

GENERAL QUANTUM THEORY — UNIFICATION OF CLASSICAL AND MODAL QUANTUM THEORIES

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ABSTRACT. Inspired by classical (“actual”) Quantum Theory over \mathbb{C} and Modal Quantum Theory (MQT), which is a model of Quantum Theory over certain finite fields, we introduce General Quantum Theory as a Quantum Theory — in the K benhavn interpretation — over general division rings with involution, in which the inner product “is” a $(\sigma, 1)$ -Hermitian form φ . This unites all known such approaches in one and the same theory, and we show that many of the known results such as no-cloning, no-deleting, quantum teleportation and super-dense quantum coding, which are known in classical Quantum Theory over \mathbb{C} and in some MQTs, hold for any General Quantum Theory. Our approach is totally different (and more general) than Hardy’s axiomatic approach [15], and generalized probability theories. On the other hand, in many General Quantum Theories, a geometrical object which we call “quantum kernel” arises, which is invariant under the unitary group $\mathbf{U}(V, \varphi)$, and which carries the geometry of a so-called polar space. This object cannot be seen in classical Quantum Theory over \mathbb{C} , but it is present, for instance, in all known MQTs. We use this object to construct new quantum (teleportation) coding schemes, which mix quantum theory with the geometry of the quantum kernel (and the action of the unitary group). We also show that in characteristic 0, every General Quantum Theory over an algebraically closed field behaves like classical Quantum Theory over \mathbb{C} at many levels, and that all such theories share one model, which we pin down as the “minimal model,” which is countable and defined over $\overline{\mathbb{Q}}$. Moreover, to make the analogy with classical Quantum Theory even more striking, we show that Born’s rule holds in any such theory. So all such theories are not modal at all. Finally, we obtain an extension theory for General Quantum Theories in characteristic 0 which allows one to extend any such theory over algebraically closed fields (such as classical complex Quantum Theory) to larger theories in which a quantum kernel is present. In this sense, these singular objects are always virtually around in abundance.

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1. INTRODUCTION

“One can give good reasons why reality cannot at all be represented by a continuous field. From the quantum phenomena it appears to follow with certainty that a finite system of finite energy can be completely described by a finite set of numbers (quantum numbers).”
(A. Einstein, from “The Meaning of Relativity”)

In classical Quantum Theory following the K benhavn interpretation — in some papers called “Actual Quantum Theory” (AQT) — the state space is a Hilbert space (foreseen with the standard inner product). More precisely:

- (*) a physical quantum system is represented by a Hilbert space $\mathcal{H} = ((\mathbb{C}^\omega, +, \cdot), \langle \cdot, \cdot \rangle)$, with $\langle \cdot, \cdot \rangle$ the standard inner product and ω allowed to be non-finite;
- (*) the standard inner product $\langle \cdot, \cdot \rangle$ sends $((x_1, \dots, x_\omega), (y_1, \dots, y_\omega))$ to $\bar{x}_1 y_1 + \dots + \bar{x}_\omega y_\omega$ (or $x_1 \bar{y}_1 + \dots + x_\omega \bar{y}_\omega$), where \bar{c} is the complex conjugate of $c \in \mathbb{C}$; complex conjugation is an involutory automorphism of the field \mathbb{C} ;
- (*) up to complex scalars, pure states (wave functions) are represented by nonzero vectors in \mathbb{C}^ω ; usually, one considers normalized vectors;
- (*) time evolution operators are represented by linear operators of \mathbb{C}^ω that preserve $\langle \cdot, \cdot \rangle$, that is, *unitary operators*. If ω is finite, unitary operators correspond to nonsingular complex $(\omega \times \omega)$ -matrices U such that $UU^* = \text{id}$;
- (*) measuring an observable A in a system described by the wave function $|\psi\rangle$, amounts to collapsing $|\psi\rangle$ into one of the orthogonal eigenvectors $|\psi_i\rangle$ of the Hermitian operator A , yielding as measurement the corresponding eigenvalue λ_i ;
- (*) composite product states correspond to tensor products $|\psi_1\rangle \otimes |\psi_2\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2$; if a state in $\mathcal{H}_1 \otimes \mathcal{H}_2$ is not a product state, it is entangled;
- (*) one follows Born’s rule, which says that $|\langle \psi, \psi_i \rangle|^2$ is the probability that the measurement λ_i will be made.

But *why* complex Hilbert spaces? On the algebraic level, states are solutions of a Schr dinger-type differential equation such as

$$(1) \quad \frac{\partial |\psi(t)\rangle}{\partial t} = \widehat{H}(|\psi(t)\rangle),$$

and solutions of such an equation satisfy a superposition principle: “linear combinations” of solutions are also solutions! Here, already some confusion turns up: eq. (1) implicitly is devised for a theory over the complex numbers — so that it encapsulates an *assumption* — but there seems to be no real reason *why* this assumption holds “in Nature.” In any case, the superposition principle is true for linear combinations *over any field*, so in any case: if Nature is described by such differential equations, the set of all solutions forms a vector space (over some field, or more general algebraic structure). Secondly, since any polynomial in one variable with complex coefficients has a complex root (since \mathbb{C} is algebraically closed), it follows that square complex matrixes always have eigenvalues, and on the level of observables (cf. the enumeration of properties above), this is an important property. Also, again because \mathbb{C} is algebraically closed, the inner product defined above allows us to have a normalization process and probabilities at our disposal. And of course, elegance and simplicity of the model adds to its value.

But what about Nature itself? It is usually argued that our universe has a finite number of particles, and so it some makes sense to consider finite models of Quantum Theory (over finite fields; see the next subsection). On the other hand, finiteness of our observable universe does not contradict the complex model at all: finite theories can obviously be embedded in infinite theories. Still, we would obtain situations on the formal level, in which vector spaces over finite fields are embedded in vector spaces over \mathbb{C} (or some other infinite field k), and at the very least, such embeddings could be of a nonstandard nature. The situation would be less problematic if k would *contain* that finite field, and hence would have

a positive characteristic. And still on the other hand, if one believes in the *multiverse* (which this author is sceptic about, since it appears to violate certain general variations of no-cloning results [31]), and then maybe also in a model with a nonfinite number of sub-universes, then the assumption of infinite fields seems to bear less obstructions than in one “local universe.”

If one accepts the fact that Schrödinger-type equations describe state spaces, then one needs vector spaces, and hence projective spaces. From a conceptual, synthetic–geometric viewpoint — a viewpoint which one *has* to take if one does not want to make further premeditated assumptions on the underlying algebraic structure — and taken that the projective dimension of such a space is at least 3, an old fundamental result reads that such an axiomatic projective space is always associated to a vector space over a *division ring* (see section 3).

As we will expose in this paper, Quantum Theory over division rings provides an extremely rich and powerful unifying theory, of which classical complex Quantum Theory is just one tiny piece. As finite fields are precisely the finite examples of division rings, finite Modal Quantum Theories come out as the finite models.

Once this general algebraic viewpoint is taken, we will explore the role of inner products in these theories, and create the natural formalism which yields all the known K benhavn Quantum Theories as very special cases.

And many other things which were completely hidden in the known theories of today.

1.1. Modal Quantum Theories. In [25] the authors propose to consider “Modal Quantum Theory” (MQT) as a toy model for AQT, in which \mathbb{C} is replaced by a finite field \mathbb{F}_q . Since an inner product is not defined on vector spaces over a finite field, the authors drop this aspect of the theory, but still build an interesting theory, obtaining for instance a no-cloning result, see [25]. The authors claim that by dropping inner products or variations, one does not have notions such as “orthogonality” at hand, and that can’t be what one wants. Also, one does not have probabilities at one’s disposal in the same way as in AQT. In [25] the authors focus primarily on the case $\mathbb{F}_q = \mathbb{F}_2$. In other papers such as [19], the authors consider vector spaces over finite prime fields \mathbb{F}_p with the property that -1 is not a square in \mathbb{F}_p , but it is in \mathbb{F}_{p^2} , so as to have, besides the similarity between the fields \mathbb{R} and \mathbb{C} , a Hermitian bilinear form at hand which shares many important aspects with the inner product $\langle \cdot, \cdot \rangle$. For one, the corresponding natural operators in that setting are also unitary operators, and now orthogonal vectors are well defined.

In various other parts of literature, allusions have been made on Quantum Theory over other rings (commutative or not), but mostly the discussions are speculative, and in any case not detailed. In this paper, we wish to address both MQT and variations over other rings, and formalize both ideas into one theory, namely Quantum Theory over division rings.

1.2. The present paper. We introduce General Quantum Theory (GQT) as a Quantum Theory in the K benhavn interpretation over general division rings with involution, in which the inner product is replaced by a $(\sigma, 1)$ -Hermitian form φ . This unites all known such approaches in one and the same theory, and we show that many of the known results such as:

- ★ no-cloning (a result first obtained for AQT by Wootters and Zurek [40, 41] and Dieks [12] in 1982, and later adapted to MQT over some finite fields in [25]),
- ★ no-deleting (obtained by Pati and Braunstein in [21] in 2000),
- ★ quantum teleportation (obtained in AQT by Bennett, Brassard, Cr peau, Jozsa and Wootters in [7] in 1993), and
- ★ super-dense quantum coding (described for AQT in Bennett and Wiesner [6] in 1992),

hold for any General Quantum Theory. One has to be a bit more careful in the proofs since multiplication is not necessarily commutative anymore, and often we have to make different proofs when the characteristic is 2.

On the other hand, in some General Quantum Theories, a geometrical object which we call “quantum kernel” arises, which is invariant under the unitary group $U(V, \varphi)$, and which carries the geometry of a so-called “polar space.” This is a combinatorial object which has a long history of study, and which is, if not trivial, the natural geometric module on which the unitary group acts. This object cannot be seen in classical Quantum Theory over \mathbb{C} , but it is present, for instance, in all known MQTs (but also many others

over algebraically closed fields or other division rings). We use this object to construct new quantum (teleportation) coding schemes, which mix quantum theory with the geometry of the quantum kernel (and the action of the unitary group). It is precisely this object (over finite fields) which has been used over and over again in Quantum Theory to construct and understand, e.g., maximal sets of mutually unbiased bases [29, 30].

We observe that in MQT there is no need to only consider prime fields with the aforementioned arithmetic property, and provide an elegant model in which one can see all MQTs in a fixed characteristic at once. We also show that in characteristic 0, every General Quantum Theory over an algebraically closed field behaves at many levels as classical Quantum Theory over the complex numbers, and that all such theories share one model, which we pin down as the “minimal model,” which is countable and defined over \mathbb{Q} . Moreover, to make the analogy with AQT even more striking, we show that Born’s rule holds in any such theory. (So these theories are not modal at all.)

Because of these results, we argue that one can replace classical Quantum Theory over \mathbb{C} by other Quantum Theories which carry the same information, but come with extra geometric tools, or by finite Quantum Theories (in which the quantum kernel is very well understood), or in the minimal model, which is countable. On the other hand, we hope to provide a deeper insight in these theories by introducing the present unification. Also, it is important to mention Wootters’s paper [36], in which the underlying finite geometrical nature of several fundamental quantum theoretical problems is investigated. The existence of the quantum kernel certainly adds a foundational aspect to the general theory.

1.3. Short overview of the sections. I have divided this paper in three larger parts: Part 1: Theory and foundations; Part 2: Applications in Quantum Information Theory; Part 3: Comparison to other theories, and conclusion. The titles speak for themselves.

In Part 1, the sections are as follows. In section 2, we give a very short overview of Modal Quantum Theories, with a number of corrections and generalizations of the current known state, prior to this paper. Then, in section 3, we motivate our use of division rings as the basic coordinatizing algebraic structures underlying Quantum Theories. In section 4, we explain our approach to “generalized inner products.” In the following section (section 5), we finally introduce *General Quantum Theories* (GQTs). In section 6, we show that the classic matrix formalism also works in GQTs, as such giving rise to tensor products and exotic “mixed Hilbert spaces.” In section 7, we show that many GQTs are endowed with a singular geometric object “quantum kernel” which in classical complex Quantum Theory cannot be observed. It is a very powerful object in code construction (as we will show in Part 2). In section 8, we explain how to geometrically characterize the quantum kernel by one axiom by using an old combinatorial result. In section 9, we give an example of a GQT over the complex numbers, with totally different properties than the classical theory. In section 10, we discover a large class of GQTs which resemble classical complex Quantum Theory, and we consider the problem of *extending* a given GQT to a larger GQT. This idea gives rise to a minimal (countable) model. And finally, in section 11, we study Born rules, and normalization.

In Part 2, we apply our theory. In section 12 we obtain a unified no cloning result for all GQTs; in section 13 a very short proof is obtained of a no quantum deletion result in all GQTs. Then, in section 14, we obtain a quantum teleportation scheme in all GQTs. And in section 15, a super-dense quantum-coding scheme. Finally, in section 16, we use the geometry of the quantum kernel in a low-dimensional GQT over finite fields, to obtain new strong quantum-coding schemes.

Finally, in Part 3, we compare GQT with other approaches to Quantum Theory. In section 17, we compare to Hardy’s axiomatic approach, and to Barret’s generalized probabilistic theories. In the last section, we conclude the paper.

In section A, we recall some basic notions and facts about field theory which is used throughout (without further notice).

1.4. Acknowledgments. I wish to thank Lucien Hardy for a number of very helpful communications on the subject of his axiomatic approach to Quantum Theory [15].

Part 1: Theory and foundations

2. MODAL QUANTUM THEORY AND VARIATIONS IN THE GENERAL SETTING

We recall the inception of “Modal Quantum Theory.” We describe a more general setting than the initial one in [25], and correct some ideas of [25].

In [25] the authors introduce *Modal Quantum Theory* (MQT) as a finite model for AQT, in which \mathbb{C} is replaced by a finite field \mathbb{F}_q . Inner products are not defined on vector spaces over a finite field, and hence the authors ignore this aspect of the theory. As such, one cannot speak of “orthogonal states.” In [19], the authors consider vector spaces over finite prime fields \mathbb{F}_p with the property that -1 is not a square in \mathbb{F}_p , but it is in \mathbb{F}_{p^2} , so as to have, besides the similarity between the fields \mathbb{R} and \mathbb{C} , a Hermitian bilinear form at hand which shares many important aspects with the inner product $\langle \cdot, \cdot \rangle$.

There is no need for restricting the theory to primes with the aforementioned property, as we will see below.

Let q be any prime power; then up to isomorphism \mathbb{F}_q has a unique extension of degree 2, namely \mathbb{F}_{q^2} . The map

$$(2) \quad \gamma : \mathbb{F}_{q^2} \mapsto \mathbb{F}_{q^2} : a \mapsto a^q$$

sends each element of \mathbb{F}_q to itself, while being an involutory automorphism of \mathbb{F}_{q^2} (that is, $u^{\gamma^2} = u$ for each $u \in \mathbb{F}_{q^2}$).

Let $n \in \mathbb{N}$ be any positive integer different from 0; then if $V = V(n, q^2)$ is the n -dimensional vector space over \mathbb{F}_{q^2} , define for $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ in V ,

$$(3) \quad \langle x, y \rangle := x_1^\gamma y_1 + \dots + x_n^\gamma y_n.$$

Then for $\rho, \rho' \in \mathbb{F}_{q^2}$ we have that

$$(4) \quad \langle \rho x, \rho' y \rangle = \rho^\gamma \langle x, y \rangle \rho', \text{ and } \langle x, y \rangle^\gamma = \langle y, x \rangle.$$

The linear $(n \times n)$ -matrices U which preserve the form $\langle \cdot, \cdot \rangle$ precisely are unitary matrices: $(n \times n)$ -matrices U for which $U^* U$ is the $(n \times n)$ -identity matrix, where $U^* := (U^\gamma)^T$.

In this model of QT, \mathbb{F}_{q^2} plays the role of \mathbb{C} , \mathbb{F}_q the role of \mathbb{R} , γ the role of complex conjugation, and $\langle \cdot, \cdot \rangle$ the role of inner product.

No-cloning and no-deleting can be obtained in these MQTs, but we will handle this in a much more general context in sections 12 and 13. By choosing any element κ in $\mathbb{F}_{q^2} \setminus \mathbb{F}_q$, we can represent each element of \mathbb{F}_{q^2} uniquely as

$$(5) \quad a + \kappa b,$$

with $a, b \in \mathbb{F}_q$. So viewed from this representation, the situation at least looks “a little classical.”

3. WHY DIVISION RINGS?

In this section, we motivate our use of division rings. The essential reason originates from the geometry of state rays, and an old result of Veblen and Young.

3.1. Projective geometries over division rings. Let D be a division ring, let $n \in \mathbb{N}$, and let $V = V(n, D)$ be the n -dimensional (left or right) vector space over D . We define the $(n-1)$ -dimensional (left or right) projective space $\mathbf{PG}(n-1, D)$ as being the geometry of vector subspaces of V . For example: the points of $\mathbf{PG}(n-1, D)$ correspond to the vector lines of $V(n, D)$, and the lines of $\mathbf{PG}(n-1, D)$ correspond to the vector planes of $V(n, D)$. A point of $\mathbf{PG}(n-1, D)$ lies on a line of $\mathbf{PG}(n-1, D)$ if the corresponding vector line lies in the corresponding vector plane. And so on. The choice of “left” or “right” does not affect the isomorphism class of the space.

If $D = \mathbb{F}_q$ is the finite field with q elements (q a prime power), one also writes $\mathbf{PG}(n-1, q)$ instead of $\mathbf{PG}(n-1, \mathbb{F}_q)$.

Geometrical properties of a $\mathbf{PG}(n-1, D)$ depend on the choice of division ring (see [26] for much more).

3.2. Veblen-Young result. In AQT, each quantum state is represented by an element $|\psi\rangle \neq \bar{0}$ of \mathbb{C}^n up to a scalar, that is, $|\psi\rangle$ and $c \cdot |\psi\rangle$ represent the same state ($c \neq 0$). So the geometry of quantum states in \mathbb{C}^n is in fact the geometry of its rays, that is, the corresponding projective geometry $\mathbf{PG}(n-1, \mathbb{C})$.

There is a notion of *axiomatic projective space*, which is defined to be a geometry of points and lines which satisfies certain axioms, of which the following are the essential ones:

- Two arbitrary different points are incident with precisely one line.
- Given two different intersecting lines U, V with $U \cap V = \{w\}$, and given two different lines X, Y meeting both U and V , and in points different from w , it holds that X and Y also meet in precisely one point.

Using only these axioms, one naturally defines a notion of *linear subspace*, and then one defines the dimension of the geometry in a natural way as well. The details can be found in [35], and will not be needed here.

A remarkable result of Veblen and Young [35] states that if the dimension $n-1$ of such a space is at least three, it is isomorphic to some $\mathbf{PG}(n-1, D)$ with D a division ring. This is not true when the dimension is less than three.

3.3. Setting. Because of the superposition principle, each “general Hilbert space” should include a vector space (as generalization of vector space over \mathbb{C}) which serves as the state space; on the other hand we do not want to make assumptions on the underlying algebraic structure of the vector space (since we want to avert assumptions on the algebraic properties of Nature). Our suggestion is to replace algebra by geometry, and as such circumvent any reference to linearity, or algebraic structure. The way to do this is through the geometric axioms of Veblen and Young for projective geometries, which have to hold for the ray spaces of the quantum systems. Then Veblen–Young theory leads us to vector spaces over division rings, and this is the start of our general setting.

In the next section, we introduce the most natural candidates for replacing the inner product we need to set up the theory.

4. LEXICON FOR $(\sigma, 1)$ -HERMITIAN FORMS

In this second lexicon, we introduce some terminology of Hermitian forms, needed to set the final stage in the context of inner products and orthogonality relations. We illustrate these notions through various concrete examples. The very important examples of *standard Hermitian forms* are explained, and the unitary group arises in the general context.

Let k be a division ring. An *anti-automorphism* of k is a map $\gamma : k \mapsto k$ such that

- γ is bijective;
- for any $u, v \in k$, we have $\gamma(u+v) = \gamma(u) + \gamma(v)$;
- for any $a, b \in k$, we have $\gamma(ab) = \gamma(b)\gamma(a)$.

4.1. Examples.

- If k is a commutative field, then anti-automorphisms and automorphisms coincide.
- Let \mathbb{H} be the quaternions, i.e., the set $\{a+bi+cj+dk \mid a, b, c, d \in \mathbb{R}\}$ with symbols i, j, k satisfying $i^2 = j^2 = k^2 = ijk = -1$. Then

$$(6) \quad \gamma : \mathbb{H} \mapsto \mathbb{H} : a + bi + cj + dk \mapsto a - bi - cj - dk$$

is an anti-automorphism. Moreover, the restriction to the subfield \mathbb{C} (generated by 1 and i) precisely is complex conjugation.

- The fields \mathbb{Q} and \mathbb{R} do not admit nontrivial automorphisms.

4.2. Hermitian forms. Suppose that k is a division ring, and suppose σ is an anti-automorphism of k . Let V be a right vector space over k . A σ -sesquilinear form on V is a map $\nu : V \times V \mapsto k$ for which we have the following properties:

- for all $a, b, c, d \in V$ we have that $\nu(a + b, c + d) = \nu(a, c) + \nu(b, c) + \nu(a, d) + \nu(b, d)$;
- for all $a, b \in V$ and $\alpha, \beta \in k$, we have that $\nu(a\alpha, b\beta) = \sigma(\alpha)\nu(a, b)\beta$.

We have that ν is reflexive if and only if there exists an $\epsilon \in k$ such that for all $a, b \in V$, we have

$$(7) \quad \nu(b, a) = \sigma\left(\nu(a, b)\right)\epsilon.$$

Such sesquilinear forms are called (σ, ϵ) -Hermitian. If $\epsilon = 1$ and $\sigma^2 = \text{id} \neq \sigma$, then we speak of a Hermitian form.

Clearly, the standard inner product in a classical Hilbert space over \mathbb{C} is a Hermitian form. In fact, any inner product on a complex Hilbert space is a Hermitian form.

4.3. Standard $(\sigma, 1)$ -Hermitian forms. If k is a division ring with involution σ , the *standard $(\sigma, 1)$ -Hermitian form* on the right vector space $V(d, k)$, is given by

$$(8) \quad \langle x, y \rangle := x_1^\sigma y_1 + \cdots + x_d^\sigma y_d,$$

where $x = (x_1, \dots, x_d)$ and $y = (y_1, \dots, y_d)$.

In the case that $\sigma = \text{id}$, we obtain a form which is usually called *symmetric*; it is not a proper Hermitian form, but still comes in handy in some situations (for example in cases of field reduction: “real Hilbert spaces” have often been considered in Quantum Theory; see e.g. [37, 2, 38]).

4.4. Morphisms of $(\sigma, 1)$ -Hermitian forms. An *automorphism* of a $(\sigma, 1)$ -Hermitian form φ on the k -vector space V , is a bijective linear operator $\omega : V \mapsto V$ which preserves φ , that is, for which

$$(9) \quad \varphi(\omega(x), \omega(y)) = \varphi(x, y)$$

for all $(x, y) \in V \times V$. The group of all such automorphisms is called the *unitary group*, and denoted $\mathbf{U}(V, \varphi)$.

Example. Let $k = \mathbb{C}$, σ be complex conjugation, and $V = V(n, \mathbb{C})$. Then $\mathbf{U}(V, \varphi) = \mathbf{GU}_n(\mathbb{C}) = \mathbf{U}(n)$.

5. GQT

In this short section, we finally pin down the notion of “General Quantum Theory” (GQT).

Throughout this paper, if we speak of “division ring with involution,” we mean a division ring with an involutory anti-automorphism.

$$\left\{ \begin{array}{l} \text{From now on, we propose to depict a physical quantum system by a general Hilbert space } \\ \mathcal{H} = \left((V(\omega, k), +, \cdot), \langle \cdot, \cdot \rangle \right), \text{ with } k \text{ a division ring with involution } \sigma, \text{ and } \langle \cdot, \cdot \rangle \text{ a } (\sigma, 1)\text{-Hermit-} \\ \text{ian form.} \end{array} \right\}$$

If we speak of “standard GQT,” we mean that given σ , the general Hilbert space comes with the standard $(\sigma, 1)$ -Hermitian form. Also, as some fields such as the reals and the rational numbers do not admit nontrivial involutions, they only can describe “improper” quantum systems. By extension of Quantum Theories (cf. section 10.2), this is no problem (as often has been the case when switching between AQT over \mathbb{C} and \mathbb{R}).

6. MATRIX REPRESENTATION, TENSOR PRODUCTS AND MIXED HILBERT SPACES

In this section, we show that the classical matrix formalism also works in any GQT, and in particular we define tensor (Kronecker) products. A new phenomenon arises in which we can build *mixed* generalized Hilbert spaces, and hence mixed GQTs, which arise from tensoring GQTs with different Hermitian forms.

Consider general Hilbert spaces \mathcal{H}_1 and \mathcal{H}_2 , with respective $(\sigma, 1)$ -Hermitian forms f_1 and f_2 defined over the same division ring D with involution σ . In the same way as for bilinear forms, we can represent f_1 and f_2 by using matrices when both vector spaces are finite-dimensional. Let the matrices be A_1 and A_2 respectively; then these are Hermitian matrices. If $x, y \in \mathcal{H}_1$ and $u, v \in \mathcal{H}_2$, then

$$(10) \quad f_1(x, y) = xA_1\sigma(y^T) \quad \text{and} \quad f_2(u, v) = uA_2\sigma(v^T).$$

One easily verifies that when D is commutative,

$$(11) \quad (x \otimes y)(A_1 \otimes A_2)(\sigma((y \otimes v)^T)) = (xA_1\sigma(y^T)) \otimes (uA_2\sigma(v^T)).$$

Here, all products are Kronecker products. The last equality does *not* necessarily hold when D is not commutative; in the commutative case, one knows that for matrices U, V, W, X for which the products UW and VX are defined, then

$$(U \otimes V)(W \otimes X) = (UW) \otimes (VX),$$

and this is not necessarily true over noncommutative D . On the level of tensor products though, one can make sense of the last equality in eq. (11), and we will come back to this aspect of the theory in a forthcoming paper. In any case, for any division ring D , we have that $\sigma((A_1 \otimes A_2)^T) = \sigma(A_1^T) \otimes \sigma(A_2^T) = A_1 \otimes A_2$, so $A_1 \otimes A_2$ is Hermitian as well. By linear expansion, it follows that $f_1 \otimes f_2$ defines a $(\sigma, 1)$ -Hermitian form over D on the tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2$. Similar reasoning can be done in the infinite-dimensional case. One has to be careful though with how one approaches the tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2$ when D is not commutative.

INTERMEZZO: Tensor products of Hilbert spaces in the noncommutative setting. If one considers (left) vector spaces \mathcal{H} (of dimension d) and \mathcal{H}' (of dimension d') over a noncommutative division ring k , it is hardly possible to define a tensor product $\mathcal{H} \otimes \mathcal{H}'$ in a natural way.

On the other hand, we have a natural tensor product (the Kronecker product) between matrices, so e.g. between operators or states, which perfectly works in the noncommutative case as well.

If $|\Psi\rangle$ and $|\Psi'\rangle$ are states in respectively the general Hilbert spaces \mathcal{H} and \mathcal{H}' , we consider their tensor product $|\Psi\rangle \otimes |\Psi'\rangle$ naturally as a state in the (left) vector space $k^{d \cdot d'}$. A similar remark goes for operators. Still, if $|\Psi\rangle$ and $|\Psi'\rangle$ are as above, and A and B operators in \mathcal{H} respectively \mathcal{H}' , then as already remarked, in general

$$(12) \quad (A|\Psi\rangle) \otimes (B|\Psi'\rangle) \neq (A \otimes B)(|\Psi\rangle \otimes |\Psi'\rangle).$$

Also, one needs to be very careful with the notion of “linear operator” in the noncommutative context: if we consider a d -dimensional left vector space \mathcal{H} over the noncommutative division ring D , and a nonsingular $(d \times d)$ -matrix A over D , then we have that

$$(13) \quad A(\mathbf{v}\lambda + \mathbf{w}\rho) = (A\mathbf{v})\lambda + (A\mathbf{w})\rho.$$

Here $\mathbf{v}, \mathbf{w} \in \mathcal{H}$ and $\lambda, \rho \in D$. In other words, if $f_A : \mathcal{H} \mapsto \mathcal{H}$ is the operator associated to A , then

$$(14) \quad f_A(\mathbf{v}\lambda + \mathbf{w}\rho) = f_A(\mathbf{v})\lambda + f_A(\mathbf{w})\rho,$$

and so f_A is *right* D -linear.

In conclusion, we can say that in every GQT, composite systems are represented in tensor products of the (generalized) Hilbert spaces (with $(\sigma, 1)$ -forms), as in AQT. Note that the above matrix formalism is not violated if one considers *different* $(\sigma, 1)$ -Hermitian forms in \mathcal{H}_A and \mathcal{H}_B . In such a way, one can construct mixed composite systems in a strong sense of the word, exposing mixed properties of the Hermitian forms from the components.

This is a new phenomenon which is not present in classical complex Quantum Theory.

6.1. **Example: AQT.** If we put $k = \mathbb{C}$ and take the standard inner products on \mathcal{H}_1 and \mathcal{H}_2 , then A_1 and A_2 are identity matrices, and so is $A_1 \otimes A_2$, so in that case $f_1 \otimes f_2$ is the standard inner product on $\mathcal{H}_1 \otimes \mathcal{H}_2$.

6.2. **Standard GQT.** Given any division ring D with involution σ , the standard $(\sigma, 1)$ -Hermitian form (with respect to a chosen basis) gives rise to the identity matrix.

7. THE QUANTUM KERNEL

In this large section we introduce a new singular quantum invariant which is present in each GQT. It is defined by self-orthogonal elements, and yields extra and deep geometric information about quantum systems. This object is a direct bridge to combinatorial incidence geometry (where it is well known and studied in a non-quantum context). We will obtain further information about orthogonality relations, provide a number of concrete examples, and mention a unitary action which is important for quantum information applications (as we will later see). This section is highly geometric in nature.

Let k be any division ring with involution γ , and let $\langle \cdot, \cdot \rangle_\gamma$ be a $(\sigma, 1)$ -Hermitian form on the right k -vector space V . To these data there is associated a geometrical object which we will describe below, and which we will call the *quantum kernel* relative to the given data. This geometric object is not visible over \mathbb{C} . Also, it essentially consists of the vectors which are orthogonal to themselves, and thus they cannot be normalized. This object was not described in [25], despite its truly different nature than its complement in the modal Hilbert space.

7.1. **Definition.** For any $v \in V$, define

$$(15) \quad \pi(v) := \{w \in V \mid \langle v, w \rangle_\gamma = 0\}.$$

Note that $\langle \cdot, \cdot \rangle_\gamma$ is reflexive. Further, for any vector subspace W of V , define

$$(16) \quad \pi(W) := \bigcap_{w \in W} \pi(w).$$

On the associated projective space $\mathbf{P}(V)$, we obtain a map π , denoted in the same way, which maps points to hyperplanes and hyperplanes to points. Moreover, we have that $\pi^2(v) = v$ for any vector $v \in V$. By definition, this means that π is a *polarity* of V , or $\mathbf{P}(V)$. We associate the following geometric structure, called *polar space* and denoted $\mathcal{P}(V, \pi)$, to $(\mathbf{P}(V), \pi)$ (and refer to the books [26, 33] for all the details):

- The points of $\mathcal{P}(V, \pi)$ are those points corresponding to self-orthogonal vectors v in V (so for which $\langle v, v \rangle_\gamma = 0$).
- The subspaces of $\mathcal{P}(V, \pi)$ are those projective subspaces α of $\mathbf{P}(V)$ such that $\alpha \subseteq \pi(\alpha)$; so, if W is the vector subspace of V corresponding to α , then $W \subseteq \bigcap_{w \in W} \pi(w)$.

We now investigate some basic properties of $\mathcal{Q} := \mathcal{P}(V, \pi)$.

7.1.1. *Structure of $\pi(v)$.* To fix ideas, we let $\langle \cdot, \cdot \rangle_\gamma$ be the standard $(\gamma, 1)$ -Hermitian form; everything in this paragraph is easily adaptable to general $(\gamma, 1)$ -Hermitian forms.

Let $v = (v_1, \dots, v_d)$; then the vectors $w = (w_1, \dots, w_d)$ of $\pi(v)$ are the solutions of $v_1 x_1^\gamma + \dots + v_d x_d^\gamma = 0$, so after letting γ act on both sides of the equation, they are the solutions of

$$(17) \quad x_1 v_1^\gamma + \dots + x_d v_d^\gamma = 0.$$

If v is not the zero vector, this means that $\pi(v)$ is a hyperplane in V (= a vector subspace of dimension $d - 1$).

7.1.2. *Polar inequality.* Let A and B be subsets of V such that $A \subseteq B$; then obviously $\pi(B) \subseteq \pi(A)$. In particular, if A and B are subspaces of V and B is absolute, then

$$(18) \quad A \subseteq B \subseteq \pi(B) \subseteq \pi(A),$$

so that A is absolute as well.

7.1.3. *Collinearity.* Let $\mathcal{P}(V, \pi)$ be as above. We say that points x, y of $\mathcal{P}(V, \pi)$ are *collinear* if they verify $\langle x, y \rangle = 0$. This is equivalent to saying that such points are collinear if there is some line of $\mathcal{P}(V, \pi)$ which contains both x and y .

7.1.4. *Self-orthogonality.* By the previous subsection, we have that if B is absolute, then all its vectors are also absolute, that is, for each $b \in B$ we have that $b \in \pi(b)$. In particular, $\langle b|b \rangle = 0$, so that b is self-orthogonal. The following observation is a direct corollary:

Proposition 7.1. *$\mathcal{P}(V, \pi)$ is empty if and only if there are no nonzero self-orthogonal vectors.*

In general Modal Quantum Theory, the situation is very different, and hence the quantum kernel carries extra information about the quantum system. We review some important examples in detail, starting with complex AQT.

7.2. **Example: the classical Hilbert space.** In the classical AQT-setting, so with $V \cong \mathbb{C}^d$, γ complex conjugation and $\langle \cdot, \cdot \rangle$ standard, the corresponding Hermitian form is also an inner product, so there are no non-trivial self-orthogonal vectors.

Proposition 7.2. *The quantum kernel for AQT is the empty space.*

7.3. **Example: general modal Quantum Theory.** Now we pass to finite fields. We use the setting of section 2. In that case, the Hermitian form is not an inner product, and the quantum kernel has a highly nontrivial structure. We consider $k = \mathbb{F}_{q^2}$, $V = V(4, q^2)$, and π as above; the involution γ is given by $\gamma : \mathbb{F}_{q^2} \mapsto \mathbb{F}_{q^2} : r \mapsto r^q$. Consider the polar space $\mathcal{Q} = \mathcal{P}(V, \pi)$ in $\mathbf{PG}(3, q^2)$; once chosen suitable homogeneous coordinates, its points satisfy the equation

$$(19) \quad X_0^{q+1} + X_1^{q+1} + X_2^{q+1} + X_3^{q+1} = 0.$$

We list some properties (all of which can be found and proved in [22, chapter 3]):

- (1) each point of \mathcal{Q} is contained in $q + 1$ lines; there are $(q + 1)(q^3 + 1)$ points;
- (2) each line of \mathcal{Q} contains $q^2 + 1$ points; there are $(q^2 + 1)(q^3 + 1)$ lines;
- (3) given a point x of \mathcal{Q} and a line U of \mathcal{Q} that does not contain x , there is a unique line V which contains x and meets U (in one point).

The case $q = 2$. Since \mathcal{Q} only has points and lines as full linear subspaces, one says it has *rank* 2. Because of properties (1)–(2)–(3), it is a *generalized quadrangle* of order (q^2, q) , and usually it is denoted as $\mathcal{H}(3, q^2)$.

Now we specialize to the case $q = 2$ to give the reader more insight in the geometry of this particular quantum kernel. We explain two alternative descriptions of $\mathcal{H}(3, 4)$, of which the first one comes from classical algebraic geometry, and the second one is combinatorial.

- (a) Consider a smooth complex cubic surface \mathcal{C} ; it is well known that \mathcal{C} contains 27 lines. These lines, together with all the intersection points, constitute the geometry of $\mathcal{H}(3, 4)$ (note the particularly interesting feature that as such, a geometry defined over \mathbb{F}_{2^2} is embedded in a projective space over \mathbb{C}).
- (b) Let \mathcal{W} be a set of points and lines defined as follows: its lines are unordered pairs $ij = ji$ with i and j distinct elements in $\{1, 2, \dots, 6\}$, and its points are sets $\{ij, kl, mn\}$ of lines such that $\{i, j, k, l, m, n\} = \{1, 2, 3, 4, 5, 6\}$. Now introduce 12 further symbols $1', 2', \dots, 6'$ and $1'', 2'', \dots, 6''$ which are by definition lines, and introduce further points as set $\{i', i'j', j''\}$ with $1 \leq i, j \leq 6$, and $i \neq j$. Then this geometry of points and lines is isomorphic to $\mathcal{H}(3, 4)$.

Still in (b), a useful exercise leads to the observation that \mathcal{W} also satisfies (3), and each point is contained in 3 lines, while each line contains 3 points. (It is a generalized quadrangle of order $(2, 2)$.)

Contrary to $\mathcal{H}(3, 4)$, we can easily draw the point-line geometry \mathcal{W} :

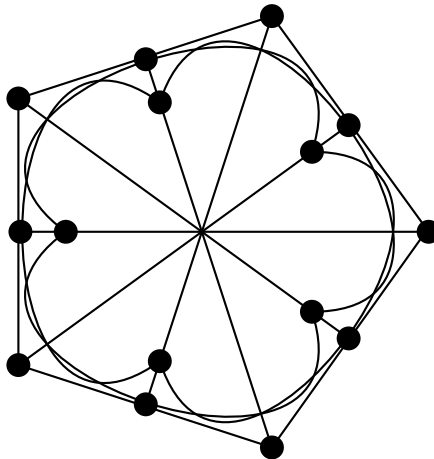


FIGURE 1. The generalized quadrangle of order $(2, 2)$. Its points are the black filled circles; its lines are the straight lines, together with the curves that contain one point c lying on the middle of some side of the pentagon, and the two closest points to that point lying in the interior, and not on the symmetry axis through c .

We will use this very important example to construct quantum coding schemes in section 16.

7.4. Automorphisms of $V \setminus \mathcal{Q}$. In AQT, the natural transformations to work with are unitary transformations; they are precisely the linear operators which preserve the inner product. In GQT, one would like to have these transformations at one's disposal as well, and this indeed remains the fact in the general theory. The fact that $\mathcal{Q} \neq \emptyset$ is no problem: in fact, it is the main geometric module to understand unitary groups in the first place.

By definition of the quantum kernel, any linear operator ω which preserves $\langle \cdot, \cdot \rangle$, also preserves self-orthogonality: if $\langle x, x \rangle = 0$, then

$$(20) \quad \langle \omega(x), \omega(x) \rangle = \langle x, x \rangle = 0.$$

So $U(V, \langle \cdot, \cdot \rangle)$ acts on \mathcal{Q} , and on $V \setminus \mathcal{Q}$. The action on the polar space \mathcal{Q} is highly nontrivial, and well known. For instance, when $D = \mathbb{F}_{q^2}$ and $d = 3$ or 4 , it acts transitively on the projective points and lines in the quantum kernel, and moreover, if we consider the stabilizer of any point v of \mathcal{Q} in the unitary group, it acts transitively on the points of \mathcal{Q} which are not orthogonal to v . For any division ring D (with involution) and any dimension, similar transitivity properties are known.¹

It goes without saying that such properties are highly suited for designing quantum codes. In AQT, for instance in quantum super dense-coding, the quantum gates operators are also used in setting up

¹One way of understanding this action on \mathcal{Q} is through the theory of “BN-pairs,” see for instance [32] and [8] for the general case of division rings, and [22] and [28] for the modal case of finite fields in low dimensions (in which case \mathcal{Q} is denoted by $\mathcal{H}(d, q^2)$; see section 7.3).

the coding scheme; see sections 15 and 16 for more details. We also note that since the unitary group $\mathbf{U}(V, \langle \cdot, \cdot \rangle)$ acts on $V \setminus \Omega$, it is not possible to transport elements of $V \setminus \Omega$ to Ω through unitary operators.

7.5. Inner–Polar dichotomy. The Inner–Polar dichotomy is the following principle:

$$\left\{ \begin{array}{l} \text{although we might partially lose the interpretation of probabilities if the Hermitian form of the} \\ \text{GQT is not an inner product, in general we gain extra structure because the quantum kernel —} \\ \text{the geometry of the vectors or projective points which are self-orthogonal — has a nontrivial} \\ \text{geometric structure.} \end{array} \right\}$$

A highly valuable and extreme example of this principle will be obtained in section 16: in that section, we will consider standard GQTs over finite fields \mathbb{F}_{q^2} which have a nontrivial quantum kernel for each value of q . Moreover, we will use the geometry of the quantum kernel to construct powerful quantum-codes in that section. As we will see in section 11, the probabilistic interpretation of Quantum Theory over finite fields is very different than in the case of real probabilities, because there do not exist total orders on finite fields. If we specialize to the case $q = 2$, then we only have the possible probabilities 0 (“NO”) and 1 (“YES”) (see section 11), but on the other hand we *do* have a valuable quantum kernel. We refer to section 16 for the concrete applications.

The word “partially” displayed in the principle, is important here: as we will see in section 11, in all GQTs of algebraically closed fields in characteristic 0 such as \mathbb{C} , we *do* have a probabilistic interpretation. Those theories in which there is a nontrivial quantum kernel, hence appear to contain *extra* information (due to the geometric structure of the quantum kernel).

In the next section, we give a first example of GQT over the complex numbers which enjoys this principle in a most interesting way. Others will arise in due course.

Question. What is the physical interpretation of the quantum kernel?

We offer some thoughts and speculation below, and hope to come back to this fundamental question in a forthcoming paper.

- It has been proposed by many that $|\langle \Psi(x), \Psi(x) \rangle|^2$ cannot be expected to describe the exact probability density for being detected at x . Quoting [20, section 3]:

“When a particle has been prepared in an ion trap (and hence is there with certainty), Born’s rule implies a tiny but positive probability that at an arbitrarily short time afterwards it is detected a light year away. In a similar spirit, Heisenberg wrote in 1930: “This result is stranger than it seems at first glance. As is well known, $\langle \Psi, \Psi \rangle$ diminishes exponentially with increasing distance from the nucleus; there is thus always a small but finite probability of finding the electron at a great distance from the center of the atom.” Thus $|\Psi(x)|^2$ cannot be the exact probability density for being detected at x .”

One of the main principles in [20] is that Born’s rule cannot be valid universally, and must be considered as a scientific law with a restricted domain of validity. In this point of view, nontrivial states $|\Psi\rangle$ for which $\langle \Psi, \Psi \rangle = 0$ might not seem to be so contradictory after all?

- Although we not necessarily claim that the standpoint of Neumaier [20] is correct in the context of classical AQT, we *do* think it might be applicable to (other) GQTs. Born’s criterion might only apply to the affine state set with equation

$$(21) \quad \langle \Psi, \Psi \rangle \neq 0,$$

the closed set of states which meet the equality representing singularities for the Physics of Born’s philosophy/law. We refer to section 11 for more on Born’s rule, and probabilities.

- Classical AQT can be embedded in a standard GQT over an AC field containing \mathbb{C} (according to section 10.2) so as to obtain a nontrivial quantum kernel in an extended setting. In other words:

“you can try to run, but you can’t hide.”

From the viewpoint of \mathbb{C} , the states in the quantum kernel cannot be seen, but they are indeed revealed after an appropriate field extension, and may be used (a philosophy we will apply in the applications part of this paper).

- The full unitary group $U_{n+1}(k)$, with k any field with involution, fixes the quantum kernel and its complement, so it naturally distinguishes between the “physical states” (those with $\langle \Psi, \Psi \rangle \neq 0$), and the “nonphysical states” (those with $\langle \Psi, \Psi \rangle = 0$). (A time evolution operator cannot send a physical state to a nonphysical state, and vice versa.)

- As we will see, the “nonphysical states” (maybe we should call them *phantom states* instead), reveal new and very strong structural tools on the level of Quantum Information Theory, as we will clearly show in the applications part of this paper.

8. GEOMETRIC AXIOM FOR THE QUANTUM KERNEL

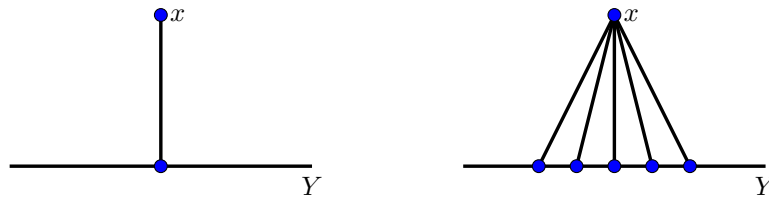
Several authors — such as Hardy [15] — have devised alternative axiomatic settings to define Quantum Theory. In this section, we derive the quantum kernel in all GQTs in dimension ≥ 3 from one — and only one — geometric axiom. We do ask that the quantum kernels are not trivial.

Let k be an arbitrary division ring with involution γ , and let $\langle \cdot, \cdot \rangle_\gamma$ be a $(\gamma, 1)$ -Hermitian form on the right k -vector space V (which we suppose to be of finite dimension, for the sake of convenience). In this section, we suppose that $\Omega \neq \emptyset$. Let \mathcal{P} be the point set of Ω , and let \mathcal{L} be the line set. The following defining axiom can be verified [33]:

One-or-All (OoA) If $x \in \mathcal{P}$ and $U \in \mathcal{L}$, and U does not contain x , then either all points of U are collinear with x , or precisely one point is.

Note that property (3) for the case $\Omega = \mathcal{H}(3, q^2)$ is a special case of the “One-or-All” axiom. A spectacular result of Buekenhout and Shult [9] characterizes polar spaces as those point-line spaces that satisfy (OoA).

Theorem 8.1 (Buekenhout and Shult [9]). *Let $\Gamma = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ be a point-line geometry which satisfies (OoA). Then there exists a division ring k with involution γ , a right k -vector space V and a $(\gamma, 1)$ -Hermitian form, such that $\Gamma = \mathcal{P}(V, \pi)$.*



Buekenhout-Shult One-or-All axiom.

9. OTHER GQTs OVER \mathbb{C} WITH NONTRIVIAL QUANTUM KERNEL

Even over the complex numbers, we can consider GQTs for which a nontrivial quantum kernel arises. In this section, we give such an example in vectorial dimension ≥ 5 .

Let $\ell \geq 4$ be a positive integer. Define the following $(\sigma, 1)$ -Hermitian form on $V(\ell + 1, \mathbb{C})$ or $\mathbf{PG}(\ell, \mathbb{C})$, where σ is complex conjugation (which we will denote by an overbar below):

$$(22) \quad \langle x, y \rangle = -(\overline{x_0}y_0 + \overline{x_1}y_1) + \overline{x_2}y_2 + \dots + \overline{x_\ell}y_\ell.$$

This form is not an inner product, but a nontrivial quantum kernel arises for each ℓ ; usually it is denoted by $\mathcal{H}(\ell, \mathbb{C})$, and as a projective variety, it has Witt index 1. This means it only contains points and lines as full linear subspaces. It is a generalized quadrangle for each ℓ (see [34]).

This GQT behaves much like AQT (it is obviously “complex-like” in the terminology of below, see section 10.1), and also obeys the Born rule — see section 11. On the other hand, the geometry of \mathcal{Q} (and the action of the unitary group upon \mathcal{Q}) yields additional information.

Even if one wants to work with classical AQT over \mathbb{C} , one could still use the quantum kernel as follows. Embed AQT in a larger GQT (see section 10.2 for more details) over for instance an algebraically closed field k that strictly contains \mathbb{C} , and extend the $(\sigma, 1)$ -Hermitian form over \mathbb{C} to k in such a way that a nontrivial quantum kernel arises over k . Classical AQT cannot see this kernel, but we can use it nevertheless, for instance in coding schemes such as the one presented in the final section of this paper.

10. COMPLEX-LIKE QUANTUM THEORIES, EXTENSION OF QUANTUM THEORIES, AND THE MINIMAL MODEL

This very important section might be the most challenging on a technical algebraic level, for the reading physicist. We study GQTs which deeply resemble classical complex Quantum Theory (all of them are defined over algebraically closed fields). We introduce “extension theory,” which focuses on GQTs which are contained in a natural way in larger GQTs (a situation which is crucially important in order to understand the bigger picture), and finally, we introduce a *minimal model* in characteristic 0, which is contained in *all* GQTs in characteristic 0.

As \mathbb{C} is an algebraically closed field, the algebraically closed fields amongst the division rings deserve separate interest in GQT. The following theorem is especially handy in this discussion.

Theorem 10.1 (Baer). *Let k be an algebraically closed field. Then k has nontrivial involutory automorphisms if and only if k has nontrivial automorphisms of finite order if and only if its characteristic is 0.*

So, each algebraically closed field k of characteristic 0 has a nontrivial involution σ in $\mathbf{Aut}(k)$, and hence any vector space over such a field has Hermitian forms, while vector spaces over fields such as $\overline{\mathbb{F}_p}$ with p any prime, cannot, and hence only have symmetric σ -sesquilinear forms.

We focus on characteristic 0 now.

10.1. Complex-like Quantum Theories. Let k be any algebraically closed field in characteristic 0. By Theorem 10.1, we know that there is an involution γ in $\mathbf{Aut}(k)^\times$. Now consider the set

$$(23) \quad k_\gamma := \{\kappa \in k \mid \kappa^\gamma = \kappa\}.$$

One easily shows that k_γ , endowed with the addition and multiplication coming from k , is also a field. There is however more (see [3]).

Theorem 10.2 ((\mathbb{C}, \mathbb{R}) -Analogy). *Let k be any algebraically closed field in characteristic 0. Let γ be an involution in $\mathbf{Aut}(k)^\times$. Then -1 is not a square in k_γ . Suppose $i \in k$ is such that $i^2 = -1$. Then $k = k_\gamma + i \cdot k_\gamma$ and $[k : k_\gamma] = 2$.*

So each element of k has a unique representation as $a + bi$, with $a, b \in k_\gamma$ and i a fixed solution of $x^2 = -1$. Fields which have index 2 in their algebraic closure are called *real-closed fields*, and can always be constructed as a k_γ of some involution γ . Real-closed fields share many properties with the reals \mathbb{R} : each such field is *elementarily equivalent* to the reals, which by definition means that it has the same first-order properties as the reals. We call a GQT *complex-like* if it is defined over an algebraically closed field k with nontrivial involution γ , where the elements of k are represented in Theorem 10.2 with respect to the field k_γ .

The analogy goes even further: once we have defined k_σ as above, and we represent each element x in k as $x = u + iv$, it can be shown that the automorphism σ is given by

$$(24) \quad \sigma : k \mapsto k : u + iv \mapsto u - iv.$$

(This is easy to see: as $u, v \in k_\sigma$, they are fixed by σ , and as $-1 \in k_\sigma$, it is also fixed. So either $\sigma(i) = i$ or $\sigma(i) = -i$. Since σ is not trivial, we have the second possibility, and this concludes the proof.)

10.2. Extension of Quantum Theories. If we consider a GQT over a field k in characteristic 0, the fundamental question arises if it is *embeddable* in a complex-like theory (or in any other GQT, for that matter). Here, the notion of “embeddable” is obvious: if k comes with the involution γ , we ask for a field extension ℓ/k , where ℓ is algebraically closed, and an involution $\bar{\gamma}$ of ℓ , such that the restriction of $\bar{\gamma}$ to k is γ . Since any GQT is only dependent on the Hermitian matrix of the $(\sigma, 1)$ -Hermitian form with respect to a chosen basis (with suitable adaptation to the infinite-dimensional case), it is clear that if the aforementioned GQT comes with matrix A over k , then the same matrix A defines a $(\bar{\gamma}, 1)$ -Hermitian form over ℓ which induces the initial form over k . So if we fix the dimension of the Hilbert space, then any GQT over k (and with respect to γ) is part of the GQT over ℓ (with involution $\bar{\gamma}$). Unfortunately, such a general result cannot hold. Consider for instance the field $\mathbb{Q}(\sqrt{2})$ with involution γ generated by

$$(25) \quad \sqrt{2} \mapsto -\sqrt{2}.$$

Then in [24] Schnor shows that although it does extend to an automorphism of \mathbb{C} , every extension of γ would have infinite order. We adapt his proof to our setting, as follows. Suppose ℓ/k is an algebraically closed field with involution $\sigma \neq \text{id}$, such that σ extends γ . Then ℓ_σ is a subfield of index 2 in ℓ , and ℓ_σ is real-closed. Let $i \in \ell \setminus \ell_\sigma$ be such that $i^2 = -1$ (noting that $-1 \in \ell_\sigma$). As $\ell = \ell_\sigma(i)$, it follows that $\sigma(i) = -i$. So $\mathbb{Q}(\sqrt{2}i) \subseteq \ell_\sigma$, as $\sigma(\sqrt{2}i) = \sigma(\sqrt{2})\sigma(i) = \sqrt{2}i$. In $\mathbb{Q}(\sqrt{2}i)$ we have the identity

$$(26) \quad -1 = -2 + 1 = (\sqrt{2}i)^2 + 1^2,$$

so -1 is a sum of squares in the real-closed field ℓ_σ , contradiction.

However, as Schnor points out in [24], the result is true if we suppose k to be algebraically closed to begin with.

Theorem 10.3 (Embedding Theorem of Quantum Theories). *Any GQT over an algebraically closed field k with involution γ is embeddable in a GQT over ℓ , where ℓ is any algebraically closed field extension of k .*

So even if one prefers to keep working with AQT over \mathbb{C} , it is a fundamental fact that AQT is embedded in a universe of GQTs which extend AQT, in many of which quantum kernels arise which cannot be seen in AQT, but which are virtually there. They should supply an extremely rich and powerful source for new tools in, e.g., Quantum Information Theory. Again, we should not only look for complex-like extensions of a given GQT — we only did that in this section because they look similar as AQT, and because we already have considered quite some details about them.

Even if there is one datum (division ring, $(\sigma, 1)$ – Hermitian form) best suited to describe a Quantum Theory resembling Nature (call the corresponding GQT “ \mathcal{T} ”), should we *only* consider this GQT, or simultaneously all GQTs which extend this theory? The set of all these extensions contains an enormous amount of hidden information which cannot be grasped from inside \mathcal{T} . If \mathcal{T} would happen to be classical complex Quantum Theory for example, we could extend \mathcal{T} to a GQT with quantum kernel \mathcal{Q} , and perform operations using \mathcal{Q} which cannot be observed from inside \mathcal{T} . We given one example involving \mathbb{C} and \mathbb{H} .

We let $m \in \{3, 4\}$, and consider the following anti-involutory automorphism μ of \mathbb{H} :

$$(27) \quad \mu : a + ib + jc + kd \mapsto a - ib + jc + kd.$$

Here, the notation of section 4 is in force. Then define $\langle \cdot, \cdot \rangle_\mu$ to be the standard $(\mu, 1)$ -Hermitian form on $V(m+1, \mathbb{H})$. The quantum kernel \mathcal{Q} is not empty, and carries the point-line geometry of an infinite generalized quadrangle which is usually denoted by $\mathcal{H}(m, \mathbb{H}, \mathbb{C})$ [34]. Of course, if we restrict $\langle \cdot, \cdot \rangle_\mu$ to $V(m+1, \mathbb{C})$, then we recover classical complex Quantum Theory in vectorial dimension $m+1 \in \{4, 5\}$. In other words, the GQT over \mathbb{H} is an extension of the classical GQT over \mathbb{C} , and over \mathbb{H} we have a nontrivial quantum kernel. Now quantum-coding schemes can be made in the Hilbert space $V(m+1, \mathbb{C})$ which use the geometry \mathcal{Q} ; we will explicitly demonstrate such a scheme in the case of finite fields, in section 16.

Note that nothing forbids us to consider values of m in the set $\{5, 6, \dots\}$: in all these cases a nontrivial quantum kernel arises over \mathbb{H} which is not to be seen over \mathbb{C} . In these cases, not only points and lines occur, but also higher dimensional vector subspaces.

10.3. The minimal model: $\overline{\mathbb{Q}}$. The following result is a well-known standard result on sizes of algebraically closed fields.

Theorem 10.4. *Let k be any field. If k is not finite, its algebraic closure \overline{k} has the same size as k ; if k is finite, \overline{k} is countable.*

Each field of characteristic 0 contains the characteristic 0 prime field: the field of rationals \mathbb{Q} . So each algebraically closed field k in characteristic 0 contains the algebraically closed field $\overline{\mathbb{Q}}$:

$$(28) \quad \overline{\mathbb{Q}} \subseteq k.$$

By Theorem 10.4, $\overline{\mathbb{Q}}$ is countable, and hence also minimal in size with respect to being algebraically closed.

Each GQT over $\overline{\mathbb{Q}}$ can hence be seen as a minimal model for other GQTs in characteristic 0. By the Embedding Theorem 10.3, any minimal GQT can be embedded in a GQT over any other given algebraically closed field in characteristic 0. In the other direction, if k is algebraically closed in characteristic 0, and we consider a GQT over k with involution σ , σ fixes the prime field \mathbb{Q} , so also $\overline{\mathbb{Q}}$ (the induced action might be trivial, of course, and then the induced quantum theory over $\overline{\mathbb{Q}}$ is orthogonal/symmetric).

11. ALL THE GENERAL BORN IDENTITIES, AND NORMALIZATION

In this section, we study the Born probability rule in various GQTs, and consider normalization in such (and other) theories.

11.1. Born identities in complex-like GQTs. As we have seen in the previous section, each GQT over an algebraically closed field k in characteristic 0 is complex-like, and all such theories essentially contain the minimal model. In this section we note that even more, all such theories enjoy precisely the same Born rule as AQT does.

First, we make a number of notes on observables and eigenbases in GQTs. We do not claim any originality in the next section.

11.2. Hermitian eigenbases. If k is an algebraically closed field, every square Hermitian matrix yields an orthogonal base of eigenvectors. The proof is exactly the same as for the complex case, and we sketch the proof here.

So we fix an algebraically closed field k with involution σ , and let $\langle \cdot, \cdot \rangle$ be the standard inproduct on the n -dimensional Hilbert space \mathcal{H} over k . Let A be a square Hermitian ($n \times n$)-matrix. Then for states $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$, we have that $\langle xA^T, y \rangle = \sigma(xA^T)y^T = \sigma(x)Ay^T = \langle x, yA^T \rangle$.

If λ is an eigenvalue of A and $u = (u_1, \dots, u_n)$ is an eigenvector coming with λ , we have that

$$(29) \quad \langle uA^T, u \rangle = \langle \lambda u, u \rangle = \lambda^\sigma \langle u, u \rangle = \langle u, uA^T \rangle = \lambda \langle u, u \rangle,$$

so that $\lambda = \lambda^\sigma$, taken that $\langle u, u \rangle \neq 0$.

If λ, λ' are different eigenvalues of A with respective eigenvectors x and y , then (as $\lambda^\sigma = \lambda$ and $\lambda'^\sigma = \lambda'$) we have that

$$(30) \quad \lambda \langle x, y \rangle = \langle xA^T, y \rangle = \langle x, yA^T \rangle = \langle x, \lambda' y \rangle = \lambda' \langle x, y \rangle,$$

so that $\langle x, y \rangle = 0$.

Now let λ be an eigenvalue of A with eigenvector x ; then x^\perp is an $(n-1)$ -dimensional subspace of \mathcal{H} , and A fixes x^\perp as $\sigma(x)(Ay^T) = \sigma(xA^T)y^T = \lambda^\sigma \sigma(x)y^T = 0$ for all $y \in x^\perp$, so that A acts as a Hermitian operator on x^\perp . Applying induction on x^\perp enables us to conclude that \mathcal{H} has an orthogonal eigenbase of eigenvectors for A . (Here, we use the fact that A is AC.)

In conclusion, we have the next theorem.

Theorem 11.1 (Spectral Theorem). *Let k be an algebraically closed field with involution σ , and let $\langle \cdot, \cdot \rangle$ be the standard inproduct on the n -dimensional Hilbert space \mathcal{H} over k . Let A be a Hermitian ($n \times n$)-matrix. Then:*

- (a) *for all eigenvalues λ of A , we have that $\lambda^\sigma = \lambda$ if there is some eigenvector u for λ such that $\langle u, u \rangle \neq 0$, so that λ is part of the fixed elements field k_σ of σ ;*

- (b) if x and y are eigenvectors corresponding to different eigenvalues, then they are orthogonal;
(c) \mathcal{H} has an orthogonal bases of eigenvectors of A .

■

The last point implies that in any standard GQT over an algebraically closed field with arbitrary involution, the theory of observables is well defined.

If k is a field (or division ring, for that matter) which is not necessarily algebraically closed, we cannot conclude a priori that the characteristic polynomial of a given square Hermitian matrix has even one root, so we cannot control the eigenvalues. This means that (c) of the previous theorem is not true for arbitrary observables. (And also (a) and (b) run into trouble when multiplication is not commutative.)

Two cases have to be kept in mind (still for k not algebraically closed):

- suppose A is a Hermitian operator for which there are no eigenvalues in k ; then A cannot produce measurements, so such Hermitian matrices are excluded as observables;
- if A does have roots, we can construct a base of eigenvectors of A , but possibly of a proper subspace of \mathcal{H} .

Among the observables in the latter case, one has the operators which have an orthogonal eigenbase for the entire Hilbert space \mathcal{H} . They play the same role as “classical observables.”

By Theorem 11.1(a), the measurements are elements of k_σ (in case k is AC), and we refer to the next section for more details on Born’s rules in this context.

11.3. The general Born identities. So suppose we consider such a GQT, and let σ be the corresponding involution. We represent each element in the algebraically closed field k as $a + ib$, with $i^2 = -1$ and a, b in the fixed field k_σ . Now let a quantum system be described by the wave function $|\psi\rangle$, and consider an observable which corresponds to the Hermitian operator A , with orthonormal base of eigenvectors $\{|\phi_1\rangle, \dots, |\phi_d\rangle\}$. To each $|\phi_i\rangle$ is associated an eigenvalue λ_i of A . Then the wave function $|\psi\rangle$ assigns a probability amplitude $\langle\phi_i|\psi\rangle \in k$ to the possible outcome measurement λ_i . If $\langle\phi_i|\psi\rangle = u + iv$ with $u, v \in k_\sigma$, then we define the *probability* of the measurement λ_i to be

$$(31) \quad \left| \langle\phi_i|\psi\rangle \right|^2 = u^2 + v^2.$$

The latter quantity is an element of k_σ . Now since k_σ is real-closed, it is well known that squaring naturally induces a total order \leq on k_σ , in which the set of all squares precisely is the set of positive numbers relative to \leq . In this total order, we say that $a < b$ if and only if we can find some square c^2 such that $a + c^2 = b$. So each quantity coming from the general Born rule (31) is a positive number, and two such quantities can be compared in the naturally defined total order. In the particular case of

$$(32) \quad k = \mathbb{C}, \sigma = \text{complex conjugation}, k_\sigma = \mathbb{R},$$

the induced total order is the standard one in \mathbb{R} , and we then obtain the classical Born rule. As to whether we can also normalize states, we refer to subsection 11.6.

11.4. Finite fields. We cannot have the same interpretation in arbitrary other division rings with involution. For instance, in the case of finite fields \mathbb{F}_{q^2} (section 2), we can indeed define, once we have chosen κ as in that section, $|\langle\phi_i|\psi\rangle|^2$ to be $a^2 + b^2 \in \mathbb{F}_q$, but finite fields do not admit a total order, so we lose our classical interpretation of probabilities. On the other hand, we might interpret the quantity $a^2 + b^2$ as some kind of probability by which we can distinguish between different probabilities, but not say which one is the more probable than the other, unless one is 0 (or 1). (The extreme case is $\mathbb{F}_q = \mathbb{F}_2$, where we only have the probabilities 0 and 1.) We hope to circumvent this issue in upcoming work.

11.5. Other fields. For the sake of completeness, we mention a large number of fields (besides the ones already mentioned) which *do* admit a total order:

- the field of *real rational functions* $p(x)/q(x)$ (where $p(x)$ and $q(x)$ are real polynomials in the variable x , and $q(x) \neq 0$);
- the field $\mathbb{R}((x))$ of *formal Laurent series* (with x taken infinitesimal and positive);

- the fields of *superreal numbers* and *hyperreal numbers*;
- every *subfield* of a totally ordered field.

In general, one can prove that a field k can be totally ordered if and only if 0 is **not** a sum of nonzero squares. Clearly, this criterion immediately rules out the complex numbers, finite fields, and more generally fields of characteristic $\neq 0$ (since those contain finite subfields).

11.6. Normalization. If we consider a complex-like GQT as in subsection 11.1, we can easily normalize states; in particular, we recover normalization in the classical complex case. The idea is of course very simple: if $|\psi\rangle = (u_1, \dots, u_m)$ with $u_r = a_r + ib_r$ for each r , then $\langle\psi|\psi\rangle$ is a sum of squares, say $(a_1^2 + b_1^2) + \dots + (a_m^2 + b_m^2) = R$, and R is a square in k . Put $R = Q^2$. Then $\frac{|\psi\rangle}{Q}$ is a normalized vector (one can take Q to be positive in the setting of complex-like GQTs). Of course, one notes that $|\psi\rangle$ must be taken outside the quantum kernel (since otherwise $R = 0$).

If we consider a not necessarily standard GQT over an algebraically closed field, we can do the exact same thing, and hence normalization also works.

More generally, we can also perform normalization in a (not necessarily standard) GQT over a field in which every element is a square.

Part 2: Applications in Quantum Information Theory

12. NO CLONING IN GQT

We obtain a no cloning result in any GQT over any division ring, as such generalizing many known results on both classical and modal quantum theories in a unified manner.

We repeat that whenever $\mathcal{H}_A \otimes \mathcal{H}_B$ is mentioned, with \mathcal{H}_A of dimension d_A and \mathcal{H}_B of dimension d_B over division ring k , it denotes a vector space over k of dimension $d_A \cdot d_B$ (cf. section 6), foreseen with the standard inproduct.

In this section we will obtain a no-cloning result, similarly as in AQT and MQT — see [25]. We will consider a GQT $\mathcal{Q} = ((k^n, +, \cdot), \langle \cdot, \cdot \rangle)$ over any division ring k ; the result, which generalizes the results of [25, 40], is proven in a slightly different manner, since k is not necessarily commutative.

Let \mathcal{H}_A and \mathcal{H}_B be copies of a Hilbert space \mathcal{H} , and consider two states $|\phi\rangle_A$ and $|\psi\rangle_A$ in \mathcal{H}_A . Suppose U is a cloning unitary operator so that, for any $\alpha, \beta \in k$, we have

$$(33) \quad U \left[(\alpha|\phi\rangle_A + \beta|\psi\rangle_A) \otimes |e\rangle_B \right] = (\alpha|\phi\rangle_A + \beta|\psi\rangle_A) \otimes (\alpha|\phi\rangle_B + \beta|\psi\rangle_B),$$

where $|e\rangle_B$ is an unknown blanco state (and \otimes is the usual matrix tensor (Kronecker) product). Using linearity of the operator on the left-hand side, and writing out the right-hand side, we obtain that

$$(34) \quad \alpha|\phi\rangle_A \otimes |\phi\rangle_B + \beta|\psi\rangle_A \otimes |\psi\rangle_B = (\alpha|\phi\rangle_A) \otimes (\alpha|\phi\rangle_B) + (\alpha|\phi\rangle_A) \otimes (\beta|\psi\rangle_B) + (\beta|\psi\rangle_A) \otimes (\alpha|\phi\rangle_B) + (\beta|\psi\rangle_A) \otimes (\beta|\psi\rangle_B).$$

First note that if k is commutative and ϕ_A and ψ_A would happen to be orthogonal, then (34) leads to the conditions $\alpha^2 = \alpha$ and $\beta^2 = \beta$ (for all $\alpha, \beta \in k$), and this only is true when k is the finite field \mathbb{F}_2 . We proceed with the general case.

Let $\alpha = 1 = \beta$; then we get that

$$(35) \quad |\phi\rangle_A \otimes |\psi\rangle_B + |\psi\rangle_A \otimes |\phi\rangle_B = 0.$$

Put $|\phi\rangle_A = (a_1 \dots a_n)^T$ and $|\psi\rangle_A = (b_1 \dots b_n)^T$. Then (35) implies that (and is equivalent with)

$$(36) \quad a_i b_j = -b_i a_j \quad \forall i, j \in \{1, \dots, n\}.$$

Suppose w.l.o.g. that $a_1 \neq 0$. Then $(a_1^{-1}(-b_1))a_j = b_j$ for all $j = 1, \dots, n$. So either $b_1 = 0$, and then $|\psi\rangle_A$ is the zero-vector, or $b_1 \neq 0$ and $|\phi\rangle_A$ and $|\psi\rangle_A$ are members of the same ray (when we multiply by the left). (Note that in a division ring, there are no zero divisors.)

12.1. Rays. Suppose k is a field, and suppose $|\psi\rangle_A$ and $|\phi\rangle_A$ are in the same ray. Write, for a fixed $\rho \in k^\times$, and using the same notation as in the previous paragraphs,

$$(37) \quad \rho a_i = b_i \quad \forall i.$$

Then (36) implies that

$$(38) \quad a_i b_j + b_i a_j = 2\rho a_i a_j = 0 \quad \forall i, j.$$

So either $2 = 0$, and k has characteristic 2, or $a_i a_j = 0$ for all i, j . In the latter case, we may take $i = j$, so $|\phi\rangle_A$ is the zero vector. (If k has characteristic 2, and $|\psi\rangle_A \neq 0$, $|\phi\rangle_A \neq 0$ are cloned, and $|\psi\rangle_A + |\phi\rangle_A$ as well, $|\phi\rangle_A$ is in the same ray as $|\psi\rangle_A$.)

Suppose k is a division ring which is not a field, and suppose $|\psi\rangle_A$ and $|\phi\rangle_A$ are in the same ray; let ρ be as in the previous section.

Then for all i, j we obtain from (36):

$$(39) \quad a_i \rho a_j = -\rho a_i a_j.$$

So for all i , we obtain that

$$(40) \quad a_i \rho = -\rho a_i,$$

and this condition is sufficient to guarantee that if $|\phi\rangle_A$ and $|\psi\rangle_A$ are cloned, then also $|\phi\rangle_A + |\psi\rangle_A$ is.

Whence for division rings, some nontrivial subsets of rays can be cloned in principle. In particular, by the above, we have that if $|\psi\rangle_A$, $|\phi\rangle_A$, and $|\psi\rangle_A + |\phi\rangle_A$ are cloned, then $|\psi\rangle_A = 0$ or $\rho|\psi\rangle_A = |\phi\rangle_A$ for some $\rho \neq 0$, and (40) holds. Note that $|\psi\rangle_A + |\phi\rangle_A$ is of the form $(1 + \rho)|\psi\rangle_A$.

Once one passes to projective space, rays become points.

Remark 12.1 (Permutation cloning in GQT and AQT). *Let k a division ring with involution, consider a GQT $\mathcal{Q} = ((k^n, +, \cdot), \langle \cdot, \cdot \rangle)$ over k . Let S be any subset of k^n , and let $|e\rangle$ be a blank state. We say that a unitary operator U permutation clones S if*

$$(41) \quad U : k^n \otimes k^n \mapsto k^n \otimes k^n : |\psi\rangle \otimes |e\rangle \mapsto U(|\psi\rangle \otimes |e\rangle)$$

induces a bijection between the set $S \otimes |e\rangle := \{|\psi\rangle \otimes |e\rangle \mid |\psi\rangle \in S\}$ and the set $\mathcal{C}(S) := \{|\varphi\rangle \otimes |\varphi\rangle \mid |\varphi\rangle \in S\}$. Clearly, if U permutation clones S , it induces a permutation on the elements of S in a natural way, and we say that it is the associated permutation.

Examples, special cases.

- Let $S = k^n$. As U is unitary, it is nonsingular, so permutation clones S .
- Put $S = \{|\psi\rangle\}$; then U clones $|\psi\rangle$.
- Suppose U is a unitary operator which clones each element of some set S ; then U permutation clones S with associated permutation id (the trivial permutation).

13. NO QUANTUM DELETING IN GQT

In this section, we obtain a very short approach to no quantum deleting. Again, we unify various known results in the known sub-quantum theories with different proofs.

In a similar fashion as for cloning, we now obtain a no-deleting result, as in AQT and MQT — see [25, 21]. We will again consider GQT over any division ring k ; we will give a short treatment that only uses the invertibility of the deleting operator.

Let \mathcal{H}_A and \mathcal{H}_B be copies of a general Hilbert space \mathcal{H} , and consider two states $|\phi_A\rangle$ and $|\psi_A\rangle$ in \mathcal{H}_A , and copies $|\phi_B\rangle$ and $|\psi_B\rangle$ in \mathcal{H}_B . Suppose U is an invertible operator (contained in $\mathbf{GL}_{d^2}(k)$) so that

$$(42) \quad U : \begin{cases} |\psi\rangle_A \otimes |\psi\rangle_B & \mapsto |\psi\rangle_A \otimes |e\rangle_B \\ |\phi\rangle_A \otimes |\phi\rangle_B & \mapsto |\phi\rangle_A \otimes |e\rangle_B, \end{cases}$$

where $|e\rangle_B$ is some unknown blank state in \mathcal{H}_B . So $\alpha|\psi\rangle_A \otimes |\psi\rangle_B + \beta|\phi\rangle_A \otimes |\phi\rangle_B$ is mapped by U to $\alpha|\psi\rangle_A \otimes |e\rangle_B + \beta|\phi\rangle_A \otimes |e\rangle_B$.

Since U is invertible and linear, we have that for every $\alpha, \beta \in k$, that

$$(43) \quad U^{-1} : \begin{cases} (\alpha|\psi\rangle_A + \beta|\phi\rangle_A) \otimes |e\rangle_B & \mapsto (\alpha|\psi\rangle_A + \beta|\phi\rangle_A) \otimes (\alpha|\psi\rangle_B + \beta|\phi\rangle_B) \\ & = \alpha|\psi\rangle_A \otimes \alpha|\psi\rangle_B + \alpha|\psi\rangle_A \otimes \beta|\phi\rangle_B + \beta|\phi\rangle_A \otimes \alpha|\psi\rangle_B + \beta|\phi\rangle_A \otimes \beta|\phi\rangle_B. \end{cases}$$

It follows that

$$(44) \quad \alpha|\psi\rangle_A \otimes |\psi\rangle_B + \beta|\phi\rangle_A \otimes |\phi\rangle_B = \alpha|\psi\rangle_A \otimes \alpha|\psi\rangle_B + \alpha|\psi\rangle_A \otimes \beta|\phi\rangle_B + \beta|\phi\rangle_A \otimes \alpha|\psi\rangle_B + \beta|\phi\rangle_A \otimes \beta|\phi\rangle_B.$$

In particular, substituting $\alpha = \beta = 1$, we get

$$(45) \quad 0 = |\psi\rangle_A \otimes |\phi\rangle_B + |\phi\rangle_A \otimes |\psi\rangle_B.$$

The detailed analysis of this equation has been done in §12.

14. QUANTUM TELEPORTATION IN GQT

In this section, we adapt the Bennett–Brassard–Crépeau–Jozsa–Wootters quantum teleportation scheme to the general case of GQTs. We also provide a new unified method which also works in characteristic 2.

As in Bennett, Brassard, Crépeau, Jozsa and Wootters [7], one can show that quantum teleportation works in every GQT in much the same way as in [7]. By quantum teleportation, we mean that Alice wants to send the information about some unknown state $|\phi\rangle$ to Bob, without sending the particle itself; she lets it interact with some other system (the “ancilla”) in such a way that the particle then is in a standard state and the ancilla is in an unknown state containing all the information about $|\phi\rangle$. Now Alice sends Bob that ancilla, and Bob recovers a replica of $|\phi\rangle$ using unitary transformations. As in most papers on quantum teleportation, we will work with (variations of) Einstein-Podolsky-Rosen (EPR) states [14] to perform the teleportation.

There is a catch though: if we have a division ring of characteristic 2, the method fails. We describe another way to handle this class of GQTs, which also works for the others.

Let us first mention that the *characteristic* of a division ring, is the smallest positive integer $n > 0$ such that $1 + 1 + \dots + 1$ (n times) equals 0. If such an n does not exist, by definition the characteristic is 0. It is a basic fact that the characteristic of a division ring is always 0 or a prime number.

Let the unknown state be the qubit $|\phi\rangle = \alpha|0\rangle + \beta|1\rangle$, relative to the computational base $\{|0\rangle, |1\rangle\}$. We put $|0\rangle = (1, 0)$ and $|1\rangle = (0, 1)$.

- Define the Bell state \mathcal{B} as

$$(46) \quad \mathcal{B} := |00\rangle + |11\rangle.$$

Here, we use the notation $|ab\rangle = |a\rangle \otimes |b\rangle$. We omit a factor $\sqrt{1/2}$, since otherwise we need to consider division rings in which this expression is well defined.

- Distribute \mathcal{B} to Alice and Bob; Alice and Bob each get two qubits, so that we write \mathcal{B} as

$$(47) \quad \mathcal{B} := |0_A 0_B\rangle + |1_A 1_B\rangle.$$

Now we describe the three qubit state of our system as

$$(48) \quad |\phi_A\rangle \otimes \mathcal{B} = \alpha(|0_A 0_A 0_B\rangle + |0_A 1_A 1_B\rangle) + \beta(|1_A 0_A 0_B\rangle + |1_A 1_A 1_B\rangle).$$

It is important to remark that this expression is indeed correct: each of the considered Kronecker products only contain 0s and 1s, and these commute with any element of the division ring D .

- Now we introduce the states

$$(49) \quad \begin{cases} |\phi^+\rangle = |00\rangle + |11\rangle \\ |\phi^-\rangle = |00\rangle - |11\rangle \\ |\psi^+\rangle = |01\rangle + |10\rangle \\ |\psi^-\rangle = |01\rangle - |10\rangle. \end{cases}$$

Notice that if the characteristic of D is 2, $|\phi^+\rangle = |\phi^-\rangle$ and $|\psi^+\rangle = |\psi^-\rangle$! If the characteristic is not 2, then we can write

$$(50) \quad \begin{cases} |00\rangle = \frac{1}{2}(|\phi^+\rangle + |\phi^-\rangle) \\ |11\rangle = \frac{1}{2}(|\phi^+\rangle - |\phi^-\rangle) \\ |01\rangle = \frac{1}{2}(|\psi^+\rangle + |\psi^-\rangle) \\ |10\rangle = \frac{1}{2}(|\psi^+\rangle - |\psi^-\rangle). \end{cases}$$

Since $2 \neq 0$, $\frac{1}{2}$ is the well-defined and unique multiplicative inverse of $1/2$.

Substituting in (48) gives

$$(51) \quad |\phi_A\rangle \otimes \mathcal{B} = \frac{1}{2}(|\phi^+\rangle \otimes (\alpha|0_B\rangle + \beta|1_B\rangle)) + \frac{1}{2}(|\phi^-\rangle \otimes (\alpha|0_B\rangle - \beta|1_B\rangle)) + \frac{1}{2}(|\psi^+\rangle \otimes (\alpha|1_B\rangle + \beta|0_B\rangle)) + \frac{1}{2}(|\psi^-\rangle \otimes (\alpha|1_B\rangle - \beta|0_B\rangle)).$$

Note that $\frac{1}{2}$ commutes with all elements of D .

- Alice subsequently performs a measurement, and she obtains one of $|\phi^+\rangle, |\phi^-\rangle, |\psi^+\rangle, |\psi^-\rangle$. Depending on the outcome, Bob's qubit is in the state $\alpha|0\rangle + \beta|1\rangle, \alpha|0\rangle - \beta|1\rangle, \alpha|1\rangle + \beta|0\rangle, \alpha|1\rangle - \beta|0\rangle$.
- Now Alice classically sends two bits to Bob which describe the outcome of her measurement.
- Bob uses this information to perform a unitary operation (one of the quantum logic gates) which transforms his qubit in the required state.

THE CHARACTERISTIC 2 CASE

When the characteristic of D is 2, the 4 Bell states collapse into 2 Bell states, and they generate a D -plane (the "Bell-plane") instead of $D^4 = V(4, D)$. The identities in (50) also fail to be true (even with the factor $1/2$). So we need a slightly different approach in the remaining cases.

- Introduce a state $\tilde{\mathcal{B}}$ as

$$(52) \quad \tilde{\mathcal{B}} := |01\rangle + |10\rangle.$$

As above, Alice and Bob share \mathcal{B} and $\tilde{\mathcal{B}}$.

- We describe the state of our system as

$$(53) \quad |\phi_A\rangle \otimes \mathcal{B} + \mathbf{I}^- (|\phi_A\rangle \otimes \tilde{\mathcal{B}}).$$

Here, \mathbf{I}^- is the (8×8) -matrix with 1's on the anti-diagonal, and otherwise zeroes (it is unitary).

- Introduce the states

$$(54) \quad \begin{cases} |\phi^+\rangle = |00\rangle + |11\rangle \\ |\psi^+\rangle = |01\rangle + |10\rangle. \end{cases}$$

- Re-write $|\phi_A\rangle \otimes \mathcal{B} + \mathbf{I}^- \left(|\phi_A\rangle \otimes \tilde{\mathcal{B}} \right)$ as
- $$(55) \quad |\phi_A\rangle \otimes \mathcal{B} + \mathbf{I}^- \left(|\phi_A\rangle \otimes \tilde{\mathcal{B}} \right) = |\phi^+\rangle \otimes (\alpha|0\rangle + \beta|1\rangle) + |\psi^+\rangle \otimes (\alpha|1\rangle + \beta|0\rangle).$$
- Now proceed as before.

Note that this procedure works for all division rings, including all fields and thus also \mathbb{C} . As a code, it is less secure though.

15. SUPER-DENSE QUANTUM-CODING IN GQT

In this section, we obtain a super dense quantum-coding scheme in all GQTs. In characteristic 2, a different scheme arises.

Due to the same properties as in the previous section, the classical super-dense quantum-coding scheme [6] works for all GQTs. We use the following notation:

$$(56) \quad X := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad Z := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \text{id}_2 := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The general scheme goes as follows.

- Alice starts with a classical message of two bits: 00, 10, 01 or 11.
- Together with Bob she prepares the state $|0_A 0_B\rangle + |1_A 1_B\rangle$ (where we use the notation as before).
- After applying one of the four unitary quantum gate operations $\{X, Z, ZX, \text{id}_2\}$ to her part of the state, Alice transfers her new qubit, which is one of the four states from eq. (49), through a noiseless qubit channel, to Bob. She applies the operators as follows:

- If the classical message is 00, she leave the state invariant.
 - If it is 10, she (matrix) multiplies it on the left with $Z \otimes \text{id}_2$.
 - If it is 01, she multiplies it on the left with $X \otimes \text{id}_2$.
 - Finally, if the message is 11, she multiplies it on the left with $ZX \otimes \text{id}_2$.

- Bob performs a Bell measurement in the Bell base.
- Since the Bell base vectors are mutually orthogonal, one is singled out.
- Knowing the shared coding scheme of Alice, Bob reconstructs the unitary operation which was applied and as such deduces what the classical message was. *Example:* if Bob receives $|1_A 0_B\rangle + |0_A 1_B\rangle$, he knows she applied $X \otimes \text{id}_2$, and this corresponds to the classical message 01.

THE CHARACTERISTIC 2 CASE

If the characteristic of D is 2, the set of quantum gates operators collapses into a set of only two operators

$$(57) \quad \{X, Z, ZX, \text{id}_2\} \mapsto \{X, \text{id}_2\}.$$

So the collapsed Bell basis now generates, as before, a “Bell plane”

$$(58) \quad V(2, D) \cong \langle |00\rangle + |11\rangle, |01\rangle + |10\rangle \rangle.$$

The same super-dense coding scheme carries over without change (with the collapsed Bell states).

16. QUANTUM CODING SCHEMES USING THE QUANTUM KERNEL

In this last section of applications in Quantum Information Theory, we will explore properties of the quantum kernel in a concrete standard GQT over the finite field \mathbb{F}_{q^2} , and construct a new coding scheme using super-dense coding.

In this section we combine the super-dense coding scheme, with geometric properties of the quantum kernel, if nontrivial, to produce new coding schemes. We concentrate on the GQT over \mathbb{F}_{q^2} of section 7.3 (but many many variations are possible). The quantum kernel then is $\mathcal{Q} = \mathcal{H}(3, q^2)$, and the projective space $\mathbf{P} = \mathbf{PG}(3, q^2)$. The involution γ is given by $\gamma : \mathbb{F}_{q^2} \mapsto \mathbb{F}_{q^2} : r \mapsto r^q$. If we choose suitable homogeneous coordinates, the points of \mathcal{Q} satisfy the equation

$$(59) \quad X_0^{q+1} + X_1^{q+1} + X_2^{q+1} + X_3^{q+1} = 0.$$

In the scheme explained below, Alice and Bob agree on three lines U, V, W in \mathcal{Q} which do not intersect mutually, and also on some element η of the unitary group $\mathbf{PGU}_4(q^2)$ (or $\mathbf{GU}_4(q^2)$ — the result is of course the same). Since we cannot transport elements of $V \setminus \mathcal{Q}$ to \mathcal{Q} (cf. subsection 7.4), we will use geometric methods to realize this transport.

- Alice considers an arbitrary state $|\psi\rangle$, in $V(4, q^2)$ which is not self-orthogonal, and considers the corresponding projective point x in $\mathbf{P} \setminus \mathcal{Q}$.

- Alice applies the map π (cf. section 1) to x , to obtain a projective plane $\pi(x)$ (the dimension of $\pi(x)$ is $3 - \dim(x)$).

- She intersects $\pi(x)$ with \mathcal{Q} to obtain a *Hermitian curve* \mathcal{C} (see [17, §7.4]) (this is the point set of a nonsingular Hermitian variety in $\mathbf{PG}(2, q^2)$; it can always be given the equation $Y_0^{q+1} + Y_1^{q+1} + Y_2^{q+1} = 0$ after suitably changing coordinates).

- Any line of \mathcal{Q} has precisely one point in common with \mathcal{C} (see [22, §3.4]), so intersecting U, V, W with \mathcal{C} , we obtain three distinct points u, v, w .

- Then Alice applies η to obtain three distinct points u', v', w' , which are still contained in \mathcal{Q} since the unitary group acts on \mathcal{Q} .

- Alice sends the points to Bob through a super-dense coding scheme.

- Bob reconstructs u', v', w' .

- Bob applies η^{-1} to obtain three points u, v, w . He then constructs the projective plane Π generated by these points.

- Bob applies π to Π and obtains $x = \pi(\Pi)$ (since π^2 acts as the identity).

Part 3: Comparison to other theories, and conclusion

17. COMPARISON WITH HARDY'S AXIOMS AND GENERALIZED PROBABILITY THEORIES (GPTs)

In this penultimate section, we compare GQT with Hardy's axiomatic approach to Quantum Theory [15], and Barret's generalized probabilistic theories (GPTs) [5].

Both Hardy's approach and Barret's generalized probabilistic theories see states as "probability vectors" in some vector space V . Probability vectors consist, by definition, of real numbers $0 \leq r \leq 1$, and V is assumed to be a real vector space to begin with. In other words: implicitly, it is *assumed* that the underlying field contains the real numbers \mathbb{R} . For us, probabilities as described in section 11 are manifestations of the Hermitian form (through the generalized Born rule, e.g.), and the field we use as underlying algebraic structure. This is very different in [15] and [5].

In Hardy’s paper [15], two integer parameters are used — denoted K and N — and Hardy shows that $K = N^2$. If one considers underlying algebraic structures such as the reals \mathbb{R} , the complex numbers \mathbb{C} or the quaternions \mathbb{H} , only \mathbb{C} appears to have this property. Hardy argues that this shows that his axioms show the “need of the complex numbers.” More precisely, he sees the expression $K = N^2$ as a *suggestion* which points towards \mathbb{C} , rather than having a formal proof [16]. Barret’s GPTs are based on Hardy’s axiomatic approach, so he also implicitly uses \mathbb{C} as coordinatizing structure (and at the very least works with real vector spaces).

Our approach is much more general than Hardy’s theory: we also work with vector spaces, but any division ring (with involution) is allowed to be the coordinatizing agent; that is, no assumptions on the underlying algebraic structure are made. This “complexifies” the theory due to its great generality, but on the other hand it also adds numerous extra layers and flexibility to the theory, even if one only accepts to work over \mathbb{C} ! Besides these facts, there is also a powerful degree of unification which further motivates GQT.

In particular, Barret shows that the no-cloning result is true in all nonclassical GPTs; but the underlying reason is very different from our no-cloning result, and is an expression of properties of real probabilities. Our result solely follows from the concept of linearity/superposition, and works for all division rings, and in this respect, Barret’s no-cloning result is a more particular instance which has no consequences for our more general viewpoint. (Of course, Barret’s approach also works for non-quantum GPTs, so it seeks generalization in another direction. Note that this also true in our case: even for a fixed field, there are many theories given by taking different Hermitian forms, and as we have seen in section 9, those GQTs can bear very different structures.)

Finally, it is also relevant to mention the work of Cassinelli and Lahti [11], where a programme is outlined for an axiomatic reconstruction of quantum mechanics based on the statistical duality of states, and considerations based on symmetry. In particular, they discuss the choice of the complex numbers in [11, section 4] (and decide that from their viewpoint of *orthomodular spaces* and using Solér’s theorem, one is coordinatizing over a member of $\{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$, with preferences for the complex numbers due to its ability to admit more elegant and convenient calculations/formulations in comparison to the others). (Also the references [1, 4, 10, 27] (from that paper) are worth mentioning.)

Once again, this conclusion differs quite a bit from ours.

18. CONCLUDING REMARKS

In this paper, we proposed a unified way to consider classical and modal Quantum Theories, as governed by vector spaces over division rings D , endowed with $(\sigma, 1)$ -Hermitian forms (σ an involution of D). We have shown that many foundational results such as no-cloning and no-deleting results, quantum teleportation and super-dense coding can be obtained in one unified way in all these theories. Extra attention has been drawn to Quantum Theories over finite fields, and over algebraically closed fields. In case of the latter fields, we have shown that in characteristic 0, the Quantum Theories behave much like classical Quantum Theory over the complex numbers, and that a fundamental extension theorem of theories applies. This provides the new possibility to enlarge classical Quantum Theory to larger algebraically closed fields and division rings, in which a singular geometrical object — the “quantum kernel” — arises, which carries the geometry of a polar space, and on which the full unitary group of the Quantum Theory acts. Modulo extension of theories, quantum kernels are always virtually around, depending on how one extends the theory. In this way, one can also easily switch between Quantum Theories when the nature of a physical situation would require a different angle of reasoning. We have used the quantum kernel in a new coding scheme over finite fields which mixes quantum teleportation together with geometric properties of the quantum kernel. All Quantum Theories over algebraically closed fields in characteristic 0 share one model — the minimal model — which is defined over the algebraic closure of \mathbb{Q} (so which is a countable model). At the end of the paper, we have also compared our approach to other axiomatic approaches, and observed its unique features in this very context.

APPENDIX A. LEXICON FOR FIELD THEORY

In this section, we introduce some basic field-theoretical notions and results which are needed to proceed. Most of the material covered here can be found in a standard work such as Serre's book [23] (the first chapter). The reader can skip this section for now, and come back to the lexicon in due course if some notion about fields needs to be further clarified.

We suppose that the reader is acquainted with the notion of "field."

A.1. Characteristic. Let k be a field. The *characteristic* of k is the minimal positive integer M for which

$$(60) \quad M = 1 + 1 + \cdots + 1 = 0,$$

where the sum contains M terms. If no such finite M exists for k , we say that the characteristic of k is 0. For example, the characteristic of \mathbb{C} , \mathbb{R} and \mathbb{Q} is 0. If $M > 0$ is finite, then M is a prime.

A.2. Isomorphisms and automorphisms. An *isomorphism* α between fields k and k' is a bijection $\alpha : k \rightarrow k'$ which preserves addition and multiplication. Isomorphic fields are considered to be the same fields, possibly in a different guise. If $k = k'$, an isomorphism is called *automorphism*, and the set of all automorphisms foreseen with composition of maps, carries the structure of a group, which we denote by $\mathbf{Aut}(k)$. It is called the *automorphism group* of k , and is an extremely important algebraic invariant.

Call an automorphism φ of a field k *involutory* if φ^2 is the identity automorphism of k . Usually, we also suppose that φ is not the identical map, although sometimes it is convenient to allow the identity (we will always be clear on this issue). Sometimes we simply call φ an *involution* as well.

A.3. Algebraically closed fields. We say that a field k is *algebraically closed* if every polynomial in one variable, and with coefficients in k , has a root in k . If ℓ is a field, then up to isomorphism there is a unique smallest field that contains ℓ , and which is algebraically closed; we call it the *algebraic closure* of ℓ , and denote it by $\bar{\ell}$.

For example, \mathbb{R} is not algebraically closed, but its algebraic closure $\mathbb{C} = \bar{\mathbb{R}}$ is.

A.4. Field extensions. If ℓ is a field which contains a field k , then we say that ℓ is a (field) *extension* of k . Sometimes one denotes this by ℓ/k . If ℓ and k are as such, one can naturally see ℓ as a vector space over k : $(\ell, +)$ already is an abelian group since ℓ is a field, and the scalar multiplication \odot is given by

$$(61) \quad \kappa \odot \rho := \kappa \rho,$$

with $\kappa \in k$ and $\rho \in \ell$.

The dimension of this vector space is called the *degree* of the field extension.

For example, \mathbb{C} is a field extension of degree 2 of \mathbb{R} .

A.5. Finite fields. If a field k has a finite number N of elements, then it can be shown that N is the power of a prime, say $N = p^n$ with p a prime and n a positive integer, and up to isomorphism, there is only one such field, denoted \mathbb{F}_{p^n} . Also, \mathbb{F}_{p^m} is a subfield of \mathbb{F}_{p^n} (or otherwise put: \mathbb{F}_{p^n} is a field extension of \mathbb{F}_{p^m}) if and only if m is a divisor of n .

For example, the finite field \mathbb{F}_2 consists of two elements 0 and 1, and we have the rules

$$\begin{array}{lll} 0 + 0 = 0 & 0 + 1 = 1 & 1 + 1 = 0 \\ 0 \cdot 0 = 0 & 0 \cdot 1 = 0 & 1 \cdot 1 = 1. \end{array}$$

Note that the characteristic of \mathbb{F}_2 is 2.

More generally, for each prime we have a unique finite field $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ with p elements (the integers modulo p), of which the characteristic is p .

Given a finite field \mathbb{F}_q with $q = p^m$ and p a prime, any automorphism γ is of the form:

$$(62) \quad \gamma : x \mapsto x^{p^j},$$

where j is contained in the set $\{0, 1, \dots, m-1\}$. It easily follows that $\mathbf{Aut}(\mathbb{F}_q)$ is isomorphic to the cyclic group C_m which has m elements.

A.6. The fields $\overline{\mathbb{F}_q}$. Let p be any prime, and let $\overline{\mathbb{F}_p}$ be an algebraic closure of \mathbb{F}_p .

For any power q of p , one has that $\overline{\mathbb{F}_q} = \overline{\mathbb{F}_p}$. For any positive integer m , $\overline{\mathbb{F}_p}$ contains *precisely one* subfield which is isomorphic to \mathbb{F}_{p^m} . So all finite degree extensions of \mathbb{F}_p are subfields of $\overline{\mathbb{F}_p}$ (up to isomorphism).

Now let α be any element in the automorphism group $\mathbf{Aut}(\overline{\mathbb{F}_p})$ of $\overline{\mathbb{F}_p}$; then for each positive integer n , α fixes the unique subfield \mathbb{F}_{p^n} of order p^n , and α induces a (possibly trivial) automorphism α_n of \mathbb{F}_{p^n} . So each such α defines a sequence $(\alpha_1, \alpha_2, \dots)$.

Note that if a and b are positive integers and a divides b , then α_b induces α_a in the field \mathbb{F}_{p^a} (recall from the previous subsection that \mathbb{F}_{p^b} is a field extension of \mathbb{F}_{p^a}).

Vice versa, if $(\alpha_1, \alpha_2, \dots)$ is a sequence of automorphisms of $\mathbb{F}_p, \mathbb{F}_{p^2}, \dots$ such that this property for all positive integers a and b for which a divides b holds, then this sequence defines a unique automorphism α in $\mathbf{Aut}(\overline{\mathbb{F}_p})$.

A.7. Real-closed fields. We say that a field k is *real-closed* if its algebraic closure \overline{k} is a field extension of degree 2 of k .

For example, \mathbb{R} is real-closed, since $\mathbb{C} = \overline{\mathbb{R}}$ and \mathbb{C} is two-dimensional over \mathbb{R} .

A.8. Division rings. A *division ring* is a field for which the multiplication is not necessarily commutative. Sometimes they are called “skew fields,” but we prefer the name division ring.

For example, the *quaternions* \mathbb{H} form a division ring which is not a field — see subsection 4.1.

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