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COMPOSITIONS OF TERNARY RELATIONS

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In this paper, we introduce six basic types of composition of ternary relations, four of which are associative. These compositions are based on two types of composition of a ternary relation with a binary relation recently introduced by Zedam et al. We study the properties of these compositions, in particular the link with the usual composition of binary relations through the use of the operations of projection and cylindrical extension.

Keywords: binary relation, ternary relation, relational compositions

Classification: 03E20, 97E60

1. INTRODUCTION

By far the most important operation on binary relations is the composition of relations, dating back to the nineteenth century (see, e. g., [25]). In the twentieth century, Bandler and Kohout [2] introduced two additional relational compositions, arising from the substitution of the underlying existential quantifier by the universal quantifier. Some decades earlier, already in his seminal work on fuzzy sets [37], Zadeh realized that also crisp relations (allowing to model relationship and non-relationship only) lacked expressivity, giving rise to the study of fuzzy relations. Goguen further generalized fuzzy relations to the lattice-theoretical framework [18]. This notion of a fuzzy relation spread very quickly and it is thus not surprising that Bandler and Kohout presented fuzzy versions of their relational compositions in tandem with the crisp versions. Some further modifications were suggested by De Baets and Kerre [10]. More recently, Štěpnička and Holčapek [35] revisited the fuzzy relational compositions by employing more general fuzzy quantifiers than the existential and universal ones. Noteworthy is also the recent work on fuzzy relations in the context of fuzzy class theory [5, 6].

In addition to the role of relational compositions in the study and characterization of various properties of binary crisp or fuzzy relations [13, 16, 36], they also appear in many branches of mathematics, for instance in the study of fuzzy relational equations [11, 12], formal concept analysis [7, 17] and relation algebras (e. g., in temporal and spatial reasoning [15]). Compositions of crisp and fuzzy relations also appear in applications, for instance, in medical diagnosis [3], fuzzy inference systems [33, 34] and relational databases [28].

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Surprisingly, in contrast to binary relations, ternary and, more generally, n-ary relations, have received far less attention. However, in recent years, the interest in ternary relations is on the rise, for instance in the theory of dependence spaces [24] and (fuzzy) triadic formal concept analysis [8, 19, 22]. From a theoretical point of view, ternary relations have been studied in algebra (e. g., in group theory [9]), order theory (e. g., in the study of cyclic orders [23]) and logic (e. g., in the Routley–Meyer semantics of relevant logic [4, 14]). Recently, betweenness relations, a specific type of ternary relation, also came to play a pivotal role in models for decision making [27] and aggregation [26]. Other applications can be found in computational biology (e. g., modelling of phylogenies [32]), qualitative spatial reasoning [20] and string matching [21]. Also, there are many different uses of ternary relations in the field of information modelling (e. g., in the Resource Description Framework (RDF) [30], entity-relationship or class diagrams [1] and the Ternary Relations Model [29]).

Motivated by the usefulness of relational compositions of binary relations and the importance of ternary relations, in this paper we introduce several types of composition of ternary relations. More specifically, we introduce six types of composition of ternary relations based on the composition of a ternary relation with a binary relation introduced by Zedam et al. [38], and investigate their properties. This paper is organized as follows. In Section 2, we recall the necessary basic concepts and properties of binary relations and ternary relations. In Section 3, we introduce six types of composition of ternary relations and, in Section 4, we investigate the properties of these compositions. We study the interaction of the compositions with the basic set operations and permutations in Section 5, and with the binary projections and cylindrical extensions in Section 6. In Section 7, we study the compositions in the context of traces of ternary relations. Finally, we present some concluding remarks in Section 8.

2. PRELIMINARIES

In this section, we recall some basic notions on binary and ternary relations that will be needed throughout this paper.

2.1. Binary relations

A binary relation R on a set X is a subset of X^2. Inclusion, intersection and union of binary relations on X are defined through the corresponding notions for subsets of X^2. For a given binary relation R on X, we denote the *transpose* of R by R^t, i. e., for any $x, y \in X$, $(x, y) \in R^t$ means that $(y, x) \in R$. The *composition* of two relations R_1 and R_2 on X is the relation $R_1 \circ R_2$ on X defined as

$$R_1 \circ R_2 = \{(x, z) \in X^2 \mid (\exists y \in X)((x, y) \in R_1 \wedge (y, z) \in R_2)\}\,.$$

For more details on binary relations, we refer to [31].

2.2. Ternary relations

A ternary relation T on a set X is a subset of X^3. Three special ternary relations on X are the empty relation $\emptyset$, the ternary identity relation $I_{X^3} = \{(x, x, x) \mid x \in X\}$

and the universal ternary relation X^3. Here, we recall the ternary relations obtained by permutation. A permutation σ of a 3-element set $U = \{u, v, w\}$ is a bijection from U to itself. We use the shorthand notation $\sigma(u, v, w)$ instead of $(\sigma(u), \sigma(v), \sigma(w))$. The six permutations of U are given by:

$$\sigma_0(u,v,w) = (u,v,w), \quad \sigma_1(u,v,w) = (u,w,v), \quad \sigma_2(u,v,w) = (v,u,w)\,,$$

$$\sigma_3(u,v,w) = (v,w,u), \quad \sigma_4(u,v,w) = (w,u,v), \quad \sigma_5(u,v,w) = (w,v,u)\,.$$

For a ternary relation T on X and any of the above six permutations σ, the ternary relation T^σ is defined as in [38]:

$$T^\sigma = \{\sigma(x,y,z) \in X^3 \mid (x,y,z) \in T\}\,.$$

Note that

$$T^\sigma = \{(x,y,z) \in X^3 \mid \sigma^{-1}(x,y,z) \in T\}\,,$$

with $\sigma_i^{-1} = \sigma_i$ for any $i \in \{0,1,2,5\}$ and $\sigma_3^{-1} = \sigma_4$. It is clear that $T^{\sigma_0} = T$ and $T^{\sigma_5} = T^t$.

Definition 2.1. (Zedam et al. [38]) Let T be a ternary relation on a set X.

(i) The right-converse of T is the ternary relation $T^{\dashv}$ on X defined as $T^{\dashv} = T^{\sigma_1}$;

(ii) The left-converse of T is the ternary relation $T^{\vdash}$ on X defined as $T^{\vdash} = T^{\sigma_2}$;

(iii) The right-rotation of T is the ternary relation T^{+} on X defined as $T^{+} = T^{\sigma_3}$;

(iv) The left-rotation of T is the ternary relation T^{-} on X defined as $T^{-} = T^{\sigma_4}$.

The following properties are straightforward, but will be useful further on.

Remark 2.1. For any family of ternary relations $(T_i)_{i\in I}$ on a set X, the following equalities hold:

$$\left(\bigcup_{i\in I} T_i\right)^{\sigma_j} = \bigcup_{i\in I} T_i^{\sigma_j} \quad \text{and} \quad \left(\bigcap_{i\in I} T_i\right)^{\sigma_j} = \bigcap_{i\in I} T_i^{\sigma_j}\,,$$

for any $j \in \{0, \ldots, 5\}$.

For more details on ternary relations, we refer to [9, 24, 38].

3. COMPOSITIONS OF TERNARY RELATIONS

In the theory of binary relations, a major role is played by the composition of relations, as it is the most important operation that allows to combine relations. In this section, based on the compositions of a ternary relation with a binary relation introduced by Zedam et al. [38], we introduce several types of composition of ternary relations and investigate their properties. First, we recall the definition of the binary projections of a ternary relation.

3.1. Binary projections of a ternary relation

In this subsection, we recall the definition and some properties of the binary projections of a ternary relation.

Definition 3.1. (Zedam et al. [38]) Let T be a ternary relation on a set X.

(i) The left projection of T is the binary relation $P_\ell(T)$ on X defined as

$$P_\ell(T) = \{(x, y) \in X^2 \mid (\exists z \in X)((z, x, y) \in T)\}\,;$$

(ii) The middle projection of T is the binary relation $P_m(T)$ on X defined as

$$P_m(T) = \{(x, y) \in X^2 \mid (\exists z \in X)((x, z, y) \in T)\}\,;$$

(iii) The right projection of T is the binary relation $P_r(T)$ on X defined as

$$P_r(T) = \{(x, y) \in X^2 \mid (\exists z \in X)((x, y, z) \in T)\}\,.$$

For a ternary relation T, we write $P(T) = P_\ell(T) \cup P_m(T) \cup P_r(T)$.

The following proposition shows the interaction of the projections of a ternary relation with the inclusion and basic set-theoretical operations.

Proposition 3.1. Let T_1 and T_2 be two ternary relations on a set X. For any $\lambda \in \{\ell, m, r\}$, the following statements hold:

(i) If $T_1 \subseteq T_2$, then $P_\lambda(T_1) \subseteq P_\lambda(T_2)$;

(ii) $P_\lambda(T_1 \cap T_2) \subseteq P_\lambda(T_1) \cap P_\lambda(T_2)$;

(iii) $P_\lambda(T_1 \cup T_2) = P_\lambda(T_1) \cup P_\lambda(T_2)$.

P r o o f. We only give the proof for the case $\lambda = \ell$, as the other cases can be proved analogously.

(i) Suppose that $T_1 \subseteq T_2$ and let $(x, y) \in P_\ell(T_1)$. Then there exists $z \in X$ such that $(z, x, y) \in T_1$. This implies that $(z, x, y) \in T_2$. Hence, $(x, y) \in P_\ell(T_2)$. Thus, $P_\ell(T_1) \subseteq P_\ell(T_2)$.

(ii) Let $(x, y) \in P_\ell(T_1 \cap T_2)$. Then there exists $z \in X$ such that $(z, x, y) \in T_1 \cap T_2$. This implies that $(z, x, y) \in T_1$ and $(z, x, y) \in T_2$. Hence, $(x, y) \in P_\ell(T_1) \cap P_\ell(T_2)$. Thus, $P_\ell(T_1 \cap T_2) \subseteq P_\ell(T_1) \cap P_\ell(T_2)$.

(iii) We easily verify that

$$\begin{aligned} P_\ell(T_1 \cup T_2) &= \{(x, y) \in X^2 \mid (\exists z \in X)((z, x, y) \in T_1 \cup T_2)\} \\ &= \{(x, y) \in X^2 \mid (\exists z \in X)((z, x, y) \in T_1 \vee (z, x, y) \in T_2)\} \\ &= P_\ell(T_1) \cup P_\ell(T_2)\,. \end{aligned}$$

□

The following proposition shows the interaction of $P(T)$ with the inclusion and basic set-theoretical operations.

Proposition 3.2. Let T_1 and T_2 be two ternary relations on a set X. The following statements hold:

(i) If $T_1 \subseteq T_2$, then $P(T_1) \subseteq P(T_2)$;

(ii) $P(T_1 \cap T_2) \subseteq P(T_1) \cap P(T_2)$;

(iii) $P(T_1 \cup T_2) = P(T_1) \cup P(T_2)$.

Proof.

(i) Suppose that $T_1 \subseteq T_2$, Proposition 3.1 then guarantees that $P_\ell(T_1) \cup P_m(T_1) \cup P_r(T_1) \subseteq P_\ell(T_2) \cup P_m(T_2) \cup P_r(T_2)$. Thus, $P(T_1) \subseteq P(T_2)$.

(ii) From Proposition 3.1, it follows that

$$\begin{aligned} P(T_1 \cap T_2) &= P_\ell(T_1 \cap T_2) \cup P_m(T_1 \cap T_2) \cup P_r(T_1 \cap T_2) \\ &\subseteq (P_\ell(T_1) \cap P_\ell(T_2)) \cup (P_m(T_1) \cap P_m(T_2)) \cup (P_r(T_1) \cap P_r(T_2)) \,. \end{aligned}$$

The distributivity of $\cap$ and $\cup$ guarantees that

$$\begin{aligned} P(T_1 \cap T_2) &\subseteq (P_\ell(T_1) \cup P_m(T_1) \cup P_r(T_1)) \cap (P_\ell(T_2) \cup P_m(T_2) \cup P_r(T_2)) \\ &= P(T_1) \cap P(T_2) \,. \end{aligned}$$

(iii) Also, from Proposition 3.1, we easily verify that

$$\begin{aligned} P(T_1 \cup T_2) &= P_\ell(T_1 \cup T_2) \cup P_m(T_1 \cup T_2) \cup P_r(T_1 \cup T_2) \\ &= (P_\ell(T_1) \cup P_\ell(T_2)) \cup (P_m(T_1) \cup P_m(T_2)) \cup (P_r(T_1) \cup P_r(T_2)) \\ &= (P_\ell(T_1) \cup P_m(T_1) \cup P_r(T_1)) \cup (P_\ell(T_2) \cup P_m(T_2) \cup P_r(T_2)) \\ &= P(T_1) \cup P(T_2) \,. \end{aligned}$$

□

3.2. Compositions of ternary relations

In this subsection, we introduce six types of composition of ternary relations. First we recall the definition of two types of composition of a ternary relation with a binary relation introduced in [38].

Definition 3.2. (Zedam et al. [38]) Let T be a ternary relation and R be a binary relation on X.

(i) The $\ltimes$-composition of T and R is the ternary relation $T \ltimes R$ on X defined as

$$T \ltimes R = \{(x, y, z) \in X^3 \mid (\exists t \in X)((x, y, t) \in T \wedge (t, z) \in R)\} \,;$$

(ii) The $\rtimes$-composition of R and T is the ternary relation $R \rtimes T$ on X defined as

$$R \rtimes T = \{(x,y,z) \in X^3 \mid (\exists t \in X)((x,t) \in R \wedge (t,y,z) \in T)\}\,.$$

Based on the above compositions of a ternary relation with a binary relation, we introduce six types of composition of ternary relations through the use of the binary projections.

Definition 3.3. Let T and S be two ternary relations on a set X. The $\circ_i$-compositions of T and S, with $i \in \{1,\ldots,6\}$, are defined as

(i) $T \circ_1 S := T \ltimes P_\ell(S) = \{(x,y,z) \in X^3 \mid (\exists t,s \in X)((x,y,t) \in T \wedge (s,t,z) \in S)\}$;

(ii) $T \circ_2 S := T \ltimes P_m(S) = \{(x,y,z) \in X^3 \mid (\exists t,s \in X)((x,y,t) \in T \wedge (t,s,z) \in S)\}$;

(iii) $T \circ_3 S := T \ltimes P_r(S) = \{(x,y,z) \in X^3 \mid (\exists t,s \in X)((x,y,t) \in T \wedge (t,z,s) \in S)\}$;

(iv) $T \circ_4 S := P_\ell(T) \rtimes S = \{(x,y,z) \in X^3 \mid (\exists t,s \in X)((s,x,t) \in T \wedge (t,y,z) \in S)\}$;

(v) $T \circ_5 S := P_m(T) \rtimes S = \{(x,y,z) \in X^3 \mid (\exists t,s \in X)((x,s,t) \in T \wedge (t,y,z) \in S)\}$;

(vi) $T \circ_6 S := P_r(T) \rtimes S = \{(x,y,z) \in X^3 \mid (\exists t,s \in X)((x,s,t) \in T \wedge (s,y,z) \in S)\}$.

Next, we introduce a composition that is composed of all of the above six compositions.

Definition 3.4. Let T and S be two ternary relations on a set X. The $\circ$-composition of T and S is defined as

$$T \circ S = (T \ltimes P(S)) \cup (P(T) \rtimes S)\,.$$

Clearly, it holds that $T \circ S = \bigcup_{i=1}^{6} T \circ_i S$.

The following example shows that the compositions introduced are different.

Example 3.1. Let T and S be the ternary relations on $X = \{x_1,x_2,x_3,x_4\}$ given by:

$$\begin{aligned} T &= \{(x_1,x_1,x_2),(x_1,x_2,x_3)\}\,,\\ S &= \{(x_1,x_1,x_1),(x_2,x_4,x_1),(x_3,x_2,x_2)\}\,. \end{aligned}$$

One easily verifies that

$$\begin{aligned} T \circ_1 S &= \{(x_1,x_1,x_2)\}\,,\\ T \circ_2 S &= \{(x_1,x_1,x_1),(x_1,x_2,x_2)\}\,,\\ T \circ_3 S &= \{(x_1,x_1,x_4),(x_1,x_2,x_2)\}\,,\\ T \circ_4 S &= \{(x_1,x_4,x_1),(x_2,x_2,x_2)\}\,,\\ T \circ_5 S &= \{(x_1,x_4,x_1),(x_1,x_2,x_2)\}\,,\\ T \circ_6 S &= \{(x_1,x_1,x_1),(x_1,x_4,x_1)\}\,, \end{aligned}$$

and

$$T \circ S = \{(x_1,x_4,x_1),(x_2,x_2,x_2),(x_1,x_2,x_2),(x_1,x_1,x_1),(x_1,x_1,x_2),(x_1,x_1,x_4)\}\,.$$

4. PROPERTIES OF THE COMPOSITIONS OF TERNARY RELATIONS

In this section, we investigate some properties of the compositions of ternary relations. First of all, we show that four of these compositions are associative.

Proposition 4.1. The compositions $\circ_i$, with $i \in \{1, 2, 5, 6\}$, are associative, i. e., for any ternary relations T_1, T_2 and T_3 on a set X, it holds that

$$(T_1 \circ_i T_2) \circ_i T_3 = T_1 \circ_i (T_2 \circ_i T_3)\,.$$

Proof. We only give the proof for the case $i = 1$, as the other cases can be proved analogously.

$$\begin{aligned}
&(T_1 \circ_1 T_2) \circ_1 T_3 = \{(x, y, z) \in X^3 \mid (\exists t, s \in X)((x, y, t) \in T_1 \circ_1 T_2 \wedge (s, t, z) \in T_3)\} \\
&= \{(x, y, z) \in X^3 \mid (\exists t, s, m, n \in X)((x, y, m) \in T_1 \wedge (n, m, t) \in T_2 \wedge (s, t, z) \in T_3)\} \\
&= \{(x, y, z) \in X^3 \mid (\exists m, n \in X)((x, y, m) \in T_1 \wedge (n, m, z) \in T_2 \circ_1 T_3)\} \\
&= \{(x, y, z) \in X^3 \mid (x, y, z) \in T_1 \circ_1 (T_2 \circ_1 T_3)\} \\
&= T_1 \circ_1 (T_2 \circ_1 T_3)\,.
\end{aligned}$$

□

In the following example, we show that the compositions $\circ_3$ and $\circ_4$ are not associative.

Example 4.1. Let T_1, T_2 and T_3 be the ternary relations on $X = \{x_1, x_2, x_3, x_4\}$ given by:

$$\begin{aligned}
T_1 &= \{(x_1, x_1, x_2), (x_1, x_2, x_3)\}\,, \\
T_2 &= \{(x_1, x_1, x_1), (x_2, x_4, x_1), (x_3, x_2, x_2)\}\,, \\
T_3 &= \{(x_1, x_1, x_1), (x_4, x_4, x_2), (x_2, x_3, x_1)\}\,.
\end{aligned}$$

One easily verifies that

$$\begin{aligned}
(T_1 \circ_3 T_2) \circ_3 T_3 &= \{(x_1, x_1, x_4), (x_1, x_2, x_3)\}\,, \\
T_1 \circ_3 (T_2 \circ_3 T_3) &= \{(x_1, x_1, x_4), (x_1, x_2, x_2)\}\,, \\
(T_1 \circ_4 T_2) \circ_4 T_3 &= \{(x_4, x_1, x_1), (x_2, x_3, x_1)\}\,, \\
T_1 \circ_4 (T_2 \circ_4 T_3) &= \{(x_1, x_3, x_1)\}\,.
\end{aligned}$$

It is clear that

$$(T_1 \circ_3 T_2) \circ_3 T_3 \neq T_1 \circ_3 (T_2 \circ_3 T_3) \quad \text{and} \quad (T_1 \circ_4 T_2) \circ_4 T_3 \neq T_1 \circ_4 (T_2 \circ_4 T_3)\,.$$

Obviously, since $\circ_3$ and $\circ_4$ are not associative, the composition $\circ$ is not associative as well.

The following proposition shows that the ternary identity relation I_{X^3} is a right neutral element of three of the compositions of ternary relations, while it is a left neutral element of the other compositions. The proof is straightforward.

Proposition 4.2. Let T be a ternary relation on a set X. It holds that

(i) $T \circ_i I_{X^3} = T$, for any $i \in \{1, 2, 3\}$;

(ii) $I_{X^3} \circ_i T = T$, for any $i \in \{4, 5, 6\}$.

Next, we give a counterexample showing that I_{X^3} is not a left (resp. right) neutral element of the $\circ_i$-composition, for any $i \in \{1, 2, 3\}$ (resp. for any $i \in \{4, 5, 6\}$).

Example 4.2. Let T be the ternary relation on $X = \{x_1, x_2, x_3, x_4\}$ given by:

$$T = \{(x_1, x_1, x_2), (x_1, x_2, x_3)\}\,.$$

One easily verifies that

$$\begin{aligned}
I_{X^3} \circ_1 T &= \{(x_1, x_1, x_2), (x_2, x_2, x_3)\}\,,\\
I_{X^3} \circ_2 T &= \{(x_1, x_1, x_2), (x_1, x_1, x_3)\}\,,\\
I_{X^3} \circ_3 T &= \{(x_1, x_1, x_1), (x_1, x_1, x_2)\}\,,\\
T \circ_4 I_{X^3} &= \{(x_1, x_2, x_2), (x_2, x_3, x_3)\}\,,\\
T \circ_5 I_{X^3} &= \{(x_1, x_2, x_2), (x_1, x_3, x_3)\}\,,\\
T \circ_6 I_{X^3} &= \{(x_1, x_1, x_1), (x_1, x_2, x_2)\}\,.
\end{aligned}$$

It is clear that

$$I_{X^3} \circ_1 T \neq T, I_{X^3} \circ_2 T \neq T \quad \text{and} \quad I_{X^3} \circ_3 T \neq T\,,$$

$$T \circ_4 I_{X^3} \neq T, T \circ_5 I_{X^3} \neq T \quad \text{and} \quad T \circ_6 I_{X^3} \neq T\,,$$

and

$$I_{X^3} \circ T \neq T \circ I_{X^3} \neq T\,.$$

In the following proposition, we show that the empty relation $\emptyset$ is an absorbing element of the compositions of ternary relations. The proof is straightforward.

Proposition 4.3. Let T be a ternary relation on a set X. It holds that

$$T \circ_i \emptyset = \emptyset \circ_i T = \emptyset\,,$$

for any $i \in \{1, \ldots, 6\}$, and hence also $T \circ \emptyset = \emptyset \circ T = \emptyset$.

5. INTERACTION OF THE COMPOSITIONS WITH THE BASIC SET OPERATIONS AND PERMUTATIONS

In this section, we study the interaction of the compositions of ternary relations with the basic set operations and permutations.

5.1. Interaction of the compositions with inclusion and basic set operations

The following proposition shows the interaction of the $\circ_i$-compositions with inclusion and set-theoretical operations, for any $i \in \{1, \ldots, 6\}$.

Proposition 5.1. Let T_1, T_2, S_1, S_2 and S be ternary relations on a set X. For any $i \in \{1, \ldots, 6\}$, the following statements hold:

(i) If $T_1 \subseteq T_2$ and $S_1 \subseteq S_2$, then $T_1 \circ_i S_1 \subseteq T_2 \circ_i S_2$;

(ii) $(T_1 \cap T_2) \circ_i S = (T_1 \circ_i S) \cap (T_2 \circ_i S)$ and $S \circ_i (T_1 \cap T_2) = (S \circ_i T_1) \cap (S \circ_i T_2)$;

(iii) $(T_1 \cup T_2) \circ_i S = (T_1 \circ_i S) \cup (T_2 \circ_i S)$ and $S \circ_i (T_1 \cup T_2) = (S \circ_i T_1) \cup (S \circ_i T_2)$.

Proof. We only give the proof for the $\circ_1$-composition and property (i) as the other results are analogous. Suppose that $T_1 \subseteq T_2$ and $S_1 \subseteq S_2$. Let $(x, y, z) \in T_1 \circ_1 S_1$, then it holds that $(x, y, z) \in T_1 \ltimes P_\ell(S_1)$. Then there exists $t \in X$ such that $(x, y, t) \in T_1$ and $(t, z) \in P_\ell(S_1)$. Since $T_1 \subseteq T_2$ and $S_1 \subseteq S_2$, it follows that $(x, y, z) \in T_2 \ltimes P_\ell(S_2)$. Hence, $(x, y, z) \in T_2 \circ_1 S_2$. Thus, $T_1 \circ_1 S_1 \subseteq T_2 \circ_1 S_2$. □

The following proposition shows the interaction of the $\circ$-composition with inclusion and set-theoretical operations.

Proposition 5.2. Let T_1, T_2, S_1, S_2 and S be ternary relations on a set X. The following statements hold:

(i) If $T_1 \subseteq T_2$ and $S_1 \subseteq S_2$, then $T_1 \circ S_1 \subseteq T_2 \circ S_2$;

(ii) $(T_1 \cap T_2) \circ S \subseteq (T_1 \circ S) \cap (T_2 \circ S)$;

(iii) $(T_1 \cup T_2) \circ S \subseteq (T_1 \circ S) \cup (T_2 \circ S)$.

Proof.

(i) Suppose that $T_1 \subseteq T_2$ and $S_1 \subseteq S_2$. Let $(x, y, z) \in T_1 \circ S_1$, then it holds that $(x, y, z) \in \bigcup_{i=1}^{6} T_1 \circ_i S_1$. Then there exists $j \in \{1, \ldots, 6\}$ such that $(x, y, z) \in T_1 \circ_j S_1$. Since $T_1 \subseteq T_2$ and $S_1 \subseteq S_2$, it follows from Proposition 5.1 that $T_1 \circ_j S_1 \subseteq T_2 \circ_j S_2$. Hence, $(x, y, z) \in T_2 \circ_j S_2$. Thus, $(x, y, z) \in T_2 \circ S_2$.

(ii) Let $(x, y, z) \in (T_1 \cap T_2) \circ S$. It holds that $(x, y, z) \in \bigcup_{i=1}^{6} (T_1 \cap T_2) \circ_i S$. Then there exists $j \in \{1, \ldots, 6\}$ such that $(x, y, z) \in (T_1 \cap T_2) \circ_j S$. From Proposition 5.1, it follows that $(x, y, z) \in (T_1 \circ_j S) \cap (T_2 \circ_j S)$, and, hence, $(x, y, z) \in \bigcup_{i=1}^{6} T_1 \circ_i S$ and $(x, y, z) \in \bigcup_{i=1}^{6} T_2 \circ_i S$. Hence, $(x, y, z) \in (T_1 \circ S) \cap (T_2 \circ S)$. Thus, $(T_1 \cap T_2) \circ S \subseteq (T_1 \circ S) \cap (T_2 \circ S)$.

(iii) The proof is analogous to that of (ii).

□

The following example shows that the equality in properties (ii) and (iii) of Proposition 5.2 does not hold in general.

Example 5.1. Let T_1, T_2 and T_3 be the ternary relations on $X = \{x_1, x_2, x_3, x_4\}$ given by:

$$\begin{aligned} T_1 &= \{(x_1, x_1, x_2), (x_1, x_2, x_3)\}\,, \\ T_2 &= \{(x_1, x_1, x_1), (x_2, x_4, x_1), (x_3, x_2, x_2)\}\,, \\ T_3 &= \{(x_1, x_1, x_2), (x_4, x_4, x_2), (x_2, x_3, x_1)\}\,. \end{aligned}$$

One easily verifies that

$$\begin{aligned} (T_1 \cap T_3) \circ T_2 &= \{(x_1, x_1, x_1), (x_1, x_1, x_2), (x_1, x_4, x_1), (x_1, x_1, x_4)\}\,, \\ (T_1 \circ T_2) \cap (T_3 \circ T_2) &= \{(x_1, x_1, x_1), (x_1, x_1, x_2), (x_1, x_4, x_1), (x_1, x_1, x_4), (x_2, x_2, x_2)\}\,. \end{aligned}$$

It is clear that $(T_1 \cap T_3) \circ T_2 \neq (T_1 \circ T_2) \cap (T_3 \circ T_2)$.

5.2. Interaction of the compositions with the permutations

In this subsection, we investigate the interaction of the compositions with the permutations.

Proposition 5.3. Let T and S be two ternary relations on a set X. The following equalities hold:

(i) $(T \circ_i S)^{\dashv} = T \circ_i S^{\dashv}$, for any $i \in \{4, 5, 6\}$;

(ii) $(T \circ_i S)^{\vdash} = T^{\vdash} \circ_i S$, for any $i \in \{1, 2, 3\}$;

(iii) $(T \circ_i S)^{+} = (S^t \circ_{7-i} T^t)^{\vdash}$, for any $i \in \{1, \ldots, 6\}$;

(iv) $(T \circ_i S)^{-} = (S^t \circ_{7-i} T^t)^{\dashv}$, for any $i \in \{1, \ldots, 6\}$;

(v) $(T \circ_i S)^t = S^t \circ_{7-i} T^t$, for any $i \in \{1, \ldots, 6\}$.

Proof.

(i) We only prove that $(T \circ_4 S)^{\dashv} = T \circ_4 S^{\dashv}$, as the other cases can be proved analogously.

$$\begin{aligned} (T \circ_4 S)^{\dashv} &= \big\{(x, y, z) \in X^3 \mid (x, z, y) \in T \circ_4 S\big\} \\ &= \big\{(x, y, z) \in X^3 \mid (x, z, y) \in P_\ell(T) \rtimes S\big\} \\ &= \big\{(x, y, z) \in X^3 \mid (\exists t \in X)((x, t) \in P_\ell(T) \wedge (t, z, y) \in S)\big\} \\ &= \big\{(x, y, z) \in X^3 \mid (\exists t \in X)((x, t) \in P_\ell(T) \wedge (t, y, z) \in S^{\dashv})\big\} \\ &= \big\{(x, y, z) \in X^3 \mid (x, y, z) \in P_\ell(T) \rtimes S^{\dashv}\big\} \\ &= T \circ_4 S^{\dashv}\,. \end{aligned}$$

(ii) We only prove that $(T \circ_1 S)^{\vdash} = T^{\vdash} \circ_1 S$, as the other cases can be proved analogously.

$$\begin{aligned}
(T \circ_1 S)^{\vdash} &= \big\{(x,y,z) \in X^3 \mid (y,x,z) \in T \circ_1 S\big\} \\
&= \big\{(x,y,z) \in X^3 \mid (y,x,z) \in T \ltimes P_\ell(S)\big\} \\
&= \big\{(x,y,z) \in X^3 \mid (\exists t \in X)((y,x,t) \in T \wedge (t,z) \in P_\ell(S))\big\} \\
&= \big\{(x,y,z) \in X^3 \mid (\exists t \in X)((x,y,t) \in T^{\vdash} \wedge (t,z) \in P_\ell(S))\big\} \\
&= \big\{(x,y,z) \in X^3 \mid (x,y,z) \in T^{\vdash} \ltimes P_\ell(S)\big\} \\
&= T^{\vdash} \circ_1 S\,.
\end{aligned}$$

(iii) We only prove that $(T \circ_1 S)^{+} = (S^t \circ_6 T^t)^{\vdash}$, as the other cases can be proved analogously.

$$\begin{aligned}
(T \circ_1 S)^{+} &= \big\{(x,y,z) \in X^3 \mid (z,x,y) \in T \circ_1 S\big\} \\
&= \big\{(x,y,z) \in X^3 \mid (z,x,y) \in T \ltimes P_\ell(S)\big\} \\
&= \big\{(x,y,z) \in X^3 \mid (\exists t \in X)((z,x,t) \in T \wedge (t,y) \in P_\ell(S))\big\} \\
&= \big\{(x,y,z) \in X^3 \mid (\exists t \in X)((y,t) \in P_r(S^t) \wedge (t,x,z) \in T^t)\big\} \\
&= \big\{(x,y,z) \in X^3 \mid (y,x,z) \in P_r(S^t) \rtimes T^t\big\} \\
&= \big\{(x,y,z) \in X^3 \mid (x,y,z) \in (S^t \circ_6 T^t)^{\vdash}\big\} \\
&= (S^t \circ_6 T^t)^{\vdash}\,.
\end{aligned}$$

(iv) We only prove that $(T \circ_1 S)^{-} = (S^t \circ_6 T^t)^{\dashv}$, as the other cases can be proved analogously.

$$\begin{aligned}
(T \circ_1 S)^{-} &= \big\{(x,y,z) \in X^3 \mid (y,z,x) \in T \circ_1 S\big\} \\
&= \big\{(x,y,z) \in X^3 \mid (y,z,x) \in T \ltimes P_\ell(S)\big\} \\
&= \big\{(x,y,z) \in X^3 \mid (\exists t \in X)((y,z,t) \in T \wedge (t,x) \in P_\ell(S))\big\} \\
&= \big\{(x,y,z) \in X^3 \mid (\exists t \in X)((x,t) \in P_r(S^t) \wedge (t,z,y) \in T^t)\big\} \\
&= \big\{(x,y,z) \in X^3 \mid (x,z,y) \in P_r(S^t) \rtimes T^t\big\} \\
&= \big\{(x,y,z) \in X^3 \mid (x,y,z) \in (S^t \circ_6 T^t)^{\dashv}\big\} \\
&= (S^t \circ_6 T^t)^{\dashv}\,.
\end{aligned}$$

(v) We only prove that $(T \circ_1 S)^t = S^t \circ_6 T^t$, as the other cases can be proved analogously.

$$\begin{aligned}
(T \circ_1 S)^t &= (T \ltimes P_\ell(S))^t \\
&= (P_\ell(S))^t \rtimes T^t \\
&= P_r(S^t) \rtimes T^t \\
&= S^t \circ_6 T^t\,.
\end{aligned}$$

$\square$

In the following proposition, we investigate the interaction of the $\circ$-composition with the permutations.

Proposition 5.4. Let T and S be two ternary relations on a set X. The following equalities hold:

(i) $(T \circ S)^{+} = (S^{t} \circ T^{t})^{\vdash}$;

(ii) $(T \circ S)^{-} = (S^{t} \circ T^{t})^{\dashv}$;

(iii) $(T \circ S)^{t} = S^{t} \circ T^{t}$.

Proof. We only prove (i), as the other cases are analogous. From Remark 2.1 and Proposition 5.3, it follows that

$$\begin{aligned}(T \circ S)^{+} = (\bigcup_{i=1}^{6} T \circ_i S)^{+} &= \bigcup_{i=1}^{6} (T \circ_i S)^{+} = \bigcup_{i=1}^{6} (S^{t} \circ_{7-i} T^{t})^{\vdash} \\ &= (\bigcup_{i=1}^{6} S^{t} \circ_{7-i} T^{t})^{\vdash} = (S^{t} \circ T^{t})^{\vdash} .\end{aligned}$$

$\square$

6. INTERACTION OF THE COMPOSITIONS WITH THE BINARY PROJECTIONS AND CYLINDRICAL EXTENSIONS

In this section, we study the interaction of the compositions of ternary relations with the binary projections and cylindrical extensions.

6.1. Interaction of the compositions with the projections

In this subsection, we investigate the projections of the compositions of ternary relations in terms of the compositions of their binary projections.

Proposition 6.1. Let T and S be two ternary relations on a set X. The left, middle and right projections of the compositions $(T \circ_i S)$, for any $i \in \{1, \ldots, 6\}$, are listed in the following table:

Comp. \ Proj.	$P_\ell(\cdot)$	$P_m(\cdot)$	$P_r(\cdot)$
$T \circ_1 S$	$P_\ell(T) \circ P_\ell(S)$	$P_m(T) \circ P_\ell(S)$	–
$T \circ_2 S$	$P_\ell(T) \circ P_m(S)$	$P_m(T) \circ P_m(S)$	–
$T \circ_3 S$	$P_\ell(T) \circ P_r(S)$	$P_m(T) \circ P_r(S)$	–
$T \circ_4 S$	–	$P_\ell(T) \circ P_m(S)$	$P_\ell(T) \circ P_r(S)$
$T \circ_5 S$	–	$P_m(T) \circ P_m(S)$	$P_m(T) \circ P_r(S)$
$T \circ_6 S$	–	$P_r(T) \circ P_m(S)$	$P_r(T) \circ P_r(S)$

Proof. We only prove that $P_\ell(T\circ_1 S) = P_\ell(T)\circ P_\ell(S)$, as the other cases are analogous.

$$\begin{aligned}P_\ell(T\circ_1 S) &= \{(x,y)\in X^2 \mid (\exists z\in X)((z,x,y)\in T\circ_1 S)\}\\ &= \{(x,y)\in X^2 \mid (\exists z\in X)((z,x,y)\in T\ltimes P_\ell(S))\}\\ &= \{(x,y)\in X^2 \mid (\exists z,t\in X)((z,x,t)\in T\wedge (t,y)\in P_\ell(S))\}\\ &= \{(x,y)\in X^2 \mid (\exists t\in X)((x,t)\in P_\ell(T)\wedge (t,y)\in P_\ell(S))\}\\ &= \{(x,y)\in X^2 \mid (x,y)\in P_\ell(T)\circ P_\ell(S)\}\\ &= P_\ell(T)\circ P_\ell(S)\,.\end{aligned}$$

□

Remark 6.1. The following example shows that the left projection of the $\circ_4$-composition of two ternary relations is not equal to the composition of any of their binary projections. Indeed, let T and S be the ternary relations on $X=\{x_1,x_2,x_3,x_4\}$ given by:

$$\begin{aligned}T &= \{(x_1,x_1,x_2),(x_1,x_2,x_3)\}\,,\\ S &= \{(x_1,x_1,x_1),(x_2,x_4,x_1),(x_3,x_2,x_2)\}\,.\end{aligned}$$

It holds that

$$T\circ_4 S = \{(x_1,x_4,x_1),(x_2,x_2,x_2)\}\,,$$

and thus

$$P_\ell(T\circ_4 S) = \{(x_4,x_1),(x_2,x_2)\}\,.$$

Further, it holds that

	$P_\ell(\cdot)$	$P_m(\cdot)$	$P_r(\cdot)$
T	$\{(x_1,x_2),(x_2,x_3)\}$	$\{(x_1,x_2),(x_1,x_3)\}$	$\{(x_1,x_1),(x_1,x_2)\}$
S	$\{(x_1,x_1),(x_2,x_2),(x_4,x_1)\}$	$\{(x_1,x_1),(x_2,x_1),(x_3,x_2)\}$	$\{(x_1,x_1),(x_2,x_4),(x_3,x_2)\}$

and

$\circ$	$P_\ell(S)$	$P_m(S)$	$P_r(S)$
$P_\ell(T)$	$\{(x_1,x_2)\}$	$\{(x_1,x_1),(x_2,x_2)\}$	$\{(x_1,x_4),(x_2,x_2)\}$
$P_m(T)$	$\{(x_1,x_2)\}$	$\{(x_1,x_1),(x_1,x_2)\}$	$\{(x_1,x_2),(x_1,x_4)\}$
$P_r(T)$	$\{(x_1,x_1),(x_1,x_2)\}$	$\{(x_1,x_1)\}$	$\{(x_1,x_1),(x_1,x_4)\}$

It is clear that for any $\lambda_1,\lambda_2\in\{\ell,m,r\}$ it holds that

$$P_\ell(T\circ_4 S)\neq P_{\lambda_1}(T)\circ P_{\lambda_2}(S)\,.$$

In a similar way, one can easily prove the other cases.

6.2. Interaction of the compositions with the cylindrical extensions

In this subsection, we investigate the interaction of the compositions of ternary relations with the left, middle and right cylindrical extensions of a binary relation. First, we recall the definition of the cylindrical extensions of a binary relation.

Definition 6.1. (Zedam et al. [38]) Let R be a binary relation on a set X.

(i) The left cylindrical extension of R is the ternary relation $C_\ell(R)$ on X defined as

$$C_\ell(R) = \{(x, y, z) \in X^3 \mid (y, z) \in R\}\,;$$

(ii) The middle cylindrical extension of R is the ternary relation $C_m(R)$ on X defined as

$$C_m(R) = \{(x, y, z) \in X^3 \mid (x, z) \in R\}\,;$$

(iii) The right cylindrical extension of R is the ternary relation $C_r(R)$ on X defined as

$$C_r(R) = \{(x, y, z) \in X^3 \mid (x, y) \in R\}\,.$$

The following proposition shows the interaction of the cylindrical extensions of a binary relation with inclusion and set-theoretical operations.

Proposition 6.2. Let R_1 and R_2 be two binary relations on a set X. For any $\lambda \in \{\ell, m, r\}$, the following statements hold:

(i) If $R_1 \subseteq R_2$, then $C_\lambda(R_1) \subseteq C_\lambda(R_2)$;

(ii) $C_\lambda(R_1 \cap R_2) = C_\lambda(R_1) \cap C_\lambda(R_2)$;

(iii) $C_\lambda(R_1 \cup R_2) = C_\lambda(R_1) \cup C_\lambda(R_2)$.

Proof. We only give the proof for the case $\lambda = \ell$, as the other cases can be proved analogously.

(i) Suppose that $R_1 \subseteq R_2$. Let $(x, y, z) \in C_\ell(R_1)$, then it holds that $(y, z) \in R_1$. Since $R_1 \subseteq R_2$, it follows that $(y, z) \in R_2$. Hence, $(x, y, z) \in C_\ell(R_2)$.

(ii) We easily verify that

$$\begin{aligned} C_\ell(R_1 \cap R_2) &= \{(x, y, z) \in X^3 \mid (y, z) \in R_1 \cap R_2\} \\ &= \{(x, y, z) \in X^3 \mid (y, z) \in R_1 \wedge (y, z) \in R_2\} \\ &= \{(x, y, z) \in X^3 \mid (x, y, z) \in C_\ell(R_1) \wedge (x, y, z) \in C_\ell(R_2)\} \\ &= C_\ell(R_1) \cap C_\ell(R_2). \end{aligned}$$

(iii) The proof is analogous to that of (ii).

□

The following proposition shows that any binary relation coincides with the binary projections of its cylindrical extensions and any ternary relation is included in the cylindrical extensions of its projections.

Proposition 6.3. Let T be a ternary relation and R be a binary relation on a set X. For any $\lambda \in \{\ell, m, r\}$, the following statements hold:

(i) $R = P_\lambda(C_\lambda(R))$;

(ii) $T \subseteq C_\lambda(P_\lambda(T))$.

Proof. We only give the proof for the case $\lambda = \ell$, as the other cases can be proved analogously.

(i) We easily verify that

$$P_\ell(C_\ell(R)) = \{(x, y) \in X^2 \mid (\exists z \in X)((z, x, y) \in C_\ell(R))\} = R\,.$$

(ii) Let $(x, y, z) \in T$, then it holds that $(y, z) \in P_\ell(T)$. Hence, $(x, y, z) \in C_\ell(P_\ell(T))$. Thus, $T \subseteq C_\ell(P_\ell(T))\,.$

□

Remark 6.2. The following example shows that in Proposition 6.3 (ii), the equality does not hold in general. Indeed, let T be the ternary relation on $X = \{x_1, x_2, x_3, x_4\}$ given by:

$$T = \{(x_1, x_1, x_2), (x_1, x_2, x_3)\}\,.$$

It holds that

$$P_\ell(T) = \{(x_1, x_2), (x_2, x_3)\}\,,$$

and thus

$$C_\ell(P_\ell(T)) = \{(x_1, x_1, x_2), (x_2, x_1, x_2), (x_3, x_1, x_2), (x_4, x_1, x_2),$$

$$(x_1, x_2, x_3), (x_2, x_2, x_3), (x_3, x_2, x_3), (x_4, x_2, x_3)\}\,.$$

It is clear that $C_\ell(P_\ell(T)) \nsubseteq T$.

The following proposition expresses the compositions of a ternary relation with a binary relation introduced in [38] in terms of the six compositions of ternary relations introduced in this paper.

Proposition 6.4. Let T be a ternary relation and R be a binary relation on a set X. The following equalities hold:

(i) $T \ltimes R = T \circ_1 C_\ell(R)$;

(ii) $T \ltimes R = T \circ_2 C_m(R)$;

(iii) $T \ltimes R = T \circ_3 C_r(R)$;

(iv) $R \rtimes T = C_\ell(R) \circ_4 T$;

(v) $R \rtimes T = C_m(R) \circ_5 T$;

(vi) $R \rtimes T = C_r(R) \circ_6 T$.

P r o o f. We only give the proof for the first equality, as the other equalities can be proved analogously.

$$\begin{aligned} T \ltimes R &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((x,y,t) \in T \wedge (t,z) \in R)\} \\ &= \{(x,y,z) \in X^3 \mid (\exists t, s \in X)((x,y,t) \in T \wedge (s,t,z) \in C_\ell(R))\} \\ &= T \circ_1 C_\ell(R)\,. \end{aligned}$$

□

The following proposition investigates the cylindrical extensions of the composition of binary relations in terms of the compositions of their cylindrical extensions.

Proposition 6.5. Let R_1 and R_2 be two binary relations on a set X. The left, middle and right cylindrical extensions of the composition $R_1 \circ R_2$ are listed in the following table:

Cyl. ext. / Comp.	$C_\ell(\cdot)$	$C_m(\cdot)$	$C_r(\cdot)$
$R_1 \circ R_2$	$C_\ell(R_1) \circ_1 C_\ell(R_2)$ $C_\ell(R_1) \circ_2 C_m(R_2)$ $C_\ell(R_1) \circ_3 C_r(R_2)$	$C_m(R_1) \circ_1 C_\ell(R_2)$ $C_m(R_1) \circ_2 C_m(R_2)$ $C_m(R_1) \circ_3 C_r(R_2)$ $C_\ell(R_1) \circ_4 C_m(R_2)$ $C_m(R_1) \circ_5 C_m(R_2)$ $C_r(R_1) \circ_6 C_m(R_2)$	$C_\ell(R_1) \circ_4 C_r(R_2)$ $C_m(R_1) \circ_5 C_r(R_2)$ $C_r(R_1) \circ_6 C_r(R_2)$

P r o o f. We only prove that $C_\ell(R_1 \circ R_2) = C_\ell(R_1) \circ_1 C_\ell(R_2)$, as the other cases can be proved analogously.

$$\begin{aligned} C_\ell(R_1 \circ R_2) &= \{(x,y,z) \in X^3 \mid (y,z) \in R_1 \circ R_2\} \\ &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((y,t) \in R_1 \wedge (t,z) \in R_2)\} \\ &= \{(x,y,z) \in X^3 \mid (\exists t, s \in X)((x,y,t) \in C_\ell(R_1) \wedge (s,t,z) \in C_\ell(R_2))\} \\ &= \{(x,y,z) \in X^3 \mid (x,y,z) \in C_\ell(R_1) \circ_1 C_\ell(R_2)\} \\ &= C_\ell(R_1) \circ_1 C_\ell(R_2)\,. \end{aligned}$$

□

Remark 6.3. The following example shows that the left cylindrical extension of the composition of two binary relations is not equal to the $\circ_4$-composition of any of their cylindrical extensions. Indeed, let R_1 and R_2 be the binary relations on $X = \{x_1, x_2, x_3\}$ given by:

$$R_1 = \{(x_1, x_2)\}\,,$$
$$R_2 = \{(x_2, x_2), (x_3, x_1)\}\,.$$

It holds that

$$R_1 \circ R_2 = \{(x_1, x_2)\}\,,$$

and thus

$$C_\ell(R_1 \circ R_2) = \{(x_1, x_1, x_2), (x_2, x_1, x_2), (x_3, x_1, x_2)\}\,.$$

Further, it holds that

	$C_\ell(\cdot)$	$C_m(\cdot)$	$C_r(\cdot)$
R_1	$\{(x_1, x_1, x_2), (x_2, x_1, x_2), (x_3, x_1, x_2)\}$	$\{(x_1, x_1, x_2), (x_1, x_2, x_2), (x_1, x_3, x_2)\}$	$\{(x_1, x_2, x_1), (x_1, x_2, x_2), (x_1, x_2, x_3)\}$
R_2	$\{(x_1, x_2, x_2), (x_1, x_3, x_1), (x_2, x_2, x_2), (x_2, x_3, x_1), (x_3, x_2, x_2), (x_3, x_3, x_1)\}$	$\{(x_2, x_1, x_2), (x_2, x_2, x_2), (x_2, x_3, x_2), (x_3, x_1, x_1), (x_3, x_2, x_1), (x_3, x_3, x_1)\}$	$\{(x_2, x_2, x_1), (x_2, x_2, x_2), (x_2, x_2, x_3), (x_3, x_1, x_1), (x_3, x_1, x_2), (x_3, x_1, x_3)\}$

and

$\circ_4$	$C_\ell(R_2)$	$C_m(R_2)$	$C_r(R_2)$
$C_\ell(R_1)$	$\{(x_1, x_2, x_2), (x_1, x_3, x_1)\}$	$\{(x_1, x_1, x_2), (x_1, x_2, x_2), (x_1, x_3, x_2)\}$	$\{(x_1, x_2, x_1), (x_1, x_2, x_2), (x_1, x_2, x_3)\}$
$C_m(R_1)$	$\{(x_1, x_2, x_2), (x_1, x_3, x_1), (x_2, x_2, x_2), (x_2, x_3, x_1), (x_3, x_2, x_2), (x_3, x_3, x_1)\}$	$\{(x_1, x_1, x_2), (x_1, x_2, x_2), (x_1, x_3, x_2), (x_2, x_1, x_2), (x_2, x_2, x_2), (x_2, x_3, x_2), (x_3, x_1, x_2), (x_3, x_2, x_2), (x_3, x_3, x_2)\}$	$\{(x_1, x_2, x_1), (x_1, x_2, x_2), (x_1, x_2, x_3), (x_2, x_2, x_1), (x_2, x_2, x_2), (x_2, x_2, x_3), (x_3, x_2, x_1), (x_3, x_2, x_2), (x_3, x_2, x_3)\}$
$C_r(R_1)$	$\{(x_2, x_2, x_2), (x_2, x_3, x_1)\}$	$\{(x_2, x_1, x_1), (x_2, x_1, x_2), (x_2, x_2, x_1), (x_2, x_2, x_2), (x_2, x_3, x_1), (x_2, x_3, x_2)\}$	$\{(x_2, x_1, x_1), (x_2, x_1, x_2), (x_2, x_1, x_3), (x_2, x_2, x_1), (x_2, x_2, x_2), (x_2, x_2, x_3)\}$

It is clear that for any $\lambda_1, \lambda_2 \in \{\ell, m, r\}$ it holds that

$$C_\ell(R_1 \circ R_2) \neq C_{\lambda_1}(R_1) \circ_4 C_{\lambda_2}(R_2)\,.$$

In a similar way, one can easily prove the other cases.

7. INTERACTION OF THE COMPOSITIONS WITH THE TRACES

In this section, we study the interaction of the compositions of ternary relations with the left, middle and right traces. First, we recall the notions of traces of a ternary relation introduced in [38].

Definition 7.1. (Zedam et al. [38]) Let T be a ternary relation on a set X.

(i) The left trace of T is the binary relation T^ℓ on X defined as

$$T^\ell = \{(x,y) \in X^2 \mid (\forall (a,b) \in X^2)((x,a,b) \in T \Rightarrow (y,a,b) \in T)\}\,;$$

(ii) The middle trace of T is the binary relation T^m on X defined as

$$T^m = \{(x,y) \in X^2 \mid (\forall (a,b) \in X^2)((a,x,b) \in T \Rightarrow (a,y,b) \in T)\}\,;$$

(iii) The right trace of T is the binary relation T^r on X defined as

$$T^r = \{(x,y) \in X^2 \mid (\forall (a,b) \in X^2)((a,b,x) \in T \Rightarrow (a,b,y) \in T)\}\,.$$

The following proposition shows the interaction of the compositions of ternary relations with the left, middle and right traces.

Proposition 7.1. Let T and S be two ternary relations on a set X. The following inclusions hold:

(i) $T^\ell \subseteq (T \circ_i S)^\ell$, for any $i \in \{1,2,3,5,6\}$;

(ii) $T^m \subseteq (T \circ_i S)^m$, for any $i \in \{1,2,3\}$;

(iii) $S^m \subseteq (T \circ_i S)^m$, for any $i \in \{4,5,6\}$;

(iv) $T^r \subseteq (T \circ_i S)^r$, for any $i \in \{1,2,4,5,6\}$.

Proof. We only give the proof for first inclusion and $i = 1$, as the other inclusions can be proved analogously. Let $(x,y) \in T^\ell$ and $(x,a,b) \in T \circ_1 S$. Then there exists $t, s \in X$ such that $(x,a,t) \in T$ and $(s,t,b) \in S$. Since $(x,y) \in T^\ell$, it follows that $(y,a,t) \in T$. This implies that $(y,a,b) \in T \circ_1 S$. Hence, $(x,y) \in (T \circ_1 S)^\ell$. Thus, $T^\ell \subseteq (T \circ_1 S)^\ell$. $\square$

Next, we will show that the traces of a ternary relation are the greatest binary relations that satisfy some relational inclusions. First, we need to recall the following two theorems.

Theorem 7.1. (Zedam et al. [38]) Let T be a ternary relation on a set X. It holds that

(i) T^ℓ is the greatest binary relation R that satisfies $R^t \rtimes T \subseteq T$;

(ii) T^r is the greatest binary relation R that satisfies $T \ltimes R \subseteq T$.

Theorem 7.2. (Zedam et al. [38]) Let T be a ternary relation on a set X. It holds that

$$(T^\ell)^t \rtimes T = T \ltimes T^r = T\,.$$

The following result shows that the traces of a ternary relation are the greatest binary relations that satisfy the following relational inclusions corresponding to the relational compositions introduced above.

Proposition 7.2. Let T be a ternary relation on a set X. It holds that

(i) T^ℓ is the greatest binary relation R that satisfies the following inclusions:

(a) $C_\ell(R^t) \circ_4 T \subseteq T$;

(b) $C_m(R^t) \circ_5 T \subseteq T$;

(c) $C_r(R^t) \circ_6 T \subseteq T$.

(ii) T^r is the greatest binary relation R that satisfies the following inclusions:

(a) $T \circ_1 C_\ell(R) \subseteq T$;

(b) $T \circ_2 C_m(R) \subseteq T$;

(c) $T \circ_3 C_r(R) \subseteq T$.

Proof. We only give the proof for the inclusion (i) (a), as the other inclusions can be proved analogously. Note that Proposition 6.4 implies that $C_\ell(R^t) \circ_4 T = R^t \rtimes T$. Theorem 7.1 guarantees that T^ℓ is the greatest binary relation R that satisfies $R^t \rtimes T \subseteq R$. Hence, T^ℓ is the greatest binary relation R that satisfies $C_\ell(R^t) \circ_4 T \subseteq T$. $\square$

Combining Proposition 6.4 and Theorem 7.2 leads to the following result.

Corollary 7.1. Let T be a ternary relation on a set X. The following equalities hold:

(i) $T \circ_1 C_\ell(T^r) = T$;

(ii) $T \circ_2 C_m(T^r) = T$;

(iii) $T \circ_3 C_r(T^r) = T$;

(iv) $C_\ell((T^\ell)^t) \circ_4 T = T$;

(v) $C_m((T^\ell)^t) \circ_5 T = T$;

(vi) $C_r((T^\ell)^t) \circ_6 T = T$.

Along the same lines of Proposition 7.2, we obtain the following equivalences.

Proposition 7.3. Let T and S be two ternary relations on a set X. The following equivalences hold:

(i) $T \circ_1 S \subseteq T$ if and only if $S \subseteq C_\ell(T^r)$;

(ii) $T \circ_2 S \subseteq T$ if and only if $S \subseteq C_m(T^r)$;

(iii) $T \circ_3 S \subseteq T$ if and only if $S \subseteq C_r(T^r)$;

(iv) $S \circ_4 T \subseteq T$ if and only if $S \subseteq C_\ell((T^\ell)^t)$;

(v) $S \circ_5 T \subseteq T$ if and only if $S \subseteq C_m((T^\ell)^t)$;

(vi) $S \circ_6 T \subseteq T$ if and only if $S \subseteq C_r((T^\ell)^t)$.

P r o o f . We only prove the first equivalence, as the other equivalences can be proved analogously. Suppose that $T \circ_1 S \subseteq T$, then we need to prove that $S \subseteq C_\ell(T^r)$. From Proposition 6.4, it follows that $T \circ_1 S = T \circ_1 C_\ell(P_\ell(S))$. Since $T \circ_1 S \subseteq T$, it follows that $T \circ_1 C_\ell(P_\ell(S)) \subseteq T$. We know from Proposition 7.2 (ii) that T^r is the greatest binary relation R that satisfies $T \circ_1 C_\ell(R) \subseteq T$, and, hence, $P_\ell(S) \subseteq T^r$, which implies that $C_\ell(P_\ell(S)) \subseteq C_\ell(T^r)$. Proposition 6.3 guarantees that $S \subseteq C_\ell(P_\ell(S))$. Thus, $S \subseteq C_\ell(T^r)$. Conversely, assume that $S \subseteq C_\ell(T^r)$. From Proposition 5.1, it follows that $T \circ_1 S \subseteq T \circ_1 C_\ell(T^r)$. Proposition 7.2 guarantees that $T \circ_1 C_\ell(T^r) \subseteq T$. Since $T \circ_1 S \subseteq T \circ_1 C_\ell(T^r)$ and $T \circ_1 C_\ell(T^r) \subseteq T$, it follows that $T \circ_1 S \subseteq T$. □

8. CONCLUSION

In this work, we have extended the composition of binary relations to the setting of ternary relations. More specifically, we have introduced six types of composition of ternary relations based on the composition of a ternary relation with a binary relation and vice versa, and we have investigated their properties in detail. Moreover, we have studied the interaction of these compositions with the binary projections of ternary relations, cylindrical extensions of binary relations and traces of ternary relations. We anticipate that these compositions of ternary relations will facilitate our future study of the different notions of transitivity of a ternary relation, among others. Given the importance of fuzzy relations, as amply illustrated in the introduction for binary relations, future efforts will be directed to the study of fuzzy ternary relations as well.

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