A full discretization for the saddle-point approach of a degenerate parabolic problem involving a moving body

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Abstract

This paper aims to study a degenerate parabolic problem for a solenoidal vector field in which the time derivative acts on a moving body. We propose a fully-discrete finite element scheme combined with backward Euler’s method for the saddle-point variational formulation. The convergence of this numerical scheme is proved and error estimates for some stable finite element pairs are also established.

Keywords: Degenerate parabolic problem, moving body, saddle-point formulation, finite element method

2010 MSC: 35K65, 65M60

1. Introduction

Let $\Omega$ be an open bounded connected polyhedral and Lipschitz domain in $\mathbb{R}^d$, with $d = 2$ or 3. Inside the domain $\Omega$, a body $\Sigma_0 \in C^{1,1}$ is considered, which occupies different regions during the time interval $[0, T]$. We refer to $\Sigma_0$ as the reference configuration and describe the motion of $\Sigma_0$ by a $C^3$-function $\Phi: \Sigma_0 \times [0, T] \to \mathbb{R}^d$, where $\Phi_t := \Phi(\cdot, t)$ for each $t \in [0, T]$ is a deformation of $\Sigma_0$ to $\Sigma(t) := \Phi(\Sigma_0, t)$, which is the space occupied by $\Sigma_0$ at time $t$, cf. [1]. We make the following assumptions throughout the article

$$\tilde{\Sigma} := \bigcup_{\tau \in [0, T]} \Sigma(\tau) \subset \Omega; \quad \det \nabla \Phi(x, t) > 0, \quad \forall (x, t) \in \Sigma_0 \times [0, T].$$

The trajectory of the motion, which is a subset of the space-time domain $Q := \Omega \times (0, T)$, is specified by $T := \{ (x, t) : x \in \Sigma(t), t \in [0, T] \}$. Since $\Phi_t$ is a bijective mapping for each $t$, the velocity vector of the moving body is defined by $v(x, t) := \Phi(\Phi_t^{-1}(x), t)$. From now on, we assume that there exists an extension of $v$ from $T$ to $Q$ such that $v \in C^1(\overline{Q})$. We denote further by $n$ the unit outward normal vector associated to the boundary of $\Omega$ and $\Sigma(t)$.

In this paper, we aim to investigate the following initial-boundary value problem for the solenoidal vector field $u$

$$\begin{align*}
\alpha \partial_t u - \beta \Delta u + \chi_{\Sigma} Au &= f & \text{in } \Omega \times (0, T), \\
\nabla \cdot u &= 0 & \text{in } \Omega \times (0, T), \\
u &= 0 & \text{on } \partial \Omega \times (0, T), \\
\| (\nabla u)n \| &= 0 & \text{on } \partial \Sigma \times (0, T), \\
u(\cdot, 0) &= u_0 & \text{in } \Sigma_0.
\end{align*}$$

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Preprint submitted to Applied Mathematics Letters March 29, 2022
where \( u(x,t) = \left[u'(x,t)\right]_{i=1}^d \), \( \beta > 0 \) is a constant and \( f \in \text{Lip}([0,T],L^2(\Omega)) \). Moreover, \( \alpha \) and \( A \) are defined by

\[
\alpha(t) = \alpha\chi_{\Sigma(t)} = \begin{cases} \alpha_1 > 0 & \text{in } \Sigma(t) \\ 0 & \text{in } \Omega \setminus \Sigma(t) \end{cases}, \quad A(x,t)u(x,t) = \left[ \sum_{j,k=1}^d a^{jk}(x,t) \frac{\partial u'(x,t)}{\partial x^j} + \sum_{i=1}^d b^i(x,t)u^i(x,t) \right]_{i=1}^d,
\]
where \( a^{jk}, b^i \in \text{Lip}([0,T],L^\infty(\Omega)) \) with \( i, j, k = 1, \ldots, d \). For example, the operator \( A \) can play the role of the convection, i.e. \( Au = (\nu \cdot \nabla)u \), which is a part of the material derivative \( \frac{Du}{Dt} = \partial_t u + (\nu \cdot \nabla)u \).

The system (2) is a degenerate parabolic problem. Moreover, it can be achieved from two different problems separately, namely a parabolic one on the trajectory \( T \) where \( \alpha > 0 \) and an elliptic one otherwise, which are combined via the interface condition. Therefore, (2) is also called a parabolic-elliptic problem. The given initial guest \( u_0 \) is supposed to satisfy that \( u \in H^2(\Sigma_0) \). If \( \Sigma_0 \) is of the class \( C^{1,1} \), there exists an extension of \( u_0 \) such that \( u_0 \in H^1_0(\Omega) \cap H^2(\Omega) \) and \( \nabla \cdot u_0 = 0 \) in \( \Omega \), see [2, Proposition 4.1].

The motivation of studying the system (2) comes from the eddy current model describing an electromagnetic problem with a moving non-magnetic conductor considered in [3, 4, 5]. More specifically, the paper [4] introduces a time discretization for the saddle-point formulation of a degenerate parabolic problem for the divergence-free vector potential. In this setting, \( \alpha \) and \( \beta \) stand for the electrical conductivity and the magnetic permeability constant of vacuum, respectively. The operator \( A \) is defined as \( Au = \sigma \nu \times (\nabla \times u) \). A full space-time discretization of this problem has not been discussed yet. In [5], we propose a fully-discrete finite element scheme by incorporating a penalty term (Coulomb gauge) into the governing partial differential equation (PDE). However, the purpose of this paper is to introduce a fully-discrete finite element scheme for the saddle-point formulation of (2).

Let us mention also some other relevant recent results to the governing problem (2). In the paper [6], the author studied regularity of the solution to a parabolic-elliptic problem with moving parabolic subdomain, which was also motivated by an eddy current model with moving conductors. Nevertheless, the divergence-free condition and the convection-type term arising from the movement of conductors were not taken into account. The goal of [7] is to present an abstract framework for analyzing a family of linear degenerate parabolic mixed equations, then the paper [8] aims at introducing a fully-discrete approximation for this kind of problems. As stated in [8], the discrete inf-sup condition plays an important role for finite element analysis of the mixed problems, which allowed the authors to get quasi-optimal error estimate \( O(\sqrt{\tau} + h/\sqrt{\tau}) \). However, in these articles, all concerned domain and subdomains were fixed during the time process.

In the present paper, we propose a full discretization based on the finite element scheme and the backward Euler method for the variational formulation, see Section 3. The discrete inf-sup condition required for the existence of a discrete solution to the saddle-point approach together with handling terms acting on the moving body makes it challenging to establish an error estimate (with independent \( h \) and \( \tau \)) for this numerical scheme, which are the highlights of this contribution. In the future, we aim to study the stability and to establish error estimates for the full discretization of the problem (2) with a jumping (non-Lipschitz) coefficient \( \beta \), which still remains as a challenge at the moment. In the next section, we derive the mixed variational formulation for the degenerate parabolic problem (2).

### 2. Variational formulation

By means of the saddle-point approach, the variational formulation of the system (2) reads as follows:

Find \( u(t) \in H^1_0(\Omega) \) with \( \partial_t u(t) \in L^2(\Sigma(t)) \) and \( p(t) \in L^2(\Omega) \) such that for a.a. \( t \in (0,T) \), it holds that

\[
\alpha_1 \left( \partial_t u(t), \varphi \right)_{\Sigma(t)} + \beta (\nabla u(t), \nabla \varphi)_{\Omega} + (A(t)u(t), \varphi)_{\Sigma(t)} + (p(t), \nabla \cdot \varphi)_{\Omega} = (f(t), \varphi)_{\Omega} \quad \forall \varphi \in H^1_0(\Omega),
\]

\[
(\nabla \cdot u(t), q)_{\Omega} = 0 \quad \forall q \in L^2_0(\Omega).
\]

Please note that the additional unknown \( p \) plays the role of the divergence of \( u \) (see [4] for more details on the interpretation of \( p \)). Since \( p(t) \in L^2(\Omega) \), the inf-sup condition is satisfied following from [9, Theorem 5.1 on p. 80]. Throughout this paper, we consider the following subspace of \( H^1_0(\Omega) \):

\[
H^1_0(\text{div}) = \left\{ \varphi \in H^1_0(\Omega) : \nabla \cdot \varphi = 0 \right\}.
\]
The well-posedness of the mixed variational problem (3-4) can be proved by performing similar arguments as presented in [10, Theorem 5.1]. Thus, we can obtain the following result without the proof.

**Theorem 2.1** (Well-posedness). Let \( u_0 \in H^1_0(\Omega) \cap H^2_0(\Sigma) \) satisfying \( \Delta u_0 = 0 \) on \( \Omega \setminus \Sigma \), \( v \in C^1(\overline{\Omega}) \) and \( f \in \text{Lip}([0,T], L^2(\Omega)) \). Moreover, we assume that \( d^i \in \text{Lip}([0,T], L^{\infty}(\Omega)) \) with \( i,j,k = 1,\ldots,d \). Then the system (3-4) admits exactly one solution \((u,p)\) satisfying \( p \in L^2((0,T), L^2_0(\Omega)) \) and \( u \in C([0,T], H^1_0(\Omega)) \) with \( \partial_t u \in L^2((0,T), H^1_0(\Omega)) \).

The following local regularity result provided by [11, Theorem 8.8] will play a crucial role for the error estimate of the full discretization scheme in the next section.

**Corollary 2.1.** Let the assumptions of Theorem 2.1 be fulfilled. Then for any subdomain \( \Sigma' \subset \subset \Omega \) (i.e. \( \Sigma' \subset \Omega \)), we have \( u \in L^2((0,T), H^2(\Sigma')) \).

We mention here the well-known Reynolds transport theorem, which will be helpful for further analysis of PDEs with time-dependent domains. Let \( \omega(t) \) be a Lipschitz moving body whose velocity vector \( v \) is of class \( C^1 \) and \( f \) an abstract function satisfying \( f(t) \in W^{1,1}(\omega(t)) \) and \( \partial_t f(t) \in L^1(\omega(t)) \) for all \( t \in (0,T) \). Then it holds that

\[
\frac{d}{dt} \int_{\omega(t)} f \, dx = \int_{\partial \omega(t)} \partial_t f \, d\omega + \int_{\omega(t)} f \cdot \nabla \omega \, dx.
\]

Let \( \mathbf{V}_0^h \) and \( V^h \) be two finite-dimensional subspaces of \( H^1_0(\Omega) \) and \( L^2_0(\Omega) \), respectively. These spaces are equipped with two orthogonal projection operators \( P^h \in L(H^1_0(\Omega), \mathbf{V}_0^h) \) and \( P^h \in L(L^2_0(\Omega), V^h) \). The time interval \([0,T]\) is divided into \( n \) equidistant subintervals with length \( \tau = \frac{T}{n} \). The fully-discrete approximations of \( u \) and \( p \) at time \( t_i = i\tau \) (\( 0 \leq i \leq n \)) are denoted by \( u^h_i \) and \( p^h_i \), respectively. We also introduce the following notations

\[
\delta u^h_i = \frac{u^h_i - u^h_{i-1}}{\tau}, \quad u^h_0 = P^h u_0, \quad A_i = A(t_i), \quad \Sigma_i = \Sigma(t_i).
\]

The full discretization of the mixed variational formulation (3-4) is defined as:

Find \( u^h_i \in V^h_i \) and \( p^h_i \in V^h_i \) such that for any \( i = 1,2,\ldots,n \), it holds that

\[
\alpha \left( \delta u^h_0, \varphi_i \right) + \beta \left( \nabla u^h_i, \nabla \varphi_i \right) + \left( A_i u^h_i, \varphi_i \right) + \left( \partial_t \varphi_i, \varphi_i \right)_{\Omega} = \left( f_i, \varphi_i \right)_{\Omega}, \quad \forall \varphi_i \in V^h_i, \quad (6)
\]

\[
\left( \nabla \cdot u^h_i, q_i^h \right) = 0, \quad \forall q_i^h \in V^h_i. \quad (7)
\]

The solvability of the system (6-7) on every time step follows from the Brezzi theorem, cf. [12, Corollary 1.1].

**Lemma 3.1** (Solvability). Let the assumptions of Theorem 2.1 be fulfilled. Moreover, we assume that the discrete inf-sup condition is satisfied, i.e. there exists a constant \( C > 0 \) such that

\[
\sup_{\varphi_i \in V^h_i, \varphi_i \neq 0} \frac{\left( \nabla \cdot \varphi_i, q_i^h \right)_{\Omega}}{\|\varphi_i\|_{H^1(\Omega)}} \geq C \left\| q_i^h \right\|_{L^2(\Omega)} \quad \forall q_i^h \in V^h_i. \quad (8)
\]

Then, for any \( i = 1,2,\ldots,n \) and any \( \tau < \tau_0 \), there exists a unique couple \((u^h_i, p^h_i)\) in \( V^h_i \times V^h_i \) solving (6-7).

The following basic a priori estimate for iterates can be obtained in the same way as in [4, Lemma 4.3]. This estimate is obtained in the main result of this paper.

**Lemma 3.2** (A priori estimate). Let the assumptions of Lemma 3.1 be fulfilled. In addition, we assume that \( u^h_0 \) solves the equation (7). Then there exists a constant \( C > 0 \) such that the following relation holds true for any \( \tau < \tau_0 \)

\[
\max_{1 \leq i \leq n} \left\| \Delta u^h_i \right\|_{L^2(\Omega)}^2 + \sum_{i=1}^n \left\| \nabla \Delta u^h_i \right\|_{L^2(\Omega)}^2 \tau + \sum_{i=1}^n \left\| \Delta u^h_i - \Delta u^h_{i-1} \right\|_{L^2(\Sigma_i-1)}^2 + \max_{1 \leq i \leq n} \left\| p^h_i \right\|_{L^2(\Omega)}^2 \leq C. \quad (9)
\]
We define the following piecewise-constant and piecewise-affine in time functions, operator and domain
\[ \overline{u}_n^h(t) = u_i^h, \quad u_i^h(t) = u_{(i-1)}^h + (t - t_{i-1}) \delta u_i^h, \quad \overline{p}_n^h(t) = p_i^h, \quad \overline{f}_n(t) = f_i, \quad \overline{A}_n(t) = A_i, \quad \overline{\Sigma}_n(t) = \Sigma_i, \]
for every \( t \in (t_{i-1}, t_i], 1 \leq i \leq n \), with the initial data
\[ \overline{u}_n^h(0) = u_i^h(0) = u_0, \quad \overline{p}_n^h(0) = 0, \quad \overline{f}_n(0) = f(0), \quad \overline{A}_n(0) = A(0), \quad \overline{\Sigma}_n(0) = \Sigma_0. \]

The following relation between Rothe’s functions \( \overline{u}_n^h \) and \( u_i^h \) comes from the a priori estimate (9):
\[
\int_0^T \| \nabla \overline{u}_n^h(t) - \nabla u_i^h(t) \|_{L^2(\Omega)}^2 \, dt + \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \| \nabla \overline{u}_n^h(t) - \nabla u_i^h(t) \|_{L^2(\Omega)}^2 \, dt \leq \sum_{i=1}^{n} \| \nabla \partial_t u_i^h(t) \|_{L^2(\Omega)}^2 \leq \epsilon^2. \tag{10}
\]

Hence, the equations (6) and (7) can be rewritten in the following form
\[
\alpha \left( \partial_t u_i^h(t), \varphi \right)_{\xi_i(t)} + \beta \left( \nabla \overline{u}_n^h(t), \nabla \varphi \right)_{\Omega} + \left( \overline{A}_n(t) \overline{u}_n^h(t), \varphi \right)_{\xi_i(t)} + \left( \overline{p}_n^h(t), \nabla \cdot \varphi \right)_{\Omega} = \left( \overline{f}_n(t), \varphi \right)_{\Omega} \quad \forall \varphi \in V_0. \tag{11}
\]
\[
\left( \nabla \cdot \overline{u}_n^h(t), q^h \right)_{\Omega} = 0 \quad \forall q^h \in V^h. \tag{12}
\]

Now, we are in the position to investigate the convergence of the full discretization scheme.

**Theorem 3.1.** Let the assumptions of Lemma 3.2 be fulfilled. Then there exists a constant \( C > 0 \) such that the following relation holds true for every \( \xi \in [0, T] \)
\[
\int_0^\xi \| \partial_t u_i^h(t) - \partial_t u(t) \|_{L^2(\Omega)}^2 \, dt + \int_0^\xi \| \nabla \overline{u}_n^h(t) - \nabla u(t) \|_{L^2(\Omega)}^2 \, dt + \int_0^\xi \| \overline{p}_n^h(t) - p(t) \|_{L^2(\Omega)}^2 \, dt \\
\leq C \tau + \int_0^\xi \| \nabla \partial_t u(t) - \nabla P^h \partial_t u(t) \|_{L^2(\Omega)}^2 \, dt + \int_0^\xi \| \nabla \partial_t u(t) - \nabla P^h \partial_t u(t) \|_{L^2(\Omega)}^2 \, dt \leq C \tau + \int_0^\xi \| \nabla \partial_t u(t) - \nabla P^h \partial_t u(t) \|_{L^2(\Omega)}^2 \, dt. \tag{13}
\]

**Proof.** Subtracting (3) for \( \varphi = \varphi^h \) from (11), then rewriting the result by the Reynolds transport theorem (5), we get for a.a. \( t \in (0, T) \) that
\[
\alpha \left( \partial_t u_i^h(t) - \partial_t u(t), \varphi \right)_{\xi_i(t)} + \beta \left( \nabla \overline{u}_n^h(t) - \nabla u(t), \nabla \varphi \right)_{\Omega} + \left( \overline{A}_n(t) \overline{u}_n^h(t), \varphi \right)_{\xi_i(t)} + \left( \overline{p}_n^h(t), \nabla \cdot \varphi \right)_{\Omega} + \left( \overline{A}_n(t) \overline{u}_n^h(t), \varphi \right)_{\xi_i(t)} + \left( \overline{p}_n^h(t), \nabla \cdot \varphi \right)_{\Omega} = \left( \overline{f}_n(t) - f(t), \varphi \right)_{\Omega}, \tag{14}
\]
where \( \tau = \int_0^\xi \tau \). Setting \( \varphi^h = \partial_t u_i^h(t) - P^h \partial_t u(t) \) in (14), then integrating in time over \( (0, \xi) \subset (0, T) \) and rearranging the result give us that
\[
\alpha \int_0^\xi \| \partial_t u_i^h(t) - \partial_t u(t) \|_{L^2(\Omega)}^2 \, dt + \frac{\beta}{2} \| \nabla \overline{u}_n^h(t) - \nabla u(t) \|_{L^2(\Omega)}^2 - \frac{\beta}{2} \| \nabla \partial_t u(t) - \nabla u(t) \|_{L^2(\Omega)}^2 \\
= -\alpha \int_0^\xi \left( \partial_t u_i^h(t) - \partial_t u(t), \partial_t u(t) - P^h \partial_t u(t) \right)_{\xi_i(t)} \, dt + \int_0^\xi \left( \overline{f}_n(t) - f(t), \partial_t u_i^h(t) - P^h \partial_t u(t) \right)_{\Omega} \, dt \\
- \beta \int_0^\xi \left( \nabla \overline{u}_n^h(t) - \nabla u(t), \nabla \partial_t u(t) - \nabla P^h \partial_t u(t) \right)_{\Omega} \, dt - \beta \int_0^\xi \left( \nabla \overline{u}_n^h(t) - \nabla u(t), \nabla \partial_t u(t) - \nabla P^h \partial_t u(t) \right)_{\Omega} \, dt \]
\[-\int_0^\xi (\overline{p}_h^h(t) - p(t), \nabla \cdot \partial_t u_h^h(t) - \nabla \cdot \partial_t u(t))_{\Omega} \, dt - \int_0^\xi (\overline{p}_h^h(t) - p(t), \nabla \cdot \partial_t u(t) - \nabla \cdot P^h \partial_t u(t))_{\Omega} \, dt \]
\[-\int_0^\xi \left( (\overline{A}_h^h(t) - A(t)) \overline{u}_h^h(t), \partial_t u_h^h(t) - P^h \partial_t u(t) \right)_{\Sigma(t)} \, dt - \int_0^\xi \left( A(t) \overline{u}_h^h(t) - u(t), \partial_t u_h^h(t) - P^h \partial_t u(t) \right)_{\Sigma(t)} \, dt \]
\[-\int_0^\xi \int_0^{\xi_t} \int_{\partial \Sigma(\eta)} \left[ (\alpha \bar{\partial}_t u_h^h(t) + A(t)u(t)) \cdot (\bar{\partial}_t u_h^h(t) - P^h \partial_t u(t)) \right] (\nu \cdot \eta) \, d\sigma \, d\eta \, dt =: \sum_{i=1}^9 S_i. \]

The Cauchy-Schwarz and $\varepsilon$-Young inequalities are used to estimate $S_1$ and $S_3$ as follows:

$$|S_1| \leq \varepsilon \int_0^\xi \|\partial_t u_h^h(t) - \partial_t u(t)\|_{L^2(\Sigma(t))}^2 \, dt + C_\varepsilon \int_0^\xi \|\partial_t u(t) - P^h \partial_t u(t)\|_{L^2(\Omega)}^2 \, dt,$$

$$|S_3| \leq \int_0^\xi \|\nabla u_h^h(t) - \nabla u(t)\|_{L^2(\Omega)}^2 \, dt + \int_0^\xi \|\nabla \partial_t u_h^h(t) - \nabla P^h \partial_t u(t)\|_{L^2(\Omega)}^2 \, dt.$$

We invoke the properties of $f$ and $a^{ijk}, b^{ij}$ ($i, j, k = 1, \ldots, d$) together with Friedrichs’s inequality and (10) to obtain that

$$|S_2| + |S_4| + |S_7| \leq \int_0^\xi \left[ \int_0^\xi \|\nabla \partial_t u_h^h(t) - \nabla P^h \partial_t u(t)\|_{L^2(\Omega)}^2 \, dt \right]^{\frac{9}{2}}.$$

To estimate $S_8$, we need the following auxiliary estimate

$$\left[ \int_0^\xi \|\partial_t u_h^h(t) - \partial_t u(t)\|_{L^2(\Sigma(t))}^2 \, dt - \int_0^\xi \|\partial_t u_h^h(t) - \partial_t u(t)\|_{L^2(\Sigma(t))}^2 \, dt \right] \leq \int_0^\xi \left[ \int_0^{\xi_t} \int_{\partial \Sigma(\eta)} \left| \partial_t u_h^h(t) - \partial_t u(t) \right|^2 (\nu \cdot \eta) \, d\sigma \, d\eta \right] \leq \int_0^\xi \|\nabla \partial_t u_h^h(t) - \nabla \partial_t u(t)\|_{L^2(\Omega)}^2 \, dt \leq \tau.$$

Therefore, we arrive at

$$|S_8| \leq C_\varepsilon \tau + C_\varepsilon \int_0^\xi \|\nabla u_h^h(t) - \nabla u(t)\|_{L^2(\Omega)}^2 \, dt + C_\varepsilon \int_0^\xi \|\partial_t u_h^h(t) - \partial_t u(t)\|_{L^2(\Sigma(t))}^2 \, dt + C_\varepsilon \int_0^\xi \|\partial_t u(t) - P^h \partial_t u(t)\|_{L^2(\Omega)}^2 \, dt.$$

The most challenges lie on handling the terms $S_5, S_6$ and $S_9$, which arise from the saddle-point approach and the movement of the body $\Sigma_0$. Using the equations (4) and (12), we have that

$$|S_5| \leq \int_0^\xi \left( \overline{p}_h^h(t) - p(t), \nabla \cdot \partial_t u_h^h(t) \right)_{\Omega} \, dt \leq \int_0^\xi \left( p(t) - P^h p(t), \nabla \cdot \partial_t u_h^h(t) \right)_{\Omega} \, dt \leq \frac{1}{\varepsilon} \sup_{\varphi \in \mathcal{V}_h^p} \frac{\|\nabla \varphi \cdot \overline{p}_h^h(t) - P^h p(t)\|_{L^2(\Omega)}}{\|\varphi\|_{L^2(\Omega)}} \leq \int_0^\xi \left( \overline{p}_h^h(t) - p(t), \nabla \cdot \partial_t u_h^h(t) \right)_{\Omega} \, dt \leq \int_0^\xi \|p(t) - P^h p(t)\|_{L^2(\Omega)}^2 \, dt.$$
we denote by $P$ which together with Corollary 2.1 and Lemma 3.2 allow us to conclude that Céa's lemma and the standard interpolation error of the finite element spaces.

### 3.1. Error estimates for some finite element spaces

**Theorem 3.2.** Let the assumptions of Lemma 3.2 be fulfilled. Then the following convergences hold true:

$$
\|\theta_h(t) - \theta(t)\|_{L^2(\Omega)}^2 + \|\nabla \theta_h(t) - \nabla \theta(t)\|_{L^2(\Omega)}^2 + \|\nabla^2 \theta_h(t) - \nabla^2 \theta(t)\|_{L^2(\Omega)}^2 \leq C(\tau + h^2).
$$

The term $S_\|t\|$ can be handled by applying the local regularity on the weak solution $u$ presented in Corollary 2.1, i.e.

$$
|S_\|t\|| \lesssim \tau \int_0^\xi \int \left( \|\nabla \theta_h(t)\|_{L^2(\Omega)}^2 + \|\theta(t)\|_{H^2(\Omega)}^2 \right) \, dt + \int_0^\xi \left( \|\nabla \theta_h(t) - \nabla \theta(t)\|_{L^2(\Omega)}^2 + \|\theta(t) - \theta_h(t)\|_{H^2(\Omega)}^2 \right) \, dt \lesssim \tau.
$$

Finally, taking all considerations into account, then fixing a sufficiently small $\varepsilon > 0$ and using a Grönwall argument, we can achieve the desired estimate for $u_h$. The error estimate for $\tilde{P}_h$ can be acquired following the relation (15).

The convergence of the proposed full discretization scheme is a consequence of Theorem 3.1 and Céa’s lemma.

**Theorem 3.2.** Let the assumptions of Lemma 3.2 be fulfilled. Then the following convergences hold true: $\partial u_h \to \partial u$ in $L^2(\Omega)$, $u_h \to u$ in $C([0, T], H^1(\Omega))$ and $\tilde{P}_h \to p$ in $L^2((0, T), L^2(\Omega))$.

### 3.1. Error estimates for some finite element spaces

In this section, we examine the error estimate (13) for some finite element space pairs $(V_h, \mathbf{V}_h)$, which satisfy the discrete inf-sup condition (8). Let $(T^h)_{h>0}$ be a regular family of triangulations of $\Omega$. For each integer $k \geq 0$, we denote by $P_k$ the space of all polynomials of degree at most $k$. We introduce the following spaces of piecewise polynomials

$$
P_0 = \{ \varphi \in L^2(\Omega) : \varphi|_K \in P_0 \quad \forall K \in T \}, \quad P_k = \{ \varphi \in C(\Omega) : \varphi|_K \in P_k \quad (k \geq 1) \quad \forall K \in T \}.
$$

There are some stable finite element pairs satisfying the discrete inf-sup condition (8), see [13, Section VI.3]. Now, we present two simple examples, namely $(P_2 - P_0)$ for discontinuous element space of $p$ and the Taylor-Hood pair $(P_2 - P_1)$ for continuous elements. The following error estimates are the results of the relation (13) combined with Céa’s lemma and the standard interpolation error of the finite element spaces.

**Corollary 3.1.** Let the assumptions of Lemma 3.2 be fulfilled. Moreover, we assume that $V_h = (P_2)^d \cap H^1_0(\Omega)$ and $u \in C([0, T], H^1_0(\Omega)) \cap L^2((0, T), H^2(\Omega))$.

(i) If $V_h = P_0 \cap L^2_0(\Omega)$ and $p \in L^2((0, T), H^1(\Omega) \cap L^2_0(\Omega))$, then there exists a constant $C > 0$ such that

$$
\int_0^\xi \|\nabla^2 \theta_h(t) - \nabla^2 \theta(t)\|_{L^2(\Omega)}^2 \, dt + \|\nabla \theta_h(t) - \nabla \theta(t)\|_{L^2(\Omega)}^2 \leq C(\tau + h^2).
$$

(ii) If $V_h = P_1 \cap L^2_0(\Omega)$ and $p \in L^2((0, T), L^2(\Omega) \cap L^2_0(\Omega))$, then there exists a constant $C > 0$ such that

$$
\int_0^\xi \|\nabla^2 \theta_h(t) - \nabla^2 \theta(t)\|_{L^2(\Omega)}^2 \, dt + \|\nabla \theta_h(t) - \nabla \theta(t)\|_{L^2(\Omega)}^2 \leq C(\tau + h^2).
$$
Acknowledgments

Van Chien Le is supported by the grant number G016618N of the Fund for Scientific Research - Flanders. Karel Van Bockstal is supported by a postdoctoral fellowship of the Research Foundation - Flanders (106016/12P2919N).

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