TRACE CLASS AND HILBERT-SCHMIDT PSEUDO DIFFERENTIAL OPERATORS ON STEP TWO NILPOTENT LIE GROUPS

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Abstract. Let G be a step two nilpotent Lie group. In this paper, we give necessary and sufficient conditions on the operator valued symbols σ such that the associated pseudo-differential operators T_σ on G are in the class of Hilbert-Schmidt operators. As a key step to prove this, we define (μ,ν)-Weyl transform on G and derive a trace formula for (μ,ν)-Weyl transform with symbols in $L^2\left(\mathbb{R}^{2n}\right)$. We show that Hilbert-Schmidt pseudo-differential operators on $L^2(G)$ are same as Hilbert-Schmidt (μ,ν)-Weyl transform with symbol in $L^2(\mathbb{R}^{2n+r+k}\times\mathbb{R}^{2n+r+k})$. Further, we present a characterization of the trace class pseudo-differential operators on G and provide a trace formula for these trace class operators.

1. Introduction

The theory of pseudo-differential operators is one of the essential tools in modern contemporary mathematics. Pseudo-differential operators are widely used in harmonic analysis, PDE, geometry, mathematical physics, time-frequency analysis, imaging, and computations [14]. Kohn and Nirenberg [15] first introduced the theory of pseudo-differential operators and later used by Hörmander [14] for solving the problems in partial differential equations.

Let σ be a measurable function on $\mathbb{R}^n\times\mathbb{R}^n$. Then the (global) pseudo-differential operator T_σ associated with the symbol σ is defined by

$$\left(T_\sigma f\right)(x)=(2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{ix\cdot\xi}\sigma(x,\xi)\hat{f}(\xi)d\xi,\quad x\in\mathbb{R}^n$$

for all f in the Schwartz space $\mathcal{S}\left(\mathbb{R}^n\right)$ on $\mathbb{R}^n$, provided that the integral exists. Here $\hat{f}$ denotes the Euclidean Fourier transform of f and is defined by

$$\hat{f}(\xi)=(2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{-ix\cdot\xi}f(x)dx,\quad \xi\in\mathbb{R}^n.$$

The formation of a pseudo-differential operator is mainly based on the Fourier inversion formula given by

$$f(x)=(2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{ix\cdot\xi}\widehat{f}(\xi)d\xi,\quad x\in\mathbb{R}^n$$

for all f in $\mathcal{S}\left(\mathbb{R}^n\right)$. To define the pseudo-differential operators on other noncommutative groups, we first observe that $\mathbb{R}^n$ is a locally compact abelian group and its dual groups is also $\mathbb{R}^n$ and a pseudo-differential operators can be defined using the inverse Fourier transform on $\mathbb{R}^n$. These observations allow one to extend the definition of pseudo-differential operators to other noncommutative groups provided that we have an Fourier inversion formula for the Fourier transform on the groups. Using this idea, pseudo-differential operators on different classes of groups such as $\mathbb{S}^1,\mathbb{Z}$, finite abelian groups, locally compact abelian

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groups, affine groups, compact groups, compact Lie groups, homogeneous spaces of compact groups, Heisenberg group, and on general locally compact type I groups have been defined and studied broadly by several researchers. We refer to [2, 4, 12, 13, 16, 20, 30, 17, 18, 26, 27] and references therein.

Ruzhansky and Fischer developed the global theory of pseudo-differential operators on Heisenberg group, more generally on graded Lie groups [12, 13]. Also, Ruzhansky and Mantoiu investigated the global quantization on locally compact unimodular type I groups and on nilpotent Lie groups [21, 22]. Particularly in [21], the authors presented two approaches to justify the global quantization formula for unimodular type I groups. The first approach is based on the cross product C^*-algebra associated with certain C^*-dynamical systems and the second way based on the suitably defined Wigner transform and Weyl system. They introduced and studied the τ-quantization, where τ is any measurable function on a unimodular group. Further, using suitably defined Fourier-Wigner τ-transform, they explained quantization by Weyl system which actually coincides with τ-quantization. As an application of this quantization, they proved the belongingness of these pseudo-differential operators to the Schatten class, in particular, Hilbert-Schmidt class. Our quantization can be seen as a particular case of the quantization defined by Ruzhansky and Mantoiu [21].

Over the years, a considerable attention has been devoted by several researchers for finding the criteria for Schatten class of pseudo-differential operators in terms of symbols. Ruzhansky and Delgado investigated this in details in many different settings; for example, using the matrix-valued symbols on compact Lie groups in [7, 6, 10, 9] they successfully characterized these classes of operators on compact Lie groups (see also [19]). Later, they with their collaborators extended these results to compact manifolds and to more general on Hilbert spaces [8, 10] using the non-harmonic analysis developed by Ruzhansky and Tokmagambetov [28].

A well known-result in the theory of pseudo-differential operators on $\mathbb{R}^n$ is that if $\sigma \in L^2(\mathbb{R}^n \times \mathbb{R}^n)$ then T_σ is a bounded linear operator from $L^2(\mathbb{R}^n)$ into $L^2(\mathbb{R}^n)$. Furthermore, the resulting bounded linear operator T_σ is in Hilbert-Schimdt class as explained in [30]. In this direction, a characterization of trace class pseudo-differential operators on compact and Hausdorff groups and on homogeneous space of compact and Hausdorff group obtained in [23] and [16] respectively. Further, this result has been extended to non-compact non-abelian groups. Dasgupta and Wong in [3, 5] provided necessary and sufficient conditions on the symbols such that the corresponding pseudo-differential operators on the Heisenberg group are in Hilbert-Schmidt class. Mingkai and Jianxun [31] studied the properties of pseudo-differential on H-type group. Recently, a similar result was established by Dasgupta and Kumar for pseudo-differential operators on the abstract Heisenberg group [4]. In this paper, we extend these results to step two nilpotent Lie group G. Note that, Heisenberg group and H-type groups are particular type of step two nilpotent Lie groups.

We obtain conditions on the symbol σ such that the corresponding pseudo-differential operator T_σ is a bounded linear operator on $L^2(G)$. Further, we show that under some additional conditions on the symbol the corresponding pseudo-differential operators on G is a Hilbert-Schmidt operator. We define (μ,ν)-Weyl transform on G and show that these class of Hilbert-Schmidt operators can be identified with (μ,ν)-Weyl transforms with symbols in $L^2\left(\mathbb{R}^{2n+r+k} \times \mathbb{R}^{2n+r+k}\right)$. Also, we derive a trace formula for the (μ,ν)-Weyl transform. Further, we present a characterization of the trace class pseudo-differential operators on G and provide a trace formula for these trace class operators.

The presentation of this manuscript is divided into five sections apart from the introduction: In Section 2, we recall basic harmonic analysis on the step two nilpotent Lie group G and define the pseudo-differential operators on the group G. In Section 3, we study L^2-boundedness property of pseudo-differential operators on G. We also prove that if two symbols with some conditions give arise to same pseudo-differential operator then the symbols must be same. In Section 4, we define and obtain a trace formula for (μ,ν)-Weyl transform. We provide a necessary and sufficient condition on the symbol σ such that the corresponding pseudo-differential operator T_σ on G is a Hilbert-Schmidt operator. Finally, we characterize the trace class pseudo-differential operators on G and find a trace formula for these trace class operators in Section 5.

2. Preliminary

In this section we recall some basics of harmonic analysis on step two nilpotent Lie groups to make the paper self contained. A complete account of representation theory for two step connected, simply connected nilpotent Lie groups can be found in [24, 1, 25].

2.1. **Step two nilpotent Lie groups.** Let G be a two step connected, simply connected nilpotent Lie group. Then its Lie algebra $\mathfrak{g}$ has the decomposition $\mathfrak{g}=\mathfrak{v}\oplus\mathfrak{z}$, where $\mathfrak{z}$ is the centre of $\mathfrak{g}$ and $\mathfrak{v}$ is any subspace of $\mathfrak{g}$ complementary to $\mathfrak{z}$. Let us choose an inner product on $\mathfrak{g}$ so that $\mathfrak{v}$ and $\mathfrak{z}$ are orthogonal. Fix an orthonormal basis $\mathcal{B}=\{V_1,V_2\cdots,V_m,T_1,\cdots,T_k\}$ such that $\mathfrak{v}=\operatorname{span}_{\mathbb{R}}\{V_1,V_2\cdots,V_m\}$ and $\mathfrak{z}=\operatorname{span}_{\mathbb{R}}\{T_1,\cdots,T_k\}$. We can identify G with $\mathfrak{v}\oplus\mathfrak{z}$ and write $(V+T)$ for $\exp(V+T)$ and denote it by (V,T), where $V\in\mathfrak{v}$ and $T\in\mathfrak{z}$. By the Baker-Campbell-Hausdorff formula, the group product law on G is given by

$$(V,T)\left(V',T'\right)=\left(V+V',T+T'+\frac{1}{2}\left[V,V'\right]\right)$$

for all $V,V'\in\mathfrak{v}$ and $T,T'\in\mathfrak{z}$. Let $\mathfrak{g}^*$ and $\mathfrak{z}^*$ be the real dual of $\mathfrak{g}$ and $\mathfrak{z}$ respectively. For each $\nu\in\mathfrak{z}^*$, consider the bilinear form B_ν on $\mathfrak{v}$ defined by

$$B_\nu(V,V')=\nu([V,V'])$$

for all $V,V'\in\mathfrak{v}$. Let

$$r_\nu=\{V\in\mathfrak{v}:\nu([V,V'])=0\text{ for all }V'\in\mathfrak{v}\}$$

and m_ν denote the orthogonal complement of r_ν in $\mathfrak{v}$. Then the set

$$\mathcal{U}=\{\nu\in\mathfrak{z}^*:\dim(m_\nu)\text{ is maximum}\}$$

is a Zariski open subset of $\mathfrak{z}^*$. If $r_\nu=\{0\}$ for each $\nu\in\mathcal{U}$, then the Lie algebra is called an MW algebra and the corresponding Lie group is called an MW group.

2.2. **Without MW-condition.** In the case $r_\nu\neq\{0\}$ for each $\nu\in\mathcal{U}$ and $B_\nu|_{m_\nu}$ is non-degenerate and hence $\dim m_\nu$ is $2n$. Then there exists an orthonormal basis

$$\{X_1(\nu),Y_1(\nu),\cdots,X_n(\nu),Y_n(\nu),Z_1(\nu),\cdots,Z_r(\nu)\}$$

of $\mathfrak{v}$ and positive numbers $d_i(\nu)>0$ such that

(1) $r_\nu=\operatorname{span}_{\mathbb{R}}\{Z_1(\nu),\cdots,Z_r(\nu)\}$,

(2) $\nu([X_i(\nu),Y_j(\nu)])=\delta_{i,j}d_j(\nu),1\leq i,j\leq n$ and
$\nu([X_i(\nu),X_j(\nu)])=0,\nu([Y_i(\nu),Y_j(\nu)])=0$ for $1\leq i,j\leq n$,

(3) $\operatorname{span}_{\mathbb{R}}\{X_1(\nu),\cdots,X_n(\nu),Z_1(\nu),\cdots,Z_r(\nu),T_1,\cdots,T_k\}=\mathfrak{h}_\nu$ is a polarization for ν.

We call the basis

$$\{X_1(\nu),\cdots,X_n(\nu),Y_1(\nu),\cdots,Y_n(\nu),Z_1(\nu),\cdots,Z_r(\nu),T_1,\cdots,T_k\}$$

of $\mathfrak{g}$ as almost symplectic basis. Let

$$\xi_\nu=\operatorname{span}_{\mathbb{R}}\{X_1(\nu)\cdots,X_n(\nu)\}\quad\text{and}\quad\eta_\nu=\operatorname{span}_{\mathbb{R}}\{Y_1(\nu),\cdots,Y_n(\nu)\}.$$

Then we have the decomposition $\mathfrak{g}=\xi_\nu \oplus \eta_\nu \oplus r_\nu \oplus \mathfrak{z}$. For $X \in \xi_\nu, Y \in \eta_\nu, Z \in r_\nu$, and $T \in \mathfrak{z}$, we denote the element $\exp(X+Y+Z+T)$ of G by (X,Y,Z,T). Moreover, we can express

$$(X,Y,Z,T)=\sum_{j=1}^{n} x_j(\nu)X_j(\nu)+\sum_{j=1}^{n} y_j(\nu)Y_j(\nu)+\sum_{j=1}^{r} z_j(\nu)Z_j(\nu)+\sum_{j=1}^{k} t_jT_j$$

and denote it by (x,y,z,t) suppressing the dependence of ν which will be understood from the context.

Since $\nu|[\mathfrak{h}_\nu,\mathfrak{h}_\nu]=0$; hence for $\mu \in r_\nu^*$ we define character $\sigma_{\mu,\nu}$ of $H_\nu=\exp\left(\mathfrak{h}_\nu\right)$ by

$$\sigma_{\mu,\nu}(X,Z,T)=e^{i\mu(Z)+i\nu(T)} \quad \text{for all } (X,Z,T)\in H_\nu.$$

The irreducible unitary representations $\pi_{\mu,\nu}$ of G realized on $L^2\left(\eta_\nu\right)$ can be described as follows:

$$\left(\pi_{\mu,\nu}(X,Y,Z,T)\phi\right)\left(Y'\right)=e^{i\nu(T+[Y'+(1/2)Y,X-Y'+Z])}e^{i\mu(z)}\phi\left(Y+Y'\right)$$

for all $\phi\in L^2\left(\eta_\nu\right)$. Using the almost symplectic basis we have the following description

$$\left(\pi_{\mu,\nu}(x,y,z,t)\phi\right)(\xi)=e^{i\sum_{j=1}^{k}\nu_jt_j+i\sum_{j=1}^{r}\mu_jz_j+i\sum_{j=1}^{n}d_j(\nu)\left(x_j\xi_j+\frac{1}{2}x_jy_j\right)}\phi(\xi+y)$$

for all $\phi\in L^2\left(\eta_\nu\right)$.

The Fourier transform of $f\in L^1(G)$ is defined by

$$\widehat{f}(\mu,\nu)=\int_{\mathfrak{z}}\int_{r_\nu}\int_{\eta_\nu}\int_{\xi_\nu} f(x,y,z,t)\pi_{\mu,\nu}(x,y,z,t)\; dxdydzdt$$

for all $\nu\in\mathcal{U},\mu\in r_\nu^*$. Let

$$f^\nu(x,y,z)=\int_{\mathfrak{z}} e^{i\sum_{j=1}^{k}\nu_jt_j} f(x,y,z,t)\; dt$$

and

$$f^{\mu,\nu}(x,y)=\int_{r_\nu}\int_{\mathfrak{z}} e^{i\sum_{j=1}^{k}\nu_jt_j+i\sum_{j=1}^{r}\mu_jz_j} f(x,y,z,t)\; dtdz \tag{1}$$

for all $\nu\in\mathcal{U}$ and $\mu\in r_\nu^*$. For $\nu\in\mathcal{U}$, $\mathrm{P}f(\nu)=\prod_{j=1}^{n}d_j(\nu)$ is called the Pfaffian of ν. For $f\in L^1\cap L^2(G)$, $\widehat{f}(\mu,\nu)$ is an Hilbert-Schmidt operator and

$$\mathrm{P}f(\nu)\|\widehat{f}(\mu,\nu)\|_{S_2}^2=(2\pi)^n\int_{\eta_\nu}\int_{\xi_\nu}\left|f^{\mu,\nu}(x,y)\right|^2 dxdy,$$

where $\|\cdot\|_{S_2}$ stands for the norm in the Hilbert space S_2 of all Hilbert-Schmidt operators on $L^2\left(\eta_\nu\right)$. Moreover, the Plancherel formula reads as

$$(2\pi)^{-(n+r+k)}\int_{\mathcal{U}}\int_{r_\nu^*}\|\widehat{f}(\mu,\nu)\|_{S_2}^2\mathrm{P}f(\nu)d\mu d\nu=\int_G|f(x,y,z,t)|^2\; dxdydzdt$$

for all L^2-functions by density argument. For $f\in\mathcal{S}(G)$, the Schwartz space of G, the following inversion formula holds:

$$f(x,y,z,t)=(2\pi)^{-(n+r+k)}\int_{\mathcal{U}}\int_{r_\nu^*}\operatorname{tr}\left(\pi_{\mu,\nu}(x,y,z,t)^*\hat{f}(\mu,\nu)\right)\mathrm{Pf}(\nu)\; d\mu d\nu.$$

Let $B\left(L^2\left(\eta_\nu\right)\right)$ denote the C^* -algebra of all bounded linear operators on $L^2\left(\eta_\nu\right)$. We call the mapping $\sigma: G\times\widehat{G}\rightarrow B(L^2(\eta_\nu))$ an operator valued symbol. We define the pseudo-differential operator $T_\sigma : L^2(G) \to L^2(G)$ corresponding to the symbol σ by

$$(T_\sigma f)(x,y,z,t) = (2\pi)^{-(n+r+k)} \int_{\mathcal{U}} \int_{r_\nu^*} \operatorname{tr}\left(\pi_{\mu,\nu}(x,y,z,t)^* \sigma(x,y,z,t,\mu,\nu) \hat{f}(\mu,\nu)\right) \operatorname{Pf}(\nu)\, d\mu d\nu$$

for all $f \in \mathcal{S}(G)$.

2.3. **With MW condition.** In this case $r_\nu = \{0\}$ and the the irreducible unitary representations are parametrized by the Zariski open set $\mathcal{U} = \{\nu \in \mathfrak{z}^* : B_\nu \text{ is nondegenerate}\}$ and is given by

$$(\pi_\nu(x,y,t)\phi)(\xi) = e^{i\sum_{j=1}^k \nu_j t_j + i\sum_{j=1}^n d_j(\nu)\left(x_j\xi_j + \frac{1}{2}x_j y_j\right)} \phi(\xi + y)$$

for all $\phi \in L^2(\eta_\nu)$. The Fourier transform of $f \in L^1(G)$ is defined by

$$\widehat{f}(\nu) = \int_{\mathfrak{z}} \int_{\eta_\nu} \int_{\xi_\nu} f(x,y,t) \pi_\nu(x,y,t)\; dxdydt$$

for all $\nu \in \mathcal{U}$. Also let

$$f^\nu(x,y) = \int_{\mathfrak{z}} e^{i\sum_{j=1}^k \nu_j t_j} f(x,y,t)\; dxdydt$$

for all $\nu \in \mathcal{U}$. If $f \in L^1 \cap L^2(G)$ then $\widehat{f}(\nu)$ is an Hilbert-Schmidt operator and

$$\mathrm{P}f(\nu) \|\widehat{f}(\nu)\|_{S_2}^2 = (2\pi)^n \int_{\eta_\nu} \int_{\xi_\nu} |f^\nu(x,y)|^2 \; dxdy.$$

The Plancherel formula takes the following form

$$(2\pi)^{-(n+k)} \int_{\mathcal{U}} \|\widehat{f}(\nu)\|_{S_2}^2 Pf(\nu) d\nu = \int_G |f(x,y,t)|^2 \; dxdydt$$

for all L^2-functions by density argument. Moreover, for $f \in \mathcal{S}(G)$, we have the following inversion formula:

$$f(x,y,t) = (2\pi)^{-(n+k)} \int_{\mathcal{U}} \operatorname{tr}\left(\pi_{\mu,\nu}(x,y,t)^* \hat{f}(\nu)\right) \operatorname{Pf}(\nu) d\nu.$$

Let $B\left(L^2(\eta_\nu)\right)$ denote the C^* -algebra of all bounded linear operators on $L^2(\eta_\nu)$. We call the mapping $\sigma : G \times \widehat{G} \to B(L^2(\eta_\nu))$ an operator valued symbol. We define the pseudo-differential operator $T_\sigma : L^2(G) \to L^2(G)$ corresponding to the symbol σ by

$$(T_\sigma f)(x,y,t) = (2\pi)^{-(n+k)} \int_{\mathcal{U}} \operatorname{tr}\left(\pi_{\mu,\nu}(x,y,t)^* \sigma(x,y,t,\nu) \hat{f}(\nu)\right) \operatorname{Pf}(\nu) d\nu$$

for all $f \in \mathcal{S}(G)$.

Remark 2.1. The step two nilpotent Lie group G (without MW condition) can be realized (as a set) by $\mathbb{R}^{2n+r+k}$ and we can identify r_ν^* with $\mathbb{R}^r$ and $\mathcal{U}$ with a full measure set in $\mathbb{R}^k$. Therefore the set of all irreducible unitary representation of G that participate in the Plancherel formula can be identified with $\mathbb{R}^{r+k}$. In this paper, we will only consider G to be a step two nilpotent Lie group without MW-condition. However, for MW-condition, the calculation will be similar and one can look at [31].

3. Boundedness

This section is devoted to study the L^2-boundedness of pseudo-differential operators on step two nilpotent Lie group G. We begin with the definition of r-Schatten-von Neumann class of operators. If $\mathcal{H}$ is a complex Hilbert space, a linear compact operator $A : \mathcal{H} \to \mathcal{H}$ belongs to the r-Schatten-von Neumann class $S_r(\mathcal{H})$ if

$$\sum_{n=1}^{\infty} (s_n(A))^r < \infty,$$

where $s_n(A)$ denote the singular values of A, i.e. the eigenvalues of $|A| = \sqrt{A^*A}$ with multiplicities counted. For $1 \leq r < \infty$, the class $S_r(\mathcal{H})$ is a Banach space endowed with the norm

$$\|A\|_{S_r} = \left(\sum_{n=1}^{\infty} (s_n(A))^r \right)^{\frac{1}{r}}.$$

For $0 < r < 1$, the $\|\cdot\|_{S_r}$ as above only defines a quasi-norm with respect to which $S_r(\mathcal{H})$ is complete. An operator belongs to the class $S_1(\mathcal{H})$ is known as *Trace class* operator. Also, an operator belongs to $S_2(\mathcal{H})$ is known as *Hilbert-Schmidt* operator.

Now, we are ready to state the following result on L^2-boundedness of pseudo-differential operators on G. Indeed, we have the following proposition.

Proposition 3.1. *Let $\sigma : G \times \widehat{G} \to S_2$ be a symbol such that*

$$\int_{\mathcal{U}} \int_{r_\nu^*} \int_{\mathfrak{g}} \|\sigma(x, y, z, t, \mu, \nu)\|_{S_2}^2 \, dxdydzdt \, \mathrm{Pf}(\nu) d\mu d\nu < \infty.$$

Then, the corresponding pseudo-differential operator T_σ is bounded on $L^2(G)$.

Proof. Let $f \in L^2(G)$. Then by Minkowski's integral inequality and Plancherel theorem, we have

$$\|T_\sigma f\|_{L^2(G)} = \left\{ \int_{\mathfrak{g}} |(T_\sigma f)(x, y, z, t)|^2 \, dxdydzdt \right\}^{1/2}$$

$$= (2\pi)^{-(n+r+k)} \left\{ \int_{\mathfrak{g}} \left| \int_{\mathcal{U}} \int_{r_\nu^*} \mathrm{tr} \left(\pi_{\mu,\nu}(x, y, z, t)^* \sigma(x, y, z, t, \mu, \nu) \hat{f}(\mu, \nu) \right) \mathrm{Pf}(\nu) d\mu d\nu \right|^2 dxdydzdt \right\}^{1/2}$$

$$\leq (2\pi)^{-(n+r+k)} \int_{\mathcal{U}} \int_{r_\nu^*} \left\{ \int_{\mathfrak{g}} \left| \mathrm{tr} \left(\pi_{\mu,\nu}(x, y, z, t)^* \sigma(x, y, z, t, \mu, \nu) \hat{f}(\mu, \nu) \right) \right|^2 dxdydzdt \right\}^{1/2} \mathrm{Pf}(\nu) d\mu d\nu$$

$$\leq (2\pi)^{-(n+r+k)} \int_{\mathcal{U}} \int_{r_\nu^*} \left\{ \int_{\mathfrak{g}} \|\sigma(x, y, z, t, \mu, \nu)\|_{S_2}^2 \|\hat{f}(\mu, \nu)\|_{S_2}^2 \, dxdydzdt \right\}^{1/2} \mathrm{Pf}(\nu) d\mu d\nu$$

$$= (2\pi)^{-(n+r+k)} \int_{\mathcal{U}} \int_{r_\nu^*} \|\hat{f}(\mu, \nu)\|_{S_2} \left\{ \int_{\mathfrak{g}} \|\sigma(x, y, z, t, \mu, \nu)\|_{S_2}^2 \, dxdydzdt \right\}^{1/2} \mathrm{Pf}(\nu) d\mu d\nu$$

$$\leq (2\pi)^{-(n+r+k)} \left\{ \int_{\mathcal{U}} \int_{r_\nu^*} \|\hat{f}(\mu, \nu)\|_{S_2}^2 \, \mathrm{Pf}(\nu) d\mu d\nu \right\}^{\frac{1}{2}}$$

$$\times \left\{ \int_{\mathcal{U}} \int_{r_\nu^*} \int_{\mathfrak{g}} \|\sigma(x, y, z, t, \mu, \nu)\|_{S_2}^2 \, dxdydzdt \, \mathrm{Pf}(\nu) d\mu d\nu \right\}^{1/2}$$

$$= \|f\|_{L^2(G)} \left\{ \int_{\mathcal{U}} \int_{r_\nu^*} \int_{\mathfrak{g}} \|\sigma(x,y,z,t,\mu,\nu)\|_{S_2}^2 \, dxdydzdt \, \mathrm{Pf}(\nu) d\mu d\nu \right\}^{1/2}.$$

This shows that $T_\sigma : L^2(G) \to L^2(G)$ is a bounded operator. □

We presented the proof of Proposition 3.1 due to simplicity of the proof in this case. A more general result in terms of Schatten-von-Neumann class follows from Corollary 3.18 of [21]. Thus, Proposition 3.1 is a particular case of Theorem 3.2 below.

Theorem 3.2. *Let $1 \le p \le 2$ with Lebesgue conjugate p' and let $\sigma : G \times \widehat{G} \to S_p$ be a operator-valued symbol such that*

$$\int_{\mathcal{U}} \int_{r_\nu^*} \int_{\mathfrak{g}} \|\sigma(x,y,z,t,\mu,\nu)\|_{S_p}^p \, dxdydzdt \, \mathrm{Pf}(\nu) d\mu d\nu < \infty.$$

Then the pseudo-differential operator $T_\sigma : L^2(G) \to L^2(G)$ is in the p'-Schatten class $S_{p'}(G)$.

In order to prove our main result, we need to observe the following fact. If two symbols with some conditions give arise to same pseudo-differential operator then the symbols must be same. We prove this result in the following theorem.

Theorem 3.3. *Let $\sigma : G \times \widehat{G} \to S_2$ be a symbol such that it satisfies the following properties:*

(i) $\displaystyle\int_{\mathcal{U}} \int_{r_\nu^*} \int_{\mathfrak{g}} \|\sigma(x,y,z,t,\mu,\nu)\|_{S_2}^2 \, dxdydzdt \, \mathrm{Pf}(\nu) d\mu d\nu < \infty,$

(ii) $\displaystyle\int_{\mathcal{U}} \int_{r_\nu^*} \|\sigma(x,y,z,t,\mu,\nu)\|_{S_2} \, \mathrm{Pf}(\nu) d\mu d\nu < \infty, \;\; \forall\, (x,y,x,t) \in G,$

(iii) $\displaystyle\sup_{(x,y,z,t,\mu,\nu) \in G \times \widehat{G}} \|\sigma(x,y,z,t,\mu,\nu)\|_{S_2} < \infty,$

(iv) the mapping $G \times \widehat{G} \ni (x,y,z,t,\mu,\nu) \mapsto \pi_{\mu,\nu}(x,y,z,t)^* \sigma(x,y,z,t,\mu,\nu) \in S_2$ is weakly continuous.

Then, $T_\sigma f = 0$ for all f in $L^2(G)$ only if $\sigma(x,y,z,t,\mu,\nu) = 0$ for almost all $(x,y,z,t,\mu,\nu) \in G \times \widehat{G}$.

Proof. For $(x,y,z,t) \in G$, let us define the function $f_{(x,y,z,t)} \in L^2(G)$ by

$$\widehat{f_{(x,y,z,t)}}(\mu,\nu) = \sigma(x,y,z,t,\mu,\nu)^* \pi_{\mu,\nu}(x,y,z,t)$$

for all $\nu \in \mathcal{U}$ and $\mu \in r_\nu^*$. Thus, for all $(x',y',z',t') \in G$, we have

$$\begin{aligned}
&\left(T_\sigma f_{(x,y,z,t)}\right)(x',y',z',t') \\
&= (2\pi)^{-(n+r+k)} \int_{\mathcal{U}} \int_{r_\nu^*} \mathrm{tr}\left(\pi_{\mu,\nu}(x',y',z',t')^* \sigma(x',y',z',t',\mu,\nu) \widehat{f_{(x,y,z,t)}}(\mu,\nu)\right) \mathrm{Pf}(\nu) d\mu d\nu \\
&= (2\pi)^{-(n+r+k)} \int_{\mathcal{U}} \int_{r_\nu^*} \mathrm{tr}\left[\pi_{\mu,\nu}(x',y',z',t')^* \sigma(x',y',z',t',\mu,\nu)\right. \\
&\qquad \left. \times\, \sigma(x,y,z,t,\mu,\nu)^* \pi_{\mu,\nu}(x,y,z,t)\right] \mathrm{Pf}(\nu) d\mu d\nu.
\end{aligned}$$

Take $(x_0,y_0,z_0,t_0) \in G$. By the weakly continuous mapping property (iv), we have that

$$\begin{aligned}
&\mathrm{tr}\left(\pi_{\mu,\nu}(x',y',z',t')^* \sigma(x',y',z',t',\mu,\nu) \sigma(x,y,z,t,\mu,\nu)^* \pi_{\mu,\nu}(x,y,z,t)\right) \\
&\to \mathrm{tr}\left(\pi_{\mu,\nu}(x_0,y_0,z_0,t_0)^* \sigma(x_0,y_0,z_0,t_0,\mu,\nu) \sigma(x,y,z,t,\mu,\nu)^* \pi_{\mu,\nu}(x,y,z,t)\right)
\end{aligned}$$

as $(x', y', z', t') \to (x_0, y_0, z_0, t_0)$ in G. Now, using the property (iii), there exists a constant C such that for all $(x', y', z', t', \mu, \nu) \in G \times \widehat{G}$, we have

$$\begin{aligned}&\left|\operatorname{tr}\left(\pi_{\mu,\nu}(x', y', z', t')^* \sigma(x', y', z', t', \mu, \nu)\sigma(x, y, z, t, \mu, \nu)^* \pi_{\mu,\nu}(x, y, z, t)\right)\right| \\ &\leq C\|\sigma(x, y, z, t, \mu, \nu)\|_{S_2}.\end{aligned}$$

Since, for all $(x, y, x, t) \in G$,

$$\int_{\mathcal{U}}\int_{r_\nu^*} \|\sigma(x, y, z, t, \mu, \nu)\|_{S_2} \ \mathrm{Pf}(\nu) d\mu d\nu < \infty,$$

by Lebesgue's dominated convergence theorem, we have

$$\begin{aligned}&\int_{\mathcal{U}}\int_{r_\nu^*} \operatorname{tr}\left(\pi_{\mu,\nu}(x', y', z', t')^* \sigma(x', y', z', t', \mu, \nu)\widehat{f_{(x,y,z,t)}}(\mu, \nu)\right) \mathrm{Pf}(\nu) d\mu d\nu \\ &\to \int_{\mathcal{U}}\int_{r_\nu^*} \operatorname{tr}\left(\pi_{\mu,\nu}(x_0, y_0, z_0, t_0)^* \sigma(x_0, y_0, z_0, t_0, \mu, \nu)\widehat{f_{(x,y,z,t)}}(\mu, \nu)\right) \mathrm{Pf}(\nu) d\mu d\nu\end{aligned}$$

as $(x', y', z', t') \to (x_0, y_0, z_0, t_0)$ in G. Therefore, $T_\sigma f_{(x,y,z,t)}$ is continuous on G. Letting $(x_0, y_0, z_0, t_0) = (x, y, z, t)$, we obtain

$$\begin{aligned}&\left(T_\sigma f_{(x,y,z,t)}\right)(x, y, z, t) \\ &= (2\pi)^{-(n+r+k)} \int_{\mathcal{U}}\int_{r_\nu^*} \operatorname{tr}\left(\sigma(x, y, z, t, \mu, \nu)\sigma(x, y, z, t, \mu, \nu)^*\right) \mathrm{Pf}(\nu) d\mu d\nu \\ &= (2\pi)^{-(n+r+k)} \int_{\mathcal{U}}\int_{r_\nu^*} \|\sigma(x, y, z, t, \mu, \nu)\|_{S_2} \mathrm{Pf}(\nu) d\mu d\nu = 0.\end{aligned}$$

Thus, $\|\sigma(x, y, z, t, \mu, \nu)\|_{S_2} = 0$ for almost all $\nu \in \mathcal{U}$ and $\mu \in r_\nu^*$. Hence the symbol $\sigma(x, y, z, t, \mu, \nu) = 0$ for almost all $(x, y, z, t, \mu, \nu) \in G \times \widehat{G}$. □

4. Hilbert-Schmidt operators

In this section, we define (μ, ν)-Weyl transform and find a trace formula for the class of (μ, ν)-Weyl transform on G. Using the trace formula, we characterize the Hilbert-Schmidt pseudo-differential operators in terms of their corresponding symbols.

Since ξ_ν and η_ν both can be identified with $\mathbb{R}^n$, in this section, we use the notation $\mathbb{R}^n$, ξ_ν or η_ν interchangeably according to our convenience. Let $x, y \in \mathbb{R}^n$ and let $\nu \in \mathcal{U}, \mu \in r_\nu^*$. Then, for every measurable function ϕ on $\mathbb{R}^n$, the function $\pi_{\mu,\nu}(x, y)\phi$ on $\mathbb{R}^n$ is defined by

$$\pi_{\mu,\nu}(x, y)\phi(\xi) = \exp\left(i\sum_{j=1}^n d_j(\nu)\left(x_j\xi_j + \frac{1}{2}x_jy_j\right)\right)\phi(\xi + y), \quad x, y \in \mathbb{R}^n,$$

where $\pi_{\mu,\nu}(x, y)$ stands for $\pi_{\mu,\nu}(x, y, 0, 0)$.

For $f, g \in L^2(\mathbb{R}^n)$, the (μ, ν)-Fourier-Wigner transform of f and g is defined by

$$V_{\mu,\nu}(f, g)(p, q) = \mathrm{Pf}(\nu)^{1/2}(2\pi)^{-n/2}\left\langle \pi_{\mu,\nu}(p, q)f, g\right\rangle,$$

where $\langle , \rangle$ is the inner product in $L^2(\mathbb{R}^n)$. Then

$$\begin{aligned}V_{\mu,\nu}(f, g)(p, q) &= \mathrm{Pf}(\nu)^{1/2}(2\pi)^{-n/2}\left\langle \pi_{\mu,\nu}(p, q)f, g\right\rangle \\ &= \mathrm{Pf}(\nu)^{1/2}(2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{i\sum_{j=1}^n d_j(\nu)\left(p_jx_j + \frac{1}{2}p_jq_j\right)} f(x + q)\overline{g(x)}dx \\ &= \mathrm{Pf}(\nu)^{1/2}(2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{i\sum_{j=1}^n d_j(\nu)p_jx_j} f(x + \frac{q}{2})\overline{g(x - \frac{q}{2})}dx.\end{aligned}$$

We define the Fourier transform by

$$\left(\mathcal{F}_\nu(f)\right)(y)=\mathrm{Pf}(\nu)^{1/2}(2\pi)^{-n/2}\int_{\mathbb{R}^n} f(x)e^{-i\sum_{j=1}^n d_j(\nu)x_jy_j}dx, \quad y\in\mathbb{R}^n,$$

where $\nu\in\mathcal{U}, f\in L^1(\mathbb{R}^n)$ and the inverse Fourier transform is defined by

$$\left(\mathcal{F}_\nu^{-1}(f)\right)(x)=\mathrm{Pf}(\nu)^{1/2}(2\pi)^{-n/2}\int_{\mathbb{R}^n} f(x)e^{i\sum_{j=1}^n d_j(\nu)x_jy_j}dy \quad x\in\mathbb{R}^n.$$

Now, we are going to compute the Fourier transform of the (μ,ν)-Fourier-Wigner transform. Similar to [29], we define

$$\begin{aligned}
&I_\varepsilon(x,\xi)\\
&=\mathrm{Pf}(\nu)^{3/2}(2\pi)^{-3n/2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-\frac{\varepsilon^2|p|^2}{2}}e^{-i\sum_{j=1}^n d_j(\nu)(x_jp_j+q_j\xi_j-p_jy_j)}f(y+\frac{q}{2})\overline{g(y-\frac{q}{2})}dqdpdy\\
&=\mathrm{Pf}(\nu)^{3/2}(2\pi)^{-n}\int_{\mathbb{R}^n} e^{-i\sum_{j=1}^n d_jq_j\xi_j}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\varepsilon^{-n}e^{\frac{\sum_{j=1}^n d_j(\nu)^2|x_j-y_j|^2}{2\varepsilon^2}}f(y+\frac{q}{2})\overline{g(y-\frac{q}{2})}dqdpdy.
\end{aligned}$$

As $\varepsilon\to 0$, we have

$$\left(\mathcal{F}_\nu\left(V_{\mu,\nu}(f,g)\right)\right)(x,\xi)=\mathrm{Pf}(\nu)^{1/2}(2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{-i\sum_{j=1}^n d_j(\nu)q_j\xi_j}f(x+\frac{q}{2})\overline{g(x-\frac{q}{2})}dq,$$

where $f,g\in L^2(\mathbb{R}^n)$. Then the (μ,ν)-Wigner transform $W_{\mu,\nu}(f,g)$ of f and g is defined by

$$\begin{aligned}
W_{\mu,\nu}(f,g)(x,\xi)&=\left(\mathcal{F}_\nu\left(V_{\mu,\nu}(f,g)\right)\right)(x,\xi)\\
&=\mathrm{Pf}(\nu)^{1/2}(2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{-i\sum_{j=1}^n d_j(\nu)q_j\xi_j}f(x+\frac{q}{2})\overline{g(x-\frac{q}{2})}dq
\end{aligned}$$

for all $f,g\in L^2(\mathbb{R}^n)$.

Let u be a function in the Schwartz space $\mathcal{S}\left(\mathbb{R}^{2n}\right)$. For $\nu\in\mathcal{U}$ and $\mu\in r_\nu^*$, we define $W_u^{\mu,\nu}$ to be the (μ,ν)-Weyl transform associated to the function u by

$$\begin{aligned}
\langle W_u^{\mu,\nu}f,g\rangle&=\mathrm{Pf}(\nu)^{1/2}(2\pi)^{-n/2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} u(x,\xi)W_{\mu,\nu}(f,g)(x,\xi)dxd\xi\\
&=\mathrm{Pf}(\nu)^{1/2}(2\pi)^{-n/2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\left(\mathcal{F}_\nu u\right)(p,q)V_{\mu,\nu}(f,g)(p,q)dpdq\\
&=\mathrm{Pf}(\nu)(2\pi)^{-n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\left(\mathcal{F}_\nu u\right)(p,q)\left\langle\pi_{\mu,\nu}(p,q)f,g\right\rangle dpdq.
\end{aligned}$$

Thus we can also write

$$W_u^{\mu,\nu}=\mathrm{Pf}(\nu)(2\pi)^{-n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\left(\mathcal{F}_\nu u\right)(p,q)\pi_{\mu,\nu}(p,q)dpdq. \tag{2}$$

For $u\in L^2(\mathbb{R}^{2n})$, we define $D_{\mathrm{Pf}(\nu)}u(x,\xi)=u\left(x_1d_1(\nu),\cdots,x_nd_n(\nu),\xi\right)$. Then the (μ,ν)-Weyl transform also can be expressed in terms of the dialation $D_{\mathrm{Pf}(\nu)}$, which we prove in the following theorem.

Theorem 4.1. *Let u and v be two functions on the Schwartz space $\mathcal{S}(\mathbb{R}^{2n})$. Then, we have the following.*
(a) $W_u^{\mu,\nu}=W_{D_{\mathrm{Pf}(\nu)^{-1}}u}$.
(b) The trace formula for the (μ,ν)-Weyl transform is given by

$$\mathrm{tr}(W_u^{\mu,\nu})=\mathrm{Pf}(\nu)(2\pi)^{-n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} u(x,\xi)dxd\xi.$$

(c) $\mathrm{tr}\left(W_u^{\mu,\nu}W_v^{\mu,\nu}\right)=\mathrm{Pf}(\nu)(2\pi)^{-n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} u(x,\xi)v(x,\xi)dxd\xi.$

Proof. (a) For all $f \in L^2(\mathbb{R}^n)$, from (2), a direct compuutation gives

$$
\begin{aligned}
W_u^{\mu,\nu} f(x) &= \mathrm{Pf}(\nu)(2\pi)^{-n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \left(\mathcal{F}_\nu u\right)(p,q)\pi_{\mu,\nu}(p,q)f(x)dpdq. \\
&= \mathrm{Pf}(\nu)^2(2\pi)^{-2n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} u(p,q)e^{-i\sum_{j=1}^n d_j(\nu)(p_j y_j + q_j\xi_j)} \\
&\quad \times e^{i\sum_{j=1}^n d_j(\nu)\left(p_j x_j + \frac{1}{2}p_j q_j\right)} f(x+q)dyd\xi dpdq.
\end{aligned}
$$

Under the substitution $p \mapsto \frac{p}{\mathrm{Pf}(\nu)} = \left(\frac{p_1}{d_1(\nu)}, \cdots, \frac{p_n}{d_n(\nu)}\right)$ and $\xi \mapsto \left(\frac{\xi_1}{d_1(\nu)}, \cdots, \frac{\xi_n}{d_n(\nu)}\right)$, we get

$$
\begin{aligned}
W_u^{\mu,\nu} f(x) &= \mathrm{Pf}(\nu)^2(2\pi)^{-2n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} u\left(\frac{p}{\mathrm{Pf}(\nu)}, q\right) e^{-i(p\cdot y + q\cdot\xi)}dy\frac{d\xi}{\mathrm{Pf}(\nu)} \\
&\quad \times e^{i\left(p\cdot x + \frac{1}{2}p\cdot q\right)} f(x+q)\frac{dp}{\mathrm{Pf}(\nu)}dq \\
&= (2\pi)^{-n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \widehat{D_{\mathrm{Pf}(\nu)^{-1}}u}(p,q)\pi(p,q)f(x)dpdq \\
&= W_{D_{\mathrm{Pf}(\nu)^{-1}}u} f(x).
\end{aligned}
$$

(b) Using the trace formula given in [11], we have

$$
\begin{aligned}
\mathrm{tr}(W_u^{\mu,\nu}) &= \mathrm{tr}(W_{D_{\mathrm{Pf}(\nu)^{-1}}u}) \\
&= (2\pi)^{-n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} D_{\mathrm{Pf}(\nu)^{-1}}u(x,\xi)dxd\xi \\
&= \mathrm{Pf}(\nu)(2\pi)^{-n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} u(x,\xi)dxd\xi.
\end{aligned}
$$

(c) Again, form Theorem 2.1 of [11], we have

$$
\begin{aligned}
\mathrm{tr}\left(W_u^{\mu,\nu}W_v^{\mu,\nu}\right) &= \mathrm{tr}\left(W_{D_{\mathrm{Pf}(\nu)^{-1}}u}W_{D_{\mathrm{Pf}(\nu)^{-1}}v}\right) \\
&= (2\pi)^{-n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} D_{\mathrm{Pf}(\nu)^{-1}}u(x,\xi)D_{\mathrm{Pf}(\nu)^{-1}}v(x,\xi)dxd\xi \\
&= \mathrm{Pf}(\nu)(2\pi)^{-n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} u(x,\xi)v(x,\xi)dxd\xi.
\end{aligned}
$$

□

Before stating our main theorem of this section, we observe the following fact.

Theorem 4.2. *Let $f \in L^1(G)$. Then*

$$
\widehat{f}(\mu,\nu) = (2\pi)^n W^{\mu,\nu}_{\mathcal{F}_\nu^{-1}(f^{\mu,\nu})}
$$

for every $\nu \in \mathcal{U}$ and $\mu \in r_\nu^*$, where $f^{\mu,\nu}$ is defined in (1).

Proof. Let $\phi \in S(\mathbb{R}^n)$. Then

$$\begin{aligned}(\widehat{f}(\mu,\nu)\phi)(\xi) &= \int_{\mathfrak{z}}\int_{r_\nu}\int_{\eta_\nu}\int_{\xi_\nu} f(x,y,z,t)\pi_{\mu,\nu}(x,y,z,t)\phi(\xi)dxdydzdt \\ &= \int_{\mathfrak{z}}\int_{r_\nu}\int_{\eta_\nu}\int_{\xi_\nu} f(x,y,z,t)e^{i\sum_{j=1}^{k}\nu_j t_j + i\sum_{j=1}^{r}\mu_j z_j}\pi_{\mu,\nu}(x,y)\phi(\xi)dxdydzdt \\ &= \int_{\xi_\nu}\int_{\eta_\nu} f^{\mu,\nu}(x,y)\pi_{\mu,\nu}(x,y)\phi(\xi)dxdy \\ &= \int_{\xi_\nu}\int_{\eta_\nu} \left(\mathcal{F}_\nu\left(\mathcal{F}_\nu^{-1}f^{\mu,\nu}\right)\right)(x,y)\pi_{\mu,\nu}(x,y)\phi(\xi)dxdy.\end{aligned}$$

Therefore

$$\widehat{f}(\mu,\nu) = \mathrm{Pf}(\nu)^{-1}(2\pi)^n W^{\mu,\nu}_{\mathcal{F}_\nu^{-1}(f^{\mu,\nu})}.$$

□

Now we are in a position to obtain a necessary and sufficient condition on symbol such that the corresponding pseudo-differential operator is a Hilbert-Schmidt operator. Indeed, we have the following theorem.

Theorem 4.3. *Let σ be a symbol such that it satisfies the hypotheses of Theorem 3.3. Then the corresponding pseudo-differential operator T_σ is a Hilbert-Schmidt operator if and only if*

$$\sigma(x,y,z,t,\mu,\nu) = \mathrm{Pf}(\nu)^{-1}\pi_{\mu,\nu}(x,y,z,t)W^{\mu,\nu}_{\mathcal{F}_\nu(\alpha(x,y,z,t)^{-\mu,-\nu})},$$

where $(x,y,z,t,\mu,\nu) \in G\times\widehat{G}$ and $\alpha : G \to L^2(G)$ is a weakly continuous mapping such that it satisfies

(i) $\int_{\mathfrak{z}}\int_{r_\nu}\int_{\eta_\nu}\int_{\xi_\nu} \|\alpha(x,y,z,t)(\cdot,\cdot,\cdot,\cdot)\|_{L^2(G)}\, dxdydzdt < \infty,$

(ii) $\sup_{(x,y,z,t,\mu,\nu)\in G\times\widehat{G}} \mathrm{Pf}(\nu)^{-1/2}\left\|\alpha(x,y,z,t)^{-\mu,-\nu}\right\|_{L^2(\mathbb{R}^{2n})} < \infty,$

(iii) $\int_{\mathcal{U}}\int_{r_\nu^*} \left\|\alpha(x,y,z,t)^{-\mu,-\nu}\right\|_{L^2(\mathbb{R}^{2n})} \mathrm{Pf}(\nu))^{1/2}d\mu d\nu < \infty,$ *a.e.* $(x,y,z,t) \in G$.

Proof. Let $f \in \mathcal{S}(G)$. Using Theorem 4.2 and Part (c) of Theorem 4.1, we have

$$\begin{aligned}&(T_\sigma f)(x,y,z,t) \\ &= (2\pi)^{-(n+r+k)}\int_{\mathcal{U}}\int_{r_\nu^*} \mathrm{tr}\left(\pi_{\mu,\nu}(x,y,z,t)^*\sigma(x,y,z,t,\mu,\nu)\hat{f}(\mu,\nu)\right)\mathrm{Pf}(\nu)d\mu d\nu \\ &= (2\pi)^{-(r+k)}\int_{\mathcal{U}}\int_{r_\nu^*} \mathrm{tr}\left(W^{\mu,\nu}_{\mathrm{Pf}(\nu)^{-1}\mathcal{F}_\nu(\alpha(x,y,z,t)^{-\mu,-\nu})}W^{\mu,\nu}_{\mathrm{Pf}(\nu)^{-1}\mathcal{F}_\nu^{-1}(f^{\mu,\nu})}\right)\mathrm{Pf}(\nu)d\mu d\nu \\ &= (2\pi)^{-(n+r+k)}\int_{\xi_\nu}\int_{\eta_\nu}\int_{\mathcal{U}}\int_{r_\nu^*} \mathcal{F}_\nu(\alpha(x,y,z,t)^{-\mu,-\nu})(x',y')\mathcal{F}_\nu^{-1}(f^{\mu,\nu})(x',y')dx'dy'd\mu d\nu \\ &= (2\pi)^{-(n+r+k)}\int_{\xi_\nu}\int_{\eta_\nu}\int_{\mathcal{U}}\int_{r_\nu^*} \alpha(x,y,z,t)^{-\mu,-\nu}(x',y')f^{\mu,\nu}(x',y')dx'dy'd\mu d\nu \\ &= (2\pi)^{-(n+r+k)}\int_{\xi_\nu}\int_{\eta_\nu}\int_{\mathcal{U}}\int_{r_\nu^*} \alpha(x,y,z,t)(x',y',\mu,\nu)f(x',y',\mu,\nu)dx'dy'd\mu d\nu.\end{aligned}$$

Therefore T_σ is an almost everywhere integral operator with kernel

$$K\left(x,y,z,t,x',y',\mu,\nu\right) = (2\pi)^{-(n+r+k)}\alpha(x,y,z,t)\left(x',y',\mu,\nu\right), \tag{3}$$

where $(x, y, z, t), (x', y', \mu, \nu) \in G$. Using Fubini's theorem and Plancherel theorem, we get

$$\int_{\mathfrak{z}}\int_{r_\nu}\int_{\eta_\nu}\int_{\xi_\nu}\int_{\mathfrak{z}}\int_{r_\nu}\int_{\eta_\nu}\int_{\xi_\nu}\left|K\left(x, y, z, t, x', y', \mu, \nu\right)\right|^2 dxdydzdtdx'dy'd\mu d\nu$$

$$= (2\pi)^{-2(n+r+k)}\int_{\mathfrak{z}}\int_{r_\nu}\int_{\eta_\nu}\int_{\xi_\nu}\int_{\mathfrak{z}}\int_{r_\nu}\int_{\eta_\nu}\int_{\xi_\nu}\left|\alpha(x, y, z, t)\left(x', y', \mu, \nu\right)\right|^2 dxdydzdtdx'dy'd\mu d\nu$$

$$= (2\pi)^{-2(n+r+k)}\int_{\mathfrak{z}}\int_{r_\nu}\int_{\eta_\nu}\int_{\xi_\nu}\|\alpha(x, y, z, t)(\cdot,\cdot,\cdot,\cdot)\|_{L^2(G)}\, dxdydzdt < \infty.$$

Thus, $T_\sigma : L^2(G) \to L^2(G)$ is a Hilbert-Schmidt operator.

Conversely, suppose that $T_\sigma : L^2(G) \to L^2(G)$ is a Hilbert-Schmidt operator. Then there exists a function $\alpha \in L^2(G \times G)$ such that for all $f \in L^2(G)$, we have

$$T_\sigma f(x, y, z, t) = \int_{\mathfrak{z}}\int_{r_\nu}\int_{\eta_\nu}\int_{\xi_\nu}\alpha\left(x, y, z, t, x', y', \mu, \nu\right) f\left(x', y', \mu, \nu\right) dx'dy'd\mu d\nu.$$

Let $\alpha : G \to L^2(G)$ be the mapping defined by

$$\alpha(x, y, z, t)\left(x', y', \mu, \nu\right) = \alpha\left(x, y, z, t, x', y', \mu, \nu\right), \quad (x, y, z, t), (x', y', \mu, \nu) \in G.$$

From part (v) of Theorem 7.5 of [29], we have that

$$\|\sigma(x, y, z, t, \mu, \nu)\|_{S_2} = (2\pi)^{-n/2}\,\mathrm{Pf}(\nu)^{-1/2}\left\|\alpha(x, y, z, t)^{-\mu,-\nu}\right\|_{L^2(\mathbb{R}^{2n})}$$

for all $(x, y, z, t, \mu, \nu) \in G \times \widehat{G}$. Then, reversing the argument for sufficiency and using Theorem 3.3, we get the converse. □

An immediate corollary of the above theorem is the following result.

Theorem 4.4. *Let $\alpha \in L^2(G \times G)$ such that*

$$\int_{\mathfrak{z}}\int_{r_\nu}\int_{\eta_\nu}\int_{\xi_\nu}|\alpha\left(x, y, z, t, x, y, z, t\right)|dxdydzdt < \infty.$$

Let $\sigma : G \times \widehat{G} \to B\left(L^2\left(\eta_\nu\right)\right)$ be the symbol as in Theorem 4.3. Then, $T_\sigma : L^2(G) \to L^2(G)$ is a trace class operator and the trace is given by

$$\mathrm{tr}\left(T_\sigma\right) = (2\pi)^{-(2n+r+k)}\int_{\mathfrak{z}}\int_{r_\nu}\int_{\eta_\nu}\int_{\xi_\nu}\alpha\left(x, y, z, t, x, y, z, t\right) dxdydzdt.$$

Proof. The proof of Theorem 4.4 follows from the formula (3) on the kernel of the pseudo-differential operator in the proof of Theorem 4.3. □

We end this section by showing a relationship between Hilbert-Schmidt pseudo-differential operators on $L^2(G)$ and (μ, ν)-Weyl transforms with symbol in $L^2\left(\mathbb{R}^{2n+r+k}\right)$. The twisting operator $T : L^2\left(\mathbb{R}^{2n}\right) \to L^2\left(\mathbb{R}^{2n}\right)$ is defined by

$$(Tf)(x, y) = f\left(x + \frac{y}{2}, x - \frac{y}{2}\right), \quad x, y \in \mathbb{R}^n$$

for all $f \in L(\mathbb{R}^{2n})$. Clearly T is a unitary operator and its the inverse is given by

$$\left(T^{-1}f\right)(x, y) = f\left(\frac{x+y}{2}, x - y\right), \quad x, y \in \mathbb{R}^n.$$

Let us define the operator $K_\nu : L^2\left(\mathbb{R}^{2n}\right) \to L^2\left(\mathbb{R}^{2n}\right)$ by

$$\left(K_\nu f\right)(x, y) = \left(T^{-1}\mathcal{F}_\nu^2 f\right)(y, x), \quad x, y \in \mathbb{R}^n,$$

where $\mathcal{F}_\nu^2$ is the Fourier transform with respect to the second variable. From Theorem 7.5 of [29], we obtain the following theorem.

Theorem 4.5. *Let $\sigma \in L^2\left(\mathbb{R}^{2n}\right)$. Then $W_\sigma^{\mu,\nu}$ is a Hilbert-Schmidt operator with kernel $\operatorname{Pf}(\nu)^{1/2}(2\pi)^{-\frac{n}{2}}K_\nu\sigma$. More precisely,*

$$\left(W_\sigma^{\mu,\nu} f\right)(x) = \operatorname{Pf}(\nu)^{1/2}(2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n} K_\nu\sigma(x,y)f(y)dy, \quad x \in \mathbb{R}^n.$$

Theorem 4.6. *Let $\tau \in L^2\left(\mathbb{R}^{2n+r+k}\times\mathbb{R}^{2n+r+k}\right)$. Then*

$$W_\tau^{\mu,\nu} = T_\sigma,$$

where $\sigma : G\times\widehat{G}\to S_2$ is a symbol such that

(1)

$$\int_{\mathfrak{g}}\int_{\mathcal{U}}\int_{r_\nu^*}\|\sigma(x,y,z,t,\mu,\nu)\|_{S_2}^2\, dxdydzdt\operatorname{Pf}(\nu)d\mu d\nu < \infty,$$

(2)

$$\sigma(x,y,z,t,\mu,\nu) = \operatorname{Pf}(\nu)^{-1}\pi_{\mu,\nu}(x,y,z,t)W^{\mu,\nu}_{\mathcal{F}_\nu(\alpha(x,y,z,t)^{-\mu,-\nu})}$$

for all $(x,y,z,t,\mu,\nu)\in G\times\widehat{G}$ and

(3) $\alpha : G\to L^2(G)$ is related to τ by

$$\alpha(x,y,z,t)\left(x',y',z',t'\right) = \operatorname{Pf}(\nu)^{\frac{1}{2}}(2\pi)^{\frac{r+k}{2}}K_\nu\tau(x,y,z,t,x',y',z',t')$$

for all $(x,y,z,t),(x',y',z',t')\in G$.

Conversely, suppose $\sigma : G\times\widehat{G}\to S_2$ is a symbol such that

(1)

$$\int_{\mathfrak{g}}\int_{\mathcal{U}}\int_{r_\nu^*}\|\sigma(x,y,z,t,\mu,\nu)\|_{S_2}^2\, dxdydzdt\operatorname{Pf}(\nu)d\mu d\nu < \infty,$$

(2)

$$\sigma(x,y,z,t,\mu,\nu) = \operatorname{Pf}(\nu)^{-1}\pi_{\mu,\nu}(x,y,z,t)W^{\mu,\nu}_{\mathcal{F}_\nu(\alpha(x,y,z,t)^{-\mu,-\nu})}$$

for all $(x,y,z,t,\mu,\nu)\in G\times\widehat{G}$, where $\alpha : G\to L^2(G)$ is a mapping such that

$$\int_{\mathfrak{g}}\|\alpha(x,y,z,t)\|_{S_2}^2\, dxdydzdt < \infty.$$

Then $T_\sigma = W_\tau^{\mu,\nu}$, where

$$\tau = \operatorname{Pf}(\nu)^{-\frac{1}{2}}(2\pi)^{-\frac{r+k}{2}}K_\nu^{-1}\beta$$

and β is a function on $G\times G$ given by

$$\beta\left(x,y,z,t,x',y',z',t'\right) = \alpha(x,y,z,t)\left(x',y',z',t'\right), \quad (x,y,z,t),(x',y',z',t')\in G.$$

Proof. The proof of Theorem 4.6 follows from the relation (3) and Theorem 4.5. □

5. Trace class Operators

In this section, we obtain a necessary and sufficient condition on the symbol σ so that the corresponditing pseudo-differential operator T_σ is a trace class operator and we derive the trace formula of the operator T_σ. Indeed, we have the following theorem.

Theorem 5.1. *Let $\sigma : G\times\widehat{G}\to S_2$ be a symbol such that it satisfying the conditions of Theorem 3.3. Then T_σ is a trace class operator if and only if*

$$\sigma(x,y,z,t,\mu,\nu) = \operatorname{Pf}(\nu)^{-1}\pi_{\mu,\nu}(x,y,z,t)W^{\mu,\nu}_{\mathcal{F}_\nu(\alpha(x,y,z,t)^{-\mu,-\nu})}, \quad (x,y,z,t,\mu,\nu)\in G\times\widehat{G},$$

where $\alpha : G \to L^2(G)$ *is a mapping such that the conditions of Theorem 4.3 are satisfied and*

$$\alpha(x,y,z,t)\left(x',y',z',t'\right) = \int_{\mathfrak{g}} \alpha_1(x,y,z,t)\left(x'',y'',z'',t''\right)\alpha_2\left(x'',y'',z'',t''\right)\left(x',y',z',t'\right)dx''dy''dz''t''$$

for all $(x,y,z,t),(x',y',z',t') \in G$, $\alpha_1 : G \to L^2(G)$ *satisfies*

$$\int_{\mathfrak{g}} \|\alpha_1(x,y,z,t)\|^2_{L^2(G)}\, dxdydzdt < \infty$$

and $\alpha_2 : G \to L^2(G)$ *satisfies*

$$\int_{\mathfrak{g}} \|\alpha_2(x,y,z,t)\|^2_{L^2(G)}\, dxdydzdt < \infty.$$

Moreover, if $T_\sigma : L^2(G) \to L^2(G)$ *is a trace class operator, then we have the trace formula*

$$\begin{aligned}\operatorname{tr}(T_\sigma) &= \int_{\mathfrak{g}} \alpha(x,y,z,t)(x,y,z,t)dxdydzdt\\ &= \int_{\mathfrak{g}}\int_{\mathfrak{g}} \alpha_1(x,y,z,t)\left(x'',y'',z'',t''\right)\alpha_2\left(x'',y'',z'',t''\right)(x,y,z,t)dx''dy''dz''dt''dxdydzdt.\end{aligned}$$

Proof. The proof of this theorem follows from Theorem 4.3 and the fact that every trace class operator can be written as a product of two Hilbert-Schmidt operators. □

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