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RESEARCH PAPER

UNIQUENESS FOR AN INVERSE SOURCE PROBLEM OF DETERMINING A SPACE-DEPENDENT SOURCE IN A NON-AUTONOMOUS TIME-FRACTIONAL DIFFUSION EQUATION

Marian Slodička

Abstract

We study uniqueness of a solution for an inverse source problem arising in linear time-fractional diffusion equations with time-dependent coefficients. We consider source term in a separated form $h(t)f(\mathbf{x})$. The unknown source $f(\mathbf{x})$ is recovered from the final time measurement $u(\mathbf{x},T)$. A new uniqueness result is formulated in Theorem 3.1 under the assumption that $h \in C([0,T])$ and $0 \not\equiv h \geq 0$. No monotonicity in time for $h(t)$ and for coefficients of the differential operator is required.

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Key Words and Phrases: time-fractional diffusion equation; inverse source problem; uniqueness

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1. **Introduction**

Inverse source problems (ISPs) in evolutionary (parabolic or hyperbolic) partial differential equations (PDEs), attracted a lot of interest among researchers, which can be documented by many papers, see e.g. [1, 2, 3, 4, 5].

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Typical situation in the parabolic case is to find (u, F) obeying

$$\partial_t u + \mathcal{A}u = F,$$

along with an additional suitable measurement, where $\mathcal{A}$ is a general second order linear partial differential operator. There is no general closed theory for abstract case of $F(\mathbf{x}, t)$. Known results deal with separated source term $F(\mathbf{x}, t) = h(t)f(\mathbf{x})$. The appropriate choice of the overdetermination depends on the choice whether the unknown is $h(t)$ or $f(\mathbf{x})$.

When the $f(\mathbf{x})$ has to be recovered, then the commonly used additional observation is the final time measurement $u(\mathbf{x}, T)$. In a case of a self-adjoint operator $\mathcal{A}$ with time-independent coefficients it is known that uniqueness can be established for $h(t) > 0$. This can be shown using the semigroup theory [6] and the closed formula for the solution

$$u(t) = e^{\mathcal{A}t}u(0) + \int_0^t e^{\mathcal{A}(t-s)}h(s)f.$$

Once considering non-autonomous (time-dependent) operator $\mathcal{A}(t)$ we do not have such a nice expression for the solution as for time-independent operators. Isakov [7, Theorem 2.1] showed the uniqueness result under some monotonic in time assumptions on $h(t)$ and on the coefficients of $\mathcal{A}(t)$. Paper [8] considered also a time-dependent operator $\mathcal{A}(t)$ and derived uniqueness result under some monotonic assumptions (different to [7, Theorem 2.1]). Article [9] established uniqueness result for the ISP without monotonicity requirements (considering transient operator $\mathcal{A}(t)$).

Identification of $f(\mathbf{x})$ in fractional diffusion equations

$$D_t^{\beta}u + \mathcal{A}u = h(t)f(\mathbf{x}), \tag{1.1}$$

with Caputo derivative for $0 < \beta < 1$ has been studied in some articles, e.g. [10, 11, 12, 13, 14, 15]. Paper [16] shows the uniqueness result for $f(\mathbf{x})$ if $h(t) > 0$ in a case of a self-adjoint operator $\mathcal{A}$ with time-independent coefficients. In this case the solution u can be expressed in terms of the Mittag-Leffler function, which was crucial in the proofs. It would be nice to have a similar result for time-dependent operator $\mathcal{A}(t)$. Article [17, Example 3.1] shows the non-uniqueness result if $h(t)$ changes its sign. Time-dependent operator $\mathcal{A}(t)$ has been considered in [17, Theorem 3.1], where uniqueness was established under some monotonic in time assumptions on $h(t), \mathcal{A}(t)$.

We study a general multi-dimensional case for a linear time-fractional PDE with time-dependent coefficients. We use separation of variable technique combined with the maximum principle to derive the uniqueness result assuming $h \in C([0, T])$ and $0 \not\equiv h \geq 0$. We do not need any monotonic premises on $h(t)$ and/or the coefficients of $\mathcal{A}(t)$, which is the new aspect (and highlight) in this area of inverse problems.

2. Mathematical setting

Let $\Omega \subset \mathbb{R}^d,\ d \in \mathbb{N}$ be a bounded domain with a Lipschitz continuous boundary $\partial\Omega$. The final time is denoted by T and $Q_T := \Omega \times (0,T]$ and $\Sigma_T := \partial\Omega \times (0,T]$. Consider the following second-order linear differential operator given by

$$\mathcal{A}(t)u(\mathbf{x},t) = p(t)\left[\nabla \cdot (-\boldsymbol{A}(\mathbf{x})\nabla u(\mathbf{x},t)) + a(\mathbf{x})u(\mathbf{x},t)\right] + q(t)u(\mathbf{x},t), \tag{2.1}$$

where $p, q \in C([0,T])$ and

$$\left\{\begin{array}{rlll} \boldsymbol{A}(\mathbf{x}) & = \left(a_{i,j}(\mathbf{x})\right)_{i,j=1,\ldots,d}, & \mathbf{x} \in \Omega, & \boldsymbol{A} = \boldsymbol{A}^{\mathsf{T}}, \\ a_{i,j} & \in L_\infty(\Omega), & i,j = 1,\ldots,d, & \\ \boldsymbol{\xi}^T \cdot \boldsymbol{A}\boldsymbol{\xi} & = \displaystyle\sum_{i,j=1}^{d} a_{i,j}(\mathbf{x})\xi_i\xi_j \geqslant \nu \left|\boldsymbol{\xi}\right|_{\mathrm{e}}^2, & \forall \boldsymbol{\xi} \in \mathbb{R}^d, & \\ 0 & \le a(\mathbf{x}) \le C, & \mathbf{x} \in \Omega, & \\ 0 & < p(t), & t \in [0,T], & \\ 0 & \le q(t), & t \in [0,T]. & \end{array}\right. \tag{2.2}$$

The ISP here is to reconstruct $f(\mathbf{x})$ for given $h(t), F, u_D$ and u_0 such that

$$\begin{array}{rll} \left(g_{1-\beta} * \partial_t u(\mathbf{x})\right)(t) + \mathcal{A}(t)u(\mathbf{x},t) & = f(\mathbf{x})h(t) + F(\mathbf{x},t) & (\mathbf{x},t) \in Q_T, \\ u(\mathbf{x},t) & = u_D(\mathbf{x},t) & (\mathbf{x},t) \in \Sigma_T, \\ u(\mathbf{x},0) & = u_0(\mathbf{x}) & \mathbf{x} \in \Omega \end{array} \tag{2.3}$$

along with given final time measurement

$$u(\mathbf{x},T) = \psi_T(\mathbf{x}), \quad \mathbf{x} \in \Omega. \tag{2.4}$$

The symbol '$*$' stands for the convolution product defined by

$$\left(D_t^\beta u(\mathbf{x})\right)(t) := \left(g_{1-\beta} * u_t(\mathbf{x})\right)(t) = \int_0^t g_{1-\beta}(t-s)u_t(\mathbf{x},s)\ \mathrm{d}s$$

for $(\mathbf{x},t) \in \overline{Q_T}$, where the kernel $g_{1-\beta}$ is defined by

$$g_{1-\beta}(t) = \frac{t^{-\beta}}{\Gamma(1-\beta)}, \quad t > 0, \quad 0 < \beta < 1,$$

where Γ denotes the Gamma function.

The convolution term is the Caputo fractional derivative of order $\beta \in (0,1)$. The kernel $g_{1-\beta} \in \mathrm{L}^1_{\mathrm{loc}}(0,\infty)$ satisfies

$$(-1)^j g_{1-\beta}^{(j)}(t) \geqslant 0, \quad \forall t \geqslant 0, j = 0,1,2; \quad g'_{1-\beta} \not\equiv 0,$$

and thus it is strongly positive definite by [18, Corollary 2.2].

Remark 2.1 (Additional notations). Denote by $(\cdot,\cdot)$ the standard inner product in $\mathrm{L}^2(\Omega)$ and by $\|\cdot\|$ its induced norm. The norm $|\cdot|_{\mathrm{e}}$ is the Euclidian norm. Consider an abstract Banach space X with norm

$\|\cdot\|_X$. Let $p \geqslant 1$. The space $\mathrm{L}^p\left((0,T),X\right)$ consists of measurable functions $u:[0,T]\to X$ such that

$$\|u\|_{\mathrm{L}^p((0,T),X)} = \left(\int_0^T \|u(t)\|_X^p \,\mathrm{d}t\right)^{1/p} < \infty.$$

The space $\mathrm{C}\left([0,T],X\right)$ consists of continuous functions $u:[0,T]\to X$ satisfying

$$\|u\|_{\mathrm{C}([0,T],X)} = \max_{t\in[0,T]} \|u(t)\|_X < \infty.$$

3. **Uniqueness of a solution to the ISP**

The well-posedness of a direct problem can be shown my means of standard techniques under appropriate conditions on data, cf. eg. [17, Theorem 2.1].

We start with the following lemma, which will be useful in the proof of the ISP.

Lemma 3.1 (maximum principle). *Assume* $0<\beta<1$ *and* $T>0$. *Suppose that* $v,r\in C([0,T])$, $\alpha\in C^1((0,T])$ *and* α' *is Lebesgue integrable on* $(0,T)$. *Presume that the function* α *solves the following time-fractional differential equation*

$$\left(D^\beta\alpha\right)(t) + r(t)\alpha(t) = v(t) \qquad \text{for } t\in[0,T] \tag{3.1}$$

along with

$$\alpha(0)=\alpha(T)=0, \qquad 0\le r(t), \qquad \text{for } t\in[0,T].$$

Then:

(i) *If* $v(t)\le 0$ *for* $t\in[0.T]$, *then* $\alpha(t)\le 0$ *for* $t\in[0,T]$.
(ii) *If* $v(t)\ge 0$ *for* $t\in[0.T]$, *then* $\alpha(t)\ge 0$ *for* $t\in[0,T]$.

Proof. (*i*) Assume that there exists an interior point $t_0\in(0,T)$ such that $\alpha(t_0)>0$. Let us introduce an auxiliary function

$$w(t) := \alpha(t) + \frac{\alpha(t_0)}{2}\frac{T-t}{T}.$$

We can easily see that

$$w(T)=0, \qquad w(0)=\frac{\alpha(t_0)}{2}, \qquad w(t_0)=\alpha(t_0)+\frac{\alpha(t_0)}{2}\frac{T-t_0}{T} > \frac{\alpha(t_0)}{2}.$$

So, the function $w(t)$ cannot attain its maximum at $t=0$ or at $t=T$, because of $w(t_0)>\max\{w(0),w(T)\}$. Let us denote by t_1 a maximum point of $w(t)$ on $[0,T]$. We can write

$$w(t_1)\ge w(t_0) > \frac{\alpha(t_0)}{2}. \tag{3.2}$$

Following [19, Theorem 1] we have that $\left(D^\beta w\right)(t_1) \geq 0$. Applying the fractional derivative to the definition of the function $w(t)$ we get

$$\left(D^\beta \alpha\right)(t) = \left(D^\beta w\right)(t) + \frac{\alpha(t_0)}{2T}\left(D^\beta t\right)(t) = \left(D^\beta w\right)(t) + \frac{\alpha(t_0)}{2T}\frac{t^{1-\beta}}{\Gamma(2-\beta)}.$$

Thus, we are allowed to write

$$\left(D^\beta \alpha\right)(t_1) = \left(D^\beta w\right)(t_1) + \frac{\alpha(t_0)}{2T}\frac{t_1^{1-\beta}}{\Gamma(2-\beta)} > 0.$$

Let us check the validity of the time-fractional differential equation (3.1) at the point t_1:

$$\begin{aligned}
\left(D^\beta \alpha\right)(t_1) + r(t_1)\alpha(t_1) - v(t_1) \quad &= \left(D^\beta \alpha\right)(t_1) \\
&\quad + r(t_1)\left[w(t_1) - \frac{\alpha(t_0)}{2}\frac{T-t_1}{T}\right] - v(t_1) \\
&\overset{(3.2)}{>} \left(D^\beta \alpha\right)(t_1) \\
&\quad + r(t_1)\frac{\alpha(t_0)}{2}\left[1 - \frac{T-t_1}{T}\right] - v(t_1) \\
&= \left(D^\beta \alpha\right)(t_1) + r(t_1)\frac{\alpha(t_0)}{2}\frac{t_1}{T} - v(t_1) \\
&> 0.
\end{aligned}$$

This is a contradiction, which means that our assumption about the existence of $t_0 \in (0,T)$ such that $\alpha(t_0) > 0$ was wrong and therefore we have proved that $\alpha(t) \leq 0$ for all $t \in [0,T]$.

(*ii*) This part follows directly from just proved (*i*) by replacing v by $-v$ and α by $-\alpha$. □

In the following theorem, the uniqueness of a solution to the ISP is stated.

THEOREM 3.1 (Uniqueness). *Suppose (2.1), (2.2). Assume that* $h \in C([0,T])$ *and* $0 \not\equiv h \geq 0$. *Then, there exists at most one couple* $\langle u, f\rangle \in \mathrm{C}\left([0,T], \mathrm{H}_0^1(\Omega)\right) \times \mathrm{L}^2(\Omega)$ *with* $\partial_t u \in \mathrm{L}^2\left((0,T), \mathrm{L}^2(\Omega)\right)$ *solving the ISP* (2.3) *and* (2.4).

Proof. Suppose that there are two solutions $\langle u_1, f_1\rangle$ and $\langle u_2, f_2\rangle$. Then $u = u_1 - u_2$ and $f = f_1 - f_2$ satisfy

$$\begin{aligned}
\left(g_{1-\beta} * \partial_t u(\mathbf{x})\right)(t) + \mathcal{A}(t)u(\mathbf{x},t) &= f(\mathbf{x})h(t) & (\mathbf{x},t) &\in Q_T, \\
u(\mathbf{x},t) &= 0 & (\mathbf{x},t) &\in \Sigma_T, \\
u(\mathbf{x},0) &= 0 & \mathbf{x} &\in \Omega.
\end{aligned} \tag{3.3}$$

We have to show that if $u(\mathbf{x},T) = 0$, then $u = 0 = f$. The weak formulation reads as

$$\begin{aligned}((g_{1-\beta} * \partial_t u)(t), \varphi) + p(t)\left[(\boldsymbol{A}\nabla u(t), \nabla\varphi) + (au(t), \varphi)\right] & \\ +q(t)(u(t), \varphi) = h(t)(f, \varphi), &\end{aligned} \tag{3.4}$$

for all $\varphi \in \mathrm{H}_0^1(\Omega)$ with $u(\cdot, 0) = 0$ and $u(\cdot, T) = 0$.

The structure of the partial differential equation allows us to use the separation of variables technique. Consider the self-adjoint and positive-definite differential operator $\mathcal{B}v := \nabla \cdot (-\boldsymbol{A}\nabla v) + av$ acting on $\mathrm{H}_0^1(\Omega)$. It has a real discrete spectrum $0 < \lambda_n$ with corresponding eigenfunctions $e_n \in \mathrm{H}_0^1(\Omega)$, which create an orthonormal basis in $\mathrm{L}^2(\Omega)$ and $Ae_n = \lambda_n e_n$. We can write $f = \sum_{n=0}^{\infty} (f, e_n) e_n$ and $u(t) = \sum_{n=0}^{\infty} \alpha_n(t) e_n$ with $\alpha_n(0) = 0$ for all n.

Without loss of generality we may assume that $(f, e_n) \geq 0$ for all n, because if not then we just replace e_n by $-e_n$. The relation (3.4) gives

$$\sum_{n=0}^{\infty} \left[\left(g_{1-\beta} * \alpha_n'\right)(t) + \alpha_n(t)\left(p(t)\lambda_n + q(t)\right)\right](e_n, \varphi) = h(t) \sum_{n=0}^{\infty} (f, e_n)(e_n, \varphi).$$

Putting $\varphi = e_j$ (for any j) we obtain

$$\left(D^{\beta}\alpha_j\right)(t) + \alpha_j(t)\left(p(t)\lambda_j + q(t)\right) = h(t)(f, e_j) \tag{3.5}$$

along with $\alpha_j(0) = 0 = \alpha_j(T)$.

The Mittag-Leffler function defined (for $\mu > 0, \gamma \in \mathbb{R}$) as

$$E_{\mu,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\mu k + \beta)}$$

plays an important role in the analysis of fractional differential equations. Consider for a while the following temporary time-fractional equation with a constant $\lambda \in \mathbb{R}_+$,

$$\left(D^{\beta} v\right)(t) + \lambda v(t) = \sigma(t), \qquad v(0) = 0. \tag{3.6}$$

Then the solution is given as (see eg. [20, (13)])

$$v(t) = \int_0^t (t-s)^{\beta-1} E_{\beta,\beta}\left(-\lambda(t-s)^{\beta}\right)\sigma(s)\ \mathrm{d}s.$$

We would like to get an interpretation of the solution in a case when $\lambda = \lambda(t)$. Assume

$$\lambda(t) < \overline{\lambda}, \qquad \forall t \in [0, T].$$

Therefore we rewrite (3.6) for $\lambda = \lambda(t)$ as

$$\left(D^{\beta} v\right)(t) + \overline{\lambda} v(t) = \sigma(t) + (\overline{\lambda} - \lambda(t)) v(t), \qquad v(0) = 0.$$

Its solution can be interpreted as

$$v(t) = \int_0^t (t-s)^{\beta-1} E_{\beta,\beta}\left(-\overline{\lambda}(t-s)^{\beta}\right) \left\{\sigma(s) + (\overline{\lambda} - \lambda(s)) v(s)\right\} \, \mathrm{d}s. \quad (3.7)$$

In light of this consideration, we rewrite (3.5) as follows

$$\begin{aligned} &\left(D^{\beta}\alpha_j\right)(t) + \alpha_j(t)\left(\overline{p}\lambda_j + \overline{q}\right) \\ &= h(t)\,(f, e_j) + \alpha_j(t)\left(\left[\overline{p} - p(t)\right]\lambda_j + \left[\overline{q} - q(t)\right]\right), \end{aligned} \quad (3.8)$$

where

$$p(t) < \overline{p}, \qquad q(t) < \overline{q}, \qquad \forall t \in [0, T],$$

along with $\alpha_j(0) = 0 = \alpha_j(T)$.

According to the interpretation formula (3.7) we may write

$$\begin{aligned} \alpha_j(t) &= \int_0^t (t-s)^{\beta-1} E_{\beta,\beta}\left(-\left(\overline{p}\lambda_j + \overline{q}\right)(t-s)^{\beta}\right) \\ &\quad \times \left\{h(s)\,(f, e_j) + \alpha_j(s)\left(\left[\overline{p} - p(s)\right]\lambda_j + \left[\overline{q} - q(s)\right]\right)\right\} \, \mathrm{d}s. \end{aligned} \quad (3.9)$$

In virtue of [21, Lemma 3.2], the function

$$t \mapsto E_{\beta,1}(-t)$$

is completely monotonic and for $\lambda > 0$ we have

$$\frac{d}{dt} E_{\beta,1}(-\lambda t^{\beta}) = -\lambda t^{\beta-1} E_{\beta,\beta}(-\lambda t^{\beta}) < 0.$$

Therefore

$$(t-s)^{\beta-1} E_{\beta,\beta}\left(-\left(\overline{p}\lambda_j + \overline{q}\right)(t-s)^{\beta}\right) > 0 \qquad \text{for } 0 < s < t.$$

We know that $(f, e_j) \geq 0$ and $h(s) \geq 0$. Then Lemma 3.1 implies that $\alpha_j(s) \geq 0$. Clearly $\left[\overline{p} - p(s)\right]\lambda_j + \left[\overline{q} - q(s)\right] > 0$. Thus, if we set $t = T$ at (3.9) we get

$$\begin{aligned} 0 &= \alpha_j(T) \\ &= \int_0^T (T-s)^{\beta-1} E_{\beta,\beta}\left(-\left(\overline{p}\lambda_j + \overline{q}\right)(T-s)^{\beta}\right) \\ &\quad \times \left\{h(s)\,(f, e_j) + \alpha_j(s)\left(\left[\overline{p} - p(s)\right]\lambda_j + \left[\overline{q} - q(s)\right]\right)\right\} \, \mathrm{d}s, \end{aligned}$$

which implies that

$$\alpha_j(s) = 0 = h(s)\,(f, e_j) \qquad \text{in } [0, T].$$

Due to the fact that $h \not\equiv 0$ we get $0 = (f, e_j)$. This is valid for any j, so we deduce that $u = 0 = f$, which concludes the proof. □

4. Conclusion

An ISP for a linear time-fractional diffusion equation with a solely space-dependent unknown source is studied. The coefficients of the differential operator are supposed to be time-dependent. New uniqueness results are derived in Theorem (3.1) under the assumption that $h \in C([0,T])$ and $0 \not\equiv h \geq 0$.

It would be nice to establish an uniqueness result for $f(\mathbf{x})$ considering a more general right-hand side of the form $f(\mathbf{x})h(\mathbf{x},t)$. Un-fortunately the proof-technique presented here is just partially applicable in this situation. If we assume that the given function h can be written as $h(\mathbf{x},t) = g(\mathbf{x})h(t)$, then our technique can be used. First, we can derive the uniqueness result for $g(\mathbf{x})f(\mathbf{x})$. This together with the condition that $g(\mathbf{x}) \neq 0$ implies the uniqueness of $f(\mathbf{x})$.

References

[1] A.I. Prilepko, D.G. Orlovsky, and I.A. Vasin, *Methods for Solving Inverse Problems in Mathematical Physics*. Chapman & Hall/CRC Pure and Applied Mathematics,Taylor & Francis, 2000.

[2] V. Isakov, *Inverse Source Problems*. Providence, RI: American Mathematical Society, 1990.

[3] A. Hasanov, Simultaneous determination of source terms in a linear parabolic problem from the final overdetermination: Weak solution approach. *J. Math. Anal. Appl.* **330**, No 2 (2007), 766–779.

[4] B.T. Johansson and D. Lesnic, A variational method for identifying a spacewise-dependent heat source. *IMA J. of Appl. Math.* **72**, No 6 (2007), 748–760.

[5] K. Van Bockstal and M. Slodička, Recovery of a space-dependent vector source in thermoelastic systems. *Inverse Problems Sci. Eng.* **23**, No 6 (2015), 956–968.

[6] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*. Ser. Applied Mathematical Sciences Vol. 44, Springer, 1983.

[7] V. Isakov, Inverse parabolic problems with the final overdetermination. *Commun. on Pure and Appl. Math.* **44**, No 2 (1991), 185–209.

[8] M. Slodička and B.T. Johansson, Uniqueness and counterexamples in some inverse source problems. *Appl. Math. Letters* **58** (2016), 56–61.

[9] M. Slodička, Uniqueness for an inverse source problem of determining a space dependent source in a non-autonomous parabolic equation. *Appl. Math. Letters* **107** (2020), Article 106395.

[10] M. Kirane, S.A. Malik, and M.A. Al-Gwaiz, An inverse source problem for a two dimensional time fractional diffusion equation with nonlocal

boundary conditions. *Math. Methods in Appl. Sci.* **36**, No 9 (2013), 1056–1069.

[11] M.I. Ismailov and M. Çiçek, Inverse source problem for a time-fractional diffusion equation with nonlocal boundary conditions. *Appl. Math. Modelling* **40**, No 7 (2016), 4891–4899.

[12] T. Wei and J. Wang, A modified quasi-boundary value method for an inverse source problem of the time-fractional diffusion equation. *Appl. Numer. Math.* **78** (2014), 95–111.

[13] K. Sakamoto and M. Yamamoto, Inverse source problem with a final overdetermination for a fractional diffusion equation. *Math. Control Relat. Fields* **1**, No 4 (2011), 509–518.

[14] F. Yang, X. Liu, and X.-X. Li, Landweber iterative regularization method for identifying the unknown source of the modified Helmholtz equation. *Boundary Value Problems* **2017**, No 1 (Jun 2017), Art. 91.

[15] J.-G. Wang, Y.-B. Zhou, and T. Wei, Two regularization methods to identify a space-dependent source for the time-fractional diffusion equation. *Appl. Numer. Math.* **68** (2013), 39–57.

[16] P. Niu, T. Helin, and Z. Zhang, An inverse random source problem in a stochastic fractional diffusion equation. *Inverse Problems* **36**, No 4 (Feb 2020), Art. 045002.

[17] M. Slodička, K. Šišková, and K. Van Bockstal, Uniqueness for an inverse source problem of determining a space dependent source in a time-fractional diffusion equation. *Appl. Math. Letters* **91** (2019), 15–21.

[18] J.A. Nohel and D.F. Shea, Frequency domain methods for volterra equations. *Advances in Math.* **22**, No 3 (1976), 278–304.

[19] Y. Luchko, Maximum principle for the generalized time-fractional diffusion equation. *J. Math. Anal. Appl.* **351**, No 1 (2009), 218–223.

[20] J.-S. Duan and L. Chen, Solution of fractional differential equation systems and computation of matrix Mittag–Leffler functions. *Symmetry* **10** (2018), Art. 503.

[21] K. Sakamoto and M. Yamamoto, Initial value/boundary value problems for fractional diffusion-wave equations and applications to some inverse problems. *J. Math. Anal. Appl.* **382**, No 1 (2011), 426–447.

Research Group Numerical Analysis and Mathematical Modelling
Department of Electronics and Information Systems
Ghent University
Krijgslaan 281, 9000 Ghent, BELGIUM
e-mail: marian.slodicka@ugent.be

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