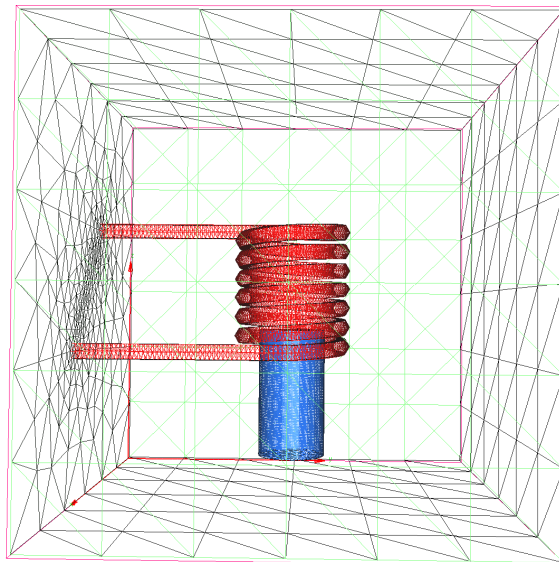


Figure 3: The mesh of the domain Ω at $t = 0$.

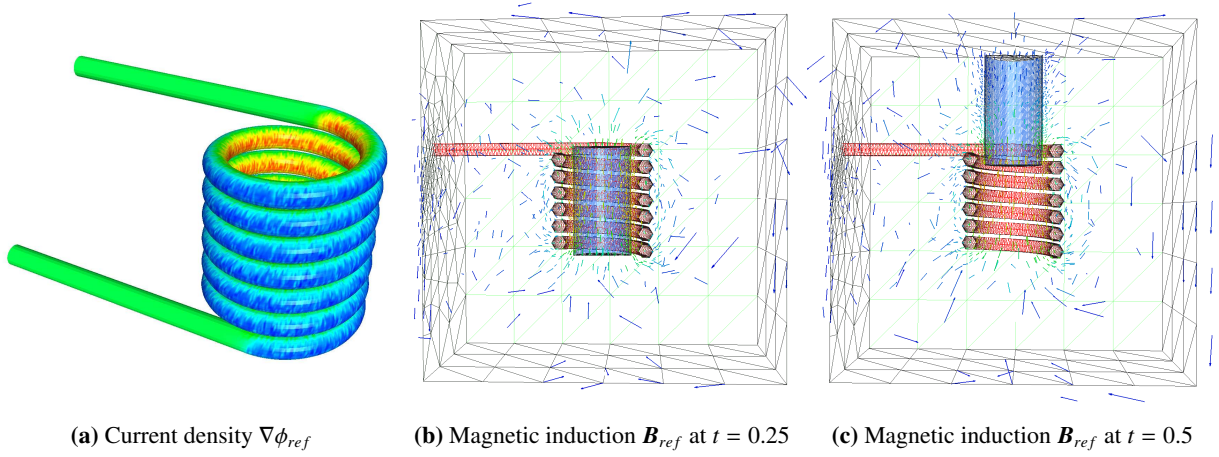


Figure 4: Reference solution for the numerical simulation at different time points.

\mathbf{B}_{ref} at two different time points are visualized by the software package MEDIT [22] on Figure 4. Then, we involve some discretizations of the domain with $N = 3, 4, 5, 6, 7, 8, 9, 10$ to assess the error corresponding to h . The following relative errors are used for this setting to calculate the errors between the fully-discrete solution and the reference one

$$\tilde{E}_\phi = \frac{\left| \left\| \nabla \bar{\phi}_n^h \right\|_{\mathbf{L}^2(\Pi)} - \left\| \nabla \phi_{ref} \right\|_{\mathbf{L}^2(\Pi)} \right|}{\left\| \nabla \phi_{ref} \right\|_{\mathbf{L}^2(\Pi)}}, \quad \tilde{E}_A = \frac{\left| \left\| \nabla \bar{\mathbf{A}}_n^h \right\|_{\mathbf{L}^2((0,T),\mathbf{L}^2(\Omega))} - \left\| \nabla \mathbf{A}_{ref} \right\|_{\mathbf{L}^2((0,T),\mathbf{L}^2(\Omega))} \right|}{\left\| \nabla \mathbf{A}_{ref} \right\|_{\mathbf{L}^2((0,T),\mathbf{L}^2(\Omega))}}.$$

Please note that the convergence rate $O(h)$ should be obtained for both relative errors. However, since these meshes are not uniform, the relative errors on ϕ and \mathbf{A} shown in Figure 5 are with respect to N , and the corresponding regression lines present the convergence rate $O(N^{-2})$. This result again confirms the convergence of the proposed scheme with respect to the space discretization.

6. Conclusion

In this paper, we have proposed a fully-discrete finite element scheme combined with the backward Euler method for a multi-component electromagnetic problem with moving non-magnetic workpiece. The workpiece is kept at a safe distance from the coil and the external boundary of the domain Ω . The convergence of the numerical scheme has been proved in the sense of the error of orthogonal interpolation operators and the time step. We have established the error of the proposed scheme for the first-order Lagrangian finite elements, and we have also performed some numerical experiments. These experiments confirm the convergence of the scheme with respect to the space discretization.

In the future, we would like to deal with a finite element analysis of the saddle-point formulation for this electromagnetic problem with a moving non-magnetic conductor. Another research question is whether the suboptimal convergence order in time can be improved.

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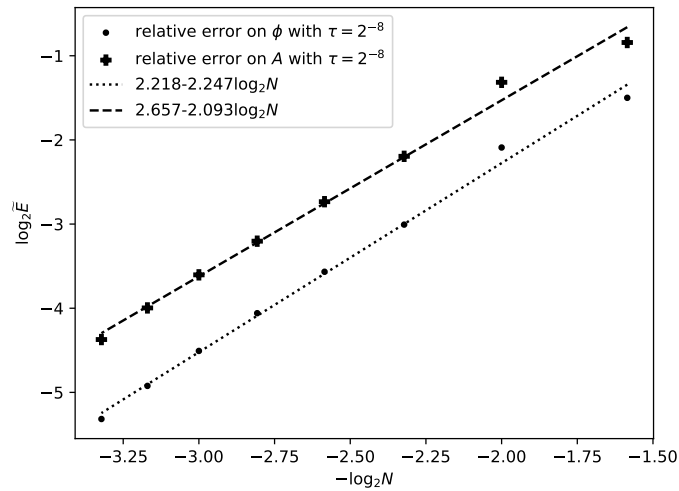


Figure 5: Convergence rate with respect to N for the numerical simulation on a logarithmic scale.

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