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Preface

This dissertation is devoted to study of fractional functional and geometric inequalities on homogeneous Lie groups. More precisely, we develop inequalities of the fractional calculus for non-commutative analysis, i.e., we combined two directions in mathematics. This perspective turned out to be extremely useful on both a conceptual and a technical level.

In Chapter 2, we give main definitions and preliminary results from [1], [2] as well as open access books [3] and [4], which both received the "Ferran Sunyer i Balauguer Award" in 2016 and 2019, respectively. Also, we briefly present definitions of the fractional Sobolev space on homogeneous Lie groups and integer order of the Sobolev space on graded, stratified Lie groups.

In Chapter 3, we develop the theory of fractional functional and geometric inequalities on homogeneous Lie groups. We obtain fractional Hardy, Sobolev, Gagliardo-Nirenberg, Caffarelli-Kohn-Nirenberg inequalities on homogeneous Lie groups and their logarithmic analogues which are even new on Euclidean settings. For the Riesz potential (or a fractional integral), we get the Hardy-Littlewood-Sobolev inequality on homogeneous Lie groups, which means boundedness of the Riesz operator in $L^q - L^p$ spaces. Also, we obtain the Stein-Weiss inequality (or a radially weighted Hardy-Littlewood-Sobolev inequality) for the Riesz potential. In addition, we show the integer order logarithmic Sobolev-Folland-Stein inequality on stratified Lie groups. This chapter is based on the papers [5], [6], [7] (joint works with M. Ruzhansky and D. Suragan), [8], [9], [10] (joint works with D. Suragan) and [11] (joint work with A. Kashkynbayev and D. Suragan).

In Chapter 4, we study reverse functional inequalities. Firstly, we start to study reverse integral Hardy inequalities on metric measure space. We note that in work [12], the authors introduced polar decomposition on metric measure space, which plays a key role in their proofs. In this chapter, we obtain reverse integral Hardy inequalities on metric measure space with parameters $q < 0$ and $p \in (0, 1)$. As consequences, we get integral reverse Hardy inequalities on homogeneous Lie groups, hyperbolic space and Cartan-Hadamard manifolds with parameters $q < 0$ and $p \in (0, 1)$. Also, we show integral reverse Hardy inequalities on metric measure space with parameters $\infty < q \leq p < 0$ and as a consequences we show reverse integral Hardy inequality on homogeneous Lie groups. Then we obtain reverse Hardy-Littlewood-Sobolev, Stein-Weiss and improved Stein-Weiss inequalities on homogeneous Lie groups with parametres $q < 0$ and $p \in (0, 1)$. Also, we obtain reverse Hardy-Littlewood-Sobolev, Stein-Weiss type and improved Stein-Weiss type inequalities with parameters $\infty < q \leq p < 0$, which are even new in Euclidean settings. In addition, we obtain reverse Hardy, L^p-Sobolev and L^p- Caffarelli-Kohn-Nirenberg inequalities with the radial derivative on homogeneous Lie groups. This chapter is based on the papers [13], [14] (joint works with M. Ruzhansky and D. Suragan), [15] (joint work with D. Suragan) and [16].

In Chapter 5, we give applications of functional inequalities in PDE. Firstly, we obtain Lyapunov inequalities for the fractional p-sub-Laplacian equation and systems on homogeneous Lie groups. As for an application of the Lyapunov inequality, we give lower estimate for the first eigenvalue of the fractional p-sub-Laplacian on homogeneous Lie groups. Then, we show existence of a weak solution for a nonlinear equation with the p-sub-Laplacian on Heisenberg and stratified groups. Also, we show existence of a weak solution for a nonlinear equation with the fractional sub-Laplacian and the Hardy potential on homogeneous Lie groups and we also show multiplicity of weak solutions with the first stratum Hardy potential on Heisenberg and stratified groups. Then, we discuss blow-up results for the heat equation with the fractional sub-Laplacian and logarithmic nonlinearity on homogeneous Lie groups and for the heat equation with sub-Laplacian and logarithmic nonlinearity on stratified group. Also, we show blow-up results for viscoelastic equations with sub-Laplacian on stratified groups, heat and wave Rockland equations on graded groups. This chapter is based on papers [5], [7] (joint works with M. Ruzhansky and D. Suragan), [8], [9], [17], [18] (joint works with D. Suragan), [11], [19] (joint works with A. Kashkynbayev and D. Suragan), [20] (joint work with B. Torebek and N. Tokmagambetov), [21] (joint work with B. Bekbolat and N. Tokmagambetov) and [22].

In Appendix, we consider one-dimensional functional inequalities on Euclidean settings. Firstly, we obtain fractional Hardy, Poincaré type, Gagliardo-Nirenberg type and Caffarelli-Kohn-Nirenberg inequalities for fractional order differential operators as Caputo, Riemann-Liouville and Hadamard fractional derivatives. Also, we show applications of these inequalities. In addition, we show Lyapunov and Hartman-Wintner-type inequalities for a fractional partial differential equation with Dirichlet condition, we give an application of this inequalities for the first eigenvalue and we show de La Vallée Poussin-type inequality for fractional elliptic boundary value problem. Appendix is based on papers [23] (joint work with M. Ruzhansky, B. Torebek and N. Tokmagambetov) and [24] (joint work with M. Kirane and B. Torebek).

Almaty, Ghent, April 2020 Aidyn Kassymov

Summary

In this PhD dissertation, we study functional and geometric inequalities on homogeneous Lie groups. As for direct inequalities we obtain fractional Hardy, Sobolev, Hardy-Sobolev, Gagliardo-Nirenberg, Caffarelli-Kohn-Nirenberg, logarithmic inequalities, Hardy-Littlewood-Sobolev and Stein-Weiss inequalities on homogeneous Lie groups. Also, we obtain the integer order Sobolev-Folland-Stein inequality on stratified groups.

For reverse inequalities, we prove reverse integral Hardy inequalities with parameters $q < 0$, $p \in (0,1)$ and $-\infty < q \leq p < 0$. Also, we show reverse integral Hardy inequalities on homogeneous Lie groups, hyperbolic spaces and Cartan-Hadamard manifolds with $q < 0$, $p \in (0,1)$. As consequences, we show reverse Hardy-Littlewood-Sobolev, Stein-Weiss and improved version Stein-Weiss inequalities for cases $q < 0$, $p \in (0,1)$ and $-\infty < q \leq p < 0$. In addition, we obtain reverse Hardy, L^p-Sobolev and L^p- Caffarelli-Kohn-Nirenberg inequalities with radial derivative on homogeneous Lie groups.

Then we show some applications of these inequalities for linear and nonlinear PDEs on homogeneous groups.

Also, we consider one-dimensional functional inequalities in Euclidean settings. We establish fractional Hardy, Poincaré type, Gagliardo-Nirenberg type and Caffarelli-Kohn-Nirenberg inequalities for fractional order differential operators as Caputo, Riemann-Liouville and Hadamard fractional derivatives. Also, we discuss applications of these inequalities. In addition, we show Lyapunov and Hartman-Wintner-type inequalities for a fractional partial differential equation with Dirichlet condition, we give an application of these inequalities for the first eigenvalue and we show a de La Vallée Poussin-type inequality for fractional elliptic boundary value problem.

Samenvatting

In dit proefschrift bestuderen we functionele en geometrische ongelijkheden bij homogene Lie-groepen. Voor de directe ongelijkheden verkrijgen we fractionele Hardy, Sobolev, Hardy-Sobolev, Gagliardo-Nirenberg, Caffarelli-Kohn-Nirenberg, logaritmische ongelijkheden, Hardy-Littlewood-Sobolev en Stein-Weiss ongelijkheden op homogene Lie-groepen. We verkrijgen ook een geheel aantal Sobolev-Folland-Stein-ongelijkheid voor gelaagde groepen.

Voor de omgekeerde ongelijkheden, bewijzen we omgekeerde integrale Hardy ongelijkheden met parameters $q < 0,\ p \in (0, 1)$ en $-\infty < q \leq p < 0$. We tonen ook omgekeerde integrale Hardy-ongelijkheden op homogene Lie-groepen, hyperbolische ruimte en Cartan-Hadamard-spruitstukken met $q < 0,\ p \in (0, 1)$. Als gevolg hiervan tonen we omgekeerde Hardy-Littlewood-Sobolev, Stein-Weiss en verbeterde versie Stein-Weiss ongelijkheden voor de gevallen $q < 0,\ p \in (0, 1)$ en $-\infty < q \leq p < 0$. Bovendien verkrijgen we de omgekeerde Hardy, L^p - Sobolev en L^p - Caffarelli-Kohn-Nirenberg ongelijkheden met de radiale derivaat op homogene Lie-groepen.

Vervolgens tonen we enkele toepassingen van deze ongelijkheden in lineaire en niet-lineaire PDE op homogene groepen.

We hebben ook rekening gehouden met eendimensionale functionele ongelijkheden in Euclidisch geval. We hebben fractionele Hardy, Poincaré type, Gagliardo-Nirenberg en Caffarelli-Kohn-Nirenberg ongelijkheden vastgesteld voor de fractionele orde differentiële operatoren als Caputo, Riemann-Liouville en Hadamard fractionele derivaten. Ook tonen we toepassingen van deze ongelijkheden. Daarnaast tonen we Lyapunov en Hartman-Wintner-type ongelijkheden voor een fractionele partiële differentiaalvergelijking met Dirichlet-voorwaarde, geven we een toepassing van deze ongelijkheden voor de eerste eigenwaarde en tonen we de La Vallé Poussin-type ongelijkheid voor probleem met fractionele elliptische grenswaarden.

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1. Introduction

Pioneering mathematicians who studied subelliptic analysis on the Heisenberg group were Folland and Stein in [25], who consistently created a generalisation of the analysis for more general stratified groups [26]. It should also be noted that Rothschild and Stein extended these results to general vector fields satisfying the Hörmander's condition. These results were published in the famous book by Folland and Stein [1], which laid the anisotropic analysis. And it is worth noting that a homogeneous Lie group is nilpotent.

The history of fractional calculus originates from the original works of Riemann and Liouville. And in their works, the concept of the fractional derivative was introduced for the first time, which was named after Riemann and Liouville. Later, Hadamard introduced another definition of the fractional derivative. And it is also worth noting that Caputo introduced a new definition of a fractional derivative that in particular cases coincides with the Riemann-Liouville derivative. These operators are one-dimensional and non-local operators. For the multidimensional case, the concept of a multidimensional fractional Laplacian is introduced via the Laplace symbol. The theory of fractional calculus is currently a rapidly developing mathematical field. The main aim of this dissertation is to combine non-commutative analysis on groups with fractional calculus.

Nowadays, functional and geometric inequalities on Lie groups are one of rapidly developing fields of mathematics. Many nonlinear differential equations in mechanics and problems in physics, to which global solvability of problems can be proved through functional inequalities. Here, one of the most important tools in PDEs is functional inequalities. For example, the integer order multi-dimensional Hardy inequality states the following inequality:

$$\int_{\mathbb{R}^n} \frac{|u(x)|^p}{|x|^p} dx \leq \left(\frac{p}{n-p}\right)^p \int_{\mathbb{R}^n} |\nabla u(x)|^p dx, \quad 1 < p < n, \quad \forall u \in C_0^\infty(\mathbb{R}^n), \tag{1.1}$$

where $|\cdot|$ is the Euclidean distance and the constant $\left(\frac{p}{n-p}\right)^p$ is sharp. This inequality has applications in many of mathematics, for example, in spectral theory. Also, via this inequality one obtains the Heisenberg-Pauli uncertainly principle, which has important interpretation in quantum theory. Historically, in group settings the Hardy inequality was obtained by Garofalo and Lanconelli on Heisenberg group in [27]. On stratified groups, Hardy's inequality was obtained in [28], [29] and [30], on homogeneous groups it was obtained in [31] and on graded groups in [32]. In [33] the authors studied the fractional p-Laplacian and established the following fractional L^p-Hardy inequality

$$C \int_{\mathbb{R}^N} \frac{|u(x)|^p}{|x|^{ps}} dx \leq \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}} dx dy, \tag{1.2}$$

where $u \in C_0^\infty(\mathbb{R}^N)$ and $C > 0$. Also, the best constant was obtained in [33]. The generalisation of this inequality was established in [34].

The classical Sobolev inequality (or a continuous Sobolev embedding) is one of the most popular functional inequality. The Sobolev inequality has many applications in the theory of PDEs and variational principles. Let $\Omega \subset \mathbb{R}^N$ be a measurable set and

$1 < p < N$, then the (classical) Sobolev inequality is formulated as

$$\|u\|_{L^{p^*}(\Omega)} \leq C\|\nabla u\|_{L^p(\Omega)}, \quad u \in C_0^\infty(\Omega), \tag{1.3}$$

where $C = C(N,p) > 0$, $p^* = \frac{Np}{N-p}$ and ∇ is a standard gradient in $\mathbb{R}^N$ (see e.g., [35]). The logarithmic Sobolev inequality was proved in [36] and it has the following form

$$\int_{\mathbb{R}^N} \frac{|u|^p}{\|u\|^p_{L^p(\mathbb{R}^N)}} \log\left(\frac{|u|^p}{\|u\|^p_{L^p(\mathbb{R}^N)}}\right) dx \leq \frac{N}{p} \log\left(C\frac{\|\nabla u\|^p_{L^p(\mathbb{R}^N)}}{\|u\|^p_{L^p(\mathbb{R}^N)}}\right), \; 1 \leq p < \infty, \tag{1.4}$$

where $u, \nabla u \in L^p(\mathbb{R}^N)$. In case of Heisenberg groups, the Sobolev inequality was obtained by Folland and Stein, on stratified groups by Garofalo and Vassilev in [37], on graded groups by Fischer and Ruzhansky in [3]. Also, the best constant of the Sobolev inequality for general hypoelliptic (Rockland operators) on general graded Lie groups was obtained in [38]. The fractional order Sobolev inequality was obtained in [39] when $N > sp$, $1 < p < \infty$, and $s \in (0,1)$, which states for any measurable and compactly supported function u one has

$$\|u\|_{L^{p^*}(\mathbb{R}^N)} \leq C[u]_{s,p}, \tag{1.5}$$

where $C = C(N,p,s) > 0$ is some suitable constant, $[u]^p_{s,p} = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} dxdy$ and $p^* = \frac{Np}{N-sp}$. There is a number of generalisations and extensions of above Sobolev's inequality. For example, in [34] the authors proved the following weighted fractional Sobolev inequality: Let $1 < p < \frac{N}{s}$ and $0 < \beta < \frac{N-ps}{2}$, then for all $u \in C_0^\infty(\mathbb{R}^N)$ one has

$$C\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}|x|^\beta|y|^\beta} dxdy \geq \left(\int_{\mathbb{R}^N} \frac{|u|^{p^*}}{|x|^{\frac{2\beta p^*}{p}}} dx\right)^{\frac{p}{p^*}}, \tag{1.6}$$

where $C = C(N,p,s) > 0$ and $p^* = \frac{Np}{N-sp}$.

E. Gagliardo and L. Nirenberg independently obtained following (interpolation) inequality

$$\|u\|^p_{L^p(\mathbb{R}^N)} \leq C\|\nabla u\|^{N(p-2)/2}_{L^2(\mathbb{R}^N)}\|u\|^{(2p-N(p-2))/2}_{L^2(\mathbb{R}^N)}, \; u \in H^1(\mathbb{R}^N), \tag{1.7}$$

where

$$\begin{cases} 2 \leq p \leq \infty \text{ for } N = 2, \\ 2 \leq p \leq \frac{2N}{N-2} \text{ for } N > 2. \end{cases}$$

In particular, from (1.7) one can obtain the Sobolev inequality. In addition, the logarithmic Gagliardo-Nirenberg inequality was proved in [36] and its fractional version was proved in [40]. On Heisenberg group, the Gagliardo-Nirenberg inequality has the following form

$$\|u\|^p_{L^p(\mathbb{H}^n)} \leq C\|\nabla_{\mathbb{H}^n} u\|^{Q(p-2)/2}_{L^2(\mathbb{H}^n)}\|u\|^{(2p-Q(p-2))/2}_{L^2(\mathbb{H}^n)}, \tag{1.8}$$

where $\nabla_{\mathbb{H}}$ is a horizontal gradient and Q is a homogeneous dimension of $\mathbb{H}^n$. Also, in [38], the authors obtained Gagliardo-Nirenberg inequality and its best constants on general hypoelliptic (Rockland operators) on general graded Lie groups. The fractional version of the Gagliardo-Nirenberg was established in [41]:

$$\|u\|_{L^\tau(\mathbb{R}^N)} \leq C[u]^a_{s,p}\|u\|^{1-a}_{L^\alpha(\mathbb{R}^N)}, \; \forall u \in C^1_c(\mathbb{R}^N), \tag{1.9}$$

for $N \geq 1,\ s \in (0,1),\ p > 1,\ \alpha \geq 1,\ \tau > 0$, and $a \in (0,1]$ is such that

$$\frac{1}{\tau} = a\left(\frac{1}{p} - \frac{s}{N}\right) + \frac{1-a}{\alpha}.$$

In the fundamental work of L. Caffarelli, R. Kohn and L. Nirenberg (see [42]), they obtained:

Theorem 1.1. *Let* $N \geq 1$, *and let* $l_1,\ l_2,\ l_3,\ a,\ b,\ d,\ \delta \in \mathbb{R}$ *be such that* $l_1, l_2 \geq 1$, $l_3 > 0,\ \ 0 \leq \delta \leq 1$, *and*

$$\frac{1}{l_1} + \frac{a}{N},\ \frac{1}{l_2} + \frac{b}{N},\ \frac{1}{l_3} + \frac{\delta d + (1-\delta)b}{N} > 0. \tag{1.10}$$

Then,

$$\||x|^{\delta d + (1-\delta)b} u\|_{L^{l_3}(\mathbb{R}^N)} \leq C \||x|^a \nabla u\|^{\delta}_{L^{l_1}(\mathbb{R}^N)} \||x|^b u\|^{1-\delta}_{L^{l_2}(\mathbb{R}^N)},\ \ u \in C_c^\infty(\mathbb{R}^N), \tag{1.11}$$

if and only if

$$\frac{1}{l_3} + \frac{\delta d + (1-\delta)b}{N} = \delta\left(\frac{1}{l_1} + \frac{a-1}{N}\right) + (1-\delta)\left(\frac{1}{l_2} + \frac{b}{N}\right),$$

$$a - d \geq 0,\quad \text{if } \delta > 0,$$

$$a - d \leq 1,\quad \text{if } \delta > 0 \ \text{ and } \ \frac{1}{l_3} + \frac{\delta d + (1-\delta)b}{N} = \frac{1}{l_1} + \frac{a-1}{N}, \tag{1.12}$$

where C *is a positive constant independent of* u.

The logarithmic analogue of the Caffarelli-Kohn-Nirenberg inequality was obtained in [43]. Recently many different versions of Caffarelli-Kohn-Nirenberg inequalities have been obtained, namely, in [44] on the Heisenberg groups, in [45] and [29] on stratified groups, in [46] on (general) homogeneous Lie groups. In [41] the authors obtained fractional analogues of the Caffarelli-Kohn-Nirenberg inequality in weighted fractional Sobolev spaces. Also, the fractional Caffarelli-Kohn-Nirenberg inequality for an admissible weight in $\mathbb{R}^N$ was obtained in [34].

In one of the pioneering work of Hardy and Littlewood (see [47]), they considered the 1D fractional integral operator on $(0, \infty)$ given by

$$T_\lambda u(x) = \int_0^\infty \frac{u(y)}{|x-y|^\lambda} dy,\quad 0 < \lambda < 1, \tag{1.13}$$

and proved the following theorem:

Theorem 1.2. *Let* $1 < p < q < \infty$ *and* $u \in L^p(0,\infty)$ *with* $\frac{1}{q} = \frac{1}{p} + \lambda - 1$, *then*

$$\|T_\lambda u\|_{L^q(0,\infty)} \leq C \|u\|_{L^p(0,\infty)}, \tag{1.14}$$

where C *is a positive constant independent of* u.

The multi-dimensional analogue of (1.13) can be expressed by the formula:

$$I_\lambda u(x) = \int_{\mathbb{R}^N} \frac{u(y)}{|x-y|^\lambda} dy,\quad 0 < \lambda < N. \tag{1.15}$$

The multi-dimensional case of Theorem 1.2 was extended by Sobolev in [48]:

Theorem 1.3. *Let* $1 < p < q < \infty$, $u \in L^p(\mathbb{R}^N)$ *with* $\frac{1}{q} = \frac{1}{p} + \frac{\lambda}{N} - 1$, *then*

$$\|I_\lambda u\|_{L^q(\mathbb{R}^N)} \leq C\|u\|_{L^p(\mathbb{R}^N)}, \tag{1.16}$$

where C *is a positive constant independent of* u.

Then, in [49] Stein and Weiss obtained the radially weghted extention of the Hardy-Littlewood-Sobolev inequality, which is known as the Stein-Weiss inequality.

Theorem 1.4. *Let* $0 < \lambda < N$, $1 < p < \infty$, $\alpha < \frac{N(p-1)}{p}$, $\beta < \frac{N}{q}$, $\alpha + \beta \geq 0$ *and* $\frac{1}{q} = \frac{1}{p} + \frac{\lambda+\alpha+\beta}{N} - 1$. *If* $1 < p \leq q < \infty$, *then*

$$\||x|^{-\beta} I_\lambda u\|_{L^q(\mathbb{R}^N)} \leq C\||x|^\alpha u\|_{L^p(\mathbb{R}^N)}, \tag{1.17}$$

where C *is a positive constant independent of* u.

On Heisenberg group, the Hardy-Littlewood-Sobolev inequality was proved by Folland and Stein in [25] and an analogue of Stein-Weiss inequality was proved in [50]. In [51], the authors studied the Stein-Weiss inequality on Carnot groups. We also note that the best constant in the Hardy-Littlewood-Sobolev inequality on Heisenberg group is now known (see Frank and Lieb [52]) and in the Euclidean case was obtained earlier by Lieb in [53].

The reverse Stein-Weiss inequality in Euclidean setting has the following form:

Theorem 1.5 ([54], Theorem 1)**.** *For* $n \geq 1, p \in (0,1), q < 0, \lambda > 0, 0 \leq \alpha < -\frac{n}{q}$, *and* $0 \leq \beta < -\frac{n}{p'}$ *satisfying* $\frac{1}{p} + \frac{1}{q'} - \frac{\alpha+\beta+\lambda}{n} = 2$, *there is a constant* $C = C(n, \alpha, \beta, \lambda, p, q) > 0$ *such that for any non-negative functions* $f \in L^{q'}(\mathbb{R}^n)$ *and* $0 < \int_{\mathbb{R}^n} g^p(y)dy < \infty$,

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |x|^\alpha |x-y|^\lambda f(x)g(y)|y|^\beta dydx \geq C\left(\int_{\mathbb{R}^n} f^{q'}(x)dx\right)^{\frac{1}{q'}}\left(\int_{\mathbb{R}^n} g^p(y)dy\right)^{\frac{1}{p}}, \tag{1.18}$$

where $\frac{1}{q} + \frac{1}{q'} = 1$ *and* $\frac{1}{p} + \frac{1}{p'} = 1$.

The inequality (1.18) is equivalent to

$$\left(\int_{\mathbb{R}^n} |x|^{\alpha q}\left(\int_{\mathbb{R}^n} |x-y|^\lambda |y|^\beta g(y)dy\right)^q dx\right)^{\frac{1}{q}} \geq C\left(\int_{\mathbb{R}^n} g^p(y)dy\right)^{\frac{1}{p}}. \tag{1.19}$$

From the latter, if $\alpha = \beta = 0$ we obtain the reverse Hardy-Littlewood-Sobolev inequality. Improved Stein-Weiss inequality was obtained in [55] on Euclidean upper half-space. For more results about the reverse Hardy–Littlewood–Sobolev inequality in Euclidean space, we refer the reader to [56] [57], [58], [59] and the references therein.

By summarising above facts, in this dissertation we developed direct and reverse inequalities on homogeneous groups. In Chapter 3, we obtain fractional Hardy, Sobolev, Gagliardo-Nirenberg, Caffarelli-Kohn-Nirenberg inequalities on homogeneous Lie groups and its logarithmic analogues. For the Riesz potential (or a fractional integral), we get the Hardy-Littlewood-Sobolev inequality on homogeneous Lie groups, which implies $L^p - L^q$ boundedness of the Riesz operator. Also, we obtain the Stein-Weiss inequality for the Riesz potential. In addition, we show the integer order logarithmic Sobolev-Folland-Stein inequality on stratified Lie groups.

In Chapter 4, we prove the reverse integral Hardy inequality on metric measure space with $q < 0 < p < 1$ and $\infty < q \leq p < 0$, the integral reverse Hardy inequality on homogeneous Lie groups, hyperbolic space and Cartan-Hadamard manifolds. As consequences we show Hardy-Littlewood-Sobolev, Stein-Weiss and improved Stein-Weiss inequalities on homogeneous Lie groups with parametres $q < 0$, $p \in (0, 1)$ and $\infty < q \leq p < 0$. In addition, we obtain reverse Hardy, L^p-Sobolev and L^p- Caffarelli-Kohn-Nirenberg inequalities with radial derivative on homogeneous Lie groups.

In Chapter 5, we give applications of functional inequalities to PDE. Firstly, we obtain Lyapunov inequalities for the fractional p-sub-Laplacian equation and systems on homogeneous Lie groups. Then, we show the existence of a weak solution for the nonlinear equation with the p-sub-Laplacian on the Heisenberg and stratified groups and we show the existence of a weak solution for the nonlinear equation with the fractional sub-Laplacian and Hardy potential on homogeneous Lie groups. Then we discuss blow-up results for heat equation with fractional sub-Laplacian and logarithmic nonlinearity on homogeneous Lie groups, for heat equation with sub-Laplacian and logarithmic nonlinearity on stratified groups, viscoelastic equation on stratified groups, heat and wave Rockland equations on graded groups. We give introduction in every section of this chapter.

In Appendix, we consider one-dimensional functional inequalities on Euclidean settings. Firstly, we obtain fractional Hardy, Poincaré type, Gagliardo-Nirenberg and Caffarelli-Kohn-Nirenberg inequalities for the fractional order differential operators as Caputo, Riemann-Liouville and Hadamard fractional derivatives. Also, we show applications of these inequalities. In addition, we show Lyapunov and Hartman-Wintner-type inequalities for a fractional partial differential equation with Dirichlet condition, we give an application of these inequalities for the first eigenvalue and we show the de La Vallée Poussin-type inequality for a fractional elliptic boundary value problem.

I want to note with pleasure, some of the results of this dissertation were included in the monograph of Prof. M.Ruzhansky and Assoc.Prof. D.Suragan, which received the award. Basic results of this dissertation were published in the following journals:

- A. Kassymov, M. Ruzhansky and D. Suragan. Fractional logarithmic inequalities and blow-up results with logarithmic nonlinearity on homogeneous groups. *Nonlinear Differ. Equ. Appl.*, 27:7, 2020;
- A. Kassymov and D. Suragan. Multiplicity of positive solutions for a nonlinear equation with the Hardy potential on the Heisenberg group. *Bulletin des Sciences Mathématiques*, 165:102916, 2020;
- A. Kassymov, M. Ruzhansky and D. Suragan. Hardy-Littlewood-Sobolev and Stein-Weiss inequalities on homogeneous Lie groups. *Integral Transform. Spec. Funct.,* 30(8):643–655, 2019;
- A. Kassymov and D. Suragan. Existence of solutions for p-sub-Laplacians with nonlinear sources on the Heisenberg group. *Complex Variables and Elliptic Equations*, doi: 10.1080/17476933.2020.1731737, 2020;
- A. Kassymov, M. Ruzhansky, B. Torebek and N. Tokmagambetov. Sobolev, Hardy, Gagliardo-Nirenberg and Caffarelli-Kohn-Nirenberg type inequalities for the fractional derivatives. *Banach Journal of Mathematical Analysis*, 15(1): 1–24. 2020;

- A. Kassymov, M. Kirane and B. Torebek. Lyapunov, Hartman-Wintner and De La Vallée Poussin-type inequalities for fractional elliptic boundary value problems. *Complex Variables and Elliptic Equations*, doi: 10.1080/17476933.2020.1825393, 2020;
- A. Kassymov, B. Torebek and N. Tokmagambetov. Nonexistence Results for the Hyperbolic-Type Equations on Graded Lie Groups. *Bulletin of the Malaysian Mathematical Sciences Society*, doi: 10.1007/s40840-020-00919-6, 2020;
- Bekbolat B., Kassymov A., Tokmagambetov N. Blow-up of Solutions of Nonlinear Heat Equation with Hypoelliptic Operators on Graded Lie Groups. *Complex Analysis and Operator Theory*, 13(7):3347-3357, 2019;
- A. Kassymov and D. Suragan. Fractional Hardy-Sobolev inequalities and existence results for fractional sub-Laplacians. *Journal of Mathematical Sciences*, 250(2):337–350, 2020. (Scopus, Q3);
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- A. Kassymov and Suragan D. An analogue of the fractional Sobolev inequality on the homogenous Lie groups. *Mathematical Journal*, 18(1):99-110, 2018;
- A. Kassymov and Suragan D. Reversed Hardy–Littlewood–Sobolev inequality on homogeneous Lie groups. *Kazakh Mathematical Journal*, 19(1):50-57, 2019;
- A. Kassymov. Blow-up of solutions for nonlinear pseudo-parabolic Rockland equation on graded Lie groups. *Kazakh Mathematical Journal*, 19(3):89-100, 2019.

2. PRELIMINARIES

In this chapter, we briefly give definitions, main properties and theorems on homogeneous, graded, stratified and Heisenberg groups. Also, we will fix the main notations in this dissertation. All main definitions were taken from [1], [2] and open access books [3] and [4].

2.1. **Homogeneous Lie groups.**

In whole of this dissertations, any Lie algebra $\mathfrak{g}$ is assumed to be real and finite dimensional. The lower central series of $\mathfrak{g}$ is defined inductively by

$$\mathfrak{g}_{(1)} := \mathfrak{g}, \;\; \mathfrak{g}_{(j)} := [\mathfrak{g}, \mathfrak{g}_{(j-1)}].$$

If the lower central series of the Lie algebra $\mathfrak{g}$ terminates at 0 in a finite number of steps, then this Lie algebra is called nilpotent. Then, if $\mathfrak{g}_{(s+1)} = \{0\}$ and $\mathfrak{g}_{(s)} \neq \{0\}$, then $\mathfrak{g}$ is said to be nilpotent of step s. A Lie groups $\mathbb{G}$ is nilpotent (of step s) whenever its Lie algebra is nilpotent (of step s). If $\exp : \mathfrak{g} \to \mathbb{G}$ is the exponential map, by the Campbell-Hausdorff formula for $X, Y \in \mathbb{G}$ sufficiently close to 0, we have

$$\exp X \exp Y = \exp H(X, Y), \tag{2.1}$$

where $H(X, Y)$ is the Campbell-Hausdorff series which is an infinite linear combination of X and X and their iterated commutators and H is universal, i.e. independent of $\mathfrak{g}$, and that

$$H(X, Y) = X + Y + \frac{1}{2}[X, Y] + \ldots, \tag{2.2}$$

where the dots indicate terms of order ≥ 3. If $\mathfrak{g}$ is nilpotent, the Campbell-Hausdorff series terminates after finitely many terms and defines a polynomial map from $V \times V$ to V, where V is the underlying vector space of $\mathfrak{g}$. Let us state the following property about Haar measure (see e.g., [3] and [4]).

Proposition 2.1 ([4, Proposition 1.1.1], [3, Proposition 1.6.6] and [1, Proposition 1.2])**.** *Let $\mathbb{G}$ be a connected and simply-connected nilpotent Lie group with Lie algebra $\mathfrak{g}$. Then if μ denotes a Lebesgue measure on $\mathfrak{g}$, then $\mu \circ \exp^{-1}$ is a bi-invariant Haar measure on $\mathbb{G}$.*

From [3] and [4], a family of dilations of a Lie algebra $\mathfrak{g}$ is a family of linear mappings of the form

$$D_\lambda = \mathrm{Exp}(A \ln \lambda) = \sum_{k=0}^{\infty} \frac{(\ln(\lambda) A)^k}{k!}, \tag{2.3}$$

where A is a diagonalisable linear operator on $\mathfrak{g}$ with positive eigenvalues, and D_λ is a morphism of the Lie algebra $\mathfrak{g}$, that is, a linear mapping from $\mathfrak{g}$ to itself which respects to the Lie bracket:

$$\forall X, Y \in \mathfrak{g}, \;\; \lambda > 0, \;\; [D_\lambda X, D_\lambda Y] = D_\lambda [X, Y]. \tag{2.4}$$

Let us give the definition of the homogeneous Lie groups, (see e.g., [4, Definition 1.1.6] and [3, Definition 3.1.7]):

Definition 2.2 (Homogeneous Lie group)**.** A *homogeneous (Lie) group* is a connected simply connected Lie group whose Lie algebra is equipped with dilations.

Also, we have another definition of a homogeneous Lie group (see [2]):

Definition 2.3 (Homogeneous Lie group)**.** A Lie group (on $\mathbb{R}^N$) $\mathbb{G}$ with the dilation

$$D_\lambda(x) := (\lambda^{\nu_1}x_1, \ldots, \lambda^{\nu_N}x_N),\ \nu_1, \ldots, \nu_n > 0,\ D_\lambda : \mathbb{R}^N \to \mathbb{R}^N,$$

which is an automorphism of the group $\mathbb{G}$ for each $\lambda > 0$, is called a *homogeneous (Lie) group.*

For simplicity, in this dissertation we use the notation λx for the dilation D_λ. We denote by

$$Q := \nu_1 + \ldots + \nu_N, \tag{2.5}$$

the homogeneous dimension of a homogeneous group $\mathbb{G}$. Let dx denote the Haar measure on $\mathbb{G}$ and let $|S|$ denote the corresponding volume of a measurable set $S \subset \mathbb{G}$. Then we have

$$|D_\lambda(S)| = \lambda^Q|S| \quad \text{and} \quad \int_{\mathbb{G}} f(\lambda x)dx = \lambda^{-Q}\int_{\mathbb{G}} f(x)dx. \tag{2.6}$$

Then we have the following widely used proposition in our dissertation.

Proposition 2.4 ([4, p. 19])**.** *Let $\mathbb{G}$ be a homogeneous Lie group with homogeneous dimension Q, $r > 0$ and dx be a Haar measure. Then, we have*

$$d(rx) = r^Q dx.$$

Definition 2.5 ([4, Definition 1.2.1])**.** For any homogeneous group $\mathbb{G}$ there exists homogeneous quasi-norm, which is a continuous non-negative function

$$\mathbb{G} \ni x \mapsto |x| \in [0, \infty), \tag{2.7}$$

with the properties

a) $|x| = |x^{-1}|$ for all $x \in \mathbb{G}$,
b) $|\lambda x| = \lambda|x|$ for all $x \in \mathbb{G}$ and $\lambda > 0$,
c) $|x| = 0$ if and only if $x = 0$.

Let us define quasi-ball centered at x with radius r in the following form:

$$B(x, r) := \{x \in \mathbb{G} : |x^{-1}y| < r\}. \tag{2.8}$$

Then we have the following proposition about triangle inequality of the quasi-norm, which is widely used in our proofs.

Proposition 2.6 ([4, Proposition 1.2.4])**.** *Let $\mathbb{G}$ be a homogeneous Lie group. Then there exists a homogeneous quasi-norm on $\mathbb{G}$ which is a norm, that is, a homogeneous quasi-norm $|\cdot|$ which satisfies the triangle inequality*

$$|xy| \leq |x| + |y|,\ \ \forall x, y \in \mathbb{G}. \tag{2.9}$$

Furthermore, all homogeneous quasi-norms on $\mathbb{G}$ are equivalent.

Also, let us also recall a well-known fact about quasi-norms.

Proposition 2.7 ([3], Proposition 3.1.38 and [4], Proposition 1.2.4)**.** *If $|\cdot|$ is a homogeneous quasi-norm on $\mathbb{G}$, there exists $C > 0$ such that for every $x, y \in \mathbb{G}$, we have*

$$|xy| \leq C(|x| + |y|). \tag{2.10}$$

Moreover, the following polarisation formula on homogeneous Lie groups will be used in our proofs.

Proposition 2.8 ([4, Proposition 1.2.10] and [3, Proposition 3.1.42])**.** *Let* $\mathbb{G}$ *be a homogeneous Lie group and* $\mathfrak{S} := \{x \in \mathbb{G} : |x| = 1\}$*, be the unit sphere with respect to the homogeneous quasi-norm* $|\cdot|$*. Then there is a unique Radon measure* σ *on* $\mathfrak{S}$ *such that for all* $f \in L^1(\mathbb{G})$*, we have*

$$\int_{\mathbb{G}} f(x)dx = \int_0^{\infty} \int_{\mathfrak{S}} f(ry) r^{Q-1} d\sigma(y) dr. \tag{2.11}$$

Let us give main definitions of the fractional Sobolev space on homogeneous Lie groups. Assume that $p \geq 1$, for any measurable function $u : \mathbb{G} \to \mathbb{R}$ we define the following quasi-seminorm, which is called the Gagliardo quasi-seminorm in the form

$$[u]_{s,p} := \left(\int_{\mathbb{G}} \int_{\mathbb{G}} \frac{|u(x) - u(y)|^p}{|y^{-1}x|^{Q+sp}} dxdy \right)^{\frac{1}{p}}, \quad s \in (0,1), \ Q > 1, \tag{2.12}$$

where $|\cdot|$ is a quasi-norm, see Definition 2.5. By $W^{s,p}(\mathbb{G})$ we call the fractional Sobolev spaces on homogeneous groups. For $p \geq 1$ and $s \in (0,1)$, the functional space

$$W^{s,p}(\mathbb{G}) = \{u : u \in L^p(\mathbb{G}), [u]_{s,p} < +\infty\}, \tag{2.13}$$

is called the fractional Sobolev space on $\mathbb{G}$.

If $\Omega \subset \mathbb{G}$ is a Haar measurable set, we define the Sobolev space

$$W^{s,p}(\Omega) = \{u : u \in L^p(\Omega), [u]_{s,p,\Omega} = \left(\int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^p}{|y^{-1}x|^{Q+sp}} dxdy \right)^{\frac{1}{p}} < +\infty\}. \tag{2.14}$$

Let us define $W_0^{s,p}(\Omega)$ as the completion of $C_0^{\infty}(\Omega)$ with respect to the norm

$$\|u\|_{W_0^{s,p}(\Omega)} = [u]_{s,p,\Omega}. \tag{2.15}$$

Let us define a weighted fractional Sobolev space on homogeneous Lie groups in the following form

$$\begin{aligned} W^{s,p,\beta}(\mathbb{G}) = \{u &: u \in L^p(\mathbb{G}), \\ &[u]_{s,p,\beta} = \left(\int_{\mathbb{G}} \int_{\mathbb{G}} \frac{|x|^{\beta_1 p} |y|^{\beta_2 p} |u(x) - u(y)|^p}{|y^{-1}x|^{Q+sp}} dxdy \right)^{\frac{1}{p}} < +\infty\}, \end{aligned} \tag{2.16}$$

where $\beta_1, \ \beta_2 \in \mathbb{R}$ with $\beta = \beta_1 + \beta_2$, that is, it depends on both β_1 and β_2.

Similarly, for a Haar measurable set $\Omega \subset \mathbb{G}$, $p \geq 1$, $s \in (0,1)$ and $\beta_1, \ \beta_2 \in \mathbb{R}$ with $\beta = \beta_1 + \beta_2$, we define the weighted fractional Sobolev space

$$\begin{aligned} W^{s,p,\beta}(\Omega) = \{u &: u \in L^p(\Omega), \\ &[u]_{s,p,\beta,\Omega} = \left(\int_{\Omega} \int_{\Omega} \frac{|x|^{\beta_1 p} |y|^{\beta_2 p} |u(x) - u(y)|^p}{|y^{-1}x|^{Q+sp}} dxdy \right)^{\frac{1}{p}} < +\infty\}. \end{aligned} \tag{2.17}$$

Obviously, taking $\beta = \beta_1 = \beta_2 = 0$ in (2.17), we recover (2.14).

Then, let us give the main definition of the fractional p-sub-Lapalcian. For a (Haar) measurable and compactly supported function u the fractional p-sub-Laplacian $(-\Delta_p)^s$ on $\mathbb{G}$ can be defined as

$$(-\Delta_p)^s u(x) = 2 \lim_{\delta \searrow 0} \int_{\mathbb{G} \setminus B(x,\delta)} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|y^{-1}x|^{Q+sp}} dy, \quad x \in \mathbb{G}, \tag{2.18}$$

where $|\cdot|$ is a quasi-norm on $\mathbb{G}$ and $B(x,\delta)$ is a quasi-ball with respect to $|\cdot|$, with radius δ centered at $x \in \mathbb{G}$. If $p = 2$, then we have $(-\Delta_2)^s = (-\Delta_s)$.

If $p > 1$, for all $\varphi \in W_0^{s,p}(\Omega)$, we have

$$\langle (-\Delta_p)^s u, \varphi \rangle := \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))(\varphi(x) - \varphi(y))}{|y^{-1}x|^{Q+sp}} dx dy. \tag{2.19}$$

2.2. **Graded Lie group.** In this section, we present a brief summary of basic definitions and properties of graded Lie groups.

Definition 2.9 (Graded Lie group and graded Lie algebra (see e.g., [4, Definition 1.1.4] and [3, Definition 3.1.1]))**.** A Lie algebra $\mathfrak{g}$ is called *graded* if it is endowed with a vector space decomposition (where all but finitely many of the V_j's are 0)

$$\mathfrak{g} = \oplus_{j=1}^{\infty} V_j, \quad \text{s.t. } [V_i, V_j] \subset V_{i+j}. \tag{2.20}$$

Consequently, a Lie group is called *graded* if it is a connected and simply-connected Lie group whose Lie algebra is graded.

Before defining the Rockland operator, let us define the Rockland condition. By π and $\widehat{\mathbb{G}}$ we define representation and unitary dual of $\mathbb{G}$, respectively and by $\mathcal{H}_\pi^\infty$ we define smooth vectors of representation $\pi \in \widehat{\mathbb{G}}$. Let us give a definition of the Rockland condition (see [3, Definition 4.1.1]):

Definition 2.10 (Rockland condition)**.** Let A be a left-invariant differential operator on a Lie group $\mathbb{G}$. Then A satisfies the *Rockland condition* when

(Rockland condition) for each representation $\pi \in \widehat{\mathbb{G}}$, except for the trivial representation, the operator $\pi(A)$ is injective on $\mathcal{H}_\pi^\infty$, that is,

$$\forall v \in \mathcal{H}_\pi^\infty, \quad \pi(A)v = 0 \Rightarrow v = 0. \tag{2.21}$$

Then let us denote a Rockland operator on homogeneous Lie groups $\mathbb{G}$ (see e.g., [3, Definition 4.1.2]).

Definition 2.11 (Rockland operator)**.** Let $\mathbb{G}$ be a homogeneous Lie group. A *Rockland operator* $\mathcal{R}$ on $\mathbb{G}$ is a left-invariant differential operator which is homogeneous of positive degree and satisfies the Rockland condition.

Then let us give proposition which relates homogeneous Lie groups and Rockland operators.

Proposition 2.12 ([3, Proposition 4.1.3])**.** *Let $\mathbb{G}$ be a homogeneous Lie group. If there exists a Rockland operator on $\mathbb{G}$ then the $\mathbb{G}$ is a graded.*

Then let us give some example for the Rockland operator on graded Lie group.

Lemma 2.13 ([3, Lemma 4.1.8])**.** *Let* $\mathbb{G}$ *be a graded Lie group on* $\mathbb{R}^n$*. We denote by* $\{D_r\}_{r>0}$ *the natural family of dilations on its Lie algebra* $\mathfrak{g}$*, and by* $v_1, ..., v_n$ *its weights. We fix a basis* $\{X_1, ..., X_n\}$ *of* $\mathfrak{g}$ *satisfying*

$$D_r X_j = r^{v_j} X_j, \ \ j = 1, \dots, n, \ r > 0.$$

If ν_0 *is any common multiple of* $v_1, \dots, v_n$*, the operator*

$$\sum_{j=1}^{n} (-1)^{\frac{\nu_0}{v_j}} c_j X_j^{2\frac{\nu_0}{v_j}}, \ \ c_j = const, \tag{2.22}$$

is a Rockland operator of homogeneous degree $2\nu_0$*.*

By combining Proposition 2.12 and Lemma 2.13, we have that the in homogeneous Lie group $\mathbb{G}$, if there exists the Rockland operator of the form (2.22) as in Lemma 2.13, then $\mathbb{G}$ is graded. In Chapter 5, we will widely use the Rockland operator as in Lemma 2.13. Let us give a definition of fractional power of the Rockland operator (see, [3, Definition 4.3.1]).

Definition 2.14. Let $\mathcal{R}$ be a positive Rockland operator on a graded Lie group $\mathbb{G}$. For $p \in [1, \infty)$, we denote by $\mathcal{R}_p$ the operator such that $-\mathcal{R}_p$ is the infinitesimal generator of the semi-group of operators $f \mapsto f * h_t$, $t > 0$, on $L^p(\mathbb{G})$.

Then let us give a definition of the Sobolev space on graded Lie groups. Assume that $\mathcal{R}$ is a positive Rockland with homogeneous degree ν and $\mathcal{R}_p$ fractional power of $\mathcal{R}$ on graded Lie group $\mathbb{G}$, which is defined in Definitions 2.11 and 2.14, respectively.

Definition 2.15 (Inhomogeneous Sobolev space ([3, Definition 4.2.2]))**.** Let $\mathcal{R}$ be a positive Rockland operator on a graded Lie group $\mathbb{G}$ and $s \in \mathbb{R}$. If $p \in [1, \infty)$, the Sobolev space $L^p_s(\mathbb{G})$ is the subspace of $S'(\mathbb{G})$ obtained by completion of $S(\mathbb{G})$ with respect to the Sobolev norm

$$\|f\|_{L^p_s(G)} := \|(I + \mathcal{R}_p)^{\frac{s}{\nu}} f\|_{L^p(\mathbb{G})}, \quad \forall f \in S(\mathbb{G}).$$

Let us give a definition of the homogeneous Sobolev space on graded Lie groups.

Definition 2.16 ([3, Definition 4.4.12])**.** Let $\mathcal{R}$ be a Rockland operator of homogeneous degree ν on a graded Lie group $\mathbb{G}$, and let $p \in (1, \infty)$. We denote by $\dot{L}^p_s(\mathbb{G})$ the space of tempered distribution obtained by the completion of $S(\mathbb{G}) \cap Dom(\mathcal{R}_p^{\frac{s}{\nu}})$ for the norm

$$\|f\|_{\dot{L}^p_s(\mathbb{G})} := \|\mathcal{R}_p^{\frac{s}{\nu}} f\|_{L^p(\mathbb{G})}, \quad \forall f \in S(\mathbb{G}) \cap Dom(\mathcal{R}_p^{\frac{s}{\nu}}).$$

Then let us give the following theorem concerning the independence of spaces $L^p_s(\mathbb{G})$ and $\dot{L}^p_s(\mathbb{G})$ of a particular choice of the Rockland operator $\mathcal{R}$.

Theorem 2.17 ([3, Theorem 4.4.20])**.** *Let* $\mathbb{G}$ *be a graded Lie group and* $p \in (1, \infty)$*. The homogeneous* L^p-*Sobolev spaces on* $\mathbb{G}$ *associated with any positive Rockland operators coincide. The inhomogeneous* L^p-*Sobolev spaces on* $\mathbb{G}$ *associated with any positive Rockland operators coincide.*

Then by using the last theorem, norms of inhomogeneous and homogeneous Sobolev spaces on graded Lie groups, respectively, have the following forms:

$$\|f\|_{L^p_s(\mathbb{G})} = \left(\int_{\mathbb{G}} |\mathcal{R}^{\frac{s}{\nu}} f|^p dx + \int_{\mathbb{G}} |f|^p dx\right)^{\frac{1}{p}}, \tag{2.23}$$

and

$$\|f\|_{\dot{L}^p_s(\mathbb{G})} = \left(\int_{\mathbb{G}} |\mathcal{R}^{\frac{s}{\nu}} f|^p dx\right)^{\frac{1}{p}}. \tag{2.24}$$

In this dissertation, we use different notations of the Sobolev space interchangeably on graded Lie groups $L^p_s(\mathbb{G}) = H^s(\mathbb{G})$.

2.3. **Stratified Lie group.** In this section, we give a definition of stratified group (homogeneous Carnot group) and basic propositions. Let us briefly recall the definition of the stratified Lie group. We refer [2], [3] and [4] for further discussions in this direction.

Definition 2.18. A Lie group $\mathbb{G} = (\mathbb{R}^n, \circ)$ is called a stratified Lie group if it satisfies the following assumptions:

(a) For some natural numbers $n_1 + ... + n_r = n$ the decomposition $\mathbb{R}^n = \mathbb{R}^{n_1} \times ... \times \mathbb{R}^{n_r}$ is valid, and for every $\lambda > 0$ the dilation $\delta_\lambda : \mathbb{R}^n \to \mathbb{R}^n$ given by

$$\delta_\lambda(x) \equiv \delta_\lambda(x^{(1)}, ..., x^{(r)}) := (\lambda x^{(1)}, ..., \lambda^r x^{(r)})$$

is an automorphism of the group $\mathbb{G}$. Here $x^{(k)} \in \mathbb{R}^{n_k}$ for $k = 1, ..., r$.

(b) Let n_1 be as in (a) and let $X_1, ..., X_{n_1}$ be the left invariant vector fields on $\mathbb{G}$ such that $X_k(0) = \frac{\partial}{\partial x_k}|_0$ for $k = 1, ..., n_1$. Then

$$\operatorname{rank}(\operatorname{Lie}\{X_1, ..., X_{n_1}\}) = n,$$

for every $x \in \mathbb{R}^n$, i.e. the iterated commutators of $X_1, ..., X_{n_1}$ span the Lie algebra of $\mathbb{G}$.

Also, by [3] and [4] we have the following definition of the stratified Lie group:

Definition 2.19. A graded Lie algebra $\mathfrak{g}$ is called stratified if V_1 generates $\mathfrak{g}$ as an algebra. In this case, if $\mathfrak{g}$ is nilpotent of step m we have

$$\mathfrak{g} = \oplus_{j=1}^{\infty} V_j, \quad \text{s.t. } [V_i, V_1] \subset V_{i+1}, \tag{2.25}$$

and the natural dilations $\mathfrak{g}$ are given by

$$D_r\left(\sum_{k=1}^{m} X_k\right) = \sum_{k=1}^{m} r^k X_k, \quad (X_k \in V_k). \tag{2.26}$$

Consequently, a Lie group is called stratified if it is connected and simply-connected Lie group whose Lie algebra is stratified.

As in homogeneous groups, by dx we understand Haar measure on stratified Lie group $\mathbb{G}$.

Then let us give an example of stratified Lie groups, which is called the Heisenberg group. Let us briefly give a definition of the Heisenberg group. By $\mathbb{H}^n := (\mathbb{R}^{2n+1}, \circ)$, we define Heisenberg group with group law:

$$\widetilde{\xi} \circ \xi' = (\widetilde{x} + x', \widetilde{y} + y', t + t' + 2(x'\widetilde{y} - \widetilde{x}y')), \quad \forall \xi = (\widetilde{x}, \widetilde{y}, t) \text{ and } \forall \xi' = (x', y', t'), \tag{2.27}$$

where $\widetilde{\xi} = (\widetilde{x}, \widetilde{y}, t) \in \mathbb{R}^{2n+1}$ with $\widetilde{x} \in \mathbb{R}^n$, $\widetilde{y} \in \mathbb{R}^n$ and $t \in \mathbb{R}$. The family of dilations has the following form

$$\delta_\lambda(\widetilde{\xi}) := (\lambda\widetilde{x}, \lambda\widetilde{y}, \lambda^2 t), \quad \forall\lambda > 0. \tag{2.28}$$

Then, homogeneous dimension of $\mathbb{H}^n$ is $Q = 2n + 2$ and the topological dimension is $2n + 1$. The Lie algebra $\mathfrak{g}$ of the left-invariant vector fields on the Heisenberg group $\mathbb{H}^n$ is spanned by

$$X_i = \partial_i + 2\widetilde{y}_i\partial_t, \quad i = 1, \ldots, n,$$

$$Y_i = \partial_{n+i} - 2\widetilde{x}_i\partial_t, \quad i = 1, \ldots, n,$$

with their non-zero commutator

$$[X_i, Y_i] = -4\partial_t.$$

Let us define the Sobolev space on stratified Lie groups. By the notation

$$\nabla_{\mathbb{G}} := (X_1, \ldots, X_{N_1})$$

we called (horizontal) gradient. Let Ω be an open subset $\mathbb{G}$. Let us consider the Sobolev space

$$S^{1,p}(\Omega) := \{u : u \in L^p(\Omega),\ |\nabla_{\mathbb{G}}u| \in L^p(\Omega), \quad p \geq 1\}, \tag{2.29}$$

supplemented with the norm

$$\|u\|_{S^{1,p}(\Omega)} := \left(\int_\Omega |u|^p + |\nabla_{\mathbb{G}}u|^p dx\right)^{\frac{1}{p}}.$$

Then, we define the functional class $S_0^{1,p}(\Omega)$ to be the completion of $C_0^1(\Omega)$ in the norm

$$\|u\|_{S_0^{1,p}(\Omega)} := \left(\int_\Omega |\nabla_{\mathbb{G}}u|^p dx\right)^{\frac{1}{p}}.$$

So, the sub-Laplacian on stratified groups is given by

$$\Delta_{\mathbb{G}} := \nabla_{\mathbb{G}} \cdot \nabla_{\mathbb{G}},$$

and the p-sub-Laplacian is given by

$$\mathcal{L}_p := \nabla_{\mathbb{G}} \cdot (|\nabla_{\mathbb{G}}|^{p-2}\nabla_{\mathbb{G}}).$$

On Heisenberg group, the sub-Laplacian is given by

$$\Delta_H := \nabla_H \cdot \nabla_H,$$

where $\nabla_H = (X_1, \ldots, Y_n)$, and the p-sub-Laplacian is given by

$$\Delta_{H,p} := \nabla_H \cdot (|\nabla_H|^{p-2}\nabla_H). \tag{2.30}$$

For simplicity, throughout this dissertation we use any of the notation ∇_H and $\nabla_{\mathbb{H}^n}$ for the horizontal gradient and for the sub-Laplacian we use any of the notation Δ_H and $\Delta_{\mathbb{H}^n}$. It is well known that the class of the Heisenberg group is a subclass of stratified Lie groups, that is, obviously, the above definition is valid in Heisenberg group settings.

2.4. **Metric measure space, Hyperbolic space and Cartan-Hadamard manifolds.** Let us introduce, main definitions of the metric measure space, hyperbolic and Cartan-Hadamard manifolds. Definitions of this sections will be widely used in Chapter 4.

Definition 2.20 ([12])**.** Let $(\mathbb{X}, d)$ be a metric space where d is a metric and dx be a Borel measure. Then the triple $(\mathbb{X}, d, dx)$ is called the metric measure space.

By [12], let us consider metric measure space $(\mathbb{X}, d, dx)$ allowing for the following polar decomposition at $a \in \mathbb{X}$: we assume that there is a locally integrable function $\lambda \in L^1_{loc}$ such that for all $f \in L^1(\mathbb{X})$ we have

$$\int_{\mathbb{X}} f(x)dx = \int_0^\infty \int_{\Sigma_r} f(r,\omega)\lambda(r,\omega)d\omega dr, \tag{2.31}$$

for the set $\Sigma_r = \{x \in \mathbb{X} : d(x,a) = r\} \subset \mathbb{X}$ with a measure on it denoted by $d\omega$, and $(r,\omega) \to a$ as $r \to 0$. This polar decomposition will play a key role in proofs of our results in Chapter 4.

Let us give a definition of the hyperbolic space.

Definition 2.21. The hyperbolic space $\mathbb{H}^n$ $(n \geq 2)$ is a complete and simply connected Riemannian manifold having constant sectional curvature equal to -1.

Let us denote that by $d(0,x)$ the hyperbolic distance in the ball model between the origin and x in the following form: $d(0,x) = \ln \frac{1+|x|}{1-|x|}$. So then let us give a definition of the Cartan-Hadamard manifolds:

Definition 2.22 ([12])**.** Let K_M be the sectional curvature on (M, g). A Riemannian manifold (M, g) is called a Cartan-Hadamard manifold if it is complete, simply connected and has non-positive sectional curvature, i.e., the sectional curvature $K_M \leq 0$ along each plane section at each point of M.

By [12], the condition (2.31) is rather general since we allow the function λ to depend on the whole variable $x = (r,\omega)$. The reason to assume (2.31) is that since $\mathbb{X}$ does not have to have a differentiable structure, the function $\lambda(r,\omega)$ cannot be, in general, obtained as the Jacobian of the polar change of coordinates. However, if such a differentiable structure exists on $\mathbb{X}$, the condition (2.31) can be obtained as the standard polar decomposition formula. In particular, let us give several examples of $\mathbb{X}$ for which the condition (2.31) is satisfied with different expressions for $\lambda(r,\omega)$:

(I) Euclidean space $\mathbb{R}^n$: $\lambda(r,\omega) = r^{n-1}$.

(II) Homogeneous groups: $\lambda(r,\omega) = r^{Q-1}$, where Q is the homogeneous dimension of the group. Such groups have been consistently developed by Folland and Stein [1], see also an up-to-date exposition in [3].

(III) Hyperbolic spaces $\mathbb{H}^n$: $\lambda(r,\omega) = (\sinh r)^{n-1}$.

(IV) Cartan-Hadamard manifolds: Let us fix a point $a \in M$ and denote by $\rho(x) = d(x,a)$ the geodesic distance from x to a on M. The exponential map $\exp_a : T_aM \to M$ is a diffeomorphism, see e.g. Helgason [60]. Let $J(\rho,\omega)$ be the density function on M. Then we have the following polar decomposition:

$$\int_M f(x)dx = \int_0^\infty \int_{\mathbb{S}^{n-1}} f(\exp_a(\rho\omega))J(\rho,\omega)\rho^{n-1}d\rho d\omega,$$

that is, we have (2.31) with $\lambda(\rho, \omega) = J(\rho, \omega)\rho^{n-1}$.

3. Direct Inequalities

In this chapter, we show basic direct fractional functional and geometric inequalities on homogeneous Lie groups.

3.1. Fractional Hardy inequality.

In this section, we obtain the fractional Hardy inequality. For showing fractional Hardy inequality, we need some preliminary results.

Lemma 3.1 ([33], Lemma 2.6)**.** *Assume that $p>1$, then for all $t\in[0,1]$ and $a\in\mathbb{C}$, we have*

$$|a-t|^p \geq (1-t)^{p-1}(|a|^p - t). \tag{3.1}$$

In all following lemma, we assume that $Q>2$, $p>1$ and $s\in(0,1)$ is such that $Q>sp$.

Lemma 3.2 (Picone-type inequality)**.** *Let $\omega\in W_0^{s,p}(\Omega)$ be $\omega>0$ in $\Omega\subset\mathbb{G}$ and suppose that $(-\Delta_p)^s\omega=\nu>0$ with $\nu\in L^1_{loc}(\Omega)$, then for all $u\in C_0^\infty(\Omega)$, we have*

$$\frac{1}{2}\int_\Omega\int_\Omega \frac{|u(x)-u(y)|^p}{|y^{-1}x|^{Q+ps}}dxdy \geq \left\langle (-\Delta_p)^s\omega, \frac{|u|^p}{\omega^{p-1}}\right\rangle. \tag{3.2}$$

Proof. Proof of this lemma is based on [34] and [61]. By setting $v=\frac{|u|^p}{|\omega|^{p-1}}$ and $k(x,y)=\frac{1}{|y^{-1}x|^{Q+ps}}$, then we obtain

$$\begin{aligned}\langle(-\Delta_p)^s\omega, v\rangle &= \int_\Omega v(x)dx\int_\Omega |\omega(x)-\omega(y)|^{p-2}(\omega(x)-\omega(y))k(x,y)dy\\ &= \int_\Omega \frac{|u|^p}{|\omega|^{p-1}}dx\int_\Omega |\omega(x)-\omega(y)|^{p-2}(\omega(x)-\omega(y))k(x,y)dy,\end{aligned}$$

where $\langle\cdot,\cdot\rangle$ is the inner product in $L^2(\Omega)$, By using the definition of a quasi-norm we have $|x^{-1}|=|x|$ for all $x\in\mathbb{G}$, we get

$$\begin{aligned}k(x,y) &= \frac{1}{|y^{-1}x|^{Q+ps}} = \frac{1}{|z|^{Q+ps}} = \frac{1}{|z^{-1}|^{Q+ps}}\\ &= \frac{1}{|(y^{-1}x)^{-1}|^{Q+ps}} = \frac{1}{|x^{-1}y|^{Q+ps}} = k(y,x),\end{aligned}$$

for all $x,y\in\mathbb{G}$. Since $k(x,y)$ is symmetric, we obtain that

$$\langle(-\Delta_p)^s\omega, v\rangle_{L^2(\Omega)} =$$

$$\frac{1}{2}\int_\Omega\int_\Omega\left(\frac{|u(x)|^p}{|\omega(x)|^{p-1}} - \frac{|u(y)|^p}{|\omega(y)|^{p-1}}\right)|\omega(x)-\omega(y)|^{p-2}(\omega(x)-\omega(y))k(x,y)dydx.$$

Let $g=\frac{u}{\omega}$ and

$$R(x,y) = |u(x)-u(y)|^p - (|g(x)|^p\omega(x) - |g(y)|^p\omega(y))|\omega(x)-\omega(y)|^{p-2}(\omega(x)-\omega(y)),$$

then we have

$$\langle(-\Delta_p)^s\omega, v\rangle_{L^2(\Omega)} + \frac{1}{2}\int_\Omega\int_\Omega R(x,y)k(x,y)dydx = \frac{1}{2}\int_\Omega\int_\Omega |u(x)-u(y)|^p k(x,y)dydx.$$

By the symmetry argument, we can assume that $\omega(x) \geq \omega(y)$. By using Lemma 3.1 with $t = \frac{\omega(y)}{\omega(x)}$ and $a = \frac{g(x)}{g(y)}$ and we establish that $R(x, y) \geq 0$. Thus, we have establishrd the inequality

$$\langle(-\Delta_p)^s\omega, v\rangle_{L^2(\Omega)} \leq \frac{1}{2}\int_\Omega\int_\Omega \frac{|u(x)-u(y)|^p}{|y^{-1}x|^{Q+ps}}dydx.$$

Lemma 3.2 is proved. □

Lemma 3.3. *Let* $\omega = |x|^{-\gamma}$ *with* $\gamma \in \left(0, \frac{Q-ps}{p-1}\right)$ *where* $Q > sp$*, then there exists a positive constant* $\mu(\gamma) > 0$ *such that*

$$(-\Delta_p)^s(|x|^{-\gamma}) = \mu(\gamma)\frac{1}{|x|^{ps+\gamma(p-1)}} \text{ a.e. in } \mathbb{G}\setminus\{0\}. \tag{3.3}$$

Proof. Let us set $r = |x|$ and $\rho = |y|$ with $x = rx'$ and $y = \rho y'$, where $|x'| = |y'| = 1$. Then by using the polar decomposition (see (2.11)), we have

$$\begin{aligned}&(-\Delta_p)^s\omega\\ &= \int_0^{+\infty}|r^{-\gamma}-\rho^{-\gamma}|^{p-2}(r^{-\gamma}-\rho^{-\gamma})\rho^{Q-1}\left(\int_{\mathfrak{S}}\frac{d\sigma(y')}{|(\rho y')^{-1}(rx')|^{Q+ps}}\right)d\rho\\ &= \frac{1}{|x|^{ps+\gamma(p-1)}}\int_0^{+\infty}\left|1-\frac{\rho^{-\gamma}}{r^{-\gamma}}\right|^{p-2}\times\\ &\times\left(1-\frac{\rho^{-\gamma}}{r^{-\gamma}}\right)\frac{\rho^{Q-1}}{r^Q}\left(\int_{\mathfrak{S}}\frac{d\sigma(y')}{|\left(\frac{\rho}{r}y'\right)^{-1}x'|^{Q+ps}}\right)d\rho.\end{aligned}$$

Let $\widetilde{\rho} = \frac{\rho}{r}$ and $L(\widetilde{\rho}) = \int_{\mathfrak{S}}\frac{d\sigma(y')}{|(\widetilde{\rho}y')^{-1}x')|^{Q+ps}}$, we get

$$(-\Delta_p)^s\omega = \frac{1}{|x|^{ps+\gamma(p-1)}}\int_0^{+\infty}|1-\widetilde{\rho}^{-\gamma}|^{p-2}(1-\widetilde{\rho}^{-\gamma})L(\widetilde{\rho})\widetilde{\rho}^{Q-1}d\widetilde{\rho}.$$

Then it easy to see

$$\mu(\gamma) = \int_0^{+\infty}\phi(\widetilde{\rho})d\widetilde{\rho} \tag{3.4}$$

with $\phi(\widetilde{\rho}) = |1-\widetilde{\rho}^{-\gamma}|^{p-2}(1-\widetilde{\rho}^{-\gamma})L(\widetilde{\rho})\widetilde{\rho}^{Q-1}$.

We need to show that $\mu(\gamma)$ is a positive and bounded. Firstly, let us show boundedness of $\mu(\gamma)$. We get

$$\mu(\gamma) = \int_0^1\phi(\widetilde{\rho})d\widetilde{\rho} + \int_1^{+\infty}\phi(\widetilde{\rho})d\widetilde{\rho} = I_1 + I_2. \tag{3.5}$$

Switching to the new variable $\zeta = \frac{1}{\widetilde{\rho}}$ we obtain $L(\widetilde{\rho}) = L\left(\frac{1}{\zeta}\right) = \zeta^{Q+ps}L(\zeta)$ for any $\zeta > 0$. Thus, we establish

$$\mu(\gamma) = \int_1^{+\infty}(\rho^{-\gamma}-1)^{p-1}(\rho^{Q-1-\gamma(p-1)}-\rho^{ps-1})L(\rho)d\rho. \tag{3.6}$$

For $\rho \to 1$ we get

$$(\rho^{-\gamma}-1)^{p-1}(\rho^{Q-1-\gamma(p-1)}-\rho^{ps-1})L(\rho) \simeq (\rho-1)^{-1-ps+p} \in L^1(1,2). \tag{3.7}$$

Similarly, for $\rho \to \infty$ we have

$$(\rho^{-\gamma}-1)^{p-1}(\rho^{Q-1-\gamma(p-1)}-\rho^{ps-1})L(\rho) \simeq \rho^{-1-ps} \in L^1(2,\infty). \tag{3.8}$$

It means $\mu(\gamma)$ is bounded. By (3.6) with $\gamma \in \left(0, \frac{Q-ps}{p-1}\right)$ we see that $\mu(\gamma)$ is positive.
Lemma 3.3 is proved. ☐

Finally, we obtain the following analogue of the fractional Hardy inequality on $\mathbb{G}$.

Theorem 3.4 (Fractional Hardy inequality). *Assume $Q > 2$, $p > 1$ and $s \in (0,1)$ is such that $Q > sp$. Then for all $u \in C_0^\infty(\mathbb{G})$ we have*

$$\int_{\mathbb{G}} \frac{|u(x)|^p}{|x|^{sp}} dx \leq C[u]^p_{s,p}, \tag{3.9}$$

where C is positive constant.

Proof. Let $u \in C_0^\infty(\mathbb{G})$ and $\gamma < \frac{Q-ps}{p-1}$. By using Lemma 3.3 and Lemma 3.2 we have

$$\begin{aligned}\frac{1}{2}[u]^p_{s,p} = \frac{1}{2}\int_{\mathbb{G}}\int_{\mathbb{G}} \frac{|u(x)-u(y)|^p}{|y^{-1}x|^{Q+ps}} dxdy &\geq \left\langle (-\Delta_p)^s(|x|^{-\gamma}), \frac{|u(x)|^p}{|x|^{-\gamma(p-1)}} \right\rangle \\ &= \mu(\gamma)\int_{\mathbb{G}} \frac{|u(x)|^p}{|x|^{ps}} dx,\end{aligned} \tag{3.10}$$

completing proof. ☐

3.2. **Fractional Sobolev inequality.** In this section, we prove the fractional Sobolev inequality on homogeneous Lie groups.

To show an analogue of the fractional Sobolev inequality, firstly we need some preliminary results.

Lemma 3.5. *Let $p > 1$, $s \in (0,1)$ and $K \subset \mathbb{G}$ be a Haar measurable set. Fix $x \in \mathbb{G}$ and a quasi-norm $|\cdot|$ on $\mathbb{G}$, then we have*

$$\int_{K^c} \frac{dy}{|y^{-1}x|^{Q+sp}} \geq C|K|^{-sp/Q}, \tag{3.11}$$

where $C = C(Q,s,p)$ is a positive constant, $K^c = \mathbb{G} \setminus K$ and $|K|$ is the Haar measure of K.

Proof. Set $\delta := \left(\frac{Q|K|}{\omega_Q}\right)^{1/Q}$, where ω_Q is a surface measure of the unit quasi-ball on $\mathbb{G}$. Let us fix $x \in \mathbb{G}$ such that $K \cap B(x,\delta) \neq \emptyset$ where $B(x,\delta)$ is a quasi-ball centered at x with radius δ. Then, we get

$$\begin{aligned}|K^c \cap B(x,\delta)| &= |B(x,\delta)| - |K \cap B(x,\delta)| \\ &= |K| - |K \cap B(x,\delta)| = |K \cap B^c(x,\delta)|,\end{aligned} \tag{3.12}$$

where $|\cdot|$ is the Haar measure on $\mathbb{G}$. Then,

$$\begin{aligned}\int_{K^c}\frac{dy}{|y^{-1}x|^{Q+sp}} &= \int_{K^c\cap B(x,\delta)}\frac{dy}{|y^{-1}x|^{Q+sp}}+\int_{K^c\cap B^c(x,\delta)}\frac{dy}{|y^{-1}x|^{Q+sp}}\\ &\geq \int_{K^c\cap B(x,\delta)}\frac{dy}{\delta^{Q+sp}}+\int_{K^c\cap B^c(x,\delta)}\frac{dy}{|y^{-1}x|^{Q+sp}}\\ &= \frac{|K^c\cap B(x,\delta)|}{\delta^{Q+sp}}+\int_{K^c\cap B^c(x,\delta)}\frac{dy}{|y^{-1}x|^{Q+sp}}.\end{aligned}$$

By using (3.12) we get

$$\begin{aligned}\int_{K^c}\frac{dy}{|y^{-1}x|^{Q+sp}} &\geq \frac{|K^c\cap B(x,\delta)|}{\delta^{Q+sp}}+\int_{K^c\cap B^c(x,\delta)}\frac{dy}{|y^{-1}x|^{Q+sp}}\\ &= \frac{|K\cap B^c(x,\delta)|}{\delta^{Q+sp}}+\int_{K^c\cap B^c(x,\delta)}\frac{dy}{|y^{-1}x|^{Q+sp}}\\ &= \int_{K\cap B^c(x,\delta)}\frac{dy}{\delta^{Q+sp}}+\int_{K^c\cap B^c(x,\delta)}\frac{dy}{|y^{-1}x|^{Q+sp}}\\ &\geq \int_{K\cap B^c(x,\delta)}\frac{dy}{|y^{-1}x|^{Q+sp}}+\int_{K^c\cap B^c(x,\delta)}\frac{dy}{|y^{-1}x|^{Q+sp}}\\ &= \int_{B^c(x,\delta)}\frac{dy}{|y^{-1}x|^{Q+sp}}.\end{aligned}$$

By using the polarisation formula (2.11) with center at x, we have

$$\int_{K^c}\frac{dy}{|y^{-1}x|^{Q+sp}} \geq C|K|^{-sp/Q}. \tag{3.13}$$

□

Lemma 3.6 ([39], Lemma 6.2). *Fix $T>1$. Let $p>1$ and $s\in(0,1)$ be such that $Q>sp$, $m\in\mathbb{Z}$ and a_k be a bounded, decreasing, nonnegative sequence with $a_k=0$ for any $k\geq m$. Then*

$$\sum_{k\in\mathbb{Z}} a_k^{(Q-sp)/Q}T^k \leq C\sum_{k\in\mathbb{Z},\,a_k\neq 0} a_{k+1}a_k^{-sp/Q}T^k,$$

for a positive constant $C=C(Q,s,p,T)>0$.

Lemma 3.7. *Suppose that $p>1$, $s\in(0,1)$, $Q>sp$ and $|\cdot|$ be a quasi-norm on $\mathbb{G}$. Assume that $u\in L^\infty(\mathbb{G})$ is compactly supported and $a_k:=|\{|u|>2^k\}|$ for any $k\in\mathbb{Z}$. Then,*

$$C\sum_{k\in\mathbb{Z},\,a_k\neq 0} a_{k+1}a_k^{-sp/Q}2^{kp} \leq [u]_{s,p}^p, \tag{3.14}$$

where $C=C(Q,p,s)$ is a positive constant and $[u]_{s,p}$ is defined by (2.12).

Proof. Let us define

$$A_k:=\{|u|>2^k\},\ k\in\mathbb{Z}, \tag{3.15}$$

and

$$D_k:=A_k\setminus A_{k+1}=\{2^k<|u|\leq 2^{k+1}\}\ \text{ and }\ d_k=|D_k|. \tag{3.16}$$

Since $A_{k+1} \subseteq A_k$, it is easy to see

$$a_{k+1} \leq a_k. \tag{3.17}$$

By the assumption that $u \in L^\infty(\mathbb{G})$ is compactly supported, a_k and d_k are bounded and vanish when k is large enough. Also, we notice that the D_k's are disjoint, therefore,

$$\bigcup_{l\in\mathbb{Z},\, l\leq k} D_l = A_{k+1}^c \tag{3.18}$$

and

$$\bigcup_{l\in\mathbb{Z},\, l\geq k} D_l = A_k. \tag{3.19}$$

By using (3.19) we establish that

$$\sum_{l\in\mathbb{Z},\, l\geq k} d_l = a_k \tag{3.20}$$

and

$$d_k = a_k - \sum_{l\in\mathbb{Z},\, l\geq k+1} d_l. \tag{3.21}$$

Since a_k and d_k are bounded and vanish when k is large enough, (3.20) and (3.21) are convergent. Let us define the convergent series

$$S := \sum_{l\in\mathbb{Z},\, a_{l-1}\neq 0} 2^{lp} a_{l-1}^{-sp/Q} d_l. \tag{3.22}$$

We have that $D_k \subseteq A_k \subseteq A_{k-1}$, then, $a_{i-1}^{-sp/Q} d_l \leq a_{i-1}^{-sp/Q} a_{l-1}$. Thus,

$$\{(i,l) \in \mathbb{Z} \text{ s.t. } a_{i-1} \neq 0 \text{ and } a_{i-1}^{-sp/Q} d_l \neq 0\} \subseteq \{(i,l) \in \mathbb{Z} \text{ s.t. } a_{l-1} \neq 0\}. \tag{3.23}$$

By combining (3.23) and (3.17), we compute that

$$\begin{aligned}
\sum_{i\in\mathbb{Z},\, a_{i-1}\neq 0} \sum_{l\in\mathbb{Z},\, l\geq i+1} 2^{ip} a_{i-1}^{-sp/Q} d_l &= \sum_{i\in\mathbb{Z},\, a_{i-1}\neq 0} \sum_{l\in\mathbb{Z},\, l\geq i+1,\, a^{sp/Q} d_l\neq 0} 2^{ip} a_{i-1}^{-sp/Q} d_l \\
\leq \sum_{i\in\mathbb{Z}} \sum_{l\in\mathbb{Z},\, l\geq i+1,\, a_{l-1}\neq 0} 2^{ip} a_{i-1}^{-sp/Q} d_l &= \sum_{l\in\mathbb{Z},\, a_{l-1}\neq 0} \sum_{i\in\mathbb{Z},\, i\leq l-1} 2^{ip} a_{i-1}^{-sp/Q} d_l \\
\leq \sum_{l\in\mathbb{Z},\, a_{l-1}\neq 0} \sum_{i\in\mathbb{Z},\, i\leq l-1} 2^{ip} a_{l-1}^{-sp/Q} d_l &= \sum_{l\in\mathbb{Z},\, a_{l-1}\neq 0} \sum_{k=0}^{+\infty} 2^{p(l-1-k)} a_{l-1}^{-sp/Q} d_l \leq S.
\end{aligned} \tag{3.24}$$

Notice that

$$||u(x)| - |u(y)|| \leq |u(x) - u(y)|, \quad \forall x, y \in \mathbb{G}.$$

By setting $i \in \mathbb{Z}$ and $x \in D_i$, then for all $j \in \mathbb{Z}$ with $j \leq i-2$, for any $y \in D_j$ using the last inequality, we have that

$$|u(x) - u(y)| \geq 2^i - 2^{j+1} \geq 2^i - 2^{i-1} \geq 2^{i-1}$$

and using (3.18), we get

$$\sum_{j\in\mathbb{Z},\, j\le i-2}\int_{D_j}\frac{|u(x)-u(y)|^p}{|y^{-1}x|^{Q+sp}}dy \ge 2^{(i-1)p}\sum_{j\in\mathbb{Z},\, j\le i-2}\int_{D_j}\frac{dy}{|y^{-1}x|^{Q+sp}}$$
$$= 2^{(i-1)p}\int_{A^c_{i-1}}\frac{dy}{|y^{-1}x|^{Q+sp}}. \tag{3.25}$$

By combining (3.25) and Lemma 3.5, we get

$$\sum_{j\in\mathbb{Z},\, j\le i-2}\int_{D_j}\frac{|u(x)-u(y)|^p}{|y^{-1}x|^{Q+sp}}dy \ge C2^{ip}a_{i-1}^{-sp/Q},$$

where C is a positive constant. It means, for any $i\in\mathbb{Z}$, we get

$$\sum_{j\in\mathbb{Z},\, j\le i-2}\int_{D_i}\int_{D_j}\frac{|u(x)-u(y)|^p}{|y^{-1}x|^{Q+sp}}dxdy \ge C2^{ip}a_{i-1}^{-sp/Q}d_i. \tag{3.26}$$

By combinig (3.26) and (3.21) we obtain that

$$\sum_{j\in\mathbb{Z},\, j\le i-2}\int_{D_i}\int_{D_j}\frac{|u(x)-u(y)|^p}{|y^{-1}x|^{Q+sp}}dxdy$$
$$\ge C\left(2^{ip}a_{i-1}^{-sp/Q}a_i-\sum_{l\in\mathbb{Z},\, l\ge i+1}2^{ip}a_{i-1}^{-sp/Q}d_l\right). \tag{3.27}$$

From (3.26) and (3.22) we obtain that

$$\sum_{i\in\mathbb{Z},\, a_{i-1}\ne0}\ \sum_{j\in\mathbb{Z},\, j\le i-2}\int_{D_i}\int_{D_j}\frac{|u(x)-u(y)|^p}{|y^{-1}x|^{Q+sp}}dxdy$$
$$\ge C\sum_{i\in\mathbb{Z},\, a_{i-1}\ne0}2^{ip}a_{i-1}^{-sp/Q}d_i \ge C\,S. \tag{3.28}$$

Then, by using (3.24), (3.27) and (3.28), we have

$$\sum_{i\in\mathbb{Z},\, a_{i-1}\ne0}\ \sum_{j\in\mathbb{Z},\, j\le i-2}\int_{D_i}\int_{D_j}\frac{|u(x)-u(y)|^p}{|y^{-1}x|^{Q+sp}}dxdy \ge C\sum_{i\in\mathbb{Z},\, a_{i-1}\ne0}2^{ip}a_{i-1}^{-sp/Q}a_i$$
$$-C\sum_{i\in\mathbb{Z},\, a_{i-1}\ne0}\ \sum_{l\in\mathbb{Z},\, l\ge i+1}2^{ip}a_{i-1}^{-sp/Q}d_l \ge C\sum_{i\in\mathbb{Z},\, a_{i-1}\ne0}2^{ip}a_{i-1}^{-sp/Q}a_i - C\,S$$
$$\ge C\sum_{i\in\mathbb{Z},\, a_{i-1}\ne0}2^{ip}a_{i-1}^{-sp/Q}a_i-\sum_{i\in\mathbb{Z},\, a_{i-1}\ne0}\ \sum_{j\in\mathbb{Z},\, j\le i-2}\int_{D_i}\int_{D_j}\frac{|u(x)-u(y)|^p}{|y^{-1}x|^{Q+sp}}dxdy.$$

Thus,

$$\sum_{i\in\mathbb{Z},\, a_{i-1}\ne0}\ \sum_{j\in\mathbb{Z},\, j\le i-2}\int_{D_i}\int_{D_j}\frac{|u(x)-u(y)|^p}{|y^{-1}x|^{Q+sp}}dxdy \ge \frac{C}{2}\sum_{i\in\mathbb{Z},\, a_{i-1}\ne0}2^{ip}a_{i-1}^{-sp/Q}a_i, \tag{3.29}$$

for a constant $C > 0$. By using symmetry property and (3.29), we obtain that

$$\begin{aligned}
[u]^p_{s,p} &= \int_{\mathbb{G}}\int_{\mathbb{G}} \frac{|u(x)-u(y)|^p}{|y^{-1}x|^{Q+sp}}dxdy = \sum_{i,j\in\mathbb{Z}} \int_{D_i}\int_{D_j} \frac{|u(x)-u(y)|^p}{|y^{-1}x|^{Q+sp}}dxdy \\
&\geq 2 \sum_{i,j\in\mathbb{Z},\, j<i} \int_{D_i}\int_{D_j} \frac{|u(x)-u(y)|^p}{|y^{-1}x|^{Q+sp}}dxdy \\
&\geq 2 \sum_{i\in\mathbb{Z},\, a_{i-1}\neq 0}\ \sum_{j\in\mathbb{Z},\ j\leq i-2} \int_{Di}\int_{D_j} \frac{|u(x)-u(y)|^p}{|y^{-1}x|^{Q+sp}}dxdy \\
&\geq C \sum_{i\in\mathbb{Z},\, a_{i-1}\neq 0} 2^{ip} a_{i-1}^{-sp/Q} a_i.
\end{aligned}$$

Lemma 3.7 is proved. ☐

Lemma 3.8. *Assume that $1 < p < \infty$ and $u : \mathbb{G} \to \mathbb{R}$ be a measurable function. For any $n \in \mathbb{R}$*

$$u_n := \max\{\min\{u(x), n\}, -n\}, \quad \textit{for any } x \in \mathbb{G}. \tag{3.30}$$

Then,

$$\lim_{n\to+\infty} \|u_n\|_{L^p(\mathbb{G})} = \|u\|_{L^p(\mathbb{G})}.$$

Proof. The proof is the same as in [39, Lemma 6.4]. ☐

Then, by using the above lemmas we show the following analogue of the fractional Sobolev inequality on $\mathbb{G}$:

Theorem 3.9 (Fractional Sobolev inequality)**.** *Let $p > 1$, $s \in (0, 1)$, $Q > sp$ be such that $p^* = p^*(Q, s, p) = \frac{Qp}{Q-sp}$. Let $|\cdot|$ be a quasi-norm on $\mathbb{G}$. Then for any $u \in W^{s,p}(\mathbb{G})$ and for any quasi-norm $|\cdot|$, we have*

$$\|u\|_{L^{p^*}(\mathbb{G})} \leq C[u]_{s,p}, \tag{3.31}$$

where $C = C(Q, p, s) > 0$.

Proof. Firstly, assume that $[u]_{s,p}$ (Gagliardo seminorm) is bounded, i.e.

$$[u]^p_{s,p} = \int_{\mathbb{G}}\int_{\mathbb{G}} \frac{|u(x)-u(y)|^p}{|y^{-1}x|^{Q+sp}}dxdy < +\infty, \tag{3.32}$$

and we assume that $u \in L^\infty(\mathbb{G})$.

If (3.32) is executed for bounded functions, this is also true for the function u_n obtained by cutting the function u at levels $-n$ and n. Then, by combining Lemma 3.8 and (3.32) with the dominated convergence theorem, we have that

$$\begin{aligned}
\lim_{n\to+\infty} [u_n]^p_{s,p} &= \lim_{n\to+\infty} \int_{\mathbb{G}}\int_{\mathbb{G}} \frac{|u_n(x)-u_n(y)|^p}{|y^{-1}x|^{Q+sp}}dxdy \\
&= \int_{\mathbb{G}}\int_{\mathbb{G}} \frac{|u(x)-u(y)|^p}{|y^{-1}x|^{Q+sp}}dxdy = [u]^p_{s,p}.
\end{aligned} \tag{3.33}$$

As in Lemma 3.7 we define a_k and A_k, so we have

$$||u||_{L^{p^*}(\mathbb{G})} = \left(\sum_{k\in\mathbb{Z}} \int_{A_k\setminus A_{k+1}} |u(x)|^{p^*} dx\right)^{1/p^*} \leq \left(\sum_{k\in\mathbb{Z}} \int_{A_k\setminus A_{k+1}} 2^{(k+1)p^*} dx\right)^{1/p^*}$$

$$\leq \left(\sum_{k\in\mathbb{Z}} 2^{(k+1)p^*} a_k\right)^{1/p^*}. \tag{3.34}$$

Therefore, by combining Lemma 3.6 with $p/p^* = 1-sp/Q < 1$ and $T = 2^p$, we obtain

$$\begin{aligned}\|u\|^p_{L^{p^*}(\mathbb{G})} \leq 2^p \left(\sum_{k\in\mathbb{Z}} 2^{kp^*} a_k\right)^{p/p^*} &\leq 2^p \sum_{k\in\mathbb{Z}} 2^{kp} a_k^{(Q-sp)/Q} \\ &\leq C \sum_{k\in\mathbb{Z},\, a_k\neq 0} 2^{kp} a_k^{-sp/Q} a_{k+1}\end{aligned} \tag{3.35}$$

for a positive constant $C = C(Q,p,s,q) > 0$. By using Lemma 3.7 get

$$\|u\|^p_{L^{p^*}(\mathbb{G})} \leq C \sum_{k\in\mathbb{Z},\, a_k\neq 0} 2^{kp} a_k^{-sp/Q} a_{k+1} \leq C \int_{\mathbb{G}}\int_{\mathbb{G}} \frac{|u(x)-u(y)|^p}{|y^{-1}x|^{Q+sp}} dxdy = C[u]^p_{s,p}, \tag{3.36}$$

completing the proof. ☐

3.3. **Fractional Hardy-Sobolev inequality.**

In this section, we obtain the fractional Hardy-Sobolev inequality. We generalise both of the above inequalities, so the unified extension with arbitrary quasi-norm gives new inequalities even in the Euclidean (Abelian) case.

Theorem 3.10 (Fractional Hardy-Sobolev inequality)**.** *Suppose that $p > 1$, $s \in (0,1)$, $Q > 2$, $0 < \beta < sp$ and $Q > sp$ is such that $p^*_{s,\beta} = \frac{p(Q-\beta)}{Q-sp}$. Then for any $u \in W^{s,p}(\mathbb{G})$ and for any quasi-norm $|\cdot|$ of $\mathbb{G}$, we have*

$$\left(\int_{\mathbb{G}} \frac{|u(x)|^{p^*_{s,\beta}}}{|x|^\beta} dx\right)^{\frac{1}{p^*_{s,\beta}}} \leq C[u]_{s,p}, \tag{3.37}$$

where C is a positive constant.

Proof. By using Hölder's inequality with $\frac{\beta}{sp} + \frac{sp-\beta}{sp} = 1$, we get

$$\begin{aligned}\int_{\mathbb{G}} \frac{|u(x)|^{p^*_{s,\beta}}}{|x|^\alpha} dx &= \int_{\mathbb{G}} \frac{|u(x)|^{\frac{\beta}{s}} |u(x)|^{p^*_{s,\beta}-\frac{\beta}{s}}}{|x|^\beta} dx \\ &\leq \left(\int_{\mathbb{G}} \frac{|u(x)|^p}{|x|^{sp}} dx\right)^{\frac{\beta}{sp}} \left(\int_{\mathbb{G}} |u(x)|^{\left(p^*_{s,\beta}-\frac{\beta}{s}\right)\frac{sp}{sp-\beta}} dx\right)^{\frac{sp-\beta}{sp}}.\end{aligned} \tag{3.38}$$

By some calculation, we have

$$\left(p^*_{s,\beta} - \frac{\beta}{s}\right)\frac{sp}{sp-\beta} = \left(\frac{p(Q-\beta)}{Q-sp} - \frac{\beta}{s}\right)\frac{sp}{sp-\beta}$$

$$= \frac{Qsp - \beta sp - Q\beta + \beta sp}{s(Q-sp)} \frac{sp}{sp-\beta} = \frac{Qp}{Q-sp} = p^*,$$

where p^* is the Sobolev exponent. By combining the fractional Hardy and Sobolev inequalities, that is, Theorems 3.4 and 3.9, we establish

$$\begin{aligned}\int_{\mathbb{G}} \frac{|u(x)|^{p^*_{s,\beta}}}{|x|^{\beta}}dx &\leq \left(\int_{\mathbb{G}} \frac{|u(x)|^{p}}{|x|^{sp}}dx\right)^{\frac{\beta}{sp}} \left(\int_{\mathbb{G}} |u(x)|^{p^*}dx\right)^{\frac{sp-\beta}{sp}} \\ &\overset{(3.9)}{\leq} C[u]_{s,p}^{\frac{\beta}{s}} \left(\int_{\mathbb{G}} |u(x)|^{p^*}dx\right)^{\frac{sp-\beta}{sp}} \\ &\overset{(3.31)}{\leq} C[u]_{s,p}^{\frac{\beta}{s}}[u]_{s,p}^{\frac{sp-\beta}{sp}p^*} \\ &= C[u]_{s,p}^{\frac{\beta}{s}+\frac{sp-\beta}{sp}p^*}.\end{aligned} \tag{3.39}$$

Let us compute the exponent of the last term

$$\begin{aligned}\frac{\beta}{s}+\frac{sp-\beta}{sp}p^* &= \frac{\beta}{s}+\frac{sp-\beta}{sp}\frac{Qp}{Q-sp} \\ = \frac{1}{s}\left(\frac{\beta Q-\beta sp+Qsp-\beta Q}{Q-sp}\right) &= \frac{1}{s}\frac{sp(Q-\beta)}{Q-sp} = p^*_{s,\beta}.\end{aligned}$$

Finally, we have

$$\int_{\mathbb{G}} \frac{|u(x)|^{p^*_{s,\beta}}}{|x|^{\beta}}dx \leq C[u]_{s,p}^{\frac{\beta}{s}+\frac{sp-\beta}{sp}p^*} = C[u]_{s,p}^{p^*_{s,\beta}},$$

completing the proof. ☐

Corollary 3.11. *In Theorem 3.10, by setting $\beta = 0$, we obtain the fractional Sobolev inequality (3.31).*

Corollary 3.12. *When $\beta = sp$ in Theorem 3.10, we have the fractional Hardy inequality (3.9).*

Remark 3.13. *In the Abelian case $(\mathbb{R}^N, +)$, $Q = N$ with $|\cdot| = |\cdot|_E$ where $|\cdot|_E$ is the standard Euclidean distance, (3.37) implies the fractional Hardy-Sobolev inequality on $\mathbb{R}^N$ (see [62]). Moreover, the inequality is valid for any quasi-norm, not necessarily the Euclidean one. Therefore, even in the Abelian (Euclidean) case it extends the results of [62].*

3.4. **Fractional Gagliardo-Nirenberg inequality.**

In this section we show fractional Gagliardo-Nirenberg inequality on homogeneous Lie groups. One of the generalisation of the fractional Sobolev inequality is the fractional Gagliardo-Nirenberg inequality.

Theorem 3.14. *Suppose that $Q \geq 2$, $s \in (0,1)$, $p > 1$, $\alpha \geq 1$, $\tau > 0$, $a \in (0,1]$, $Q > sp$ and*

$$\frac{1}{\tau} = a\left(\frac{1}{p}-\frac{s}{Q}\right)+\frac{1-a}{\alpha}.$$

Then,

$$\|u\|_{L^{\tau}(\mathbb{G})} \leq C[u]_{s,p}^{a}\|u\|_{L^{\alpha}(\mathbb{G})}^{1-a}, \ \forall\, u \in C_c^1(\mathbb{G}), \tag{3.40}$$

where $C = C(s,p,Q,a,\alpha) > 0$.

Proof of Theorem 3.14. By using the Hölder inequality with $\frac{1}{\tau} = a\left(\frac{1}{p} - \frac{s}{Q}\right) + \frac{1-a}{\alpha}$ we establish

$$\|u\|^{\tau}_{L^{\tau}(\mathbb{G})} = \int_{\mathbb{G}} |u|^{\tau} dx = \int_{\mathbb{G}} |u|^{a\tau} |u|^{(1-a)\tau} dx \leq \|u\|^{a\tau}_{L^{p^*}(\mathbb{G})} \|u\|^{(1-a)\tau}_{L^{\alpha}(\mathbb{G})}, \tag{3.41}$$

where $p^* = \frac{Qp}{Q-sp}$. By combining (3.41) and the fractional Sobolev inequality (Theorem 3.9), we have

$$\|u\|^{\tau}_{L^{\tau}(\mathbb{G})} \leq \|u\|^{a\tau}_{L^{p^*}(\mathbb{G})} \|u\|^{(1-a)\tau}_{L^{\alpha}(\mathbb{G})} \leq C[u]^{a\tau}_{s,p} \|u\|^{(1-a)\tau}_{L^{\alpha}(\mathbb{G})},$$

that is,

$$\|u\|_{L^{\tau}(\mathbb{G})} \leq C[u]^{a}_{s,p} \|u\|^{1-a}_{L^{\alpha}(\mathbb{G})}, \tag{3.42}$$

where C is a positive constant independent of u. Theorem 3.14 is proved. ☐

Remark 3.15. *In the Abelian case $(\mathbb{R}^N, +)$ with the standard Euclidean distance instead of the quasi-norm and $s \to 1^-$, from Theorem 3.14 we get the Gagliardo-Nirenberg inequality which was proved in [63] and [64].*

Remark 3.16. *In the Abelian case $(\mathbb{R}^N, +)$ with the standard Euclidean distance instead of the quasi-norm, from Theorem 3.14 we get the fractional Gagliardo-Nirenberg inequality which was showed in [41].*

3.5. Fractional Caffarelli-Kohn-Nirenberg inequality.

In this section we prove the weighted fractional Caffarelli-Kohn-Nirenberg inequality on the homogeneous Lie groups.

Let us give some notations. The mean of a function u is defined by

$$u_{\Omega} = \fint_{\Omega} u(x)dx = \frac{1}{|\Omega|} \int_{\Omega} u dx, \quad u \in L^1(\Omega), \tag{3.43}$$

where $|\Omega|$ is the Haar measure of $\Omega \subset \mathbb{G}$.

We will also use the decomposition of $\mathbb{G}$ into quasi-annuli A_k defined by

$$A_k := \{x \in \mathbb{G} : \ 2^k \leq |x| < 2^{k+1}\}, \tag{3.44}$$

where $|x|$ is a quasi-norm on $\mathbb{G}$.

To show the fractional Caffarelli-Kohn-Nirenberg inequality on $\mathbb{G}$ we will use the fractional Gagliardo-Nirenberg inequality (Theorem 3.14) in the proof of the following lemma.

Lemma 3.17. *Suppose that $Q \geq 2$, $s \in (0,1)$, $p > 1$, $\alpha \geq 1$, $\tau > 0$, $a \in (0,1]$ and*

$$\frac{1}{\tau} \geq a\left(\frac{1}{p} - \frac{s}{Q}\right) + \frac{1-a}{\alpha}.$$

Assume that $\lambda > 0$ and $0 < r < R$ and set

$$\Omega = \{x \in \mathbb{G} : \ \lambda r < |x| < \lambda R\}.$$

Then, for every $u \in C^1(\overline{\Omega})$, we have

$$\left(\fint_{\Omega} |u - u_{\Omega}|^{\tau} dx\right)^{\frac{1}{\tau}} \leq C_{r,R} \lambda^{\frac{a(sp-Q)}{p}} [u]^{a}_{s,p,\Omega} \left(\fint_{\Omega} |u|^{\alpha} dx\right)^{\frac{1-a}{\alpha}}, \tag{3.45}$$

where $C_{r,R}$ is a positive constant independent of u and λ.

Proof of Lemma 3.17. Without loss of generality, we suppose that $0 < s' \leq s$ and $\tau' \geq \tau$ are such that

$$\frac{1}{\tau'} = a\left(\frac{1}{p} - \frac{s'}{Q}\right) + \frac{1-a}{\alpha},$$

and $\lambda = 1$, then let Ω_1 be

$$\Omega_1 = \{x \in \mathbb{G} : \ r < |x| < R\}.$$

By combining the fractional Gagliardo-Nirenberg inequality (see, Theorem 3.14), Jensen's inequality and $[u]_{s',p,\Omega} \leq C[u]_{s,p,\Omega}$, we establish

$$\begin{aligned}
\left(\fint_{\Omega_1} |u - u_{\Omega_1}|^\tau dx\right)^{\frac{1}{\tau}} &= \frac{1}{|\Omega_1|^{\frac{1}{\tau}}}\|u - u_{\Omega_1}\|_\tau \\
&\leq C_{r,R}\|u - u_{\Omega_1}\|_{L^{\tau'}(\Omega_1)} \leq C_{r,R}[u - u_{\Omega_1}]^a_{s',p,\Omega_1}\|u\|^{1-a}_{L^\alpha(\Omega_1)} \\
&\leq C_{r,R}\left(\int_{\Omega_1}\int_{\Omega_1} \frac{|u(x) - u_{\Omega_1} - u(y) + u_{\Omega_1}|^p}{|y^{-1}x|^{Q+s'p}} dxdy\right)^{\frac{a}{p}} \|u\|^{1-a}_{L^\alpha(\Omega_1)} \\
&\leq C_{r,R}[u]^a_{s,p,\Omega_1}\|u\|^{1-a}_{L^\alpha(\Omega_1)} \\
&\leq C_{r,R}[u]^a_{s,p,\Omega_1}\left(\fint_{\Omega_1} |u|^\alpha dx\right)^{\frac{1-a}{\alpha}},
\end{aligned} \tag{3.46}$$

where $C_{r,R} > 0$. By setting $u(\lambda x)$ instead of $u(x)$, we have

$$\begin{aligned}
\left(\fint_{\Omega_1} \left|u(\lambda x) - \fint_{\Omega_1} u(\lambda x)dx\right|^\tau dx\right)^{\frac{1}{\tau}} \leq C_{r,R}\left(\int_{\Omega_1}\int_{\Omega_1} \frac{|u(\lambda x) - u(\lambda y)|^p}{|y^{-1}x|^{Q+sp}} dxdy\right)^{\frac{a}{p}} \\
\times \left(\frac{1}{|\Omega_1|}\int_{\Omega_1} |u(\lambda x)|^\alpha dx\right)^{\frac{1-a}{\alpha}}.
\end{aligned} \tag{3.47}$$

Then by using (3.46) and Proposition 2.4, we calculate

$$\begin{aligned}
&\left(\fint_{\Omega}\left|u(x)-\fint_{\Omega}u(x)dx\right|^{\tau}dx\right)^{\frac{1}{\tau}}=\left(\frac{1}{|\Omega|}\int_{\Omega}\left|u(x)-\frac{1}{|\Omega|}\int_{\Omega}u(x)dx\right|^{\tau}dx\right)^{\frac{1}{\tau}}\\
&=\left(\frac{1}{|\Omega|}\int_{\Omega}\left|u(\lambda y)-\frac{1}{|\Omega|}\int_{\Omega}u(\lambda y)d(\lambda y)\right|^{\tau}d(\lambda y)\right)^{\frac{1}{\tau}}\\
&=\left(\frac{1}{|\Omega_1|}\int_{\Omega_1}\frac{\lambda^{Q}}{\lambda^{Q}}\left|u(\lambda y)-\frac{\lambda^{Q}}{\lambda^{Q}|\Omega_1|}\int_{\Omega_1}u(\lambda y)dy\right|^{\tau}dy\right)^{\frac{1}{\tau}}\\
&=\left(\frac{1}{|\Omega_1|}\int_{\Omega_1}\left|u(\lambda y)-\frac{1}{|\Omega_1|}\int_{\Omega_1}u(\lambda y)dy\right|^{\tau}dy\right)^{\frac{1}{\tau}}\\
&\leq C_{r,R}\left(\int_{\Omega_1}\int_{\Omega_1}\frac{|u(\lambda x)-u(\lambda y)|^{p}}{|y^{-1}x|^{Q+sp}}dxdy\right)^{\frac{a}{p}}\left(\frac{1}{|\Omega_1|}\int_{\Omega_1}|u(\lambda x)|^{\alpha}dx\right)^{\frac{1-a}{\alpha}}\\
&=C_{r,R}\left(\int_{\Omega_1}\int_{\Omega_1}\frac{\lambda^{2Q}\lambda^{Q+sp}|u(\lambda x)-u(\lambda y)|^{p}}{\lambda^{2Q}\lambda^{Q+sp}|y^{-1}x|^{Q+sp}}dxdy\right)^{\frac{a}{p}}\left(\frac{1}{|\Omega_1|}\int_{\Omega_1}\frac{\lambda^{Q}}{\lambda^{Q}}|u(\lambda x)|^{\alpha}dx\right)^{\frac{1-a}{\alpha}}\\
&=C_{r,R}\left(\int_{\Omega}\int_{\Omega}\frac{\lambda^{sp-Q}|u(\lambda x)-u(\lambda y)|^{p}}{|(\lambda y)^{-1}(\lambda x)|^{Q+sp}}d(\lambda x)d(\lambda y)\right)^{\frac{a}{p}}\left(\frac{1}{|\Omega|}\int_{\Omega}|u(\lambda x)|^{\alpha}d(\lambda x)\right)^{\frac{1-a}{\alpha}}\\
&=C_{r,R}\left(\int_{\Omega}\int_{\Omega}\frac{\lambda^{sp-Q}|u(x)-u(y)|^{p}}{|y^{-1}x|^{Q+sp}}dxdy\right)^{\frac{a}{p}}\left(\frac{1}{|\Omega|}\int_{\Omega}|u(x)|^{\alpha}dx\right)^{\frac{1-a}{\alpha}}\\
&=C_{r,R}\lambda^{\frac{a(sp-Q)}{p}}[u]^{a}_{s,p,\Omega}\left(\frac{1}{|\Omega|}\int_{\Omega}|u(x)|^{\alpha}dx\right)^{\frac{1-a}{\alpha}},
\end{aligned} \tag{3.48}$$

completing the proof. □

Theorem 3.18 (Fractional Caffarelli-Kohn-Nirenberg inequality)**.** *Suppose that* $Q\geq 2$*,* $s\in(0,1)$*,* $p>1$*,* $\alpha\geq 1$*,* $\tau>0$*,* $a\in(0,1]$*,* $\beta_1,\ \beta_2,\ \beta,\ \mu,\ \gamma\in\mathbb{R}$*,* $\beta_1+\beta_2=\beta$ *and*

$$\frac{1}{\tau}+\frac{\gamma}{Q}=a\left(\frac{1}{p}+\frac{\beta-s}{Q}\right)+(1-a)\left(\frac{1}{\alpha}+\frac{\mu}{Q}\right). \tag{3.49}$$

Suppose in addition that, $0\leq\beta-\sigma$ *with* $\gamma=a\sigma+(1-a)\mu$*, and*

$$\beta-\sigma\leq s \text{ only if } \frac{1}{\tau}+\frac{\gamma}{Q}=\frac{1}{p}+\frac{\beta-s}{Q}. \tag{3.50}$$

Then for $u\in C^{1}_{c}(\mathbb{G})$ *we have*

$$\||x|^{\gamma}u\|_{L^{\tau}(\mathbb{G})}\leq C[u]^{a}_{s,p,\beta}\||x|^{\mu}u\|^{1-a}_{L^{\alpha}(\mathbb{G})}, \tag{3.51}$$

when $\frac{1}{\tau}+\frac{\gamma}{Q}>0$*, and for* $u\in C^{1}_{c}(\mathbb{G}\setminus\{e\})$ *we have*

$$\||x|^{\gamma}u\|_{L^{\tau}(\mathbb{G})}\leq C[u]^{a}_{s,p,\beta}\||x|^{\mu}u\|^{1-a}_{L^{\alpha}(\mathbb{G})}, \tag{3.52}$$

when $\frac{1}{\tau}+\frac{\gamma}{Q}<0$*. Here* e *is the identity element of* $\mathbb{G}$*.*

Proof. Firstly, let us consider the case (3.50), that is, $\beta - \sigma \leq s$ and $\frac{1}{\tau} + \frac{\gamma}{Q} = \frac{1}{p} + \frac{\beta - s}{Q}$. By combining Lemma 3.17, $\lambda = 2^k$, $r = 1$, $R = 2$ and $\Omega = A_k$, we obtain

$$\left(\fint_{A_{k,q}} |u - u_{A_{k,q}}|^{\tau} dx \right)^{\frac{1}{\tau}} \leq C 2^{\frac{ak(sp-Q)}{p}} [u]^{a}_{s,p,q,A_{k,q}} \left(\fint_{A_{k,q}} |u|^{\alpha} dx \right)^{\frac{1-a}{\alpha}}, \tag{3.53}$$

where A_k is defined in (3.44) and $k \in \mathbb{Z}$. From (3.53) we obtain

$$\begin{aligned} \int_{A_k} |u|^{\tau} dx &= \int_{A_k} |u - u_{A_k} + u_{A_k}|^{\tau} dx \\ &\leq C \left(\int_{A_k} |u_{A_k}|^{\tau} dx + \int_{A_k} |u - u_{A_k}|^{\tau} dx \right) \\ &= C \left(\int_{A_k} |u_{A_k}|^{\tau} dx + \frac{|A_k|}{|A_k|} \int_{A_k} |u - u_{A_k}|^{\tau} dx \right) \\ &= C \left(|A_k| |u_{A_k}|^{\tau} + |A_k| \fint_{A_k} |u - u_{A_k}|^{\tau} dx \right) \\ &\leq C \left(|A_k| |u_{A_k}|^{\tau} + 2^{\frac{ak(sp-Q)\tau}{p}} |A_k| [u]^{a\tau}_{s,p,A_k} \left(\frac{1}{|A_k|} \int_{A_k} |u|^{\alpha} dx \right)^{\frac{(1-a)\tau}{\alpha}} \right) \\ &\leq C \left(2^{Qk} |u_{A_k}|^{\tau} + 2^{\frac{ak(sp-Q)\tau}{p}} 2^{kQ} 2^{-\frac{Q(1-a)\tau k}{\alpha}} [u]^{a\tau}_{s,p,A_k} \|u\|^{(1-a)\tau}_{L^{\alpha}(A_k)} \right). \end{aligned} \tag{3.54}$$

Then, from (3.54) we establish

$$\begin{aligned} &\int_{A_k} |x|^{\gamma\tau} |u|^{\tau} dx \leq 2^{(k+1)\gamma\tau} \int_{A_k} |u|^{\tau} dx \leq C 2^{(Q+\gamma\tau)k} |u_{A_k}|^{\tau} \\ &\quad + C 2^{\gamma\tau k} 2^{kQ} 2^{\frac{ak(sp-Q)\tau}{p}} 2^{-\frac{Q(1-a)\tau k}{\alpha}} [u]^{a\tau}_{s,p,A_k} \|u\|^{(1-a)\tau}_{L^{\alpha}(A_k)} = C 2^{(Q+\gamma\tau)k} |u_{A_k}|^{\tau} \\ &\quad + C 2^{\left(\gamma\tau + Q + \frac{a(sp-Q)\tau}{p} - \frac{Q(1-a)\tau}{\alpha}\right)k} \left(\int_{A_k} \int_{A_k} \frac{2^{kp\beta_1} 2^{kp\beta_2} |u(x) - u(y)|^p}{2^{kp\beta} |y^{-1} \ x|^{Q+sp}} dx dy \right)^{\frac{a\tau}{p}} \\ &\quad \times \left(\int_{A_k} \frac{2^{k\alpha\mu}}{2^{k\alpha\mu}} |u(x)|^{\alpha} dx \right)^{\frac{(1-a)\tau}{\alpha}} \leq C 2^{(Q+\gamma\tau)k} |u_{A_k}|^{\tau} \\ &\quad + C 2^{\left(\gamma\tau + Q + \frac{a(sp-Q)\tau}{p} - \frac{Q(1-a)\tau}{\alpha} - a\beta\tau - \mu\tau(1-a)\right)k} \left(\int_{A_k} \int_{A_k} \frac{|x|^{p\beta_1} |y|^{p\beta_2} |u(x) - u(y)|^p}{|y^{-1} x|^{Q+sp}} dx dy \right)^{\frac{a\tau}{p}} \\ &\quad \times \left(\int_{A_k} |x|^{\alpha\mu} |u(x)|^{\alpha} dx \right)^{\frac{(1-a)\tau}{\alpha}} \leq C 2^{(Q+\gamma\tau)k} |u_{A_k}|^{\tau} \\ &\quad + C 2^{\left(\gamma\tau + Q + \frac{a(sp-Q)\tau}{p} - \frac{Q(1-a)\tau}{\alpha} - a\beta\tau - \mu\tau(1-a)\right)k} [u]^{a\tau}_{s,p,\beta,A_k} \| |x|^{\mu} u \|^{(1-a)\tau}_{L^{\alpha}(A_k)}. \end{aligned}$$

From (3.49), we have

$$\begin{aligned}
&\gamma\tau + Q + \frac{a(sp-Q)\tau}{p} - \frac{Q(1-a)\tau}{\alpha} - a\beta\tau - \mu\tau(1-a)\\
&\quad = Q\tau\left(\frac{\gamma}{Q} + \frac{1}{\tau} + \frac{a(sp-Q)}{Qp} - \frac{(1-a)}{\alpha} - \frac{a\beta}{Q} - \frac{\mu(1-a)}{Q}\right)\\
&\quad = Q\tau\left(a\left(\frac{1}{p} + \frac{\beta-s}{Q}\right) + (1-a)\left(\frac{1}{\alpha} + \frac{\mu}{Q}\right) + \frac{a(sp-Q)}{Qp} - \frac{(1-a)}{\alpha} - \frac{a\beta}{Q} - \frac{\mu(1-a)}{Q}\right)\\
&\quad = 0.
\end{aligned}$$

Thus, we obtain

$$\int_{A_k} |x|^{\gamma\tau}|u|^\tau dx \le C2^{(\gamma\tau+Q)k}|u_{A_k}|^\tau + C[u]^{a\tau}_{s,p,\beta,A_k}\||x|^\mu u\|^{(1-a)\tau}_{L^\alpha(A_k)}, \tag{3.55}$$

and by summing over k from m to n, we get

$$\begin{aligned}
\int_{\cup_{k=m}^n A_k} |x|^{\gamma\tau}|u|^\tau dx = \int_{\{2^m<|x|<2^{n+1}\}} |x|^{\gamma\tau}|u|^\tau dx \le C\sum_{k=m}^n 2^{(\gamma\tau+Q)k}|u_{A_k}|^\tau\\
+ C\sum_{k=m}^n [u]^{a\tau}_{s,p,\beta,A_k}\||x|^\mu u\|^{(1-a)\tau}_{L^\alpha(A_k)},
\end{aligned} \tag{3.56}$$

where $k, m, n \in \mathbb{Z}$ and $m \le n-2$.

Let us show (3.51). By choosing n such that

$$\operatorname{supp} u \subset B_{2^n}, \tag{3.57}$$

where B_{2^n} is a quasi-ball of $\mathbb{G}$ with the radius 2^n.

The following known inequality will be used in the proof.

Lemma 3.19 (Lemma 2.2, [65]). *Let $\xi > 1$ and $\eta > 1$. Then exists a positive constant C depending ξ and η such that $1 < \zeta < \xi$,*

$$(|a| + |b|)^\eta \le \zeta|a|^\eta + \frac{C}{(\zeta-1)^{\eta-1}}|b|^\eta, \quad \forall\, a, b \in \mathbb{R}. \tag{3.58}$$

Let us consider the following integral

$$\begin{aligned}
&⨍_{A_{k+1,q}\cup A_{k,q}} \left|u - ⨍_{A_{k+1,q}\cup A_{k,q}} u\right|^\tau dx\\
&\qquad = \frac{1}{|A_{k+1,q}| + |A_{k,q}|}\int_{A_{k+1,q}\cup A_{k,q}} \left|u - ⨍_{A_{k+1,q}\cup A_{k,q}} u\right|^\tau dx\\
&= \frac{1}{|A_{k+1,q}| + |A_{k,q}|}\left(\int_{A_{k+1,q}} \left|u - ⨍_{A_{k+1,q}\cup A_{k,q}} u\right|^\tau dx + \int_{A_{k,q}} \left|u - ⨍_{A_{k+1,q}\cup A_{k,q}} u\right|^\tau dx\right).
\end{aligned}$$

Then, we compute

$$
\begin{aligned}
&\fint_{A_{k+1}\cup A_k}\left|u-\fint_{A_{k+1}\cup A_k}u\right|^{\tau}dx\\
&\quad=\frac{1}{|A_{k+1}|+|A_k|}\Big(\int_{A_{k+1}}\left|u-\fint_{A_{k+1}\cup A_k}u\right|^{\tau}dx\\
&\quad+\int_{A_k}\left|u-\fint_{A_{k+1}\cup A_k}u\right|^{\tau}dx\Big)\\
&\quad\geq\frac{1}{|A_{k+1}|+|A_k|}\int_{A_k}\left|u-\fint_{A_{k+1}\cup A_k}u\right|^{\tau}dx\\
&\quad\geq\frac{1}{|A_{k+1}|+|A_k|}\left|\int_{A_k}\left(u-\fint_{A_{k+1}\cup A_k}u\right)dx\right|^{\tau}\\
&\quad=\frac{1}{|A_{k+1}|+|A_k|}\Big|\int_{A_k}udx-\frac{|A_k|}{|A_{k+1}|+|A_k|}\int_{A_k}udx\\
&\quad-\frac{|A_k|}{|A_{k+1}|+|A_k|}\int_{A_{k+1}}udx\Big|^{\tau}\\
&\quad=\frac{1}{|A_{k+1}|+|A_k|}\Big|\frac{|A_{k+1}|}{|A_{k+1}|+|A_k|}\int_{A_k}udx\\
&\quad-\frac{|A_k|}{|A_{k+1}|+|A_k|}\int_{A_{k+1}}udx\Big|^{\tau}\\
&\quad=\frac{1}{(|A_{k+1}|+|A_k|)^{\tau+1}}\left||A_{k+1}|\int_{A_k}udx-|A_k|\int_{A_{k+1}}udx\right|^{\tau}\\
&\quad=\frac{|A_{k+1}|^{\tau}|A_k|^{\tau}}{(|A_{k+1}|+|A_k|)^{\tau+1}}\left|\frac{1}{|A_k|}\int_{A_k}udx-\frac{1}{|A_{k+1}|}\int_{A_{k+1}}udx\right|^{\tau}\\
&\quad=\frac{|A_{k+1}|^{\tau}|A_k|^{\tau}}{(|A_{k+1}|+|A_k|)^{\tau+1}}|u_{A_{k+1}}-u_{A_k}|^{\tau}\\
&\quad\geq C|u_{A_{k+1}}-u_{A_k}|^{\tau}.
\end{aligned}
\tag{3.59}
$$

By combining (3.59) and Lemma 3.17, we get

$$
\begin{aligned}
|u_{A_{k+1}}-u_{A_k}|^{\tau}&\leq C\fint_{A_{k+1}\cup A_k}\left|u-\fint_{A_{k+1}\cup A_k}u\right|^{\tau}dx\\
&\leq C2^{\frac{ak(sp-Q)}{p}}[u]^{\tau a}_{s,p,A_{k+1}\cup A_k}\left(\fint_{A_{k+1}\cup A_k}|u|^{\alpha}dx\right)^{\frac{(1-a)\tau}{\alpha}}.
\end{aligned}
$$

By using this fact with $\tau = 1$, we get

$$\begin{aligned} |u_{A_k}| &\le |u_{A_{k+1}} - u_{A_k}| + |u_{A_{k+1}}| \\ &\le |u_{A_{k+1}}| + C2^{\frac{ak(sp-Q)}{p}}[u]^a_{s,p,A_{k+1}\cup A_k}\left(\fint_{A_{k+1}\cup A_k}|u|^\alpha dx\right)^{\frac{(1-a)}{\alpha}}, \end{aligned} \tag{3.60}$$

and from Lemma 3.19 and $\eta = \tau$, $\zeta = 2^{\gamma\tau+Q}c$, where $c = \frac{2}{1+2^{\gamma\tau+Q}} < 1$, since $\gamma\tau + Q > 0$, we obtain

$$2^{(\gamma\tau+Q)k}|u_{A_k}|^\tau \le c2^{(k+1)(\gamma\tau+Q)}|u_{A_{k+1}}|^\tau + C[u]^{\tau a}_{s,p,\beta,A_{k+1}\cup A_k}\||x|^\mu u\|^{(1-a)\tau}_{L^\alpha(A_{k+1}\cup A_k)}.$$

By summing over k from m to n and by using (3.57) we have

$$\begin{aligned} \sum_{k=m}^n 2^{(\gamma\tau+Q)k}|u_{A_k}|^\tau &\le \sum_{k=m}^n c2^{(k+1)(\gamma\tau+Q)}|u_{A_{k+1}}|^\tau \\ &\quad + C\sum_{k=m}^n [u]^{\tau a}_{s,p,\beta,A_{k+1}\cup A_k}\||x|^\mu(x)u\|^{(1-a)\tau}_{L^\alpha(A_{k+1}\cup A_k)}. \end{aligned} \tag{3.61}$$

From (3.61), we get

$$\begin{aligned} (1-c)\sum_{k=m}^n 2^{(\gamma\tau+Q)k}|u_{A_k}|^\tau &\le 2^{(\gamma\tau+Q)m}|u_{A_m}|^\tau + (1-c)\sum_{k=m+1}^n 2^{(\gamma\tau+Q)k}|u_{A_k}|^\tau \\ &\le C\sum_{k=m}^n [u]^{\tau a}_{s,p,\beta,A_{k+1}\cup A_k}\||x|^\mu u\|^{(1-a)\tau}_{L^\alpha(A_{k+1}\cup A_k)}. \end{aligned} \tag{3.62}$$

This yields

$$\sum_{k=m}^n 2^{(\gamma\tau+Q)k}|u_{A_k}|^\tau \le C\sum_{k=m}^n [u]^{\tau a}_{s,p,\beta,A_{k+1}\cup A_k}\||x|^\mu u\|^{(1-a)\tau}_{L^\alpha(A_{k+1}\cup A_k)}. \tag{3.63}$$

By using (3.56) and (3.63), we have

$$\int_{\{2^m<|x|<2^{n+1}\}}|x|^{\gamma\tau}|u|^\tau dx \le C\sum_{k=m}^n [u]^{\tau a}_{s,p,\beta,A_{k+1}\cup A_k}\||x|^\mu u\|^{(1-a)\tau}_{L^\alpha(A_{k+1}\cup A_k)}. \tag{3.64}$$

Assume $s, t \ge 0$ be such that $s + t \ge 1$. Then for any $x_k, y_k \ge 0$, we have

$$\sum_{k=m}^n x_k^s y_k^t \le \left(\sum_{k=m}^n x_k\right)^s \left(\sum_{k=m}^n y_k\right)^t. \tag{3.65}$$

By using this inequality in (3.64) with $s = \frac{\tau a}{p}$, $t = \frac{(1-a)\tau}{\alpha}$, $\frac{a}{p} + \frac{1-a}{\alpha} \ge \frac{1}{\tau}$ and $s \ge \beta - \sigma$, we obtain

$$\int_{\{|x|>2^m\}}|x|^{\gamma\tau}|u|^\tau dx \le C[u]^{a\tau}_{s,p,\beta,\cup_{k=m}^\infty A_k}\||x|^\mu u\|^{(1-a)\tau}_{L^\alpha(\cup_{k=m}^\infty A_k)}. \tag{3.66}$$

Inequality (3.51) is proved.

Let us show (3.52). The strategy of the proof is similar to the previous case. By choosing m such that

$$\operatorname{supp} u \cap B_{2^m} = \emptyset. \tag{3.67}$$

By using Lemma 3.17 we get

$$|u_{A_{k+1}} - u_{A_k}|^\tau \le C2^{\frac{a\tau k(sp-Q)}{p}} [u]^{\tau a}_{s,p,A_{k+1}\cup A_k} \left(\fint_{A_{k+1}\cup A_k} |u|^\alpha dx \right)^{\frac{(1-a)\tau}{\alpha}}.$$

From Lemma 3.19 and choosing $c = \frac{1+2^{\gamma\tau+Q}}{2} < 1$, since $\gamma\tau + Q < 0$, we establish

$$2^{(\gamma\tau+Q)(k+1)}|u_{A_{k+1}}|^\tau \le c2^{k(\gamma\tau+Q)}|u_{A_k}|^\tau + C[u]^{\tau a}_{s,p,\beta,A_{k+1}\cup A_k} \||x|^\mu u\|^{(1-a)\tau}_{L^\alpha(A_{k+1}\cup A_k)},$$

and by summing over k from m to n and by using (3.67) we obtain

$$\sum_{k=m}^{n} 2^{(\gamma\tau+Q)k}|u_{A_k}|^\tau \le C \sum_{k=m-1}^{n-1} [u]^{\tau a}_{s,p,\beta,A_{k+1}\cup A_k} \||x|^\mu u\|^{(1-a)\tau}_{L^\alpha(A_{k+1}\cup A_k)}. \tag{3.68}$$

By using (3.56) and (3.68), we obtain that

$$\int_{\{2^m<|x|<2^{n+1}\}} |x|^{\gamma\tau}|u|^\tau dx \le C \sum_{k=m-1}^{n-1} [u]^{\tau a}_{s,p,\beta,A_{k+1}\cup A_k} \||x|^\mu u\|^{(1-a)\tau}_{L^\alpha(A_{k+1}\cup A_k)}. \tag{3.69}$$

From (3.65) we get

$$\int_{\{|x|<2^{n+1}\}} |x|^{\gamma\tau}|u|^\tau dx \le C[u]^{\tau a}_{s,p,\beta,\cup_{k=-\infty}^n A_k} \||x|^\mu u\|^{(1-a)\tau}_{L^\alpha(\cup_{k=-\infty}^n A_k)}. \tag{3.70}$$

The proof of the case $s \ge \beta - \sigma$ is complete.

Let us prove the case of $\beta - \sigma > s$. Without loss of generality, we suppose that

$$[u]_{s,p,\beta} = \|u\|_{L^\alpha(\mathbb{G})} = 1, \tag{3.71}$$

where

$$\frac{1}{p} + \frac{\beta - s}{Q} \ne \frac{1}{\alpha} + \frac{\mu}{Q}.$$

We also suppose that $a_1 > 0,\ 1 > a_2$ and $\tau_1, \tau_2 > 0$ with

$$\frac{1}{\tau_2} = \frac{a_2}{p} + \frac{1-a_2}{\alpha}, \tag{3.72}$$

and

$$\begin{aligned} &\text{if} \quad \frac{a}{p} + \frac{1-a}{\alpha} - \frac{as}{Q} > 0, \quad \text{then} \quad \frac{1}{\tau_1} = \frac{a_1}{p} + \frac{1-a_1}{\alpha} - \frac{a_1 s}{Q}, \\ &\text{if} \quad \frac{a}{p} + \frac{1-a}{\alpha} - \frac{as}{Q} \le 0, \quad \text{then} \quad \frac{1}{\tau} > \frac{1}{\tau_1} \ge \frac{a_1}{p} + \frac{1-a_1}{\alpha} - \frac{a_1 s}{Q}. \end{aligned} \tag{3.73}$$

By taking $\gamma_1 = a_1\beta + (1-a_1)\mu$ and $\gamma_2 = a_2(\beta - s) + (1-a_2)\mu$, we have

$$\frac{1}{\tau_1} + \frac{\gamma_1}{Q} \ge a_1\left(\frac{1}{p} + \frac{\beta - s}{Q}\right) + (1-a_1)\left(\frac{1}{\alpha} + \frac{\mu}{Q}\right) \tag{3.74}$$

and

$$\frac{1}{\tau_2} + \frac{\gamma_2}{Q} = a_2\left(\frac{1}{p} + \frac{\beta - s}{Q}\right) + (1-a_2)\left(\frac{1}{\alpha} + \frac{\mu}{Q}\right). \tag{3.75}$$

Assume a_1 and a_2 be such that

$$|a - a_1| \ \text{ and } \ |a - a_2| \ \text{ are small enough,} \tag{3.76}$$

$$a_2 < a < a_1, \text{ if } \frac{1}{p} + \frac{\beta - s}{Q} > \frac{1}{\alpha} + \frac{\mu}{Q}, \tag{3.77}$$

$$a_1 < a < a_2, \text{ if } \frac{1}{p} + \frac{\beta - s}{Q} < \frac{1}{\alpha} + \frac{\mu}{Q}. \tag{3.78}$$

By combining (3.76)-(3.78) in (3.74), (3.75) and (3.49), we get

$$\frac{1}{\tau_1} + \frac{\gamma_1}{Q} > \frac{1}{\tau} + \frac{\gamma}{Q} > \frac{1}{\tau_2} + \frac{\gamma_2}{Q} > 0. \tag{3.79}$$

From (3.73) in the case $\frac{a}{p} + \frac{1-a}{\alpha} - \frac{as}{Q} > 0$ with $a > 0$, $\beta - \sigma > s$ and (3.76), we get

$$\frac{1}{\tau} - \frac{1}{\tau_1} = (a - a_1)\left(\frac{1}{p} - \frac{s}{Q} - \frac{1}{\alpha}\right) + \frac{a}{Q}(\beta - \sigma) > 0, \tag{3.80}$$

and

$$\frac{1}{\tau} - \frac{1}{\tau_2} = (a - a_2)\left(\frac{1}{p} - \frac{1}{\alpha}\right) + \frac{a}{Q}(\beta - \sigma - s) > 0. \tag{3.81}$$

By combining (3.73), (3.80) and (3.81), we get

$$\tau_1 > \tau, \quad \tau_2 > \tau.$$

Thus, by using last fact, (3.76) and Hölder's inequality, we get

$$\||x|^{\gamma} u\|_{L^{\tau}(\mathbb{G}\backslash B_1)} \leq C\||x|^{\gamma_1} u\|_{L^{\tau_1}(\mathbb{G})}, \tag{3.82}$$

and

$$\||x|^{\gamma} u\|_{L^{\tau}(B_1)} \leq C\||x|^{\gamma_2} u\|_{L^{\tau_2}(\mathbb{G})}, \tag{3.83}$$

where B_1 is the unit quasi-ball. By using the previous case, we get

$$\||x|^{\gamma_1} u\|_{L^{\tau_1}(\mathbb{G})} \leq C[u]^{a_1}_{s,p,\beta}\||x|^{\mu} u\|^{1-a_1}_{L^{\alpha}(\mathbb{G})} \leq C, \tag{3.84}$$

and

$$\||x|^{\gamma_2} u\|_{L^{\tau_2}(\mathbb{G})} \leq C[u]^{a_2}_{s,p,\beta}\||x|^{\mu} u\|^{1-a_2}_{L^{\alpha}(\mathbb{G})} \leq C. \tag{3.85}$$

The proof of Theorem 3.18 is complete. ☐

Remark 3.20. *In the Abelian case* $(\mathbb{R}^N, +)$ *with the standard Euclidean distance instead of quasi-norm in Theorem 3.18, we get the (Euclidean) fractional Caffarelli-Kohn-Nirenberg inequality (see, e.g.* [41], *Theorem 1.1).*

Remark 3.21. *In the Abelian case* $(\mathbb{R}^N, +)$ *with the standard Eucledian distance instead of the quasi-norm and* $s \to 1^-$ *in (3.52), we get classical Caffarelli-Kohn-Nirenberg inequality.*

Remark 3.22. *By taking in (3.52)* $a = 1$, $\tau = p$, $\beta_1 = \beta_2 = 0$, *and* $\gamma = -s$, *we get an analogue of the fractional Hardy inequality on homogeneous Lie groups (Theorem 3.4).*

Remark 3.23. *In the Abelian case* $(\mathbb{R}^N, +)$ *with the standard Eucledian distance instead of the quasi-norm and by taking in (3.52)* $a = 1$, $\tau = p$, $\beta_1 = \beta_2 = 0$, *and* $\gamma = -s$, *we get the fractional Hardy inequality (Theorem 1.1,* [1]*).*

Remark 3.24. *By taking in (3.51)* $a = 1$, $\tau = p^*$, $\beta_1 = \beta_2 = 0$, *and* $\gamma = 0$, *we get an analogue of the fractional Sobolev inequality on homogeneous Lie groups (Theorem 3.9).*

Now we consider the critical case $\frac{1}{\tau}+\frac{\gamma}{Q}=0$.

Theorem 3.25 (Fractional critical Caffarelli-Kohn-Nirenberg inequality)**.** *Suppose that* $Q\geq 2$, $s\in(0,1)$, $p>1$, $\alpha\geq 1$, $\tau>1$, $a\in(0,1]$, β_1, β_2, β, μ, $\gamma\in\mathbb{R}$, $\beta_1+\beta_2=\beta$,

$$\frac{1}{\tau}+\frac{\gamma}{Q}=a\left(\frac{1}{p}+\frac{\beta-s}{Q}\right)+(1-a)\left(\frac{1}{\alpha}+\frac{\mu}{Q}\right). \tag{3.86}$$

Suppose in addition that, $0\leq\beta-\sigma\leq s$ *with* $\gamma=a\sigma+(1-a)\mu$.

If $\frac{1}{\tau}+\frac{\gamma}{Q}=0$ *and* $\operatorname{supp}u\subset B_R$, *then, we have*

$$\left\|\frac{|x|^{\gamma}}{\ln\frac{2R}{|x|}}u\right\|_{L^{\tau}(\mathbb{G})}\leq C[u]^{a}_{s,p,\beta}\||x|^{\mu}u\|^{1-a}_{L^{\alpha}(\mathbb{G})},\ u\in C^{1}_{c}(\mathbb{G}), \tag{3.87}$$

where $B_R=\{x\in\mathbb{G}:|x|<R\}$ *is the quasi-ball and* $0<r<R$.

Proof of Theorem 3.25. The proof is similar to the proof of Theorem 3.18. In (3.55), by summarising over k from m to n and by fixing $\varepsilon>0$, we get

$$\begin{aligned}\int_{\{|x|>2^{m}\}}\frac{|x|^{\gamma\tau}}{\ln^{1+\varepsilon}\left(\frac{2R}{|x|}\right)}|u|^{\tau}dx\leq C\sum_{k=m}^{n}\frac{1}{(n+1-k)^{1+\varepsilon}}|u_{A_k}|^{\tau}\\ +C\sum_{k=m}^{n}[u]^{a\tau}_{s,p,\beta,A_k}\||x|^{\mu}(x)u\|^{(1-a)\tau}_{L^{\alpha}(A_k)}.\end{aligned} \tag{3.88}$$

By using Lemma 3.17, we get

$$|u_{A_{k+1}}-u_{A_k}|\leq C2^{\frac{ak(sp-Q)}{p}}[u]^{a}_{s,p,A_{k+1}\cup A_k}\left(\fint_{A_{k+1}\cup A_k}|u|^{\alpha}dx\right)^{\frac{1-a}{\alpha}}.$$

From Lemma 3.19 with $\zeta=\frac{(n+1-k)^{\varepsilon}}{(n+\frac{1}{2}-k)^{\varepsilon}}$ we establish

$$\begin{aligned}\frac{|u_{A_k}|^{\tau}}{(n+1-k)^{\varepsilon}}\leq\frac{|u_{A_{k+1}}|^{\tau}}{(n+\frac{1}{2}-k)^{\varepsilon}}\\ +C(n+1-k)^{\tau-1-\varepsilon}[u]^{a\tau}_{s,p,\beta,A_{k+1}\cup A_k}\||x|^{\mu}u\|^{(1-a)\tau}_{L^{\alpha}(A_{k+1}\cup A_k)}.\end{aligned} \tag{3.89}$$

For $\varepsilon>0$ and $n\geq k$, we have

$$\frac{1}{(n-k+1)^{\varepsilon}}-\frac{1}{(n-k+\frac{3}{2})^{\varepsilon}}\sim\frac{1}{(n-k+1)^{1+\varepsilon}}. \tag{3.90}$$

By combining this fact, (3.89), (3.90) and $\varepsilon=\tau-1$, we get

$$\sum_{k=m}^{n}\frac{|u_{A_k}|^{\tau}}{(n+1-k)^{\tau}}\leq C\sum_{k=m}^{n}[u]^{a\tau}_{s,p,\beta,A_{k+1}\cup A_k}\||x|^{\mu}u\|^{(1-a)\tau}_{L^{\alpha}(A_{k+1}\cup A_k)}. \tag{3.91}$$

By using (3.88) and (3.91), we have

$$\int_{\{|x|>2^{m}\}}\frac{|x|^{\gamma\tau}}{\ln^{\tau}\frac{2R}{|x|}}|u|^{\tau}dx\leq C\sum_{k=m}^{n}[u]^{a\tau}_{s,p,\beta,A_{k+1}\cup A_k}\||x|^{\mu}u\|^{(1-a)\tau}_{L^{\alpha}(A_{k+1}\cup A_k)}. \tag{3.92}$$

By combining (3.65) with (3.86) and $0 \leq \beta - \sigma \leq s$, where $s = \frac{\tau a}{p}$, $t = \frac{(1-a)\tau}{\alpha}$, we have $s + t \geq 1$ and we get

$$\int_{\{|x|>2^m\}} \frac{|x|^{\gamma\tau}}{\ln^{\tau} \frac{2R}{|x|}} |u|^{\tau} dx \leq C \sum_{k=m}^{n} [u]^{a\tau}_{s,p,\beta,\cup_{k=m}^{\infty} A_k} \||x|^{\mu} u\|^{(1-a)\tau}_{L^{\alpha}(\cup_{k=m}^{\infty} A_k)}, \tag{3.93}$$

completing the proof. ☐

3.6. **Fractional Logarithmic inequalities.** In this section, we show fractional logarithmic inequalities on homogeneous Lie group. By the way, we need some preliminary results. Firstly, we show weighted Hölder's inequality on $\mathbb{G}$.

Lemma 3.26. *Assume that* $1 < p \leq r \leq q \leq \infty$, $a \in [0,1]$, $\alpha \in \mathbb{R}$, $|x|^{\alpha} u \in L^p(\mathbb{G}) \cap L^q(\mathbb{G})$ *with*

$$\frac{1}{r} = \frac{a}{p} + \frac{1-a}{q}, \tag{3.94}$$

then we have

$$\||x|^{\alpha} u\|_{L^r(\mathbb{G})} \leq \||x|^{\alpha} u\|^{a}_{L^p(\mathbb{G})} \||x|^{\alpha} u\|^{1-a}_{L^q(\mathbb{G})}. \tag{3.95}$$

Proof. By using Hölder's inequality we obtain

$$\begin{aligned} \||x|^{\alpha} u\|^{r}_{L^r(\mathbb{G})} &= \int_{\mathbb{G}} |x|^{\alpha r} |u(x)|^{r} dx = \int_{\mathbb{G}} (|x|^{\alpha} |u(x)|)^{ar} (|x|^{\alpha} |u(x)|)^{(1-a)r} dx \\ &\leq \left(\int_{\mathbb{G}} |x|^{\alpha p} |u(x)|^{p} dx \right)^{\frac{ar}{p}} \left(\int_{\mathbb{G}} |x|^{\alpha q} |u(x)|^{q} dx \right)^{\frac{(1-a)r}{q}} \\ &= \||x|^{\alpha} u\|^{ar}_{L^p(\mathbb{G})} \||x|^{\alpha} u\|^{(1-a)r}_{L^q(\mathbb{G})}, \end{aligned} \tag{3.96}$$

with

$$\frac{ar}{p} + \frac{(1-a)r}{q} = 1. \tag{3.97}$$

☐

Now let us show logarithmic Hölder's inequality.

Lemma 3.27 (Logarithmic Hölder's inequality)**.** *Suppose that* $|x|^{\alpha} u \in L^p(\mathbb{G}) \cap L^q(\mathbb{G})$ *with some* $\alpha \in \mathbb{R}$, $1 < p < q \leq \infty$. *Then we have*

$$\int_{\mathbb{G}} \frac{(|x|^{\alpha p} |u|^{p})}{\||x|^{\alpha} u\|^{p}_{L^p(\mathbb{G})}} \log \left(\frac{|x|^{\alpha p} |u|^{p}}{\||x|^{\alpha} u\|^{p}_{L^p(\mathbb{G})}} \right) dx \leq \frac{q}{q-p} \log \left(\frac{\||x|^{\alpha} u\|^{p}_{L^q(\mathbb{G})}}{\||x|^{\alpha} u\|^{p}_{L^p(\mathbb{G})}} \right). \tag{3.98}$$

Proof. Let us consider the following function

$$F\left(\frac{1}{r}\right) = \log \left(\||x|^{\alpha} u\|_{L^r(\mathbb{G})} \right). \tag{3.99}$$

Firstly, we need to prove the function (3.99) is convex. By using Lemma 3.26, we obtain

$$\begin{aligned} F\left(\frac{1}{r}\right) &= \log \left(\||x|^{\alpha} u\|_{L^r(\mathbb{G})} \right) \leq \log \left(\||x|^{\alpha} u\|^{a}_{L^p(\mathbb{G})} \||x|^{\alpha} u\|^{1-a}_{L^q(\mathbb{G})} \right) \\ &= \log \left(\||x|^{\alpha} u\|^{a}_{L^p(\mathbb{G})} \right) + \log \left(\||x|^{\alpha} u\|^{1-a}_{L^q(\mathbb{G})} \right) = aF\left(\frac{1}{p}\right) + (1-a)F\left(\frac{1}{q}\right), \end{aligned} \tag{3.100}$$

with $a \in [0,1]$ and $\frac{1}{r} = \frac{a}{p} + \frac{1-a}{q}$.

Since we have

$$F(r) = r \log \int_{\mathbb{G}} |x|^{\frac{\alpha}{r}} |u(x)|^{\frac{1}{r}} dx, \tag{3.101}$$

the derivative of (3.101) is

$$\begin{aligned} F'(r) &= \log \int_{\mathbb{G}} |x|^{\frac{\alpha}{r}} |u(x)|^{\frac{1}{r}} dx + r \left(\log \int_{\mathbb{G}} |x|^{\frac{\alpha}{r}} |u(x)|^{\frac{1}{r}} dx \right)'_r \\ &= \log \int_{\mathbb{G}} |x|^{\frac{\alpha}{r}} |u(x)|^{\frac{1}{r}} dx + r \frac{\left(\int_{\mathbb{G}} (|x|^{\alpha} u(x))^{\frac{1}{r}} dx \right)'_r}{\int_{\mathbb{G}} |x|^{\frac{\alpha}{r}} |u(x)|^{\frac{1}{r}} dx} \\ &= \log \int_{\mathbb{G}} |x|^{\frac{\alpha}{r}} |u(x)|^{\frac{1}{r}} dx - \frac{1}{r} \frac{\int_{\mathbb{G}} (|x|^{\alpha} u(x))^{\frac{1}{r}} \log(|x|^{\alpha} |u(x)|) dx}{\int_{\mathbb{G}} |x|^{\frac{\alpha}{r}} |u(x)|^{\frac{1}{r}} dx}. \end{aligned} \tag{3.102}$$

From (3.100) $F(r)$ is convex, hence, we get

$$F'(r) \geq \frac{F(r') - F(r)}{r' - r}, \quad r' > r > 0. \tag{3.103}$$

With $r = \frac{1}{p}$ and $r' = \frac{1}{q}$ it yields

$$\begin{aligned} p \frac{\int_{\mathbb{G}} ||x|^{\alpha} u|^p \log |x|^{\alpha} |u| dx}{\int_{\mathbb{G}} |x|^{\alpha p} |u(x)|^p dx} - \log \int_{\mathbb{G}} |x|^{\alpha p} |u(x)|^p dx & \\ \leq \frac{qp}{q-p} \log \left(\int_{\mathbb{G}} \frac{\| |x|^{\alpha} u \|_{L^q(\mathbb{G})}}{\| |x|^{\alpha} u \|_{L^p(\mathbb{G})}} \right)&. \end{aligned} \tag{3.104}$$

We have

$$\begin{aligned} \log \int_{\mathbb{G}} |x|^{\alpha p} |u(x)|^p dx &= \frac{\log \int_{\mathbb{G}} |x|^{\alpha p} |u(x)|^p dx \int_{\mathbb{G}} |x|^{\alpha p} |u(x)|^p dx}{\int_{\mathbb{G}} |x|^{\alpha p} |u(x)|^p dx} \\ &= \frac{\int_{\mathbb{G}} |x|^{\alpha p} |u(x)|^p \log \| |x|^{\alpha} u \|^p_{L^p(\mathbb{G})} dx}{\int_{\mathbb{G}} |x|^{\alpha p} |u(x)|^p dx}. \end{aligned} \tag{3.105}$$

By using last fact in (3.104) we establish logarithmic Hölder's inequality

$$\begin{aligned} & p \frac{\int_{\mathbb{G}} ||x|^{\alpha} u|^p \log |x|^{\alpha} |u| dx}{\int_{\mathbb{G}} |x|^{\alpha p} |u(x)|^p dx} - \log \int_{\mathbb{G}} |x|^{\alpha p} |u(x)|^p dx \\ &= p \frac{\int_{\mathbb{G}} ||x|^{\alpha} u|^p \log |x|^{\alpha} |u| dx}{\int_{\mathbb{G}} |x|^{\alpha p} |u(x)|^p dx} - \frac{\int_{\mathbb{G}} |x|^{\alpha p} |u(x)|^p \log \| |x|^{\alpha} u \|^p_{L^p(\mathbb{G})} dx}{\int_{\mathbb{G}} |x|^{\alpha p} |u(x)|^p dx} \\ &= \frac{\int_{\mathbb{G}} ||x|^{\alpha} u|^p \log |x|^{\alpha p} |u|^p dx}{\int_{\mathbb{G}} |x|^{\alpha p} |u(x)|^p dx} - \frac{\int_{\mathbb{G}} |x|^{\alpha p} |u(x)|^p \log \| |x|^{\alpha} u \|^p_{L^p(\mathbb{G})} dx}{\int_{\mathbb{G}} |x|^{\alpha p} |u(x)|^p dx} \\ &= \int_{\mathbb{G}} \frac{(|x|^{\alpha p} |u|^p)}{\| |x|^{\alpha} |u| \|^p_{L^p(\mathbb{G})}} \log \left(\frac{|x|^{\alpha p} |u|^p}{\| |x|^{\alpha} u \|^p_{L^p(\mathbb{G})}} \right) \leq \frac{q}{q-p} \log \left(\frac{\| |x|^{\alpha} u \|^p_{L^q(\mathbb{G})}}{\| |x|^{\alpha} u \|^p_{L^p(\mathbb{G})}} \right). \end{aligned} \tag{3.106}$$

□

3.6.1. *Fractional Logarithmic Sobolev inequality.* In this subsection, we present the fractional logarithmic Sobolev inequality on $\mathbb{G}$.

Theorem 3.28 (Fractional Logarithmic Sobolev inequality). *Let $p > 1$, $s \in (0, 1)$, $Q > sp$ be such that $p^* = p^*(Q, s, p) = \frac{Qp}{Q-sp}$. Let $|\cdot|$ be a quasi-norm on $\mathbb{G}$. Then for any $u \in W^{s,p}(\mathbb{G})$ and for any quasi-norm $|\cdot|$, we have the fractional logarithmic Sobolev's inequality*

$$\int_{\mathbb{G}} \frac{|u|^p}{\|u\|^p_{L^p(\mathbb{G})}} \log\left(\frac{|u|^p}{\|u\|^p_{L^p(\mathbb{G})}}\right) dx \leq \frac{Q}{sp} \log\left(C\frac{[u]^p_{s,p}}{\|u\|^p_{L^p(\mathbb{G})}}\right), \tag{3.107}$$

where C is a positive constant independent on u.

Proof. By using weighted logarithmic Hölder's inequality (3.98) with $\alpha = 0$, we obtain

$$\int_{\mathbb{G}} \frac{|u|^p}{\|u\|^p_{L^p(\mathbb{G})}} \log\left(\frac{|u|^p}{\|u\|^p_{L^p(\mathbb{G})}}\right) dx \leq \frac{q}{q-p} \log\left(\frac{\|u\|^p_{L^q(\mathbb{G})}}{\|u\|^p_{L^p(\mathbb{G})}}\right). \tag{3.108}$$

By the assumption we have $1 \leq p < q = p^* = \frac{pQ}{Q-sp}$ and by using Theorem 3.9, we get

$$\begin{aligned}\int_{\mathbb{G}} \frac{|u|^p}{\|u\|^p_{L^p(\mathbb{G})}} \log\left(\frac{|u|^p}{\|u\|^p_{L^p(\mathbb{G})}}\right) dx &\leq \frac{p^*}{p^*-p} \log\left(\frac{\|u\|^p_{L^{p^*}(\mathbb{G})}}{\|u\|^p_{L^p(\mathbb{G})}}\right)\\ &\overset{(3.31)}{\leq} \frac{p^*}{p^*-p} \log\left(C\frac{[u]^p_{s,p}}{\|u\|^p_{L^p(\mathbb{G})}}\right).\end{aligned} \tag{3.109}$$

Here we have

$$\frac{p^*}{p^*-p} = \frac{\frac{pQ}{Q-sp}}{\frac{pQ}{Q-sp} - p} = \frac{\frac{Q}{Q-sp}}{\frac{Q}{Q-sp} - 1} = \frac{Q}{sp}.$$

□

Remark 3.29. *In the Abelian (Euclidean) case $\mathbb{G} = (\mathbb{R}^N, +)$, we have $Q = N$ and $|\cdot| = |\cdot|_E$ ($|\cdot|_E$ is the Euclidean distance), if $s \to 1^-$ and from (3.107) we get the logarithmic Sobolev inequality from* [36].

3.6.2. *Fractional Logarithmic Hardy-Sobolev type inequality.* Motivated by the above result, in this section we prove the fractional logarithmic Hardy-Sobolev inequality on the homogeneous Lie groups.

Theorem 3.30. *Suppose that $p > 1$, $s \in (0, 1)$, $Q > 2$, $0 < \beta < sp$ and $Q > sp$ be such that $p^*_{s,\beta} = \frac{p(Q-\beta)}{Q-sp}$. Then for any $u \in W^{s,p}(\mathbb{G})$ and for any quasi-norm $|\cdot|$ of $\mathbb{G}$,* *the fractional logarithmic Hardy-Sobolev's type inequality:*

$$\int_{\mathbb{G}} \frac{|x|^{-\frac{\beta p}{p^*_{s,\beta}}}|u|^p}{\||x|^{-\frac{\beta}{p^*_{s,\beta}}}u\|^p_{L^p(\mathbb{G})}} \log\left(\frac{|x|^{-\frac{\beta p}{p^*_{s,\beta}}}|u|^p}{\||x|^{-\frac{\beta}{p^*_{s,\beta}}}u\|^p_{L^p(\mathbb{G})}}\right) dx \\ \leq \frac{Q-\beta}{sp-\beta} \log\left(C\frac{[u]^p_{s,p}}{\||x|^{-\frac{\beta}{p^*_{s,\beta}}}u\|^p_{L^p(\mathbb{G})}}\right), \quad (3.110)$$

where C is a positive constant and independent of u.

Proof. In the assumptions of Lemma 3.27, by taking $\alpha = -\frac{\beta p}{p^*_{s,\beta}}$.Then, it is easy to see that $p < p^*_{s,\beta} = q$. Hence by using Lemma 3.27 and Theorem 3.10 with $\alpha = -\frac{\beta p}{p^*_{s,\beta}}$, we get

$$\int_{\mathbb{G}} \frac{(|x|^{-\frac{\beta p}{p^*_{s,\beta}}}|u|^p)}{\||x|^{-\frac{\beta p}{p^*_{s,\beta}}}u\|^p_{L^p(\mathbb{G})}} \log\left(\frac{|x|^{-\frac{\beta p}{p^*_{s,\beta}}}|u|^p}{\||x|^{-\frac{\beta p}{p^*_{s,\beta}}}u\|^p_{L^p(\mathbb{G})}}\right) dx \\ \overset{(3.98)}{\leq} \frac{p^*_{s,\beta}}{p^*_{s,\beta}-p} \log\left(\frac{\||x|^{-\frac{\beta}{p^*_{s,\beta}}}u\|^p_{L^{p^*_{s,\beta}}(\mathbb{G})}}{\||x|^{-\frac{\beta}{p^*_{s,\beta}}}u\|^p_{L^p(\mathbb{G})}}\right) \quad (3.111) \\ \overset{(3.37)}{\leq} \frac{p^*_{s,\beta}}{p^*_{s,\beta}-p} \log\left(C\frac{[u]^p_{s,p}}{\||x|^{-\frac{\beta}{p^*_{s,\beta}}}u\|^p_{L^p(\mathbb{G})}}\right).$$

Finally, we compute

$$\frac{p^*_{s,\beta}}{p^*_{s,\beta}-p} = \frac{\frac{p(Q-\beta)}{Q-sp}}{\frac{p(Q-\beta)}{Q-sp}-p} = \frac{Q-\beta}{sp-\beta},$$

with $sp > \beta > 0$. ☐

Remark 3.31. *In (3.110) with $\beta = 0$, we have the fractional logarithmic Sobolev inequality on $\mathbb{G}$. However, from (3.110) it does not follow the fractional logarithmic Hardy inequality since in Lemma 3.27 we have the assumtion $p < q = p^*_{s,\beta}$. To get the fractional Hardy inequality we have to set $\beta = sp$, then $p = q = p^*_{s,sp}$.*

Remark 3.32. *In the Abelian case $(\mathbb{R}^N, +)$, $Q = N$ with $|\cdot| = |\cdot|_E$ where $|\cdot|_E$ is the standard Euclidean distance, combining (3.110) and (3.37) we obtain the following* *fractional logarithmic Hardy-Sobolev inequality:*

$$\int_{\mathbb{R}^N} \frac{|x|_E^{-\frac{\beta p}{p^*_{s,\beta}}}|u|^p}{\||x|_E^{-\frac{\beta}{p^*_{s,\beta}}}u\|^p_{L^p(\mathbb{R}^N)}} \log\left(\frac{|x|_E^{-\frac{\beta p}{p^*_{s,\beta}}}|u|^p}{\||x|_E^{-\frac{\beta}{p^*_{s,\beta}}}u\|^p_{L^p(\mathbb{R}^N)}}\right) dx \leq \frac{N-\beta}{sp-\beta}\log\left(C\frac{\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^p}{|x-y|_E^{N+sp}}dxdy}{\||x|_E^{-\frac{\beta}{p^*_{s,\beta}}}u\|^p_{L^p(\mathbb{R}^N)}}\right), \tag{3.112}$$

for all $u \in W^{s,p}(\mathbb{R}^N)$.

Remark 3.33. *In the Abelian case* $(\mathbb{R}^N,+)$, $Q = N$ *with* $|\cdot| = |\cdot|_E$ *where* $|\cdot|_E$ *is the standard Euclidean distance and* $s \to 1^-$, *combining (3.110) and (3.37) we have the following fractional logarithmic Hardy-Sobolev inequality:*

$$\int_{\mathbb{R}^N} \frac{|x|_E^{-\frac{\beta p}{p^*_{1,\beta}}}|u|^p}{\||x|_E^{-\frac{\beta}{p^*_{1,\beta}}}u\|^p_{L^p(\mathbb{R}^N)}} \log\left(\frac{|x|_E^{-\frac{\beta p}{p^*_{1,\beta}}}|u|^p}{\||x|_E^{-\frac{\beta}{p^*_{1,\beta}}}u\|^p_{L^p(\mathbb{R}^N)}}\right) dx \leq \frac{N-\beta}{p-\beta}\log\left(C\frac{\|\nabla u\|^p_{L^p(\mathbb{R}^N)}}{\||x|_E^{-\frac{\beta}{p^*_{1,\beta}}}u\|^p_{L^p(\mathbb{R}^N)}}\right), \tag{3.113}$$

and also, setting $\beta = 0$, *we get result from* [36].

3.6.3. *Fractional Logarithmic Gagliardo-Nirenberg inequality.* In this subsection, we show fractional logarithmic Gagliardo-Nirenberg inequality on $\mathbb{G}$.

Theorem 3.34 (Fractional Logarithmic Gagliardo-Nirenberg inequality)**.** *Under the assumptions of Theorem 3.14 with the parameters* $1 \leq p < \infty$, $1 < q < \infty$ *and* $q \leq p^*$, *there exists* $C = C(Q,p,s,q) > 0$ *such that for all measurable and compactly supported* u *we have*

$$\int_{\mathbb{G}} \frac{|u|^q}{\|u\|^q_{L^q(\mathbb{G})}} \log\left(\frac{|u|^q}{\|u\|^q_{L^q(\mathbb{G})}}\right) dx \leq \frac{1}{1-\frac{q}{\tau}}\log\left(C\frac{[u]^q_{s,p}}{\|u\|^q_{L^q(\mathbb{G})}}\right) dx. \tag{3.114}$$

Proof. From the fractional Gagliardo-Nirenberg inequality (3.40) and the logarithmic Hölder inequality (3.98), we get

$$\int_{\mathbb{G}} \frac{|u|^q}{\|u\|^q_{L^q(\mathbb{G})}} \log\left(\frac{|u|^q}{\|u\|^q_{L^q(\mathbb{G})}}\right) dx \leq \frac{1}{1-\frac{q}{\tau}}\log\left(\frac{\|u\|^q_{L^\tau(\mathbb{G})}}{\|u\|^q_q}\right)$$
$$\leq \frac{1}{1-\frac{q}{\tau}}\log\left(C\frac{[u]^{qa}_{s,p}\|u\|^{(1-a)q}_{L^q(\mathbb{G})}}{\|u\|^q_{L^q(\mathbb{G})}}\right) = \frac{a}{1-\frac{q}{\tau}}\log\left(C\frac{[u]^q_{s,p}}{\|u\|^q_{L^q(\mathbb{G})}}\right). \tag{3.115}$$

□

Remark 3.35. *In the Abelian (Euclidean) case* $\mathbb{G}=(\mathbb{R}^N,+)$, *we have* $Q=N$ *and* $|\cdot|=|\cdot|_E$ ($|\cdot|_E$ *is the Euclidean distance), if* $s\to 1^-$ *and from* (3.114) *we get the logarithmic Sobolev inequality in* [36].

3.6.4. *Fractional Logarithmic Caffarelli-Kohn-Nirenberg inequality.* Now we present the fractional logarithmic CKN type inequality on homogeneous groups.

Theorem 3.36 (Fractional Logarithmic CKN inequality)**.** *Under the assumptions of Theorem 3.18 with*

$$\alpha=\beta=\mu,\ \ 1<q<p^*,\ \ 1<p<Q,\ \ \beta p+Q>0,\ \ \beta q+Q>0, \tag{3.116}$$

there exists a positive constant C *such that*

$$\int_{\mathbb{G}}\frac{(|x|^{\alpha}|u|)^q}{\||x|^{\alpha}u\|^q_{L^q(\mathbb{G})}}\log\left(\frac{|x|^{\alpha q}|u|^q}{\||x|^{\alpha}u\|^q_{L^q(\mathbb{G})}}\right)dx\leq\frac{1}{1-\frac{q}{p^*}}\log\left(\frac{[u]_{s,p,\alpha}}{\||x|^{\alpha}u\|^q_{L^q(\mathbb{G})}}\right), \tag{3.117}$$

for all measurable and compactly supported u.

Proof. By taking $\alpha=\beta=\gamma$ in the assumptions of Theorem 3.18, we obtain that

$$\frac{1}{\tau}=\frac{a}{p^*}+\frac{1-a}{q}. \tag{3.118}$$

From the last fact with $q<p^*$ we have $q<\tau$. By combining these facts with weighted logarithmic Hölder's inequality and $\alpha=\beta=\gamma$ we get

$$\begin{aligned}\int_{\mathbb{G}}\frac{|x|^{\alpha q}|u|^q}{\||x|^{\alpha}u\|^q_{L^q(\mathbb{G})}}\log\left(\frac{|x|^{\alpha q}|u|^q}{\||x|^{\alpha}u\|^q_{L^q(\mathbb{G})}}\right)dx &\leq\frac{\tau}{\tau-q}\log\left(\frac{\||x|^{\alpha}u\|^q_{L^\tau(\mathbb{G})}}{\||x|^{\alpha}u\|^q_{L^q(\mathbb{G})}}\right)\\ &\leq\frac{\tau}{\tau-q}\log\left(C^a\frac{[u]^{aq}_{s,p,\alpha}\||x|^{\alpha}u\|^{(1-a)q}_{L^q(\mathbb{G})}}{\||x|^{\alpha}u\|^q_{L^q(\mathbb{G})}}\right)\\ &=\frac{a\tau}{\tau-q}\log\left(C\frac{[u]^q_{s,p,\alpha}}{\||x|^{\alpha}u\|^q_{L^q(\mathbb{G})}}\right).\end{aligned} \tag{3.119}$$

Since $\alpha=\beta=\gamma$, we have

$$\frac{a\tau}{\tau-q}=\frac{p^*}{p^*-q}. \tag{3.120}$$

□

3.7. **Hardy-Littlewood-Sobolev inequality.** In this section, we show Hardy-Littlewood-Sobolev inequality. We prove this inequality by using Marcinkiewicz interpolation theorem.

Let us consider the integral operator

$$I_{\lambda,|\cdot|}u(x)=\int_{\mathbb{G}}\frac{u(y)}{|y^{-1}x|^{\lambda}}dy,\ \ 0<\lambda<Q. \tag{3.121}$$

Note that when $Q>\alpha>0$ and $\lambda=Q-\alpha$ we get the Riesz potential $I_{\lambda,|\cdot|}=I_{Q-\alpha,|\cdot|}$. First we give a short proof of a version of the Hardy-Littlewood-Sobolev inequality on $\mathbb{G}$.

Theorem 3.37. *Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension Q and let $|\cdot|$ be an arbitrary homogeneous quasi-norm on $\mathbb{G}$. Let $1<p<q<\infty,\ 0<\lambda<Q$, $\frac{1}{q}=\frac{1}{p}+\frac{\lambda}{Q}-1$, and $u\in L^p(\mathbb{G})$. Then we have*

$$\|I_{\lambda,|\cdot|}u\|_{L^q(\mathbb{G})}\le C\|u\|_{L^p(\mathbb{G})}, \tag{3.122}$$

where C is a positive constant independent of u.

Proof. As in the Euclidean case we will show that there is a constant $C>0$, such that

$$m\{x:|K*u(x)|>\zeta\}\le C\frac{\|u\|^q_{L^p(\mathbb{G})}}{\zeta^q}, \tag{3.123}$$

where m is the Haar measure on $\mathbb{G}$, $K(x)=|x|^{-\lambda}$ and $I_{\lambda,|\cdot|}u(x)=K*u(x)$, where $*$ is convolution. By using the Marcinkiewicz interpolation theorem we will prove (3.122). Let $K(x)=K_1(x)+K_2(x)$, where

$$K_1(x):=\begin{cases}K(x), & \text{if } |x|\le\mu,\\ 0, & \text{if } |x|>\mu,\end{cases}\quad \text{and}\quad K_2(x):=\begin{cases}K(x), & \text{if } |x|>\mu,\\ 0, & \text{if } |x|\le\mu,\end{cases} \tag{3.124}$$

$\mu>0$. Then, we have $I_{\lambda,|\cdot|}u(x)=K*u(x)=K_1*u(x)+K_2*u(x)$, so

$$m\{x:|K*u(x)|>2\zeta\}\le m\{x:|K_1*u(x)|>\zeta\}+m\{x:|K_2*u(x)|>\zeta\}. \tag{3.125}$$

Therefore, it is enough to prove inequality (3.123) with 2ζ instead of ζ in the left-hand side of the inequality. Without loss of generality we can assume $\|u\|_{L^p(\mathbb{G})}=1$ and by using Chebychev's and Minkowski's inequalities, we get

$$\begin{aligned}m\{x:|K_1*u(x)|>\zeta\}&\le\frac{\int_{|K_1*u|>\zeta}|K_1*u|^p dx}{\zeta^p}\\ &\le\frac{\|K_1*u\|^p_{L^p(\mathbb{G})}}{\zeta^p}\le\frac{\|K_1\|^p_{L^1(\mathbb{G})}\|u\|^p_{L^p(\mathbb{G})}}{\zeta^p}=\frac{\|K_1\|^p_{L^1(\mathbb{G})}}{\zeta^p}.\end{aligned} \tag{3.126}$$

By combining (2.11) and (3.124), we have

$$\begin{aligned}\|K_1\|_{L^1(\mathbb{G})}&=\int_{0<|x|\le\mu}|x|^{-\lambda}dx=\int_0^\mu r^{Q-1}r^{-\lambda}dr\int_{\mathfrak{S}}|y|^{-\lambda}d\sigma(y)\\ &=|\mathfrak{S}|\int_0^\mu r^{Q-\lambda-1}dr=\frac{|\mathfrak{S}|}{Q-\lambda}(r^{Q-\lambda}|_0^\mu)=\frac{|\mathfrak{S}|}{Q-\lambda}\mu^{Q-\lambda},\end{aligned} \tag{3.127}$$

where $|\mathfrak{S}|$ is the dimensional surface measure of the unit quasi-sphere $\mathfrak{S}$. By using last fact in (3.126), we get

$$m\{x:|K_1*u(x)|>\zeta\}\le\left(\frac{|\mathfrak{S}|}{Q-\lambda}\right)^p\frac{\mu^{(Q-\lambda)p}}{\zeta^p}. \tag{3.128}$$

Then, similarly from Young's inequality, (2.11) and the assumptions, we obtain

$$\begin{aligned}\|K_2 * u\|_{L^\infty(\mathbb{G})} \leq \|K_2\|_{L^{p'}(\mathbb{G})} \|u\|_{L^p(\mathbb{G})} &= \left(\int_\mu^\infty r^{-\lambda p'} r^{Q-1} dr \int_{\mathfrak{S}} |y|^{-\lambda p'} d\sigma(y) \right)^{\frac{1}{p'}} \\ = \left(\frac{|\mathfrak{S}|}{Q-\lambda p'} \right)^{\frac{1}{p'}} \left(\int_\mu^\infty r^{Q-\lambda p'-1} dr \right)^{\frac{1}{p'}} &= \left(\frac{|\mathfrak{S}|}{Q-\lambda p'} \right)^{\frac{1}{p'}} (r^{Q-\lambda p'}|_\mu^\infty)^{\frac{1}{p'}} \\ &= \left(\frac{|\mathfrak{S}|}{\lambda p'-Q} \right)^{\frac{1}{p'}} \mu^{-\frac{Q}{q}}, \quad (3.129)\end{aligned}$$

since from the assumptions, we get $\frac{Q-\lambda p'}{p'} = \frac{Q}{p'} - \lambda = Q(1-\frac{1}{p}-\frac{\lambda}{Q}) = -\frac{Q}{q}$. Moreover, if $\left(\frac{|\mathfrak{S}|}{\lambda p'-Q}\right)^{\frac{1}{p'}} \mu^{-\frac{Q}{q}} = \zeta$, then $\mu = \left(\frac{|\mathfrak{S}|}{\lambda p'-Q}\right)^{-\frac{q}{Qp'}} \zeta^{-\frac{\theta}{Q}}$, so we have $\|K_2 * u\|_{L^\infty(\mathbb{G})} \leq \zeta$. Hence, we have $m\{x : |K_2 * u| > \zeta\} = 0$. From these facts with (3.125), $\|u\|_{L^p(\mathbb{G})} = 1$ and the assumptions we get

$$\begin{aligned}m\{x : |K * u| > 2\zeta\} &\leq \left(\frac{|\mathfrak{S}|}{Q-\lambda} \right)^p \frac{\mu^{(Q-\lambda)p}}{\zeta^p} \\ = \left(\frac{|\mathfrak{S}|}{Q-\lambda} \right)^p \left(\frac{|\mathfrak{S}|}{\lambda p'-Q} \right)^{-\frac{q(Q-\lambda)p}{Qp'}} &\zeta^{\frac{-(Q-\lambda)pq}{Q}-p} \leq C\zeta^{\frac{-(Q-\lambda)pq}{Q}-p} = C\zeta^{(\frac{\lambda}{Q}-1)pq-p} \\ &= C\zeta^{(\frac{1}{q}-\frac{1}{p})pq-p} = C\zeta^{p-q-p} = C\frac{\|u\|^q_{L^p(\mathbb{G})}}{\zeta^q}. \quad (3.130)\end{aligned}$$

For completeness, let us recall two well-known ingredients.

Definition 3.38 ([66])**.** Let $1 \leq p \leq \infty$, $1 \leq q < \infty$ and $V : L^p(\mathbb{G}) \to L^q(\mathbb{G})$ be a operator, then V is called an operator of *weak type* (p, q) if

$$m\{x : |Vu| > \zeta\} \leq C \left(\frac{\|u\|_{L^p(\mathbb{G})}}{\zeta} \right)^q, \quad \zeta > 0, \tag{3.131}$$

where C is a positive constant and independent by u.

Let us also recall the classical Marcinkiewicz interpolation theorem:

Theorem 3.39. *Let V be sublinear operator of weak type (p_k, q_k) with $1 \leq p_k \leq q_k < \infty$, $k = 0, 1$ and $q_0 < q_1$. Then V is bounded from $L^p(\mathbb{G})$ to $L^q(\mathbb{G})$ with*

$$\frac{1}{p} = \frac{1-\gamma}{p_0} + \frac{\gamma}{p_1}, \quad \frac{1}{q} = \frac{1-\gamma}{q_0} + \frac{\gamma}{q_1}, \tag{3.132}$$

for any $0 < \gamma < 1$, namely,

$$\|Vu\|_{L^q(\mathbb{G})} \leq C\|u\|_{L^p(\mathbb{G})}, \tag{3.133}$$

for any $u \in L^p(\mathbb{G})$ and C is a positive constant.

By using assumptions $\frac{1}{q} = \frac{1}{p} + \frac{\lambda}{Q} - 1 < \frac{1}{p}$, we have $q > p$. According to Definition 3.38, $I_{\lambda,|\cdot|}u$ is of weak type (p, q), so by using the Marcinkiewicz interpolation theorem, we prove (3.122).

The proof of Theorem 3.37 is complete. $\square$

Remark 3.40. *Under assumption of the Theorem 3.37 and $h \in L^{q'}(\mathbb{G})$, we have the following Hardy-Littlewood-Sobolev inequality*

$$\left| \int_{\mathbb{G}} \int_{\mathbb{G}} \frac{u(y)h(x)}{|y^{-1}x|^{\lambda}} dxdy \right| \leq C \|u\|_{L^p(\mathbb{G})} \|h\|_{L^{q'}(\mathbb{G})}, \tag{3.134}$$

where C is a positive constant independent of u and h.

3.8. **Stein-Weiss inequality.** In this section, we show the Stein-Weiss inequality on homogeneous Lie group. For showing this inequality we need some preliminary results as the integral version of Hardy inequalities on general homogeneous groups and Proposition 2.6 which is play key roles in our proof. Firstly, let us show the integral version of Hardy inequalities on general homogeneous groups.

Theorem 3.41 ([32])**.** *Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension Q and let $1 < p \leq q < \infty$. Let $W(x)$ and $U(x)$, be positive functions on $\mathbb{G}$. Then we have the following properties:*

(1) The inequality

$$\left(\int_{\mathbb{G}} \left(\int_{B(0,|x|)} f(z)dz \right)^{q} W(x)dx \right)^{\frac{1}{q}} \leq C_1 \left(\int_{\mathbb{G}} f^p(x)U(x)dx \right)^{\frac{1}{p}} \tag{3.135}$$

holds for all $f \geq 0$ a.e. on $\mathbb{G}$ if and only if

$$A_1 := \sup_{R>0} \left(\int_{\mathbb{G}\setminus B(0,|x|)} W(x)dx \right)^{\frac{1}{q}} \left(\int_{B(0,|x|)} U^{1-p'}(x)dx \right)^{\frac{1}{p'}} < \infty. \tag{3.136}$$

(2) The inequality

$$\left(\int_{\mathbb{G}} \left(\int_{\mathbb{G}\setminus B(0,|x|)} f(z)dz \right)^{q} W(x)dx \right)^{\frac{1}{q}} \leq C_2 \left(\int_{\mathbb{G}} f^p(x)U(x)dx \right)^{\frac{1}{p}}, \tag{3.137}$$

holds for all $f \geq 0$ if and only if

$$A_2 := \sup_{R>0} \left(\int_{B(0,|x|)} W(x)dx \right)^{\frac{1}{q}} \left(\int_{\mathbb{G}\setminus B(0,|x|)} U^{1-p'}(x)dx \right)^{\frac{1}{p'}} < \infty. \tag{3.138}$$

(3) If $\{C_i\}_{i=1}^2$ are the smallest constants for which (3.135) and (3.137) hold, then

$$A_i \leq C_i \leq (p')^{\frac{1}{p'}} p^{\frac{1}{q}} A_i, \quad i = 1, 2. \tag{3.139}$$

Now we formulate the Stein-Weiss inequality on $\mathbb{G}$.

Theorem 3.42. *Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension Q and let $|\cdot|$ be an arbitrary homogeneous quasi-norm on $\mathbb{G}$. Let $0 < \lambda < Q$, $1 < p < \infty$, $\alpha < \frac{Q}{p'}$, $\beta < \frac{Q}{q}$, $\alpha + \beta \geq 0$, $\frac{1}{q} = \frac{1}{p} + \frac{\alpha+\beta+\lambda}{Q} - 1$, where $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{q} + \frac{1}{q'} = 1$. Then for $1 < p \leq q < \infty$, we have*

$$\||x|^{-\beta} I_{\lambda,|\cdot|} u\|_{L^q(\mathbb{G})} \leq C \||x|^{\alpha} u\|_{L^p(\mathbb{G})}. \tag{3.140}$$

where C is positive constant and independent by u.

Proof. Let us define

$$\||x|^{-\beta}I_{\lambda,|\cdot|}u\|^q_{L^q(\mathbb{G})}=\int_{\mathbb{G}}\left(\int_{\mathbb{G}}\frac{u(y)}{|x|^{\beta}|y^{-1}x|^{\lambda}}dy\right)^q dx=I_1+I_2+I_3, \tag{3.141}$$

where

$$I_1=\int_{\mathbb{G}}\left(\int_{B\left(0,\frac{|x|}{2}\right)}\frac{u(y)}{|x|^{\beta}|y^{-1}x|^{\lambda}}dy\right)^q dx, \tag{3.142}$$

$$I_2=\int_{\mathbb{G}}\left(\int_{B(0,2|x|)\setminus B\left(0,\frac{|x|}{2}\right)}\frac{u(y)}{|x|^{\beta}|y^{-1}x|^{\lambda}}dy\right)^q dx, \tag{3.143}$$

and

$$I_3=\int_{\mathbb{G}}\left(\int_{\mathbb{G}\setminus B(0,2|x|)}\frac{u(y)}{|x|^{\beta}|y^{-1}x|^{\lambda}dy}\right)^q dx. \tag{3.144}$$

From Proposition 2.6 we can suppose that our quasi-norm is actually a norm.

Step 1. Firstly, let us consider I_1. From Proposition 2.6 and the definition of the quasi-norm with $|y|\leq\frac{|x|}{2}$, we obtain

$$\begin{aligned}|x|=|x^{-1}|&=|x^{-1}yy^{-1}|\\ &\leq|x^{-1}y|+|y^{-1}|=|y^{-1}x|+|y|\\ &\leq|y^{-1}x|+\frac{|x|}{2}.\end{aligned}$$

For any $\lambda>0$, we get

$$2^{\lambda}|x|^{-\lambda}\geq|y^{-1}x|^{-\lambda}.$$

Therefore, we get

$$\begin{aligned}I_1=\int_{\mathbb{G}}\left(\int_{B\left(0,\frac{|x|}{2}\right)}\frac{u(y)}{|x|^{\beta}|y^{-1}x|^{\lambda}}dy\right)^q dx&\leq 2^{\lambda}\int_{\mathbb{G}}\left(\int_{B\left(0,\frac{|x|}{2}\right)}\frac{u(y)}{|x|^{\beta+\lambda}}dy\right)^q dx\\ &=2^{\lambda}\int_{\mathbb{G}}\left(\int_{B\left(0,\frac{|x|}{2}\right)}u(y)dy\right)^q|x|^{-(\beta+\lambda)q}dx.\end{aligned} \tag{3.145}$$

Assume that $W(x)=|x|^{-(\beta+\lambda)q}$ and $U(y)=|y|^{\alpha p}$ and if condition (3.136) in Theorem 3.41 is satisfied, then by (3.135) we have

$$I_1\leq 2^{\lambda}\int_{\mathbb{G}}\left(\int_{B\left(0,\frac{|x|}{2}\right)}u(y)dy\right)^q|x|^{-(\beta+\lambda)q}dx\leq C_1\||x|^{\alpha}u\|^q_{L^p(\mathbb{G})}. \tag{3.146}$$

Let us check condition (3.136) with $W(x)=|x|^{-(\beta+\lambda)q}$ and $U(y)=|y|^{\alpha p}$. By the assumption we have $\alpha<\frac{Q}{p'}$, then

$$\frac{1}{q}=\frac{1}{p}+\frac{\alpha+\beta+\lambda}{Q}-1<\frac{1}{p}+\frac{\frac{Q}{p'}+\beta+\lambda}{Q}-1=\frac{1}{p}+\frac{1}{p'}+\frac{\beta+\lambda}{Q}-1=\frac{\beta+\lambda}{Q},$$

that is, $Q-(\beta+\lambda)q<0$ and by the using polar decomposition (2.11):

$$\left(\int_{\mathbb{G}\backslash B(0,|x|)} W(x)dx\right)^{\frac{1}{q}} = \left(\int_{\mathbb{G}\backslash B(0,|x|)} |x|^{-(\beta+\lambda)q}dx\right)^{\frac{1}{q}}$$

$$= \left(\int_R^\infty \int_{\mathfrak{S}} r^{Q-1}r^{-(\beta+\lambda)q}drd\sigma(y)\right)^{\frac{1}{q}} = \left(|\mathfrak{S}|\int_R^\infty r^{Q-1-(\beta+\lambda)q}dr\right)^{\frac{1}{q}} \leq CR^{\frac{Q-(\beta+\lambda)q}{q}}. \tag{3.147}$$

From $\alpha<\frac{Q}{p'}$, we get

$$\alpha p(1-p')+Q>\alpha p(1-p')+\alpha p'=\alpha p+\alpha p'(1-p)=\alpha p-\alpha p=0.$$

Finally, $\alpha p(1-p')+Q>0$. Then, let us consider

$$\left(\int_{B(0,|x|)} U^{1-p'}(x)dx\right)^{\frac{1}{p'}} = \left(\int_{B(0,|x|)} |x|^{(1-p')\alpha p}dx\right)^{\frac{1}{p'}}$$

$$= \left(\int_0^R \int_{\mathfrak{S}} r^{(1-p')\alpha p}r^{Q-1}drd\sigma(y)\right)^{\frac{1}{p'}} \leq C\left(|\mathfrak{S}|\int_0^R r^{(1-p')\alpha p+Q-1}dr\right)^{\frac{1}{p'}}$$

$$\leq CR^{\frac{(1-p')\alpha p+Q}{p'}} = CR^{\frac{Q-\alpha p'}{p'}}. \tag{3.148}$$

Moreover, the assumptions imply

$$A_1=\sup_{R>0}\left(\int_{\mathbb{G}\backslash B(0,|x|)} W(x)dx\right)^{\frac{1}{q}}\left(\int_{B(0,|x|)} U^{1-p'}(x)dx\right)^{\frac{1}{p'}} \leq CR^{\frac{Q-(\beta+\lambda)q}{q}+\frac{Q-\alpha p'}{p'}}$$

$$= CR^{Q(\frac{1}{q}-\frac{1}{p}-\frac{\alpha+\beta+\lambda}{Q}+1)} = C<\infty,$$

where $C=C(\alpha,\beta,p,\lambda)$ is a positive constant. From (3.135), we get

$$I_1 \leq C\int_{\mathbb{G}}\left(\int_{B\left(0,\frac{|x|}{2}\right)} u(y)dy\right)^q |x|^{-(\beta+\lambda)q}dx \leq C_1\||x|^\alpha u\|^q_{L^p(\mathbb{G})}. \tag{3.149}$$

Step 2. Similarly with the previous case I_1, now we consider I_3. From $2|x|\leq|y|$, we have

$$|y|=|y^{-1}|=|y^{-1}xx^{-1}|\leq|y^{-1}x|+|x|$$
$$\leq|y^{-1}x|+\frac{|y|}{2},$$

that is,

$$\frac{|y|}{2}\leq|y^{-1}x|.$$

Then, if condition (3.138) with $W(x)=|x|^{-\beta q}$ and $U(y)=|y|^{(\alpha+\lambda)p}$ is satisfied, then we have

$$I_3=\int_{\mathbb{G}}\left(\int_{\mathbb{G}\backslash B(0,2|x|)}\frac{u(y)}{|x|^\beta|y^{-1}x|^\lambda}dy\right)^q dx \leq C\int_{\mathbb{G}}\left(\int_{\mathbb{G}\backslash B(0,2|x|)}\frac{u(y)}{|x|^\beta|y|^\lambda}dy\right)^q dx$$

$$= C\int_{\mathbb{G}}\left(\int_{\mathbb{G}\backslash B(0,2|x|)} u(y)|y|^{-\lambda}dy\right)^q |x|^{-\beta q}dx \leq C\||x|^\alpha u\|^q_{L^p(\mathbb{G})}. \tag{3.150}$$

Now let us verify condition (3.138). Then, we get

$$\left(\int_{B(0,|x|)} W(x)dx\right)^{\frac{1}{q}} = \left(\int_{B(0,|x|)} |x|^{-\beta q}dx\right)^{\frac{1}{q}}$$

$$= \left(\int_0^R \int_{\mathfrak{S}} r^{-\beta q} r^{Q-1} dr d\sigma(y)\right)^{\frac{1}{q}} \leq CR^{\frac{Q-\beta q}{q}}, \quad (3.151)$$

where $Q - \beta q > 0$, and

$$\left(\int_{\mathbb{G}\backslash B(0,|x|)} U^{1-p'}(x)dx\right)^{\frac{1}{p'}} = \left(\int_{\mathbb{G}\backslash B(0,|x|)} |x|^{(\alpha+\lambda)(1-p')p}dx\right)^{\frac{1}{p'}}$$

$$= \left(\int_R^\infty \int_{\mathfrak{S}} r^{Q-1} r^{(\alpha+\lambda)(1-p')p} dr d\sigma(y)\right)^{\frac{1}{p'}} \leq CR^{\frac{Q-p'(\alpha+\lambda)}{p'}}, \quad (3.152)$$

where from $\beta < \frac{Q}{q}$, we get $Q - p'(\alpha + \lambda) < 0$.

By using these facts we have

$$A_2 := \sup_{R>0} \left(\int_{B(0,|x|)} W(x)dx\right)^{\frac{1}{q}} \left(\int_{\mathbb{G}\backslash B(0,|x|)} U^{1-p'}(x)dx\right)^{\frac{1}{p'}} \leq CR^{\frac{Q-p'(\alpha+\lambda)}{p'}+\frac{Q-\beta q}{q}}$$

$$= CR^{\frac{Q}{p'}-(\alpha+\beta+\lambda)+\frac{Q}{q}} = CR^{Q(\frac{1}{p'}-\frac{\alpha+\beta+\lambda}{Q}+\frac{1}{q})} = C < \infty, \quad (3.153)$$

where $C = C(\alpha, \beta, p, \lambda)$ is a positive constant. Then we establish

$$I_3 = \int_{\mathbb{G}} \left(\int_{\mathbb{G}\backslash B(0,2|x|)} \frac{u(y)}{|x|^\beta |y^{-1}x|^\lambda} dy\right)^q dx \leq C\||x|^\alpha u\|^q_{L^p(\mathbb{G})}. \quad (3.154)$$

Step 3. Let us estimate I_2 now.

Case 1: $p < q$. By $\frac{|x|}{2} < |y| < 2|x|$, we get

$$\frac{|y^{-1}x|}{2} \leq \frac{|x| + |y|}{2} = \frac{|x|}{2} + \frac{|y|}{2} < \frac{3}{2}|y|,$$

that is,

$$|y^{-1}x| < 3|y|.$$

For all $\alpha + \beta \geq 0$, we have

$$|y^{-1}x|^{\alpha+\beta} < 3^{\alpha+\beta}|y|^{\alpha+\beta} = 3^{\alpha+\beta}|y|^\alpha |y|^\beta \leq 3^{\alpha+\beta} 2^{|\beta|} |x|^\beta |y|^\alpha.$$

Hence,

$$I_2 = \int_{\mathbb{G}} \left(\int_{B(0,2|x|)\backslash B\left(0,\frac{|x|}{2}\right)} \frac{u(y)}{|x|^\beta |y^{-1}x|^\lambda} dy\right)^q dx$$

$$\leq C \int_{\mathbb{G}} \left(\int_{B(0,2|x|)\backslash B\left(0,\frac{|x|}{2}\right)} \frac{|y|^\alpha u(y)}{|y^{-1}x|^{\alpha+\beta+\lambda}} dy\right)^q dx$$

$$\leq C \int_{\mathbb{G}} \left(\int_{\mathbb{G}} \frac{|y|^\alpha u(y)}{|y^{-1}x|^{\alpha+\beta+\lambda}} dy\right)^q dx = C\|I_{\lambda+\alpha+\beta,|\cdot|}\tilde{u}\|^q_{L^q(\mathbb{G})},$$

where $\tilde{u}(x) = |x|^\alpha u(x)$.

From the assumption $\frac{1}{q}-\frac{1}{p}=\frac{\lambda+\alpha+\beta}{Q}-1<0$, we get $Q>\lambda+\alpha+\beta$ and by using Theorem 3.37 with $p<q$, we obtain

$$I_2 \leq C\|I_{\lambda+\alpha+\beta,|\cdot|}\tilde{u}\|^q_{L^q(\mathbb{G})} \leq C\|\tilde{u}\|^q_{L^p(\mathbb{G})} = C\||x|^\alpha u\|^q_{L^p(\mathbb{G})}. \tag{3.155}$$

Case 2: $p=q$. Let us decompose I_2 as

$$I_2=\sum_{k\in\mathbb{Z}}\int_{2^k\leq|x|\leq 2^{k+1}}\left(\int_{B(0,2|x|)\setminus B\left(0,\frac{|x|}{2}\right)}\frac{u(y)}{|x|^\beta|y^{-1}x|^\lambda}dy\right)^p dx. \tag{3.156}$$

By $|x|\leq 2|y|\leq 4|x|$ and $2^k\leq|x|\leq 2^{k+1}$, we get $2^{k-1}\leq|y|\leq 2^{k+2}$ and $0\leq |y^{-1}x|\leq 3|x|\leq 3\cdot 2^{k+1}$.

By combining Young's inequality with $\frac{1}{p}+\frac{1}{r}=1+\frac{1}{q}$ (our case $p=q$, hence $r=1$), we get

$$\begin{aligned}
I_2 &= \sum_{k\in\mathbb{Z}}\int_{2^k\leq|x|\leq 2^{k+1}}\left(\int_{B(0,2|x|)\setminus B\left(0,\frac{|x|}{2}\right)}\frac{u(y)}{|x|^\beta|y^{-1}x|^\lambda}dy\right)^p dx\\
&= \sum_{k\in\mathbb{Z}}\int_{2^k\leq|x|\leq 2^{k+1}}\left(\int_{B(0,2|x|)\setminus B\left(0,\frac{|x|}{2}\right)}\frac{u(y)}{|y^{-1}x|^\lambda}dy\right)^p \frac{dx}{|x|^{\beta p}}\\
&\leq \sum_{k\in\mathbb{Z}}2^{-\beta pk}\|u\cdot\chi_{\{2^{k-1}\leq|y|\leq 2^{k+2}\}} * |x|^{-\lambda}\|^p_{L^p(\mathbb{G})}\\
&\leq \sum_{k\in\mathbb{Z}}2^{-\beta pk}\||x|^{-\lambda}\cdot\chi_{\{0\leq|y|\leq 3\cdot 2^{k+1}\}}\|^p_{L^1(\mathbb{G})}\|u\cdot\chi_{\{2^{k-1}\leq|y|\leq 2^{k+2}\}}\|^p_{L^p(\mathbb{G})}\\
&\leq C\sum_{k\in\mathbb{Z}}2^{(Q-\lambda-\beta)kp}\|u\cdot\chi_{\{2^{k-1}\leq|y|\leq 2^{k+2}\}}\|^p_{L^p(\mathbb{G})} = C\sum_{k\in\mathbb{Z}}2^{\alpha kp}\|u\cdot\chi_{\{2^{k-1}\leq|y|\leq 2^{k+2}\}}\|^p_{L^p(\mathbb{G})}\\
&= C\sum_{k\in\mathbb{Z}}\|2^{\alpha(k-1)}u\cdot\chi_{\{2^{k-1}\leq|y|\leq 2^{k+2}\}}\|^p_{L^p(\mathbb{G})} \leq C\sum_{k\in\mathbb{Z}}\||y|^\alpha u\cdot\chi_{\{2^{k-1}\leq|y|\leq 2^{k+2}\}}\|^p_{L^p(\mathbb{G})}\\
&= C\||x|^\alpha u\|^p_{L^p(\mathbb{G})}.
\end{aligned}$$

Theorem 3.42 is proved. ☐

Remark 3.43. *With assumptions Theorem 3.42 and $h\in L^{q'}(\mathbb{G})$, we have the following Stein-Weiss inequality*

$$\left|\int_{\mathbb{G}}\frac{u(y)h(x)}{|x|^\beta|y^{-1}x|^\lambda|y|^\alpha}dxdy\right|\leq C\|u\|_{L^p(\mathbb{G})}\|h\|_{L^{q'}(\mathbb{G})}, \tag{3.157}$$

where C is a positive constant independent of u and h.

Remark 3.44. *In inequality (3.140) with $\alpha=0$ we get the weighted Hardy-Littlewood-Sobolev inequality established in* [32, Theorem 4.1]. *Thus, by setting $\alpha=\beta=0$ we get Hardy-Littlewood-Sobolev inequality on the homogeneous Lie groups. In the Abelian (Euclidean) case $\mathbb{G}=(\mathbb{R}^N,+)$, we have $Q=N$ and $|\cdot|$ can be any homogeneous quasi-norm on $\mathbb{R}^N$, so with the usual Euclidean distance, i.e. $|\cdot|=\|\cdot\|_E$, Theorem 3.42 gives the classical result of Stein and Weiss.*

3.9. **Logarithmic Sobolev-Folland-Stein inequality.** In this section, we present the logarithmic Sobolev-Folland-Stein inequality on stratified groups. Let us recall the well-known Sobolev-Folland-Stein inequality.

Theorem 3.45. *Let $\mathbb{G}$ be a stratified Lie group and $\Omega \subset \mathbb{G}$ be an open set. Then there exists a constant $C_S = C_S(\mathbb{G}) > 0$ such that for all $u \in C_0^\infty(\Omega)$ we have*

$$\|u\|_{L^{p^*}(\Omega)} \leq C_S \left(\int_\Omega |\nabla_H u|^p dx \right)^{\frac{1}{p}}, \; 1 < p < Q, \tag{3.158}$$

where $p^* = \frac{Qp}{Q-p}$. Here ∇_H is the horizontal gradient and Q is the homogeneous dimension of $\mathbb{G}$.

Now let us state the logarithmic Sobolev-Folland-Stein inequality on stratified groups.

Theorem 3.46. *Suppose that $p^* = \frac{2Q}{Q-2}$ and $a > 0$. Then*

$$2\int_{\mathbb{G}} |u(x)|^2 \ln \frac{|u(x)|}{\|u\|_{L^2(\mathbb{G})}} dx + Q(1+\ln a)\|u\|^2_{L^2(\mathbb{G})} \leq QC_S^2 a \|\nabla_{\mathbb{G}} u\|^2_{L^2(\mathbb{G})}, \tag{3.159}$$

where $u \in S_0^{1,2}(\mathbb{G})$.

Proof. By a direct calculation with $\varepsilon > 0$, we have

$$\begin{aligned} 2\int_{\mathbb{G}} |u(x)|^2 \ln \frac{|u(x)|}{\|u\|_{L^2(\mathbb{G})}} dx &= \frac{1}{\varepsilon} \int_{\mathbb{G}} |u(x)|^2 \ln \left(\frac{|u(x)|}{\|u\|_{L^2(\mathbb{G})}} \right)^\varepsilon dx \\ &= \frac{\|u\|^2_{L^2(\mathbb{G})}}{\varepsilon} \int_{\mathbb{G}} \frac{|u(x)|^2}{\|u\|^2_{L^2(\mathbb{G})}} \ln \left(\frac{|u(x)|}{\|u\|_{L^2(\mathbb{G})}} \right)^\varepsilon dx. \end{aligned} \tag{3.160}$$

From Jensen's inequality we obtain the upper estimate for the integral:

$$\begin{aligned} 2\int_{\mathbb{G}} |u(x)|^2 \ln \frac{|u(x)|}{\|u\|_{L^2(\mathbb{G})}} dx &= \frac{2\|u\|^2_{L^2(\mathbb{G})}}{\varepsilon} \int_{\mathbb{G}} \frac{|u(x)|^2}{\|u\|^2_{L^2(\mathbb{G})}} \ln \left(\frac{|u(x)|}{\|u\|_{L^2(\mathbb{G})}} \right)^\varepsilon dx \\ &\leq \frac{2\|u\|^2_{L^2(\mathbb{G})}}{\varepsilon} \ln \left(\int_{\mathbb{G}} \frac{|u(x)|}{\|u\|_{L^2(\mathbb{G})}} dx \right)^{\varepsilon+1} \\ &= \frac{2(\varepsilon+1)\|u\|^2_{L^2(\mathbb{G})}}{\varepsilon} \ln \frac{\|u\|^2_{L^{2\varepsilon+2}(\mathbb{G})}}{\|u\|^2_{L^2(\mathbb{G})}}. \end{aligned} \tag{3.161}$$

From the inequality $\ln x \leq ax - \ln(a) - 1$ for all $a, x > 0$, and by choosing $2\varepsilon + 2 = \frac{2Q}{Q-2}$ as well as using the Sobolev-Folland-Stein inequality, we get

$$\begin{aligned}
2\int_{\mathbb{G}} |u(x)|^2 \ln \frac{|u(x)|}{\|u\|_{L^2(\mathbb{G})}} dx &\leq \frac{2(\varepsilon+1)\|u\|^2_{L^2(\mathbb{G})}}{\varepsilon} \ln \frac{\|u\|^2_{L^{2\varepsilon+2}(\mathbb{G})}}{\|u\|^2_{L^2(\mathbb{G})}} \\
&\leq \frac{2(\varepsilon+1)\|u\|^2_{L^2(\mathbb{G})}}{\varepsilon} \left(a \frac{\|u\|^2_{L^{2\varepsilon+2}(\mathbb{G})}}{\|u\|^2_{L^2(\mathbb{G})}} - (\ln(a)+1) \right) \\
&= \frac{2(\varepsilon+1)}{\varepsilon} \left(a\|u\|^2_{L^{2\varepsilon+2}(\mathbb{G})} - (\ln(a)+1)\|u\|^2_{L^2(\mathbb{G})} \right) \\
&\leq \frac{2(\varepsilon+1)}{\varepsilon} \left(aC_S^2 \|\nabla_{\mathbb{G}} u\|^2_{L^2(\mathbb{G})} - (\ln(a)+1)\|u\|^2_{L^2(\mathbb{G})} \right) \\
&= \frac{\frac{2Q}{Q-2}}{\frac{2}{Q-2}} \left(aC_S^2 \|\nabla_{\mathbb{G}} u\|^2_{L^2(\mathbb{G})} - (\ln(a)+1)\|u\|^2_{L^2(\mathbb{G})} \right) \\
&= Q \left(aC_S^2 \|\nabla_{\mathbb{G}} u\|^2_{L^2(\mathbb{G})} - (\ln(a)+1)\|u\|^2_{L^2(\mathbb{G})} \right).
\end{aligned} \tag{3.162}$$

It yields that

$$2\int_{\mathbb{G}} |u(x)|^2 \ln \frac{|u(x)|}{\|u\|_{L^2(\mathbb{G})}} dx + Q(\ln(a)+1)\|u\|^2_{L^2(\mathbb{G})} \leq C_S^2 Q a \|\nabla_{\mathbb{G}} u\|^2_{L^2(\mathbb{G})}. \tag{3.163}$$

□

4. REVERSE INEQUALITIES

In this chapter, we show reverse integral Hardy inequality on metric measure space. We show the reverse integral Hardy inequality in two cases. In the first case we consider the case $q < 0$ and $p \in (0,1)$. In the second case, we consider the case $-\infty < q \leq p < 0$. For the both cases we also obtain conjugate reverse integral Hardy inequality. In the first case, as consequences we show the reverse integral Hardy inequality for the homogeneous Lie groups, hyperbolic space and Cartan-Hadamard manifolds. Also, we show reverse Hardy-Littlewood-Sobolev and Stein-Weiss inequalities for the both cases. In addition, we obtain Hardy, L^p-Sobolev and L^p-Caffarelli-Kohn-Nirenberg inequalities on homogeneous groups with radial derivative.

Firstly, we need to give some preliminary results of this chapter. Let us recall briefly the reverse Hölder's inequality.

Theorem 4.1 ([35], Theorem 2.12, p. 27)**.** *Let $\mathbb{X}$ be metric measure space. Let $p \in (0,1)$, so that $p' = \frac{p}{p-1} < 0$. If non-negative functions satisfy $f \in L^p(\mathbb{X})$ and $0 < \int_{\mathbb{X}} g^{p'}(x)dx < +\infty$, we have*

$$\int_{\mathbb{X}} f(x)g(x)dx \geq \left(\int_{\mathbb{X}} f^p(x)dx\right)^{\frac{1}{p}} \left(\int_{\mathbb{X}} g^{p'}(x)dx\right)^{\frac{1}{p'}}. \tag{4.1}$$

Let us give the reverse integral Minkowski inequality (or a continuous version of reverse Minkowski inequality) with $q < 0$ on metric measure space.

Theorem 4.2. *Let $\mathbb{X}, \mathbb{Y}$ be metric measure spaces and let $F = F(x,y) \in \mathbb{X} \times \mathbb{Y}$ be a non-negative measurable function. Then we have*

$$\left[\int_{\mathbb{X}} \left(\int_{\mathbb{Y}} F(x,y)dy\right)^q dx\right]^{\frac{1}{q}} \geq \int_{\mathbb{Y}} \left(\int_{\mathbb{X}} F^q(x,y)dx\right)^{\frac{1}{q}} dy, \; q < 0. \tag{4.2}$$

Proof. Let us consider the following function:

$$A(x) := \int_{\mathbb{Y}} F(x,y)dy, \tag{4.3}$$

so we have

$$A^q(x) = \left(\int_{\mathbb{Y}} F(x,y)dy\right)^q. \tag{4.4}$$

By integrating over $\mathbb{X}$ both sides and by using reverse Hölder's inequality (Theorem 4.1), we obtain

$$\begin{aligned}\int_{\mathbb{X}} A^q(x)dx &= \int_{\mathbb{X}} A^{q-1}(x)A(x)dx \\ &= \int_{\mathbb{X}} A^{q-1}(x)\int_{\mathbb{Y}} F(x,y)dydx \\ &= \int_{\mathbb{Y}}\int_{\mathbb{X}} A^{q-1}(x)F(x,y)dxdy \\ &\overset{(4.1)}{\geq} \int_{\mathbb{Y}}\left(\int_{\mathbb{X}} A^{q-1\frac{q}{q-1}}(x)dx\right)^{\frac{q-1}{q}}\left(\int_{\mathbb{X}} F^q(x,y)dx\right)^{\frac{1}{q}}dy \\ &= \left(\int_{\mathbb{X}} A^q(x)dx\right)^{\frac{q-1}{q}}\int_{\mathbb{Y}}\left(\int_{\mathbb{X}} F^q(x,y)dx\right)^{\frac{1}{q}}dy.\end{aligned} \tag{4.5}$$

From this, we get

$$\left[\int_{\mathbb{X}}\left(\int_{\mathbb{Y}} F(x,y)dy\right)^q dx\right]^{\frac{1}{q}} \geq \int_{\mathbb{Y}}\left(\int_{\mathbb{X}} F^q(x,y)dx\right)^{\frac{1}{q}}dy, \tag{4.6}$$

proving (4.2). ☐

Remark 4.3. *In our sense, the negative exponent $q<0$ of 0, we understand in the following form:*

$$0^q = (+\infty)^{-q} = +\infty, \quad \textit{and} \quad 0^{-q} = (+\infty)^q = 0. \tag{4.7}$$

We denote by $B(a,r)$ the ball in $\mathbb{X}$ with centre a and radius r, i.e

$$B(a,r) := \{x \in \mathbb{X} : d(x,a) < r\},$$

where d is the metric on $\mathbb{X}$. Once and for all we will fix some point $a \in \mathbb{X}$, and we will write

$$|x|_a := d(a,x). \tag{4.8}$$

4.1. Reverse integral Hardy inequality with $q<0$ and $p \in (0,1)$ on the metric measure space.

Now we prove the reverse integral Hardy inequality on a metric measure space.

Theorem 4.4 (Reverse integral Hardy inequality)**.** *Suppose that $p \in (0,1)$ and $q<0$. Let $\mathbb{X}$ be a metric measure space with a polar decomposition at $a \in \mathbb{X}$. Assume that $u, v > 0$ are locally integrable functions on $\mathbb{X}$. Then the inequality*

$$\left[\int_{\mathbb{X}}\left(\int_{B(a,|x|_a)} f(y)dy\right)^q u(x)dx\right]^{\frac{1}{q}} \geq C(p,q)\left(\int_{\mathbb{X}} f^p(x)v(x)dx\right)^{\frac{1}{p}} \tag{4.9}$$

holds for some $C(p,q)>0$ and for all non-negative real-valued measurable functions f, if and only if

$$0 < D_1 := \inf_{x \neq a}\left[\left(\int_{\mathbb{X}\setminus B(a,|x|_a)} u(y)dy\right)^{\frac{1}{q}}\left(\int_{B(a,|x|_a)} v^{1-p'}(y)dy\right)^{\frac{1}{p'}}\right]. \tag{4.10}$$

Moreover, the biggest constant $C(p,q)$ in (4.9) has the following relation to D_1:

$$D_1 \geq C(p,q) \geq \left(\frac{p'}{p'+q}\right)^{-\frac{1}{q}} \left(\frac{q}{p'+q}\right)^{-\frac{1}{p'}} D_1. \tag{4.11}$$

Proof. Let us divide proof of this theorem in several steps.

Step 1. Let us denote $g(x) := f(x)v^{\frac{1}{p}}(x)$. Let $\frac{1}{p}+\frac{1}{p'}=1$, $\alpha \in \left(0, -\frac{1}{p'}\right)$ and $z(x) = v^{-\frac{1}{p}}(x)$. Let us denote,

$$V(x) := \int_{B(a,|x|_a)} v^{-\frac{p'}{p}}(y)dy = \int_{B(a,|x|_a)} z^{p'}(y)dy, \tag{4.12}$$

$$H_1(s) := \int_{\sum_s} \lambda(s,\sigma)g(s,\sigma)z(s,\sigma)d\sigma, \tag{4.13}$$

$$H_2(s) := \int_{\sum_s} \lambda(s,\sigma)z^{p'}(s,\sigma)V^{\alpha p'}(s,\sigma)d\sigma, \tag{4.14}$$

$$H_3(s) := \int_{\sum_s} \lambda(s,\sigma)g^{p}(s,\sigma)V^{-\alpha p}(s,\sigma)d\sigma, \tag{4.15}$$

$$U(r) := \int_{\sum_r} \lambda(r,\omega)u(r,\omega)d\omega. \tag{4.16}$$

By using reverse Hölder's inequality (Theorem 4.1) with polar decomposition (2.31), we compute

$$\begin{aligned} A :&= \int_{\mathbb{X}} \left(\int_{B(a,|x|_a)} f(y)dy\right)^q u(x)dx = \int_{\mathbb{X}} \left(\int_{B(a,|x|_a)} g(y)z(y)dy\right)^q u(x)dx \\ &= \int_{\mathbb{X}} \left(\int_{B(a,|x|_a)} g(y)z(y)dy\right)^p \left(\int_{B(a,|x|_a)} g(y)z(y)dy\right)^{q-p} u(x)dx \\ &= \int_{\mathbb{X}} \left(\int_{B(a,|x|_a)} g(y)V^{-\alpha}(y)V^{\alpha}(y)z(y)dy\right)^p \left(\int_{B(a,|x|_a)} g(y)z(y)dy\right)^{q-p} u(x)dx \\ &\geq \int_{\mathbb{X}} \left(\int_{B(a,|x|_a)} g^p(y)V^{-\alpha p}(y)dy\right) \left(\int_{B(a,|x|_a)} z^{p'}(y)V^{\alpha p'}(y)dy\right)^{\frac{p}{p'}} \\ &\times \left(\int_{B(a,|x|_a)} g(y)z(y)dy\right)^{q-p} u(x)dx \\ &= \int_0^\infty U(r) \left(\int_0^r H_1(s)ds\right)^{q-p} \left(\int_0^r H_2(s)ds\right)^{\frac{p}{p'}} \left(\int_0^r H_3(s)ds\right) dr. \end{aligned} \tag{4.17}$$

Let us denote by $\tilde{H}_2(s) := \int_{\sum_s} \lambda(s,\sigma) z^{p'}(s,\sigma) d\sigma$. Then we obtain

$$
\begin{aligned}
\left(\int_0^r H_2(s)ds\right)^{\frac{p}{p'}} &= \left(\int_0^r \int_{\sum_s} \lambda(s,\sigma) z^{p'}(s,\sigma) V^{\alpha p'}(s,\sigma) ds d\sigma\right)^{\frac{p}{p'}} \\
&= \left(\int_0^r \int_{\sum_s} \lambda(s,\sigma) z^{p'}(s,\sigma) \left(\int_0^s \int_{\sum_\rho} \lambda(\rho,\sigma_1) z^{p'}(\rho,\sigma_1) d\rho d\sigma_1\right)^{\alpha p'} ds d\sigma\right)^{\frac{p}{p'}} \\
&= \left(\int_0^r \tilde{H}_2(s) \left(\int_0^s \tilde{H}_2(\rho)d\rho\right)^{\alpha p'} ds\right)^{\frac{p}{p'}} \qquad (4.18)\\
&= \left(\int_0^r \left(\int_0^s \tilde{H}_2(\rho)d\rho\right)^{\alpha p'} d_s \left(\int_0^s \tilde{H}_2(\rho)d\rho\right)\right)^{\frac{p}{p'}} \\
&\overset{1+\alpha p'>0}{=} \frac{1}{(1+\alpha p')^{\frac{p}{p'}}} \left(\left(\int_0^s \tilde{H}_2(\rho)d\rho\right)^{1+\alpha p'} \Big|_0^r\right)^{\frac{p}{p'}} \\
&\overset{1+\alpha p'>0}{=} \frac{1}{(1+\alpha p')^{\frac{p}{p'}}} \left(\int_0^r \tilde{H}_2(\rho)d\rho\right)^{\frac{p(1+\alpha p')}{p'}} \\
&= \frac{V_1^{\frac{p(1+\alpha p')}{p'}}(r)}{(1+\alpha p')^{\frac{p}{p'}}},
\end{aligned}
$$

where $V_1(r) = \int_0^r \tilde{H}_2(\rho)d\rho$. By combining this fact and reverse Hölder's inequality with $\frac{p}{q} + \frac{q-p}{q} = 1$, we get

$$
\begin{aligned}
A &\geq \int_0^\infty \left(\int_0^r H_3(s)ds\right) U(r) \left(\int_0^r H_1(s)ds\right)^{q-p} \left(\int_0^r H_2(s)ds\right)^{\frac{p}{p'}} dr \\
&\overset{(4.18)}{=} \frac{1}{(1+\alpha p')^{\frac{p}{p'}}} \int_0^\infty \left(\int_0^r H_3(s)ds\right) U(r) \left(\int_0^r H_1(s)ds\right)^{q-p} V_1^{\frac{p(1+\alpha p')}{p'}}(r)dr \\
&= \frac{1}{(1+\alpha p')^{\frac{p}{p'}}} \int_0^\infty U^{\frac{p}{q}}(r) \left(\int_0^r H_3(s)ds\right) V_1^{\frac{p(1+\alpha p')}{p'}}(r) \left(\int_0^r H_1(s)ds\right)^{q-p} U^{\frac{q-p}{q}} dr \\
&\geq \frac{1}{(1+\alpha p')^{\frac{p}{p'}}} \left(\int_0^\infty \left(\int_0^r H_3(s)ds\right)^{\frac{q}{p}} U(r) V_1^{\frac{q(1+\alpha p')}{p'}}(r)dr\right)^{\frac{p}{q}} \\
&\times \left(\int_0^\infty \left(\int_0^r H_1(s)ds\right)^{q} U(r)dr\right)^{\frac{q-p}{q}}
\end{aligned}
$$

$$
\begin{aligned}
=&\frac{1}{(1+\alpha p')^{\frac{p}{p'}}}\left(\int_0^\infty U(r)\left(\int_0^r H_3(s)ds\right)^{\frac{q}{p}} V_1^{\frac{q(1+\alpha p')}{p'}}(r)dr\right)^{\frac{p}{q}} \\
&\times\left(\int_{\mathbb{X}}\left(\int_{B(a,|x|_a)} g(y)z(y)dy\right)^q u(x)dx\right)^{\frac{q-p}{q}} \\
=&\ \frac{A^{\frac{q-p}{q}}}{(1+\alpha p')^{\frac{p}{p'}}}\left(\int_0^\infty U(r)\left(\int_0^r H_3(s)ds\right)^{\frac{q}{p}} V_1^{\frac{q(1+\alpha p')}{p'}}(r)dr\right)^{\frac{p}{q}}.
\end{aligned}
$$

Therefore,

$$
A^{\frac{p}{q}} \geq \frac{1}{(1+\alpha p')^{\frac{p}{p'}}}\left(\int_0^\infty U(r)\left(\int_0^r H_3(s)ds\right)^{\frac{q}{p}} V_1^{\frac{q(1+\alpha p')}{p'}}(r)dr\right)^{\frac{p}{q}}.
$$

By using the reverse Minkowski inequality (continuous version of the reverse Minkowski inequality) with exponent $\frac{q}{p} < 0$, we obtain

$$
\begin{aligned}
&\left(\int_0^\infty U(r)\left(\int_0^r H_3(s)ds\right)^{\frac{q}{p}} V_1^{\frac{q(1+\alpha p')}{p'}}(r)dr\right)^{\frac{p}{q}} \\
&=\left(\int_0^\infty\left(\int_0^r U^{\frac{p}{q}}(r)H_3(s)V^{\frac{(1+\alpha p')p}{p'}}(r)ds\right)^{\frac{q}{p}} dr\right)^{\frac{p}{q}} \\
&=\left(\int_0^\infty\left(\int_0^\infty U^{\frac{p}{q}}(r)H_3(s)V_1^{\frac{(1+\alpha p')p}{p'}}(r)\chi_{\{s<r\}}ds\right)^{\frac{q}{p}} dr\right)^{\frac{p}{q}} \\
&\overset{(4.2)}{\geq}\int_0^\infty H_3(s)\left(\int_s^\infty U(r)V_1^{\frac{q(1+\alpha p')}{p'}}(r)dr\right)^{\frac{p}{q}} ds \\
&=\int_{\mathbb{X}} g^p(y)V^{-\alpha p}(y)\left(\int_{\mathbb{X}\setminus B(a,|y|_a)} u(x)V^{\frac{q(1+\alpha p')}{p'}}(x)dx\right)^{\frac{p}{q}} dy \\
&\geq D^p(\alpha)\int_{\mathbb{X}} g^p(y)dy,
\end{aligned}
$$

where $D(\alpha) := \inf\limits_{x\neq a} D(x,\alpha) = \inf\limits_{x\neq a} V^{-\alpha}(x)\left(\int_{\mathbb{X}\setminus B(a,|x|_a)} u(y)V^{\frac{q(1+\alpha p')}{p'}}(y)dy\right)^{\frac{1}{q}}$ and χ is the cut-off function. Then we have

$$
\begin{aligned}
A^{\frac{p}{q}} &= \left(\int_{\mathbb{X}}\left(\int_{B(a,|x|_a)} f(y)dy\right)^q u(x)dx\right)^{\frac{p}{q}} \geq \frac{D^p(\alpha)}{(1+\alpha p')^{\frac{p}{p'}}}\int_{\mathbb{X}} g^p(y)dy \\
&= \frac{D^p(\alpha)}{(1+\alpha p')^{\frac{p}{p'}}}\int_{\mathbb{X}} f^p(y)v(y)dy.
\end{aligned}
$$

Step 2. Let us define D_1 in the following form:

$$
0 < D_1 = \inf_{x\neq a}\left[\left(\int_{\mathbb{X}\setminus B(a,|x|_a)} u(x)dx\right)^{\frac{1}{q}}\left(\int_{B(a,|x|_a)} v^{1-p'}(y)dy\right)^{\frac{1}{p'}}\right]. \tag{4.19}
$$

Let us note a relation between V and V_1,

$$\begin{aligned} V(x) = \int_{B(a,|x|_a)} v^{-\frac{p'}{p}} dx &= \int_{B(a,|x|_a)} z^{p'} dx \\ &= \int_0^{|x|_a} \int_{\sum_r} z^{p'}(r,\omega)\lambda(r,\omega) dr d\omega \\ &= \int_0^{|x|_a} \tilde{H}_2(r) dr \\ &=: V_1(|x|_a), \end{aligned} \tag{4.20}$$

where, as before, $\tilde{H}_2(r) = \int_{\sum_r} z^{p'}(r,\omega)\lambda(r,\omega)d\omega$. Then let us calculate the following integral:

$$\begin{aligned} I &= \int_{\mathbb{X}\setminus B(a,|x|_a)} u(y)V^{\frac{q(1+\alpha p')}{p'}}(y)dy = \int_{|x|_a}^{\infty}\int_{\sum_r} \lambda(r,\omega)u(r,\omega)V_1^{\frac{q(1+\alpha p')}{p'}}(r)drd\omega \\ &= \int_{|x|_a}^{\infty} U(r)V_1^{\frac{q(1+\alpha p')}{p'}}(r)dr = \int_{|x|_a}^{\infty} V_1^{\frac{q(1+\alpha p')}{p'}}(r)d_r\left(-\int_r^{\infty} U(s)ds\right) \\ &= -V_1^{\frac{q(1+\alpha p')}{p'}}(r)\int_r^{\infty} U(s)ds\Big|_{|x|_a}^{\infty} \\ &+ \frac{q(1+\alpha p')}{p'}\int_{|x|_a}^{\infty}\left(\int_r^{\infty} U(s)ds\right)V_1^{\frac{q(1+\alpha p')}{p'}-1}(r)dV_1(r) \\ &\overset{\frac{q}{p'}>0}{=} V_1^{\frac{q(1+\alpha p')}{p'}}(|x|_a)\int_{|x|_a}^{\infty} U(s)ds \\ &+ \frac{q(1+\alpha p')}{p'}\int_{|x|_a}^{\infty}\left(\int_r^{\infty} U(s)ds\right)V_1^{\frac{q(1+\alpha p')}{p'}-1}(r)dV_1(r) \\ &= V_1^{\frac{q}{p'}}(|x|_a)\left(\int_{|x|_a}^{\infty} U(s)ds\right)V_1^{\alpha q}(|x|_a) \\ &+ \frac{q(1+\alpha p')}{p'}\int_{|x|_a}^{\infty}\left(\int_r^{\infty} U(s)ds\right)V_1^{\frac{q}{p'}}(r)V_1^{\alpha q-1}(r)dV_1(r) \\ &\leq D_1^q V_1^{\alpha q}(|x|_a) + \frac{q(1+\alpha p')D_1^q}{p'}\int_{|x|_a}^{\infty} V_1^{\alpha q-1}(r)dV_1(r) \\ &= D_1^q V_1^{\alpha q}(|x|_a) + \frac{(1+\alpha p')D_1^q}{\alpha p'}V_1^{\alpha q}(r)\Big|_{|x|_a}^{\infty} \\ &= D_1^q V_1^{\alpha q}(|x|_a) + \lim_{r\to\infty}\frac{(1+\alpha p')D_1^q}{\alpha p'}V_1^{\alpha q}(r) - \frac{(1+\alpha p')D_1^q}{\alpha p'}V_1^{\alpha q}(|x|_a) \\ &\overset{\frac{(1+\alpha p')D_1^q}{\alpha p'}<0}{\leq} D_1^q V_1^{\alpha q}(|x|_a) - \frac{(1+\alpha p')D_1^q}{\alpha p'}V_1^{\alpha q}(|x|_a) \\ &\overset{(4.20)}{=} -\frac{1}{\alpha p'}D_1^q V^{\alpha q}(x). \end{aligned} \tag{4.21}$$

Then we get $I = D^q(x,\alpha)V^{\alpha q}(x) \leq -\frac{1}{\alpha p'}D_1^q V^{\alpha q}(x)$. Hence,

$$D(x,\alpha) \geq (-\alpha p')^{-\frac{1}{q}} D_1,$$

it means

$$D(\alpha) \geq (-\alpha p')^{-\frac{1}{q}} D_1.$$

Finally, we obtain

$$A^{\frac{1}{q}} \geq \frac{D_1(-\alpha p')^{-\frac{1}{q}}}{(1+\alpha p')^{\frac{1}{p'}}} \left(\int_{\mathbb{X}} f^p(y)v(y)dy \right)^{\frac{1}{p}}.$$

Let us consider the function $k(\alpha) := \frac{(-\alpha p')^{-\frac{1}{q}}}{(1+\alpha p')^{\frac{1}{p'}}} = (-\alpha p')^{-\frac{1}{q}}(1+\alpha p')^{-\frac{1}{p'}}$, where $\alpha \in \left(0, -\frac{1}{p'}\right)$. Firstly, let us find an extremum of this function. After some calculation we have

$$\begin{aligned} \frac{dk(\alpha)}{d\alpha} &= -\frac{1}{q}(-p')(-\alpha p')^{-\frac{1}{q}-1}(1+\alpha p')^{-\frac{1}{p'}} \\ &+ \left(-\frac{1}{p'}\right) p'(1+\alpha p')^{-\frac{1}{p'}-1}(-\alpha p')^{-\frac{1}{q}} \\ &= p'(-\alpha p')^{-\frac{1}{q}-1}(1+\alpha p')^{-\frac{1}{p'}-1}\left(\frac{(1+\alpha p')}{q} + \alpha\right) \\ &= \frac{p'}{q}(-\alpha p')^{-\frac{1}{q}-1}(1+\alpha p')^{-\frac{1}{p'}-1}\left(\alpha(p'+q)+1\right) \\ &= 0, \end{aligned} \tag{4.22}$$

which implies that its solution is given by

$$\alpha_1 = -\frac{1}{p'+q} \in \left(0, -\frac{1}{p'}\right).$$

After taking the second derivative of $k(\alpha)$ at the point α_1 and denoting $k_1(\alpha) = (-\alpha p')^{-\frac{1}{q}-1}(1+\alpha p')^{-\frac{1}{p'}-1}$, we get

$$\begin{aligned}
\frac{d^2k(\alpha)}{d\alpha^2}\Big|_{\alpha=\alpha_1} &= \left(\frac{p'}{q}(-\alpha p')^{-\frac{1}{q}-1}(1+\alpha p')^{-\frac{1}{p'}-1}\left(\alpha(p'+q)+1\right)\right)'\Big|_{\alpha=\alpha_1}\\
&= \frac{p'}{q}\left(k_1(\alpha)\left(\alpha(p'+q)+1\right)\right)'\Big|_{\alpha=\alpha_1}\\
&= \frac{p'}{q}\left(\frac{dk_1(\alpha)}{d\alpha}\left(\alpha(p'+q)+1\right)+(p'+q)k_1(\alpha)\right)\Big|_{\alpha=\alpha_1}\\
&= \frac{p'}{q}\left(\frac{dk_1(\alpha)}{d\alpha}\Big|_{\alpha=\alpha_1}\underbrace{\left(\alpha_1(p'+q)+1\right)}_{=0}+(p'+q)k_1(\alpha_1)\right) \qquad (4.23)\\
&= \frac{p'(p'+q)}{q}k_1(\alpha_1) = \frac{p'(p'+q)}{q}(-\alpha_1 p')^{-\frac{1}{q}-1}(1+\alpha_1 p')^{-\frac{1}{p'}-1}\\
&= \underbrace{\frac{p'(p'+q)}{q}}_{<0}\underbrace{\left(\frac{p'}{p'+q}\right)^{-\frac{1}{q}-1}}_{>0}\underbrace{\left(\frac{q}{p'+q}\right)^{-\frac{1}{p'}-1}}_{>0} < 0.
\end{aligned}$$

It means the function $k(\alpha)$ has supremum at the point $\alpha = \alpha_1$. Then, the biggest constant satisfies the following inequality $C(p,q) \geq \left(\frac{p'}{p'+q}\right)^{-\frac{1}{q}}\left(\frac{q}{p'+q}\right)^{-\frac{1}{p'}} D_1$.

Step 3. Let us give a necessary condition of inequality (4.9). By using (4.9) and $f(x) = v^{-\frac{p'}{p}}(x)\chi_{\{(0,t)\}}(|x|_a)$, we compute

$$\begin{aligned}
C(p,q) &\leq \left[\int_{\mathbb{X}}\left(\int_{B(a,|x|_a)} f(y)dy\right)^q u(x)dx\right]^{\frac{1}{q}}\left[\int_{\mathbb{X}} f^p(y)v(y)dx\right]^{-\frac{1}{p}}\\
&= \left[\int_{\mathbb{X}}\left(\int_{|y|_a\leq t} v^{1-p'}(y)dy\right)^q u(x)dx\right]^{\frac{1}{q}}\left[\int_{|y|_a\leq t} v^{-p'}(y)v(y)dx\right]^{-\frac{1}{p}}\\
&\overset{q<0}{\leq} \left[\int_{|x|_a\geq t}\left(\int_{|y|_a\leq t} v^{1-p'}(y)dy\right)^q u(x)dx\right]^{\frac{1}{q}}\left[\int_{|y|_a\leq t} v^{-p'}(y)v(y)dx\right]^{-\frac{1}{p}} \qquad (4.24)\\
&= \left[\int_{|x|_a\geq t} u(x)dx\right]^{\frac{1}{q}}\left[\int_{|y|_a\leq t} v^{-p'}(y)v(y)dx\right]^{\frac{1}{p'}},
\end{aligned}$$

which gives $D_1 \geq C(p,q)$. ☐

Let us give the conjugate reverse integral Hardy inequality.

Theorem 4.5 (Conjugate reverse integral Hardy inequality)**.** *Let $p \in (0,1)$ and $q < 0$. Let $\mathbb{X}$ be a metric measure space with a polar decomposition at a. Assume that $u, v > 0$ are locally integrable functions on $\mathbb{X}$. Then the inequality*

$$\left[\int_{\mathbb{X}}\left(\int_{\mathbb{X}\backslash B(a,|x|_a)} f(y)dy\right)^q u(x)dx\right]^{\frac{1}{q}} \geq C(p,q)\left(\int_{\mathbb{X}} f^p(x)v(x)dx\right)^{\frac{1}{p}} \qquad (4.25)$$

holds for some $C(p,q)>0$ *and for all non-negative real-valued measurable functions* f*, if and only if*

$$0<D_2:=\inf_{x\neq a}\left[\left(\int_{B(a,|x|_a)}u(y)dy\right)^{\frac{1}{q}}\left(\int_{\mathbb{X}\backslash B(a,|x|_a)}v^{1-p'}(y)dy\right)^{\frac{1}{p'}}\right]. \tag{4.26}$$

Moreover, the biggest constant $C(p,q)$ *in (4.25) has the following relation to* D_2*:*

$$D_2\geq C(p,q)\geq\left(\frac{p'}{p'+q}\right)^{-\frac{1}{q}}\left(\frac{q}{p'+q}\right)^{-\frac{1}{p'}}D_2. \tag{4.27}$$

Proof. The proof of this theorem is similar to the previous case. Let us split the proof of this theorem to several steps.

Step 1. Let us denote $g(x):=f(x)v^{\frac{1}{p}}(x)$. Let $\frac{1}{p}+\frac{1}{p'}=1$, $\alpha\in\left(0,-\frac{1}{p'}\right)$ and $z(x)=v^{-\frac{1}{p}}(x)$. Let us denote

$$G(x):=\int_{\mathbb{X}\backslash B(a,|x|_a)}v^{-\frac{p'}{p}}(y)dy=\int_{\mathbb{X}\backslash B(a,|x|_a)}z^{p'}(y)dy.$$

By using the reverse Hölder's inequality (Theorem 4.1), we get

$$\begin{aligned}
B:&=\int_{\mathbb{X}}\left(\int_{\mathbb{X}\backslash B(a,|x|_a)}f(y)dy\right)^{q}u(x)dx=\int_{\mathbb{X}}\left(\int_{\mathbb{X}\backslash B(a,|x|_a)}g(y)z(y)dy\right)^{q}u(x)dx\\
&=\int_{\mathbb{X}}\left(\int_{\mathbb{X}\backslash B(a,|x|_a)}g(y)z(y)dy\right)^{p}\left(\int_{\mathbb{X}\backslash B(a,|x|_a)}g(y)z(y)dy\right)^{q-p}u(x)dx\\
&=\int_{\mathbb{X}}\left(\int_{\mathbb{X}\backslash B(a,|x|_a)}g(y)G^{-\alpha}(y)G^{\alpha}(y)z(y)dy\right)^{p}\left(\int_{\mathbb{X}\backslash B(a,|x|_a)}g(y)z(y)dy\right)^{q-p}u(x)dx\\
&\geq\int_{\mathbb{X}}\left(\int_{\mathbb{X}\backslash B(a,|x|_a)}g^{p}(y)G^{-\alpha p}(y)dy\right)\left(\int_{\mathbb{X}\backslash B(a,|x|_a)}z^{p'}(y)G^{\alpha p'}(y)dy\right)^{\frac{p}{p'}}\\
&\times\left(\int_{\mathbb{X}\backslash B(a,|x|_a)}g(y)z(y)dy\right)^{q-p}u(x)dx\\
&=\int_0^{\infty}U(r)\left(\int_r^{\infty}H_1(s)ds\right)^{q-p}\left(\int_r^{\infty}H_2(s)ds\right)^{\frac{p}{p'}}\left(\int_r^{\infty}H_3(s)ds\right)dr,
\end{aligned} \tag{4.28}$$

where $U(r)$, $H_i(s)$, $i = 1, 2, 3$, are defined in (4.13)-(4.16). Let us denote by $\tilde{H}_2(s) := \int_{\sum_s} \lambda(s,\sigma) z^{p'}(s,\sigma) d\sigma$. Then we have

$$
\begin{align*}
\left(\int_r^\infty H_2(s)ds\right)^{\frac{p}{p'}} &= \left(\int_r^\infty \int_{\sum_s} \lambda(s,\sigma) z^{p'}(s,\sigma) V^{\alpha p'}(s,\sigma) ds d\sigma\right)^{\frac{p}{p'}} \\
&= \left(\int_r^\infty \int_{\sum_s} \lambda(s,\sigma) z^{p'}(s,\sigma) \left(\int_r^\infty \int_{\sum_\rho} \lambda(\rho,\sigma_1) z^{p'}(\rho,\sigma_1) d\rho d\sigma_1\right)^{\alpha p'} ds d\sigma\right)^{\frac{p}{p'}} \\
&= \left(\int_r^\infty \tilde{H}_2(s) \left(\int_s^\infty \tilde{H}_2(\rho) d\rho\right)^{\alpha p'} ds\right)^{\frac{p}{p'}} \tag{4.29} \\
&= \left(\int_r^\infty \left(\int_s^\infty \tilde{H}_2(\rho) d\rho\right)^{\alpha p'} d_s \left(-\int_s^\infty \tilde{H}_2(\rho) d\rho\right)\right)^{\frac{p}{p'}} \\
&= \left(-\int_r^\infty \left(\int_s^\infty \tilde{H}_2(\rho) d\rho\right)^{\alpha p'} d_s \left(\int_s^\infty \tilde{H}_2(\rho) d\rho\right)\right)^{\frac{p}{p'}} \\
&\overset{1+\alpha p'>0}{=} \frac{1}{(1+\alpha p')^{\frac{p}{p'}}} \left(-\left(\int_s^\infty \tilde{H}_2(\rho) d\rho\right)^{1+\alpha p'} \Big|_r^\infty\right)^{\frac{p}{p'}} \\
&\overset{1+\alpha p'>0}{=} \frac{1}{(1+\alpha p')^{\frac{p}{p'}}} \left(\int_r^\infty \tilde{H}_2(\rho) d\rho\right)^{\frac{p(1+\alpha p')}{p'}} \\
&= \frac{G_1^{\frac{p(1+\alpha p')}{p'}}(r)}{(1+\alpha p')^{\frac{p}{p'}}},
\end{align*}
$$

where $G_1(r) = \int_0^r \tilde{H}_2(\rho) d\rho$. By combining this fact and the reverse Hölder's inequality with $\frac{p}{q} + \frac{q-p}{q} = 1$, we get

$$
\begin{align*}
B &\geq \int_0^\infty \left(\int_r^\infty H_3(s)ds\right) U(r) \left(\int_r^\infty H_1(s)ds\right)^{q-p} \left(\int_r^\infty H_2(s)ds\right)^{\frac{p}{p'}} dr \\
&\overset{(4.29)}{=} \frac{1}{(1+\alpha p')^{\frac{p}{p'}}} \int_0^\infty \left(\int_r^\infty H_3(s)ds\right) U(r) \left(\int_r^\infty H_1(s)ds\right)^{q-p} G_1^{\frac{p(1+\alpha p')}{p'}}(r) dr \\
&= \frac{1}{(1+\alpha p')^{\frac{p}{p'}}} \int_0^\infty U^{\frac{p}{q}}(r) \left(\int_r^\infty H_3(s)ds\right) G_1^{\frac{p(1+\alpha p')}{p'}}(r) \left(\int_r^\infty H_1(s)ds\right)^{q-p} U^{\frac{q-p}{q}} dr \\
&\geq \frac{1}{(1+\alpha p')^{\frac{p}{p'}}} \left(\int_0^\infty \left(\int_r^\infty H_3(s)ds\right)^{\frac{q}{p}} U(r) G_1^{\frac{q(1+\alpha p')}{p'}}(r) dr\right)^{\frac{p}{q}} \\
&\times \left(\int_0^\infty \left(\int_r^\infty H_1(s)ds\right)^{q} U(r) dr\right)^{\frac{q-p}{q}}
\end{align*}
$$

$$
\begin{aligned}
&= \frac{1}{(1+\alpha p')^{\frac{p}{p'}}} \left(\int_0^\infty U(r) \left(\int_r^\infty H_3(s)ds \right)^{\frac{q}{p}} G_1^{\frac{q(1+\alpha p')}{p'}}(r)dr \right)^{\frac{p}{q}} \\
&\times \left(\int_{\mathbb{X}} \left(\int_{\mathbb{X}\setminus B(a,|x|_a)} g(y)z(y)dy \right)^q u(x)dx \right)^{\frac{q-p}{q}} \\
&= \frac{B^{\frac{q-p}{q}}}{(1+\alpha p')^{\frac{p}{p'}}} \left(\int_0^\infty U(r) \left(\int_r^\infty H_3(s)ds \right)^{\frac{q}{p}} G_1^{\frac{q(1+\alpha p')}{p'}}(r)dr \right)^{\frac{p}{q}}.
\end{aligned}
$$

Therefore,

$$
B^{\frac{p}{q}} \geq \frac{1}{(1+\alpha p')^{\frac{p}{p'}}} \left(\int_0^\infty U(r) \left(\int_r^\infty H_3(s)ds \right)^{\frac{q}{p}} G_1^{\frac{q(1+\alpha p')}{p'}}(r)dr \right)^{\frac{p}{q}}. \tag{4.30}
$$

From the reverse Minkowski inequality with exponent $\frac{q}{p} < 0$, we obtain

$$
\begin{aligned}
&\left(\int_0^\infty U(r) \left(\int_r^\infty H_3(s)ds \right)^{\frac{q}{p}} G_1^{\frac{q(1+\alpha p')}{p'}}(r)dr \right)^{\frac{p}{q}} \\
&= \left(\int_0^\infty \left(\int_r^\infty U^{\frac{p}{q}}(r) H_3(s) G^{\frac{(1+\alpha p')p}{p'}}(r)ds \right)^{\frac{q}{p}} dr \right)^{\frac{p}{q}} \\
&= \left(\int_0^\infty \left(\int_0^\infty U^{\frac{p}{q}}(r) H_3(s) G_1^{\frac{(1+\alpha p')p}{p'}}(r)\chi_{\{r<s\}}ds \right)^{\frac{q}{p}} dr \right)^{\frac{p}{q}} \\
&\overset{(4.2)}{\geq} \int_0^\infty H_3(s) \left(\int_0^s U(r) G_1^{\frac{q(1+\alpha p')}{p'}}(r)dr \right)^{\frac{p}{q}} ds \\
&= \int_{\mathbb{X}} g^p(y) G^{-\alpha p}(y) \left(\int_{\mathbb{X}\setminus B(a,|y|_a)} u(x) G^{\frac{q(1+\alpha p')}{p'}}(x)dx \right)^{\frac{p}{q}} dy \\
&\geq \widetilde{D}^p(\alpha) \int_{\mathbb{X}} g^p(y)dy,
\end{aligned}
$$

where $\widetilde{D}(\alpha) := \inf\limits_{x\neq a} \widetilde{D}(x,\alpha) = \inf\limits_{x\neq a} G^{-\alpha}(x) \left(\int_{B(a,|x|_a)} u(y) G^{\frac{q(1+\alpha p')}{p'}}(y)dy \right)^{\frac{1}{q}}$ and χ is the cut-off function. Then we obtain

$$
\begin{aligned}
B^{\frac{p}{q}} &= \left(\int_{\mathbb{X}} \left(\int_{\mathbb{X}\setminus B(a,|x|_a)} f(y)dy \right)^q u(x)dx \right)^{\frac{p}{q}} \geq \frac{\widetilde{D}^p(\alpha)}{(1+\alpha p')^{\frac{p}{p'}}} \int_{\mathbb{X}} g^p(y)dy \\
&= \frac{\widetilde{D}^p(\alpha)}{(1+\alpha p')^{\frac{p}{p'}}} \int_{\mathbb{X}} f^p(y)v(y)dy.
\end{aligned}
$$

Step 2. Let us define D_2 in the following way:

$$
0 < D_2 = \inf_{x\neq a} \left[\left(\int_{B(a,|x|_a)} u(x)dx \right)^{\frac{1}{q}} \left(\int_{\mathbb{X}\setminus B(a,|x|_a)} v^{1-p'}(y)dy \right)^{\frac{1}{p'}} \right]. \tag{4.31}
$$

Let us note a relation between G and G_1,

$$\begin{aligned}G(x)=\int_{\mathbb{X}\backslash B(a,|x|_a)} v^{-\frac{p'}{p}}dx &= \int_{\mathbb{X}\backslash B(a,|x|_a)} z^{p'}dx \\ &= \int_{|x|_a}^{\infty}\int_{\sum_r} z^{p'}(r,\omega)\lambda(r,\omega)drd\omega \\ &= \int_{|x|_a}^{\infty}\tilde{H}_2(r)dr \\ &=: G_1(|x|_a).\end{aligned} \tag{4.32}$$

For $|x|_a \leq |y|_a$, we have

$$G_1(|x|_a)=\int_{|x|_a}^{\infty}\tilde{H}_2(r)dr \geq \int_{|y|_a}^{\infty}\tilde{H}_2(r)dr = G_1(|y|_a),$$

which means $G(x)\geq G(y)$. By $\frac{q(1+\alpha p')}{p'}>0$, we get

$$\int_{B(a,|x|_a)} u(y)G^{\frac{q(1+\alpha p')}{p'}}(x)dy \geq \int_{B(a,|x|_a)} u(y)G^{\frac{q(1+\alpha p')}{p'}}(y)dy,$$

and by using $q<0$, we have

$$\begin{aligned}\widetilde{D}(x,\alpha) &= G^{-\alpha}(x)\left(\int_{B(a,|x|_a)} u(y)G^{\frac{q(1+\alpha p')}{p'}}(y)dy\right)^{\frac{1}{q}} \\ &\geq G^{-\alpha}(x)\left(\int_{B(a,|x|_a)} u(y)G^{\frac{q(1+\alpha p')}{p'}}(x)dy\right)^{\frac{1}{q}} \\ &= G^{-\alpha}(x)G^{\frac{(1+\alpha p')}{p'}}(x)\left(\int_{B(a,|x|_a)} u(y)dy\right)^{\frac{1}{q}} \\ &= G^{\frac{1}{p'}}(x)\left(\int_{B(a,|x|_a)} u(y)dy\right)^{\frac{1}{q}} \\ &\overset{1>(-\alpha p')^{-\frac{1}{q}}}{\geq} (-\alpha p')^{-\frac{1}{q}}G^{\frac{1}{p'}}(x)\left(\int_{B(a,|x|_a)} u(y)dy\right)^{\frac{1}{q}}.\end{aligned} \tag{4.33}$$

Consequently,

$$\widetilde{D}(x,\alpha)\geq(-\alpha p')^{-\frac{1}{q}}D_2,$$

it means

$$\widetilde{D}(\alpha)\geq(-\alpha p')^{-\frac{1}{q}}D_2.$$

Finally, we obtain

$$B^{\frac{1}{q}} \geq \frac{D_1(-\alpha p')^{-\frac{1}{q}}}{(1+\alpha p')^{\frac{1}{p'}}}\left(\int_{\mathbb{X}} f^p(y)v(y)dy\right)^{\frac{1}{p}}.$$

Then, as in the previous case we have

$$\sup_{\alpha\in\left(0,-\frac{1}{p'}\right)} \frac{(-\alpha p')^{-\frac{1}{q}}}{(1+\alpha p')^{\frac{1}{p'}}} = \left(\frac{p'}{p'+q}\right)^{-\frac{1}{q}}\left(\frac{q}{p'+q}\right)^{-\frac{1}{p'}}.$$

Therefore, we have that the biggest constant satisfies

$$C(p,q) \geq \left(\frac{p'}{p'+q}\right)^{-\frac{1}{q}} \left(\frac{q}{p'+q}\right)^{-\frac{1}{p'}} D_2.$$

Step 3. Let us give a necessary condition for inequality (4.25). By using (4.25) and $f(x) = v^{-\frac{p'}{p}}(x)\chi_{(t,\infty)}(|x|_a)$, where χ is cut-off function, we compute

$$\begin{aligned} C(p,q) &\leq \left[\int_{\mathbb{X}} \left(\int_{\mathbb{X}\backslash B(a,|x|_a)} f(y)dy\right)^q u(x)dx\right]^{\frac{1}{q}} \left(\int_{\mathbb{X}} f^p(y)v(y)dx\right)^{-\frac{1}{p}} \\ &= \left[\int_{\mathbb{X}} \left(\int_{\mathbb{X}\backslash B(a,|x|_a)} f(y)dy\right)^q u(x)dx\right]^{\frac{1}{q}} \left(\int_{|x|_a\geq t} v^{-p'}(y)v(y)dx\right)^{-\frac{1}{p}} \\ &\overset{q<0}{\leq} \left[\int_{|x|_a\leq t} \left(\int_{|x|_a\geq t} v^{-\frac{p'}{p}}(y)dy\right)^q u(x)dx\right]^{\frac{1}{q}} \left(\int_{|x|_a\geq t} v^{-p'}(y)v(y)dx\right)^{-\frac{1}{p}} \\ &= \left[\int_{|x|_a\geq t} v^{1-p'}(y)dx\right]^{\frac{1}{p'}} \left[\int_{|x|_a\leq t} u(y)dy\right]^{\frac{1}{q}}, \end{aligned} \tag{4.34}$$

which gives $D_2 \geq C(p,q)$. □

4.2. **Reverse integral Hardy inequality with $-\infty < q \leq p < 0$ on the metric measure space.** In this section, we show the reverse integral Hardy inequality and its conjugate in the case $-\infty < q \leq p < 0$.

Theorem 4.6. *Assume that $p,q<0$ such that $q \leq p < 0$. Let $\mathbb{X}$ be a metric measure space with a polar decomposition at $a \in \mathbb{X}$. Suppose that $u, v \geq 0$ are locally integrable functions on $\mathbb{X}$. Then the inequality*

$$\left[\int_{\mathbb{X}} \left(\int_{B(a,|x|_a)} f(y)dy\right)^q u(x)dx\right]^{\frac{1}{q}} \geq C_1(p,q) \left(\int_{\mathbb{X}} f^p(x)v(x)dx\right)^{\frac{1}{p}} \tag{4.35}$$

holds for all non-negative real-valued measurable functions f, if

$$0 < D_1 = \inf_{x\neq a} \mathcal{D}_1(|x|_a) = \inf_{x\neq a} \left[\left(\int_{B(a,|x|_a)} u(y)dy\right)^{\frac{1}{q}} \left(\int_{B(a,|x|_a)} v^{1-p'}(y)dy\right)^{\frac{1}{p'}}\right], \tag{4.36}$$

and $\mathcal{D}_1(|x|_a)$ is non-decreasing. Moreover, the biggest constant $C_1(p,q)$ satisfies

$$D_1 \geq C_1(p,q) \geq |p|^{\frac{1}{q}}(p')^{\frac{1}{p'}} D_1, \tag{4.37}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$.

Proof. Similarly to the previous case, let us divide the proof of this theorem to steps.

Step 1. Firstly, let us denote

$$F_n(s) := \int_{\sum_\sigma} \lambda(s,\sigma) f^p(s,\sigma) v(s,\sigma) d\sigma, \tag{4.38}$$

$$V_n(s) := \int_{\sum_\sigma} \lambda(s,\sigma) v^{1-p'}(s,\sigma) d\sigma, \tag{4.39}$$

$$h(t) := \left(\int_0^t \int_{\sum_\sigma} \lambda(s,\sigma) v^{1-p'}(s,\sigma) ds d\sigma \right)^{\frac{1}{pp'}}, \tag{4.40}$$

$$H_1(t) := \int_0^t \int_{\sum_\sigma} \lambda(s,\sigma) v^{-\frac{p'}{p}}(s,\sigma) h^{-p'}(s) d\sigma ds, \tag{4.41}$$

$$U_1(t) := \int_{\sum_\sigma} \lambda(s,\sigma) u(s,\sigma) d\sigma. \tag{4.42}$$

By using the reverse Hölder's inequality with the polar decomposition, we compute

$$\begin{aligned}
\int_{B(a,|x|_a)} f(y)dy &= \int_{B(a,|x|_a)} [f(y)v^{\frac{1}{p}}(y)h(y)][v^{\frac{1}{p}}(y)h(y)]^{-1}dy \\
&\geq \left(\int_{B(a,|x|_a)} (f(y)v^{\frac{1}{p}}(y)h(y))^p dy \right)^{\frac{1}{p}} \left(\int_{B(a,|x|_a)} (v^{\frac{1}{p}}(y)h(y))^{-p'} dy \right)^{\frac{1}{p'}} \\
&= \left(\int_0^r \int_{\sum_\sigma} h^p(s) \lambda(s,\sigma) f^p(s,\sigma) v(s,\sigma) d\sigma ds \right)^{\frac{1}{p}} \\
&\times \left(\int_0^r \int_{\sum_\sigma} v^{-\frac{p'}{p}}(s,\sigma) h^{-p'}(s) \lambda(s,\sigma) d\sigma ds \right)^{\frac{1}{p'}} \\
&= \left(\int_0^r h^p(s) F_n(s) ds \right)^{\frac{1}{p}} H_1^{\frac{1}{p'}}(r).
\end{aligned} \tag{4.43}$$

Let us calculate $H_1(t)$, then we obtain

$$\begin{aligned}
H_1(t) &= \int_0^t \int_{\sum_\sigma} \lambda(s,\sigma) v^{-\frac{p'}{p}}(s,\sigma) h^{-p'}(s) d\sigma ds \overset{(4.39)}{=} \int_0^t h^{-p'}(s) V_n(s) ds \\
&\overset{(4.40)}{=} \int_0^t \left(\int_0^s \int_{\sum_z} \lambda(z,\omega) v^{1-p'}(z,\omega) dz d\omega \right)^{-\frac{1}{p}} V_n(s) ds \\
&\overset{(4.39)}{=} \int_0^t \left(\int_0^s V_n(z) dz \right)^{-\frac{1}{p}} V_n(s) ds \\
&= \int_0^t \left(\int_0^s V_n(z) dz \right)^{-\frac{1}{p}} d_s \left(\int_0^s V_n(z) dz \right)
\end{aligned} \tag{4.44}$$

$$
\begin{aligned}
&= p'\left(\int_0^s V_n(z)dz\right)^{\frac{1}{p'}}\Big|_0^t \\
&\overset{\frac{1}{p'}>0}{=} p'\left(\int_0^t V_n(z)dz\right)^{\frac{1}{p'}} \\
&= p'h^p(t).
\end{aligned}
$$

By combining this fact with (4.43), we have

$$
\int_{B(a,|x|_a)} f(y)dy \geq \left(\int_0^r h^p(s)F_n(s)ds\right)^{\frac{1}{p}} H_1^{\frac{1}{p'}}(r) \overset{(4.44)}{=} (p')^{\frac{1}{p'}}\left(\int_0^r h^p(s)F_n(s)ds\right)^{\frac{1}{p}} h^{\frac{p}{p'}}(r), \tag{4.45}
$$

multiplying by u, integrating over $\mathbb{X}$ with $q<0$ and by using the (direct) Minkowski's inequality with $\frac{q}{p}\geq 1$, we compute

$$
\begin{aligned}
&\int_{\mathbb{X}}\left(\int_{B(a,|x|_a)} f(y)dy\right)^q u(x)dx \\
&= \int_0^\infty \int_{\sum_\omega} u(r,\omega)\lambda(r,\omega)\left(\int_0^r \int_{\sum_\sigma} \lambda(s,\sigma)f(s,\sigma)dsd\sigma\right)^q drd\omega \\
&\overset{(4.42)}{=} \int_0^\infty U_1(r)\left(\int_0^r \int_{\sum_\sigma} \lambda(s,\sigma)f(s,\sigma)dsd\sigma\right)^q dr \\
&\overset{q<0,\,(4.45)}{\leq} (p')^{\frac{q}{p'}}\int_0^\infty U_1(r)\left(\int_0^r h^p(s)F_n(s)ds\right)^{\frac{q}{p}} h^{\frac{qp}{p'}}(r)dr \\
&= (p')^{\frac{q}{p'}}\int_0^\infty U_1(r)\left(\int_0^\infty \chi_{\{0,r\}}h^p(s)F_n(s)ds\right)^{\frac{q}{p}} h^{\frac{qp}{p'}}(r)dr \\
&\leq (p')^{\frac{q}{p'}}\left[\int_0^\infty h^p(s)F_n(s)\left(\int_s^\infty U_1(r)h^{\frac{qp}{p'}}(r)dr\right)^{\frac{p}{q}} ds\right]^{\frac{q}{p}},
\end{aligned} \tag{4.46}
$$

where $\chi_{\{0,r\}}$ is the cut-off function. At the same time, we can also estimate

$$
\begin{aligned}
h^{\frac{pq}{p'}}(t) &= \left[\left(\int_0^t \int_{\sum_\sigma} \lambda(s,\sigma)v^{1-p'}(s,\sigma)dsd\sigma\right)^{\frac{q}{p'}}\right]^{\frac{1}{p'}} \\
&\overset{(4.39)}{=} \left[\left(\int_0^t V_n(s)ds\right)^{\frac{q}{p'}}\right]^{\frac{1}{p'}} \\
&= \left[\left(\int_0^t V_n(s)ds\right)^{\frac{q}{p'}}\left(\int_0^t U_1(s)ds\right)\left(\int_0^t U_1(s)ds\right)^{-1}\right]^{\frac{1}{p'}} \\
&= \mathcal{D}_1^{\frac{q}{p'}}(|t|_a)\left(\int_0^t U_1(s)ds\right)^{-\frac{1}{p'}},
\end{aligned} \tag{4.47}
$$

where $\mathcal{D}_1(|t|_a) := \left(\int_0^t V_n(s)ds\right)^{\frac{1}{p'}} \left(\int_0^t U_1(s)ds\right)^{\frac{1}{q}}$. By using this fact and the fact that $\mathcal{D}_1(|x|_a)$ is non-decreasing, we get

$$
\begin{aligned}
&\int_{\mathbb{X}} \left(\int_{B(a,|x|_a)} f(y)dy\right)^q u(x)dx \\
&\overset{(4.46)}{\leq} (p')^{\frac{q}{p'}} \left[\int_0^\infty h^p(s)F_n(s) \left(\int_s^\infty U_1(r)h^{\frac{qp}{p'}}(r)dr\right)^{\frac{p}{q}} ds\right]^{\frac{q}{p}} \\
&\overset{\frac{p}{q}>0,\,(4.47)}{\leq} (p')^{\frac{q}{p'}} \left[\int_0^\infty h^p(s)F_n(s)\mathcal{D}_1^{\frac{p}{p'}}(s) \left(\int_s^\infty U_1(r)\left(\int_0^r U_1(z)dz\right)^{-\frac{1}{p'}} dr\right)^{\frac{p}{q}} ds\right]^{\frac{q}{p}} \\
&= (p')^{\frac{q}{p'}} \left[\int_0^\infty h^p(s)F_n(s)\mathcal{D}_1^{\frac{p}{p'}}(s) \left(\int_s^\infty d_r\left[p\left(\int_0^r U_1(z)dz\right)^{\frac{1}{p}}\right]\right)^{\frac{p}{q}} ds\right]^{\frac{q}{p}} \\
&= (p')^{\frac{q}{p'}} \left[\int_0^\infty h^p(s)F_n(s)\mathcal{D}_1^{\frac{p}{p'}}(s) \left(p\left(\int_0^\infty U_1(z)dz\right)^{\frac{1}{p}} - p\left(\int_0^s U_1(z)dz\right)^{\frac{1}{p}}\right)^{\frac{p}{q}} ds\right]^{\frac{q}{p}} \\
&\overset{p<0}{\leq} (-p)(p')^{\frac{q}{p'}} \left[\int_0^\infty h^p(s)F_n(s)\mathcal{D}_1^{\frac{p}{p'}}(s) \left(\int_0^s U_1(z)dz\right)^{\frac{1}{q}} ds\right]^{\frac{q}{p}} \\
&\overset{(4.40)}{=} (-p)(p')^{\frac{q}{p'}} \left[\int_0^\infty F_n(s)\mathcal{D}_1^{1+\frac{p}{p'}}(s)ds\right]^{\frac{q}{p}} \\
&= (-p)(p')^{\frac{q}{p'}} \left[\int_0^\infty F_n(s)\mathcal{D}_1^{p}(s)ds\right]^{\frac{q}{p}} \\
&\overset{p<0}{\leq} (-p)(p')^{\frac{q}{p'}} D_1^q \left[\int_0^\infty F_n(s)ds\right]^{\frac{q}{p}} \\
&\overset{(4.38)}{=} (-p)(p')^{\frac{q}{p'}} D_1^q \left(\int_{\mathbb{X}} f^p(x)v(x)dx\right)^{\frac{q}{p}} \\
&= |p|(p')^{\frac{q}{p'}} D_1^q \left(\int_{\mathbb{X}} f^p(x)v(x)dx\right)^{\frac{q}{p}}.
\end{aligned}
\tag{4.48}
$$

Finally,

$$
\left(\int_{\mathbb{X}} \left(\int_{B(a,|x|_a)} f(y)dy\right)^q u(x)dx\right)^{\frac{1}{q}} \geq |p|^{\frac{1}{q}}(p')^{\frac{1}{p'}} D_1 \left(\int_{\mathbb{X}} f^p(x)v(x)dx\right)^{\frac{1}{p}}. \tag{4.49}
$$

Hence, it follows that (4.35) holds with $C_1(p,q) \geq |p|^{\frac{1}{q}}(p')^{\frac{1}{p'}} D_1$, which proves one of the relations in (4.37).

Step 2. In this step we show the biggest constant satisfies $C_1(p,q) \leq D_1$. Let us denote by $f(x) = v^{1-p'}\chi_{\{0,t\}}(|x|_a)$. Then we have

$$\begin{aligned}\left[\int_{\mathbb{X}}\left(\int_{B(a,|x|_a)} f(y)dy\right)^q u(x)dx\right]^{\frac{1}{q}} &\overset{\frac{1}{q}<0}{\leq} \left[\int_{|x|_a\geq t}\left(\int_{B(a,|x|_a)} f(y)dy\right)^q u(x)dx\right]^{\frac{1}{q}} \\ &= \left[\int_{|x|_a\geq t}\left(\int_{|y|_a\leq t} v^{1-p'}(y)dy\right)^q u(x)dx\right]^{\frac{1}{q}} \\ &= \left[\int_{|y|_a\leq t} v^{1-p'}(y)dy\right]\left[\int_{|x|_a\geq t} u(x)dx\right]^{\frac{1}{q}},\end{aligned} \tag{4.50}$$

and

$$C_1(p,q)\left[\int_{\mathbb{X}} f^p(x)v(x)dx\right]^{\frac{1}{p}} = C_1(p,q)\left[\int_{|y|_a\leq t} v^{1-p'}(y)dy\right]^{\frac{1}{p}}. \tag{4.51}$$

By using above facts, we obtain

$$C_1(p,q) \leq \left[\int_{|y|_a\leq t} v^{1-p'}(y)dy\right]^{\frac{1}{p'}}\left[\int_{|x|_a\geq t} u(x)dx\right]^{\frac{1}{q}}. \tag{4.52}$$

Finally, we get $C_1(p,q) \leq D_1$. ☐

Then let us give the conjugate integral Hardy inequality.

Theorem 4.7. *Assume that $p,q<0$ such that $q\leq p<0$. Let $\mathbb{X}$ be a metric measure space with a polar decomposition at $a\in\mathbb{X}$. Suppose that $u,v\geq 0$ are locally integrable functions on $\mathbb{X}$. Then the inequality*

$$\left[\int_{\mathbb{X}}\left(\int_{\mathbb{X}\setminus B(a,|x|_a)} f(y)dy\right)^q u(x)dx\right]^{\frac{1}{q}} \geq C_2(p,q)\left(\int_{\mathbb{X}} f^p(x)v(x)dx\right)^{\frac{1}{p}} \tag{4.53}$$

holds for all non-negative real-valued measurable functions f, if

$$0 < D_2 = \inf_{x\neq a}\mathcal{D}_2(|x|_a) = \inf_{x\neq a}\left[\left(\int_{\mathbb{X}\setminus B(a,|x|_a)} u(y)dy\right)^{\frac{1}{q}}\left(\int_{\mathbb{X}\setminus B(a,|x|_a)} v^{1-p'}(y)dy\right)^{\frac{1}{p'}}\right], \tag{4.54}$$

and $\mathcal{D}_2(|x|_a)$ is non-increasing. Moreover, the biggest constant $C_2(p,q)$ satisfies

$$D_2 \geq C_2(p,q) \geq |p|^{\frac{1}{q}}(p')^{\frac{1}{p'}}D_2, \tag{4.55}$$

where $\frac{1}{p}+\frac{1}{p'}=1$.

Proof. The main idea of the proof of this theorem is similar with Theorem 4.6, except we use the fact that $\mathcal{D}_2(|x|_a)$ is non-increasing. ☐

4.3. **Reverse Hardy inequality with $q<0$ and $p\in(0,1)$ on the homogeneous Lie groups.** Then we have the following reverse integral Hardy inequality on homogeneous Lie groups.

Corollary 4.8. *Let $\mathbb{G}$ be a homogeneous Lie group of homogeneous dimension Q with a quasi-norm $|\cdot|$. Suppose that $q<0$, $p\in(0,1)$ and $\alpha,\beta\in\mathbb{R}$. Then the reverse integral Hardy inequality*

$$\left[\int_{\mathbb{G}}\left(\int_{B(0,|x|)} f(y)dy\right)^{q}|x|^{\alpha}dx\right]^{\frac{1}{q}} \geq C\left(\int_{\mathbb{G}} f^{p}(x)|x|^{\beta}dx\right)^{\frac{1}{p}}, \tag{4.56}$$

holds for $C>0$ and for all non-negative measurable functions f, if and only if

$$\alpha+Q<0,\ \ \beta(1-p')+Q>0\ \ and\ \ \frac{Q+\alpha}{q}+\frac{Q+\beta(1-p')}{p'}=0. \tag{4.57}$$

Moreover, the biggest constant C for (4.56) satisfies

$$\begin{aligned}&\left(\frac{|\mathfrak{S}|}{|\alpha+Q|}\right)^{\frac{1}{q}}\left(\frac{|\mathfrak{S}|}{Q+\beta(1-p')}\right)^{\frac{1}{p'}}\geq C\\ &\geq\left(\frac{|\mathfrak{S}|}{|\alpha+Q|}\right)^{\frac{1}{q}}\left(\frac{|\mathfrak{S}|}{Q+\beta(1-p')}\right)^{\frac{1}{p'}}\left(\frac{p'}{p'+q}\right)^{-\frac{1}{q}}\left(\frac{q}{p'+q}\right)^{-\frac{1}{p'}},\end{aligned} \tag{4.58}$$

where $|\mathfrak{S}|$ is the area of unit sphere with respect to $|\cdot|$.

Proof. Let us verify condition (4.10) with $u(x)=|x|^{\alpha}$, $v(x)=|x|^{\beta}$ and with $a=0$. By calculating the first integral in (4.10), we obtain

$$\begin{aligned}\int_{\mathbb{G}\backslash B(0,|x|)} u(y)dy&=\int_{\mathbb{G}\backslash B(0,|x|)}|y|^{\alpha}dy\overset{(2.11)}{=}\int_{|x|}^{\infty}\int_{\mathfrak{S}}\rho^{\alpha}\rho^{Q-1}d\rho d\sigma(\omega)\\ &=|\mathfrak{S}|\int_{|x|}^{\infty}\rho^{Q+\alpha-1}d\rho\overset{Q+\alpha<0}{=}-\frac{|\mathfrak{S}|}{Q+\alpha}|x|^{Q+\alpha}=\frac{|\mathfrak{S}|}{|Q+\alpha|}|x|^{Q+\alpha},\end{aligned} \tag{4.59}$$

where $|\mathfrak{S}|$ is the area of the unit quasi-sphere in $\mathbb{G}$. Then,

$$\begin{aligned}\int_{B(0,|x|)} v^{1-p'}(y)dy&=\int_{B(0,|x|)}|y|^{\beta(1-p')}dy\overset{(2.11)}{=}\int_{0}^{|x|}\int_{\mathfrak{S}}\rho^{\beta(1-p')}\rho^{Q-1}d\rho d\sigma(\omega)\\ &=|\mathfrak{S}|\int_{0}^{|x|}\rho^{Q+\beta(1-p')-1}d\rho\\ &\overset{Q+\beta(1-p')>0}{=}\frac{|\mathfrak{S}|}{Q+\beta(1-p')}|x|^{Q+\beta(1-p')}.\end{aligned} \tag{4.60}$$

Finally by summarising above facts with $\frac{Q+\alpha}{q}+\frac{Q+\beta(1-p')}{p'}=0$, we get

$$\begin{aligned}D_1&=\left(\frac{|\mathfrak{S}|}{|\alpha+Q|}\right)^{\frac{1}{q}}\left(\frac{|\mathfrak{S}|}{Q+\beta(1-p')}\right)^{\frac{1}{p'}}\inf_{r>0} r^{\frac{Q+\alpha}{q}+\frac{Q+\beta(1-p')}{p'}}\\ &=\left(\frac{|\mathfrak{S}|}{|\alpha+Q|}\right)^{\frac{1}{q}}\left(\frac{|\mathfrak{S}|}{Q+\beta(1-p')}\right)^{\frac{1}{p'}}>0.\end{aligned} \tag{4.61}$$

From (4.11), we obtain

$$\begin{aligned}
&\left(\frac{|\mathfrak{S}|}{|\alpha+Q|}\right)^{\frac{1}{q}}\left(\frac{|\mathfrak{S}|}{Q+\beta(1-p')}\right)^{\frac{1}{p'}} \geq C\\
&\geq \left(\frac{|\mathfrak{S}|}{|\alpha+Q|}\right)^{\frac{1}{q}}\left(\frac{|\mathfrak{S}|}{Q+\beta(1-p')}\right)^{\frac{1}{p'}}\left(\frac{p'}{p'+q}\right)^{-\frac{1}{q}}\left(\frac{q}{p'+q}\right)^{-\frac{1}{p'}},
\end{aligned} \tag{4.62}$$

completing the proof. ☐

Similarly, we have the conjugate reverse integral Hardy inequality on homogeneous Lie groups.

Corollary 4.9. *Let $\mathbb{G}$ be a homogeneous Lie group of homogeneous dimension Q with a quasi-norm $|\cdot|$. Assume that $q<0$, $p\in(0,1)$ and $\alpha,\beta\in\mathbb{R}$. Then the conjugate reverse integral Hardy inequality*

$$\left[\int_{\mathbb{G}}\left(\int_{\mathbb{G}\setminus B(0,|x|)} f(y)dy\right)^{q}|x|^{\alpha}dx\right]^{\frac{1}{q}} \geq C\left(\int_{\mathbb{G}} f^{p}(x)|x|^{\beta}dx\right)^{\frac{1}{p}}, \tag{4.63}$$

holds for $C>0$ and for all non-negative measurable functions f, if and only if

$$\alpha+Q>0,\ \ \beta(1-p')+Q<0 \ \ \textit{and}\ \ \frac{Q+\alpha}{q}+\frac{Q+\beta(1-p')}{p'}=0. \tag{4.64}$$

Moreover, the biggest constant C for (4.63) satisfies

$$\begin{aligned}
&\left(\frac{|\mathfrak{S}|}{\alpha+Q}\right)^{\frac{1}{q}}\left(\frac{|\mathfrak{S}|}{|Q+\beta(1-p')|}\right)^{\frac{1}{p'}} \geq C\\
&\geq \left(\frac{|\mathfrak{S}|}{\alpha+Q}\right)^{\frac{1}{q}}\left(\frac{|\mathfrak{S}|}{|Q+\beta(1-p')|}\right)^{\frac{1}{p'}}\left(\frac{p'}{p'+q}\right)^{-\frac{1}{q}}\left(\frac{q}{p'+q}\right)^{-\frac{1}{p'}},
\end{aligned} \tag{4.65}$$

where $|\mathfrak{S}|$ is the area of unit sphere with respect to $|\cdot|$.

Proof. Proof of this corollary is similar to the previous case. ☐

4.4. **Reverse Hardy inequality with $\infty<q\leq p<0$ on the homogeneous Lie groups.** In this section we show the reverse integral Hardy inequality on homogeneous Lie groups.

Theorem 4.10. *Let $\mathbb{G}$ be a homogeneous Lie group of homogeneous dimension Q with a quasi-norm $|\cdot|$. Assume that $q\leq p<0$ and $\alpha,\beta\in\mathbb{R}$. Then the reverse integral Hardy inequality*

$$\left[\int_{\mathbb{G}}\left(\int_{B(0,|x|)} f(y)dy\right)^{q}|x|^{\alpha}dx\right]^{\frac{1}{q}} \geq C_1\left(\int_{\mathbb{G}} f^{p}(x)|x|^{\beta}dx\right)^{\frac{1}{p}}, \tag{4.66}$$

holds for $C_1>0$ and for all non-negative measurable functions f, if $\alpha+Q>0$, $\beta(1-p')+Q>0$ and $\frac{Q+\alpha}{q}+\frac{Q+\beta(1-p')}{p'}=0$. Moreover, the biggest constant C_1 for (4.66) satisfies

$$\left(\frac{|\mathfrak{S}|}{\alpha+Q}\right)^{\frac{1}{q}}\left(\frac{|\mathfrak{S}|}{Q+\beta(1-p')}\right)^{\frac{1}{p'}} \geq C_1 \geq |p|^{\frac{1}{q}}(p')^{\frac{1}{p'}}\left(\frac{|\mathfrak{S}|}{\alpha+Q}\right)^{\frac{1}{q}}\left(\frac{|\mathfrak{S}|}{Q+\beta(1-p')}\right)^{\frac{1}{p'}}.$$

Proof. Let us verify (4.36) with $u(x)=|x|^{\alpha}$ and $v(x)=|x|^{\beta}$. Let us calculate the first integral in (4.36):

$$\begin{aligned}\int_{B(0,|x|)} u(y)dy &= \int_{B(0,|x|)} |y|^{\alpha}dy \overset{(2.11)}{=} \int_0^{|x|}\int_{\mathfrak{S}} r^{\alpha}r^{Q-1}drd\sigma \\ &= |\mathfrak{S}|\int_0^{|x|} r^{Q+\alpha-1}dr \overset{Q+\alpha>0}{=} \frac{|\mathfrak{S}|}{Q+\alpha}|x|^{Q+\alpha},\end{aligned} \tag{4.67}$$

where $|\mathfrak{S}|$ is the area of the unit quasi-sphere in $\mathbb{G}$. Then,

$$\begin{aligned}\int_{B(0,|x|)} v^{1-p'}(y)dy &= \int_{B(0,|x|)} |y|^{\beta(1-p')}dy \overset{(2.11)}{=} \int_0^{|x|}\int_{\mathfrak{S}} r^{\beta(1-p')}r^{Q-1}drd\sigma \\ &= |\mathfrak{S}|\int_0^{|x|} r^{Q+\beta(1-p')-1}dr \overset{Q+\beta(1-p')>0}{=} \frac{|\mathfrak{S}|}{Q+\beta(1-p')}|x|^{Q+\beta(1-p')}.\end{aligned} \tag{4.68}$$

Finally by summarising above facts with $\frac{Q+\alpha}{q}+\frac{Q+\beta(1-p')}{p'}=0$, we have

$$\begin{aligned}\mathcal{D}_1(|x|) &= \left(\frac{|\mathfrak{S}|}{\alpha+Q}\right)^{\frac{1}{q}}\left(\frac{|\mathfrak{S}|}{Q+\beta(1-p')}\right)^{\frac{1}{p'}}\left[|x|^{\frac{Q+\alpha}{q}+\frac{Q+\beta(1-p')}{p'}}\right] \\ &= \left(\frac{|\mathfrak{S}|}{\alpha+Q}\right)^{\frac{1}{q}}\left(\frac{|\mathfrak{S}|}{Q+\beta(1-p')}\right)^{\frac{1}{p'}},\end{aligned} \tag{4.69}$$

it means $\mathcal{D}_1(|x|)$ is a non-decreasing function. Then

$$D_1=\inf_{x\neq a}\mathcal{D}_1(|x|)=\left(\frac{|\mathfrak{S}|}{\alpha+Q}\right)^{\frac{1}{q}}\left(\frac{|\mathfrak{S}|}{Q+\beta(1-p')}\right)^{\frac{1}{p'}}>0.$$

Therefore, by (4.37) we have

$$D_1\geq C_1\geq |p|^{\frac{1}{q}}(p')^{\frac{1}{p'}}D_1,$$

where $D_1=\left(\frac{|\mathfrak{S}|}{\alpha+Q}\right)^{\frac{1}{q}}\left(\frac{|\mathfrak{S}|}{Q+\beta(1-p')}\right)^{\frac{1}{p'}}$, completing the proof. ☐

Then we have the conjugate reverse integral Hardy inequality on homogeneous Lie groups.

Theorem 4.11. *Let $\mathbb{G}$ be a homogeneous Lie group of homogeneous dimension Q with a quasi-norm $|\cdot|$. Assume that $q\leq p<0$ and $\alpha,\beta\in\mathbb{R}$. Then the reverse conjugate integral Hardy inequality*

$$\left[\int_{\mathbb{G}}\left(\int_{\mathbb{G}\setminus B(0,|x|)} f(y)dy\right)^{q}|x|^{\alpha}dx\right]^{\frac{1}{q}}\geq C_2\left(\int_{\mathbb{G}} f^{p}(x)|x|^{\beta}dx\right)^{\frac{1}{p}}, \tag{4.70}$$

holds for $C_2>0$ and for all non-negative measurable functions f, if $\alpha+Q<0$, $\beta(1-p')+Q<0$ and $\frac{Q+\alpha}{q}+\frac{Q+\beta(1-p')}{p'}=0$. Moreover, the biggest constant C_2 for (4.70) satisfies

$$\left(\frac{|\mathfrak{S}|}{|\alpha+Q|}\right)^{\frac{1}{q}}\left(\frac{|\mathfrak{S}|}{|Q+\beta(1-p')|}\right)^{\frac{1}{p'}}\geq C_2\geq |p|^{\frac{1}{q}}(p')^{\frac{1}{p'}}\left(\frac{|\mathfrak{S}|}{|\alpha+Q|}\right)^{\frac{1}{q}}\left(\frac{|\mathfrak{S}|}{|Q+\beta(1-p')|}\right)^{\frac{1}{p'}}.$$

Proof. Proof of this theorem is similar to the previous case, but we need to use Theorem 4.7. ☐

4.5. **Reverse Hardy inequality with $q < 0$ and $p \in (0,1)$ on the hyperbolic space.** Let $\mathbb{H}^n$ be the hyperbolic space of dimension n and let $a \in \mathbb{H}^n$. Let us set

$$u(x) = (\sinh |x|_a)^{\alpha}, \quad v(x) = (\sinh |x|_a)^{\beta}. \tag{4.71}$$

Then we have the main result of this subsection.

Corollary 4.12. *Let $\mathbb{H}^n$ be the hyperbolic space of dimension n and let $a \in \mathbb{H}^n$. Assume that $q < 0$, $p \in (0,1)$ and $\alpha, \beta \in \mathbb{R}$. Then the reverse integral Hardy inequality*

$$\left[\int_{\mathbb{H}^n} \left(\int_{B(a,|x|_a)} f(y)dy\right)^q (\sinh |x|_a)^{\alpha} dx\right]^{\frac{1}{q}} \geq C \left(\int_{\mathbb{H}^n} f^p(x)(\sinh |x|_a)^{\beta} dx\right)^{\frac{1}{p}}, \tag{4.72}$$

holds for $C > 0$ and for all non-negative measurable functions f, if

$$0 \leq \alpha + n < 1, \;\; \beta(1-p') + n > 0 \;\; \textit{and} \;\; \frac{\alpha+n}{q} + \frac{\beta(1-p')+n}{p'} \geq \frac{1}{q} + \frac{1}{p'}. \tag{4.73}$$

Proof. Let us verify condition (4.10). By using polar decomposition for the hyperbolic space, we have

$$D_1 = \inf_{x \neq a} \left(\int_{|x|_a}^{\infty} (\sinh \rho)^{\alpha+n-1} d\rho\right)^{\frac{1}{q}} \left(\int_0^{|x|_a} (\sinh \rho)^{\beta(1-p')+n-1} d\rho\right)^{\frac{1}{p'}}. \tag{4.74}$$

If $\alpha + n < 1$ and $\beta(1-p') + n > 0$, then (4.74) is integrable. Let us check the finiteness and positiveness of the infimum (4.74). Let us consider two cases.

First case, $|x|_a \gg 1$. Then $\sinh |x|_a \approx \exp |x|_a$ if $|x|_a \gg 1$. Then we obtain,

$$\begin{aligned} D_1^1 &= \inf_{|x|_a \gg 1} \left(\int_{|x|_a}^{\infty} (\sinh \rho)^{\alpha+n-1} d\rho\right)^{\frac{1}{q}} \left(\int_0^{|x|_a} (\sinh \rho)^{\beta(1-p')+n-1} d\rho\right)^{\frac{1}{p'}} \\ &\simeq \inf_{|x|_a \gg 1} \left(\int_{|x|_a}^{\infty} (\exp \rho)^{\alpha+n-1} d\rho\right)^{\frac{1}{q}} \left(\int_0^{|x|_a} (\exp \rho)^{\beta(1-p')+n-1} d\rho\right)^{\frac{1}{p'}} \\ &= \inf_{|x|_a \gg 1} \left((\exp |x|_a)^{\alpha+n-1}\right)^{\frac{1}{q}} \left((\exp |x|_a)^{\beta(1-p')+n-1}\right)^{\frac{1}{p'}} \\ &= \inf_{|x|_a \gg 1} (\exp |x|_a)^{\frac{\alpha+n-1}{q} + \frac{\beta(1-p')+n-1}{p'}}, \end{aligned} \tag{4.75}$$

infimum of the last term is positive, if and only if $\frac{\alpha+n-1}{q} + \frac{\beta(1-p')+n-1}{p'} \geq 0$, i.e., $\frac{\alpha+n}{q} + \frac{\beta(1-p')+n}{p'} \geq \frac{1}{q} + \frac{1}{p'}$, then $D_1^1 > 0$.

Let us consider the second case $|x|_a \ll 1$. For $|x|_a \ll 1$ we have $\sinh \rho_{\{0\leq \rho < |x|_a\}} \approx \rho$, then we calculate

$$\begin{aligned}
&\inf_{|x|_a\ll 1}\left(\int_{|x|_a}^{\infty}(\sinh\rho)^{\alpha+n-1}d\rho\right)^{\frac{1}{q}}\left(\int_0^{|x|_a}(\sinh\rho)^{\beta(1-p')+n-1}d\rho\right)^{\frac{1}{p'}}\\
&\simeq \inf_{|x|_a\ll 1}\left(\int_{|x|_a}^{R}(\sinh\rho)^{\alpha+n-1}d\rho+\int_R^{\infty}(\sinh\rho)^{\alpha+n-1}d\rho\right)^{\frac{1}{q}}\left(\int_0^{|x|_a}\rho^{\beta(1-p')+n-1}d\rho\right)^{\frac{1}{p'}}\\
&\simeq \inf_{|x|_a\ll 1}\left(\int_{|x|_a}^{R}(\sinh\rho)^{\alpha+n-1}d\rho+\int_R^{\infty}(\sinh\rho)^{\alpha+n-1}d\rho\right)^{\frac{1}{q}}|x|_a^{\frac{\beta(1-p')+n}{p'}}.
\end{aligned}\tag{4.76}$$

Similarly, for small R we have $\sinh \rho_{\{|x|_a\leq \rho < R\}} \approx \rho$, so that we obtain

$$\begin{aligned}
&\inf_{|x|_a\ll 1}\left(\int_{|x|_a}^{\infty}(\sinh\rho)^{\alpha+n-1}d\rho\right)^{\frac{1}{q}}\left(\int_0^{|x|_a}(\sinh\rho)^{\beta(1-p')+n-1}d\rho\right)^{\frac{1}{p'}}\\
&\simeq \inf_{|x|_a\ll 1}\left(\int_{|x|_a}^{R}(\sinh\rho)^{\alpha+n-1}d\rho+\int_R^{\infty}(\sinh\rho)^{\alpha+n-1}d\rho\right)^{\frac{1}{q}}|x|_a^{\frac{\beta(1-p')+n}{p'}}\\
&\simeq \inf_{|x|_a\ll 1}\left(\int_{|x|_a}^{R}\rho^{\alpha+n-1}d\rho+\int_R^{\infty}(\sinh\rho)^{\alpha+n-1}d\rho\right)^{\frac{1}{q}}|x|_a^{\frac{\beta(1-p')+n}{p'}}\\
&\simeq \inf_{|x|_a\ll 1}\left(|x|_a^{\alpha+n}+C_R\right)^{\frac{1}{q}}|x|_a^{\frac{\beta(1-p')+n}{p'}}.
\end{aligned}\tag{4.77}$$

If $\alpha+n\geq 0$, we have $\frac{\alpha+n}{q}\leq 0$, then we get

$$\begin{aligned}
D_1^2 &= \inf_{|x|_a\ll 1}\left(\int_{|x|_a}^{\infty}(\sinh\rho)^{\alpha+n-1}d\rho\right)^{\frac{1}{q}}\left(\int_0^{|x|_a}(\sinh\rho)^{\beta(1-p')+n-1}d\rho\right)^{\frac{1}{p'}}\\
&\simeq \inf_{|x|_a\ll 1}\left(|x|_a^{\alpha+n}+C_R\right)^{\frac{1}{q}}|x|_a^{\frac{\beta(1-p')+n}{p'}}\\
&\simeq \inf_{|x|_a\ll 1}|x|_a^{\frac{\beta(1-p')+n}{p'}}>0,
\end{aligned}\tag{4.78}$$

and infimum is positive, if and only if $\frac{\beta(1-p')+n}{p'}<0$, i.e., $\beta(1-p')+n>0$. ☐

Let us give the reverse conjugate integral Hardy's inequality in hyperbolic spaces:

Corollary 4.13. *Let $\mathbb{H}^n$ be the hyperbolic space of dimension n and $a\in\mathbb{H}^n$. Assume that $q<0$, $p\in(0,1)$ and let $\alpha,\beta\in\mathbb{R}$. Then the reverse conjugate integral Hardy inequality*

$$\left[\int_{\mathbb{H}^n}\left(\int_{\mathbb{X}\backslash B(a,|x|_a)}f(y)dy\right)^q(\sinh|x|_a)^{\alpha}dx\right]^{\frac{1}{q}}\geq C\left(\int_{\mathbb{H}^n}f^p(x)(\sinh|x|_a)^{\beta}dx\right)^{\frac{1}{p}},\tag{4.79}$$

holds for all non-negative measurable functions f, if

$$\alpha+n>0,\quad 1>\beta(1-p')+n\geq 0 \;\; \textit{and} \;\; \frac{\alpha+n}{q}+\frac{\beta(1-p')+n}{p'}\geq\frac{1}{q}+\frac{1}{p'}.$$

Proof. Similarly to the previous case, we check condition (4.26) and then, we have

$$D_2=\inf_{x\neq a}\left(\int_0^{|x|_a}(\sinh\rho)^{\alpha+n-1}d\rho\right)^{\frac{1}{q}}\left(\int_{|x|_a}^{\infty}(\sinh\rho)^{\beta(1-p')+n-1}d\rho\right)^{\frac{1}{p'}}. \tag{4.80}$$

If $\alpha+n>0$ and $\beta(1-p')+n<1$, then (4.80) is integrable. If $|x|_a\gg 1$, we obtain

$$\begin{aligned} D_2^1&=\inf_{|x|_a\gg1}\left(\int_{|x|_a}^{\infty}(\sinh\rho)^{\alpha+n-1}d\rho\right)^{\frac{1}{q}}\left(\int_0^{|x|_a}(\sinh\rho)^{\beta(1-p')+n-1}d\rho\right)^{\frac{1}{p'}}\\ &\simeq\inf_{|x|_a\gg1}\left(\int_{|x|_a}^{\infty}(\exp\rho)^{\alpha+n-1}d\rho\right)^{\frac{1}{q}}\left(\int_0^{|x|_a}(\exp\rho)^{\beta(1-p')+n-1}d\rho\right)^{\frac{1}{p'}}\\ &=\inf_{|x|_a\gg1}\left((\exp|x|_a)^{\alpha+n-1}\right)^{\frac{1}{q}}\left((\exp|x|_a)^{\beta(1-p')+n-1}\right)^{\frac{1}{p'}}\\ &=\inf_{|x|_a\gg1}(\exp|x|_a)^{\frac{\alpha+n-1}{q}+\frac{\beta(1-p')+n-1}{p'}}, \end{aligned} \tag{4.81}$$

infimum of the last term is positive, if and only if $\frac{\alpha+n-1}{q}+\frac{\beta(1-p')+n-1}{p'}\geq 0$, i.e., $\frac{\alpha+n}{q}+\frac{\beta(1-p')+n}{p'}\geq\frac{1}{q}+\frac{1}{p'}$, then $D_1^1>0$.

If $|x|_a\ll 1$, we obtain

$$\begin{aligned} &\inf_{|x|_a\ll1}\left(\int_0^{|x|_a}(\sinh\rho)^{\alpha+n-1}d\rho\right)^{\frac{1}{q}}\left(\int_{|x|_a}^{\infty}(\sinh\rho)^{\beta(1-p')+n-1}d\rho\right)^{\frac{1}{p'}}\\ &\quad\simeq\inf_{|x|_a\ll1}\left(\int_0^{|x|_a}\rho^{\alpha+n-1}d\rho\right)^{\frac{1}{q}}\left(\int_{|x|_a}^{R}(\sinh\rho)^{\beta(1-p')+n-1}d\rho+\int_R^{\infty}(\sinh\rho)^{\beta(1-p')+n-1}d\rho\right)^{\frac{1}{p'}}\\ &\quad\simeq\inf_{|x|_a\ll1}\left(\int_{|x|_a}^{R}(\sinh\rho)^{\beta(1-p')+n-1}d\rho+\int_R^{\infty}(\sinh\rho)^{\beta(1-p')+n-1}d\rho\right)^{\frac{1}{p'}}|x|_a^{\frac{\alpha+n}{q}}. \end{aligned} \tag{4.82}$$

Similarly, for small R we have $\sinh\rho_{\{|x|_a\leq\rho<R\}}\approx\rho$, so that we obtain

$$\begin{aligned} &\inf_{|x|_a\ll1}\left(\int_0^{|x|_a}(\sinh\rho)^{\alpha+n-1}d\rho\right)^{\frac{1}{q}}\left(\int_{|x|_a}^{\infty}(\sinh\rho)^{\beta(1-p')+n-1}d\rho\right)^{\frac{1}{p'}}\\ &\quad\simeq\inf_{|x|_a\ll1}\left(\int_{|x|_a}^{R}(\sinh\rho)^{\beta(1-p')+n-1}d\rho+\int_R^{\infty}(\sinh\rho)^{\beta(1-p')+n-1}d\rho\right)^{\frac{1}{p'}}|x|_a^{\frac{\alpha+n}{q}}\\ &\quad\simeq\inf_{|x|_a\ll1}\left(|x|_a^{\beta(1-p')+n}+C_R'\right)^{\frac{1}{q}}|x|_a^{\frac{\alpha+n}{q}}. \end{aligned} \tag{4.83}$$

If $\beta(1-p')+n \geq 0$, we have $\frac{\beta(1-p')+n}{q} \leq 0$, then we have

$$\begin{aligned}
D_2^2 &= \inf_{|x|_a \ll 1} \left(\int_0^{|x|_a} (\sinh\rho)^{\alpha+n-1} d\rho \right)^{\frac{1}{q}} \left(\int_{|x|_a}^{\infty} (\sinh\rho)^{\beta(1-p')+n-1} d\rho \right)^{\frac{1}{p'}} \\
&\simeq \inf_{|x|_a \ll 1} \left(|x|_a^{\beta(1-p')+n} + C'_R \right)^{\frac{1}{q}} |x|_a^{\frac{\alpha+n}{q}} \\
&\simeq \inf_{|x|_a \ll 1} |x|_a^{\frac{\alpha+n}{q}},
\end{aligned} \tag{4.84}$$

and infimum is positive, if and only if $\frac{\alpha+n}{q} < 0$, i.e., $\alpha+n > 0$. $\square$

4.6. **Reverse Hardy inequality with $q < 0$ and $p \in (0,1)$ on the Cartan-Hadamard manifolds.** Let (M,g) be the Cartan-Hadamard manifold with curvature K_M. If $K_M = 0$ then $J(t,\omega) = 1$ and we set

$$u(x) = |x|_a^{\alpha}, \quad v(x) = |x|_a^{\beta}, \text{ when } K_M = 0. \tag{4.85}$$

If $K_M < 0$ then $J(t,\omega) = \left(\frac{\sinh\sqrt{b}t}{\sqrt{b}t} \right)^{n-1}$ and we set

$$u(x) = (\sinh\sqrt{-K_M}|x|_a)^{\alpha}, \quad v(x) = (\sinh\sqrt{-K_M}|x|_a)^{\beta}, \text{ when } K_M < 0. \tag{4.86}$$

Then we have the following result of this subsection.

Corollary 4.14. *Let (M,g) be the Cartan-Hadamard manifold of dimension n with curvature K_M. Assume that $q < 0$, $p \in (0,1)$ and $\alpha, \beta \in \mathbb{R}$. Then we have*

i) *if $K_M = 0$, $u(x) = |x|_a^{\alpha}, v(x) = |x|_a^{\beta}$, then*

$$\left[\int_M \left(\int_{B(a,|x|_a)} f(y)dy \right)^q |x|_a^{\alpha} dx \right]^{\frac{1}{q}} \geq C \left(\int_M f^p(x)|x|^{\beta} dx \right)^{\frac{1}{p}}, \tag{4.87}$$

holds for $C > 0$ and for non-negative measurable functions f, if and only if $\alpha+n < 0$, $\beta(1-p')+n > 0$ and $\frac{n+\alpha}{q} + \frac{n+\beta(1-p')}{p'} = 0$;

ii) *if $K_M = 0$, $u(x) = |x|_a^{\alpha}, v(x) = |x|_a^{\beta}$, then*

$$\left[\int_M \left(\int_{M\setminus B(a,|x|_a)} f(y)dy \right)^q |x|^{\alpha} dx \right]^{\frac{1}{q}} \geq C \left(\int_M f^p(x)|x|^{\beta} dx \right)^{\frac{1}{p}}, \tag{4.88}$$

holds for $C > 0$ and for non-negative measurable functions f, if and only if $\alpha+n > 0$, $\beta(1-p')+n < 0$ and $\frac{n+\alpha}{q} + \frac{n+\beta(1-p')}{p'} = 0$;

iii) *if $K_M < 0$, $u(x) = (\sinh\sqrt{-K_M}|x|_a)^{\alpha}, v(x) = (\sinh|x|_a)^{\beta}$, then*

$$\begin{aligned}
&\left[\int_M \left(\int_{B(a,|x|_a)} f(y)dy \right)^q (\sinh\sqrt{-K_M}|x|_a)^{\alpha} dx \right]^{\frac{1}{q}} \\
&\qquad \geq C \left(\int_M f^p(x)(\sinh\sqrt{-K_M}|x|_a)^{\beta} dx \right)^{\frac{1}{p}},
\end{aligned} \tag{4.89}$$

holds for $C > 0$ and for all non-negative measurable functions f, if $0 \leq \alpha+n < 1$, $\beta(1-p')+n > 0$ and $\frac{\alpha+n}{q} + \frac{\beta(1-p')+n}{p'} \geq \frac{1}{q} + \frac{1}{p'}$;

iv) *if* $K_M < 0$, $u(x) = (\sinh\sqrt{-K_M}|x|_a)^\alpha$, $v(x) = (\sinh\sqrt{-K_M}|x|_a)^\beta$, *then*

$$\left[\int_M \left(\int_{M\setminus B(a,|x|_a)} f(y)dy\right)^q (\sinh\sqrt{-K_M}|x|_a)^\alpha dx\right]^{\frac{1}{q}}$$
$$\geq C\left(\int_M f^p(x)(\sinh\sqrt{-K_M}|x|_a)^\beta dx\right)^{\frac{1}{p}}, \quad (4.90)$$

holds for $C>0$ *and for all non-negative measurable functions* f, *if* $\alpha+n>0$, $1>\beta(1-p')+n\geq 0$ *and* $\frac{\alpha+n}{q}+\frac{\beta(1-p')+n}{p'}\geq\frac{1}{q}+\frac{1}{p'}$.

4.7. Reverse Hardy-Littlewood-Sobolev, Stein-Weiss and improved Stein-Weiss inequalities with $q<0$ and $p\in(0,1)$ on the homogeneous Lie groups.

Now we formulate the reverse Stein-Weiss inequality on homogeneous Lie group.

Theorem 4.15. *Let* $\mathbb{G}$ *be a homogeneous group of homogeneous dimension* $Q\geq 1$ *and let* $|\cdot|$ *be an arbitrary homogeneous quasi-norm on* $\mathbb{G}$. *Assume that* $\lambda>0$, $p,q'\in(0,1)$, $0\leq\alpha<-\frac{Q}{q}$, $0\leq\beta<-\frac{Q}{p'}$, $\frac{1}{q'}+\frac{1}{p}=\frac{\alpha+\beta+\lambda}{Q}+2$, *where* $\frac{1}{p}+\frac{1}{p'}=1$ *and* $\frac{1}{q}+\frac{1}{q'}=1$. *Then for all non-negative functions* $f\in L^{q'}(\mathbb{G})$ *and* $h\in L^p(\mathbb{G})$ *we have*

$$\int_{\mathbb{G}}\int_{\mathbb{G}}|x|^\alpha|y^{-1}x|^\lambda f(x)h(y)|y|^\beta dxdy\geq C\|f\|_{L^{q'}(\mathbb{G})}\|h\|_{L^p(\mathbb{G})}, \quad (4.91)$$

where C *is a positive constant independent of* f *and* h.

Proof. By using the reverse Hölder's inequality with $\frac{1}{q}+\frac{1}{q'}=1$ (Theorem 4.1) in (4.91), we calculate,

$$\int_{\mathbb{G}}\int_{\mathbb{G}}|x|^\alpha f(x)|y^{-1}x|^\lambda h(y)|y|^\beta dydx=\int_{\mathbb{G}}\left(\int_{\mathbb{G}}|x|^\alpha|y^{-1}x|^\lambda h(y)|y|^\beta dy\right)f(x)dx$$
$$\overset{(4.1)}{\geq}\left(\int_{\mathbb{G}}\left(\int_{\mathbb{G}}|x|^\alpha|y^{-1}x|^\lambda h(y)|y|^\beta dy\right)^q dx\right)^{\frac{1}{q}}\|f\|_{L^{q'}(\mathbb{G})}.$$

For (4.91), it is enough to show that

$$\left(\int_{\mathbb{G}}\left(\int_{\mathbb{G}}|x|^\alpha|y^{-1}x|^\lambda h(y)|y|^\beta dy\right)^q dx\right)^{\frac{1}{q}}\geq C\|h\|_{L^p(\mathbb{G})},$$

and by changing $u(y)=h(y)|y|^\beta$, this is equivalent to

$$\int_{\mathbb{G}}\left(\int_{\mathbb{G}}|x|^\alpha|y^{-1}x|^\lambda u(y)dy\right)^q dx\leq C\||y|^{-\beta}u\|^q_{L^p(\mathbb{G})}.$$

We have that

$$\int_{\mathbb{G}}|x|^\alpha|y^{-1}x|^\lambda u(y)dy\geq\int_{B\left(0,\frac{|x|}{2}\right)}|x|^\alpha|y^{-1}x|^\lambda u(y)dy,$$

then

$$\left(\int_{\mathbb{G}}|x|^\alpha|y^{-1}x|^\lambda u(y)dy\right)^q\overset{q<0}{\leq}\left(\int_{B\left(0,\frac{|x|}{2}\right)}|x|^\alpha|y^{-1}x|^\lambda u(y)dy\right)^q.$$

Hence, we get

$$\left(\int_{\mathbb{G}}|x|^{\alpha q}\left(\int_{\mathbb{G}}|y^{-1}x|^{\lambda}u(y)dy\right)^{q}dx\right)^{\frac{1}{q}} \overset{q<0}{\geq} \left(\int_{\mathbb{G}}|x|^{\alpha q}\left(\int_{B\left(0,\frac{|x|}{2}\right)}|y^{-1}x|^{\lambda}u(y)dy\right)^{q}dx\right)^{\frac{1}{q}} := I_1^{\frac{1}{q}}. \quad (4.92)$$

Similarly with (4.92), we get

$$\left(\int_{\mathbb{G}}|x|^{\alpha q}\left(\int_{\mathbb{G}}|y^{-1}x|^{\lambda}u(y)dy\right)^{q}dx\right)^{\frac{1}{q}} \overset{q<0}{\geq} \left(\int_{\mathbb{G}}|x|^{\alpha q}\left(\int_{\mathbb{G}\setminus B(0,2|x|)}|y^{-1}x|^{\lambda}u(y)dy\right)^{q}dx\right)^{\frac{1}{q}} := I_2^{\frac{1}{q}}. \quad (4.93)$$

By summarising above facts, from (4.92)-(4.93), we have

$$\left(\int_{\mathbb{G}}|x|^{\alpha q}\left(\int_{\mathbb{G}}|y^{-1}x|^{\lambda}u(y)dy\right)^{q}dx\right)^{\frac{1}{q}} \geq \frac{I_1^{\frac{1}{q}}}{2}+\frac{I_2^{\frac{1}{q}}}{2}. \quad (4.94)$$

From now on, in view of Proposition 2.6, we can assume that our quasi-norm is actually a norm.

Step 1. Let us consider I_1. From Proposition 2.6 and the properties of the quasi-norm with $|y| \leq \frac{|x|}{2}$, we have

$$|x| = |x^{-1}| = |x^{-1}yy^{-1}| \leq |x^{-1}y| + |y^{-1}| = |y^{-1}x| + |y| \leq |y^{-1}x| + \frac{|x|}{2}. \quad (4.95)$$

For any $\lambda > 0$, we have

$$2^{-\lambda}|x|^{\lambda} \leq |y^{-1}x|^{\lambda}.$$

It means,

$$2^{-\lambda}\int_{B\left(0,\frac{|x|}{2}\right)}|x|^{\lambda}u(y)dy \leq \int_{B\left(0,\frac{|x|}{2}\right)}|y^{-1}x|^{\lambda}u(y)dy,$$

so that

$$\left(\int_{B\left(0,\frac{|x|}{2}\right)}|y^{-1}x|^{\lambda}u(y)dy\right)^{q} \leq 2^{-\lambda q}\left(\int_{B\left(0,\frac{|x|}{2}\right)}|x|^{\lambda}u(y)dy\right)^{q}.$$

Therefore, we have

$$\begin{aligned} I_1 &= \int_{\mathbb{G}}|x|^{\alpha q}\left(\int_{B\left(0,\frac{|x|}{2}\right)}|y^{-1}x|^{\lambda}u(y)dy\right)^{q}dx \\ &\leq 2^{-\lambda q}\int_{\mathbb{G}}|x|^{(\alpha+\lambda)q}\left(\int_{B\left(0,\frac{|x|}{2}\right)}u(y)dy\right)^{q}dx. \end{aligned}$$

Assume that $W(x)=|x|^{(\alpha+\lambda)q}$ and $U(y)=|y|^{-\beta p}$, if condition (4.10) in Theorem 4.4 is satisfied, then by (4.9) we have

$$I_1 \leq 2^{-\lambda q}\int_{\mathbb{G}}\left(\int_{B(0,\frac{|x|}{2})}u(y)dy\right)^q |x|^{(\alpha+\lambda)q}dx \leq C_1\||y|^{-\beta}u\|^q_{L^p(\mathbb{G})}.$$

Let us check condition (3.136). By assumption $\beta<-\frac{Q}{p'}$, we get

$$\frac{1}{p}+\frac{1}{q'}=\frac{\alpha+\beta+\lambda}{Q}+2<\frac{\alpha+\lambda}{Q}-\frac{1}{p'}+2,$$

that is, $\frac{Q+(\alpha+\lambda)q}{Qq}>0$, then $Q+(\alpha+\lambda)q<0$ and by using the polar decomposition (2.11):

$$\begin{aligned}\left(\int_{\mathbb{G}\backslash B(0,|x|)}W(y)dy\right)^{\frac{1}{q}} &= \left(\int_{\mathbb{G}\backslash B(0,|x|)}|y|^{(\alpha+\lambda)q}dy\right)^{\frac{1}{q}}\\ &= \left(\int_{|x|}^{\infty}\int_{\mathfrak{S}}r^{Q-1}r^{(\alpha+\lambda)q}drd\sigma(\omega)\right)^{\frac{1}{q}}\\ &= \left(|\mathfrak{S}|\int_{|x|}^{\infty}r^{Q-1+(\alpha+\lambda)q}dr\right)^{\frac{1}{q}}\\ &= \left(-\frac{|\mathfrak{S}|}{Q+(\alpha+\lambda)q}|x|^{Q+(\alpha+\lambda)q}\right)^{\frac{1}{q}}\\ &= \left(\frac{|\mathfrak{S}|}{|Q+(\alpha+\lambda)q|}\right)^{\frac{1}{q}}|x|^{\frac{Q+(\alpha+\lambda)q}{q}}.\end{aligned}$$

Since $\beta<-\frac{Q}{p'}$, we get

$$-\beta p(1-p')+Q>-\beta p(1-p')-\beta p'=0.$$

It means $-\beta p(1-p')+Q>0$. Let us consider

$$\begin{aligned}\left(\int_{B(0,|x|)}U^{1-p'}(y)dy\right)^{\frac{1}{p'}} &= \left(\int_{B(0,|x|)}|y|^{-\beta p(1-p')}dy\right)^{\frac{1}{p'}}\\ &= \left(\int_0^{|x|}\int_{\mathfrak{S}}r^{-\beta p(1-p')}r^{Q-1}drd\sigma(\omega)\right)^{\frac{1}{p'}}\\ &= \left(|\mathfrak{S}|\int_0^{|x|}r^{-\beta p(1-p')+Q-1}dr\right)^{\frac{1}{p'}} \qquad (4.96)\\ &= \left(\frac{|\mathfrak{S}|}{-\beta p(1-p')+Q}|x|^{-\beta p(1-p')+Q}\right)^{\frac{1}{p'}}\\ &= \left(\frac{|\mathfrak{S}|}{Q-\beta p(1-p')}\right)^{\frac{1}{p'}}|x|^{\frac{-\beta p(1-p')+Q}{p'}}.\end{aligned}$$

Hence, we get

$$\begin{aligned}
A_1 &= \inf_{x\neq a}\left(\int_{\mathbb{G}\setminus B(0,|x|)} W(y)dx\right)^{\frac{1}{q}}\left(\int_{B(0,|x|)} U^{1-p'}(y)dy\right)^{\frac{1}{p'}}\\
&= \left(\frac{|\mathfrak{S}|}{|Q+(\alpha+\lambda)q|}\right)^{\frac{1}{q}}\left(\frac{|\mathfrak{S}|}{Q-\beta p(1-p')}\right)^{\frac{1}{p'}}\inf_{x\neq a}|x|^{\frac{(\alpha+\lambda)q+Q}{q}+\frac{-\beta p(1-p')+Q}{p'}}\\
&= \left(\frac{|\mathfrak{S}|}{|Q+(\alpha+\lambda)q|}\right)^{\frac{1}{q}}\left(\frac{|\mathfrak{S}|}{Q-\beta p(1-p')}\right)^{\frac{1}{p'}}\inf_{x\neq a}|x|^{Q\left(\frac{1}{q}+\frac{1}{p'}+\frac{\alpha+\beta+\lambda}{Q}\right)} \qquad (4.97)\\
&= \left(\frac{|\mathfrak{S}|}{|Q+(\alpha+\lambda)q|}\right)^{\frac{1}{q}}\left(\frac{|\mathfrak{S}|}{Q-\beta p(1-p')}\right)^{\frac{1}{p'}}\inf_{x\neq a}|x|^{Q\left(2-\frac{1}{q'}-\frac{1}{p}+\frac{\alpha+\beta+\lambda}{Q}\right)}\\
&= \left(\frac{|\mathfrak{S}|}{|Q+(\alpha+\lambda)q|}\right)^{\frac{1}{q}}\left(\frac{|\mathfrak{S}|}{Q-\beta p(1-p')}\right)^{\frac{1}{p'}} > 0.
\end{aligned}$$

From (4.9), we have

$$I_1 \leq 2^{-\lambda q}\int_{\mathbb{G}}|x|^{(\alpha+\lambda)q}\left(\int_{B\left(0,\frac{|x|}{2}\right)} u(y)dy\right)^q dx \leq 2^{-\lambda q}C_1^q\||y|^{-\beta}u\|^q_{L^p(\mathbb{G})}, \tag{4.98}$$

so that

$$I_1^{\frac{1}{q}} \geq 2^{-\lambda}C_1\||y|^{-\beta}u\|_{L^p(\mathbb{G})} = 2^{-\lambda}C_1\|h\|_{L^p(\mathbb{G})}. \tag{4.99}$$

Step 2. As in the previous case I_1, now we consider I_2. From $2|x| \leq |y|$, we calculate

$$|y| = |y^{-1}| = |y^{-1}xx^{-1}| \leq |y^{-1}x| + |x| \leq |y^{-1}x| + \frac{|y|}{2},$$

that is,

$$\frac{|y|}{2} \leq |y^{-1}x|.$$

Assume that $W(x) = |x|^{\alpha q}$ and $U(y) = |y|^{-(\beta+\lambda)p}$ and if condition (4.26) is satisfied, then we have

$$\begin{aligned}
I_2 &= \int_{\mathbb{G}}\left(\int_{\mathbb{G}\setminus B(0,2|x|)} |x|^{\alpha}|y^{-1}x|^{\lambda}u(y)dy\right)^q dx\\
&\leq 2^{-\lambda q}\int_{\mathbb{G}}|x|^{\alpha q}\left(\int_{\mathbb{G}\setminus B(0,2|x|)} u(y)|y|^{\lambda}dy\right)^q dx \leq 2^{-\lambda q}\||y|^{-\beta}u\|^q_{L^p(\mathbb{G})}.
\end{aligned}$$

Let us verify condition (4.26). Then we get

$$\begin{aligned}
\left(\int_{B(0,|x|)} W(y)dy\right)^{\frac{1}{q}} &= \left(\int_{B(0,|x|)} |y|^{\alpha q}dy\right)^{\frac{1}{q}}\\
&= \left(\int_0^{|x|}\int_{\mathfrak{S}} r^{\alpha q}r^{Q-1}drd\sigma(\omega)\right)^{\frac{1}{q}}\\
&= \left(\frac{|\mathfrak{S}|}{Q+\alpha q}\right)^{\frac{1}{q}}|x|^{\frac{Q+\alpha q}{q}},
\end{aligned}$$

where $Q+\alpha q>0$. By using $\alpha<-\frac{Q}{q}$, we get

$$\frac{1}{q'}+\frac{1}{p}=\frac{\alpha+\beta+\lambda}{Q}+2<-\frac{1}{q}+\frac{\beta+\lambda}{Q}+2=\frac{\beta+\lambda}{Q}+1+\frac{1}{q'},$$

then

$$(\beta+\lambda)p'+Q<0. \tag{4.100}$$

By using this fact, we have

$$\begin{aligned}
\left(\int_{\mathbb{G}\backslash B(0,|x|)} U^{1-p'}(y)dy\right)^{\frac{1}{p'}} &= \left(\int_{\mathbb{G}\backslash B(0,|x|)} |y|^{-(\beta+\lambda)(1-p')p}dy\right)^{\frac{1}{p'}}\\
&= \left(\int_{|x|}^{\infty}\int_{\mathfrak{S}} r^{Q-1}r^{-(\beta+\lambda)(1-p')p}drd\sigma(\omega)\right)^{\frac{1}{p'}}\\
&= \left(|\mathfrak{S}|\int_{|x|}^{\infty} r^{-(\beta+\lambda)(1-p')p+Q-1}dr\right)^{\frac{1}{p'}}\\
&\overset{(4.100)}{=} \left(-\frac{|\mathfrak{S}|}{Q-(\beta+\lambda)(1-p')p}|x|^{Q-(\beta+\lambda)(1-p')p}\right)^{\frac{1}{p'}}\\
&= \left(-\frac{|\mathfrak{S}|}{Q+(\beta+\lambda)p'}|x|^{Q+(\beta+\lambda)p'}\right)^{\frac{1}{p'}}\\
&= \left(\frac{|\mathfrak{S}|}{|Q+(\beta+\lambda)p'|}\right)^{\frac{1}{p'}}|x|^{\frac{Q+(\beta+\lambda)p'}{p'}}.
\end{aligned}$$

Combining these facts we have

$$\begin{aligned}
A_2 &= \inf_{x\neq a}\left(\int_{B(0,|x|)} W(y)dx\right)^{\frac{1}{q}}\left(\int_{\mathbb{G}\backslash B(0,|x|)} U^{1-p'}(y)dx\right)^{\frac{1}{p'}}\\
&= \left(\frac{|\mathfrak{S}|}{Q+\alpha q}\right)^{\frac{1}{q}}\left(\frac{|\mathfrak{S}|}{|Q+(\beta+\lambda)p'|}\right)^{\frac{1}{p'}}\inf_{x\neq a}|x|^{\frac{Q+\alpha q}{q}+\frac{Q+(\beta+\lambda)p'}{p'}}\\
&= \left(\frac{|\mathfrak{S}|}{Q+\alpha q}\right)^{\frac{1}{q}}\left(\frac{|\mathfrak{S}|}{|Q+(\beta+\lambda)p'|}\right)^{\frac{1}{p'}}\inf_{x\neq a}|x|^{\frac{Q}{q}+\alpha+\frac{Q}{p'}+\beta+\lambda}\\
&= \left(\frac{|\mathfrak{S}|}{Q+\alpha q}\right)^{\frac{1}{q}}\left(\frac{|\mathfrak{S}|}{|Q+(\beta+\lambda)p'|}\right)^{\frac{1}{p'}}\inf_{x\neq a}|x|^{Q\left(\frac{1}{q}+\frac{1}{p'}+\frac{\alpha+\beta+\lambda}{Q}\right)}\\
&= \left(\frac{|\mathfrak{S}|}{Q+\alpha q}\right)^{\frac{1}{q}}\left(\frac{|\mathfrak{S}|}{|Q+(\beta+\lambda)p'|}\right)^{\frac{1}{p'}}\inf_{x\neq a}|x|^{Q\left(2-\frac{1}{q'}-\frac{1}{p}+\frac{\alpha+\beta+\lambda}{Q}\right)}\\
&= \left(\frac{|\mathfrak{S}|}{Q+\alpha q}\right)^{\frac{1}{q}}\left(\frac{|\mathfrak{S}|}{|Q+(\beta+\lambda)p'|}\right)^{\frac{1}{p'}}>0.
\end{aligned} \tag{4.101}$$

Hence, we have

$$I_2=\int_{\mathbb{G}}\left(\int_{\mathbb{G}\backslash B(0,2|x|)}|x|^{\alpha}u(y)|y^{-1}x|^{\lambda}dy\right)^{q}dx\leq 2^{-\lambda q}C_2^{q}\||y|^{-\beta}u\|^{q}_{L^p(\mathbb{G})}.$$

Then, we have

$$I_2^{\frac{1}{q}} \geq 2^{-\lambda} C_2 \||y|^{-\beta} u\|_{L^p(\mathbb{G})} = 2^{-\lambda} C_2 \|h\|_{L^p(\mathbb{G})}. \tag{4.102}$$

Finally, from (4.99) and (4.102) in (4.94), we obtain

$$\begin{aligned}\left(\int_{\mathbb{G}} |x|^{\alpha q} \left(\int_{\mathbb{G}} |y^{-1}x|^{\lambda} u(y) dy\right)^q dx\right)^{\frac{1}{q}} &\geq \frac{I_1^{\frac{1}{q}}}{2} + \frac{I_2^{\frac{1}{q}}}{2} \\ &\geq \frac{2^{-\lambda}(C_1 + C_2)}{2} \||y|^{-\beta} u\|_{L^p(\mathbb{G})} \\ &= \frac{2^{-\lambda}(C_1 + C_2)}{2} \||y|^{-\beta} u\|_{L^p(\mathbb{G})} \\ &= C_3 \||y|^{-\beta} u\|_{L^p(\mathbb{G})},\end{aligned} \tag{4.103}$$

where $C_3 = \frac{2^{-\lambda}(C_1+C_2)}{2} > 0$.

Theorem 4.15 is proved. □

Corollary 4.16. *By setting* $\alpha = \beta = 0$ *we get the reverse Hardy-Littlewood-Sobolev inequality on the homogeneous groups, in the following form:*

$$\int_{\mathbb{G}} \int_{\mathbb{G}} |y^{-1}x|^{\lambda} f(x) h(y) dx dy \geq C \|f\|_{L^{q'}(\mathbb{G})} \|h\|_{L^p(\mathbb{G})}, \tag{4.104}$$

for all non-negative functions $f \in L^{q'}(\mathbb{G})$ *and* $h \in L^p(\mathbb{G})$ *with* $\lambda > 0$, $p, q' \in (0,1)$, $\frac{1}{q'} + \frac{1}{p} = \frac{\lambda}{Q} + 2$, *where* $\frac{1}{p} + \frac{1}{p'} = 1$ *and* $\frac{1}{q} + \frac{1}{q'} = 1$.

Remark 4.17. *In the Abelian (Euclidean) case* $\mathbb{G} = (\mathbb{R}^N, +)$, *hence* $Q = N$ *and* $|\cdot|$ *can be any homogeneous quasi-norm on* $\mathbb{R}^N$, *in particular with the usual Euclidean distance, i.e.* $|\cdot| = \|\cdot\|_E$, *this was investigated in* [54].

Let us give improved reverse Stein-Weiss inequality.

Theorem 4.18. *Let* $\mathbb{G}$ *be a homogeneous group of homogeneous dimension* $Q \geq 1$ *and let* $|\cdot|$ *be an arbitrary homogeneous quasi-norm on* $\mathbb{G}$. *Suppose that* $\lambda > 0$, $p, q' \in (0,1)$ *and* $\frac{1}{q'} + \frac{1}{p} = \frac{\alpha+\beta+\lambda}{Q} + 2$, *where* $\frac{1}{p} + \frac{1}{p'} = 1$, $\frac{1}{q} + \frac{1}{q'} = 1$. *Then for all non-negative functions* $f \in L^{q'}(\mathbb{G})$ *and* $h \in L^p(\mathbb{G})$, *inequality (4.91) holds, that is,*

$$\int_{\mathbb{G}} \int_{\mathbb{G}} |x|^{\alpha} |y^{-1}x|^{\lambda} f(x) h(y) |y|^{\beta} dx dy \geq C \|f\|_{L^{q'}(\mathbb{G})} \|h\|_{L^p(\mathbb{G})},$$

if one of the following conditions is satisfied:

(a) $0 \leq \alpha < -\frac{Q}{q}$.

(b) $0 \leq \beta < -\frac{Q}{p'}$.

Proof. Firstly, let us show (a). By using some notations from proof of Theorem 4.15 and (4.94), we get

$$\left(\int_{\mathbb{G}} |x|^{\alpha q} \left(\int_{\mathbb{G}} |y^{-1}x|^{\lambda} u(y) dy\right)^q dx\right)^{\frac{1}{q}} \geq I_2^{\frac{1}{q}}, \tag{4.105}$$

and from Step 2 in the proof of Theorem 4.15 and by using (4.102), we get $I_2^{\frac{1}{q}} \geq C\||y|^{-\beta}u\|_{L^p(\mathbb{G})}$, then we get

$$\left(\int_{\mathbb{G}} |x|^{\alpha q}\left(\int_{\mathbb{G}} |y^{-1}x|^{\lambda}u(y)dy\right)^q dx\right)^{\frac{1}{q}} \geq I_2^{\frac{1}{q}} \geq C\||y|^{-\beta}u\|_{L^p(\mathbb{G})}. \tag{4.106}$$

Let us show (b). From(4.94), we get

$$\left(\int_{\mathbb{G}} |x|^{\alpha q}\left(\int_{\mathbb{G}} |y^{-1}x|^{\lambda}u(y)dy\right)^q dx\right)^{\frac{1}{q}} \geq I_1^{\frac{1}{q}}, \tag{4.107}$$

and from Step 1 in the proof of Theorem 4.15 and by using (4.92), we get $I_1^{\frac{1}{q}} \geq C\||y|^{-\beta}u\|_{L^p(\mathbb{G})}$, then we have

$$\left(\int_{\mathbb{G}} |x|^{\alpha q}\left(\int_{\mathbb{G}} |y^{-1}x|^{\lambda}u(y)dy\right)^q dx\right)^{\frac{1}{q}} \geq I_1^{\frac{1}{q}} \geq C\||y|^{-\beta}u\|_{L^p(\mathbb{G})}. \tag{4.108}$$

□

4.8. **Reverse Hardy-Littlewood-Sobolev inequality with $-\infty < q < p < 0$ on the homogeneous Lie groups.** In this section, we prove the reverse Hardy-Littlewood-Sobolev inequality and Stein-Weiss type inequality with $-\infty < q < p < 0$ on homogeneous Lie groups.

Let us present one of the main results of this section.

Theorem 4.19 (Reverse Hardy-Littlewood-Sobolev inequality)**.** *Let $\mathbb{G}$ be a homogeneous Lie group of homogeneous dimension $Q \geq 1$ with a quasi-norm $|\cdot|$. Assume that $q < p < 0$, $\lambda < 0$ and $\frac{1}{p'} + \frac{1}{q} + \frac{\lambda}{Q} = 0$, where $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{q} + \frac{1}{q'} = 1$. Then for all non-negative functions $f \in L^{q'}(\mathbb{G})$ and $0 < \int_{\mathbb{G}} h^p(x)dx < \infty$,*

$$\int_{\mathbb{G}}\int_{\mathbb{G}} f(x)|y^{-1}x|^{\lambda}h(y)dxdy \geq C\left(\int_{\mathbb{G}} f^{q'}(x)dx\right)^{\frac{1}{q'}}\left(\int_{\mathbb{G}} h^p(x)dx\right)^{\frac{1}{p}}, \tag{4.109}$$

where C is a positive constant independent of f and h.

Proof. By using reverse Hölder's inequality with $\frac{1}{q} + \frac{1}{q'} = 1$, we get

$$\begin{aligned}\int_{\mathbb{G}}\int_{\mathbb{G}} f(x)|y^{-1}x|^{\lambda}h(y)dydx &= \int_{\mathbb{G}}\left(\int_{\mathbb{G}} |y^{-1}x|^{\lambda}h(y)dy\right)f(x)dx \\ &\geq \left(\int_{\mathbb{G}}\left(\int_{\mathbb{G}} |y^{-1}x|^{\lambda}h(y)dy\right)^q dx\right)^{\frac{1}{q}}\|f\|_{L^{q'}(\mathbb{G})}.\end{aligned}$$

For (4.109), it is enough to show that

$$\left(\int_{\mathbb{G}}\left(\int_{\mathbb{G}} |y^{-1}x|^{\lambda}h(y)dy\right)^q dx\right)^{\frac{1}{q}} \geq C\left(\int_{\mathbb{G}} h^p(x)dx\right)^{\frac{1}{p}}.$$

We have that

$$\int_{\mathbb{G}} |y^{-1}x|^{\lambda}h(y)dy \geq \int_{B\left(0,\frac{|x|}{2}\right)} |x|^{\alpha}|y^{-1}x|^{\lambda}h(y)dy,$$

then

$$\left(\int_{\mathbb{G}}|y^{-1}x|^{\lambda}h(y)dy\right)^{q}\overset{q<0}{\leq}\left(\int_{B\left(0,\frac{|x|}{2}\right)}|y^{-1}x|^{\lambda}h(y)dy\right)^{q}.$$

Therefore, we obtain

$$\left(\int_{\mathbb{G}}\left(\int_{\mathbb{G}}|y^{-1}x|^{\lambda}h(y)dy\right)^{q}dx\right)^{\frac{1}{q}}\overset{q<0}{\geq}\left(\int_{\mathbb{G}}\left(\int_{B\left(0,\frac{|x|}{2}\right)}|y^{-1}x|^{\lambda}h(y)dy\right)^{q}dx\right)^{\frac{1}{q}}. \quad (4.110)$$

By using Proposition 2.7 with $|y|\leq\frac{|x|}{2}$, we get

$$|y^{-1}x|\overset{(2.10)}{\leq}C(|x|+|y|)\leq\frac{3C}{2}|x|=C_1|x|, \quad (4.111)$$

where $C>0$ and $C_1=\frac{3C}{2}$. Then for any $\lambda<0$, we have

$$C_1^{\lambda}|x|^{\lambda}\leq|y^{-1}x|^{\lambda}.$$

It means,

$$C_1^{\lambda}\int_{B\left(0,\frac{|x|}{2}\right)}|x|^{\lambda}h(y)dy\leq\int_{B\left(0,\frac{|x|}{2}\right)}|y^{-1}x|^{\lambda}h(y)dy,$$

so that

$$\left(\int_{B\left(0,\frac{|x|}{2}\right)}|y^{-1}x|^{\lambda}h(y)dy\right)^{q}\leq C_1^{\lambda q}\left(\int_{B\left(0,\frac{|x|}{2}\right)}|x|^{\lambda}h(y)dy\right)^{q}.$$

Finally,

$$\begin{aligned}\left(\int_{\mathbb{G}}\left(\int_{\mathbb{G}}|y^{-1}x|^{\lambda}h(y)dy\right)^{q}dx\right)^{\frac{1}{q}}&\overset{q<0}{\geq}\left(\int_{\mathbb{G}}\left(\int_{B\left(0,\frac{|x|}{2}\right)}|y^{-1}x|^{\lambda}h(y)dy\right)^{q}dx\right)^{\frac{1}{q}}\\&\geq C_1^{\lambda}\left(\int_{\mathbb{G}}|x|^{\lambda q}\left(\int_{B\left(0,\frac{|x|}{2}\right)}h(y)dy\right)^{q}dx\right)^{\frac{1}{q}}.\end{aligned} \quad (4.112)$$

If condition (4.36) in Theorem 4.6 with $u(x)=|x|^{\lambda q}$ and $v(x)=1$ in (4.35) is satisfied, then we have

$$\left(\int_{\mathbb{G}}|x|^{\lambda q}\left(\int_{B\left(0,\frac{|x|}{2}\right)}h(y)dy\right)^{q}dx\right)^{\frac{1}{q}}\geq C\left(\int_{\mathbb{G}}h^{p}(x)dx\right)^{\frac{1}{p}}.$$

Let us start to check condition (4.36). From assumption, we have

$$0=\frac{1}{p'}+\frac{1}{q}+\frac{\lambda}{Q}\overset{\frac{1}{p'}>0}{>}\frac{1}{q}+\frac{\lambda}{Q}, \quad (4.113)$$

it means $Q+\lambda q>0$. By using this fact, we obtain

$$\begin{aligned}\int_{B\left(0,\frac{|x|}{2}\right)}u(y)dy&=\int_{B\left(0,\frac{|x|}{2}\right)}|y|^{\lambda q}dy\overset{(2.11)}{=}\int_{0}^{\frac{|x|}{2}}\int_{\mathfrak{S}}r^{\lambda q}r^{Q-1}drd\sigma\\&=|\mathfrak{S}|\int_{0}^{\frac{|x|}{2}}r^{Q+\lambda q}dr\overset{Q+\lambda q>0}{=}\frac{|\mathfrak{S}|}{2^{Q+\lambda q}(Q+\lambda q)}|x|^{Q+\lambda q},\end{aligned} \quad (4.114)$$

and

$$\int_{B\left(0,\frac{|x|}{2}\right)} v^{1-p'}(y)dy=\int_{B\left(0,\frac{|x|}{2}\right)} 1dy=\int_0^{\frac{|x|}{2}}\int_{\mathfrak{S}} r^{Q-1}drd\sigma=\frac{|\mathfrak{S}|}{2^Q}|x|^Q. \tag{4.115}$$

Finally, by using assumption $\frac{1}{p'}+\frac{1}{q}+\frac{\lambda}{Q}=0$,

$$\mathcal{D}_1(|x|)=\left(\frac{|\mathfrak{S}|}{2^{Q+\lambda q}(Q+\lambda q)}\right)^{\frac{1}{q}}\left(\frac{|\mathfrak{S}|}{2^Q}\right)^{\frac{1}{p'}}|x|^{\frac{Q}{p'}+\frac{Q+\lambda q}{q}}=\left(\frac{|\mathfrak{S}|}{2^{Q+\lambda q}(Q+\lambda q)}\right)^{\frac{1}{q}}\left(\frac{|\mathfrak{S}|}{2^Q}\right)^{\frac{1}{p'}}, \tag{4.116}$$

it means, $\mathcal{D}_1(|x|)$ is a non-decreasing function. Then,

$$D_1=\inf_{x\neq a}\mathcal{D}_1(|x|)=\left(\frac{|\mathfrak{S}|}{2^{Q+\lambda q}(Q+\lambda q)}\right)^{\frac{1}{q}}\left(\frac{|\mathfrak{S}|}{2^Q}\right)^{\frac{1}{p'}}>0.$$

□

Remark 4.20. *Inequality (4.109) is an even new in the Abelian (Euclidean) case* $\mathbb{G}=(\mathbb{R}^n,+)$, $Q=n$ *and* $|\cdot|=|\cdot|_E$ *(*$|\cdot|_E$ *is the Euclidean distance).*

4.9. **Reverse Stein-Weiss type inequality with $-\infty<q\leq p<0$ on the homogeneous Lie groups.** Let us show, the reverse Stein-Weiss type inequality on homogeneous Lie groups.

Theorem 4.21. *Let* $\mathbb{G}$ *be a homogeneous Lie group of homogeneous dimension* $Q\geq 1$ *with any quasi-norm* $|\cdot|$. *Assume that* $q\leq p<0$, $\lambda<0$, $\beta>-\frac{Q}{p'}$, $\alpha>-\frac{Q}{q}$ *and* $\frac{1}{p'}+\frac{1}{q}+\frac{\alpha+\beta+\lambda}{Q}=0$, *where* $\frac{1}{p}+\frac{1}{p'}=1$ *and* $\frac{1}{q}+\frac{1}{q'}=1$. *Then for all non-negative functions* $f\in L^{q'}(\mathbb{G})$ *and* $0<\int_{\mathbb{G}}h^p(x)dx<\infty$,

$$\int_{\mathbb{G}}\int_{\mathbb{G}}|x|^{\alpha}f(x)|y^{-1}x|^{\lambda}h(y)|y|^{\beta}dxdy\geq C\left(\int_{\mathbb{G}}f^{q'}(x)dx\right)^{\frac{1}{q'}}\left(\int_{\mathbb{G}}h^p(x)dx\right)^{\frac{1}{p}}, \tag{4.117}$$

where C *is a positive constant independent of* f *and* h.

Proof. Similarly to Theorem 4.15, we need to show

$$\int_{\mathbb{G}}\left(\int_{\mathbb{G}}|x|^{\alpha}|y^{-1}x|^{\lambda}u(y)dy\right)^q dx\leq C\left(\int_{\mathbb{G}}|y|^{-\beta p}u^p(x)dx\right)^{\frac{q}{p}},$$

where $u(y)=h(y)|y|^{\beta}$. We have that

$$\int_{\mathbb{G}}|x|^{\alpha}|y^{-1}x|^{\lambda}u(y)dy\geq\int_{B\left(0,\frac{|x|}{2}\right)}|x|^{\alpha}|y^{-1}x|^{\lambda}u(y)dy,$$

then

$$\left(\int_{\mathbb{G}}|x|^{\alpha}|y^{-1}x|^{\lambda}u(y)dy\right)^q\overset{q<0}{\leq}\left(\int_{B\left(0,\frac{|x|}{2}\right)}|x|^{\alpha}|y^{-1}x|^{\lambda}u(y)dy\right)^q.$$

Hence, we obtain

$$\left(\int_{\mathbb{G}}|x|^{\alpha q}\left(\int_{\mathbb{G}}|y^{-1}x|^{\lambda}u(y)dy\right)^{q}dx\right)^{\frac{1}{q}}$$
$$\overset{q<0}{\geq}\left(\int_{\mathbb{G}}|x|^{\alpha q}\left(\int_{B\left(0,\frac{|x|}{2}\right)}|y^{-1}x|^{\lambda}u(y)dy\right)^{q}dx\right)^{\frac{1}{q}}:=I_1^{\frac{1}{q}}. \quad (4.118)$$

Similarly to (4.118), we have

$$\left(\int_{\mathbb{G}}|x|^{\alpha q}\left(\int_{\mathbb{G}}|y^{-1}x|^{\lambda}u(y)dy\right)^{q}dx\right)^{\frac{1}{q}}$$
$$\overset{q<0}{\geq}\left(\int_{\mathbb{G}}|x|^{\alpha q}\left(\int_{\mathbb{G}\setminus B(0,2|x|)}|y^{-1}x|^{\lambda}u(y)dy\right)^{q}dx\right)^{\frac{1}{q}}:=I_2^{\frac{1}{q}}. \quad (4.119)$$

By using (4.118)-(4.119), we get

$$\left(\int_{\mathbb{G}}|x|^{\alpha q}\left(\int_{\mathbb{G}}|y^{-1}x|^{\lambda}u(y)dy\right)^{q}dx\right)^{\frac{1}{q}}\geq\frac{I_1^{\frac{1}{q}}}{2}+\frac{I_2^{\frac{1}{q}}}{2}. \quad (4.120)$$

From now on, in view of Proposition 2.7, we can assume that our quasi-norm is actually a norm.

Step 1. Let us consider I_1. By using Proposition 2.7 with $|y|\leq\frac{|x|}{2}$, we get

$$|y^{-1}x|\overset{(2.10)}{\leq}C(|x|+|y|)\leq\frac{3C}{2}|x|=C_1|x|, \quad (4.121)$$

where $C>0$ and $C_1=\frac{3C}{2}$. Then for any $\lambda<0$, we have

$$C_1^{\lambda}|x|^{\lambda}\leq|y^{-1}x|^{\lambda}.$$

Therefore, we get

$$I_1=\int_{\mathbb{G}}|x|^{\alpha q}\left(\int_{B\left(0,\frac{|x|}{2}\right)}|y^{-1}x|^{\lambda}u(y)dy\right)^{q}dx\leq C_1^{\lambda q}\int_{\mathbb{G}}|x|^{(\alpha+\lambda)q}\left(\int_{B\left(0,\frac{|x|}{2}\right)}u(y)dy\right)^{q}dx.$$

If condition (4.36) in Theorem 4.6 with $u(x)=|x|^{(\alpha+\lambda)q}$ and $v(y)=|y|^{-\beta p}$ in (4.35) is satisfied, then we have

$$I_1\leq C_1^{\lambda q}\int_{\mathbb{G}}\left(\int_{B\left(0,\frac{|x|}{2}\right)}u(y)dy\right)^{q}|x|^{(\alpha+\lambda)q}dx\leq C\left(\int_{\mathbb{G}}|y|^{-\beta p}u^{p}(y)dy\right)^{\frac{q}{p}}.$$

Let us verify condition (4.36). By using assumption $\beta>-\frac{Q}{p'}$, we obtain

$$0=\frac{1}{p'}+\frac{1}{q}+\frac{\alpha+\beta+\lambda}{Q}>\frac{1}{q}+\frac{\alpha+\lambda}{Q},$$

that is, $\frac{Q+(\alpha+\lambda)q}{Qq} < 0$, then $Q+(\alpha+\lambda)q > 0$ and by using the polar decomposition (2.11):

$$\begin{aligned}\left(\int_{B\left(0,\frac{|x|}{2}\right)} u(y)dy\right)^{\frac{1}{q}} &= \left(\int_{B\left(0,\frac{|x|}{2}\right)} |y|^{(\alpha+\lambda)q}dy\right)^{\frac{1}{q}} \\ &= \left(\int_0^{\frac{|x|}{2}}\int_{\mathfrak{S}} r^{Q-1}r^{(\alpha+\lambda)q}drd\sigma\right)^{\frac{1}{q}} \\ &= \left(|\mathfrak{S}|\int_0^{\frac{|x|}{2}} r^{Q-1+(\alpha+\lambda)q}dr\right)^{\frac{1}{q}} \\ &= \left(\frac{|\mathfrak{S}|}{2^{(\alpha+\lambda)q}(Q+(\alpha+\lambda)q)}|x|^{Q+(\alpha+\lambda)q}\right)^{\frac{1}{q}} \\ &= \left(\frac{|\mathfrak{S}|}{2^{(\alpha+\lambda)q}(Q+(\alpha+\lambda)q)}\right)^{\frac{1}{q}}|x|^{\frac{Q+(\alpha+\lambda)q}{q}}.\end{aligned}$$

Since $\beta > -\frac{Q}{p'}$, we have

$$-\beta p(1-p') + Q = \beta p' + Q > 0.$$

So, $-\beta p(1-p') + Q > 0$. Then, let us consider

$$\begin{aligned}\left(\int_{B\left(0,\frac{|x|}{2}\right)} v^{1-p'}(y)dy\right)^{\frac{1}{p'}} &= \left(\int_{B\left(0,\frac{|x|}{2}\right)} |y|^{-\beta p(1-p')}dy\right)^{\frac{1}{p'}} \\ &= \left(\int_{B\left(0,\frac{|x|}{2}\right)} |y|^{\beta p'}dy\right)^{\frac{1}{p'}} \\ &= \left(\int_0^{\frac{|x|}{2}}\int_{\mathfrak{S}} r^{-\beta p(1-p')}r^{Q-1}drd\sigma\right)^{\frac{1}{p'}} \\ &= \left(|\mathfrak{S}|\int_0^{\frac{|x|}{2}} r^{\beta p'+Q-1}dr\right)^{\frac{1}{p'}} \\ &= \left(\frac{|\mathfrak{S}|}{2^{\beta p'+Q}(\beta p'+Q)}|x|^{\beta p'+Q}\right)^{\frac{1}{p'}} \\ &= \left(\frac{|\mathfrak{S}|}{2^{\beta p'+Q}(\beta p'+Q)}\right)^{\frac{1}{p'}}|x|^{\frac{\beta p'+Q}{p'}}.\end{aligned} \tag{4.122}$$

Therefore by using $\frac{1}{p'}+\frac{1}{q}+\frac{\alpha+\beta+\lambda}{Q}=0$, we have

$$\begin{aligned}\mathcal{D}_1(|x|) &= \left(\int_{B\left(0,\frac{|x|}{2}\right)} u(y)dy\right)^{\frac{1}{q}} \left(\int_{B\left(0,\frac{|x|}{2}\right)} v^{1-p'}(y)dy\right)^{\frac{1}{p'}} \\ &= \left(\frac{|\mathfrak{S}|}{2^{(\alpha+\lambda)q}(Q+(\alpha+\lambda)q)}\right)^{\frac{1}{q}} \left(\frac{|\mathfrak{S}|}{2^{\beta p'+Q}(\beta p'+Q)}\right)^{\frac{1}{p'}} |x|^{\frac{(\alpha+\lambda)q+Q}{q}} |x|^{\frac{\beta p'+Q}{p'}} \\ &= \left(\frac{|\mathfrak{S}|}{2^{(\alpha+\lambda)q}(Q+(\alpha+\lambda)q)}\right)^{\frac{1}{q}} \left(\frac{|\mathfrak{S}|}{2^{\beta p'+Q}(\beta p'+Q)}\right)^{\frac{1}{p'}},\end{aligned} \tag{4.123}$$

it means $\mathcal{D}_1(|x|)$ is a non-decreasing function. Therefore,

$$D_1 = \inf_{x\neq a} \mathcal{D}_1(|x|) = \left(\frac{|\mathfrak{S}|}{2^{(\alpha+\lambda)q}(Q+(\alpha+\lambda)q)}\right)^{\frac{1}{q}} \left(\frac{|\mathfrak{S}|}{2^{\beta p'+Q}(\beta p'+Q)}\right)^{\frac{1}{p'}} > 0. \tag{4.124}$$

Then by using (4.35), we obtain

$$I_1 \leq 2^{-\lambda q} \int_{\mathbb{G}} |x|^{(\alpha+\lambda)q} \left(\int_{B\left(0,\frac{|x|}{2}\right)} u(y)dy\right)^{q} dx \leq C \left(\int_{\mathbb{G}} |y|^{-\beta p} u^p(y)dy\right)^{\frac{q}{p}}, \tag{4.125}$$

so that

$$I_1^{\frac{1}{q}} \geq C \left(\int_{\mathbb{G}} |y|^{-\beta p} u^p(y)dy\right)^{\frac{1}{p}} = C \left(\int_{\mathbb{G}} h^p(y)dy\right)^{\frac{1}{p}}. \tag{4.126}$$

Step 2. As in the previous case I_1, now we consider I_2. From $2|x| \leq |y|$, we calculate

$$|y^{-1}x| \overset{(2.10)}{\leq} C(|x|+|y|) \leq \frac{3C}{2}|y| = C_1|y|,$$

then

$$|y^{-1}x|^{\lambda} \geq C|y|^{\lambda},$$

where $C>0$. Then, if condition (4.54) with $u(x)=|x|^{\alpha q}$ and $v(y)=|y|^{-(\beta+\lambda)p}$ is satisfied, then we have

$$\begin{aligned}I_2 &= \int_{\mathbb{G}} \left(\int_{\mathbb{G}\setminus B(0,2|x|)} |x|^{\alpha}|y^{-1}x|^{\lambda}u(y)dy\right)^{q} dx \\ &\leq C\int_{\mathbb{G}} |x|^{\alpha q} \left(\int_{\mathbb{G}\setminus B(0,2|x|)} u(y)|y|^{\lambda}dy\right)^{q} dx \leq C\left(\int_{\mathbb{G}} |y|^{-\beta p}u^p(y)dy\right)^{\frac{q}{p}}.\end{aligned}$$

Now let us check condition (4.54). We have

$$\begin{aligned}\left(\int_{\mathbb{G}\setminus B(0,2|x|)} u(y)dy\right)^{\frac{1}{q}} &= \left(\int_{\mathbb{G}\setminus B(0,2|x|)} |y|^{\alpha q}dy\right)^{\frac{1}{q}} \\ &= \left(\int_{2|x|}^{\infty}\int_{\mathfrak{S}} r^{\alpha q}r^{Q-1}drd\sigma\right)^{\frac{1}{q}} \\ &= \left(\frac{2|\mathfrak{S}|}{|Q+\alpha q|}\right)^{\frac{1}{q}} |x|^{\frac{Q+\alpha q}{q}},\end{aligned}$$

where $Q+\alpha q<0$. From $\alpha>-\frac{Q}{q}$, we have

$$0=\frac{1}{p'}+\frac{1}{q}+\frac{\alpha+\beta+\lambda}{Q}>\frac{1}{p'}+\frac{\beta+\lambda}{Q},$$

then

$$(\beta+\lambda)p'+Q<0. \tag{4.127}$$

By using this fact, we have

$$\begin{aligned}\left(\int_{\mathbb{G}\setminus B(0,2|x|)} v^{1-p'}(y)dy\right)^{\frac{1}{p'}} &= \left(\int_{\mathbb{G}\setminus B(0,2|x|)} |y|^{-(\beta+\lambda)(1-p')p}dy\right)^{\frac{1}{p'}}\\ &= \left(\frac{2|\mathfrak{S}|}{|Q+(\beta+\lambda)p'|}\right)^{\frac{1}{p'}}|x|^{\frac{Q+(\beta+\lambda)p'}{p'}}.\end{aligned}$$

Then by using $\frac{1}{p'}+\frac{1}{q}+\frac{\alpha+\beta+\lambda}{Q}=0$, we get

$$\begin{aligned}\mathcal{D}_2(|x|) &= \left(\frac{2|\mathfrak{S}|}{|Q+\alpha q|}\right)^{\frac{1}{q}}\left(\frac{2|\mathfrak{S}|}{|Q+(\beta+\lambda)p'|}\right)^{\frac{1}{p'}}|x|^{\frac{Q+\alpha q}{q}}|x|^{\frac{Q+(\beta+\lambda)p'}{p'}}\\ &= \left(\frac{2|\mathfrak{S}|}{|Q+\alpha q|}\right)^{\frac{1}{q}}\left(\frac{2|\mathfrak{S}|}{|Q+(\beta+\lambda)p'|}\right)^{\frac{1}{p'}},\end{aligned} \tag{4.128}$$

it means $\mathcal{D}_2(|x|)$ is a non-increasing function. Therefore, we have

$$\begin{aligned}D_2 = \inf_{x\neq a}\mathcal{D}_2(|x|) &= \inf_{x\neq a}\left(\int_{\mathbb{G}\setminus B(0,2|x|)} u(y)dy\right)^{\frac{1}{q}}\left(\int_{\mathbb{G}\setminus B(0,2|x|)} v^{1-p'}(y)dy\right)^{\frac{1}{p'}}\\ &= \left(\frac{2|\mathfrak{S}|}{|Q+\alpha q|}\right)^{\frac{1}{q}}\left(\frac{2|\mathfrak{S}|}{|Q+(\beta+\lambda)p'|}\right)^{\frac{1}{p'}}>0.\end{aligned} \tag{4.129}$$

Therefore, we have

$$I_2=\int_{\mathbb{G}}\left(\int_{\mathbb{G}\setminus B(0,2|x|)}|x|^{\alpha}u(y)|y^{-1}x|^{\lambda}dy\right)^{q}dx\leq 2^{-\lambda q}C_2^{q}\left(\int_{\mathbb{G}}|y|^{-\beta p}u^{p}(y)dy\right)^{\frac{q}{p}}.$$

Then, we have

$$I_2^{\frac{1}{q}}\geq 2^{-\lambda}C_2\left(\int_{\mathbb{G}}|y|^{-\beta p}u^{p}(y)dy\right)^{\frac{1}{p}}=2^{-\lambda}C_2\left(\int_{\mathbb{G}}h^{p}(y)dy\right)^{\frac{1}{p}}. \tag{4.130}$$

Finally, by using (4.126) and (4.130) in (4.120), we obtain

$$\begin{aligned}\left(\int_{\mathbb{G}}|x|^{\alpha q}\left(\int_{\mathbb{G}}|y^{-1}x|^{\lambda}|y|^{\beta}h(y)dy\right)^{q}dx\right)^{\frac{1}{q}}&=\left(\int_{\mathbb{G}}|x|^{\alpha q}\left(\int_{\mathbb{G}}|y^{-1}x|^{\lambda}u(y)dy\right)^{q}dx\right)^{\frac{1}{q}}\\&\geq\frac{I_1^{\frac{1}{q}}}{2}+\frac{I_2^{\frac{1}{q}}}{2}\\&\geq C\left(\int_{\mathbb{G}}|y|^{-\beta p}u^{p}(y)dy\right)^{\frac{1}{p}}\\&=C\left(\int_{\mathbb{G}}h^{p}(y)dy\right)^{\frac{1}{p}}.\end{aligned} \tag{4.131}$$

□

Remark 4.22. *Inequality (4.117) is an even new in the Abelian (Euclidean) case* $\mathbb{G}=(\mathbb{R}^N,+)$*,* $Q=N$ *and* $|\cdot|=|\cdot|_E$ *(*$|\cdot|_E$ *is the Euclidean distance).*

Remark 4.23. *Particularly, from* $\alpha>-\frac{Q}{q}>0$ *we can not obtain the reverse Hardy-Littlewood-Sobolev in (4.117).*

4.10. **Improved reverse Stein-Weiss type inequality with** $-\infty<q\leq p<0$**.** Let us present the improved reverse Stein-Weiss type inequality on homogeneous Lie groups.

Theorem 4.24. *Let* $\mathbb{G}$ *be a homogeneous group of homogeneous dimension* $Q\geq 1$ *and let* $|\cdot|$ *be an arbitrary homogeneous quasi-norm on* $\mathbb{G}$*. Assume that* $q\leq p<0$*,* $\lambda<0$*, and* $\frac{1}{p'}+\frac{1}{q}+\frac{\alpha+\beta+\lambda}{Q}=0$*, where* $\frac{1}{p}+\frac{1}{p'}=1$ *and* $\frac{1}{q}+\frac{1}{q'}=1$*. Then for all non-negative functions* $f\in L^{q'}(\mathbb{G})$ *and* $0<\int_{\mathbb{G}}h^p(x)dx<\infty$*, (4.117) holds, that is,*

$$\int_{\mathbb{G}}\int_{\mathbb{G}}|x|^{\alpha}f(x)|y^{-1}x|^{\lambda}h(y)|y|^{\beta}dxdy\geq C\left(\int_{\mathbb{G}}f^{q'}(x)dx\right)^{\frac{1}{q'}}\left(\int_{\mathbb{G}}h^{p}(x)dx\right)^{\frac{1}{p}}, \tag{4.132}$$

if one of the following conditions is satisfied:

(a) $\beta>-\frac{Q}{p'}$;
(b) $\alpha>-\frac{Q}{q}$.

Proof. Let us prove (a). By using (4.130), we have

$$\left(\int_{\mathbb{G}}|x|^{\alpha q}\left(\int_{\mathbb{G}}|y^{-1}x|^{\lambda}u(y)dy\right)^{q}dx\right)^{\frac{1}{q}}\geq\left(\int_{\mathbb{G}}|x|^{\alpha q}\left(\int_{\mathbb{G}\setminus B(0,2|x|)}|y^{-1}x|^{\lambda}u(y)dy\right)^{q}dx\right)^{\frac{1}{q}},$$

where $u(y)=|y|^{\beta}h(y)$. Then by using Step 2 in the proof of Theorem 4.21, we obtain

$$\begin{aligned}\left(\int_{\mathbb{G}}|x|^{\alpha q}\left(\int_{\mathbb{G}}|y^{-1}x|^{\lambda}u(y)dy\right)^{q}dx\right)^{\frac{1}{q}}&\geq\left(\int_{\mathbb{G}}\left(\int_{\mathbb{G}\setminus B(0,2|x|)}|x|^{\alpha}u(y)|y^{-1}x|^{\lambda}dy\right)^{q}dx\right)^{\frac{1}{q}}\\&\overset{\text{Step 2}}{\geq}C\left(\int_{\mathbb{G}}|y|^{-\beta p}u^{p}(y)dy\right)^{\frac{1}{p}}.\end{aligned}$$

Let us prove (b). By using (4.126), we have

$$\left(\int_{\mathbb{G}}|x|^{\alpha q}\left(\int_{\mathbb{G}}|y^{-1}x|^{\lambda}u(y)dy\right)^{q}dx\right)^{\frac{1}{q}}\geq\left(\int_{\mathbb{G}}|x|^{\alpha q}\left(\int_{B\left(0,\frac{|x|}{2}\right)}|y^{-1}x|^{\lambda}u(y)dy\right)^{q}dx\right)^{\frac{1}{q}},$$

where $u(y)=|y|^{\beta}h(y)$. Then by using Step 1 in the proof of Theorem 4.21, we obtain

$$\begin{aligned}\left(\int_{\mathbb{G}}|x|^{\alpha q}\left(\int_{\mathbb{G}}|y^{-1}x|^{\lambda}u(y)dy\right)^{q}dx\right)^{\frac{1}{q}}&\geq\left(\int_{\mathbb{G}}\left(\int_{B\left(0,\frac{|x|}{2}\right)}|x|^{\alpha}u(y)|y^{-1}x|^{\lambda}dy\right)^{q}dx\right)^{\frac{1}{q}}\\&\overset{\text{Step 1}}{\geq}C\left(\int_{\mathbb{G}}|y|^{-\beta p}u^{p}(y)dy\right)^{\frac{1}{p}}.\end{aligned}$$

□

Remark 4.25. *Inequality (4.132) is an even new in the Abelian (Euclidean) case $\mathbb{G}=(\mathbb{R}^{N},+)$, $Q=N$ and $|\cdot|=|\cdot|_{E}$ ($|\cdot|_{E}$ is the Euclidean distance) with conditions in Theorem 4.24.*

4.11. Reverse Hardy inequality with radial derivative on the homogeneous Lie groups.

Let us give reverse Hardy, L^p-Sobolev and L^p-Caffarelli-Kohn-Nirenberg inequalities on $\mathbb{G}$. Suppose that f is a radially decreasing function, i.e., $\mathcal{R}f:=\frac{d}{d|x|}f<0$. Let us give the reverse Hardy inequality on homogeneous Lie groups.

Theorem 4.26 (Reverse Hardy inequality)**.** *Let $\mathbb{G}$ be a homogeneous Lie group with homogeneous dimension $Q\geq 1$. Assume that $p\in(0,1)$. Then for any non-negative, real-valued and radially decreasing function $f\in C_0^{\infty}(\mathbb{G}\setminus\{0\})$, we have*

$$\left\|\frac{f}{|x|}\right\|_{L^{p}(\mathbb{G})}\geq\frac{p}{Q-p}\|\mathcal{R}f\|_{L^{p}(\mathbb{G})}. \tag{4.133}$$

Proof. By denoting $\mathcal{R}_1=-\mathcal{R}$, so that we have $\mathcal{R}_1 f>0$. By combining polar decomposition (2.11), integration by parts and reverse Hölder's inequality, we get

$$\begin{aligned}\int_{\mathbb{G}}\frac{f^{p}(x)}{|x|^{p}}dx&=\int_{0}^{\infty}\int_{\mathfrak{S}}\frac{f^{p}(ry)}{r^{p}}r^{Q-1}drd\sigma(y)\\&=-\frac{p}{Q-p}\int_{\mathbb{G}}\frac{f^{p-1}(x)}{|x|^{p-1}}\mathcal{R}f(x)dx\\&=\frac{p}{Q-p}\int_{\mathbb{G}}\frac{f^{p-1}(x)}{|x|^{p-1}}\mathcal{R}_1 f(x)dx\\&\geq\frac{p}{Q-p}\left\|\frac{f}{|x|}\right\|^{p-1}_{L^{p}(\mathbb{G})}\|\mathcal{R}_1 f\|_{L^{p}(\mathbb{G})}.\end{aligned} \tag{4.134}$$

Finally, we get

$$\left\|\frac{f}{|x|}\right\|_{L^{p}(\mathbb{G})}\geq\frac{p}{Q-p}\|\mathcal{R}_1 f\|_{L^{p}(\mathbb{G})}. \tag{4.135}$$

□

4.12. **Reverse L^p-Sobolev inequality with radial derivative on the homogeneous Lie groups.** Let us define by $\mathbb{E} = |x|\mathcal{R}$ the Euler operator. Then we have the reverse L^p-Sobolev inequality on $\mathbb{G}$.

Theorem 4.27 (Reverse L^p-Sobolev inequality)**.** *Let $\mathbb{G}$ be a homogeneous Lie group with homogeneous dimension $Q \geq 1$. Assume that $p \in (0,1)$. Then for any nonnegative, real-valued and radially decreasing function $f \in C_0^\infty(\mathbb{G}\setminus\{0\})$, we have*

$$\|f\|_{L^p(\mathbb{G})} \geq \frac{p}{Q}\|\mathbb{E}f\|_{L^p(\mathbb{G})}. \tag{4.136}$$

Proof. By denote $\mathbb{E}_1 = |x|\mathcal{R}_1$, so that $\mathbb{E}_1 f > 0$. By combining polar decomposition (2.11), integration by parts and reverse Hölder's inequality, we get

$$\begin{aligned}\int_{\mathbb{G}} f^p(x)dx &= \int_0^\infty \int_{\mathfrak{S}} f^p(ry)r^{Q-1}drd\sigma(y)\\ &= -\frac{p}{Q}\int_{\mathbb{G}} f^{p-1}(x)|x|\mathcal{R}f(x)dx\\ &= \frac{p}{Q}\int_{\mathbb{G}} f^{p-1}(x)|x|\mathcal{R}_1 f(x)dx\\ &= \frac{p}{Q}\int_{\mathbb{G}} f^{p-1}(x)\mathbb{E}_1 f(x)dx\\ &\geq \frac{p}{Q}\|f\|^{p-1}_{L^p(\mathbb{G})}\|\mathbb{E}_1 f\|_{L^p(\mathbb{G})}.\end{aligned} \tag{4.137}$$

This gives

$$\|f\|_{L^p(\mathbb{G})} \geq \frac{p}{Q}\|\mathbb{E}_1 f\|_{L^p(\mathbb{G})}, \tag{4.138}$$

implying (4.136). ☐

4.13. **Reverse L^p-Caffarelli-Kohn-Nirenberg inequality on the homogeneous Lie groups.** Let us give the reverse L^p-Caffarelli-Kohn-Nirenberg inequality on $\mathbb{G}$.

Theorem 4.28 (Reverse L^p-Caffarelli-Kohn-Nirenberg inequality)**.** *Let $\mathbb{G}$ be a homogeneous Lie group with homogeneous dimension $Q \geq 1$. Assume that $p \in (0,1)$. Then for any nonnegative, real-valued and radially decreasing function $f \in C_0^\infty(\mathbb{G}\setminus\{0\})$, we have*

$$\left\|\frac{f}{|x|^{\frac{\gamma}{p}}}\right\|^p_{L^p(\mathbb{G})} \geq \frac{p}{Q-\gamma}\left\|\frac{\mathcal{R}f}{|x|^\alpha}\right\|_{L^p(\mathbb{G})}\left\|\frac{f}{|x|^{\frac{\beta}{p-1}}}\right\|^{p-1}_{L^p(\mathbb{G})}, \tag{4.139}$$

for all $\alpha, \beta \in \mathbb{R}$ and $\gamma = \alpha+\beta+1$, such that $Q > \gamma$.

Proof. By combining polar decomposition (2.11), integration by parts and reverse Hölder's inequality, we get

$$\begin{aligned}\int_{\mathbb{G}} \frac{f^p(x)}{|x|^\gamma}dx &= \int_0^\infty \int_{\mathfrak{S}} \frac{f^p(ry)}{r^\gamma} r^{Q-1} dr d\sigma(y)\\ &= -\frac{p}{Q-\gamma}\int_{\mathbb{G}} \frac{f^{p-1}(x)}{|x|^{\gamma-1}}\mathcal{R}f(x)dx\\ &= \frac{p}{Q-\gamma}\int_{\mathbb{G}} \frac{f^{p-1}(x)}{|x|^{\alpha+\beta}}\mathcal{R}_1 f(x)dx\\ &= \frac{p}{Q-\gamma}\int_{\mathbb{G}} \frac{f^{p-1}(x)}{|x|^{\beta}}\frac{\mathcal{R}_1 f(x)}{|x|^\alpha}dx\\ &\geq \frac{p}{Q-\gamma}\left\| \frac{f}{|x|^{\frac{\beta p'}{p}}}\right\|^{p-1}_{L^p(\mathbb{G})} \left\|\frac{\mathcal{R}_1 f}{|x|^\alpha}\right\|_{L^p(\mathbb{G})}\\ &= \frac{p}{Q-\gamma}\left\| \frac{f}{|x|^{\frac{\beta\frac{p}{p-1}}{p}}}\right\|^{p-1}_{L^p(\mathbb{G})} \left\|\frac{\mathcal{R}_1 f}{|x|^\alpha}\right\|_{L^p(\mathbb{G})}\\ &= \frac{p}{Q-\gamma}\left\| \frac{f}{|x|^{\frac{\beta}{p-1}}}\right\|^{p-1}_{L^p(\mathbb{G})} \left\|\frac{\mathcal{R}_1 f}{|x|^\alpha}\right\|_{L^p(\mathbb{G})}.\end{aligned} \tag{4.140}$$

This gives

$$\left\|\frac{f}{|x|^{\frac{\gamma}{p}}}\right\|^p_{L^p(\mathbb{G})} \geq \frac{p}{Q-\gamma}\left\|\frac{\mathcal{R}_1 f}{|x|^\alpha}\right\|_{L^p(\mathbb{G})} \left\|\frac{f}{|x|^{\frac{\beta}{p-1}}}\right\|^{p-1}_{L^p(\mathbb{G})}, \tag{4.141}$$

which implies (4.139). ☐

Remark 4.29. *In (4.139), if we take $\gamma = p$ and $\alpha = 0$, then we have the reverse Hardy inequality. Also, if we take $\gamma = 0$ and $\beta = 0$, then we have the reverse L^p-Sobolev inequality.*

5. APPLICATIONS

In this chapter, we show some applications of fractional functional inequalities in PDE.

5.1. **Lyapunov-type inequality for the fractional p-sub-Laplacian.** In a popular work (see [67]), Lyapunov considered the following one-dimensional homogeneous Dirichlet boundary value problem:

$$\begin{cases} u''(x) + \omega(x)u(x) = 0, \ x \in (a,b), \\ u(a) = u(b) = 0, \end{cases} \tag{5.1}$$

and it was proved that, if u is a non-trivial solution of (5.1) and $\omega(x)$ is a real-valued and continuous function on $[a,b]$, then

$$\int_a^b |\omega(x)|dx > \frac{4}{b-a}. \tag{5.2}$$

Inequality (5.2) is called a (classical) Lyapunov inequality. This inequality has an application in spectral theory. If $\omega(x) = \lambda$, where λ is a positive constant, then we get lower estimate for the first eigenvalue of the problem (5.1) in the following form:

$$\lambda_1 > \frac{4}{(b-a)^2}.$$

Now, the Lyapunov inequality has a lot of extensions in one-dimensional and multi-dimensional cases. As example, in [68], the author obtained the Lyapunov inequality for the one-dimensional Dirichlet p-Laplacian

$$\begin{cases} (|u'(x)|^{p-2}u'(x))' + \omega(x)u^{p-1}(x) = 0, \ x \in (a,b), \ 1 < p < \infty, \\ u(a) = u(b) = 0, \end{cases} \tag{5.3}$$

where $\omega(x) \in L^1(a,b)$, so

$$\int_a^b |\omega(x)|dx > \frac{2^p}{(b-a)^{p-1}}, \ \ 1 < p < \infty. \tag{5.4}$$

Particularly, if $p = 2$ in (5.4), we recover (5.2).

In [69], the authors obtained interesting results concerning Lyapunov inequalities for the multi-dimensional fractional p-Laplacian $(-\Delta_p)^s$, $1 < p < \infty$, $s \in (0,1)$, with a homogeneous Dirichlet boundary condition, that is,

$$\begin{cases} (-\Delta_p)^s u = \omega(x)|u|^{p-2}u, \ x \in \Omega, \\ u(x) = 0, \ x \in \mathbb{R}^N \setminus \Omega, \end{cases} \tag{5.5}$$

where $\Omega \subset \mathbb{R}^N$ is a measurable set, $1 < p < \infty$, and $s \in (0,1)$. Let us recall the following result of [69].

Theorem 5.1. *Let $\omega \in L^\theta(\Omega)$ with $N > sp$, $\frac{N}{sp} < \theta < \infty$, be a non-negative weight. Suppose that problem (5.5) has a non-trivial weak solution $u \in W_0^{s,p}(\Omega)$. Then*

$$\left(\int_\Omega \omega^\theta(x)\,dx\right)^{\frac{1}{\theta}} > \frac{C}{r_\Omega^{sp-\frac{N}{\theta}}}, \tag{5.6}$$

where $C > 0$ is a universal constant and r_Ω is the inner radius of Ω.

In this section we prove a Lyapunov-type inequality for the fractional p-sub-Laplacian with a homogeneous Dirichlet boundary problem on $\mathbb{G}$. Assume $p > 1$ and $s \in (0,1)$ be such that $Q > sp$ and $\Omega \subset \mathbb{G}$ be a Haar measurable set. We denote by $r_{\Omega,q}$ the inner quasi-radius of Ω, that is,

$$r_{\Omega,q} = \max\{|x| : \ x \in \Omega\}. \tag{5.7}$$

Let us consider

$$\begin{cases} (-\Delta_p)^s u(x) = \omega |u(x)|^{p-2}u(x), \ x \in \Omega, \\ u(x) = 0, \quad x \in \mathbb{G} \setminus \Omega, \end{cases} \tag{5.8}$$

where $\omega \in L^\infty(\Omega)$.

Definition 5.2. A function $u \in W_0^{s,p}(\Omega)$ is called a weak solution of the problem (5.8) if

$$\int_\Omega \int_\Omega \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(v(x)-v(y))}{|y^{-1}x|^{Q+sp}} dxdy = \int_\Omega \omega(x)|u(x)|^{p-2}u(x)v(x)dx \tag{5.9}$$

for all $v \in W_0^{s,p}(\Omega)$.

Then we have the following theorem:

Theorem 5.3. *Let $\Omega \subset \mathbb{G}$ be a Haar measurable set. Let $\omega \in L^\theta(\Omega)$ be a non-negative weight with $\frac{Q}{sp} < \theta < \infty$. Suppose that problem (5.8) with $Q > ps$ has a non-trivial weak solution $u \in W_0^{s,p}(\Omega)$. Then, we have*

$$\|\omega\|_{L^\theta(\Omega)} \geq \frac{C}{r_{\Omega,q}^{sp-Q/\theta}}, \tag{5.10}$$

where $C = C(Q,p,s) > 0$.

Proof. By denoting

$$\beta = \alpha p + (1-\alpha)p^*,$$

where $\alpha = \frac{\theta - \theta/sp}{\theta - 1} \in (0,1)$ and p^* is the Sobolev conjugate exponent as in Theorem 3.9. Assume that $\beta = p\theta'$ with $1/\theta + 1/\theta' = 1$. Then, we have

$$\int_\Omega \frac{|u(x)|^\beta}{r_{\Omega,q}^{\alpha sp}} dx \leq \int_\Omega \frac{|u(x)|^\beta}{|x|^{\alpha sp}} dx. \tag{5.11}$$

By using Hölder's inequality with exponents $\nu = \alpha^{-1}$ and $1/\nu + 1/\nu' = 1$, we get

$$\int_\Omega \frac{|u(x)|^\beta}{|x|^{\alpha sp}} dx \leq \int_\Omega \frac{|u(x)|^{\alpha p}|u(x)|^{(1-\alpha)p^*}}{|x|^{\alpha sp}} dx \leq \left(\int_\Omega \frac{|u(x)|^p}{|x|^{sp} dx} \right)^\alpha \left(\int_\Omega |u(x)|^{p^*} dx \right)^{1-\alpha}. \tag{5.12}$$

Then, by combining Theorem 3.9 and 3.4, we have

$$\begin{aligned}\int_\Omega \frac{|u(x)|^\beta}{|x|^{\alpha sp}}dx &\le C_1^\alpha\left(\int_\Omega\int_\Omega \frac{|u(x)-u(y)|^p}{|y^{-1}x|^{Q+sp}}dxdy\right)^{\alpha/p} C_2^{(1-\alpha)p^*/p}[u]_{s,p}^{(1-\alpha)p^*/p}\\ &\le C_1^\alpha [u]_{s,p}^\alpha C_2^{(1-\alpha)p^*/p}[u]_{s,p}^{(1-\alpha)p^*/p}\\ &= C\left([u]_{s,p}^p\right)^{(\alpha p+(1-\alpha)p^*)/p}\\ &= C\left(\int_\Omega \omega(x)|u(x)|^p dx\right)^{\theta'}\\ &\le C\left(\int_\Omega \omega^\theta(x)dx\right)^{\theta'/\theta}\int_\Omega |u(x)|^{p\theta'}dx\\ &= C\|\omega\|_{L^\theta(\Omega)}^{\theta'}\int_\Omega |u(x)|^\beta dx.\end{aligned}$$

It means, we have

$$\int_\Omega \frac{|u(x)|^\beta}{|x|^{\alpha sp}}dx \le C\|\omega\|_{L^\theta(\Omega)}^{\theta'}\int_\Omega |u(x)|^\beta dx.$$

Thus, from (5.11) we get

$$\frac{1}{r_{\Omega,q}^{\alpha sp}}\int_\Omega |u(x)|^\beta dx \le \int_\Omega \frac{|u(x)|^\beta}{|x|^{\alpha sp}}dx \le C\|\omega\|_{L^\theta(\Omega)}^{\theta'}\int_\Omega |u(x)|^\beta dx. \tag{5.13}$$

Finally, we arrive at

$$\frac{C}{r_{\Omega,q}^{sp-Q/\theta}} \le \|\omega\|_{L^\theta(\Omega)}. \tag{5.14}$$

Theorem 5.3 is proved. □

Let consider the following spectral problem for the non-linear, fractional p-sub-Laplacian $(-\Delta_p)^s$, $1<p<\infty$, $s\in(0,1)$, with Dirichlet boundary condition:

$$\begin{cases}(-\Delta_p)^s u = \lambda|u|^{p-2}u, \ x\in\Omega,\\ u(x)=0, \ x\in\mathbb{G}\setminus\Omega.\end{cases} \tag{5.15}$$

We have the following Rayleigh quotient for the fractional Dirichlet p-sub-Laplacian (cf. [69])

$$\lambda_1 = \inf_{u\in W_0^{s,p}(\Omega),\, u\ne 0}\frac{[u]_{s,p}^p}{\|u\|_{L^p(\mathbb{G})}^p}. \tag{5.16}$$

As a consequence of Theorem 5.3 we obtain the following theorem:

Theorem 5.4. *Assume λ_1 be the first eigenvalue of problem (5.15) given by (5.16). Assume $Q>sp$, $s\in(0,1)$ and $1<p<\infty$. Then we have*

$$\lambda_1 \ge \sup_{\frac{Q}{sp}<\theta<\infty}\frac{C}{|\Omega|^{\frac{1}{\theta}}r_{\Omega,q}^{sp-Q/\theta}}, \tag{5.17}$$

where C is a positive constant given in Theorem 5.3, $|\cdot|$ is the Haar measure and $r_{\Omega,q}$ is the inner quasi-radius of Ω.

Proof. In Theorem 5.3, by taking $\omega = \lambda_1 \in L^{\theta}(\Omega)$ and using Lyapunov-type inequality (5.10), we get that

$$\|\omega\|_{L^{\theta}(\Omega)} = \|\lambda_1\|_{L^{\theta}(\Omega)} = \left(\int_{\Omega} \lambda_1^{\theta} dx\right)^{1/\theta} \geq \frac{C}{r_{\Omega,q}^{sp-Q/\theta}}. \tag{5.18}$$

For every $\theta > \frac{Q}{sp}$, we have

$$\lambda_1 \geq \frac{C}{|\Omega|^{\frac{1}{\theta}} r_{\Omega,q}^{sp-Q/\theta}}. \tag{5.19}$$

Thus, we get

$$\lambda_1 \geq \sup_{\frac{Q}{sp}<\theta<\infty} \frac{C}{|\Omega|^{\frac{1}{\theta}} r_{\Omega,q}^{sp-Q/\theta}}, \tag{5.20}$$

for all $\frac{Q}{sp} < \theta < \infty$.

Theorem 5.4 is proved. □

5.2. **Lyapunov-type inequality for the fractional p-sub-Laplacian system.** Historically, in the work [70], at the first time authors showed Lyapunov's inequality for the system. They considered a system of ODE for p and q-Laplacian on the interval (a, b) with the homogeneous Dirichlet condition in the following form:

$$\begin{cases} -(|u'(x)|^{p-2}u'(x))' = f(x)|u(x)|^{\alpha-2}u(x)|v(x)|^{\beta}, \\ -(|v'(x)|^{q-2}v'(x))' = g(x)|u(x)|^{\alpha}|v(x)|^{\beta-2}v(x), \end{cases} \tag{5.21}$$

on the interval (a, b), with

$$u(a) = u(b) = v(a) = v(b) = 0, \tag{5.22}$$

where $f, g \in L^1(a, b)$, $f, g \geq 0$, $p, q > 1$, α, $\beta \geq 0$ and

$$\frac{\alpha}{p} + \frac{\beta}{q} = 1.$$

So, for the system (5.21) with Dirichlet condition (5.22), we have the following estimate (Lyapunov's inequality):

$$2^{\alpha+\beta} \leq (b-a)^{\frac{\alpha}{p'}+\frac{\beta}{q'}} \left(\int_a^b f(x)dx\right)^{\frac{\alpha}{p}} \left(\int_a^b g(x)dx\right)^{\frac{\beta}{q}}, \tag{5.23}$$

where $p' = \frac{p}{p-1}$ and $q' = \frac{q}{q-1}$. For the more general Lyapunov's inequality for fractional p-Laplacian with homogeneous Dirichlet conditions was proved in [71]. In the previous section, we proved a Lyapunov-type inequality for the fractional p-sub-Laplacian with the homogeneous Dirichlet condition. Here we establish Lyapunov-type inequality for the fractional p-sub-Laplacian system for the homogeneous Dirichlet problem.

Namely, let us consider the fractional p-sub-Laplacian system:

$$\begin{cases} (-\Delta_{p_1})^{s_1}u_1(x) = \omega_1(x)|u_1(x)|^{\alpha_1-2}u_1(x)|u_2(x)|^{\alpha_2}\ldots|u_n(x)|^{\alpha_n}, \; x \in \Omega, \\ (-\Delta_{p_2})^{s_2}u_2(x) = \omega_2(x)|u_1(x)|^{\alpha_1}|u_2(x)|^{\alpha_2-2}u_2(x)\ldots|u_n(x)|^{\alpha_n}, \; x \in \Omega, \\ \qquad\qquad \ldots \\ (-\Delta_{p_n})^{s_n}u_n(x) = \omega_n(x)|u_1(x)|^{\alpha_1}|u_2(x)|^{\alpha_2}\ldots|u_n(x)|^{\alpha_n-2}u_n(x), \; x \in \Omega, \end{cases} \tag{5.24}$$

with homogeneous Dirichlet conditions

$$u_i(x) = 0, \quad x \in \mathbb{G} \setminus \Omega, \quad i = 1, \ldots, n, \tag{5.25}$$

where $\Omega \subset \mathbb{G}$ is a Haar measurable set, $\omega_i \in L^1(\Omega)$, $\omega_i \geq 0$, $s_i \in (0,1)$, $p_i \in (1,\infty)$ and $(-\Delta_p)^s$ is the fractional p-sub-Laplacian on $\mathbb{G}$. Here $B(x,\delta)$ is a quasi-ball with respect to q, with radius δ, centred at $x \in \mathbb{G}$, and α_i are positive parameters such that

$$\sum_{i=1}^{n} \frac{\alpha_i}{p_i} = 1. \tag{5.26}$$

We denote by $r_{\Omega,q}$ the inner quasi-radius of Ω.

Definition 5.5. We say that $(u_1, \ldots, u_n) \in \prod_{i=1}^n W_0^{s_i,p_i}(\Omega)$ is a weak solution of (5.24)-(5.25) if for all $(v_1, \ldots, v_n) \in \prod_{i=1}^n W_0^{s_i,p_i}(\Omega)$, we have

$$\begin{aligned}\int_{\mathbb{G}} \int_{\mathbb{G}} & \frac{|u_i(x) - u_i(y)|^{p_i-2}(u_i(x) - u_i(y))(v_i(x) - v_i(y))}{|y^{-1}x|^{Q+s_ip_i}} dxdy \\ &= \int_{\Omega} \omega_i(x) \left(\prod_{j=1}^{i-1} |u_j(x)|^{\alpha_j} \right) \left(\prod_{j=i+1}^{n} |u_j(x)|^{\alpha_j} \right) |u_i(x)|^{\alpha_i-2} u_i(x) v_i(x) dx, \end{aligned} \tag{5.27}$$

for every $i = 1, \ldots, n$.

Now we present the following analogue of a Lyapunov-type inequality for the fractional p-sub-Laplacian system on $\mathbb{G}$.

Theorem 5.6. *Let $s_i \in (0,1)$ and $p_i \in (1,\infty)$ be such that $Q > s_ip_i$ for all $i = 1, \ldots, n$. Let $\omega_i \in L^\theta(\Omega)$ be a non-negative weight and assume that*

$$1 < \max_{i=1,\ldots,n} \left\{ \frac{Q}{s_ip_i} \right\} < \theta < \infty.$$

If (5.24)-(5.25) admits a nontrivial weak solution, then

$$\prod_{i=1}^{n} \|\omega_i\|_{L^\theta(\Omega)}^{\frac{\theta\alpha_i}{p_i}} \geq C r_{\Omega,q}^{Q-\theta\sum_{j=1}^n s_j\alpha_j}, \tag{5.28}$$

where $C > 0$ is a positive constant.

Remark 5.7. *In Theorem 5.6, by taking $n = 1$ and $\alpha_1 = p$, we establish the Lyapunov-type inequality for the fractional p-sub-Laplacian on $\mathbb{G}$ (see, e.g. Theorem 5.3).*

Proof of Theorem 5.6. For all $i = 1, \ldots, n$, let us denote

$$\xi_i = \gamma_i p_i + (1-\gamma_i)p_i^*, \tag{5.29}$$

and

$$\gamma_i = \frac{\theta - \frac{Q}{s_ip_i}}{\theta - 1}, \tag{5.30}$$

where $p_i^* = \frac{Q}{Q-s_ip_i}$ is the Sobolev conjugate exponent as in Theorem 3.9. For all $i=1,\ldots,n$ we have $\gamma_i \in (0,1)$ and $\xi_i = p_i\theta'$, where $\theta' = \frac{\theta}{\theta-1}$. Then for every $i \in \{1,\ldots,n\}$ we have

$$\int_\Omega \frac{|u_i(x)|^{\xi_i}}{r_{\Omega,q}^{\gamma_i s_i p_i}}dx \leq \int_\Omega \frac{|u_i(x)|^{\xi_i}}{|x|^{\gamma_i s_i p_i}}dx,$$

and from Hölder's inequality with the following exponents $\nu_i = \frac{1}{\gamma_i}$ and $\frac{1}{\nu_i} + \frac{1}{\nu_i'} = 1$, we get

$$\int_\Omega \frac{|u_i(x)|^{\xi_i}}{|x|^{\gamma_i s_i p_i}}dx = \int_\Omega \frac{|u_i(x)|^{\gamma_i p_i}|u_i(x)|^{(1-\gamma_i)p_i^*}}{|x|^{\gamma_i s_i p_i}}dx$$
$$\leq \left(\int_\Omega \frac{|u_i(x)|^{p_i}}{|x|^{s_i p_i}}dx\right)^{\gamma_i}\left(\int_\Omega |u_i(x)|^{p_i^*}dx\right)^{1-\gamma_i}. \quad (5.31)$$

On the other hand, from Theorem 3.9, we obtain

$$\left(\int_\Omega |u_i(x)|^{p_i^*}dx\right)^{1-\gamma_i} \leq C[u_i]_{s_i,p_i}^{p_i^*(1-\gamma_i)},$$

and from Theorem 3.4, we have

$$\left(\int_\Omega \frac{|u_i(x)|^{p_i}}{|x|^{s_i p_i}}dx\right)^{\gamma_i} \leq C[u_i]_{s_i,p_i}^{p_i\gamma_i}.$$

Thus, from (5.31) and by taking $u_i(x) = v_i(x)$ in (5.27), we get

$$\int_\Omega \frac{|u_i(x)|^{\xi_i}}{|x|^{\gamma_i s_i p_i}} \leq C([u_i]_{s_i,p_i,\Omega}^{p_i})^{\frac{\xi_i}{p_i}} \leq C([u_i]_{s_i,p_i}^{p_i})^{\frac{\xi_i}{p_i}}$$

$$= C\left(\int_\Omega \omega_i(x)\prod_{j=1}^n |u_j|^{\alpha_j}dx\right)^{\frac{\xi_i}{p_i}} = C\left(\int_\Omega \omega_i(x)\prod_{j=1}^n |u_j|^{\alpha_j}dx\right)^{\theta'},$$

for every $i=1,\ldots,n$. Hence, by using Hölder's inequality with exponents θ and θ', we obtain

$$\int_\Omega \frac{|u_i(x)|^{\xi_i}}{|x|^{\gamma_i s_i p_i}}dx \leq C\|\omega_i\|_{L^\theta(\Omega)}^{\frac{\theta}{\theta-1}} \int_\Omega \prod_{j=1}^n |u_j(x)|^{\alpha_j\theta'}dx.$$

By using Hölder's inequality and (5.26), we get

$$\int_\Omega \prod_{j=1}^n |u_j(x)|^{\alpha_j\theta'}dx \leq \prod_{j=1}^n \left(\int_\Omega |u_j|^{\theta' p_j}dx\right)^{\frac{\alpha_j}{p_j}}.$$

This implies that

$$\int_\Omega \frac{|u_i(x)|^{\xi_i}}{|x|^{\gamma_i s_i p_i}}dx \leq C\|\omega_i\|_{L^\theta(\Omega)}^{\frac{\theta}{\theta-1}} \prod_{j=1}^n \left(\int_\Omega |u_j|^{\theta' p_j}dx\right)^{\frac{\alpha_j}{p_j}}.$$

So we establish

$$\int_\Omega \frac{|u_i(x)|^{\xi_i}}{r_{\Omega,q}^{\gamma_i s_i p_i}}dx \leq \int_\Omega \frac{|u_i(x)|^{\xi_i}}{|x|^{\gamma_i s_i p_i}}dx$$

$$\leq C\|\omega_i\|^{\frac{\theta}{\theta-1}}_{L^\theta(\Omega)} \prod_{j=1}^{n} \left(\int_\Omega |u_j|^{\theta' p_j} dx\right)^{\frac{\alpha_j}{p_j}}.$$

Thus, for every $e_i > 0$ we have

$$\left(\int_\Omega \frac{|u_i(x)|^{\xi_i}}{r_{\Omega,q}^{\gamma_i s_i p_i}} dx\right)^{e_i} = \frac{1}{r_{\Omega,q}^{e_i \gamma_i s_i p_i}} \left(\int_\Omega |u_i(x)|^{\xi_i} dx\right)^{e_i}$$

$$\leq C\|\omega_i\|^{\frac{e_i\theta}{\theta-1}}_{L^\theta(\Omega)} \prod_{j=1}^{n} \left(\int_\Omega |u_j|^{\theta' p_j} dx\right)^{\frac{e_i\alpha_j}{p_j}},$$

so that

$$\frac{1}{r_{\Omega,q}^{\sum_{j=1}^n \gamma_j s_j p_j e_j}} \prod_{i=1}^{n} \left(\int_\Omega |u_i(x)|^{\theta' p_i} dx\right)^{e_i}$$

$$\leq C \left(\prod_{i=1}^{n} \|\omega_i\|^{\frac{e_i\theta}{\theta-1}}_{L^\theta(\Omega)}\right) \left(\prod_{i=1}^{n} \left(\int_\Omega |u_i(x)|^{\theta' p_i} dx\right)^{\frac{\alpha_i \sum_{j=1}^n e_j}{p_i}}\right).$$

This yields

$$\frac{1}{r_{\Omega,q}^{\sum_{j=1}^n \gamma_j s_j p_j e_j}} \leq C \left(\prod_{i=1}^{n} \|\omega_i\|^{\frac{e_i\theta}{\theta-1}}_{L^\theta(\Omega)}\right) \left(\prod_{i=1}^{n} \left(|u_i(x)|^{\theta' p_i} dx\right)^{\frac{\alpha_i \sum_{j=1}^n e_j}{p_i} - e_i}\right), \tag{5.32}$$

where C is a positive constant. Then, let us choose e_i, $i = 1, \ldots, n$, such that

$$\frac{\alpha_i \sum_{j=1}^n e_j}{p_i} - e_i = 0, \quad i = 1, \ldots, n.$$

Consequently, by using (5.26) we have the solution of this system

$$e_i = \frac{\alpha_i}{p_i}, \quad i = 1, \ldots, n. \tag{5.33}$$

By combining (5.32), (5.30) and (5.33) we establish

$$\prod_{i=1}^{n} \|\omega_i\|^{\frac{\theta\alpha_i}{p_i}}_{L^\theta(\Omega)} \geq C r_{\Omega,q}^{Q-\theta\sum_{j=1}^n s_j\alpha_j}. \tag{5.34}$$

Theorem 5.3 is proved. ☐

Now, let us discuss an application of the Lyapunov-type inequality for the fractional p-sub-Laplacian system on $\mathbb{G}$. In order to do it we consider the spectral problem for the fractional p-sub-Laplacian system in the following form:

$$\begin{cases} (-\Delta_{p_1})^{s_1} u_1(x) = \lambda_1 \alpha_1 \varphi(x) |u_1(x)|^{\alpha_1 - 2} u_1(x) |u_2(x)|^{\alpha_2} \ldots |u_n(x)|^{\alpha_n}, \ x \in \Omega, \\ (-\Delta_{p_2})^{s_2} u_2(x) = \lambda_2 \alpha_2 \varphi(x) |u_1(x)|^{\alpha_1} |u_2(x)|^{\alpha_2 - 2} u_2(x) \ldots |u_n(x)|^{\alpha_n}, \ x \in \Omega, \\ \qquad \ldots \\ (-\Delta_{p_n})^{s_n} u_n(x) = \lambda_n \alpha_n \varphi(x) |u_1(x)|^{\alpha_1} |u_2(x)|^{\alpha_2} \ldots |u_n(x)|^{\alpha_n - 2} u_n(x), \ x \in \Omega, \end{cases} \tag{5.35}$$

with

$$u_i(x) = 0, \quad x \in \mathbb{G} \setminus \Omega, \ \ i = 1, \ldots, n, \tag{5.36}$$

where $\Omega \subset \mathbb{G}$ is a Haar measurable set, $\varphi \in L^1(\Omega)$, $\varphi \geq 0$ and $s_i \in (0,1)$, $p_i \in (1,\infty)$, $i = 1, \ldots, n$.

Definition 5.8. We say that $\lambda = (\lambda_1, \ldots, \lambda_n)$ is an eigenvalue if the problem (5.35)-(5.36) admits at least one nontrivial weak solution $(u_1, \ldots, u_n) \in \prod_{i=1}^n W_0^{s_i,p_i}(\Omega)$.

Theorem 5.9. *Let* $s_i \in (0,1)$ *and* $p_i \in (1,\infty)$ *be such that* $Q > s_i p_i$, *for all* $i = 1, \ldots, n$, *and*

$$1 < \max_{i=1,\ldots,n}\left\{\frac{Q}{s_i p_i}\right\} < \theta < \infty.$$

Let $\varphi \in L^\theta(\Omega)$ *with* $\|\varphi\|_{L^\theta(\Omega)} \neq 0$. *Then, we have*

$$\lambda_k \geq \frac{C}{\alpha_k}\left(\frac{1}{\prod_{i=1,i\neq k}^n \lambda_i^{\frac{\alpha_i}{p_i}}}\right)^{\frac{p_k}{\alpha_k}}\left(\frac{1}{r_{\Omega,q}^{\theta\sum_{i=1}^n \alpha_i s_i - Q}\prod_{i=1,i\neq k}^n \alpha_i^{\frac{\theta\alpha_i}{p_i}}\int_\Omega \varphi^\theta(x)dx}\right)^{\frac{p_k}{\theta\alpha_k}}, \tag{5.37}$$

where C *is a positive constant and* $k = 1, \ldots, n$.

Proof of Theorem 5.9. In Theorem 5.6 by taking $\omega_k = \lambda_k \alpha_k \varphi(x)$, $k = 1, \ldots, n$, we have

$$\alpha_k^{\frac{\theta\alpha_k}{p_k}} \lambda_k^{\frac{\theta\alpha_k}{p_k}} \prod_{i=1,i\neq k}^n (\alpha_i\lambda_i)^{\frac{\theta\alpha_i}{p_i}} \prod_{i=1}^n \|\varphi\|_{L^\theta(\Omega)}^{\frac{\theta\alpha_i}{p_i}} \geq C r_{\Omega,q}^{Q-\theta\sum_{j=1}^n s_j\alpha_j}.$$

Thus, using (5.26) we obtain

$$\alpha_k^{\frac{\theta\alpha_k}{p_k}} \lambda_k^{\frac{\theta\alpha_k}{p_k}} \prod_{i=1,i\neq k}^n (\alpha_i\lambda_i)^{\frac{\theta\alpha_i}{p_i}} \int_\Omega \varphi^\theta(x)dx \geq C r_{\Omega,q}^{Q-\theta\sum_{j=1}^n s_j\alpha_j}.$$

This implies

$$\lambda_k^{\frac{\theta\alpha_k}{p_k}} \geq \frac{C}{\alpha_k^{\frac{\theta\alpha_k}{p_k}} r_{\Omega,q}^{\theta\sum_{j=1}^n s_j\alpha_j - Q}\prod_{i=1,i\neq k}^n (\alpha_i\lambda_i)^{\frac{\theta\alpha_i}{p_i}}\int_\Omega \varphi^\theta(x)dx}, \quad k = 1, \ldots, n.$$

Finally, we obtain that

$$\lambda_k \geq \frac{C}{\alpha_k}\left(\frac{1}{\prod_{i=1,i\neq k}^n \lambda_i^{\frac{\alpha_i}{p_i}}}\right)^{\frac{p_k}{\alpha_k}}\left(\frac{1}{r_{\Omega,q}^{\theta\sum_{i=1}^n \alpha_i s_i - Q}\prod_{i=1,i\neq k}^n \alpha_i^{\frac{\theta\alpha_i}{p_i}}\int_\Omega \varphi^\theta(x)dx}\right)^{\frac{p_k}{\theta\alpha_k}},$$
$$k = 1, \ldots, n. \tag{5.38}$$

Theorem 5.9 is proved. ☐

5.3. **Existence of weak solutions with nonlocal source on the Heisenberg and stratified groups.** In [72], under certain assumptions on f (classically, this condition is called Ambrosetti-Rabinowitz condition), for the following semilinear equation

$$\begin{cases} -\Delta u = f(x,u), \ x \in \Omega \subset \mathbb{R}^n, \\ u(x) = 0, \ x \in \partial\Omega, \end{cases} \tag{5.39}$$

the authors proved existence of solutions by the mountain pass theorem. Mountain pass theorem is using to show critical points of the some differentiable functional. Here and after by $\partial\Omega$ we denote the boundary of a smooth bounded set Ω. After the work of Ambrosetti and Rabinowitz [72], a number of extensions and generalisations of their result has been published. Also, for fractional nonlinear problems, for the fractional $p-$Laplacian, fractional Schrödinger–Kirchhoff type and Choquard-Kirchhoff existence of weak solutions were proved in [73], [74], [75] and [76]. One of the main aim of this section is to extend the above ideas to non-commutative analysis, it means using our functional inequalities. Hence, we will consider analogues problems on the Heisenberg group, which is the most popular example of the non-Abelian nilpotent Lie groups. On Heisenberg group, there is already a number of results related to the existence of solutions to the semilinear equations starting the pioneering works (see e.g., [77] and [78]). In this section we show existence of the weak solution by mountain pass theorem on Heisenberg group, which can be easily extended to the general stratified Lie groups.

Firstly, let us give definition of the Palais-Smale sequence (shortly, $(PS)_c$) sequence.

Definition 5.10 ([72])**.** Let E be a Banach space. A sequence $\{u_n\}$ is a $(PS)_c$ sequence for a functional $\Phi \in (\Phi, \mathbb{R})$, if every $\{u_n\} \subset E$ satisfies:

$$\Phi(u_n) \to c, \quad \text{for} \quad n \to \infty, \tag{5.40}$$

and

$$\Phi'(u_n) \to 0, \quad \text{for} \quad n \to \infty \quad \text{in } E^*, \tag{5.41}$$

where $'$ is the Fréchet differential and E^* is the dual space of E.

Then let us give a version of the (minimax) mountain pass theorem (see, e.g. [79]).

Theorem 5.11. *Suppose that X be a Banach space and $\Phi : X \to \mathbb{R}$ a C^1-functional with a $(PS)_c$ sequence. Let Γ be a class of paths joining $u = 0$ with $u = \omega$:*

$$\Gamma := \{\gamma \in C([0,1], X) \mid \gamma(0) = 0, \ \gamma(1) = \omega\}, \tag{5.42}$$

where $\omega \in X$, $\|\omega\| > r > 0$, Φ is bounded from below on $S(0,\rho) = \{u \in X : \|u\| \leq \rho\}$, that is,

$$\alpha = \max\{\Phi(0), \Phi(\omega)\} < \inf_{u \in S(0,\rho)} \Phi(u) = \beta. \tag{5.43}$$

Then Φ possesses a critical value $c \geq \beta$ which can be characterised as

$$c = \inf_{\gamma \in \Gamma} \max_{u \in \gamma(0,1)} \Phi(u).$$

5.3.1. *On Heisenberg group.* It is well-known that the class of Heisenberg groups is a subclass of the stratified Lie groups, that is, obviously, the above theorem is valid for the Heisenberg group setting. Firstly, we show our result on Heisenberg group.

Assume that $f(x,\xi)$ is a Carathéodory function $f : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfying the following assumptions (Ambrosetti-Rabinowitz condition):

- $p1)$ There exist constants a_1, $a_2 > 0$ such that $|f(x,\xi)| \leq a_1 + a_2|\xi|^s$, a.e. $x \in \Omega$ and $\xi \in \mathbb{R}$ with $p < s < \frac{Qp}{Q-p} - 1$;
- $p2)$ $\lim_{|\xi| \to 0} \frac{f(x,\xi)}{|\xi|^{p-1}} = 0$, uniformly in $x \in \Omega$;

$p3)$ There exist $\mu > p$ and $r > 0$ such that $0 < \mu F(x,\xi) < \xi f(x,\xi)$ with $|\xi| > r$, a.e. $x \in \mathbb{H}^n$, $\xi \in \mathbb{R}$. Here $F(x,\xi) = \int_0^\xi f(x,t)dt$.

$p4)$ $f(x,\xi) \in C(\overline{\Omega}, \mathbb{R})$.

As the model case, the function $f(x,\xi) = a(x)|\xi|^{s-2}\xi$ with $a \in L^\infty(\Omega)$ and $s \in [p, p^*)$ can be considered as a Carathéodory function satisfying the assumptions $p1)$-$p4)$.

Then, under above assumptions on the right hand side, we consider the following homogeneous Dirichlet boundary value problem for the p-sub-Laplacian with the nonlinear source (or the nonlinear right hand side) on the Heisenberg group:

$$\begin{cases} -\Delta_{H,p} u = f(x,u), \ x \in \Omega \subset \mathbb{H}^n, \ 1 < p < Q, \\ u(x) = 0, \ x \in \partial\Omega, \end{cases} \tag{5.44}$$

where $\Delta_{H,p}$ is defined in (2.30). Let us recall from Section 2.3 the Sobolev space in the following form:

$$S^{1,p}(\Omega) := \{u \in L^p(\Omega) : X_i u \in L^p(\Omega) \text{ and } Y_i u \in L^p(\Omega), \ i = 1, \dots, n\} \tag{5.45}$$

with the norm

$$\|u\|_{S^{1,p}(\Omega)} = \left(\int_\Omega |u(x)|^p + |\nabla_H u(x)|^p dx \right)^{\frac{1}{p}}. \tag{5.46}$$

Let $S_0^{1,p}(\Omega)$ defined as the completion of $C_0^\infty(\Omega)$ with the norm

$$\|u\|_{S_0^{1,p}(\Omega)} = \left(\int_\Omega |\nabla_H u(x)|^p dx \right)^{\frac{1}{p}}. \tag{5.47}$$

For simplicity, we also use the notation $W := S_0^{1,p}(\Omega)$.

Note that the above integral measure is indeed the standard Lebesque measure since it can be considered as a Haar measure on $\mathbb{H}^n$, that is, the Lebesque measure is also translation invariant with respect to the group law of $\mathbb{H}^n$.

To introduce a variational structure for problem (5.44), we introduce $I : W \to \mathbb{R}$ as follows

$$I(u) := \frac{1}{p} \int_\Omega |\nabla_H u|^p dx - \int_\Omega F(x,u) dx, \tag{5.48}$$

where

$$F(x,u) = \int_0^u f(x,\xi) d\xi.$$

We note I is a Fréchet differentiable functional with respect to $u \in W$ for any $\varphi \in W$, so we have

$$\langle I'(u), \varphi \rangle = \int_\Omega |\nabla_H u|^{p-2} \nabla_H u \cdot \nabla_H \varphi dx - \int_\Omega f(x,u)\varphi(x) dx, \tag{5.49}$$

where $\langle \cdot, \cdot \rangle$ is the dual product between W and its dual space W^*. Let us give the definition of a weak solution.

Definition 5.12. We say $u : \Omega \to \mathbb{R}$ is a weak solution of (5.44), if $u \in W$, such that

$$\int_\Omega |\nabla_H u|^{p-2} \nabla_H u \cdot \nabla_H \varphi dx = \int_\Omega f(x,u)\varphi(x) dx, \ \ \forall \varphi \in C_c^\infty(\Omega). \tag{5.50}$$

Then we have the following properties of Carathéodory functions:

Lemma 5.13. *Let Ω be a measurable set in $\mathbb{H}^n$. Assume that f is a Carathéodory function and assumption p3) holds true, then there exist constants $a_3, a_4 > 0$ such that*

$$a_3|\xi|^{\mu} - a_4 \leq F(x, \xi), \quad \forall x \in \Omega, \tag{5.51}$$

where $\mu > p$.

Lemma 5.14. *Let Ω be a measurable set in $\mathbb{H}^n$. Assume that f be a Carathéodory function satisfying assumptions p1) and p4). Then for any $\xi \in \mathbb{R}$, we have*

$$|f(x, \xi)| \leq \varepsilon|\xi|^{p-1} + (s+1)\kappa(\varepsilon)|\xi|^{s}, \tag{5.52}$$

and

$$|F(x, \xi)| \leq \varepsilon|\xi|^{p} + \kappa(\varepsilon)|\xi|^{s+1}, \tag{5.53}$$

where ε and $\kappa(\varepsilon)$ are some positive small numbers. Here the numbers s and p are defined as in p1).

Note that the proofs of Lemma 5.13 and Lemma 5.14 are exactly the same as the Euclidean case in [72].

Let us check the first assumption of the mountain pass theorem.

Lemma 5.15. *Let Ω be a measurable set in $\mathbb{H}^n$. Assume that f be a Carathéodory function satisfying the assumptions p1) and p2). Then there exist positive constants $\rho, \alpha > 0$ such that $\|u\|_W = \rho$ and $I(u) \geq \alpha$ for all $u \in W$.*

Proof. By using Lemma 5.14 in (5.48), we get

$$\begin{aligned} I(u) = \frac{1}{p}\int_{\Omega}|\nabla_H u(x)|^p dx - \int_{\Omega} F(x, u(x))dx \geq \frac{1}{p}\int_{\Omega}|\nabla_H u(x)|^p dx - \varepsilon\int_{\Omega}|u(x)|^p dx \\ - \kappa(\varepsilon)\int_{\Omega}|u(x)|^{s+1}dx. \end{aligned} \tag{5.54}$$

From $1 < p < p^*$ and Ω is a measurable domain, we have the continuous embedding $L^{p^*}(\Omega) \hookrightarrow L^p(\Omega)$ in $\Omega \subset \mathbb{H}^n$. For $s + 1 < p^*$ we also have the following continuous embedding $L^{p^*}(\Omega) \hookrightarrow L^{s+1}(\Omega)$, then

$$\begin{aligned} I(u) &= \frac{1}{p}\int_{\Omega}|\nabla_H u(x)|^p dx - \int_{\Omega} F(x, u(x))dx \\ &\geq \frac{1}{p}\int_{\Omega}|\nabla_H u(x)|^p dx - \varepsilon\int_{\Omega}|u(x)|^p dx - \kappa(\varepsilon)\int_{\Omega}|u(x)|^{s+1}dx \\ &= \frac{1}{p}\int_{\Omega}|\nabla_H u(x)|^p dx - \varepsilon\|u\|^p_{L^p(\Omega)} - \kappa(\varepsilon)\|u\|^{s+1}_{L^{s+1}(\Omega)} \\ &\geq \frac{1}{p}\int_{\Omega}|\nabla_H u(x)|^p dx - C_1\varepsilon\|u\|^p_{L^{p^*}(\Omega)} - \kappa(\varepsilon)C_2\|u\|^{s+1}_{L^{p^*}(\Omega)}. \end{aligned} \tag{5.55}$$

Moreover, by using Folland-Stein's continuous embedding $W \hookrightarrow L^{p^*}(\Omega)$ (Theorem 3.45), we have

$$I(u) \geq \frac{1}{p}\int_\Omega |\nabla_H u(x)|^p dx - C_1\varepsilon \left(\int_\Omega |u(x)|^{p^*} dx\right)^{\frac{p}{p*}} - C_2\kappa(\varepsilon)\left(\int_\Omega |u(x)|^{p*} dx\right)^{\frac{s+1}{p*}}$$
$$\geq \|u\|_W^p \left(\frac{1}{p} - C_1\varepsilon - C_2\kappa(\varepsilon)\|u\|_W^{s+1-p}\right). \quad (5.56)$$

Assume that $u \in W$ and $\|u\|_W = \rho > 0$. From assumption $s+1 > p$, choosing ρ sufficiently small and choosing ε such that $\alpha := \rho^p\left(\frac{1}{p} - C_1\varepsilon - C_2\kappa(\varepsilon)\rho^{s+1-p}\right) > 0$, we get

$$\inf_{u\in W, \|u\|_W = \rho} I(u) \geq \rho^p\left(\frac{1}{p} - C_1\varepsilon - C_2\kappa(\varepsilon)\rho^{s+1-p}\right) > 0. \quad (5.57)$$

Lemma 5.15 is proved. ☐

Now let us check the second assupmtion of the mountain pass theorem.

Lemma 5.16. *Assume that f be a Carathe*ó*dory function satisfying p1)-p4). Then there exists $v > 0$ a.e. in W, $\|v\|_W > \rho$ and $I(v) < \alpha$, where the constants α and ρ are given as in Lemma 5.15.*

Proof. By fixing $\|u\|_W = 1$ and $u \geq 0$ a.e. in $\mathbb{H}^n$ with $t > 0$. From Lemma 5.13, we calculate

$$I(tu) = \frac{1}{p}\int_\Omega |\nabla_H(tu)|^p dx - \int_\Omega F(x, tu(x))dx \leq \frac{t^p}{p}\int_\Omega |\nabla_H u|^p dx$$
$$- a_4 t^\mu \int_\Omega |u|^\mu dx + a_3|\Omega| = \frac{t^p}{p} - a_4 t^\mu \int_\Omega |u|^\mu dx + a_3|\Omega|. \quad (5.58)$$

From the assumption $\mu > p$ and by taking $t \to +\infty$, we have $I(tu) \to -\infty$. Consequently, by taking $v = \beta u$, with β sufficiently large, we obtain the desired result. ☐

From the above lemmas follow that the assumptions of the mountain pass theorem are fulfilled by the functional (5.48). Then we need to show the $(PS)_c$ compactness condition for the functional (5.48).

Lemma 5.17. *Assume that f be a Carathéodory function satisfying p1)-p4). Let $\{u_n\}$ be a sequence satisfying $I(u_n) \to c$ and*

$$\sup\{|\langle I'(u_n), \varphi\rangle| : \varphi \in W, \ \|\varphi\|_W = 1\} \to 0 \quad n \to \infty. \quad (5.59)$$

Then the sequence $\{u_n\} \subset W$ is bounded in W.

Proof. Assume that $\{u_n\} \subset W$ be a $(PS)_c$ sequence. Then for every $\varphi \in W$ we have

$$\langle I'(u_n), \varphi\rangle = \int_\Omega |\nabla_H u_n|^{p-2}\nabla_H u_n \cdot \nabla_H \varphi dx - \int_\Omega f(x, u_n)\varphi dx, \quad (5.60)$$

and

$$I(u_n) = \frac{1}{p}\int_\Omega |\nabla_H u_n|^p dx - \int_\Omega F(x, u_n)dx. \quad (5.61)$$

Hence, we have

$$\begin{aligned} I(u_n) - \frac{1}{\mu}\langle I'(u_n), u_n\rangle &= \left(\frac{1}{p} - \frac{1}{\mu}\right)\int_\Omega |\nabla_H u_n|^p dx - \int_\Omega \left(F(x,u_n) - \frac{f(x,u_n)u_n}{\mu}\right) dx \\ &= \left(\frac{1}{p} - \frac{1}{\mu}\right)\int_\Omega |\nabla_H u_n|^p dx - \int_{\Omega\cap|u_n|\leq r} \left(F(x,u_n) - \frac{f(x,u_n)u_n}{\mu}\right) dx \\ &\qquad - \int_{\Omega\cap|u_n|>r} \left(F(x,u_n) - \frac{f(x,u_n)u_n}{\mu}\right) dx, \end{aligned} \tag{5.62}$$

where $\mu > p$.

Let us consider the second term on the right hand side. From Lemma 5.14 we calculate

$$\begin{aligned} &\left|\int_{\Omega\cap|u_n|\leq r} F(x,u_n) - \frac{f(x,u_n)u_n}{\mu} dx\right| \\ &\qquad\leq \left(\varepsilon r^p + \kappa(\varepsilon)r^{s+1} + \frac{1}{\mu}(\varepsilon r^2 + q\kappa(\varepsilon)r^{s+1})\right)|\Omega|. \end{aligned} \tag{5.63}$$

Let us denote the right hand side by

$$\widetilde{\theta} := \left(\varepsilon r^p + \kappa(\varepsilon)r^{s+1} + \frac{1}{\mu}(\varepsilon r^2 + q\kappa(\varepsilon)r^{s+1})\right)|\Omega|$$

By combining (5.63) and assumption $p3$), we get

$$I(u_n) - \frac{1}{\mu}\langle I'(u_n), u_n\rangle \geq \left(\frac{1}{p} - \frac{1}{\mu}\right)\int_\Omega |\nabla_H u_n|^p dx - \widetilde{\theta}. \tag{5.64}$$

By the assumption in (5.59) with $\varphi := \frac{u_n}{\|u_n\|_W}$ for any n there exists a number $\lambda > 0$, such that

$$\left|\left\langle I'(u_n), \left(\frac{u_n}{\|u_n\|_W}\right)\right\rangle\right| \leq \lambda,$$

with $I(u_n) \leq \lambda$. Hence, we have

$$I(u_n) - \frac{1}{\mu}\langle I'(u_n), u_n\rangle \leq \lambda(1 + \|u_n\|_W), \tag{5.65}$$

combining this with (5.64) we arrive at

$$\left(\frac{1}{p} - \frac{1}{\mu}\right)\|u_n\|_W^p \leq \lambda(1 + \|u_n\|_W) + \widetilde{\theta}.$$

Finally,

$$\begin{aligned} \|u_n\|_W^p &\leq \left(\frac{1}{p} - \frac{1}{\mu}\right)^{-1}(\lambda(1 + \|u_n\|_W) + \widetilde{\theta}) \\ &\leq \left(\frac{1}{p} - \frac{1}{\mu}\right)^{-1} C_1(1 + \|u_n\|_W) \leq C(1 + \|u_n\|_W). \end{aligned}$$

where C is a positive constant. □

Now we have to show that the $(PS)_c$ sequence of I has a strong convergent subsequence, so we can say I satisfies the $(PS)_c$ condition.

Lemma 5.18. *Under assumptions p1)-p4), if $\{u_n\} \subset W$ is a $(PS)_c$ sequence of I, then $\{u_n\}$ has a strong convergent subsequence in W.*

Proof. Since W is a Banach space, we have $u_n \rightharpoonup u$ weakly in W. Hence,

$$\begin{aligned}\langle I'(u_n), (u_n - u)\rangle = &\int_\Omega |\nabla_H u_n|^{p-2}\nabla_H u_n \cdot \nabla_H(u_n - u)dx \\ &- \int_\Omega f(x, u_n)(u_n - u)dx \to 0, \;\; n \to \infty.\end{aligned} \tag{5.66}$$

Also, we have $u_n \to u$ strongly convergence in $L^{s+1}(\Omega)$, where $s \in [p, p^* - 1)$. Then,

$$f(x, u_n)(u_n - u) \to 0, \;\; \text{a.e. in } \Omega, \;\; n \to \infty. \tag{5.67}$$

Moreover, by using the Vitali convergence theorem, we obtain

$$\lim_{n\to\infty} \int_\Omega f(x, u_n)(u_n - u)dx = 0. \tag{5.68}$$

Plugging (5.68) in (5.66), we have

$$\int_\Omega |\nabla_H u_n|^{p-2}\nabla_H u_n \cdot \nabla_H(u_n - u)dx \to 0, \;\; n \to \infty. \tag{5.69}$$

Since $\{u_n\}$ weakly converges in W, we arrive at

$$\int_\Omega (|\nabla_H u_n|^{p-2}\nabla_H u_n - |\nabla_H u|^{p-2}\nabla_H u) \cdot \nabla_H(u_n - u)dx \to 0, \;\; n \to \infty. \tag{5.70}$$

Now let us give some useful vector inequalities. Let C_1, C_2 be positive constants depending only on p. Then, we have

$$|a - b|^p \le C_1(|a|^{p-2}a - |b|^{p-2}b) \cdot (a - b), \quad p \ge 2, \tag{5.71}$$

and

$$|a - b|^2 \le C_2(|a| + |b|)^{2-p}(|a|^{p-2}a - |b|^{p-2}b) \cdot (a - b), \quad 1 < p < 2, \tag{5.72}$$

for all vectors $a, b \in \mathbb{R}^N$. Firstly, let us consider the case $p \ge 2$. By applying (5.71) to (5.70), we have

$$\begin{aligned}\|u_n - u\|_W^p &= \int_\Omega |\nabla_H(u_n - u)|^p dx = \int_\Omega |\nabla_H u_n - \nabla_H u|^p dx \\ &\le C_1 \int_\Omega \left(|\nabla_H u_n|^{p-2}\nabla_H u_n - |\nabla_H u|^{p-2}\nabla_H u\right) \cdot (\nabla_H u_n - \nabla_H u)dx \\ &= C_1 \int_\Omega \left(|\nabla_H u_n|^{p-2}\nabla_H u_n - |\nabla_H u|^{p-2}\nabla_H u\right) \cdot \nabla_H(u_n - u)dx \to 0,\end{aligned} \tag{5.73}$$

as $n \to \infty$. It means for $p \ge 2$, we have

$$\|u_n - u\|_W \to 0, \quad n \to \infty.$$

Let us consider the case $1 < p < 2$. By using the fact $\{u_n\}$ is bounded in W, applying (5.70) to (5.72), we have

$$\begin{aligned}
\|u_n - u\|_W^p &= \int_\Omega |\nabla_H (u_n - u)|^p dx = \int_\Omega |\nabla_H u_n - \nabla_H u|^p dx \\
&\leq C_2 \left(\int_\Omega \left(|\nabla_H u_n|^{p-2} \nabla_H u_n - |\nabla_H u|^{p-2} \nabla_H u \right) \cdot (\nabla_H u_n - \nabla_H u) dx \right)^{\frac{p}{2}} \\
&\times \left(\int_\Omega |\nabla_H u_n| + |\nabla_H u| dx \right)^{\frac{(2-p)p}{2}} \\
&\leq C_3 \left(\int_\Omega \left(|\nabla_H u_n|^{p-2} \nabla_H u_n - |\nabla_H u|^{p-2} \nabla_H u \right) \cdot (\nabla_H u_n - \nabla_H u) dx \right)^{\frac{p}{2}} \\
&\to 0,
\end{aligned} \tag{5.74}$$

as $n \to \infty$. hence, we get

$$\|u_n - u\|_W \to 0, \quad n \to \infty, \quad 1 < p < \infty.$$

□

Theorem 5.19. *Let f be a Carathéodory function satisfying p1)-p4). Then there exists a non-trivial weak solution of problem (5.44).*

Proof. By using Lemma 5.18, any $(PS)_c$ subsequence of I has strong convergence in W. Also, we have that

$$I(0) = 0,$$

and by taking ρ as in Lemma 5.16, there exists α such that $I(u) \geq \alpha > 0 = I(0)$, where

$$u \in W, \text{ and } \|u\|_W = \rho.$$

Therefore, now applying the mountain pass theorem, we get a critical point of the functional $I(u)$ which is a non-trivial weak solution of problem (5.44). □

5.3.2. *On Stratified groups.* Then let us extend previous result on the case of stratified groups. Now let us consider the Dirichlet boundary value problem on stratified Lie groups $\mathbb{G}$:

$$\begin{cases} -\mathcal{L}_p u = f(x, u), \ x \in \Omega \subset \mathbb{G}, \ 1 < p < Q, \\ u(x) = 0, \ x \in \partial\Omega, \end{cases} \tag{5.75}$$

where f is a Carathéodory function satisfying the assumptions $p1) - p4)$ on $\mathbb{G}$. Then we have the following theorem:

Theorem 5.20. *There exists a non-trivial weak solution of problem (5.75).*

The proof is the same as the one of Theorem 5.19.

5.4. **Multiplicity of the weak solutions for the sub-Laplacian with Hardy potential.** In [80], Ghoussoub and Yuan considered the following problem with the Hardy-Sobolev potential:

$$\begin{cases} -\Delta_p u(x) - \lambda \frac{u(x)}{|x|^p} = |u|^{p-2}u, \ x \in \Omega \subset \mathbb{R}^n, \\ u(x) = 0, \ x \in \partial\Omega, \end{cases} \tag{5.76}$$

and the authors obtained existence and multiplicity of weak solutions. Since then, analogues of the problem with the Hardy potential have been considered by many authors, see [81, 82, 83] and [84], for example.

In [85], Ghoussoub and Shakerian considered the following problem with fractional Laplacian and the Hardy-Sobolev potential:

$$(-\Delta_s)u - \gamma \frac{u}{|x|^{2s}} = |u|^{2_s^*-2}u + \frac{|u|^{2_s^*(\alpha)-2}u}{|x|^{\alpha}}, \quad u > 0, \quad x \in \mathbb{R}^n,$$

where the authors showed existence of nontrivial weak solutions. In this direction, most of studies have been dedicated to the single Hardy-Sobolev nonlinearity. In [86], the author investigated the following problem:

$$(-\Delta_s)u - \gamma \frac{u}{|x|^{2s}} = \frac{|u|^{2_s^*(\beta)-2}u}{|x|^{\beta}} + \frac{|u|^{2_s^*(\alpha)-2}u}{|x|^{\alpha}}, \quad u > 0, \quad x \in \mathbb{R}^n,$$

where the authors showed multiplicity of weak solutions with the doubling Hardy-Sobolev potential, which generalises previous cases. In this section, we show multiplicity of weak solutions with first stratum Hardy potential on Heisenberg and stratified groups.

5.4.1. *On Heisenberg group.* Let us recall the "horizontal" L^p-Caffarelli–Kohn–Nirenberg inequality on Heisenberg group.

Theorem 5.21 (Theorem 3.1., [29]). *For any $f \in C_0^\infty(\mathbb{H}^n \setminus \{z = 0\})$, and all $1 < p < \infty$, we have*

$$\frac{|2n-\gamma|}{p} \left\| \frac{f}{|z|^{\frac{\gamma}{p}}} \right\|_{L^p(\mathbb{H}^n)} \leq \left\| \frac{\nabla_H f}{|z|^{\alpha}} \right\|_{L^p(\mathbb{H}^n)} \left\| \frac{f}{|z|^{\frac{\beta}{p-1}}} \right\|_{L^p(\mathbb{H}^n)}, \ \alpha, \beta \in \mathbb{R}, \tag{5.77}$$

where $\gamma = \alpha + \beta + 1$. If $\gamma \neq 2n$ then the constant $\frac{|2n-\gamma|}{p}$ is sharp.

When $\alpha = 0$ and $\beta = p - 1$, inequality (5.77) implies the first stratum Hardy inequality, that is, for all $f \in C_0^\infty(\mathbb{H}^n \setminus \{z = 0\})$, we have

$$\frac{|2n-p|}{p} \left\| \frac{f}{|z|} \right\|_{L^p(\mathbb{H}^n)} \leq \|\nabla_H f\|_{L^p(\mathbb{H}^n)}, \quad z = (x, y) \in \mathbb{R}^{2n}, \tag{5.78}$$

where $|z| = \sqrt{x_1^2 + \ldots + x_n^2 + y_1^2 + \ldots + y_n^2}$.

Similarly with previous section, we also use the notation $W := S_0^{1,2}(\Omega)$. Also, let us define the Sobolev space with the norm:

$$\|u\|_X := \left(\|\nabla_H u\|_W^2 - \lambda \int_\Omega \frac{|u|^2}{|z|^2} d\xi \right)^{\frac{1}{2}}, \quad 0 < \lambda < \overline{\lambda} = (n-1)^2 = \frac{(Q-4)^2}{4}. \tag{5.79}$$

Indeed, $\| \cdot \|_W$ and $\| \cdot \|_X$ are equivalent norms.

Let $\Omega \subset \mathbb{H}^n$ be a measurable set with sufficiently smooth boundary $\partial\Omega$ such that $(0,0,t) \notin \Omega$. Assume that $n > 1$ (that is, $Q > 4$), $0 < \lambda < \overline{\lambda} = (n-1)^2 = \frac{(Q-4)^2}{4}$ and $1 < p < 2^* - 1 = \frac{2Q}{Q-2} - 1$. In this subsection, we show multiplicity of positive weak solutions to the problem:

$$\begin{cases} -\Delta_H u(\xi) - \lambda \frac{u(\xi)}{|z|^2} = u^p(\xi), \ \xi \in \Omega \subset \mathbb{H}^n, \\ u(\xi) > 0, \quad \xi \in \Omega, \\ u(\xi) = 0, \ \xi \in \partial\Omega, \end{cases} \tag{5.80}$$

where $|z| = \sqrt{x_1^2 + \ldots + x_n^2 + y_1^2 + \ldots + y_n^2}, \quad z = (x,y) \in \mathbb{R}^{2n}$.

To present a variational structure for problem (5.80), we introduce $I : W \to \mathbb{R}$ as follows

$$I(u) := \frac{1}{2}\int_\Omega |\nabla_H u|^2 d\xi - \frac{\lambda}{2}\int_\Omega \frac{u_+^2}{|z|^2} d\xi - \frac{1}{p+1}\int_\Omega u_+^{p+1} d\xi, \tag{5.81}$$

where $u_+ = \max\{u, 0\}$.

Note that I is a Fréchet differentiable functional with respect to $u \in W$ for any $\varphi \in W$, so we have

$$\langle I'(u), \varphi\rangle = \int_\Omega \nabla_H u \cdot \nabla_H \varphi d\xi - \lambda \int_\Omega \frac{u_+\varphi}{|z|^2} d\xi - \int_\Omega u_+^p \varphi d\xi. \tag{5.82}$$

For the functional I, let us verify the assumptions of Theorem 5.11.

Lemma 5.22. *Let Ω be a Haar measurable set in $\mathbb{H}^n$. Then there exist positive constants $\rho, \alpha > 0$ such that $\|u\|_W = \rho$ and $I(u) \geq \alpha$ for all $u \in W$.*

Proof. Firstly, by the Folland-Stein-Sobolev inequality (Theorem 3.45), by using the facts that the norms (5.47) and (5.79) are equivalent, $2 < p+1 < 2^* = \frac{2Q}{Q-2}$ and $L^{2^*}(\Omega) \hookrightarrow L^{p+1}(\Omega)$, we have

$$\|u\|_{L^{p+1}(\Omega)} \leq C\|u\|_{L^{2^*}(\Omega)} \leq C\|u\|_W. \tag{5.83}$$

Now we give an estimate to the functional $I(u)$. So, using the above embedding and first stratum Hardy inequality we compute

$$\begin{aligned} I(u) &= \frac{1}{2}\int_\Omega |\nabla_H u|^2 d\xi - \frac{\lambda}{2}\int_\Omega \frac{u_+^2}{|z|^2} d\xi - \frac{1}{p+1}\int_\Omega u_+^{p+1} d\xi \\ &\overset{(5.78)}{\geq} \frac{1}{2}\|u\|_W^2 - \frac{\lambda}{2\overline{\lambda}}\|u\|_W^2 - \frac{1}{p+1}\int_\Omega u_+^{p+1} d\xi \\ &\overset{(5.83)}{\geq} \left(\frac{1}{2} - \frac{\lambda}{2\overline{\lambda}}\right)\|u\|_W^2 - \frac{C_2}{p+1}\|u\|_W^{p+1} \\ &= C_1\|u\|_W^2 - \frac{C_2}{p+1}\|u\|_W^{p+1}, \end{aligned} \tag{5.84}$$

where $C_1, C_2 > 0$. Let $u \in W$ and $\|u\|_W = \rho > 0$. By choosing ρ sufficiently small, we have $\alpha := \frac{C_1\rho^2}{2} - \frac{C_2\rho^{p+1}}{p+1} > 0$, thus, we arrive at

$$\inf_{u \in W, \|u\|_W = \rho} I(u) \geq \frac{C_1\rho^2}{2} - \frac{C_2\rho^{p+1}}{p+1} > 0. \tag{5.85}$$

□

Lemma 5.23. *Under assumptions of Lemma 5.22, there exists $v > 0$ a.e. in W, $\|v\|_W > \rho$ and $I(v) < \alpha$, where the constants α and ρ are given as in Lemma 5.22.*

Proof. Let us fix $\|u\|_W = 1$ and $u \geq 0$ a.e. in $\mathbb{H}^n$ with $t > 0$. Then we get

$$\begin{aligned} I(tu) &= \frac{1}{2}\|tu\|_W^2 - \frac{\lambda}{2}\int_\Omega \frac{(tu_+)^2}{|z|^2}d\xi - \frac{1}{p+1}\int_\Omega (tu_+)^{p+1}d\xi \\ &\leq \frac{1}{2}\|tu\|_W^2 - \frac{1}{p+1}\int_\Omega (tu_+)^{p+1}d\xi \qquad (5.86)\\ &= \frac{t^2}{2} - \frac{t^{p+1}}{p+1}\int_\Omega u_+^{p+1}d\xi. \end{aligned}$$

By the assumption $p > 1$ and by taking $t \to +\infty$, we get $I(tu) \to -\infty$. Thus, by setting $v = \beta u$, with β sufficiently large, we arrive at the desired result. ☐

Finally, we need to check $(PS)_c$ condition for our functional. But we need to show some preliminary result.

Lemma 5.24. *Let $\{u_n\}$ be a bounded sequence in W such that $I'(u_n) \to 0$, as $n \to \infty$. Then there exists $u \in W$ such that, up to a subsequence, $\|u_n - u\|_W \to 0$, as $n \to \infty$.*

Proof. Since the norm $\|\cdot\|_W$ is equivalent to $\|\cdot\|_X$ and $\{u_n\}$ is a bounded in W with the norm $\|\cdot\|_W$, then we have

$$\|u_n\|_X = \|u_n\|_W - \lambda\int_\Omega \frac{u^2}{|z|^2}d\xi \overset{\lambda>0}{\leq} \|u_n\|_W \leq C. \qquad (5.87)$$

By [3, Theorem 4.4.28], W is a Banach and reflexive space, so we have

$$u_n \rightharpoonup u, \text{ in } W, \text{ with the norm, } \|\cdot\|_X \qquad (5.88)$$

and

$$u_n \to u, \text{ in } L^r(\mathbb{H}^n), 1 \leq r < 2^*, \ u_n \to u, \text{a.e. in } \mathbb{H}^n. \qquad (5.89)$$

From this fact for $p+1 < 2^*$ and $I'(u_n) \to 0$ as $n \to \infty$, we have

$$\lim_{n\to\infty}\|u_n\|_X^2 = \lim_{n\to\infty}\|(u_n)_+\|_{L^{p+1}(\Omega)}^{p+1}. \qquad (5.90)$$

Similarly, we get

$$\lim_{n\to\infty}\left(\int_\Omega \nabla_H u\cdot\nabla_H u_n - \lambda\int_\Omega \frac{u(u_n)_+}{|z|^2}d\xi\right) = \|(u)_+\|_{L^{p+1}(\Omega)}^{p+1}. \qquad (5.91)$$

By combining above facts, we obtain

$$\|u_n - u\|_X \to 0, \ \ n \to \infty.$$

By using property of norm's equivalence, we have

$$\|u_n - u\|_W \to 0, \ \ n \to \infty.$$

☐

Lemma 5.25. *Assume that $\{u_n\}$ be a $(PS)_c$ sequence such that Definition 5.10. Then there exists $u \in W$ such that*

$$\lim_{n\to\infty}\|u_n - u\|_W = 0. \qquad (5.92)$$

Proof. By using Definition 5.10, we obtain

$$\begin{aligned} c+o(1) = I(u_n) - \frac{1}{p+1}\langle I'(u_n), u_n\rangle = \left(\frac{1}{2}-\frac{1}{p+1}\right)\|u_n\|_W^2 \\ -\lambda\left(\frac{1}{2}-\frac{1}{p+1}\right)\int_\Omega \frac{(u_n)_+^2}{|z|^2}d\xi \geq \left(\frac{1}{2}-\frac{1}{p+1}\right)\|u_n\|_W^2 \\ -C\|u_n\|_W^2 = C\|u_n\|_W^2, \end{aligned} \tag{5.93}$$

with $p+1>2$. Thus, we have $\|u_n\|_W \leq C$. Therefore, by Lemma 5.24, we have strong convergence of $\{u_n\}$ in W. ☐

Finally, let us give main result of this section.

Theorem 5.26. *Problem (5.80) has at least two positive solutions.*

Proof. Let us construct two solutions of the problem (5.80). By using Lemma 5.25, any $(PS)_c$ subsequence of I has strong convergence in W. Also, we have that

$$I(0)=0,$$

and by taking ρ as in Lemma 5.23, there exists α such that $I(u) \geq \alpha > 0 = I(0)$, where

$$u \in W, \text{ and } \|u\|_W = \rho.$$

Therefore, now applying the mountain pass theorem, we get a critical point of the functional $I(u)$ which is a positive weak solution of problem (5.80).

Now let us construct another solution of (5.80). By Lemma 5.22, there exist positive constants $\rho, \alpha > 0$ such that $\|u\|_W = \rho$ and $I(u) \geq \alpha$ for all $u \in W$. Hence, we can choose

$$\rho_1 = \{\inf_{\rho \in \mathbb{R}} : I(u) > 0, \ \forall u \in W, \text{ with } \|u\|_X = \rho\}.$$

From this, we have $\rho_1 > 0$, then $I(u) < 0$. Assume that $\rho_2 > \rho_1$, s.t. $I(u)$ is a non-decreasing functional with $\rho_1 < \|u\|_X < \rho_2$. Then let us define the following smooth function $\theta(\eta)$ in the following form: $\theta(\eta) = 1$ if $\eta \leq \rho_1$, and $\theta(\eta) = 0$ if $\eta \geq \rho_2$.

We define the following energy functional:

$$I_2(u) := \frac{1}{2}\int_\Omega |\nabla_H u|^2 d\xi - \frac{\lambda\theta(\|u\|_W)}{2}\int_\Omega \frac{u_+^2}{|z|^2}d\xi - \frac{1}{p+1}\int_\Omega u_+^{p+1}d\xi. \tag{5.94}$$

If $\|u\|_W \leq \rho_1$, then $I_2(u) = I(u)$ and $\|u\|_W \geq \rho_2$, so we have

$$I_2(u) := \frac{1}{2}\int_\Omega |\nabla_H u|^2 d\xi - \frac{1}{p+1}\int_\Omega u_+^{p+1}d\xi.$$

It easy to see that I_2 is a coercive functional. By W is a Hilbert space, we have that the functional lower semi-continuity. Then we can say there exists minimum point of I_2 with negative energy, it means I_2 has a minimum point. It gives the second solution. ☐

5.4.2. *On stratified groups.* It is well-known fact, that the Heisenberg group is the most popular example of stratified groups. In this subsection extended results on stratified groups.

Let us give the L^p-Caffarelli–Kohn–Nirenberg inequality on stratified groups.

Theorem 5.27 (Theorem 3.1., [29])**.** *Let $\mathbb{G}$ be a stratified group with N_1 being the dimension of the first stratum, and let $\alpha, \beta \in \mathbb{R}$. Then for any $f \in C_0^\infty(\mathbb{G}\setminus\{x' = 0\})$, and all $1 < p < \infty$, we have*

$$\frac{|N_1-\gamma|}{p}\left\|\frac{f}{|x'|^{\frac{\gamma}{p}}}\right\|_{L^p(\mathbb{G})} \leq \left\|\frac{\nabla_{\mathbb{G}} f}{|x'|^{\alpha}}\right\|_{L^p(\mathbb{G})}\left\|\frac{f}{|x'|^{\frac{\beta}{p-1}}}\right\|_{L^p(\mathbb{G})}, \tag{5.95}$$

where $\gamma = \alpha+\beta+1$ and $|\cdot|$ is the Euclidean norm on $\mathbb{R}^N$. If $\gamma = N_1$then the constant $\frac{|N_1-\gamma|}{p}$ is sharp.

If $\alpha = 0$ and $\beta = p-1$, we obtain the first stratum Hardy inequality on $\mathbb{G}$.

Let $\Omega \subset \mathbb{G}$ be a measurable set with sufficiently smooth boundary $\partial\Omega$ such that $\{x' = 0\} \notin \Omega$. Assume that dimension of the first stratum $N_1 > 2$, $0 < \lambda < \overline{\lambda} = \frac{(N_1-2)^4}{2}$ and $1 < p < 2^* - 1 = \frac{2Q}{Q-2} - 1$. Let us consider the following problem:

$$\begin{cases} \mathcal{L}u(x) - \lambda\frac{u(x)}{|x'|^2} = u^p(x), \; x \in \Omega \subset \mathbb{G}, \\ u > 0, \quad x \in \Omega, \\ u(x) = 0, \; x \in \partial\Omega. \end{cases} \tag{5.96}$$

Theorem 5.28. *Problem (5.96) has at least two positive solutions.*

Proof. The proof follows the almost same lines of the proof of Theorem 5.26. Only difference is that now we use, that is, stratified group versions of Theorem 3.45 and Theorem 5.27 instead of Theorem 3.45 and Theorem 5.21, respectively. ☐

5.5. **Existence of the weak solution for the fractional sub-Laplacian with Hardy potential.** Let us continue our studying of the existence of the weak solution. In this section, we show existence of the weak solution for semilinear equation with fractional sub-Laplacian and Hardy potential. Since then, fractional analogues of this problem on Euclidean setting have been considered by many different authors, for example, in [87, 88, 89] and [90]. In addition, we refer to [91, 92] and [93] as well as references therein for fractional Laplacian problems with the Hardy potential.

Let us consider the following problem with Hardy potential on $\mathbb{G}$:

$$\begin{cases} (-\Delta_s)u(x) - \lambda\frac{u(x)}{|x|^{2s}} = u^p, \quad x \in \Omega\setminus\{0\} \subset \mathbb{G}, \\ u(x) = 0, \; x \in \mathbb{G}\setminus\Omega, \end{cases} \tag{5.97}$$

where Ω is an open bounded domain in $\mathbb{G}$ with smooth boundary, $0 \leq \lambda < \overline{\lambda}$ is the best constant of the fractional Hardy inequality on $\mathbb{G}$, $1 < p < 2^* - 1$ and $2s < Q$.

Setting $S := W_0^{s,2}(\Omega)$, let us define the fractional Sobolev space on $\mathbb{G}$ with the norm

$$\|u\|_S^2 := [u]_{s,2}^2 - \lambda\int_\Omega \frac{|u|^2}{|x|^{2s}}dx, \tag{5.98}$$

which is equivalent (by the fractional Hardy inequality) to the norm

$$\|u\|_{W_0^{s,2}(\Omega)} = [u]_{s,2}, \tag{5.99}$$

where $[\cdot]_{s,2} = [\cdot]_{s,2,\Omega}$ is the Gagliardo semi-norm which is defined in (2.14).

Definition 5.29. We say $u : \Omega \to \mathbb{R}$ is a weak solution of (5.97), if $u \in S$, such that

$$\int_\Omega \int_\Omega \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|y^{-1}x|^{Q+2s}} dxdy - \lambda \int_\Omega \frac{u_+\varphi}{|x|^{2s}} dx - \int_\Omega u_+^p \varphi dx = 0, \tag{5.100}$$

for all $\varphi \in S$, where $u_+ = \max\{u, 0\}$.

The energy functional corresponding to (5.97) can be given by the expression

$$I(u) = \frac{1}{2}\left(\int_\Omega \int_\Omega \frac{|u(x)-u(y)|^2}{|y^{-1}x|^{Q+2s}} dxdy - \lambda \int_\Omega \frac{(u_+)^2}{|x|^{2s}} dx\right) - \frac{1}{p+1}\int_\Omega u_+^{p+1} dx. \tag{5.101}$$

Note that I is a Fréchet differentiable functional with respect to $u \in S$ for any $\varphi \in S$, so we have

$$\begin{aligned}\langle I'(u), \varphi\rangle = \int_\Omega \int_\Omega \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|y^{-1}x|^{Q+2s}} dxdy - \lambda \int_\Omega \frac{u_+\varphi}{|x|^{2s}} dx \\ - \int_\Omega u_+^p \varphi dx.\end{aligned} \tag{5.102}$$

For the functional I, let us verify the assumptions of Theorem 5.11.

Lemma 5.30. *Let Ω be a Haar measurable set in $\mathbb{G}$. Then there exist positive constants $\rho, \alpha > 0$ such that $\|u\|_S = \rho$ and $I(u) \geq \alpha$ for all $u \in S$.*

Proof. Firstly, by Sobolev embedding theorem, by using the facts that the norms (5.98) and (5.99) are equivalent, $2 < p+1 < 2^* = \frac{2Q}{Q-2s}$ and $L^{2^*}(\Omega) \hookrightarrow L^{p+1}(\Omega)$, we have

$$\|u\|_{L^{p+1}(\Omega)} \leq C\|u\|_{L^{2^*}(\Omega)} \overset{(3.37)\ with\, \beta=0}{\leq} C\|u\|_S. \tag{5.103}$$

Now we give an estimate to the functional $I(u)$. So, using above embedding we compute

$$\begin{aligned} I(u) &= \frac{1}{2}\left(\int_\Omega \int_\Omega \frac{|u(x)-u(y)|^2}{|y^{-1}x|^{Q+2s}} dxdy - \lambda \int_\Omega \frac{|u|^2}{|x|^{2s}} dx\right) - \frac{1}{p+1}\int_\Omega u_+^{p+1} dx \\ &= \frac{1}{2}\|u\|_S^2 - \frac{1}{p+1}\int_\Omega u_+^{p+1} dx \\ &\overset{(5.103)}{\geq} \frac{1}{2}\|u\|_S^2 - \frac{C}{p+1}\|u\|_S^{p+1}. \end{aligned} \tag{5.104}$$

Let $u \in W$ and $\|u\|_S = \rho > 0$. By choosing ρ sufficiently small, we have $\alpha := \frac{\rho^2}{2} - \frac{C\rho^{p+1}}{p+1} > 0$, thus, we arrive at

$$\inf_{u\in W, \|u\|_W = \rho} I(u) \geq \frac{\rho^2}{2} - \frac{C\rho^{p+1}}{p+1} > 0. \tag{5.105}$$

$\square$

Lemma 5.31. *Under assumptions of Lemma 5.30, there exists* $0 < v \in S$ *a.e. in* W, $\|v\|_S > \rho$ *and* $I(v) < \alpha$, *where the constants* α *and* ρ *are given as in Lemma 5.30.*

Proof. Let us fix $\|u\|_W = 1$ and $u \geq 0$ a.e. in $\mathbb{G}$ with $t > 0$. Then we calculate

$$I(tu) = \frac{1}{2}\|tu\|_S^2 - \frac{1}{p+1}\int_\Omega (tu_+)^{p+1}dx = \frac{t^2}{2} - \frac{t^{p+1}}{p+1}\int_\Omega u_+^{p+1}dx. \tag{5.106}$$

By the assumption $p > 1$ and by taking $t \to +\infty$, we get $I(tu) \to -\infty$. Thus, by setting $v = \beta u$, with β sufficiently large, we arrive at the desired result. □

Lemma 5.32. *Let* $\{u_n\}$ *be a bounded sequence in* S *such that* $I'(u_n) \to 0$, *as* $n \to \infty$. *Then there exists* $u \in S$ *such that, up to a subsequence,* $\|u_n - u\|_S \to 0$, *as* $n \to \infty$.

Proof. Since the norm $\|\cdot\|_S$ is equivalent to $\|\cdot\|_{S_*}$, for the norm $\|\cdot\|_{S_*}$ there exists $u \in S$ and a subsequence $\{u_n\}$, such that,

$$u_n \rightharpoonup u, \text{ in } S, \tag{5.107}$$

and

$$u_n \to u, \text{ in } L^r(\mathbb{G}), 1 \leq r < 2^*, \ u_n \to u, \text{a.e. in } \mathbb{G}. \tag{5.108}$$

From this fact for $p+1 < 2^*$ and $I'(u_n) \to 0$ as $n \to \infty$ we have

$$\lim_{n\to\infty} \|u_n\|_S^2 = \lim_{n\to\infty} \|u_n\|_{L^{p+1}(\Omega)}^{p+1} = \|u\|_{L^{p+1}(\Omega)}^{p+1}. \tag{5.109}$$

Similarly, we get

$$\lim_{n\to\infty}\left(\int_\Omega\int_\Omega \frac{(u_n(x)-u_n(y))(u(x)-u(y))}{|y^{-1}x|^{Q+2s}}dxdy - \lambda\int_\Omega \frac{u(u_n)_+}{|x|^{2s}}dx\right) = \|u\|_{L^{p+1}(\Omega)}^{p+1}. \tag{5.110}$$

Thus, we have

$$\|u_n - u\|_S \to 0, \ \ n \to \infty.$$

□

Lemma 5.33. *Assume that* $\{u_n\}$ *be a* $(PS)_c$ *sequence such that Definition 5.10. Then there exists* $u \in S$ *such that*

$$\lim_{n\to\infty} \|u_n - u\|_S = 0. \tag{5.111}$$

Proof. By using Definition 5.10, we obtain

$$\begin{aligned} c + o(1) &= I(u_n) - \frac{1}{p+1}\langle I'(u_n), u_n\rangle = \left(\frac{1}{2} - \frac{1}{p+1}\right)[u_n]_{s,2}^2 \\ &- \lambda\left(\frac{1}{2} - \frac{1}{p+1}\right)\int_\Omega \frac{(u_n)_+^2}{|x|^{2s}}dx \overset{(3.37)\text{ with } \beta=2s}{\geq} \left(\frac{1}{2} - \frac{1}{p+1}\right)[u_n]_{s,2}^2 \\ &- C[u_n]_{s,2}^2 = C[u_n]_{s,2}^2, \end{aligned} \tag{5.112}$$

with $p+1 > 2$. Thus, we have $\|u_n\|_S \leq C$. Therefore, by Lemma 5.32, we have strong convergence of $\{u_n\}$ in S. □

We are now in a position to present the main result of this section.

Theorem 5.34. *Assume that* $\Omega \subset \mathbb{G}$ *be a Haar measurable set. Then there exists a non-trivial weak solution of problem (5.97).*

Proof. By using Lemma 5.33, any $(PS)_c$ subsequence of $I(u_n)$ has strong convergence in S. Also, we have that

$$I(0) = 0,$$

and by taking ρ as in Lemma 5.30, there exists α such that $I(u) \geq \alpha > 0 = I(0)$, where

$$u \in S, \text{ and } \|u\|_S = \rho.$$

Therefore, now applying the mountain pass theorem, we have a critical point of the functional $I(u)$ which is a non-trivial weak solution of problem (5.97). ☐

5.6. **Blow-up result to heat equation with fractional sub-Laplacian and logarithmic nonlinearity on homogeneous groups.** Firstly, the heat equation with logarithmic nonlinearity with Cauchy-Dirichlet problem was considered in [94]:

$$\begin{cases} u_t(x,t) - \Delta_x u(x,t) = u \log|u|, \quad (x,t) \in \Omega \times (0,+\infty), \\ u(x,0) = u_0(x), \quad x \in \Omega, \\ u(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,+\infty). \end{cases} \tag{5.113}$$

Also, the authors showed global solvability of solution by potential wells method and the following blow-up theorem (in the Euclidean setting):

Theorem 5.35 ([94]). *Assume that $u_0 \in H_0^1(\Omega)$ and*

$$J(u_0) = \frac{1}{2}\int_\Omega |\nabla u_0|^2 dx - \frac{1}{2}\int_\Omega |u_0|^2 \log|u_0| dx + \frac{1}{4}\int_\Omega |u_0|^2 dx \leq M, \tag{5.114}$$

and

$$I(u_0) = \int_\Omega |\nabla u_0|^2 dx - \int_\Omega |u_0|^2 \log|u_0| dx < 0. \tag{5.115}$$

Then the weak solution of the problem (5.113) blows up at $+\infty$.

Moreover, in [95], the author showed the condition $J(u_0) \leq M$ is unnecessary to blow-up at infinity to a solution of the problem (5.113). In this section, we considered the heat equation with the fractional sub-Laplacian with logarithmic nonlinearity and we obtain the blow-up result. That is, we extend the blow-up theorem from [95] to general homogeneous groups.

Let us consider the following Cauchy-Dirichlet fractional heat equation on the homogeneous group:

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} + (-\Delta_s)u(x,t) = u(x,t)\log|u(x,t)|, \quad (x,t) \in \Omega \times (0,+\infty), \ \Omega \subset \mathbb{G}, \\ u(x,t) = 0, \quad (x,t) \in \mathbb{G} \setminus \Omega \times (0,+\infty), \\ u(x,0) = u_0(x), \end{cases} \tag{5.116}$$

where $(-\Delta_s)$ is the fractional sub-Laplacian with $s \in (0,1)$.

For simplicity, we introduce the notations $H_0^s(\Omega) := W_0^{s,2}(\Omega)$ and $[u]_s := [u]_{s,2,\Omega}$. Let us give the definition of a weak solution.

Definition 5.36. Let $T > 0$. A function $u : \Omega \times [0,+\infty) \to \mathbb{R}$, $u = u(x,t) \in L^\infty(0,T; H_0^s(\Omega))$ with $\frac{\partial u}{\partial t} \in L^2(0,T; L^2(\Omega))$ is a called a weak solution of problem

(5.116) in $\Omega \times [0, +\infty)$, if $u_0 \in H_0^s(\Omega)$ and u satisfies (5.116) in the sense of distribution,

$$\int_\Omega u_t \varphi dx + \langle (-\Delta_s)u, \varphi \rangle = \int_\Omega u \log |u| \varphi dx, \tag{5.117}$$

for any $\varphi \in H_0^s(\Omega)$, $t \in (0, +\infty)$.

Let us introduce the definition of the blow-up in infinite time.

Definition 5.37. Let $u(x,t)$ be a weak solution of (5.116). We say that $u(x,t)$ blows up at $+\infty$ if

$$\lim_{t\to+\infty} \|u(\cdot,t)\|^2_{L^2(\Omega)} = +\infty. \tag{5.118}$$

Let us consider the following energy functionals

$$J(u) = \frac{1}{2}[u]_s^2 - \frac{1}{2}\int_\Omega u^2 \log |u| dx + \frac{1}{4}\int_\Omega |u|^2 dx, \tag{5.119}$$

and

$$I(u) = [u]_s^2 - \int_\Omega u^2 \log |u| dx. \tag{5.120}$$

By combining last facts, we have relation between two functionals in the following form:

$$J(u) = \frac{1}{2} I(u) + \frac{1}{4}\int_\Omega |u|^2 dx. \tag{5.121}$$

We have the following energy identity for (5.116).

Lemma 5.38. *Assume that u is a weak solution of the problem (5.116). Then we have*

$$\int_0^t \|u_\tau\|^2_{L^2(\Omega)} d\tau + J(u) = J(u_0), \quad \forall t \in (0, +\infty). \tag{5.122}$$

Proof. By taking inner product between (5.116) and u_t over Ω, we get

$$\int_\Omega |u_t|^2 dx + \langle (-\Delta_s)u, u_t \rangle = \int_\Omega u_t u \log |u| dx. \tag{5.123}$$

For the second term on the left hand side of (5.123), we have

$$\begin{aligned} \langle (-\Delta_s)u, u_t \rangle &= \int_\Omega \int_\Omega \frac{(u(x,t) - u(y,t))(u_t(x,t) - u_t(y,t))}{|y^{-1}x|^{Q+2s}} dxdy \\ &= \frac{1}{2}\frac{d}{dt}\int_\Omega \int_\Omega \frac{|u(x,t) - u(y,t)|^2}{|y^{-1}x|^{Q+2s}} dxdy = \frac{1}{2}\frac{d[u]_s^2}{dt}. \end{aligned} \tag{5.124}$$

On the right hand side of (5.123), we get

$$\frac{du^2 \log |u|}{dt} = 2u_t u \log |u| + u u_t, \tag{5.125}$$

then

$$\begin{aligned} \int_\Omega u_t u \log |u| dx &= \frac{1}{2}\frac{d}{dt}\int_\Omega u^2 \log |u| dx - \frac{1}{2}\int_\Omega u u_t dx \\ &= \frac{1}{2}\frac{d}{dt}\int_\Omega u^2 \log |u| dx - \frac{1}{4}\frac{d}{dt}\int_\Omega u^2 dx. \end{aligned} \tag{5.126}$$

By using (5.124) and (5.126) in (5.123), we obtain

$$\int_\Omega |u_t|^2 dx + \frac{d}{dt}\left(\frac{1}{2}[u]_s^2 - \frac{1}{2}\int_\Omega u^2 \log|u| dx + \frac{1}{4}\int_\Omega u^2 dx\right)$$
$$= \int_\Omega |u_t|^2 dx + \frac{d}{dt}J(u) = 0. \quad (5.127)$$

Integrating over $(0, t)$, we arrive at

$$\int_0^t \|u_\tau\|^2_{L^2(\Omega)} d\tau + \int_0^t \frac{dJ(u)}{d\tau} d\tau = 0, \quad (5.128)$$

that is,

$$\int_0^t \|u_\tau\|^2_{L^2(\Omega)} d\tau + J(u) = J(u_0). \quad (5.129)$$

□

Now we are in the position to present the main result of this section.

Theorem 5.39. *Assume that u is a weak solution of (5.116) with $u_0 \in H_0^s(\Omega)$ and $I(u_0) < 0$. Then*

$$\lim_{t\to+\infty} \|u(\cdot, t)\|^2_{L^2(\Omega)} = +\infty. \quad (5.130)$$

Proof. Firstly, by combining (5.117) with $u = \varphi$ we have

$$\frac{d}{dt}\|u\|^2_{L^2(\Omega)} = \frac{d}{dt}\int_\Omega u^2 dx = 2\int_\Omega u u_t dx$$
$$= -2\left(\langle(-\Delta_s)u, u\rangle - \int_\Omega u^2 \log|u| dx\right) = -2I(u). \quad (5.131)$$

From last fact, (5.117) and (5.120), we get

$$\begin{aligned}
\frac{dI(u)}{dt} &= \frac{d}{dt}\left([u]_s^2 - \int_\Omega u^2 \log|u| dx\right)\\
&= 2\int_\Omega\int_\Omega \frac{(u(x,t)-u(y,t))(u_t(x,t)-u_t(y,t))}{|y^{-1}x|^{Q+2s}} dxdy\\
&\quad - 2\int_\Omega u(x,t)u_t(x,t)\log|u(x,t)| dx - \int_\Omega u(x,t)u_t(x,t) dx\\
&= 2\langle(-\Delta_s)u, u_t\rangle - 2\int_\Omega u(x,t)u_t(x,t)\log|u(x,t)| dx - \int_\Omega u(x,t)u_t(x,t) dx\\
&= \int_\Omega |u_t(x,t)|^2 dx - \int_\Omega u(x,t)u_t(x,t) dx\\
&= -2\|u_t\|^2_{L^2(\Omega)} - \int_\Omega u(x,t)u_t(x,t) dx\\
&= -2\|u_t\|^2_{L^2(\Omega)} + [u]_s^2 - \int_\Omega u^2(x,t)\log|u(x,t)| dx\\
&= -2\|u_t\|^2_{L^2(\Omega)} + I(u) \leq I(u).
\end{aligned} \quad (5.132)$$

Then by combining Grönwall–Bellman's inequality and $I(u_0) < 0$ in the last fact we have

$$I(u) \leq I(u_0)e^t \leq I(u_0) < 0, \;\; \forall t \in (0, +\infty). \tag{5.133}$$

It means that $I(u(x,t))$ is decreasing functional with respect to the argument t. By setting

$$A(t) = \int_0^t \|u\|^2_{L^2(\Omega)} dt, \;\; A'(t) = \|u\|^2_{L^2(\Omega)}, \tag{5.134}$$

and by Definition 5.36 we have

$$A''(t) = 2\int_\Omega u u_t dx = -2[u]_s + 2\int_\Omega u^2 \log|u| dx = -2I(u). \tag{5.135}$$

A simple calculation gives

$$(\log A(t))' = \frac{A'(t)}{A(t)}, \;\; (\log A(t))'' = \frac{A''(t)A(t) - (A'(t))^2}{A^2(t)}. \tag{5.136}$$

Now let us estimate $\frac{A''(t)A(t)-(A'(t))^2}{A^2(t)}$. By using (5.134), (5.120) and Lemma 5.38, we obtain

$$A''(t) = -2I(u) = -4J(u) + A'(t) = -4J(u_0) + 4\int_0^t \|u_\tau\|^2_{L^2(\Omega)} d\tau + A'(t). \tag{5.137}$$

Similarly, from (5.134) we obtain

$$\begin{aligned}
(A'(t))^2 &= \|u\|^4_{L^2(\Omega)} = \|u\|^4_{L^2(\Omega)} + 2\|u\|^2_{L^2(\Omega)}\|u_0\|^2_{L^2(\Omega)} - 2\|u\|^2_{L^2(\Omega)}\|u_0\|^2_{L^2(\Omega)} \\
&+ \|u_0\|^4_{L^2(\Omega)} - \|u_0\|^4_{L^2(\Omega)} = \left(\int_\Omega \left(u^2 - u_0^2\right) dx\right)^2 + 2\|u\|^2_{L^2(\Omega)}\|u_0\|^2_{L^2(\Omega)} - \|u_0\|^4_{L^2(\Omega)} \\
&= \left(\int_\Omega \int_0^t \frac{\partial u^2}{\partial \tau} d\tau dx\right)^2 + 2\|u\|^2_{L^2(\Omega)}\|u_0\|^2_{L^2(\Omega)} - \|u_0\|^4_{L^2(\Omega)} = 4\left(\int_0^t \int_\Omega u_\tau u dx d\tau\right)^2 \\
&\qquad + 2\|u\|^2_{L^2(\Omega)}\|u_0\|^2_{L^2(\Omega)} - \|u_0\|^4_{L^2(\Omega)}.
\end{aligned} \tag{5.138}$$

Finally, we obtain

$$(A'(t))^2 = 4\left(\int_0^t \int_\Omega u_\tau u dx d\tau\right)^2 + 2\|u\|^2_{L^2(\Omega)}\|u_0\|^2_{L^2(\Omega)} - \|u_0\|^4_{L^2(\Omega)}. \tag{5.139}$$

It follows that

$$\begin{aligned}
A''(t)A(t) - (A'(t))^2 &= -4J(u_0)A(t) + 4\int_0^t \|u_\tau\|^2_{L^2(\Omega)} d\tau A(t) + A'(t)A(t) \\
&- 4\left(\int_0^t \int_\Omega u_\tau u dx d\tau\right)^2 - 2\|u\|^2_{L^2(\Omega)}\|u_0\|^2_{L^2(\Omega)} + \|u_0\|^4_{L^2(\Omega)} \\
&= 4\left(\int_0^t \|u_\tau\|^2_{L^2(\Omega)} d\tau \int_0^t \|u\|^2_{L^2(\Omega)} - \left(\int_0^t \int_\Omega u_\tau u dx d\tau\right)^2\right) \\
&- 4J(u_0)A(t) + A'(t)A(t) - 2\|u_0\|^2_{L^2(\Omega)}A'(t) + \|u_0\|^4_{L^2(\Omega)}.
\end{aligned} \tag{5.140}$$

By using the Cauchy-Bunyakovsky-Schwarz inequality, we have

$$\begin{aligned} A''(t)A(t)-(A'(t))^2 &= 4\left(\int_0^t \|u_\tau\|^2_{L^2(\Omega)}d\tau \int_0^t \|u\|^2_{L^2(\Omega)} - \left(\int_0^t\int_\Omega u_\tau u dx d\tau\right)^2\right) \\ &\quad - 4J(u_0)A(t) + A'(t)A(t) - 2\|u_0\|^2_{L^2(\Omega)}A'(t) + \|u_0\|^4_{L^2(\Omega)} \\ &\geq A'(t)\left(\frac{A(t)}{2} - \|u_0\|^2_{L^2(\Omega)}\right) + A(t)\left(\frac{A'(t)}{2} - 4J(u_0)\right). \end{aligned} \tag{5.141}$$

By using (5.134), (5.135) and $I(u) \leq I(u_0) < 0$, we get

$$\begin{aligned} &A'(t) = A'(0) - 2\int_0^t I(u(x,\tau))d\tau = -2I(u_0)t \geq 0, \quad t \geq 0, \\ &A(t) = -I(u_0)t^2 \geq 0, \quad t \geq 0. \end{aligned} \tag{5.142}$$

By combining (5.142) and (5.120) in (5.141), we compute

$$\begin{aligned} A''(t)A(t)-(A'(t))^2 &\geq A'(t)\left(\frac{A(t)}{2} - \|u_0\|^2_{L^2(\Omega)}\right) + A(t)\left(\frac{A'(t)}{2} - 4J(u_0)\right) \\ &\geq A'(t)\left(\frac{-I(u_0)t^2}{2} - \|u_0\|^2_{L^2(\Omega)}\right) + A(t)\left(-I(u_0)t - 4J(u_0)\right) \\ &\geq A'(t)\left(\frac{-I(u_0)t^2}{2} - \|u_0\|^2_{L^2(\Omega)}\right) + A(t)\left(-I(u_0)t - 2I(u_0) - \|u_0\|^2_{L^2(\Omega)}\right) \\ &\geq A'(t)\left(\frac{-I(u_0)t^2}{2} - \|u_0\|^2_{L^2(\Omega)}\right) + A(t)\left(-I(u_0)(t+2) - \|u_0\|^2_{L^2(\Omega)}\right). \end{aligned} \tag{5.143}$$

From Definition 5.36, we have that $u_0 \in H_0^s(\Omega)$ and let

$$t > t_0 = \max\left\{\frac{\|u_0\|^2_{L^2(\Omega)}}{-I(u_0)} - 2,\ \frac{\sqrt{2}\|u_0\|_{L^2(\Omega)}}{\sqrt{-I(u_0)}}\right\} \geq 0. \tag{5.144}$$

Firstly, let us consider the case

$$t_0 = \max\left\{\frac{\|u_0\|^2_{L^2(\Omega)}}{-I(u_0)} - 2,\ \frac{\sqrt{2}\|u_0\|_{L^2(\Omega)}}{\sqrt{-I(u_0)}}\right\} = \frac{\sqrt{2}\|u_0\|_{L^2(\Omega)}}{\sqrt{-I(u_0)}}. \tag{5.145}$$

By using this fact in (5.143), we get

$$\begin{aligned}
&A''(t)A(t) - (A'(t))^2 \\
&\qquad \geq A'(t)\left(\frac{-I(u_0)t^2}{2} - \|u_0\|^2_{L^2(\Omega)}\right) + A(t)\left(-I(u_0)(t+2) - \|u_0\|^2_{L^2(\Omega)}\right) \\
&\qquad \geq A'(t)\left(\frac{-I(u_0)t_0^2}{2} - \|u_0\|^2_{L^2(\Omega)}\right) + A(t)\left(-I(u_0)(t_0+2) - \|u_0\|^2_{L^2(\Omega)}\right) \\
&\qquad = A'(t)\left(\|u_0\|^2_{L^2(\Omega)} - \|u_0\|^2_{L^2(\Omega)}\right) + A(t)\left(-I(u_0)(t_0+2) - \|u_0\|^2_{L^2(\Omega)}\right) \\
&\qquad = A(t)\left(-I(u_0)(t_0+2) - \|u_0\|^2_{L^2(\Omega)}\right) \\
&\qquad \geq A(t)\left(-I(u_0)\left(\frac{\|u_0\|^2_{L^2(\Omega)}}{-I(u_0)} - 2 + 2\right) - \|u_0\|^2_{L^2(\Omega)}\right) \\
&\qquad = 0.
\end{aligned} \tag{5.146}$$

Hence, we obtain

$$A''(t)A(t) - (A'(t))^2 \geq 0. \tag{5.147}$$

Similarly, in the other case

$$t_0 = \max\left\{\frac{\|u_0\|^2_{L^2(\Omega)}}{-I(u_0)} - 2, \frac{\|u_0\|_{L^2(\Omega)}}{\sqrt{-I(u_0)}}\right\} = \frac{\|u_0\|^2_{L^2(\Omega)}}{-I(u_0)} - 2, \tag{5.148}$$

we have

$$A''(t)A(t) - (A'(t))^2 \geq 0. \tag{5.149}$$

So we get

$$(\log A(t))'' = \frac{A''(t)A(t) - (A'(t))^2}{A^2(t)}, \tag{5.150}$$

and integrating over (t_0, t), we have

$$(\log A(t))' - (\log A(t))'|_{t=t_0} = \int_{t_0}^{t} \frac{A''(\tau)A(\tau) - (A'(\tau))^2}{A^2(\tau)} d\tau \geq 0. \tag{5.151}$$

Hence, we have

$$(\log A(t))' \geq (\log A(t))'|_{t=t_0}. \tag{5.152}$$

Similarly, we have

$$\frac{A'(t_0)}{A(t_0)}(t - t_0) = (\log A(t))'|_{t=t_0}(t - t_0) \leq \int_{t_0}^{t} \log(A(\tau))' d\tau = \log(A(t)) - \log(A(t_0)). \tag{5.153}$$

Finally, we arrive at

$$A(t_0)e^{\frac{A'(t_0)}{A(t)}(t-t_0)} \leq A(t). \tag{5.154}$$

By summarising above facts (5.152)-(5.154) with $t \geq t_0$, we compute

$$\|u\|^2_{L^2(\Omega)} = A'(t) = (\log A(t))' A(t) \geq (\log A(t))'|_{t=t_0} A(t) = \frac{A'(t_0)}{A(t_0)} A(t) = \frac{A(t)}{A(t_0)} A'(t_0)$$
$$\geq A'(t_0) e^{\frac{A'(t_0)}{A(t_0)}(t-t_0)} = \|u(\cdot, t_0)\|^2_{L^2(\Omega)} e^{\frac{A'(t_0)}{A(t_0)}(t-t_0)} \geq \|u_0\|^2_{L^2(\Omega)} e^{\frac{A'(t_0)}{A(t_0)}(t-t_0)}. \tag{5.155}$$

That is,

$$\lim_{t\to+\infty} \|u(\cdot, t)\|^2_{L^2(\Omega)} = +\infty. \tag{5.156}$$

□

Remark 5.40. *In the Abelian (Euclidean) case $\mathbb{G} = (\mathbb{R}^N, +)$, we have $Q = N$ and $|\cdot| = |\cdot|_E$ ($|\cdot|_E$ is the Euclidean distance), if $s \to 1^-$ we get blow-up result at infinity in [94] and [95].*

5.7. **Non blow-up and blow-up results for the heat equation on stratified groups.** Similarly to previous section, we prove non blow-up and blow-up results for the heat equation on stratified groups.

Firstly, let us give Green's formulae which is play a key role in our proof.

Theorem 5.41 (Green's identity, [96])**.** *Let $Q \geq 3$ be a homogeneous dimension of a stratified group $\mathbb{G}$ and dx be the volume element on $\mathbb{G}$. Let $v \in C^1(\Omega) \cap C(\overline{\Omega})$ and $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$. Then the following Green's identity holds*

$$\int_\Omega \left((\widetilde{\nabla} v) u + v \Delta_{\mathbb{G}} u \right) dx = \int_{\partial\Omega} |u|^{p-2} v \langle \widetilde{\nabla} u, dx \rangle, \tag{5.157}$$

where

$$\widetilde{\nabla} u = \sum_{k=1}^{N_1} (X_k u) X_k.$$

In this section, we obtain a non-blow-up result for the following problem on stratified group:

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} - \mu \Delta_{\mathbb{G}} u(x,t) = u(x,t) \ln |u(x,t)|, & (x,t) \in \Omega \times (0,T), \ \Omega \subset \mathbb{G}, \\ u(x,t)\Big|_{\partial\Omega} = 0, \ \ t \in (0,T), \\ u(x,0) = u_0(x) \ \ x \in \Omega, \end{cases} \tag{5.158}$$

where $\Delta_{\mathbb{G}}$ is the sub-Laplacian, μ is a positive constant and Ω is a bounded domain with smooth boundary.

Let us recall the definition of a weak solution.

Definition 5.42. Let $T > 0$. A function $u : \Omega \times [0, +\infty) \to \mathbb{R}$, $u = u(x,t) \in L^\infty(0, T; S^{1,2}_0(\Omega))$ with $\frac{\partial u}{\partial t} \in L^2(0, T; L^2(\Omega))$ is a called a weak solution of problem (5.158) in $\Omega \times [0, +\infty)$, if $u_0 \in S^{1,2}_0(\Omega)$ and u satisfies (5.158) in the sense of distribution

$$\int_\Omega u_t \varphi dx - \mu \int_\Omega \varphi \Delta_{\mathbb{G}} u dx = \int_\Omega u \ln |u| \varphi dx, \tag{5.159}$$

for any $\varphi \in S^{1,2}_0(\Omega)$, $t \in (0, T)$.

Let us also recall the definition of blow-up at finite time.

Definition 5.43. Let $u(x,t)$ be a weak solution of (5.158). We say that $u(x,t)$ blows up at $T < +\infty$ if

$$\lim_{t\to T^-} \|u(\cdot,t)\|^2_{L^2(\Omega)} = +\infty. \tag{5.160}$$

We use the following notations for energy functionals

$$J(u) = \frac{\mu}{2}\int_\Omega |\nabla_{\mathbb{G}} u|^2 dx - \frac{1}{2}\int_\Omega u^2 \ln|u| dx + \frac{1}{4}\int_\Omega |u|^2 dx, \tag{5.161}$$

and

$$I(u) = \mu \int_\Omega |\nabla_{\mathbb{G}} u|^2 dx - \int_\Omega u^2 \ln|u| dx, \tag{5.162}$$

where $\mu > 0$. Thus, we have

$$J(u) = \frac{1}{2}I(u) + \frac{1}{4}\int_\Omega |u|^2 dx. \tag{5.163}$$

Also, one of the main tool is the logarithmic Sobolev-Follan-Stein inequality which is defined in Theorem 3.45.

Theorem 5.44. *Suppose that u is a weak solution of (5.158) with $u_0 \in S^{1,2}_0(\Omega)$ and $\mu \geq QC_S$, where C_S is the Sobolev-Folland-Stein constant. Then u does not blow-up at finite time.*

Proof. Let us define the following function:

$$A(t) := \int_0^t \|u(\cdot,\tau)\|^2_{L^2(\Omega)} d\tau,$$

then we obtain

$$A'(t) = \|u(\cdot,t)\|^2_{L^2(\Omega)},$$

$$A''(t) = -2I(u).$$

By using the logarithmic Sobolev-Folland-Sobolev inequality (Theorem 3.45) with $a = 1$, we get

$$\begin{aligned}
A'(t)\ln A'(t) - A''(t) &= \|u\|^2_{L^2(\Omega)} \ln \|u\|^2_{L^2(\Omega)} + 2I(u) \\
&= 2\|u\|^2_{L^2(\Omega)} \ln \|u\|_{L^2(\Omega)} + 2I(u) \\
&= 2\|u\|^2_{L^2(\Omega)} \ln \|u\|_{L^2(\Omega)} + 2\mu\|\nabla_{\mathbb{G}} u\|^2_{L^2(\Omega)} - 2\int_\Omega u^2 \ln |u| dx \\
&\geq 2\|u\|^2_{L^2(\Omega)} \ln \|u\|_{L^2(\Omega)} + \mu\|\nabla_{\mathbb{G}} u\|^2_{L^2(\Omega)} - 2\int_\Omega u^2 \ln |u| dx \\
&\geq 2\|u\|^2_{L^2(\Omega)} \ln \|u\|_{L^2(\Omega)} + QC_S\|\nabla_{\mathbb{G}} u\|^2_{L^2(\Omega)} - 2\int_\Omega u^2 \ln |u| dx \\
&= QC_S\|\nabla_{\mathbb{G}} u\|^2_{L^2(\Omega)} - 2\int_\Omega |u|^2 \ln \frac{|u|}{\|u\|_{L^2(\Omega)}} dx \\
&\overset{(3.159)}{\geq} 2\int_{\mathbb{G}} |u|^2 \ln \frac{|u|}{\|u\|_{L^2(\mathbb{G})}} dx + Q\|u\|^2_{L^2(\mathbb{G})} - 2\int_\Omega |u|^2 \ln \frac{|u|}{\|u\|_{L^2(\Omega)}} dx \\
&= Q\|u\|^2_{L^2(\mathbb{G})}.
\end{aligned} \tag{5.164}$$

It implies

$$A'(t)\ln A'(t) - A''(t) \geq Q\|u\|^2_{L^2(\mathbb{G})} \geq 0. \tag{5.165}$$

That is, $A'(t)\ln A'(t) \geq A''(t)$ which yields

$$\ln A'(t) \geq (\ln A'(t))'.$$

Now by integrating it over $(0, t)$, we obtain

$$\ln \|u(\cdot, t)\|^2_{L^2(\Omega)} = \ln A'(t) \leq e^t \ln A'(0) = e^t \ln \|u_0\|^2_{L^2(\Omega)}.$$

Finally, we arrive at

$$\|u(\cdot, t)\|_{L^2(\Omega)} \leq \|u_0\|^{e^t}_{L^2(\Omega)}. \tag{5.166}$$

It means $\|u(\cdot, t)\|^2_{L^2(\Omega)}$ is bounded at finite time $T^* \in (0, \infty)$. ☐

Then let us show blow-up result in infinite time. Let us consider the following initial-boundary (Cauchy-Dirichlet) heat equation on stratified groups:

$$\begin{cases}
\frac{\partial u(x,t)}{\partial t} - \mu\Delta_{\mathbb{G}} u(x,t) = u(x,t)\ln|u(x,t)|, & (x,t) \in \Omega \times (0,+\infty),\ \Omega \subset \mathbb{G}, \\
u(x,t) = 0, \ \ (x,t) \in \Omega \times (0,+\infty), \\
u(x,0) = u_0(x),
\end{cases} \tag{5.167}$$

where $\Delta_{\mathbb{G}}$ is the sub-Laplacian and $\mu > 0$.

Definition 5.45. Assume that $u(x,t)$ be a weak solution of (5.167). We say that $u(x,t)$ blows up at $+\infty$ if

$$\lim_{t\to+\infty} \|u(\cdot, t)\|^2_{L^2(\Omega)} = +\infty. \tag{5.168}$$

We have the following energy identity for problem (5.167).

Lemma 5.46. *Suppose that u is a weak solution of the problem (5.167). Then we have*

$$\int_0^t \|u_\tau\|^2_{L^2(\Omega)} d\tau + J(u) = J(u_0), \;\; \forall t \in (0, +\infty), \tag{5.169}$$

where the functional J is defined by (5.161).

Proof. As usual, multiplying by u_t and integrating over Ω in (5.167), we get

$$\int_\Omega |u_t|^2 dx - \mu \int_\Omega \Delta_{\mathbb{G}} u \, u_t dx = \int_\Omega u_t u \ln |u| dx. \tag{5.170}$$

By using Green's identity to the second term on the left hand side of (5.170), we obtain

$$-\int_\Omega \Delta_{\mathbb{G}} u(x,t) \, u_t(x,t) dx = \int_\Omega \nabla_{\mathbb{G}} u(x,t) \cdot \nabla_{\mathbb{G}} u_t(x,t) dx = \frac{1}{2}\frac{d}{dt}\|\nabla_{\mathbb{G}} u\|^2_{L^2(\Omega)}. \tag{5.171}$$

On the other hand, we have

$$\frac{du^2 \ln |u|}{dt} = 2u_t u \ln |u| + uu_t, \tag{5.172}$$

that is,

$$\begin{aligned} \int_\Omega u_t u \ln |u| dx &= \frac{1}{2}\frac{d}{dt}\int_\Omega u^2 \ln |u| dx - \frac{1}{2}\int_\Omega uu_t dx \\ &= \frac{1}{2}\frac{d}{dt}\int_\Omega u^2 \ln |u| dx - \frac{1}{4}\frac{d}{dt}\int_\Omega u^2 dx. \end{aligned} \tag{5.173}$$

By combining (5.171) and (5.173) with (5.170), we get

$$\begin{aligned} \int_\Omega |u_t|^2 dx + \frac{d}{dt}\left(\frac{\mu}{2}\|\nabla_{\mathbb{G}} u\|^2_{L^2(\Omega)} - \frac{1}{2}\int_\Omega u^2 \ln |u| dx + \frac{1}{4}\int_\Omega u^2 dx\right) \\ = \int_\Omega |u_t|^2 dx + \frac{d}{dt}J(u) = 0. \end{aligned} \tag{5.174}$$

Now integrating over $(0,t)$, we arrive at

$$\int_0^t \|u_\tau\|^2_{L^2(\Omega)} d\tau + \int_0^t \frac{dJ(u)}{d\tau} d\tau = 0, \tag{5.175}$$

that is,

$$\int_0^t \|u_\tau\|^2_{L^2(\Omega)} d\tau + J(u) = J(u_0). \tag{5.176}$$

□

Now we are in the position to present one of the main result of this section.

Theorem 5.47. *Assume that u be a weak solution of (5.167) with $u_0 \in S_0^{1,2}(\Omega)$ and $I(u_0) < 0$. Then*

$$\lim_{t \to +\infty} \|u(\cdot,t)\|^2_{L^2(\Omega)} = +\infty. \tag{5.177}$$

Proof. Firstly, by taking (5.159) with $u = \varphi$ we get

$$\begin{aligned}\frac{d}{dt}\|u\|^2_{L^2(\Omega)} = \frac{d}{dt}\int_\Omega u^2 dx = 2\int_\Omega uu_t dx \\ = -2\left(\mu\|\nabla_{\mathbb{G}} u\|^2_{L^2(\Omega)} - \int_\Omega u^2 \ln|u| dx\right) = -2I(u).\end{aligned} \tag{5.178}$$

By combining last fact with (5.159) and (5.162), we get

$$\begin{aligned}\frac{dI(u)}{dt} &= \frac{d}{dt}\left(\mu\|\nabla_{\mathbb{G}} u\|^2_{L^2(\Omega)} - \int_\Omega u^2 \ln|u| dx\right) \\ &= 2\mu\int_\Omega \nabla_{\mathbb{G}} u \cdot \nabla_{\mathbb{G}} u_t dx - 2\int_\Omega u(x,t)u_t(x,t)\ln|u(x,t)|dx - \int_\Omega u(x,t)u_t(x,t)dx \\ &= -2\mu\int_\Omega u_t \Delta_{\mathbb{G}} u dx - 2\int_\Omega u(x,t)u_t(x,t)\ln|u(x,t)|dx - \int_\Omega u(x,t)u_t(x,t)dx \\ &= -2\int_\Omega |u_t(x,t)|^2 dx - \int_\Omega u(x,t)u_t(x,t)dx \\ &= -2\|u_t\|^2_{L^2(\Omega)} - \int_\Omega u(x,t)u_t(x,t)dx \\ &= -2\|u_t\|^2_{L^2(\Omega)} + \mu\|\nabla_{\mathbb{G}} u\|^2_{L^2(\Omega)} - \int_\Omega u^2(x,t)\ln|u(x,t)|dx \\ &= -2\|u_t\|^2_{L^2(\Omega)} + I(u) \leq I(u).\end{aligned} \tag{5.179}$$

From Grönwall–Bellman's inequality and $I(u_0) < 0$ we have

$$I(u) \leq I(u_0)e^t \leq I(u_0) < 0, \quad \forall t \in (0, T). \tag{5.180}$$

It shows that $I(u(x,t))$ is a decreasing functional with respect to t.

By setting

$$A(t) = \int_0^t \|u(\cdot,\tau)\|^2_{L^2(\Omega)} d\tau, \;\; A'(t) = \|u(\cdot,t)\|^2_{L^2(\Omega)}, \tag{5.181}$$

and by Definition 5.159 we have

$$A''(t) = 2\int_\Omega uu_t dx = -2\|\nabla_{\mathbb{G}} u\|^2_{L^2(\Omega)} + 2\int_\Omega u^2 \ln|u| dx = -2I(u). \tag{5.182}$$

Now let us estimate

$$(\ln A(t))'' = \frac{A''(t)A(t) - (A'(t))^2}{A^2(t)}.$$

From (5.181), (5.162) and Lemma 5.46, we get

$$A''(t) = -2I(u) = -4J(u) + A'(t) = -4J(u_0) + 4\int_0^t \|u_\tau\|^2_{L^2(\Omega)} d\tau + A'(t). \tag{5.183}$$

Similarly, from (5.181) we obtain

$$\begin{aligned}(A'(t))^2 &= \|u\|^4_{L^2(\Omega)}+2\|u\|^2_{L^2(\Omega)}\|u_0\|^2_{L^2(\Omega)}-2\|u\|^2_{L^2(\Omega)}\|u_0\|^2_{L^2(\Omega)}+\|u_0\|^4_{L^2(\Omega)}-\|u_0\|^4_{L^2(\Omega)}\\ &= 4\left(\int_0^t\int_\Omega u_\tau u dx d\tau\right)^2 + 2\|u\|^2_{L^2(\Omega)}\|u_0\|^2_{L^2(\Omega)} - \|u_0\|^4_{L^2(\Omega)}. \qquad (5.184)\end{aligned}$$

Hence, we have

$$(A'(t))^2 = 4\left(\int_0^t\int_\Omega u_\tau u dx d\tau\right)^2 + 2\|u\|^2_{L^2(\Omega)}\|u_0\|^2_{L^2(\Omega)} - \|u_0\|^4_{L^2(\Omega)}. \qquad (5.185)$$

It follows that

$$\begin{aligned}A''(t)A(t) - (A'(t))^2 &= 4\left(\int_0^t \|u_\tau\|^2_{L^2(\Omega)}d\tau\int_0^t\|u\|^2_{L^2(\Omega)} - \left(\int_0^t\int_\Omega u_\tau u dx d\tau\right)^2\right)\\ &\quad - 4J(u_0)A(t) + A'(t)A(t) - 2\|u_0\|^2_{L^2(\Omega)}A'(t) + \|u_0\|^4_{L^2(\Omega)}.\end{aligned} \qquad (5.186)$$

From the Cauchy-Bunyakovsky-Schwarz inequality, we obtain

$$A''(t)A(t) - (A'(t))^2 \geq A'(t)\left(\frac{A(t)}{2} - \|u_0\|^2_{L^2(\Omega)}\right) + A(t)\left(\frac{A'(t)}{2} - 4J(u_0)\right). \qquad (5.187)$$

By using (5.181), (5.182) and $I(u) \leq I(u_0) < 0$, we have

$$\begin{aligned}&A'(t) = A'(0) - 2\int_0^t I(u(x,\tau))d\tau = -2I(u_0)t \geq 0, \quad t \geq 0,\\ &A(t) = -I(u_0)t^2 \geq 0, \quad t \geq 0.\end{aligned} \qquad (5.188)$$

By using (5.188) and (5.162) in (5.187), we calculate

$$\begin{aligned}A''(t)A(t) - (A'(t))^2 &\geq A'(t)\left(\frac{A(t)}{2} - \|u_0\|^2_{L^2(\Omega)}\right) + A(t)\left(\frac{A'(t)}{2} - 4J(u_0)\right)\\ &\geq A'(t)\left(\frac{-I(u_0)t^2}{2} - \|u_0\|^2_{L^2(\Omega)}\right) + A(t)\left(-I(u_0)t - 4J(u_0)\right)\\ &\geq A'(t)\left(\frac{-I(u_0)t^2}{2} - \|u_0\|^2_{L^2(\Omega)}\right) + A(t)\left(-I(u_0)t - 2I(u_0) - \|u_0\|^2_{L^2(\Omega)}\right)\\ &\geq A'(t)\left(\frac{-I(u_0)t^2}{2} - \|u_0\|^2_{L^2(\Omega)}\right) + A(t)\left(-I(u_0)(t+2) - \|u_0\|^2_{L^2(\Omega)}\right).\end{aligned} \qquad (5.189)$$

From Definition 5.42, we have that $u_0 \in S_0^{1,2}(\Omega)$ and let

$$t > t_0 = \max\left\{\frac{\|u_0\|^2_{L^2(\Omega)}}{-I(u_0)} - 2, \frac{\sqrt{2}\|u_0\|_{L^2(\Omega)}}{\sqrt{-I(u_0)}}\right\} \geq 0. \qquad (5.190)$$

Let us consider the case

$$t_0 = \max\left\{\frac{\|u_0\|^2_{L^2(\Omega)}}{-I(u_0)} - 2, \frac{\sqrt{2}\|u_0\|_{L^2(\Omega)}}{\sqrt{-I(u_0)}}\right\} = \frac{\sqrt{2}\|u_0\|_{L^2(\Omega)}}{\sqrt{-I(u_0)}}. \qquad (5.191)$$

By combining this fact in (5.189), we obtain

$$\begin{aligned} &A''(t)A(t)-(A'(t))^2 \\ &\qquad \geq A'(t)\left(\frac{-I(u_0)t^2}{2}-\|u_0\|^2_{L^2(\Omega)}\right)+A(t)\left(-I(u_0)(t+2)-\|u_0\|^2_{L^2(\Omega)}\right) \\ &\qquad \geq A'(t)\left(\frac{-I(u_0)t_0^2}{2}-\|u_0\|^2_{L^2(\Omega)}\right)+A(t)\left(-I(u_0)(t_0+2)-\|u_0\|^2_{L^2(\Omega)}\right) \\ &\qquad \geq A(t)\left(-I(u_0)\left(\frac{\|u_0\|^2_{L^2(\Omega)}}{-I(u_0)}\right)-\|u_0\|^2_{L^2(\Omega)}\right)=0. \end{aligned} \tag{5.192}$$

Hence, we obtain

$$A''(t)A(t)-(A'(t))^2 \geq 0. \tag{5.193}$$

Similarly, in the other case

$$t_0=\max\left\{\frac{\|u_0\|^2_{L^2(\Omega)}}{-I(u_0)}-2,\ \frac{\|u_0\|_{L^2(\Omega)}}{\sqrt{-I(u_0)}}\right\}=\frac{\|u_0\|^2_{L^2(\Omega)}}{-I(u_0)}-2, \tag{5.194}$$

we have

$$A''(t)A(t)-(A'(t))^2 \geq 0. \tag{5.195}$$

So we have

$$(\ln A(t))''=\frac{A''(t)A(t)-(A'(t))^2}{A^2(t)},$$

and integrating over (t_0,t), we have

$$(\ln A(t))'-(\ln A(t))'|_{t=t_0}=\int_{t_0}^{t}\frac{A''(\tau)A(\tau)-(A'(\tau))^2}{A^2(\tau)}d\tau \geq 0. \tag{5.196}$$

Thus, we have

$$(\ln A(t))' \geq (\ln A(t))'|_{t=t_0}. \tag{5.197}$$

Similarly, we obtain

$$\frac{A'(t_0)}{A(t_0)}(t-t_0)=(\ln A(t))'|_{t=t_0}(t-t_0)\leq\int_{t_0}^{t}\ln(A(\tau))'d\tau=\ln(A(t))-\ln(A(t_0)). \tag{5.198}$$

Finally, we arrive at

$$A(t_0)e^{\frac{A'(t_0)}{A(t)}(t-t_0)} \leq A(t). \tag{5.199}$$

By using above facts (5.197)-(5.199) with $t \geq t_0$, we compute

$$\begin{aligned} \|u\|^2_{L^2(\Omega)}&=A'(t)=(\ln A(t))'A(t)\overset{(5.197)}{\geq}(\ln A(t))'|_{t=t_0}A(t)=\frac{A(t)}{A(t_0)}A'(t_0) \\ &\overset{(5.154)}{\geq}A'(t_0)e^{\frac{A'(t_0)}{A(t_0)}(t-t_0)}=\|u(\cdot,t_0)\|^2_{L^2(\Omega)}e^{\frac{A'(t_0)}{A(t_0)}(t-t_0)}\geq\|u_0\|^2_{L^2(\Omega)}e^{\frac{A'(t_0)}{A(t_0)}(t-t_0)}. \end{aligned} \tag{5.200}$$

That is,

$$\lim_{t\to+\infty}\|u(\cdot,t)\|^2_{L^2(\Omega)}=+\infty. \tag{5.201}$$

□

5.8. **Blow-up results for the viscoelastic equation on stratified groups.** The following viscoelastic wave equation with weak damping was considered by Messaoudi in [97].

$$\begin{cases} u_{tt} - \Delta u + \displaystyle\int_0^t k(t-\tau)\Delta u d\tau + a|u_t|^{q-2}u_t = |u|^{p-2}u, \ \ (x,t) \in \Omega \times [0,T], \\ u(x,t) = 0, \quad x \in \partial\Omega, \\ u(x,0) = u_0(x), \ u_t(x,0) = u_1(x), \end{cases} \tag{5.202}$$

where $u_0 \in W_0^{1,2}(\Omega)$, $u_1 \in L^2(\Omega)$ and $k \in C^1[0,T]$ satisfying $1 - \int_0^\infty k(\tau)d\tau = r > 0$. The author proved that any solution with negative initial energy $p > q$ blows up in finite-time and extended the result by considering positive initial energy in [98]. We refer [99] and [100] for the further discussions in this topic. Further, let us recall $L^p(\Omega)$-Poincaré inequality on stratified Lie groups (see [29]).

Theorem 5.48. *Assume that* $\Omega \subset \mathbb{G}$ *and* $f \in C_0^\infty(\Omega \setminus \{x' = 0\})$ *and* $R' = \sup_{x\in\Omega} |x'|$. *Then we have*

$$R\|f\|_p \leq \|\nabla_{\mathbb{G}} f\|_p, \quad 1 < p < \infty, \tag{5.203}$$

where $R = \frac{|N-p|}{R'p}$.

5.8.1. *Blow-up with strong damping.* In this subsection, we consider the following nonlinear viscoelastic wave equation on stratified Lie groups:

$$\begin{cases} u_{tt} - \Delta_{\mathbb{G}} u + \displaystyle\int_0^t k(t-\tau)\Delta_{\mathbb{G}} u d\tau - a\Delta_{\mathbb{G}} u_t = |u|^{p-2}u, \ \ (x,t) \in \Omega \times [0,T], \\ u(x,t) = 0, \quad x \in \partial\Omega, \\ u(x,0) = u_0(x), \ u_t(x,0) = u_1(x), \end{cases} \tag{5.204}$$

where $\Omega \subset \mathbb{G}$ is a Haar measurable set with a smooth boundary $\partial\Omega$, $N \geq 3$, where N is defined in (i), $u_0 \in S_0^{1,2}(\Omega)$, $u_1 \in L^2(\Omega)$, a is a positive constant and $p > 2$ satisfies the following condition.

$$\frac{2Q}{Q-2} > p > 2, \ \ Q \geq 3. \tag{5.205}$$

We assume that the function $C^1(0,\infty) \ni k : \mathbb{R}_+ \to \mathbb{R}_+$ has the following properties:

$$1 - \int_0^{+\infty} k(s)ds = r > \frac{1}{(p-1)^2} \tag{5.206}$$

and

$$k(s) \geq 0, \ \ k'(s) \leq 0. \tag{5.207}$$

Let us define the following functional

$$I(t) = \frac{1}{2}\left(\|u_t(t)\|_2^2 + \left(1 - \int_0^t k(s)ds\right)\|\nabla_{\mathbb{G}} u(t)\|_2^2 + k \circ \nabla_{\mathbb{G}} u\right) - \frac{1}{p}\|u(t)\|_p^p, \tag{5.208}$$

where $k \circ \nabla_{\mathbb{G}} u = \displaystyle\int_0^t k(t-\tau)\|\nabla_{\mathbb{G}} u(\cdot,t) - \nabla_{\mathbb{G}} u(\cdot,\tau)\|_2^2 d\tau$.

Let us give the main tools for obtaining blow-up result.

Lemma 5.49. *Assume that (5.206)-(5.207) hold true. Let u be a weak solution of (5.204), then we have*

(a) *$I(t)$ is a non-increasing function, i.e.,*

$$I'(t) \leq 0, \quad \forall t \in [0, T]; \tag{5.209}$$

(b)

$$I(t) + a \int_0^t \|\nabla_{\mathbb{G}} u_t(\tau)\|^2 d\tau \leq I(0), \quad t \in [0, T], \ \ a > 0. \tag{5.210}$$

Proof. Let us rewrite the equation in (5.204) as follows

$$u_{tt} - \Delta_{\mathbb{G}} u + \int_0^t k(t-\tau) \Delta_{\mathbb{G}} u d\tau - a \Delta_{\mathbb{G}} u_t - |u|^{p-2} u = 0.$$

Multiplying both sides by u_t and integrating over Ω , we compute

$$\begin{aligned}
0 &= \int_\Omega u_{tt} u_t dx - \int_\Omega u_t \Delta_{\mathbb{G}} u dx + \int_0^t k(t-\tau) \int_\Omega u_t \Delta_{\mathbb{G}} u dx d\tau - a \int_\Omega u_t \Delta_{\mathbb{G}} u_t dx \\
&- \int_\Omega u_t |u|^{p-2} u dx \\
&\overset{(5.157)}{=} \int_\Omega u_{tt} u_t dx - \int_\Omega \nabla_{\mathbb{G}} u_t \cdot \nabla_{\mathbb{G}} u dx - \int_0^t k(t-\tau) \int_\Omega \nabla_{\mathbb{G}} u_t \cdot \nabla_{\mathbb{G}} u dx d\tau \\
&+ a \int_\Omega |\nabla_{\mathbb{G}} u_t|^2 dx - \int_\Omega u_t |u|^{p-2} u dx \\
&= \frac{d}{dt} \left(\frac{1}{2} \int_\Omega |u_t|^2 dx + \frac{1}{2} \int_\Omega |\nabla_{\mathbb{G}} u|^2 dx - \frac{1}{p} \int_\Omega |u|^p dx \right) \\
&- \int_0^t k(t-\tau) \int_\Omega \nabla_{\mathbb{G}} u_t \cdot \nabla_{\mathbb{G}} u dx d\tau + a \int_\Omega |\nabla_{\mathbb{G}} u_t|^2 dx \\
&= \frac{d}{dt} \left(\frac{1}{2} \|u_t\|^2 + \frac{1}{2} \|\nabla_{\mathbb{G}} u\|^2 - \frac{1}{p} \|u\|_p^p dx \right) - \int_0^t k(t-\tau) \int_\Omega \nabla_{\mathbb{G}} u_t \cdot \nabla_{\mathbb{G}} u dx d\tau \\
&+ a \|\nabla_{\mathbb{G}} u_t\|^2.
\end{aligned} \tag{5.211}$$

Let us calculate the following integral.

$$\begin{aligned}
&\int_0^t k(t-\tau)\int_\Omega \nabla_{\mathbb{G}}u_t(t)\cdot\nabla_{\mathbb{G}}u(\tau)dxd\tau\\
&\qquad=\int_0^t k(t-\tau)\int_\Omega \nabla_{\mathbb{G}}u_t(t)\cdot(\nabla_{\mathbb{G}}u(\tau)-\nabla_{\mathbb{G}}u(t)+\nabla_{\mathbb{G}}u(t))dxd\tau\\
&\qquad=\int_0^t k(t-\tau)\int_\Omega \nabla_{\mathbb{G}}u_t(t)\cdot(\nabla_{\mathbb{G}}u(\tau)-\nabla_{\mathbb{G}}u(t))dxd\tau\\
&\qquad+\int_0^t k(t-\tau)\int_\Omega \nabla_{\mathbb{G}}u_t(t)\cdot\nabla_{\mathbb{G}}u(t)dxd\tau\\
&\qquad=-\frac{1}{2}\int_0^t k(t-\tau)\frac{d}{dt}\int_\Omega |\nabla_{\mathbb{G}}u(\tau)-\nabla_{\mathbb{G}}u(t)|^2dxd\tau\\
&\qquad+\frac{1}{2}\int_0^t k(t-\tau)\frac{d}{dt}\int_\Omega |\nabla_{\mathbb{G}}u(t)|^2dxd\tau\\
&\qquad=-\frac{1}{2}\int_0^t k(t-\tau)\frac{d}{dt}\int_\Omega |\nabla_{\mathbb{G}}u(\tau)-\nabla_{\mathbb{G}}u(t)|^2dxd\tau\\
&\qquad+\frac{1}{2}\int_0^t k(\tau)\frac{d}{dt}\int_\Omega |\nabla_{\mathbb{G}}u(t)|^2dxd\tau.
\end{aligned}\tag{5.212}$$

By direct calculation shows

$$\begin{aligned}
&-\frac{1}{2}\int_0^t k(t-\tau)\frac{d}{dt}\int_\Omega |\nabla_{\mathbb{G}}u(\tau)-\nabla_{\mathbb{G}}u(t)|^2dxd\tau\\
&\qquad=-\frac{1}{2}\frac{d}{dt}\int_0^t k(t-\tau)\|\nabla_{\mathbb{G}}u(\tau)-\nabla_{\mathbb{G}}u(t)\|_2^2d\tau\\
&\qquad+\frac{1}{2}\int_0^t k'(t-\tau)\int_\Omega |\nabla_{\mathbb{G}}u(\tau)-\nabla_{\mathbb{G}}u(t)|^2dxd\tau\\
&\qquad=-\frac{1}{2}\frac{dk\circ\nabla_{\mathbb{G}}u}{dt}+\frac{k'\circ\nabla_{\mathbb{G}}u}{2}
\end{aligned}\tag{5.213}$$

and

$$\frac{1}{2}\int_0^t k(\tau)\frac{d}{dt}\int_\Omega |\nabla_{\mathbb{G}}u(t)|^2dxd\tau=\frac{1}{2}\frac{d}{dt}\left(\int_0^t k(\tau)\|\nabla_{\mathbb{G}}u\|_2^2d\tau\right)-\frac{1}{2}k(t)\|\nabla_{\mathbb{G}}u\|^2. \tag{5.214}$$

By changing the last expressions in (5.212), we have

$$\begin{aligned}
&\int_0^t k(t-\tau)\int_\Omega \nabla_{\mathbb{G}}u_t(t)\cdot\nabla_{\mathbb{G}}u(\tau)dxd\tau\\
&\qquad=-\frac{1}{2}\int_0^t k(t-\tau)\frac{d}{dt}\int_\Omega |\nabla_{\mathbb{G}}u(\tau)-\nabla_{\mathbb{G}}u(t)|^2dxd\tau\\
&\qquad+\frac{1}{2}\int_0^t k(\tau)\frac{d}{dt}\int_\Omega |\nabla_{\mathbb{G}}u(t)|^2dxd\tau\\
&\qquad=-\frac{1}{2}\frac{dk\circ\nabla_{\mathbb{G}}u}{dt}+\frac{k'\circ\nabla_{\mathbb{G}}u}{2}+\frac{1}{2}\frac{d}{dt}\left(\int_0^t k(\tau)\|\nabla_{\mathbb{G}}u\|^2d\tau\right)\\
&\qquad-\frac{1}{2}k(t)\|\nabla_{\mathbb{G}}u\|^2.
\end{aligned}\tag{5.215}$$

Next, by using (5.215) in (5.211) yields

$$\begin{aligned}
0&=\frac{d}{dt}\left(\frac{1}{2}\|u_t\|_2^2+\frac{1}{2}\|\nabla_{\mathbb{G}}u\|_2^2-\frac{1}{p}\|u\|_p^pdx\right)-\int_0^t k(t-\tau)\int_\Omega \nabla_{\mathbb{G}}u_t\cdot\nabla_{\mathbb{G}}udxd\tau\\
&+a\|\nabla_{\mathbb{G}}u_t\|^2\\
&=\frac{d}{dt}\left(\frac{1}{2}\|u_t\|^2+\frac{1}{2}\|\nabla_{\mathbb{G}}u\|^2-\frac{1}{p}\|u\|_p^pdx\right)+\frac{1}{2}\frac{dk\circ\nabla_{\mathbb{G}}u}{dt}-\frac{k'\circ\nabla_{\mathbb{G}}u}{2}\\
&-\frac{1}{2}\frac{d}{dt}\left(\int_0^t k(\tau)\|\nabla_{\mathbb{G}}u\|^2d\tau\right)+\frac{1}{2}k(t)\|\nabla_{\mathbb{G}}u\|^2+a\|\nabla_{\mathbb{G}}u_t\|^2\\
&=\frac{d}{dt}\left(\frac{1}{2}\|u_t\|^2+\frac{1}{2}\|\nabla_{\mathbb{G}}u\|^2-\frac{1}{2}\int_0^t k(\tau)\|\nabla_{\mathbb{G}}u\|^2d\tau+\frac{1}{2}k\circ\nabla_{\mathbb{G}}u-\frac{1}{p}\|u\|_p^pdx\right)\\
&+\frac{1}{2}k(t)\|\nabla_{\mathbb{G}}u\|^2+a\|\nabla_{\mathbb{G}}u_t\|^2-\frac{k'\circ\nabla_{\mathbb{G}}u}{2}\\
&=\frac{dI}{dt}+\frac{1}{2}k(t)\|\nabla_{\mathbb{G}}u\|_2^2+a\|\nabla_{\mathbb{G}}u_t\|^2-\frac{k'\circ\nabla_{\mathbb{G}}u}{2},
\end{aligned}\tag{5.216}$$

that is,

$$\begin{aligned}
\frac{dI}{dt}&=-\frac{1}{2}k(t)\|\nabla_{\mathbb{G}}u\|^2-a\|\nabla_{\mathbb{G}}u_t\|^2+\frac{k'\circ\nabla_{\mathbb{G}}u}{2}=-\frac{1}{2}k(t)\|\nabla_{\mathbb{G}}u\|^2\\
&+\frac{1}{2}\int_0^t k'(t-\tau)\|\nabla_{\mathbb{G}}u(t)-\nabla_{\mathbb{G}}u(\tau)\|^2d\tau-a\|\nabla_{\mathbb{G}}u_t\|^2\\
&\overset{(5.207)}{\leq}-a\|\nabla_{\mathbb{G}}u_t\|^2.
\end{aligned}\tag{5.217}$$

Hence, we get

$$\frac{dI}{dt}\leq -a\|\nabla_{\mathbb{G}}u_t\|^2\leq 0,\tag{5.218}$$

that is,

$$I'(t)\leq 0.$$

It means we proved the statement (a). The part (b) follows from integrating (5.218) over $(0,t)$

$$I(t)-I(0)\leq -a\int_0^t \|\nabla_{\mathbb{G}}u_t\|^2 d\tau, \tag{5.219}$$

which is equivalent to

$$I(t)+a\int_0^t \|\nabla_{\mathbb{G}}u_t\|^2 d\tau \leq I(0).$$

□

Now, we present the main result of this subsection.

Theorem 5.50. *Assume that $p>2$ satisfies (5.205), $a>0$ and $k\in C^1[0,T]$ satisfies the conditions (5.206) and (5.207). Let u be a solution of (5.204), satisfying*

$$\left(2(u,u_t)+a\|\nabla_{\mathbb{G}}u\|_2^2\right)|_{t=0} > \frac{2p}{\theta}I(0), \tag{5.220}$$

where $\theta=\max\limits_{\mu_1\in(0,1)}\theta(\mu_1)=\theta(\mu_1^*)$ with

$$\theta(\mu_1)=\min\left(((p+2)a\alpha\mu_1 R)^{\frac{1}{2}}, \frac{\alpha(1-\mu_1)}{a}\right). \tag{5.221}$$

Then, u blows up at a finite time.

Proof. Let us denote the following function:

$$Z(t)=2(u_t,u)+a\|\nabla_{\mathbb{G}}u(t)\|^2-\mu I(0), \tag{5.222}$$

where μ is a positive constant to be specified. By multiplying $u(t)$ the equation (5.204) and integrating over Ω, we have

$$(u_{tt},u)+a(\nabla_{\mathbb{G}}u,\nabla_{\mathbb{G}}u_t)=-\|\nabla_{\mathbb{G}}u\|^2-\int_0^t\int_\Omega k(t-\tau)\Delta_{\mathbb{G}}u(\tau)d\tau u(t)dx+\|u\|_p^p. \tag{5.223}$$

Then by using this fact, we get

$$\begin{aligned} Z'(t)&=2\|u_t\|^2+2(u_{tt},u)+2a(\nabla_{\mathbb{G}}u,\nabla_{\mathbb{G}}u_t)\\ &=2\|u_t\|^2-2\|\nabla_{\mathbb{G}}u\|^2-2\int_\Omega k(t-\tau)\Delta_{\mathbb{G}}u(\tau)d\tau u(t)dx+2\|u\|_p^p. \end{aligned} \tag{5.224}$$

By using the first Green's identity, we compute

$$\begin{aligned} \int_0^t\int_\Omega k(t-\tau)u(t)\Delta_{\mathbb{G}}u d\tau dx &= -\int_0^t\int_\Omega k(t-\tau)(\nabla_{\mathbb{G}}u(t)\cdot\nabla_{\mathbb{G}}u(\tau))dxd\tau\\ &=-\int_0^t\int_\Omega k(t-\tau)\nabla_{\mathbb{G}}u(t)\cdot(\nabla_{\mathbb{G}}(u(\tau)-u(t)))dxd\tau-\int_0^t k(t-\tau)\|\nabla_{\mathbb{G}}u(t)\|^2d\tau\\ &=-\int_0^t\int_\Omega k(t-\tau)\nabla_{\mathbb{G}}u(t)\cdot(\nabla_{\mathbb{G}}(u(\tau)-u(t)))dxd\tau-\|\nabla_{\mathbb{G}}u(t)\|^2\int_0^t k(\tau)d\tau\\ &=-\int_0^t k(t-\tau)(\nabla_{\mathbb{G}}u(t),\nabla_{\mathbb{G}}(u(\tau)-u(t)))d\tau-\|\nabla_{\mathbb{G}}u(t)\|^2\int_0^t k(\tau)d\tau. \end{aligned} \tag{5.225}$$

This yields

$$\begin{aligned}
Z'(t) &= 2\|u_t\|^2 + 2(u_{tt}, u) + 2a(\nabla_{\mathbb{G}} u, \nabla_{\mathbb{G}} u_t) \\
&= 2\|u_t\|^2 - 2\|\nabla_{\mathbb{G}} u\|^2 - 2\int_\Omega k(t-\tau)\Delta_{\mathbb{G}} u(\tau) d\tau u(t) dx + 2\|u\|_p^p \\
&= 2\|u_t\|^2 - 2\|\nabla_{\mathbb{G}} u\|^2 + 2\int_0^t k(t-\tau)(\nabla_{\mathbb{G}} u(t), \nabla_{\mathbb{G}}(u(\tau) - u(t))) d\tau \\
&+ 2\|\nabla_{\mathbb{G}} u(t)\|^2 \int_0^t k(\tau) d\tau + 2\|u\|_p^p.
\end{aligned} \tag{5.226}$$

On the other hand, by using Young's inequality, we have

$$\begin{aligned}
\int_0^t k(t-\tau)(\nabla_{\mathbb{G}} u(t), \nabla_{\mathbb{G}}(u(t) - u(\tau))) d\tau \leq \frac{p}{2}\int_0^t k(t-\tau)\|\nabla_{\mathbb{G}} u(\tau) - \nabla_{\mathbb{G}} u(t)\|^2 d\tau \\
+ \frac{1}{2p}\|\nabla_{\mathbb{G}} u(t)\|^2 \int_0^t k(\tau) d\tau,
\end{aligned} \tag{5.227}$$

that is,

$$\begin{aligned}
&\int_0^t k(t-\tau)(\nabla_{\mathbb{G}} u(t), \nabla_{\mathbb{G}}(u(\tau) - u(t))) d\tau \geq \\
&\quad -\frac{p}{2}\int_0^t k(t-\tau)\|\nabla_{\mathbb{G}} u(\tau) - \nabla_{\mathbb{G}} u(t)\|^2 d\tau - \frac{1}{2p}\|\nabla_{\mathbb{G}} u(t)\|^2 \int_0^t k(\tau) d\tau.
\end{aligned} \tag{5.228}$$

Hence, in the view of (5.228), we have

$$\begin{aligned}
Z'(t) &= 2\|u_t\|^2 - 2\|\nabla_{\mathbb{G}} u\|^2 + 2\int_0^t k(t-\tau)(\nabla_{\mathbb{G}} u(t), \nabla_{\mathbb{G}}(u(\tau) - u(t))) d\tau \\
&+ 2\|\nabla_{\mathbb{G}} u(t)\|^2 \int_0^t k(\tau) d\tau + \|u\|_p^p \\
&\geq 2\|u_t\|^2 - 2\|\nabla_{\mathbb{G}} u\|^2 - p\int_0^t k(t-\tau)\|\nabla_{\mathbb{G}} u(\tau) - \nabla_{\mathbb{G}} u(t)\|^2 d\tau \\
&- \frac{1}{p}\|\nabla_{\mathbb{G}} u\|^2 \int_0^t k(\tau) d\tau + 2\|\nabla_{\mathbb{G}} u\|^2 \int_0^t k(\tau) d\tau + 2\|u\|_p^p \\
&= (p+2)\|u_t\|^2 + (p-2)\left(1 - \int_0^t k(\tau) d\tau\right)\|\nabla_{\mathbb{G}} u(t)\|^2 \\
&- p\|u_t\|^2 - p\left(1 - \int_0^t k(\tau) d\tau\right)\|\nabla_{\mathbb{G}} u(t)\|^2 + 2\|u\|_p^p \\
&+ 2\left(-\frac{p}{2}\int_0^t k(t-\tau)\|\nabla_{\mathbb{G}} u(\tau) - \nabla_{\mathbb{G}} u(t)\|^2 d\tau - \frac{1}{2p}\|\nabla_{\mathbb{G}} u\|^2 \int_0^t k(\tau) d\tau\right) \\
&\geq (p+2)\|u_t\|^2 + (p-2)\left(1 - \int_0^t k(\tau) d\tau\right)\|\nabla_{\mathbb{G}} u(t)\|^2 \\
&- \frac{1}{p}\|\nabla_{\mathbb{G}} u\|^2 \int_0^t k(\tau) d\tau - 2pI(t).
\end{aligned} \tag{5.229}$$

By using the part (b) of Lemma 5.49 it follows that

$$
\begin{aligned}
Z'(t) &\geq (p+2)\|u_t\|^2 + (p-2)\left(1 - \int_0^t k(\tau)d\tau\right)\|\nabla_{\mathbb{G}} u(t)\|^2 \\
&- \frac{1}{p}\|\nabla_{\mathbb{G}} u\|^2 \int_0^t k(\tau)d\tau - 2pI(t) \\
&\overset{(5.210)}{\geq} (p+2)\|u_t\|^2 + (p-2)\left(1 - \int_0^t k(\tau)d\tau\right)\|\nabla_{\mathbb{G}} u(t)\|^2 \\
&- \frac{1}{p}\|\nabla_{\mathbb{G}} u\|^2 \int_0^t k(\tau)d\tau + 2ap\int_0^t \|\nabla_{\mathbb{G}} u_\tau(\tau)\|^2 d\tau - 2pI(0) \\
&\overset{(5.206)}{\geq} (p+2)\|u_t\|^2 + \left((p-2)r - \frac{1}{p}(1-r)\right)\|\nabla_{\mathbb{G}} u\|^2 \\
&- 2pI(0) + 2ap\int_0^t \|\nabla_{\mathbb{G}} u_t(\tau)\|^2 d\tau \\
&= (p+2)\|u_t\|^2 + \alpha\|\nabla_{\mathbb{G}} u\|^2 - 2pI(0) + 2ap\int_0^t \|\nabla_{\mathbb{G}} u_t\|^2 d\tau \\
&\overset{a>0}{\geq} (p+2)\|u_t\|^2 + \alpha\|\nabla_{\mathbb{G}} u\|^2 - 2pI(0),
\end{aligned}
\tag{5.230}
$$

where $\alpha = \left((p-2)r - \frac{1}{p}(1-r)\right)$. Note that $\alpha > 0$ since the condition (5.206). Further, by using Young's inequality, we get

$$
2\left((p+2)a\alpha\mu_1 R\right)^{\frac{1}{2}} |(u_t, u)| \leq (p+2)\|u_t\|^2 + a\alpha\mu_1 R\|u\|^2. \tag{5.231}
$$

Combining Theorem 5.48 with this fact, we get

$$
\begin{aligned}
Z'(t) &\geq (p+2)\|u_t\|^2 + \alpha\|\nabla_{\mathbb{G}} u\|^2 - 2pI(0) + 2ap\int_0^t \|\nabla_{\mathbb{G}} u_t\|^2 d\tau \\
&= (p+2)\|u_t\|^2 + \alpha a\mu_1\|\nabla_{\mathbb{G}} u\|^2 + (1 - a\mu_1)\alpha\|\nabla_{\mathbb{G}} u\|^2 - 2pI(0) \\
&+ 2ap\int_0^t \|\nabla_{\mathbb{G}} u_t\|^2 d\tau \\
&\overset{(5.203)}{\geq} (p+2)\|u_t\|^2 + \alpha Ra\mu_1\|u\|^2 + (1 - a\mu_1)\alpha\|\nabla_{\mathbb{G}} u\|^2 - 2pI(0) \\
&+ 2ap\int_0^t \|\nabla_{\mathbb{G}} u_t\|^2 d\tau \\
&\overset{(5.231)}{\geq} 2\left((p+2)\alpha\mu_1 R\right)^{\frac{1}{2}} |(u_t, u)| + (1 - a\mu_1)\alpha\|\nabla_{\mathbb{G}} u\|^2 - 2pI(0) \\
&+ 2ap\int_0^t \|\nabla_{\mathbb{G}} u_t\|^2 d\tau \\
&\overset{a>0}{\geq} 2\left((p+2)a\alpha\mu_1 R\right)^{\frac{1}{2}} |(u_t, u)| + (1 - a\mu_1)\alpha\|\nabla_{\mathbb{G}} u\|^2 - 2pI(0) \\
&\geq \theta(\mu_1)\left(2(u_t, u) + a\|\nabla_{\mathbb{G}} u\|^2 - \frac{2p}{\theta(\mu_1)}I(0)\right),
\end{aligned}
\tag{5.232}
$$

where R is defined in Theorem 5.48, $\mu_1 \in (0,1)$ is to be specified later and

$$\theta(\mu_1) = \min\left(((p+2)a\alpha\mu_1 R)^{\frac{1}{2}}, \frac{\alpha(1-\mu_1)}{a}\right). \tag{5.233}$$

Then we need to show that $K_1(\mu_1) = ((p+2)a\alpha\mu_1 R)^{\frac{1}{2}}$ is strictly increasing function for $\mu_1 \in [0,1]$ with $K_1(0) = 0$ and $K_1(1) = ((p+2)a\alpha R)^{\frac{1}{2}}$. Similarly, $K_2(\mu_2) = \frac{\alpha(1-\mu_1)}{a}$ is strictly decreasing function for $\mu_1 \in [0,1]$ with $K_2(0) = \frac{\alpha}{a}$ and $K_2(1) = 0$. Thus, $\theta(\mu_1)$ attains its maximum at the point $\mu_1 = \mu_1^*$, where μ_1^* is the root of the $((p+2)a\alpha\mu_1 R)^{\frac{1}{2}} = \frac{\alpha(1-\mu_1)}{a}$. Setting

$$\theta = \sup_{\mu_1 \in (0,1)} \theta(\mu_1) = \theta(\mu_1^*) \quad \text{and} \quad \mu = \frac{2p}{\theta}$$

in (5.222) implies that $Z(0) \geq 0$. Hence, we get

$$Z'(t) \geq \theta Z(t),$$

which implies

$$Z(t) \geq Z(0)\exp(\theta t),$$

that is,

$$Z(t) \to +\infty \quad \text{as} \quad t \to +\infty.$$

By introducing a new function

$$\xi(t) = \|u\|^2 + a\int_0^t \|\nabla_{\mathbb{G}} u(\tau)\|^2 d\tau + a(T-t)\|\nabla_{\mathbb{G}} u_0\|^2, \quad t \in [0,T], \tag{5.234}$$

we compute

$$\begin{aligned} \xi'(t) = 2(u_t, u) + a\|\nabla_{\mathbb{G}} u\|^2 - a\|\nabla_{\mathbb{G}} u_0\|^2 &= 2(u_t,u) + a\int_0^t \frac{d}{d\tau}\|\nabla_{\mathbb{G}} u\|^2 d\tau \\ &= 2(u_t,u) + 2a\int_0^t (\nabla_{\mathbb{G}} u_\tau(\tau), \nabla_{\mathbb{G}} u(\tau))d\tau. \end{aligned} \tag{5.235}$$

It easy to see that $\xi''(t) = Z'(t)$, so we have

$$\begin{aligned} \xi''(t) &\geq (p+2)\|u_t\|^2 + \alpha\|\nabla_{\mathbb{G}} u\|^2 - 2pI(0) + 2ap\int_0^t \|\nabla_{\mathbb{G}} u_t\|^2 d\tau \\ &\geq (p+2)\|u_t\|^2 + \alpha\|\nabla_{\mathbb{G}} u\|^2 - 2pI(0). \end{aligned} \tag{5.236}$$

Let $0 < \gamma < 1, \varepsilon > 0, T_B > 0$ be such that $\gamma(p+2) > 4 + \varepsilon = \nu$, and

$$(p+2)\|u_t\|^2 + \alpha\|\nabla_{\mathbb{G}} u\|^2 - 2pI(0) \geq \gamma((p+2)\|u_t\|^2 + \alpha\|\nabla_{\mathbb{G}} u\|^2) \quad t > T_B. \tag{5.237}$$

Thus, by using these facts, we have

$$\begin{aligned}\xi''(t) &\geq (p+2)\|u_t\|^2 + \alpha\|\nabla_{\mathbb{G}} u\|^2 - 2pI(0) + 2ap\int_0^t \|\nabla_{\mathbb{G}} u_t\|^2 d\tau \\ &\geq (p+2)\|u_t\|^2 + \alpha\|\nabla_{\mathbb{G}} u\|^2 - 2pI(0) + 2ap\int_0^t \|\nabla_{\mathbb{G}} u_t\|^2 d\tau \\ &\geq \gamma((p+2)\|u_t\|^2 + \alpha\|\nabla_{\mathbb{G}} u\|^2) + a\gamma(p+2)\int_0^t \|\nabla_{\mathbb{G}} u_t\|^2 d\tau \\ &\geq (4+\varepsilon)(\|u_t\|^2 + a\int_0^t \|\nabla_{\mathbb{G}} u_t\|^2 d\tau) \\ &= \nu(\|u_t\|^2 + a\int_0^t \|\nabla_{\mathbb{G}} u_t\|^2 d\tau), \ \ t > T_B.\end{aligned} \tag{5.238}$$

Next, from Cauchy-Schwarz-Bunyakovsky inequality yields the following estimates:

$$|(u_t, u)| \leq \|u_t\|^2\|u\|^2, \tag{5.239}$$

$$\left(\int_0^t (\nabla_{\mathbb{G}} u_t, \nabla_{\mathbb{G}} u) d\tau\right)^2 \leq \int_0^t \|\nabla_{\mathbb{G}} u_t\|^2 d\tau \int_0^t \|\nabla_{\mathbb{G}} u\|^2 d\tau. \tag{5.240}$$

Hence, we get

$$\begin{aligned}&2(u_t,u)\int_0^t (\nabla_{\mathbb{G}} u_t, \nabla_{\mathbb{G}} u) d\tau \\ &\overset{(5.239),(5.240)}{\leq} 2\|u_t\|\|u\| \left(\int_0^t \|\nabla_{\mathbb{G}} u_t\|^2 d\tau\right)^{\frac{1}{2}} \left(\int_0^t \|\nabla_{\mathbb{G}} u\|^2 d\tau\right)^{\frac{1}{2}} \\ &\leq \|u\|^2 \left(\int_0^t \|\nabla_{\mathbb{G}} u_t\|^2 d\tau\right) + \|u_t\|^2 \left(\int_0^t \|\nabla_{\mathbb{G}} u\|^2 d\tau\right).\end{aligned} \tag{5.241}$$

Hence, for $t > T_B$ we get

$$\begin{aligned}\xi''(t)\xi(t) - \frac{\nu}{4}(\xi'(t))^2 &> \nu\left(\|u_t\|^2 + a\int_0^t \|\nabla_{\mathbb{G}} u_t\|^2 d\tau\right)\left(\|u\|^2 + a\int_0^t \|\nabla_{\mathbb{G}} u(\tau)\|^2 d\tau\right) \\ &- \nu\left(2(u_t,u) + 2a\int_0^t (\nabla_{\mathbb{G}} u_t(\tau), \nabla_{\mathbb{G}} u(\tau)) d\tau\right)^2 \\ &= \nu\Big(\|u_t\|^2\|u\|^2 + a\|u_t\|^2\int_0^t \|\nabla_{\mathbb{G}} u(\tau)\|^2 d\tau + a\|u\|^2\int_0^t \|\nabla_{\mathbb{G}} u_t\|^2 d\tau \\ &+ a^2\int_0^t \|\nabla_{\mathbb{G}} u_t\|^2 d\tau \int_0^t \|\nabla_{\mathbb{G}} u(\tau)\|^2 d\tau\Big)\end{aligned}$$

$$\begin{aligned}
&-\nu\Big((u_t,u)^2+2a(u_t,u)\int_0^t(\nabla_{\mathbb{G}}u_t(\tau),\nabla_{\mathbb{G}}u(\tau))d\tau\\
&\quad+a^2\left(\int_0^t(\nabla_{\mathbb{G}}u_t(\tau),\nabla_{\mathbb{G}}u(\tau))d\tau\right)^2\Big)\\
&\overset{(5.239)}{\geq}\ \nu\Big(a\|u_t\|^2\int_0^t\|\nabla_{\mathbb{G}}u(\tau)\|^2d\tau+a\|u\|^2\int_0^t\|\nabla_{\mathbb{G}}u_t\|^2d\tau\\
&\quad+a^2\int_0^t\|\nabla_{\mathbb{G}}u_t\|^2d\tau\int_0^t\|\nabla_{\mathbb{G}}u(\tau)\|^2d\tau\Big)\\
&\quad-\gamma\Big(2a(u_t,u)\int_0^t(\nabla_{\mathbb{G}}u_t(\tau),\nabla_{\mathbb{G}}u(\tau))d\tau \qquad\qquad (5.242)\\
&\quad+a^2\left(\int_0^t(\nabla_{\mathbb{G}}u_t(\tau),\nabla_{\mathbb{G}}u(\tau))d\tau\right)^2\Big)\\
&\overset{(5.240)}{\geq}\ a\nu\left(\|u_t\|^2\int_0^t\|\nabla_{\mathbb{G}}u(\tau)\|^2d\tau+\|u\|^2\int_0^t\|\nabla_{\mathbb{G}}u_t\|^2d\tau\right)\\
&\quad-2\nu a(u_t,u)\int_0^t(\nabla_{\mathbb{G}}u_t(\tau),\nabla_{\mathbb{G}}u(\tau))d\tau\\
&\overset{(5.241)}{\geq}\ 0.
\end{aligned}$$

By setting $\phi(s)=\xi(t-T_B)$, where $s=t-T_B$, it is easy to see that

$$\phi''\phi-\frac{\gamma}{4}(\phi')^2\geq 0.$$

Thus, there exists $T_B<t<T$ such that

$$\lim_{t\to T_B}\phi(s)=+\infty, \tag{5.243}$$

i.e.,

$$\lim_{t\to T_B}\left(\|u\|^2+a\int_0^t\|\nabla_{\mathbb{G}}u(\tau)\|^2d\tau+(T-t)\|\nabla_{\mathbb{G}}u_0\|^2\right)=+\infty. \tag{5.244}$$

Hence, in the view of the last expression we have

$$\|\nabla_{\mathbb{G}}u\|^2\to+\infty,\quad t\to T_B.$$

□

5.8.2. *Blow-up with weak damping.* In this subsection, we consider the viscoelastic wave equation with weak damping for the sub-Laplacian:

$$\begin{cases}u_{tt}-\Delta_{\mathbb{G}}u+\int_0^t k(t-\tau)\Delta_{\mathbb{G}}u d\tau+a|u_t|^{q-2}u_t=|u|^{p-2}u, & (x,t)\in\Omega\times[0,T],\\ u(x,t)=0, & x\in\partial\Omega,\\ u(x,0)=u_0(x), & u_t(x,0)=u_1(x),\end{cases} \tag{5.245}$$

where $\Omega\subset\mathbb{G}$, is a Haar measurable set with a smooth boundary $\partial\Omega$, $a>0$, $p>2$ $q\geq 1$, $u_0\in S_0^{1,2}(\Omega)$, and $u_1\in L^2(\Omega)$. The function $I(t)$ is defined as in (5.208) and the function k satisfies (5.206)-(5.207). Further, let p and q be such that

$$\max\{p,q\} \leq \frac{2(Q-1)}{Q-2}. \tag{5.246}$$

We state the following lemmas which will be useful in proving blow-up result for (5.245).

Lemma 5.51. *Assume that p, q satisfy (5.246). Then, we have*

$$\|u\|_p^\gamma \leq C\left(\|u\|_p^p + \|\nabla_{\mathbb{G}} u\|^2\right), \quad 2 \leq \gamma \leq p, \tag{5.247}$$

where C is a positive constant which depends only on the Haar measure of Ω.

Proof. Suppose that $\|u\|_p > 1$. Since $2 \leq \gamma \leq p$, Sobolev Embedding Theorem 3.45 with $2^* = \frac{2Q}{Q-2}$ yields

$$\|u\|_p^\gamma \leq \|u\|_p^p \leq \|u\|_p^p + \|u\|_{2^*}^2 \leq \|u\|_p^p + C\|\nabla_{\mathbb{G}} u\|_2^2 \leq C\left(\|u\|_p^p + \|\nabla_{\mathbb{G}} u\|^2\right). \tag{5.248}$$

Now suppose $\|u\|_p \leq 1$. Let $p = \frac{Qp'}{Q-p'}$ with $1 < p' < Q$. Then we have the $1 < p' < p$ yielding continuous embedding, i.e., $L^p(\Omega) \hookrightarrow L^{p'}(\Omega)$. Hence, we have

$$\|\nabla_{\mathbb{G}} u\|_{p'} \leq C\|\nabla_{\mathbb{G}} u\|_p. \tag{5.249}$$

Since $2 \leq \gamma$, we get

$$\begin{aligned} \|u\|_p^\gamma \leq \|u\|_p^2 \leq C\|\nabla_{\mathbb{G}} u\|_{p'}^2 \overset{(5.249)}{\leq} C\|\nabla_{\mathbb{G}} u\|_p^2 \leq C\|\nabla_{\mathbb{G}} u\|_p^2 + \|u\|_p^p \\ \leq C\left(\|u\|_p^p + \|\nabla_{\mathbb{G}} u\|^2\right). \end{aligned} \tag{5.250}$$

□

Lemma 5.52. *Assume that u be a weak solution of (5.245) with (5.246). Then we get*

$$\|u\|_p^\gamma \leq C\left(I(t) - \|u_t\|^2 - (k \circ \nabla_{\mathbb{G}} u) + \|u\|_p^p\right), \quad \forall t \in [0,T], \tag{5.251}$$

where $2 \leq \gamma \leq p$ and C is a positive constant.

Proof. The function $I(t)$ is given by

$$I(t) = \frac{1}{2}\left(\|u_t(t)\|_2^2 + \left(1 - \int_0^t k(s)ds\right)\|\nabla_{\mathbb{G}} u(t)\|_2^2 + k \circ \nabla_{\mathbb{G}} u\right) - \frac{1}{p}\|u(t)\|_p^p. \tag{5.252}$$

Therefore, by combining (5.206) and (5.207), we compute

$$\begin{aligned} r\|\nabla_{\mathbb{G}} u(t)\|_2^2 &\overset{(5.206)}{=} \left(1 - \int_0^\infty k(s)ds\right)\|\nabla_{\mathbb{G}} u(t)\|_2^2 \\ &\overset{(5.207)}{\leq} \left(1 - \int_0^t k(s)ds\right)\|\nabla_{\mathbb{G}} u(t)\|_2^2 \\ &= 2I(t) - \|u_t(t)\|_2^2 - k \circ \nabla_{\mathbb{G}} u + \frac{2}{p}\|u\|_p^p. \end{aligned} \tag{5.253}$$

Now we apply Lemma 5.51 with $2 \leq \gamma \leq p$, to obtain

$$\|u\|_p^\gamma \leq C\left(\|u\|_p^p + \|\nabla_{\mathbb{G}} u\|^2\right) \overset{(5.253)}{\leq} C\left(I(t) - \|u_t\|^2 - (k \circ \nabla_{\mathbb{G}} u) + \|u\|_p^p\right). \tag{5.254}$$

□

Lemma 5.53. *Assume that (5.206)-(5.207) are satisfied. Suppose u be a weak solution of (5.245), then $I(t)$ is a non-increasing function for $t \in [0,T]$, i.e.,*

$$I'(t) \leq 0, \;\; \forall t \in [0,T]. \tag{5.255}$$

We omit the proof of Lemma 5.53 since it is similar to that of Lemma 5.49. The main result of this section is the following theorem.

Theorem 5.54. *Suppose that $q > 1$ and $p > \max\{2,q\}$ satisfy the condition (5.246). If (5.206) and (5.207) hold with $I(0) < 0$, then solution u of (5.245) blows up at a finite time.*

Proof. From Lemma 5.53, we have

$$I'(t) \leq 0, \tag{5.256}$$

therefore,

$$I(t) \leq I(0), \;\; \forall t \in [0,T].$$

Let us denote by $Z(t) = -I(t)$. Then, we have

$$\begin{aligned}
0 &< Z(0) \leq Z(t) = -I(t) \\
&= -\frac{1}{2}\left(\|u_t(t)\|_2^2 + \left(1 - \int_0^t k(s)ds\right)\|\nabla_{\mathbb{G}} u(t)\|_2^2 + k \circ \nabla_{\mathbb{G}} u\right) + \frac{1}{p}\|u(t)\|_p^p \\
&= -\frac{1}{2}\|u_t(t)\|_2^2 - \frac{1}{2}\left(1 - \int_0^t k(s)ds\right)\|\nabla_{\mathbb{G}} u(t)\|_2^2 - \frac{1}{2}k \circ \nabla_{\mathbb{G}} u + \frac{1}{p}\|u(t)\|_p^p \\
&\overset{(5.206),(5.207)}{\leq} \frac{1}{p}\|u(t)\|_p^p.
\end{aligned} \tag{5.257}$$

Similarly by Lemma 5.49, we get

$$Z'(t) = -I'(t) = a\|u_t\|_q^q - \frac{1}{2}(k' \cdot \nabla_{\mathbb{G}} u) + \frac{1}{2}k(t)\|\nabla_{\mathbb{G}} u\|^2 \overset{(5.206),(5.207)}{\geq} 0. \tag{5.258}$$

Let us also define the following function

$$A(t) = Z^{1-\beta}(t) - \varepsilon(u_t, u), \tag{5.259}$$

where $0 < \beta \leq \min\{\frac{p-2}{2p}, \frac{p-q}{p(q-1)}\}$. By means of direct calculations and the Cauchy-Bunyakovsy-Schwarz inequality, we have

$$\begin{aligned}
A'(t) &= (1-\beta)Z^{-\beta}(t)Z'(t) - \varepsilon(u_{tt}, u) - \varepsilon\|u_t\|^2 \\
&= (1-\beta)Z^{-\beta}(t)Z'(t) - \varepsilon\|\nabla_{\mathbb{G}}u\|^2 + \varepsilon\int_0^t k(t-\tau)(\nabla_{\mathbb{G}}u(\tau), \nabla_{\mathbb{G}}u(t))d\tau \\
&+ \varepsilon\|u\|_p^p - a\varepsilon\int_\Omega |u_t|^{q-2}u_t u dx + \varepsilon\|u_t\|^2 \\
&\overset{(5.225),(5.258)}{\geq} a(1-\beta)Z^{-\beta}(t)\|u_t\|_q^q - \varepsilon\|\nabla_{\mathbb{G}}u\|^2 + \varepsilon\|u\|_p^p - a\varepsilon\int_\Omega |u_t|^{q-2}u_t u dx \\
&+ \varepsilon\int_0^t\int_\Omega k(t-\tau)(\nabla_{\mathbb{G}}u(t), \nabla_{\mathbb{G}}(u(\tau)-u(t)))dxd\tau \\
&+ \varepsilon\|\nabla_{\mathbb{G}}u(t)\|^2\int_0^t k(\tau)d\tau + \varepsilon\|u_t\|^2 \\
&\overset{C-B-S}{\geq} a(1-\beta)Z^{-\beta}(t)\|u_t\|_q^q - \varepsilon\|\nabla_{\mathbb{G}}u\|^2 + \varepsilon\|u\|_p^p - a\varepsilon\int_\Omega |u_t|^{q-2}u_t u dx + \varepsilon\|u_t\|^2 \\
&- \varepsilon\int_0^t k(t-\tau)\|\nabla_{\mathbb{G}}u\|^2\|\nabla_{\mathbb{G}}(u(\tau)-u(t))\|^2 d\tau + \varepsilon\|\nabla_{\mathbb{G}}u(t)\|^2\int_0^t k(\tau)d\tau.
\end{aligned} \tag{5.260}$$

In the view of (5.208), we get

$$\frac{1}{p}\|u\|_p^p = Z(t) + \frac{1}{2}\left(\|u_t(t)\|^2 + \left(1 - \int_0^t k(s)ds\right)\|\nabla_{\mathbb{G}}u(t)\|^2 + k \circ \nabla_{\mathbb{G}}u\right). \tag{5.261}$$

On the other hand, by combining (5.260) with (5.228), we have

$$\begin{aligned}
A'(t) &\geq a(1-\beta)Z^{-\beta}(t)\|u_t\|_q^q - \varepsilon\|\nabla_{\mathbb{G}}u\|^2 + \varepsilon\|u\|_p^p - a\varepsilon\int_\Omega |u_t|^{q-2}u_t u dx + \varepsilon\|u_t\|^2 \\
&- \varepsilon\int_0^t k(t-\tau)\|\nabla_{\mathbb{G}}u\|^2\|\nabla_{\mathbb{G}}(u(\tau)-u(t))\|^2 d\tau + \varepsilon\|\nabla_{\mathbb{G}}u(t)\|^2\int_0^t k(\tau)d\tau \\
&\overset{(5.261)}{=} a(1-\beta)Z^{-\beta}(t)\|u_t\|_q^q - \varepsilon\|\nabla_{\mathbb{G}}u\|^2 - a\varepsilon\int_\Omega |u_t|^{q-2}u_t u dx + \varepsilon\|u_t\|^2 \\
&+ \frac{\varepsilon p}{2}\left(2Z(t) + \|u_t\|^2 + \left(1 - \int_0^t k(s)ds\right)\|\nabla_{\mathbb{G}}u\|^2 + k \circ \nabla_{\mathbb{G}}u\right) \\
&- \varepsilon\int_0^t k(t-\tau)\|\nabla_{\mathbb{G}}u\|^2\|\nabla_{\mathbb{G}}(u(\tau)-u(t))\|^2 d\tau + \varepsilon\|\nabla_{\mathbb{G}}u(t)\|^2\int_0^t k(\tau)d\tau \\
&\overset{(5.228)}{\geq} a(1-\beta)Z^{-\beta}(t)\|u_t\|_q^q + \left(\varepsilon + \frac{\varepsilon p}{2}\right)\|u_t\|^2 + \varepsilon p Z(t) - a\varepsilon\int_\Omega |u_t|^{q-2}u_t u dx \\
&+ \left(\frac{\varepsilon p}{2} - \varepsilon\delta\right)(k \circ \nabla_{\mathbb{G}}u) + \left(\left(\frac{p}{2} - 1\right) - \varepsilon\left(\frac{p}{2} - 1 + \frac{1}{4\delta}\right)\int_0^t k(\tau)d\tau\right)\|\nabla_{\mathbb{G}}u\|^2,
\end{aligned} \tag{5.262}$$

where $\delta \in (0, \frac{p}{2})$. By applying Young's inequality to estimate the fourth term on the right hand side of the (5.262) to obtain

$$
\begin{aligned}
A'(t) \geq{}& (1-\beta)Z^{-\beta}(t)\|u_t\|_q^q + \left(\varepsilon + \frac{\varepsilon p}{2}\right)\|u_t\|^2 + \varepsilon p Z(t) - a\varepsilon \int_\Omega |u_t|^{q-2}u_t u dx \\
&+ \left(\frac{\varepsilon p}{2} - \varepsilon\delta\right)(k \circ \nabla_{\mathbb{G}} u) + \left(\left(\frac{p}{2}-1\right) - \left(\frac{p}{2} - 1 + \frac{1}{4\delta}\right)\int_0^t k(\tau)d\tau\right)\|\nabla_{\mathbb{G}} u\|^2 \\
={}& (1-\beta)Z^{-\beta}(t)\|u_t\|_q^q + \left(\varepsilon + \frac{\varepsilon p}{2}\right)\|u_t\|^2 + \varepsilon p Z(t) - a\varepsilon \int_\Omega |u_t|^{q-2}u_t u dx \\
&+ \varepsilon C_1 (k \circ \nabla_{\mathbb{G}} u) + \varepsilon C_2 \|\nabla_{\mathbb{G}} u\|^2 + \varepsilon p Z(t) \\
\geq{}& a\left((1-\beta)Z^{-\beta} - \frac{\varepsilon\lambda^{-q'}}{q'}\right)\|u_t\|_q^q + \left(\varepsilon + \frac{\varepsilon p}{2}\right)\|u_t\|^2 + \varepsilon C_1(k \circ \nabla_{\mathbb{G}} u) \\
&+ \varepsilon C_2 \|\nabla_{\mathbb{G}} u\|^2 - \frac{\varepsilon a \lambda^q}{q}\|u\|_q^q + \varepsilon p Z(t), \quad \forall \lambda > 0,
\end{aligned}
\tag{5.263}
$$

where

$$
C_1 = \frac{\varepsilon p}{2} - \varepsilon\delta > 0, \quad C_2 = \left(\frac{p}{2} - 1\right) - \left(\frac{p}{2} - 1 + \frac{1}{4\delta}\right)\int_0^t k(\tau)d\tau > 0.
$$

Then, by setting $\lambda^{-q'} = \chi Z^{-\beta}(t)$ we get

$$
\begin{aligned}
A'(t) \geq{}& a\left((1-\beta)Z^{-\beta} - \frac{\varepsilon\lambda^{-q'}}{q'}\right)\|u_t\|_q^q + \left(\varepsilon + \frac{\varepsilon p}{2}\right)\|u_t\|^2 \\
&+ \varepsilon C_1(k \circ \nabla_{\mathbb{G}} u) + \varepsilon p Z(t) + \varepsilon C_2 \|\nabla_{\mathbb{G}} u\|^2 - \frac{\varepsilon a \lambda^q}{q}\|u\|_q^q \\
={}& a\left((1-\beta) - \frac{\varepsilon\chi}{q'}\right)Z^{-\beta}\|u_t\|_q^q + \left(\varepsilon + \frac{\varepsilon p}{2}\right)\|u_t\|^2 + \varepsilon C_1(k \circ \nabla_{\mathbb{G}} u) \\
&+ \varepsilon C_2 \|\nabla_{\mathbb{G}} u\|^2 + \varepsilon\left(pZ(t) - \frac{\varepsilon a \chi^{1-q}}{q} Z^{-\beta(1-q)}\|u\|_q^q\right).
\end{aligned}
\tag{5.264}
$$

Next, from (5.257) and the fact that $L^p(\Omega) \hookrightarrow L^q(\Omega)$ for $p > q$, we have

$$
\|u\|_q^q \leq C\left(\frac{1}{p}\right)^{\beta(q-1)} \|u\|_p^{q+\beta p(q-1)}. \tag{5.265}
$$

The last inequality applied to (5.264) yields

$$
\begin{aligned}
A'(t) \geq{}& a\left((1-\beta) - \frac{\varepsilon\chi}{q'}\right)Z^{-\beta}\|u_t\|_q^q + \left(\varepsilon + \frac{\varepsilon p}{2}\right)\|u_t\|^2 + \varepsilon C_1(k \circ \nabla_{\mathbb{G}} u) \\
&+ \varepsilon C_2 \|\nabla_{\mathbb{G}} u\|^2 + \varepsilon\left(pZ(t) - \frac{a\chi^{1-q}}{q}\|u\|_q^q\right) \\
\geq{}& a\left((1-\beta) - \frac{\varepsilon\chi}{q'}\right)Z^{-\beta}\|u_t\|_q^q + \left(\varepsilon + \frac{\varepsilon p}{2}\right)\|u_t\|^2 + \varepsilon C_1(k \circ \nabla_{\mathbb{G}} u) \\
&+ \varepsilon C_2 \|\nabla_{\mathbb{G}} u\|^2 + \varepsilon\left(pZ(t) - C\frac{a\chi^{1-q}}{q}\left(\frac{1}{p}\right)^{\beta(q-1)}\|u\|_p^{q+\beta p(q-1)}\right).
\end{aligned}
\tag{5.266}
$$

Now, by applying Lemma 5.52 with $\gamma = q + \beta p(q-1) \leq p$.

$$\begin{aligned}
A'(t) &\geq a\left((1-\beta) - \frac{\varepsilon\chi}{q'}\right) Z^{-\beta}\|u_t\|_q^q + \left(\varepsilon + \frac{\varepsilon p}{2}\right)\|u_t\|^2 + \varepsilon C_1(k \circ \nabla_{\mathbb{G}} u) \\
&+ \varepsilon C_2\|\nabla_{\mathbb{G}} u\|^2 + \varepsilon\left(pZ(t) - C\frac{a\chi^{1-q}}{q}\left(\frac{1}{p}\right)^{\beta(q-1)}\|u\|_p^{\gamma}\right) \\
&\overset{(5.251)}{\geq} a\left((1-\beta) - \frac{\varepsilon\chi}{q'}\right) Z^{-\beta}\|u_t\|_q^q + \left(\varepsilon + \frac{\varepsilon p}{2}\right)\|u_t\|^2 + \varepsilon C_1(k \circ \nabla_{\mathbb{G}} u) \\
&+ \varepsilon C_2\|\nabla_{\mathbb{G}} u\|^2 + \varepsilon\left(pZ(t) + C_1'\chi^{1-q}\left(Z(t) + \|u_t\|^2 + (k \circ \nabla_{\mathbb{G}} u) - \|u\|_p^p\right)\right) \\
&\geq a\left((1-\beta) - \frac{\varepsilon\chi}{q'}\right) Z^{-\beta}\|u_t\|_q^q + \varepsilon\left(\frac{p}{2} + 1 + C_1\chi^{1-q}\right)\|u_t\|^2 - \varepsilon C_1'\chi^{1-q}\|u\|_p^p \\
&+ \varepsilon\left(C_1 + C_1'\chi^{1-q}\right)(k \circ \nabla_{\mathbb{G}} u) + \varepsilon C_2\|\nabla_{\mathbb{G}} u\|^2 + \varepsilon\left(p + C_1'\chi^{1-q}\right)Z(t),
\end{aligned} \tag{5.267}$$

where $C_1' = \frac{aC\left(\frac{1}{p}\right)^{\beta(q-1)}}{q}$. From assumption $I(t) < 0$, that is,

$$Z(t) \geq -\frac{1}{2}\left(\|u_t(t)\|_2^2 + \left(1 - \int_0^t k(s)ds\right)\|\nabla_{\mathbb{G}} u(t)\|_2^2 + k \circ \nabla_{\mathbb{G}} u\right) + \frac{1}{p}\|u(t)\|_p^p. \tag{5.268}$$

By setting $p = 2b + (p-2b)$ where $b = \min\{C_1, C_2\}$ and letting χ to be large enough in (5.267) we have

$$A'(t) \geq a\left((1-\beta) - \frac{\varepsilon\chi}{q'}\right) Z^{-\beta}\|u_t\|_q^q + \varepsilon\sigma\left(Z(t) + \|u_t\|^2 + \|u\|_p^p + k \circ \nabla_{\mathbb{G}} u\right), \tag{5.269}$$

where $\sigma > 0$. Next, we choose sufficiently small ε so that $(1-\beta) - \frac{\varepsilon\chi}{q'} > 0$. Thus, we have

$$A'(t) > \varepsilon\sigma\left(Z(t) + \|u_t\|^2 + \|u\|_p^p + k \circ \nabla_{\mathbb{G}} u\right), \tag{5.270}$$

and

$$A(0) = Z^{1-\beta}(0) + \varepsilon(u_0, u_1) > 0.$$

Hence,

$$0 < A(0) \leq A(t), \quad \forall t \in [0, T].$$

Now, by using the Cauchy-Bunyakovsky-Schwarz inequality, embedding of spaces and Young's inequalities, we have

$$|(u_t, u)|^{\frac{1}{1-\beta}} \leq \|u_t\|^{\frac{1}{1-\beta}}\|u\|^{\frac{1}{1-\beta}} \leq C\|u_t\|^{\frac{1}{1-\beta}}\|u\|_p^{\frac{1}{1-\beta}} \leq C\left(\|u\|_p^{\gamma} + \|u_t\|^2\right), \tag{5.271}$$

with $\frac{1}{(1-\beta)\gamma} + \frac{1}{2(1-\beta)} = 1$. By Lemma 5.52, we obtain

$$|(u_t, u)|^{\frac{1}{1-\beta}} \leq C\left(Z(t) + \|u\|_p^p + \|u_t\|^2 + k \circ \nabla_{\mathbb{G}} u\right). \tag{5.272}$$

From this fact, we calculate

$$\begin{aligned} A(t) &= \left(Z^{1-\beta}(t) + \varepsilon(u_t, u)\right)^{\frac{1}{1-\beta}} \le 2^{\frac{1}{1-\beta}} \left(Z(t) + |(u_t, u)|^{\frac{1}{1-\beta}}\right) \\ &\le 2^{\frac{1}{1-\beta}} \left(Z(t) + C\left(Z(t) + \|u\|_p^p + \|u_t\|^2 + k \circ \nabla_{\mathbb{G}} u\right)\right) \\ &\le C\left(Z(t) + \|u\|_p^p + \|u_t\|^2 + k \circ \nabla_{\mathbb{G}} u\right) \\ &\le CA'(t), \quad \forall t \in [0, T]. \end{aligned} \tag{5.273}$$

Hence,

$$A^{\frac{\beta}{1-\beta}}(t) \ge \frac{C(1-\beta)}{C(1-\beta)A^{-\frac{\beta}{1-\beta}}(0) - t\beta}, \tag{5.274}$$

therefore, we arrive at

$$T_B \le \frac{C(1-\beta)}{\beta (A(0))^{\frac{\beta}{1-\beta}}}. \tag{5.275}$$

Therefore, $A(t)$ blows up in finite time. That is,

$$\lim_{t \to T_B} \|\nabla_{\mathbb{G}} u\| = +\infty. \tag{5.276}$$

□

5.9. **Kato type exponents for the wave Rockland equations.** In one of the most popular works of Kato he considered the following problem:

$$\frac{\partial^2 u(x,t)}{\partial t^2} - \Delta u(x,t) = |u(x,t)|^p, \quad (x,t) := \mathbb{R}^N \times (0, +\infty), \tag{5.277}$$

for $N > 1$ and $p > 1$, with the Cauchy data

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \mathbb{R}^N.$$

For the wave problem (5.277), Kato's result states that if u is a generalised solution of the problem (5.277) with $u_0, u_1 \in C_0^\infty(\mathbb{R}^N)$, $\operatorname{supp} u \subset \{|x| \le R + t\}$ and

$$\int\limits_{\mathbb{R}^N} |x|^{\eta - 1} u_0(x) dx > 0, \quad \int\limits_{\mathbb{R}^N} u_1(x) dx > 0,$$

where

$$\eta(N) = \begin{cases} 0 & \text{if } N \text{ is odd}, \\ \frac{1}{2} & \text{if } N \text{ is even}, \end{cases}$$

then the solution u cannot be globally (in time) defined if

$$1 < p \le \frac{N+1}{N-1}. \tag{5.278}$$

The exponent $p^* = \frac{N+1}{N-1}$ is usually called the Kato critical exponent for the problem (5.277).

The wave equation on the Heisenberg group $\mathbb{H}^n$ studied in [101], where the authors concerned the following problem

$$\frac{\partial^2 u(x,t)}{\partial t^2} - \Delta_{\mathbb{H}^n} u(x,t) = |u(x,t)|^p, \quad (x,t) := \mathbb{H}^n \times (0, +\infty),$$

with the Cauchy data

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x),$$

where $p > 1$. Also, in the paper [102] it is considered a space–fractional analogue of the non-linear wave equation on the Heisenberg group

$$\frac{\partial^2 u(x,t)}{\partial t^2} + (-\Delta_{\mathbb{H}^n})^s |u(x,t)|^m = |u(x,t)|^p, \quad (x,t) := \mathbb{H}^n \times (0,+\infty),$$

with the initial conditions

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x),$$

where $(-\Delta_{\mathbb{H}^n})^s$ is the fractional sub-Laplacian on $\mathbb{H}^n$, $s \in (0,2)$, $m \neq 1$, $p > 1$.

In this dissertation we are not only interested in studying of the wave equations, also, the pseudo-hyperbolic equations and systems on graded Lie groups are in the field of our interest. In particular, we extend nonexistence results obtained by Véron and Pohozaev [101] for the hyperbolic equation and by Kirane and Ragoub [103] for the pseudo-hyperbolic equation and system on the Heisenberg group to the case of the graded Lie groups.

5.9.1. *Wave equation case.* Assume that $m > 0$ and let consider the Cauchy problem for the nonlinear Rockland wave equation

$$\begin{cases} u_{tt}(x,t) + \mathcal{R}|u(x,t)|^m = |u(x,t)|^p, & (x,t) \in \mathbb{G} \times (0,+\infty) := \Omega, \\ u(x,0) = u_0(x) \geq 0, & x \in \mathbb{G}, \\ u_t(x,0) = u_1(x), & x \in \mathbb{G}. \end{cases} \tag{5.279}$$

where $\mathcal{R}$ is the Rockland operator in the following form:

$$\mathcal{R} = \sum_{j=1}^{n} (-1)^{\frac{\nu_0}{\nu_j}} c_j X_j^{2\frac{\nu_0}{\nu_j}},$$

where $\nu_j \in \mathbb{N}$, $c_j \in \mathbb{R}$, $j = 1,\dots,n$, and ν_0 is any common multiple of $\nu_1,\dots,\nu_n$, ([3, Lemma 4.1.8]). By $c_j \in \mathbb{R}$, $j = 1,\dots,n$, we can choose $\mathcal{R}$ such that it will be positive. Also, we introduce this operator in the Section 2.2.

Let us give definition of the weak solution of the Rockland wave equation (5.279).

Definition 5.55. Assume that $u_1, u_0 \in L^1_{loc}(\mathbb{G})$. We say that the function $u \in L^{\max\{m,p\}}_{loc}(\Omega_T)$ $(\Omega_T = \mathbb{G} \times (0,T))$ is a local weak solution of (5.279) if the identity

$$\begin{aligned} &\int_{\Omega_T} u(x,t) \frac{\partial^2 \varphi(x,t)}{\partial t^2} dxdt + \int_{\Omega_T} |u(x,t)|^m \mathcal{R}\varphi(x,t) dxdt \\ &= -\int_{\mathbb{G}} u_0(x) \frac{\partial \varphi(x,0)}{\partial t} dx + \int_{\mathbb{G}} u_1(x)\varphi(x,0) dx + \int_{\Omega_T} |u(x,t)|^p \varphi(x,t) dxdt, \end{aligned} \tag{5.280}$$

holds for all test functions

$$\varphi \in C^2((0,T]; L^2(\mathbb{G})) \cap C([0,T]; H^\gamma(\mathbb{G})),$$

such that $\gamma = 2 \max\limits_{j=1,\dots,n} \frac{\nu_0}{\nu_j}$, $\varphi(x,T) = 0$ and $\varphi \geq 0$. If $T = +\infty$ then u is called a global weak solution.

Here, the space $H^{\gamma}(\mathbb{G})$ is the homogeneous Sobolev space related to the Rockland operator $\mathcal{R}$, for more details, see [3, 104] and Section 2.2.

Theorem 5.56. *Assume that $\mathbb{G}$ be a graded Lie group with homogeneous dimension $Q \geq 2$ and $\mu = \max\limits_{j=1,\ldots,n} \frac{\nu_j}{\nu_0}$ be such that $\mu Q > 1$. Assume that $p > 1$, and $\int_{\mathbb{G}} u_1(x)dx \geq 0$. Then if*

$$1 \leq m < p < p_c = \frac{\mu Q m + 1}{\mu Q - 1}, \tag{5.281}$$

the Cauchy problem (5.279) admits no non-negative global weak solution other than trivial.

Proof. We prove this theorem by contradiction. Suppose that there exists a weak solution u for some $T > 0$. By using (5.279) and Definition 5.55, we get

$$\begin{aligned} -\int_{\mathbb{G}} u_0(x)\frac{\partial\varphi(x,0)}{\partial t}dx + \int_{\mathbb{G}} u_1(x)\varphi(x,0)dx + \int_{\Omega_T} |u(x,t)|^p\varphi(x,t)dxdt \\ = \int_{\Omega_T} \frac{\partial^2\varphi(x,t)}{\partial t^2}u(x,t)dxdt + \int_{\Omega_T} |u(x,t)|^m\mathcal{R}\varphi(x,t)dxdt. \end{aligned} \tag{5.282}$$

By choosing $\varphi(x,t)$ such that

$$\frac{\partial\varphi}{\partial t}(x,0) = 0.$$

By using ε-Young's inequality

$$ab \leq \varepsilon a^p + C(\varepsilon)b^{p'}, \ \ \frac{1}{p} + \frac{1}{p'} = 1, \ \ a, b \geq 0,$$

we obtain

$$\begin{aligned} \int_{\mathbb{G}} u_1(x)\varphi(x,0)dx &+ \int_{\Omega_T} |u(x,t)|^p\varphi(x,t)dxdt \\ &= \int_{\Omega_T} \frac{\partial^2\varphi(x,t)}{\partial t^2}u(x,t)dxdt + \int_{\Omega_T} |u(x,t)|^m\mathcal{R}\varphi(x,t)dxdt \\ &\leq \frac{1}{4}\int_{\Omega_T} |u|^p\varphi dxdt + C\int_{\Omega_T} \varphi^{-\frac{1}{p-1}}\left|\frac{\partial^2\varphi}{\partial t^2}\right|^{\frac{p}{p-1}} dxdt \\ &\quad + \frac{1}{4}\int_{\Omega_T} |u|^p\varphi dxdt + C\int_{\Omega_T} |\mathcal{R}\varphi|^{\frac{p}{p-m}}\varphi^{-\frac{m}{p-m}}dxdt. \end{aligned} \tag{5.283}$$

Then, we have

$$\begin{aligned} \int_{\Omega_T} |u|^p\varphi dxdt &\leq \int_{\mathbb{G}} u_1(x)\varphi(x,0)dx + \int_{\Omega_T} |u|^p\varphi dxdt \\ &\leq C\int_{\Omega_T} \varphi^{-\frac{1}{p-1}}\left|\frac{\partial^2\varphi}{\partial t^2}\right|^{\frac{p}{p-1}} dxdt + C\int_{\Omega_T} |\mathcal{R}\varphi|^{\frac{p}{p-m}}\varphi^{-\frac{m}{p-m}}dxdt. \end{aligned} \tag{5.284}$$

Assume that $\Phi : \mathbb{R}_+ \to [0, 1]$ be a smooth nonincreasing function such that

$$\Phi(z) := \begin{cases} 1, \ \text{if } 0 \leq z \leq 1, \\ 0, \ \text{if } z \geq 2. \end{cases} \tag{5.285}$$

For $R>0$, we define

$$\varphi(x,t)=\Phi\left(\frac{|x|^2}{R^{\mu}}\right)\Phi\left(\frac{t^2}{R^{2\frac{p-1}{p-m}}}\right),$$

where $\Phi\in C^{\infty}[0,+\infty)$. By Denoting the following vector fields acting to the variable $X_j={}_xX_j$. By denoting $\Omega_1:=\{x\in\mathbb{G}:0\leq|\widetilde{x}|\leq 2\}$ and $\Omega_2:=\{t:0\leq\widetilde{t}\leq 2\}$. By substituting $x=R^{\mu}\widetilde{x}$ and $t=R^{\frac{p-1}{p-m}}\widetilde{t}$ and from Proposition 2.4, we get

$$\begin{aligned}
&\int_{\Omega_T}|\mathcal{R}\varphi(x,t)|^{\frac{p}{p-m}}\varphi^{-\frac{m}{p-m}}dxdt\\
&=\int_{\Omega_T}\sum_{j=1}^{n}\left|(-1)^{\frac{\nu_0}{\nu_j}}\,{}_xX_j^{2\frac{\nu_0}{\nu_j}}\varphi(x,t)\right|^{\frac{p}{p-m}}\varphi^{-\frac{m}{p-m}}dxdt\\
&=\int_{\Omega_T}\sum_{j=1}^{n}R^{-2\frac{\nu_0 p}{\nu_j(p-m)}\mu}\left|(-1)^{\frac{\nu_0}{\nu_j}}\,{}_{\widetilde{x}}X_j^{2\frac{\nu_0}{\nu_j}}\varphi(x,t)\right|^{\frac{p}{p-m}}\varphi^{-\frac{m}{p-m}}dxdt\\
&\overset{\mu=\max\limits_{j=1,\dots,n}\frac{\nu_j}{\nu_0}}{\leq}\; R^{-\frac{2p}{p-m}}\int_{\Omega_T}\sum_{j=1}^{n}\left|{}_{\widetilde{x}}X_j^{2\frac{\nu_0}{\nu_j}}\varphi(x,t)\right|^{\frac{p}{p-m}}\varphi^{-\frac{m}{p-m}}dxdt\\
&=R^{\frac{-2p}{p-m}}R^{\mu Q}R^{\frac{p-1}{p-m}}\int_{\Omega_{TR^{-\frac{p-1}{p-m}}}}|\mathcal{R}_{\widetilde{x}}\varphi(R^{\mu}\widetilde{x},R^{\frac{p-1}{p-m}}\widetilde{t})|^{\frac{p}{p-m}}\varphi^{-\frac{m}{p-m}}d\widetilde{x}d\widetilde{t}\\
&\leq CR^{\mu Q-\frac{p+1}{p-m}},
\end{aligned}\tag{5.286}$$

and also,

$$\begin{aligned}
&\int_{\Omega_T}\varphi^{-\frac{p+1}{p-1}}(x,t)\left|\varphi(x,t)\frac{\partial^2\varphi(x,t)}{\partial t^2}\right|^{p'}dxdt\\
&\quad=\int_{\Omega_T}\varphi^{-\frac{p+1}{p-1}}(R\widetilde{x},R^{\frac{p-1}{p-m}}\widetilde{t})\left|\varphi(R\widetilde{x},R^{\frac{p-1}{p-m}}\widetilde{t})\frac{\partial^2\varphi(R\widetilde{x},R^{\frac{p-1}{p-m}}\widetilde{t})}{\partial\widetilde{t}^2}\right|^{p'}R^{\frac{-2p'(p-1)}{p-m}}dxdt\\
&\quad\leq R^{\frac{-2p'(p-1)}{p-m}+\mu Q+\frac{p-1}{p-m}}\int_{\Omega_{TR^{-\frac{p-1}{p-m}}}}\varphi^{-\frac{p+1}{p-1}}(R\widetilde{x},R^{\frac{p-1}{p-m}}\widetilde{t})\\
&\quad\times\left|\varphi(R\widetilde{x},R^{\frac{p-1}{p-m}}\widetilde{t})\frac{\partial^2\varphi(R\widetilde{x},R^{\frac{p-1}{p-m}}\widetilde{t})}{\partial\widetilde{t}^2}\right|^{p'}d\widetilde{x}d\widetilde{t}\\
&\quad\leq CR^{\mu Q-\frac{p+1}{p-m}}.
\end{aligned}\tag{5.287}$$

Hence, we have

$$\int_{\Omega_T}|u|^p\varphi dxdt\leq CR^{\mu Q-\frac{p+1}{p-m}}.\tag{5.288}$$

If $1<m<p<p_c=\frac{\mu Qm+1}{\mu Q-1}$ with $\mu Q-1>0$ and $R\to\infty$, we get

$$\int_{\Omega_T}|u|^p\varphi dxdt\leq 0.\tag{5.289}$$

Therefore, we get $u = 0$. That is a contradiction, completing the proof. $\square$

Corollary 5.57. *In the Abelian case* $(\mathbb{R}^n, +)$ *with* $Q = n$, $\mathcal{R} = -\Delta$, *and by taking Euclidean distance instead of the quasi-norm, we claim the well-known results by Kato* [105].

Corollary 5.58. *By Lemma 4.1.7 in* [3], *if* $\mathbb{G}$ *is a stratified Lie groups with* $\mathcal{R} = -\Delta_{\mathbb{G}} = -\sum_1^n X_i^2$, *where* $\Delta_{\mathbb{G}}$ *is a sub-Laplacian (i.e.,* $\nu_0 = \nu_1 = \ldots = \nu_n$, *then* $\mu = 1$*), we obtain Kato's exponent for the wave equation with the sub-Laplacian on stratified Lie groups.*

Corollary 5.59. *That is a well-known one of the particular case of the stratified Lie groups is the Heisenberg group (*[3, p.174]*). So, then in the case of the Heisenberg group with* $m = 1$ *and* $m \in \mathbb{N}$ *we obtain the results by* [101] *and* [102], *respectively.*

5.9.2. *Wave equation with linear damping term.* In this section we consider the initial problem for the wave equation on the graded Lie group

$$\begin{cases} u_{tt}(x,t) + \mathcal{R}|u(x,t)|^m + u_t(x,t) = |u(x,t)|^p, \; (x,t) \in \mathbb{G} \times (0,+\infty) := \Omega, \\ u(x,0) = u_0(x) \geq 0, \quad x \in \mathbb{G}, \\ u_t(x,0) = u_1(x), \quad x \in \mathbb{G}, \end{cases} \tag{5.290}$$

where $\mathcal{R}$ is the Rockland operator in the form

$$\mathcal{R} = \sum_{j=1}^{n} (-1)^{\frac{\nu_0}{\nu_j}} c_j X_j^{2\frac{\nu_0}{\nu_j}},$$

and $m, p > 0$.

Definition 5.60. Suppose that $u_1, u_0 \in L^1_{loc}(\mathbb{G})$. We call that $u \in L^{\max\{m,p\}}_{loc}(\Omega_T)$ $(\Omega_T = \mathbb{G} \times (0,T))$ is a local weak solution of the equation (5.290) if the identity

$$\begin{aligned} \int_{\Omega_T} u(x,t) \frac{\partial^2 \varphi(x,t)}{\partial t^2} dxdt &+ \int_{\Omega_T} |u(x,t)|^m \mathcal{R}\varphi(x,t) dxdt \\ &- \int_{\Omega_T} u(x,t) \frac{\partial \varphi(x,t)}{\partial t} dxdt \\ &= \int_{\mathbb{G}} u_0(x) \left(\varphi(x,0) + \frac{\partial \varphi(x,0)}{\partial t} \right) dx \\ &+ \int_{\mathbb{G}} u_1(x) \varphi(x,0) dx \\ &+ \int_{\Omega_T} |u(x,t)|^p \varphi(x,t) dxdt, \end{aligned} \tag{5.291}$$

holds for all nonnegative test functions

$$\varphi \in C^2((0,T]; L^2(\mathbb{G})) \cap C^1([0,T]; H^\gamma(\mathbb{G})),$$

such that $\varphi(x,T) = 0$. In the case $T = +\infty$, the solution of the equation (5.291) u is called a global weak solution.

Theorem 5.61. *Let $\mathbb{G}$ be graded groups with the homogeneous dimension $Q \geq 2$ and $\mu = \max\limits_{j=1,\dots,n} \frac{\nu_j}{\nu_0}$. Assume that $p > 1$ and $\int_{\mathbb{G}} u_1(x)dx \geq 0$. If*

$$1 < m < p < p_c = m + \frac{2}{\mu Q}, \tag{5.292}$$

then the Cauchy problem (5.279) admits no global weak nonnegative solution other than trivial.

Proof. Similarly to the previous theorem, we have

$$\begin{aligned}
\int_{\Omega_T} |u|^p \varphi dxdt \leq \int_{\mathbb{G}} u_1(x)\varphi(x,0)dx + \int_{\Omega_T} |u|^p \varphi dxdt \\
\leq C \int_{\Omega_T} \varphi^{-\frac{p+1}{p-1}} \left|\varphi \frac{\partial^2 \varphi}{\partial t^2}\right|^{p'} dxdt \\
+ C \int_{\Omega_T} |\mathcal{R}\varphi|^{\frac{p}{p-m}} \varphi^{-\frac{m}{p-m}} dxdt \\
+ C \int_{\Omega_T} \left|\frac{\partial \varphi}{\partial t}\right|^{p'} \varphi^{-\frac{1}{p-1}} dxdt.
\end{aligned}$$

By assuming

$$\int_{\Omega_T} \left|\frac{\partial \varphi}{\partial t}\right|^{p'} \varphi^{-\frac{1}{p-1}} dxdt < \infty,$$

we obtain

$$\int_{\Omega_T} |u|^p \varphi dxdt \leq C R^{\mu Q - \frac{2}{p-m}}. \tag{5.293}$$

By substituting $x = R^{\mu}\widetilde{x}$ and $t = R^{\frac{2(p-1)}{p-m}}\widetilde{t}$ with $1 < m < p < p_c = m + \frac{2}{\mu Q}$ and, letting $R \to \infty$, we get

$$\int_{\Omega_T} |u|^p \varphi dxdt \leq 0. \tag{5.294}$$

Hence, we get $u = 0$. ☐

Corollary 5.62. *In the case, if $\mathbb{G}$ is a stratified Lie groups with $\mathcal{R} = -\Delta_{\mathbb{G}} = -\sum_1^n X_i^2$, where $\Delta_{\mathbb{G}}$ is a sub-Laplacian (i.e., $\nu_0 = \nu_1 = \ldots = \nu_n$, then $\mu = 1$), we obtain Kato's type exponent for the linear damping wave equation with the sub-Laplacian on stratified Lie groups.*

Corollary 5.63. *In the case of the Heisenberg group we obtain the result by* [102].

5.9.3. *Pseudo-hyperbolic equation case.* In this subsection we show blow-up result for the pseudo-hyperbolic equation with Rockland operator on graded Lie groups in the following form:

$$\begin{cases} u_{tt} + \mathcal{R}u_{tt} + \mathcal{R}u = |u|^p, \ (x,t) \in \mathbb{G} \times (0,T) := \Omega_T, \ p > 1, \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), \ \ x \in \mathbb{G}. \end{cases} \tag{5.295}$$

Let us give definition of the weak solution of the (5.295).

Definition 5.64. We say that u is a local weak solution to (5.295) on Ω with initial data $u(x,0)=u_0(x)\in L^1_{loc}(\mathbb{G})$, if $u\in L^p_{loc}(\Omega_T)$ and satisfies

$$\begin{aligned}\int_{\Omega_T}|u|^p\varphi dxdt &+ \int_{\mathbb{G}}u_1(x)(\varphi(x,0)+\mathcal{R}\varphi(x,0))dx\\ &-\int_{\mathbb{G}}u_0(x)(\varphi_t(x,0)+\mathcal{R}\varphi_t(x,0))dx\\ &=\int_{\Omega_T}u\varphi_{tt}dxdt+\int_{\Omega_T}u\mathcal{R}\varphi_{tt}dxdt+\int_{\Omega_T}u\mathcal{R}\varphi dx,dt\end{aligned}$$

for any test function φ with $\varphi(x,T)=\varphi_t(x,T)=0$. The solution u is said global if it exists on $(0;\infty)$.

Theorem 5.65. *Let $\mathbb{G}$ be a graded Lie group with homogeneous dimension Q and $\mu=\max\limits_{j=1,\ldots,n}\frac{\nu_j}{\nu_0}$ be such that $\mu Q>1$. Assume that $u_1\in L^1(\mathbb{G})$ and*

$$\int_{\mathbb{G}}u_1dx>0. \tag{5.296}$$

If

$$1<p\le p_c=1+\frac{2}{\mu Q-1}, \tag{5.297}$$

then there exists no nontrivial global weak solution of (5.295).

Proof. Firstly, we consider the case $1<p<p_c$. From Definition 5.64 with $\varphi_t(x,0)=0$, we obtain

$$\begin{aligned}\int_{\Omega}|u|^p\varphi dxdt+\int_{\mathbb{G}}u_1(x)\varphi(x,0)dx &=\int_{\Omega}u\varphi_{tt}dxdt\\ &+\int_{\Omega}\mathcal{R}u\varphi_{tt}dxdt+\int_{\Omega}u\mathcal{R}\varphi dxdt-\int_{\mathbb{G}}u_1(x)\mathcal{R}\varphi(x,0)dx\\ &\le\Big|\int_{\Omega}u\varphi_{tt}dxdt+\int_{\Omega}u\mathcal{R}\varphi_{tt}dxdt\\ &+\int_{\Omega}u\mathcal{R}\varphi dxdt-\int_{\mathbb{G}}u_1(x)\mathcal{R}\varphi(x,0)dx\Big|\\ &\le\int_{\Omega}|u\varphi_{tt}|dxdt+\int_{\Omega}|u\mathcal{R}\varphi_{tt}|dxdt\\ &+\int_{\Omega}|u\mathcal{R}\varphi|dxdt+\int_{\mathbb{G}}|u_1(x)\mathcal{R}\varphi(x,0)|dx.\end{aligned} \tag{5.298}$$

Then from the Young inequality, we have

$$\begin{aligned}\int_{\Omega}|u||\varphi_{tt}|dxdt&=\int_{\Omega}|u|\varphi^{\frac{1}{p}}\varphi^{-\frac{1}{p}}|\varphi_{tt}|dxdt\\ &\le\varepsilon\int_{\Omega}|u|^p\varphi dxdt+c_\varepsilon\int_{\Omega}\varphi^{-\frac{1}{p-1}}|\varphi_{tt}|^{\frac{p}{p-1}}dxdt,\end{aligned}$$

$$\int_\Omega |u||\mathcal{R}\varphi_{tt}|dtdx \leq \varepsilon \int_\Omega |u|^p \varphi dtdx + c_\varepsilon \int_\Omega \varphi^{-\frac{1}{p-1}} |\mathcal{R}\varphi_{tt}|^{\frac{p}{p-1}} dxdt, \tag{5.299}$$

and

$$\int_\Omega |u||\mathcal{R}\varphi|dtdx \leq \varepsilon \int_\Omega |u|^p \varphi dtdx + c_\varepsilon \int_\Omega \varphi^{-\frac{1}{p-1}} |\mathcal{R}\varphi|^{\frac{p}{p-1}} dxdt, \tag{5.300}$$

for some positive constant c_ε. By using above facts, we get

$$\begin{aligned} &\int_\Omega |u|^p \varphi dxdt + \int_{\mathbb{G}} u_1(x)\varphi(x,0)dx \\ &\qquad \leq C\left(A_p(\varphi) + B_p(\varphi) + C_p(\varphi) + \int_{\mathbb{G}} |u_1(x)||\mathcal{R}\varphi(x,0)|dx\right), \end{aligned} \tag{5.301}$$

where

$$A_p(\varphi) = \int_\Omega \varphi^{-\frac{1}{p-1}} |\varphi_{tt}|^{\frac{p}{p-1}} dxdt, \tag{5.302}$$

$$B_p(\varphi) = \int_\Omega \varphi^{-\frac{1}{p-1}} |\mathcal{R}\varphi_{tt}|^{\frac{p}{p-1}} dxdt, \tag{5.303}$$

$$C_p(\varphi) = \int_\Omega \varphi^{-\frac{1}{p-1}} |\mathcal{R}\varphi|^{\frac{p}{p-1}} dxdt. \tag{5.304}$$

Let us choose the following test function

$$\varphi_R(x,t) = \Phi\left(\frac{|x|^2}{R^\mu}\right)\Phi\left(\frac{t^2}{R^2}\right), \ R > 0, \tag{5.305}$$

with the following property

$$\Phi(r) = \begin{cases} 1, & \text{if } 0 \leq r < 1, \\ \searrow, & \text{if } 1 \leq r < 2, \\ 0, & \text{if } r \geq 2, \end{cases}$$

where $\Phi : \mathbb{R}_+ \to [0,1]$ is a sufficiently smooth nonincreasing function. We note that

$$\frac{\partial \varphi_R(x,t)}{\partial t} = \frac{2t}{R^2}\Phi\left(\frac{|x|^2}{R^\mu}\right)\Phi_t'\left(\frac{t^2}{R^2}\right), \tag{5.306}$$

we have

$$\frac{\partial \varphi_R(x,0)}{\partial t} = 0. \tag{5.307}$$

Let us estimate $A_p(\varphi_R), B_p(\varphi_R), C_p(\varphi_R)$. By choosing variables $x = R^\mu \widetilde{x}$ and $t = R\widetilde{t}$, then

$$\widetilde{\Omega} := \{\overline{x} \in \mathbb{G} : 0 \leq |x| \leq 2\} \text{ and } \widehat{\Omega} := \{\overline{t} : 0 \leq t^2 \leq 2\}. \tag{5.308}$$

By using Proposition 2.4, we calculate

$$A_p(\varphi_R) = \int_0^\infty \int_{\mathbb{G}} |\varphi_R(x,t)|^{-\frac{p}{p-1}} \left|\frac{\partial^2 \varphi_R(x,t)}{\partial t^2}\right|^{\frac{p}{p-1}} dxdt \leq CR^{\mu Q + 1 - \frac{2p}{p-1}}, \tag{5.309}$$

$$B_p(\varphi_R) \leq CR^{\mu Q + 1 - \frac{4p}{p-1}}, \tag{5.310}$$

and

$$C_p(\varphi_R) \leq CR^{\mu Q + 1 - \frac{2p}{p-1}}. \tag{5.311}$$

Also, we get that

$$|\mathcal{R}\varphi_R(x,t)| \leq CR^{-2}. \tag{5.312}$$

By combining (5.309)-(5.311) and (5.301), we get

$$\begin{aligned}\int_{\Omega} |u|^p \varphi_R dxdt + \int_{\mathbb{G}} u_1(x)\varphi_R(x,0)dx \\ &\leq C\left(R^{\mu Q+1-\frac{2p}{p-1}} + R^{\mu Q+1-\frac{4p}{p-1}} + R^{\mu Q+1-\frac{4p}{p-1}}\right. \\ &+ \left.\int_{\widetilde{\Omega}} |u_1(x)||\mathcal{R}\varphi_R(x,0)|dx\right) \\ &\leq C\left(R^{\mu Q+1-\frac{2p}{p-1}} + \int_{\widetilde{\Omega}} |u_1(x)||\mathcal{R}\varphi_R(x,0)|dx\right).\end{aligned} \tag{5.313}$$

On the other hand, we get

$$\begin{aligned}\liminf_{R\to\infty} \int_{\Omega} |u|^p \varphi_R dxdt + \int_{\mathbb{G}} u_1(x)\varphi_R(x,0)dx \\ &\geq \lim\inf_{R\to\infty} \int_{\Omega} |u|^p \varphi_R dxdt \\ &+ \lim\inf_{R\to\infty} \int_{\mathbb{G}} u_1(x)\varphi_R(x,0)dx.\end{aligned}$$

By using the monotone convergence theorem, we obtain

$$\lim\inf_{R\to\infty} \int_{\Omega} |u|^p \varphi_R dxdt = \int_{\Omega} |u|^p dxdt.$$

Since $u_1 \in L^1(\mathbb{G})$, by the dominated convergence theorem, we have

$$\lim\inf_{R\to\infty} \int_{\mathbb{G}} u_1(x)\varphi_R(x,0)dx = \int_{\mathbb{G}} u_1(x)dx.$$

Now, we have

$$\lim\inf_{R\to\infty} \left(\int_{\Omega} |u|^p \varphi_R dxdt + \int_{\mathbb{G}} u_1(x)\varphi_R(x,0)dx\right) \geq \int_{\Omega} |u|^p dxdt + d,$$

where

$$d = \int_{\mathbb{G}} u_1(x)dx > 0.$$

For every $\varepsilon > 0$ exists $R_0 > 0$ such that

$$\begin{aligned}\int_{\Omega} |u|^p \varphi_R dxdt + \int_{\mathbb{G}} u_1(x)\varphi_R(x,0)dx \\ &> \lim\inf_{R\to\infty} (\int_{\Omega} |u|^p \varphi_R dxdt + \int_{\mathbb{G}} u_1(x)\varphi_R(x,0)dx) - \varepsilon \\ &\geq \int_{\Omega} |u|^p dxdt + d - \varepsilon,\end{aligned}$$

for every $R \geq R_0$. By taking $\varepsilon = \frac{d}{2}$, we have

$$\int_{\Omega} |u|^p \varphi_R dxdt + \int_{\mathbb{G}} u_1(x)\varphi_R(x,0)dx \geq \int_{\Omega} |u|^p dxdt + \frac{d}{2},$$

for every $R \geq R_0$. Then from (5.313), (5.312) and $u_1 \in L^1(\mathbb{G})$, we have

$$\begin{aligned}\int_\Omega |u|^p dxdt + \frac{l}{2} &\leq C\left(R^{\mu Q+1-\frac{2p}{p-1}} + \int_{\widetilde{\Omega}} |u_1(x)||\mathcal{R}\varphi_R(x,0)|dx\right)\\ &\leq C\left(R^{\mu Q+1-\frac{2p}{p-1}} + R^{-2}\int_{\widetilde{\Omega}} |u_1(x)|dx\right)\\ &\leq C\left(R^{\mu Q+1-\frac{2p}{p-1}} + R^{-2}\right).\end{aligned} \tag{5.314}$$

Thus, we obtain

$$\mu Q + 1 - \frac{2p}{p-1} < 0,$$

or

$$p < 1 + \frac{2}{\mu Q - 1}.$$

If $R \to \infty$, we get

$$\int_\Omega |u|^p dxdt + \frac{l}{2} \leq 0.$$

This is a contradiction. Finally, we obtain

$$\int_\Omega |u|^p dxdt + \frac{l}{2} = 0.$$

Let us consider the case $p = 1 + \frac{2}{\mu Q-1}$. By using (5.314), we have

$$\int_\Omega |u|^p dxdt \leq C < \infty, \tag{5.315}$$

then

$$\lim_{R\to\infty} \int_\Omega |u|^p \varphi_R dxdt = 0. \tag{5.316}$$

Using the Hölder inequality instead of Young's inequality in (5.298), we get

$$\int_\Omega |u|^p \varphi_R dxdt + \frac{d}{2} \leq C\left(\int_{\widetilde{\Omega}} |u|^p \varphi_R dxdt\right)^{\frac{1}{p}}.$$

If $R \to \infty$ then by combining the above facts, we have

$$\int_\Omega |u|^p \varphi_R dxdt + \frac{d}{2} = 0.$$

This contradiction completes the proof. □

Corollary 5.66. *In the case, if $\mathbb{G}$ is a stratified Lie groups with $\mathcal{R} = -\Delta_{\mathbb{G}} = -\sum_1^n X_i^2$, where $\Delta_{\mathbb{G}}$ is a sub-Laplacian (i.e., $\nu_0 = \nu_1 = \ldots = \nu_n$, then $\mu = 1$) and $c_j = 1$, $j = 1, \ldots, n$, we obtain Kato-type exponent for the linear damping wave equation with the sub-Laplacian on stratified Lie groups.*

Corollary 5.67. *In the case of the Heisenberg group, in particular, we obtain the results of the paper* [103].

5.9.4. *The case of system.* Let us consider the system of the pseudo-hyperbolic Rockland equations with the Cauchy conditions:

$$\begin{cases} u_{tt} + \mathcal{R}u_{tt} + \mathcal{R}u = |v|^q, \ (x,t) \in \mathbb{G} \times (0,T) := \Omega_T, \ q > 1, \\ v_{tt} + \mathcal{R}v_{tt} + \mathcal{R}v = |u|^p, \ (x,t) \in \mathbb{G} \times (0,T) := \Omega_T, \ p > 1, \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), \ \ x \in \mathbb{G}, \\ v(x,0) = v_0(x), v_t(x,0) = v_1(x), \ \ x \in \mathbb{G}. \end{cases} \tag{5.317}$$

Firstly, let us give a definition of the weak solution of (5.317) in the following form:

Definition 5.68. We say that the pair $(u;v)$ is a local weak solution of (5.317) on $\mathbb{G}$ with the Cauchy data $(u(x,0);v(x,0)) = (u_0;v_0) \in L^1_{loc}(\mathbb{G}) \times L^1_{loc}(\mathbb{G})$, if $(u,v) \in L^p_{loc}(\Omega_T) \times L^q_{loc}(\Omega_T)$ satisfies

$$\begin{aligned} \int_{\Omega_T} |v|^q \varphi dx dt &+ \int_{\mathbb{G}} u_1(x)(\varphi(x,0) + \mathcal{R}\varphi(x,0))dx \\ &- \int_{\mathbb{G}} u_0(x)(\varphi_t(x,0) + \mathcal{R}\varphi_t(x,0))dx \\ &= \int_{\Omega_T} u\varphi_{tt} dx dt + \int_{\Omega_T} u\mathcal{R}\varphi_{tt} dx dt + \int_{\Omega_T} u\mathcal{R}\varphi dx dt, \end{aligned}$$

and

$$\begin{aligned} \int_{\Omega_T} |u|^p \varphi dx dt &+ \int_{\mathbb{G}} v_1(x)(\varphi(x,0) + \mathcal{R}\varphi(x,0))dx \\ &- \int_{\mathbb{G}} v_0(x)(\varphi_t(x,0) + \mathcal{R}\varphi_t(x,0))dx \\ &= \int_{\Omega_T} v\varphi_{tt} dx dt + \int_{\Omega_T} v\mathcal{R}\varphi_{tt} dx dt + \int_{\Omega_T} v\mathcal{R}\varphi dx dt, \end{aligned}$$

for any test function φ with $\varphi(\cdot,T) = \varphi_t(\cdot,T) = 0$. The solution is said to be a global if it exists for $T = +\infty$.

Now we present the main result in the system case.

Theorem 5.69. *Let $\mathbb{G}$ be a graded Lie group with homogeneous dimension Q and $\mu = \max\limits_{j=1,\dots,n} \frac{\nu_j}{\nu_0}$ be such that $\mu Q > 1$.Assume that $(u_1, v_1) \in L^1(\mathbb{G}) \times L^1(\mathbb{G})$ with*

$$\int_{\mathbb{G}} u_1 dx > 0, \ \ \textit{and} \ \ \int_{\mathbb{G}} v_1 dx > 0. \tag{5.318}$$

If $1 < pq \le (pq)^* = 1 + \frac{2}{\mu Q - 1} \max\{p+1; q+1\}$ then there exists no nontrivial weak solution to (5.317).

Proof. Similarly to the single equation with $\varphi_t(x,0)=0$, we get

$$\begin{aligned}\int_\Omega |v|^q\varphi dxdt &+ \int_{\mathbb{G}} u_1(x)\varphi(x,0)dx\\ &\leq \int_\Omega |u||\varphi_{tt}|dxdt + \int_\Omega |u||\mathcal{R}\varphi_{tt}|dxdt\\ &+ \int_\Omega |u||\mathcal{R}\varphi|dxdt + \int_{\mathbb{G}} |u_1(\vartheta)||\mathcal{R}\varphi(x,0)|dx\end{aligned}$$

and

$$\begin{aligned}\int_\Omega |u|^p\varphi dxdt &+ \int_{\mathbb{G}} v_1(x)\varphi(x,0)dx\\ &\leq \int_\Omega |v||\varphi_{tt}|dxdt + \int_\Omega |v||\mathcal{R}\varphi_{tt}|dxdt\\ &+ \int_\Omega |v||\mathcal{R}\varphi|dxdt + \int_{\mathbb{G}} |v_1(x)||\mathcal{R}\varphi(x,0)|dx.\end{aligned}$$

By choosing $\varphi=\varphi_R$, the test function given by (5.305) and from the Hölder inequality, we calculate

$$\begin{aligned}\int_\Omega |v|^q\varphi_R dxdt &+ \int_{\mathbb{G}} u_1(x)\varphi(x,0)dx - \int_{\mathbb{G}} |u_1(x)||\mathcal{R}\varphi(x,0)|dx\\ &\leq (A_p(\varphi_R)^{\frac{p-1}{p}} + B_p(\varphi_R)^{\frac{p-1}{p}} + C_p(\varphi_R)^{\frac{p-1}{p}})\left(\int_\Omega |u|^p\varphi_R dxdt\right)^{\frac{1}{p}},\end{aligned}$$

where $A_p(\varphi)$, $B_p(\varphi)$ and $C_p(\varphi)$ are given in the single equation case. Similarly, by the Hölder inequality, we have

$$\begin{aligned}\int_\Omega |u|^p\varphi_R dxdt &+ \int_{\mathbb{G}} v_1(x)\varphi(x,0)dx - \int_{\mathbb{G}} |v_1(x)||\mathcal{R}\varphi(x,0)|dx\\ &\leq (A_q(\varphi_R)^{\frac{q-1}{q}} + B_q(\varphi_R)^{\frac{q-1}{q}} + C_q(\varphi_R)^{\frac{q-1}{q}})\left(\int_\Omega |v|^q\varphi_R dxdt\right)^{\frac{1}{q}}.\end{aligned}$$

Assume that for the large R, we get

$$\begin{aligned}\int_{\mathbb{G}} u_1(x)\varphi_R(x,0)dx - \int_{\mathbb{G}} |u_1(x)||\mathcal{R}\varphi_R(x,0)|dx \geq 0,\\ \int_{\mathbb{G}} v_1(x)\varphi_R(x,0)dx - \int_{\mathbb{G}} |v_1(x)||\mathcal{R}\varphi_R(x,0)|dx \geq 0.\end{aligned} \tag{5.319}$$

Then, we have

$$\int_\Omega |v|^q\varphi_R dxdt \leq (A_p(\varphi_R)^{\frac{p-1}{p}} + B_p(\varphi_R)^{\frac{p-1}{p}} + C_p(\varphi_R)^{\frac{p-1}{p}})\left(\int_\Omega |u|^p\varphi_R dxdt\right)^{\frac{1}{p}}, \tag{5.320}$$

and

$$\int_\Omega |u|^p\varphi_R dxdt \leq (A_q(\varphi_R)^{\frac{q-1}{q}} + B_q(\varphi_R)^{\frac{q-1}{q}} + C_q(\varphi_R)^{\frac{q-1}{q}})\left(\int_\Omega |v|^q\varphi_R dxdt\right)^{\frac{1}{q}}. \tag{5.321}$$

By choosing variables $\overline{t} = R^{-1}t$ and $\overline{x} = R^{-\mu}x$, we get

$$\int_\Omega |v|^q \varphi_R dxdt \leq CR^{\frac{\mu Q(p-1)-(p+1)}{p}} \left(\int_\Omega |u|^p \varphi_R dxdt\right)^{\frac{1}{p}}, \tag{5.322}$$

and

$$\int_\Omega |u|^p \varphi_R dxdt \leq CR^{\frac{\mu Q(q-1)-(q+1)}{q}} \left(\int_\Omega |v|^q \varphi_R dxdt\right)^{\frac{1}{q}}. \tag{5.323}$$

By using the last two inequalities, we have

$$\left(\int_\Omega |u|^p \varphi_R dxdt\right)^{1-\frac{1}{pq}} \leq CR^{\alpha_1}, \tag{5.324}$$

and

$$\left(\int_\Omega |v|^q \varphi_R dxdt\right)^{1-\frac{1}{pq}} \leq CR^{\alpha_2}, \tag{5.325}$$

where

$$\alpha_1 = \frac{\mu Q(pq-1) - pq - 2q - 1}{pq}$$

and

$$\alpha_2 = \frac{\mu Q(pq-1) - pq - 2p - 1}{pq}.$$

Then we need $\alpha_1, \alpha_2 < 0$.

Secondly, let us consider the case $1 < pq < 1 + \frac{2}{\mu Q-1}\max\{p+1; q+1\}$.

Case 1: $1 < pq < 1 + \frac{2}{\mu Q-1}\max\{p+1; q+1\}$. By letting $R \to \infty$ in (5.324) with $1 < q \leq p$, we have

$$\int_\Omega |u|^p dxdt = 0,$$

which is a contradiction. Similarly, in the case $1 < p \leq q$, from (5.325), we have

$$\int_\Omega |v|^q dxdt = 0.$$

Case 2: $pq = 1 + \frac{2}{\mu Q-1}\max\{p+1; q+1\}$. This case is similar with the proof of Theorem 5.65.

□

Corollary 5.70. *In the case $p = q$ and $u = v$ in Theorem 5.69, we arrive at a single equation given by Theorem 5.65.*

Proof. From Theorem 5.69, we get

$$p^2 \leq 1 + \frac{2(p+1)}{\mu Q - 1},$$

and

$$p^2 - 1 \leq \frac{2(p+1)}{\mu Q - 1}.$$

Then dividing both sides by $p+1$, we obtain

$$p-1 \leq \frac{2}{\mu Q-1}. \tag{5.326}$$

□

Corollary 5.71. *In the case, if $\mathbb{G}$ is a stratified Lie groups with $\mathcal{R}=-\Delta_{\mathbb{G}}=-\sum_1^n X_i^2$, where $\Delta_{\mathbb{G}}$ is a sub-Laplacian (i.e., $\nu_0=\nu_1=\ldots=\nu_n$, then $\mu=1$), we obtain Kato's type exponent for the linear damping wave equation with the sub-Laplacian on stratified Lie groups.*

Corollary 5.72. *In the case of the Heisenberg group, in particular, we obtain the result by* [103].

5.10. **Fujita type exponents for the heat Rockland equations.** In [106], Fujita considered the nonlinear heat equation

$$\begin{cases} u_t(x,t)-\Delta u(x,t)=u^{1+p}, \ \ (x,t)\in\mathbb{R}^N\times(0,\infty), \\ u(x,0)=u_0(x)\geq 0, \ \ x\in\mathbb{R}^N, \end{cases} \tag{5.327}$$

for the subject of blowing up. He obtained that if $0<p<\frac{2}{N}$, then a solution of the problem (5.327) blows up in finite time for some $x_0\in\mathbb{R}^N$, $N\in\mathbb{N}$. One of the further generalisations of the problem (5.327) is considering the fractional Laplacian $(-\Delta)^s$ instead of the classical Laplacian $-\Delta$. Namely, in [107, 108, 109] the authors considered the following Cauchy problem. In this dissertation, we show Fujita's exponent for the heat Rockland operator and we show necessary condition for the global solvability.

That is well-known that the critical Fujita exponent determined as $p^*=1+\frac{2}{N}$ for the pseudo-parabolic equation and in [110], [111], the authors obtained the critical Fujita exponents. In [112], the authors studied the nonexistence of the global solutions for the following nonlinear pseudo-parabolic equation on Heisenberg group:

$$u_t+(-\Delta_{\mathbb{H}^n})^m u_t+(-\Delta_{\mathbb{H}^n})^m u=|u|^p, (\eta,t)\in\mathbb{H}^n\times(0,\infty), \tag{5.328}$$

with the Cauchy data

$$u(\eta,0)=u_0(\eta), \eta\in\mathbb{H}^n, \tag{5.329}$$

where $m>1, p>1$, $\Delta_{\mathbb{H}^n}$ is the Kohn-Laplace operator on (2×2)-dimensional Heisenberg group $\mathbb{H}^n$. For more details, the reader referred to [112] and references therein, [113], [114].

5.10.1. *Fujita exponent for the Heat Rockland equation.* Let us consider the Cauchy problem for the nonlinear heat Rockland equation in the following form:

$$\begin{cases} u_t(x,t)+\mathcal{R}^\alpha\{u\}^m(x,t)=u^p(x,t), \ \ (x,t)\in\mathbb{G}\times(0,+\infty):=\Omega_\infty, \\ u(x,0)=u_0(x)\geq 0, \ \ \ x\in\mathbb{G}, \end{cases} \tag{5.330}$$

where $\alpha>0$, $m\in\mathbb{N}$, and $\mathcal{R}$ is a Rockland operator of the k-th order, that is,

$$\mathcal{R}=\sum_{j=1}^n(-1)^{\frac{\nu_0}{\nu_j}}c_jX_j^{2\frac{\nu_0}{\nu_j}}.$$

By $\mathcal{R}^\alpha$ we understand fractional Rockland operator as Proposition 2.14. Let us denote by $\mathfrak{C}^{\alpha,1}_{x,t}(\Omega_T)$ the space of test functions φ with a compact support $\operatorname{supp}\varphi \subset \Omega_T$ such that $\varphi, \partial_t\varphi$ and $\mathcal{R}^\alpha\varphi$ are continuous functions on Ω_T with compact supports $\operatorname{supp}\partial_t\varphi, \operatorname{supp}\mathcal{R}^\alpha\varphi \subset \Omega_T$, where $\Omega_T := \mathbb{G}\times(0,T)$ for some $T>0$.

Let us give a definition of the weak solution to the equation (5.330).

Definition 5.73. Fix $T>0$. Assume that $u_0 \in L^1(\Omega_T)$ ($\Omega_T = \mathbb{G}\times(0,T)$). Then we call the function $u \in L^p(\Omega_T)\cap L^m(\Omega_T)$ a local weak solution of (5.330) if the identity

$$-\int_{\Omega_T} u(x,t)\,\frac{\partial\varphi(x,t)}{\partial t}\,dxdt + \int_{\Omega_T} \{u\}^m(x,t)\mathcal{R}^\alpha\varphi(x,t)dxdt = \int_{\mathbb{G}} u_0(x)\varphi(x,0)dx + \int_{\Omega_T} u^p(x,t)\varphi(x,t)dxdt, \quad (5.331)$$

holds for all positive test functions φ from $\mathfrak{C}^{\alpha,1}_{x,t}(\Omega_T)$ such that $\varphi(x,T)=0$.

If it is allowed to be $T=+\infty$ then u is called a global weak solution of the equation (5.330).

Theorem 5.74. *Assume that $\mathbb{G}$ be the graded Lie group with homogeneous dimension $Q\geq 2$. Assume that*

$$1 < p \leq p_c = m + \frac{k\alpha}{Q}. \quad (5.332)$$

Then the Cauchy problem (5.330), admits no global weak nonnegative solutions other than trivial.

Proof. We prove this theorem by contradiction. By using (5.330) and Definition 5.55, we have

$$\int_{\Omega_T} |u|^p\varphi dxdt \leq \int_{\Omega_T} |u|^p\varphi dxdt + \int_{\mathbb{G}} u_0(x)\varphi(x,0)dx = -\int_{\Omega_T} u(x,t)\frac{\partial\varphi(x,t)}{\partial t}dxdt + C\int_{\Omega_T} |u(x,t)|^m\mathcal{R}^\alpha\varphi(x,t)dxdt, \quad (5.333)$$

for some constant $C>0$.

From s-Young's inequality

$$ab \leq sa^l + \frac{1}{s}b^{l'},\ \ \frac{1}{l}+\frac{1}{l'} = 1,\ \ a,b\geq 0,$$

we get

$$\begin{aligned}\int_{\Omega_T} u^p\varphi dxdt &\le -\int_{\Omega_T} u(x,t)\frac{\partial\varphi(x,t)}{\partial t}dxdt + C\int_{\Omega_T} |u(x,t)|^m\mathcal{R}^\alpha\varphi(x,t)dxdt\\ &= \int_{\Omega_T}\varphi^{-\frac{1}{p}}\frac{\partial\varphi(x,t)}{\partial t}(-u(x,t))\varphi^{\frac{1}{p}}dxdt + C\int_{\Omega_T}\varphi^{\frac{m}{p}}|u(x,t)|^m\,\mathcal{R}^\alpha\varphi(x,t)\varphi^{-\frac{m}{p}}dxdt\\ &\le \frac{1}{4}\int_{\Omega_T}u^p\varphi dxdt + C_1\int_{\Omega_T}\varphi^{-\frac{p'}{p}}\left|\frac{\partial\varphi}{\partial t}\right|^{p'}dxdt + \frac{1}{4}\int_{\Omega_T}u^p\varphi dxdt\\ &\qquad\qquad + C_2\int_{\Omega_T}\varphi^{\frac{-m}{p-m}}|\mathcal{R}^\alpha\varphi|^{\frac{p}{p-m}}dxdt\\ &= \frac{1}{2}\int_{\Omega_T}u^p\varphi dxdt + C_1\int_{\Omega_T}\varphi^{-\frac{p'}{p}}\left|\frac{\partial\varphi}{\partial t}\right|^{p'}dxdt + C_2\int_{\Omega_T}\varphi^{\frac{-m}{p-m}}|\mathcal{R}^\alpha\varphi|^{\frac{p}{p-m}}dxdt,\end{aligned}\tag{5.334}$$

where

$$C_1\int_{\Omega_T}\varphi^{-\frac{p'}{p}}\left|\frac{\partial\varphi}{\partial t}\right|^{p'}dxdt + C_2\int_{\Omega_T}\varphi^{\frac{-m}{p-m}}|\mathcal{R}^\alpha\varphi|^{\frac{p}{p-m}}dxdt < \infty,\tag{5.335}$$

and C, C_1, C_2 are positive constants, then

$$\int_{\Omega_T}u^p\varphi dxdt \le C_1\int_{\Omega_T}\varphi^{-\frac{p'}{p}}\left|\frac{\partial\varphi}{\partial t}\right|^{p'}dxdt + C_2\int_{\Omega_T}\varphi^{\frac{-m}{p-m}}|\mathcal{R}^\alpha\varphi|^{\frac{p}{p-m}}dxdt.\tag{5.336}$$

Let $\Phi_1, \Phi_2 : \mathbb{R}_+ \to [0,1]$ be smooth nonincreasing functions such that

$$\Phi(z) := \begin{cases}1, & \text{if } 0\le z\le 1,\\ 0, & \text{if } z\ge 2.\end{cases}\tag{5.337}$$

For $R>0$, we define

$$\varphi(x,t) = \Phi\left(\frac{|x|}{R}\right)\Phi\left(\frac{t}{R^\beta}\right).$$

By substituting variables $x = R\widetilde{x}$ and $t = R^\beta\widetilde{t}$ and by using Proposition 2.4 and (5.335), we get

$$\int_{\Omega_T}\varphi^{-\frac{p'}{p}}(x,t)\left|\frac{\partial\varphi(x,t)}{\partial t}\right|^{\frac{p}{p-1}}dxdt \le CR^{-\beta\frac{p}{p-1}+Q+\beta},\tag{5.338}$$

and

$$\int_{\Omega_T}\varphi^{\frac{-m}{p-m}}|\mathcal{R}^\alpha\varphi|^{\frac{p}{p-m}}dxdt \le CR^{-k\alpha\frac{p}{p-m}+Q+\beta}.\tag{5.339}$$

Then by from (5.338) and (5.339), we have

$$\int_{\Omega_T}|u|^p\varphi dxdt \le C(R^{-\beta\frac{p}{p-1}+Q+\beta} + R^{-k\alpha\frac{p}{p-m}+Q+\beta}).\tag{5.340}$$

Let us choose $\beta = Q(m-1)+k\alpha$. Then, if $1<p<m+\frac{k\alpha}{Q}$, we obtain

$$\int_{\Omega_T}u^p dxdt = \lim_{R\to\infty}\int_{\Omega_T}u^p\varphi dxdt \le 0.\tag{5.341}$$

Hence, $u = 0$. This is a contradiction. ☐

5.10.2. *Necessary conditions for local and global existence.* In this subsection, we present necessary conditions for existence of local and global solutions to the equation (5.330).

Theorem 5.75. *Suppose $p > m$ and $\alpha > 0$. Assume that u be a local solution to (5.330) for $T < \infty$. Then we have the estimate*

$$\lim_{|x|\to\infty} \inf u_0(x) \leq C T^{1-p'}, \tag{5.342}$$

for a positive constant C, where $\frac{1}{p} + \frac{1}{p'} = 1$.

Proof. By denoting the following test function

$$\varphi(x,t) = \Phi\left(\frac{|x|}{R}\right)\Phi\left(\frac{t}{T}\right), \tag{5.343}$$

where Φ is a smooth nonnegative function with a compact support and

$$\Phi_2\left(\frac{t}{T}\right) := \begin{cases} \left(1 - \frac{t}{T}\right)^l, & 0 < t \leq T, \\ 0, & t > T, \end{cases} \tag{5.344}$$

where $l > p' - 1$. By combining Definition 5.73 and s-Young's inequality, we get

$$\begin{aligned} &\int_{\mathbb{G}} u_0(x)\varphi(x,0)dx + \int_{\Omega_T} u^p \varphi dxdt \\ &\quad\leq -\int_{\Omega_T} \frac{\partial\varphi(x,t)}{\partial t} u(x,t)dxdt + \int_{\Omega_T} \{u\}^m(x,t)\mathcal{R}^\alpha\varphi(x,t)dxdt \\ &\quad= \int_{\Omega_T} \varphi^{-\frac{1}{p}} \frac{\partial\varphi(x,t)}{\partial t}(-u(x,t))\varphi^{\frac{1}{p}}dxdt + \int_{\Omega_T} \varphi^{\frac{1}{p}}\{u\}^m(x,t)\mathcal{R}^\alpha\varphi(x,t)\varphi^{-\frac{1}{p}}dxdt \\ &\quad\leq \frac{1}{2}\int_{\Omega_T} u^p\varphi dxdt + C\int_{\Omega_T} \varphi^{-\frac{p'}{p}}\left|\frac{\partial\varphi}{\partial t}\right|^{p'} dxdt + \frac{1}{2}\int_{\Omega_T} |u|^p\varphi dxdt \\ &\qquad\qquad + C\int_{\Omega_T} \varphi^{\frac{-m}{p-m}}|\mathcal{R}^\alpha\varphi|^{\frac{p}{p-m}}dxdt \\ &\quad= \int_{\Omega_T} u^p\varphi dxdt + C\int_{\Omega_T} \varphi^{-\frac{p'}{p}}\left|\frac{\partial\varphi}{\partial t}\right|^{p'} dxdt + C\int_{\Omega_T} \varphi^{\frac{-m}{p-m}}|\mathcal{R}^\alpha\varphi|^{\frac{p}{p-m}}dxdt. \end{aligned} \tag{5.345}$$

Finally, we have

$$\int_{\mathbb{G}} u_0(x)\varphi(x,0)dx \leq C\int_{\Omega_T} \varphi^{-\frac{p'}{p}}\left|\frac{\partial\varphi}{\partial t}\right|^{p'} dxdt + C\int_{\Omega_T} \varphi^{\frac{-m}{p-m}}|\mathcal{R}^\alpha\varphi|^{\frac{p}{p-m}}dxdt. \tag{5.346}$$

By substituting $t = T\widetilde{t}$ and $x = R\widetilde{x}$ and by using Proposition 2.4, we get

$$R^Q\int_{\mathbb{G}} u_0(R\widetilde{x})\Phi_1(\widetilde{x})d\widetilde{x} \leq CR^QT^{\frac{-p'}{p}}\int_{\mathbb{G}}\Phi_1(\widetilde{x})d\widetilde{x} + CTR^{Q-\frac{k\alpha p}{p-m}}\int_{\mathbb{G}}\Phi_1(\widetilde{x})d\widetilde{x}, \tag{5.347}$$

and, we obtain

$$\int_{\mathbb{G}} u_0(R\widetilde{x})\Phi_1(\widetilde{x})d\widetilde{x} \leq CT^{\frac{-p'}{p}} \int_{\mathbb{G}} \Phi_1(\widetilde{x})d\widetilde{x} + CTR^{-\frac{k\alpha p}{p-m}} \int_{\mathbb{G}} \Phi_1(\widetilde{x})d\widetilde{x}$$
$$= C(T^{\frac{-p'}{p}} + TR^{-\frac{k\alpha p}{p-m}}) \int_{\mathbb{G}} \Phi_1(\widetilde{x})d\widetilde{x}. \quad (5.348)$$

Hence, we obtain

$$C(T^{\frac{-p'}{p}} + TR^{-\frac{k\alpha p}{p-m}}) \int_{\mathbb{G}} \Phi_1(\widetilde{x})d\widetilde{x} \geq \int_{\mathbb{G}} u_0(R\widetilde{x})\Phi_1(\widetilde{x})d\widetilde{x}$$
$$= \inf_{q(\widetilde{x})>1} (u_0(R\widetilde{x})) \int_{\mathbb{G}} \Phi_1(\widetilde{x})d\widetilde{x}, \quad (5.349)$$

and by dividing to $\int_{\mathbb{G}} \Phi_1(\widetilde{x})d\widetilde{x}$ both sides, we get

$$\inf_{q(\widetilde{x})>1} u_0(R\widetilde{x}) \leq C(T^{\frac{-p'}{p}} + TR^{-\frac{k\alpha p}{p-m}}). \quad (5.350)$$

By letting $R \to \infty$, we have

$$\lim_{|x|\to\infty} \inf u_0(x) \leq CT^{\frac{-p'}{p}}. \quad (5.351)$$

□

Now, we show a necessary condition of existence of the global solution.

Theorem 5.76. *Assume that $p > m$ and $\alpha > 0$ be such that $0 < \gamma < \frac{k\alpha}{p-m}$. Suppose that the problem (5.330) has a nontrivial and nonnegative global weak solution. Then the initial function u_0 satisfies the condition*

$$\lim_{|x|\to\infty} \inf(u_0(x)|x|^{\gamma}) \leq C, \quad (5.352)$$

where C is a positive constant independent of u.

Proof. Continuing discussions of the proof of the previous theorem, by (5.348), we have

$$\int_{\mathbb{G}} u_0(R\widetilde{x})\Phi(\widetilde{x})d\widetilde{x} \leq C(T^{\frac{-p'}{p}} + TR^{-\frac{k\alpha p}{p-m}}) \int_{\mathbb{G}} \Phi(\widetilde{x})d\widetilde{x}. \quad (5.353)$$

From $\operatorname{supp}\Phi \subset \{x : R < |x| < 2R\}$, we obtain

$$\begin{aligned}
&\inf_{|x|>R}(u_0(x)|x|^{\gamma}) \int_{\mathbb{G}} \Phi(\widetilde{x})|R\widetilde{x}|^{-\gamma}d\widetilde{x} \\
&\leq \int_{\mathbb{G}} u_0(R\widetilde{x})|R\widetilde{x}|^{p'-1}\Phi(\widetilde{x})|R\widetilde{x}|^{1-p'}d\widetilde{x} \\
&\leq C(T^{\frac{-p'}{p}} + TR^{-\frac{k\alpha p}{p-m}}) \int_{\mathbb{G}} |R\widetilde{x}|^{\gamma}\Phi(\widetilde{x})|R\widetilde{x}|^{-\gamma}d\widetilde{x} \\
&\leq C(T^{\frac{-p'}{p}} + TR^{-\frac{k\alpha p}{p-m}})R^{\gamma} \int_{\mathbb{G}} \Phi(\widetilde{x})|R\widetilde{x}|^{-\gamma}d\widetilde{x}.
\end{aligned} \quad (5.354)$$

Since $0 < \gamma < \frac{k\alpha}{p-m}$, we have

$$\inf_{|x|>R}(u_0(x)|x|^\gamma) \leq C(T^{\frac{-p'}{p}} + TR^{-\frac{k\alpha p}{p-m}})R^\gamma. \tag{5.355}$$

By changing $T = R^{\gamma(p-1)}$, we get

$$\inf_{|x|>R}(u_0(x)|x|^\gamma) \leq C(1 + R^{-(\frac{k\alpha}{p-m}-\gamma)p}), \tag{5.356}$$

and as $R \to \infty$, we have

$$\inf_{|x|\to\infty}(u_0(x)|x|^\gamma) \leq C. \tag{5.357}$$

□

5.10.3. *Fujita exponent for the pseudo-parabolic Rockland equation.* In this subsection, we concern nonexistence of global weak solutions to the following nonlinear pseudo–parabolic equation

$$u_t(x,t)+\mathcal{R}u_t(x,t)+\mathcal{R}u(x,t) = |u(x,t)|^p+f(x,t), \quad (x,t) \in \mathbb{G}\times(0,\infty) := \Omega, \tag{5.358}$$

under the initial condition

$$u(x,0) = u_0(x), \quad x \in \mathbb{G}. \tag{5.359}$$

Similarly, with the heat Rockland equation case, We denote by $\mathfrak{C}^{\alpha,1}_{x,t}(\Omega_T)$ the space of test functions φ with a compact support $supp\,\varphi \subset \Omega_T$ such that $\varphi, \partial_t\varphi$, $\mathcal{R}\varphi$ and $\partial_t\mathcal{R}\varphi$ are continuous functions on Ω_T with compact supports $supp\,\partial_t\varphi$, $supp\,\mathcal{R}\varphi$, $supp\,\partial_t\mathcal{R}\varphi \subset \Omega_T$.

Definition 5.77. We say that u is a global weak solution to the problem (5.358)–(5.359) on Ω with the initial data $u(\cdot,0) = u_0(\cdot) \in L^1_{loc}(\mathbb{G})$, if $u \in L^p_{loc}(\Omega)$ and satisfies

$$\begin{aligned}\int_\Omega |u|^p\varphi dxdt &+ \int_{\mathbb{G}} u_0(x)\varphi(x,0)dx + \int_\Omega f\varphi dxdt \\ = -\int_\Omega u\varphi_t dxdt &+ \int_\Omega u(\mathcal{R}\varphi)_t dxdt - \int_\Omega u\mathcal{R}\varphi dxdt + \int_{\mathbb{G}} u_0(x)\mathcal{R}\varphi(x,0)dx\end{aligned} \tag{5.360}$$

for any regular test function φ with $\varphi(\cdot,t) = 0$ for large enough t.

For $R > 0$, we define

$$\Gamma_R = \{(x,t) \in \Omega : 0 \leq t \leq R^\alpha,\ 0 \leq |x| \leq R\}.$$

Theorem 5.78. *Suppose that $\mathcal{R}$ is a Rockland operator of k-th order. Let $u_0 \in L^1(\mathbb{G})$ and $f^- \in L^1(\Omega)$, where $f^- = \max\{-f,0\}$. Suppose that*

$$\int_{\mathbb{G}} u_0 dx + \lim_{R\to\infty}\inf\int_{\Gamma_R} f dxdt > 0. \tag{5.361}$$

If $1 < p \leq p^* = 1 + \frac{k}{Q}$, then the problem (5.358)–(5.359) does not admit any global weak solution.

Proof. Suppose that u is a global weak solution to the problem (5.358)–(5.359). Then, we have

$$\begin{aligned}\int_\Omega |u|^p\varphi dxdt &+ \int_{\mathbb{G}} u_0(x)\varphi(x,0)dx + \int_\Omega f\varphi dxdt \\ &\leq \int_\Omega |u||\varphi_t|dxdt + \int_\Omega |u||(\mathcal{R}\varphi)_t|dxdt - \int_\Omega |u||\mathcal{R}\varphi|dxdt \\ &\quad + \int_{\mathbb{G}} |u_0(x)||\mathcal{R}\varphi(x,0)|dx.\end{aligned} \tag{5.362}$$

By using the ε-Young's inequality

$$ab \leq \varepsilon a^p + C(\varepsilon)b^{p'}, \;\; \frac{1}{p}+\frac{1}{p'}=1, \;\; a,b\geq 0,$$

with parameters p and $p/(p-1)$, we obtain

$$\int_\Omega |u||\varphi_t|dxdt \leq \varepsilon\int_\Omega |u|^p\varphi dxdt + c_\varepsilon\int_\Omega \varphi^{\frac{-1}{p-1}}|\varphi_t|^{\frac{p}{p-1}}dxdt, \tag{5.363}$$

for some positive constant c_ε.

Similarly, we have

$$\int_\Omega |u||(\mathcal{R}\varphi)_t|dxdt \leq \varepsilon\int_\Omega |u|^p\varphi dxdt + c_\varepsilon\int_\Omega \varphi^{\frac{-1}{p-1}}|(\mathcal{R}\varphi)_t|^{\frac{p}{p-1}}dxdt, \tag{5.364}$$

and

$$\int_\Omega |u||\mathcal{R}\varphi|dxdt \leq \varepsilon\int_\Omega |u|^p\varphi dtdt + c_\varepsilon\int_\Omega \varphi^{\frac{-1}{p-1}}|\mathcal{R}\varphi|^{\frac{p}{p-1}}dtdt. \tag{5.365}$$

By using (5.362)-(5.365), for $\varepsilon > 0$ small enough, we have

$$\begin{aligned}\int_\Omega |u|^p\varphi dxdt &+ \int_\Omega u_0(x)\varphi(x,0)dx + \int_\Omega f\varphi dxdt \\ &\leq C\Big(A_p(\varphi)+B_p(\varphi)+C_p(\varphi)+\int_{\mathbb{G}} |u_0(x)||\mathcal{R}\varphi(x,0)|dx\Big),\end{aligned} \tag{5.366}$$

where

$$A_p(\varphi) = \int_\Omega \varphi^{\frac{-1}{p-1}}|\varphi_t|^{\frac{p}{p-1}}dxdt, \tag{5.367}$$

$$B_p(\varphi) = \int_\Omega \varphi^{\frac{-1}{p-1}}|(\mathcal{R}\varphi)_t|^{\frac{p}{p-1}}dxdt, \tag{5.368}$$

$$C_p(\varphi) = \int_\Omega \varphi^{\frac{-1}{p-1}}|\mathcal{R}\varphi|^{\frac{p}{p-1}}dxdt. \tag{5.369}$$

Let $\Phi_1, \Phi_2 : \mathbb{R}_+ \to [0,1]$ be smooth nonincreasing functions such that

$$\Phi_i(\rho) := \begin{cases} 1, & \text{if } 0\leq\rho\leq 1, \\ 0, & \text{if } \rho\geq 2,\end{cases} \tag{5.370}$$

for $i = 1, 2$.

Now, for $R > 0$, let us consider the test function

$$\varphi_R(x,t) = \Phi_1\left(\frac{|x|}{R}\right)\Phi_2\left(\frac{t}{R^\alpha}\right),$$

for some $\alpha > 0$ to be defined later.

We observe that *supp* φ_R is a subset of

$$\Omega_R = \{(x,t) \in \Omega : 0 \leq t \leq 2R^{\alpha},\ 0 \leq |x| \leq 2R\},$$

while *supp* $\partial_t \varphi_R$, *supp* $\mathcal{R}\varphi_R$ and *supp* $\partial_t \mathcal{R}\varphi_R$ are subsets of

$$\Theta_R = \{(x,t) \in \Omega : R^{\alpha} \leq t \leq 2R^{\alpha},\ R \leq |x| \leq 2R\},$$

also, we put

$$\Gamma_R = \{(x,t) \in \Omega : 0 \leq t \leq R^{\alpha},\ 0 \leq |x| \leq R\}.$$

It follows that there is a positive constant $C > 0$, independent of R, such that for all $(x,t) \in \Omega_R$, we have

$$|\mathcal{R}_x \varphi_R(t,x)| \leq CR^{-k}\chi(t,x), \tag{5.371}$$

where $\chi(t,x)$ is a nonnegative function with a compact support in Ω_R, and

$$|\partial_t \mathcal{R}\varphi_R(t,x)| \leq CR^{-k-\alpha}\xi(t,x), \tag{5.372}$$

where $\xi(t,x)$ is a nonnegative function with a compact support in Ω_R.

Using (5.371) and (5.372), we get

$$A_p(\varphi) \leq CR^{\frac{-\alpha p}{p-1}}, \tag{5.373}$$

$$B_p(\varphi_R) \leq CR^{\frac{-(k+\alpha)p}{p-1}}, \tag{5.374}$$

$$C_p(\varphi_R) \leq CR^{\frac{-kp}{p-1}}. \tag{5.375}$$

Let us consider now the change of variables

$$\tilde{t} = R^{-\alpha}t, \quad \tilde{x} = R^{-1}x.$$

Put $\Sigma_R = \{x \in \mathbb{G} : R \leq |x| \leq 2R\}$.

By combining Proposition 2.4, (5.373), (5.374) and (5.375) in (5.366) we get

$$\begin{aligned} &\int_{\Omega} |u|^p \varphi_R dxdt + \int_{\Omega} u_0(x)\varphi_R(x,0)dxdt + \int_{\Omega} f\varphi_R dxdt \\ &\leq C\Big(R^{\lambda_1} + R^{\lambda_2} + R^{\lambda_3} + \int_{\Sigma_R} |u_0(v)||\mathcal{R}\varphi_R(0,v)|dv\Big), \end{aligned} \tag{5.376}$$

where

$$\lambda_1 = Q + \alpha - \frac{\alpha p}{p-1}$$

and

$$\lambda_2 = Q + \alpha - \frac{(k+\alpha)p}{p-1}$$

and

$$\lambda_3 = Q + \alpha - \frac{kp}{p-1}.$$

On the other hand, we have

$$\lim_{R\to\infty}\inf\Big(\int_{\Omega}|u|^p\varphi_R dxdt+\int_{\mathbb{G}}u_0(x)\varphi_R(x,0)dx+\int_{\Omega}f\varphi_R dxdt\Big)$$
$$\geq \lim_{R\to\infty}\inf\int_{\Omega}|u|^p\varphi_R dxdt+\lim_{R\to\infty}\inf\int_{\Omega}u_0(x)\varphi_R(x,0)dx+\lim_{R\to\infty}\inf\int_{\Omega}f\varphi_R dxdt.$$

From the monotone convergence theorem, we get

$$\lim_{R\to\infty}\inf\int_{\Omega}|u|^p\varphi_R dxdt=\int_{\Omega}|u|^p dxdt.$$

Since $u_0\in L^1(\Omega)$, by the dominated convergence theorem, we have

$$\lim_{R\to\infty}\inf\int_{\mathbb{G}}u_0(x)\varphi_R(x,0)dx=\int_{\mathbb{G}}u_0(x)dx.$$

By denoting $f=f^+-f^-$, where $f^+=\max\{f,0\}$, we have

$$\begin{aligned}\int_{\Omega}f\varphi_R dxdt&=\int_{\Gamma_R}f dxdt+\int_{\Theta_R}f^+\varphi_R dxdt-\int_{\Theta_R}f^-\varphi_R dxdt\\&\geq\int_{\Gamma_R}f dxdt-\int_{\Theta_R}f^-\varphi_R dxdt.\end{aligned}$$

Since $f^-\in L^1(\Omega)$, by the dominated convergence theorem we have

$$\lim_{R\to\infty}\int_{\Theta_R}f^-\varphi_R dxdt=0$$

Then

$$\lim_{R\to\infty}\inf\int_{\Omega}f\varphi_R dxdt\geq\lim_{R\to\infty}\inf\int_{\Gamma_R}f dxdt.$$

Then, we get

$$\lim_{R\to\infty}\inf\Big(\int_{\Omega}|u|^p\varphi_R dxdt+\int_{\Omega}u_0(x)\varphi_R(x,0)dx+\int_{\Omega}f\varphi_R dxdt\Big)$$
$$\geq\int_{\Omega}|u|^p dxdt+\ell.$$

where from (5.361), we establish

$$\ell=\int_{\Omega}u_0(x)dx+\lim_{R\to\infty}\inf\int_{\Gamma_R}f dxdt>0.$$

By the definition of the limit inferior, for every $\varepsilon>0$, there exists $R_0>0$ such that

$$\int_{\Omega}|u|^p\varphi_R dxdt+\int_{\Omega}u_0(x)\varphi_R(x,0)dx+\int_{\Omega}f\varphi_R dxdt$$

$$> \lim_{R\to\infty} \inf\Big(\int_\Omega |u|^p\varphi_R dxdt + \int_\Omega u_0(x)\varphi_R(x,0)dx + \int_\Omega f\varphi_R dxdt\Big) - \varepsilon$$
$$\geq \int_\Omega |u|^p dxdt + \ell - \varepsilon,$$

for every $R \geq R_0$. Taking $\varepsilon = \ell/2$, we get

$$\int_\Omega |u|^p\varphi_R dxdt + \int_\Omega u_0(x)\varphi_R(x,0)dx + \int_\Omega f\varphi_R dxdt$$
$$\geq \int_\Omega |u|^p dxdt + \frac{\ell}{2},$$

for every $R \geq R_0$. From (5.376), we have

$$\int_\Omega |u|^p dxdt + \frac{\ell}{2} \leq C\Big(R^{\lambda_1} + R^{\lambda_2} + R^{\lambda_3} + \int_{\Sigma_R} |u_0(x)||\mathcal{R}_x\varphi_R(x,0)|dx\Big), \tag{5.377}$$

for R large enough.

Now, we take $\alpha = k$ and require that $\lambda = \max\{\lambda_1, \lambda_2, \lambda_3\} \leq 0$, which is equivalent to $1 < p \leq 1 + \frac{k}{Q}$. Let us split the proof to two cases.

- Case 1. If $1 < p < 1 + \frac{k}{Q}$.

By, letting $R \to \infty$ in (5.377) and using the dominated convergence theorem, we obtain

$$\int_\Omega |u|^p dxdt + \frac{\ell}{2} \leq 0,$$

which is a contradiction with $\ell > 0$.

- Case 2. If $p = 1 + \frac{k}{Q}$.

From (5.377), we obtain

$$\int_\Omega |u|^p dxdt \leq C < \infty. \tag{5.378}$$

By using the Hölder inequality with parameters p and $p/(p-1)$ and from (5.362), we get

$$\int_\Omega |u|^p dxdt + \frac{\ell}{2} \leq C\Big(\int_{\Theta_R} |u|^p\varphi_R dxdt\Big)^{\frac{1}{p}}.$$

By letting $R \to \infty$ in the above inequality and using (5.378), we have

$$\int_\Omega |u|^p dxdt + \frac{\ell}{2} = 0$$

This contradiction completes the proof of the theorem. $\square$

6. APPENDIX

In this appendix, we deal with new inequalities related to the fractional order differential operators. Especially, the Caputo derivative analogues of the above inequalities are in the field of our interest. Here, we derive generalisations of classical Sobolev, Hardy, Gagliardo-Nirenberg and Caffarelli-Kohn-Nirenberg inequalities. Note that in this direction systematic studies of different functional inequalities on general homogeneous (Lie) groups were initiated by the book [4]. Also, we obtain these inequalities for Hadamard fractional derivative.

One of Lyapunov's classical result (see [67]), Lyapunov established that, if $q \in C\left([a,b];\mathbb{R}\right)$, for the boundary value problem

$$\begin{cases} u''(t) + q(t)u(t) = 0, \; x \in (a,b), \\ u(a) = u(b) = 0, \end{cases} \tag{6.1}$$

has a nontrivial classical solution, then we have

$$\int\limits_a^b |q(s)|\, ds > \frac{4}{b-a}. \tag{6.2}$$

In [115], Hartman and Wintner generalised Lyapunov's inequality, that is, if (6.1) has a nontrivial solution, then

$$\int\limits_a^b (b-s)(s-a)q^+(s)ds > b-a, \tag{6.3}$$

where $q^+(s) = \max\{q(s), 0\}$. Generalisation of the Lyapunov's inequality (6.2) can be obtained from (6.3) using the fact that $\max\limits_{a\leq s\leq b}(b-s)(s-a) = \frac{(b-a)^2}{4}$. Recently, some Hartman-Wintner-type inequalities were obtained for different fractional boundary value problems [116, 117].

In the [118], De La Vallée Poussin showed the following result:

Theorem 6.1. *Suppose that $u \in C^2([a,b])$ is a nontrivial solution to*

$$\begin{cases} -u''(x) - g(x)u'(x) = f(x)u(x), \; x \in (a,b), \\ u(a) = 0, \; u(b) = 0, \end{cases} \tag{6.4}$$

for $f, g \in C([a,b])$. Then

$$1 < M_1(b-a) + M_2\frac{(b-a)^2}{2}, \tag{6.5}$$

where $M_1 = \max\limits_{x\in[a,b]} |g(x)|$ and $M_2 = \max\limits_{x\in[a,b]} |f(x)|$.

As example, generalisation of the inequality (6.5) can be found in[119, 120]. Also, generalisation of above inequalities to the multidimensional case were obtained in the works [121, 122]. Motivated by the above cited works, using the approach introduced in [121, 122], some generalisations of above mentioned inequalities are established for fractional partial differential equations with Dirichlet conditions. Our results are natural generalizations of results in [122, 121]. In this dissertation, we established these inequalities for fractional order derivatives.

Let us recall the Riemann–Liouville fractional integrals and derivatives. Also, we give definitions of the Caputo fractional derivatives. In ([123], p. 394) the sequential differentiation was formulated in a way that we will use in the further investigations. We refer to [124, 123] and references therein for further properties.

Definition 6.2. The left Riemann–Liouville fractional integral I_{a+}^{α} of order $\alpha > 0$, and right Riemann–Liouville I_{b-}^{α} of order $0 < \alpha \leq 1$ are given by

$$I_{a+}^{\alpha}\left[f\right](t) = \frac{1}{\Gamma\left(\alpha\right)} \int\limits_{a}^{t} \left(t-s\right)^{\alpha-1} f\left(s\right) ds, \ \ t \in (a,b],$$

and

$$I_{b-}^{\alpha}\left[f\right](t) = \frac{1}{\Gamma\left(\alpha\right)} \int\limits_{t}^{b} \left(s-t\right)^{\alpha-1} f\left(s\right) ds, \ \ t \in [a,b),$$

respectively. Here Γ denotes the Euler gamma function.

The left Riemann–Liouville fractional derivative D_{a+}^{α} of order $\alpha \in \mathbb{R}$ $(0 < \alpha < 1)$ of a continuous function f on $[a,b]$ is defined by

$$D_{a+}^{\alpha}\left[f\right](t) = \frac{d}{dt} I_{a+}^{1-\alpha}\left[f\right](t), \ \ \text{for any} \ \ t \in (a,b].$$

Similarly, the right Riemann–Liouville fractional derivative D_{b-}^{α} of order $\alpha \in \mathbb{R}$ $(0 < \alpha < 1)$ of a continuous function f on $[a,b]$ is given by

$$D_{b-}^{\alpha}\left[f\right](t) = -\frac{d}{dt} I_{b-}^{1-\alpha}\left[f\right](t), \ \ \text{for any} \ \ t \in [a,b).$$

and

$$D_{a+}^{\alpha}\left[f\right](t) = \frac{d}{dt} I_{a+}^{1-\alpha}\left[f\right](t), \ \ t \in (a,b],$$

respectively and $f \in AC[a,b]$. Here Γ denotes the Euler gamma function.

Since $I^{\alpha} f(t) \to f(t)$ almost everywhere as $\alpha \to 0$, then by definition we suppose that $I^{0} f(t) = f(t)$. Hence $D_{a+}^{1} f(t) = f'(t)$.

Definition 6.3. The left and right Caputo fractional derivatives of order $\alpha \in \mathbb{R}$ $(0 < \alpha < 1)$ of a differentiable function f on $[a,b]$ are defined by

$$\mathcal{D}_{a+}^{\alpha}\left[f\right](t) = D_{a+}^{\alpha}\left[f\left(t\right) - f\left(a\right)\right], \ \ t \in (a,b],$$

and

$$\mathcal{D}_{b-}^{\alpha}\left[f\right](t) = D_{b-}^{\alpha}\left[f\left(t\right) - f\left(b\right)\right], \ \ t \in [a,b),$$

respectively.

Remark 6.4. *From Definition 6.3, if* $f(a) = 0$*, then* $\mathcal{D}_{a+}^{\alpha} = D_{a+}^{\alpha}$.

Proposition 6.5. *If* $f \in L^{1}([a,b])$ *and* $\alpha > 0$, $\beta > 0$, *then the following equality holds*

$$I_{a+}^{\alpha} I_{a+}^{\beta} f(t) = I_{a+}^{\alpha+\beta} f(t).$$

Proposition 6.6 ([123])**.** *If* $f \in L^{1}([a,b])$ *and* $f' \in L^{1}([a,b])$, *then the equality*

$$I_{a}^{\alpha} \mathcal{D}_{a+}^{\alpha} f(t) = f(t) - f(a), \ 0 < \alpha \leq 1,$$

holds almost everywhere on $[a,b]$.

Let us give some defintion of the Hadamard fractional derivative.

Definition 6.7. The left Hadamard fractional integrals $\mathfrak{J}^{\alpha}_{a+}$ of order $\alpha > 0$ and derivatives $\mathfrak{D}^{\alpha}_{a+}$ of order $0 < \alpha < 1$ are given by

$$\mathfrak{J}^{\alpha}_{a+}\left[f\right](t) = \frac{1}{\Gamma(\alpha)}\int_a^t \left(\log\frac{t}{s}\right)^{\alpha-1} f(s)\,\frac{ds}{s},\;\; t \in (a,b],$$

and

$$\mathfrak{D}^{\alpha}_{a+}\left[f\right](t) = \frac{1}{\Gamma(1-\alpha)}\int_a^t \left(\log\frac{t}{s}\right)^{-\alpha} f'(s)\,\frac{ds}{s},\;\; t \in (a,b].$$

Here Γ denotes the Euler gamma function.

Proposition 6.8 ([123])**.** *If $f \in L^1(a,b)$ and $f' \in L^1_{\frac{1}{x}}(a,b)$, then the equality*

$$\mathfrak{J}^{\alpha}_{a}\mathfrak{D}^{\alpha}_{a+}f(t) = f(t) - f(a),\; 0 < \alpha < 1,$$

holds almost everywhere on $[a,b]$.

Let us define weighted Lebesgue space with the norm:

$$\|u\|_{L^p_{\frac{1}{x}}(a,b)} := \left(\int_a^b |u(x)|^p \frac{dx}{x}\right)^{\frac{1}{p}}. \tag{6.6}$$

Proposition 6.9 ([123])**.** *If $f \in L^1_{\frac{1}{x}}(a,b)$ and $\alpha > 0$, $\beta > 0$, then the following equality holds*

$$\mathfrak{J}^{\alpha}_{a+}\mathfrak{J}^{\beta}_{a+}f(t) = \mathfrak{J}^{\alpha+\beta}_{a+}f(t).$$

Let us define fractional and fractional p-Laplacian on $\mathbb{R}^N$:

Definition 6.10. The fractional Laplacian operator of order $s \in (0,1)$ of a function $u \in C_0^\infty\left(\mathbb{R}^N\right)$ is defined by

$$(-\Delta)^s u(x) = 2\lim_{\delta \searrow 0}\int_{\mathbb{R}^N\setminus B(x,\delta)} \frac{u(x)-u(y)}{|x-y|^{N+2s}}dy,\;\; x \in \mathbb{R}^N, \tag{6.7}$$

where $B(x,\delta)$ is a ball at centered at $x \in \mathbb{R}^N$ with radius δ.

Definition 6.11. The fractional p-Laplacian operator of order $s \in (0,1)$ and $1 < p < \infty$ of a function $u \in C_0^\infty\left(\mathbb{R}^N\right)$ is defined by

$$(-\Delta)^s_p u(x) = 2\lim_{\delta \searrow 0}\int_{\mathbb{R}^N\setminus B(x,\delta)} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+sp}}dy,\;\; x \in \mathbb{R}^N. \tag{6.8}$$

In this appendix, we derive one-dimensional fractional functional inequalities.

6.1. Poincaré–Sobolev type inequality for the Caputo fractional derivative.

In this subsection, we show fractional order Poincaré–Sobolev type inequality.

Theorem 6.12. *Let $u \in L^p(a,b)$, $u(a) = 0$, $\mathcal{D}^{\alpha}_{a+}u \in L^p(a,b)$ and $p > 1$. Then for the $\mathcal{D}^{\alpha}_{a+}$ Caputo fractional derivative of order $\alpha \in \left(\frac{1}{p}, 1\right]$ we have the inequality*

$$\|u\|_{L^\infty(a,b)} \le \frac{(b-a)^{\alpha-\frac{1}{p}}}{\left(\frac{\alpha p}{p-1} - \frac{1}{p-1}\right)^{\frac{p-1}{p}}\Gamma(\alpha)}\left\|\mathcal{D}^{\alpha}_{a+}u\right\|_{L^p(a,b)}. \tag{6.9}$$

Proof. Let $u \in L^p(a,b)$, $u(a)=0$, $\mathcal{D}^{\alpha}_{a+}u \in L^p(a,b)$ and consider the function

$$u(t) = I^{\alpha}_{a+}\mathcal{D}^{\alpha}_{a+}u(t). \tag{6.10}$$

Using the Hölder inequality with $\frac{1}{p}+\frac{1}{q}=1$, we obtain

$$\begin{aligned}
\left|I^{\alpha}_{a+}\mathcal{D}^{\alpha}_{a+}u(t)\right| &\le \frac{1}{\Gamma(\alpha)}\int_a^t \left|(t-s)^{\alpha-1}\mathcal{D}^{\alpha}_{a+}u(s)\right| ds \\
&\le \frac{1}{\Gamma(\alpha)}\left(\int_a^t (t-s)^{\alpha q-q} ds\right)^{\frac{1}{q}}\left(\int_a^t \left|\mathcal{D}^{\alpha}_{a+}u(s)\right|^p ds\right)^{\frac{1}{p}} \\
&\overset{\alpha>\frac{1}{p}}{=} \frac{(t-a)^{\alpha-1+\frac{1}{q}}}{(\alpha q-q+1)^{\frac{1}{q}}\,\Gamma(\alpha)}\left(\int_a^t \left|\mathcal{D}^{\alpha}_{a+}u(s)\right|^p ds\right)^{\frac{1}{p}} \\
&\le \frac{(b-a)^{\alpha-1+\frac{1}{q}}}{(\alpha q-q+1)^{\frac{1}{q}}\,\Gamma(\alpha)}\left\|\mathcal{D}^{\alpha}_{a+}u\right\|_{L^p(a,b)} \\
&= \frac{(b-a)^{\alpha-\frac{1}{p}}}{(\alpha q-q+1)^{\frac{1}{q}}\,\Gamma(\alpha)}\left\|\mathcal{D}^{\alpha}_{a+}u\right\|_{L^p(a,b)} \\
&= \frac{(b-a)^{\alpha-\frac{1}{p}}}{\left(\frac{\alpha p}{p-1}-\frac{1}{p-1}\right)^{\frac{p-1}{p}}\Gamma(\alpha)}\left\|\mathcal{D}^{\alpha}_{a+}u\right\|_{L^p(a,b)},
\end{aligned}$$

where $q=\frac{p}{p-1}>1$.

Then,

$$\|u\|_{L^\infty(a,b)} = \|I^{\alpha}_{a+}\mathcal{D}^{\alpha}_{a+}u\|_{L^\infty(a,b)} \le \frac{(b-a)^{\alpha-\frac{1}{p}}}{\left(\frac{\alpha p}{p-1}-\frac{1}{p-1}\right)^{\frac{p-1}{p}}\Gamma(\alpha)}\left\|\mathcal{D}^{\alpha}_{a+}u\right\|_{L^p(a,b)}, \tag{6.11}$$

showing (6.9). □

Remark 6.13. *In Theorem 6.12, by taking $1<q<\infty$, we obtain*

$$\|u\|_{L^q(a,b)} \le \frac{(b-a)^{\alpha-\frac{1}{p}+\frac{1}{q}}}{\left(\frac{\alpha p}{p-1}-\frac{1}{p-1}\right)^{\frac{p-1}{p}}\Gamma(\alpha)}\left\|\mathcal{D}^{\alpha}_{a+}u\right\|_{L^p(a,b)}. \tag{6.12}$$

Theorem 6.14. *Also, Theorem 6.12 holds for the Riemann-Liouville derivative.*

Proof. By assumption of Theorem 6.12, we have $u(a)=0$ and by using Remark 6.4, we have $\mathcal{D}^{\alpha}_{a+} = D^{\alpha}_{a+}$. □

Let us also present the following result.

Theorem 6.15. *Let* $\mathcal{D}^{\alpha}_{a+}u \in L^p(a,b)$ *with* $p > 1$ *and let* $\beta \in [0,1)$ *be such that* $\alpha \in \left(\beta + \frac{1}{p}, 1\right]$. *Then for the Caputo fractional derivative* $\mathcal{D}^{\beta}_{a+}$, *we have*

$$\|\mathcal{D}^{\beta}_{a+}u\|_{L^\infty(a,b)} \leq \frac{(b-a)^{\alpha-\beta-\frac{1}{p}+\frac{1}{q}}}{(\alpha q - \beta q - q + 1)^{\frac{1}{q}}\,\Gamma(\alpha-\beta)} \left\|\mathcal{D}^{\alpha}_{a+}u\right\|_{L^p(a,b)}, \tag{6.13}$$

for all $1 < p \leq q < \infty$, *where* $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. By using Definition 6.3 and Properties 6.5, 6.6 we introduce the function

$$\mathcal{D}^{\beta}_{a+}u(t) = I^{1-\beta}_{a+}u'(t) = I^{\alpha-\beta}_{a+}I^{1-\alpha}_{a+}u'(t) = I^{\alpha-\beta}_{a+}\mathcal{D}^{\alpha}_{a+}u(t). \tag{6.14}$$

Using the Hölder inequality with $\frac{1}{p} + \frac{1}{q} = 1$, we get

$$\begin{aligned}
\left|I^{\alpha-\beta}_{a+}\mathcal{D}^{\alpha}_{a+}u(t)\right| &\leq \frac{1}{\Gamma(\alpha-\beta)} \int_a^t \left|(t-s)^{\alpha-\beta-1}\mathcal{D}^{\alpha}_{a+}u(s)\right| ds \\
&\leq \frac{1}{\Gamma(\alpha-\beta)} \left(\int_a^t (t-s)^{\alpha q-\beta q-q} ds\right)^{\frac{1}{q}} \left(\int_a^t \left|\mathcal{D}^{\alpha}_{a+}u(s)\right|^p ds\right)^{\frac{1}{p}} \\
&= \frac{(t-a)^{\alpha-\beta-1+\frac{1}{q}}}{(\alpha q - \beta q - q + 1)^{\frac{1}{q}}\,\Gamma(\alpha-\beta)} \left(\int_a^t \left|\mathcal{D}^{\alpha}_{a+}u(s)\right|^p ds\right)^{\frac{1}{p}} \\
&= \frac{(t-a)^{\alpha-\beta-\frac{1}{p}}}{(\alpha q - \beta q - q + 1)^{\frac{1}{q}}\,\Gamma(\alpha-\beta)} \left(\int_a^t \left|\mathcal{D}^{\alpha}_{a+}u(s)\right|^p ds\right)^{\frac{1}{p}} \\
&\leq \frac{(b-a)^{\alpha-\beta-\frac{1}{p}}}{(\alpha q - \beta q - q + 1)^{\frac{1}{q}}\,\Gamma(\alpha-\beta)} \left\|\mathcal{D}^{\alpha}_{a+}u\right\|_{L^p(a,b)},
\end{aligned}$$

where by assumption $\alpha > \beta + \frac{1}{p}$, we have $\alpha q - \beta q - q + 1 > 0$. From this, we obtain

$$\|\mathcal{D}^{\beta}_{a+}u\|_{L^\infty(a,b)} \leq \frac{(b-a)^{\alpha-\beta-\frac{1}{p}}}{(\alpha q - \beta q - q + 1)^{\frac{1}{q}}\,\Gamma(\alpha-\beta)} \left\|\mathcal{D}^{\alpha}_{a+}u\right\|_{L^p(a,b)}, \tag{6.15}$$

showing (6.13). ☐

Remark 6.16. *In (6.13), if* $\beta = 0$, *we obtain Sobolev type inequality.*

Remark 6.17. *In Theorem 6.15, by taking* $1 < q < \infty$, *we get*

$$\|\mathcal{D}^{\beta}_{a+}u\|_{L^q(a,b)} \leq \frac{(b-a)^{\alpha-\beta-\frac{1}{p}+\frac{1}{q}}}{(\alpha q - \beta q - q + 1)^{\frac{1}{q}}\,\Gamma(\alpha-\beta)} \left\|\mathcal{D}^{\alpha}_{a+}u\right\|_{L^p(a,b)}. \tag{6.16}$$

6.2. **Hardy type inequality for the Caputo fractional derivative.** Let us show Hardy inequality.

Theorem 6.18. *Let* $a>0,\ u(a)=0$ *and* $\mathcal{D}^{\alpha}_{a+}u\in L^p(a,b)$ *with* $p>1$. *Then for the* $\mathcal{D}^{\alpha}_{a+}$ *Caputo fractional derivative of order* $\alpha\in\left(\frac{1}{p},1\right]$ *we have the inequality*

$$\left\|\frac{u}{x}\right\|_{L^p(a,b)}\leq\frac{a^{-1}(b-a)^{\alpha}}{\left(\frac{\alpha p}{p-1}-\frac{1}{p-1}\right)^{\frac{p-1}{p}}\Gamma(\alpha)}\left\|\mathcal{D}^{\alpha}_{a+}u\right\|_{L^p(a,b)}. \tag{6.17}$$

Proof. From $a<x<b$ we have $\frac{1}{b}<\frac{1}{x}<\frac{1}{a}$. By using Theorem 6.12, we calculate

$$\begin{aligned}\left(\int_a^b\frac{|u(x)|^p}{x^p}dx\right)^{\frac{1}{p}}&=\left(\int_a^b x^{-p}|u(x)|^p dx\right)^{\frac{1}{p}}\\&\leq a^{-1}\|u\|_{L^p(a,b)}\\&\overset{(6.9)}{\leq}\frac{a^{-1}(b-a)^{\alpha}}{\left(\frac{\alpha p}{p-1}-\frac{1}{p-1}\right)^{\frac{p-1}{p}}\Gamma(\alpha)}\left\|\mathcal{D}^{\alpha}_{a+}u\right\|_{L^p(a,b)},\end{aligned} \tag{6.18}$$

showing (6.17). ☐

Theorem 6.19. *Also, Theorem 6.18 holds for the Riemann-Liouville derivative.*

Proof. The proof is similar with Theorem 6.14. ☐

Let us give the weighted one-dimensional Hardy type inequality.

Theorem 6.20. *Let* $a>0$, $u\in L^p(a,b)$, $u(a)=0$ *and* $\mathcal{D}^{\alpha}_{a+}u\in L^p(a,b)$ *with* $p>1$. *Then for the* $\mathcal{D}^{\alpha}_{a+}$ *Caputo fractional derivative of order* $\alpha\in\left(\frac{1}{p},1\right]$ *and* $\gamma\in\mathbb{R}$, *we have*

$$\left\|\frac{u}{x^{\gamma+1}}\right\|_{L^p(a,b)}\leq\frac{a^{-|\gamma|-1}b^{|\gamma|}(b-a)^{\alpha}}{(\alpha q-q+1)^{\frac{1}{q}}\Gamma(\alpha)}\left\|\frac{\mathcal{D}^{\alpha}_{a+}u}{x^{\gamma}}\right\|_{L^p(a,b)}. \tag{6.19}$$

Proof. Let us divide the proof in two cases $\gamma\geq0$ and $\gamma<0$. Firstly, let us prove the case $\gamma\geq0$. From $a>0$, we have $b^{-\gamma-1}<x^{-\gamma-1}<a^{-\gamma-1}$, so that

$$\begin{aligned}\left(\int_a^b\frac{|u(x)|^p}{x^{(\gamma+1)p}}dx\right)^{\frac{1}{p}}&\leq a^{-\gamma-1}\left(\int_a^b|u(x)|^p dx\right)^{\frac{1}{p}}\\&\overset{(6.9)}{\leq}\frac{a^{-\gamma-1}(b-a)^{\alpha}}{\left(\frac{\alpha p}{p-1}-\frac{1}{p-1}\right)^{\frac{p-1}{p}}\Gamma(\alpha)}\left(\int_a^b|\mathcal{D}^{\alpha}_{a+}u|^p dx\right)^{\frac{1}{p}}\\&=\frac{a^{-\gamma-1}(b-a)^{\alpha}}{\left(\frac{\alpha p}{p-1}-\frac{1}{p-1}\right)^{\frac{p-1}{p}}\Gamma(\alpha)}\left(\int_a^b\frac{x^{\gamma p}}{x^{\gamma p}}|\mathcal{D}^{\alpha}_{a+}u|^p dx\right)^{\frac{1}{p}}\\&\leq\frac{a^{-\gamma-1}b^{\gamma}(b-a)^{\alpha}}{\left(\frac{\alpha p}{p-1}-\frac{1}{p-1}\right)^{\frac{p-1}{p}}\Gamma(\alpha)}\left(\int_a^b\frac{|\mathcal{D}^{\alpha}_{a+}u|^p}{x^{\gamma p}}dx\right)^{\frac{1}{p}}\\&=\frac{a^{-\gamma-1}b^{\gamma}(b-a)^{\alpha}}{(\alpha q-q+1)^{\frac{1}{q}}\Gamma(\alpha)}\left\|\frac{\mathcal{D}^{\alpha}_{a+}u}{x^{\gamma}}\right\|_{L^p(a,b)}.\end{aligned} \tag{6.20}$$

Let us show the case $\gamma < 0$,

$$\begin{aligned}
\left(\int_a^b \frac{|u(x)|^p}{x^{(\gamma+1)p}}dx\right)^{\frac{1}{p}} &= \left(\int_a^b \frac{|u(x)|^p}{x^{(\gamma p+p)}}dx\right)^{\frac{1}{p}} \\
&\leq b^{-\gamma}\left(\int_a^b \frac{|u(x)|^p}{x^p}dx\right)^{\frac{1}{p}} \\
&\overset{(6.17)}{\leq} \frac{a^{-1}b^{-\gamma}(b-a)^{\alpha}}{\left(\frac{\alpha p}{p-1}-\frac{1}{p-1}\right)^{\frac{p-1}{p}}\Gamma(\alpha)} \left\|\mathcal{D}^{\alpha}_{a+}u\right\|_{L^p(a,b)} \\
&= \frac{a^{-1}b^{-\gamma}(b-a)^{\alpha}}{\left(\frac{\alpha p}{p-1}-\frac{1}{p-1}\right)^{\frac{p-1}{p}}\Gamma(\alpha)} \left(\int_a^b |\mathcal{D}^{\alpha}_{a+}u|^p dx\right)^{\frac{1}{p}} \\
&= \frac{a^{-1}b^{-\gamma}(b-a)^{\alpha}}{\left(\frac{\alpha p}{p-1}-\frac{1}{p-1}\right)^{\frac{p-1}{p}}\Gamma(\alpha)} \left(\int_a^b \frac{x^{\gamma p}}{x^{\gamma p}}|\mathcal{D}^{\alpha}_{a+}u|^p dx\right)^{\frac{1}{p}} \\
&\leq \frac{a^{\gamma-1}b^{-\gamma}(b-a)^{\alpha}}{\left(\frac{\alpha p}{p-1}-\frac{1}{p-1}\right)^{\frac{p-1}{p}}\Gamma(\alpha)} \left(\int_a^b \frac{|\mathcal{D}^{\alpha}_{a+}u|^p}{x^{\gamma p}} dx\right)^{\frac{1}{p}} \\
&= \frac{a^{\gamma-1}b^{-\gamma}(b-a)^{\alpha}}{\left(\frac{\alpha p}{p-1}-\frac{1}{p-1}\right)^{\frac{p-1}{p}}\Gamma(\alpha)} \left\|\frac{\mathcal{D}^{\alpha}_{a+}u}{x^{\gamma}}\right\|_{L^p(a,b)},
\end{aligned} \tag{6.21}$$

implying (6.19). ☐

Remark 6.21. *Also, Theorem 6.20 holds for the Riemann-Liouville derivative.*

6.3. **Gagliardo-Nirenberg type inequality for the Caputo fractional derivative.** In this subsection, we establish the fractional Gagliardo-Nirenberg type inequality.

Theorem 6.22. *Assume that $\alpha \in \left(\frac{1}{q}, 1\right]$, $1 \leq p, q < \infty$. Then we have the following Gagliardo-Nirenberg type inequality,*

$$\|u\|_{L^{\gamma}(a,b)} \leq C\|\mathcal{D}^{\alpha}_{a+}u\|^{s}_{L^{q}(a,b)}\|u\|^{1-s}_{L^{p}(a,b)}, \tag{6.22}$$

with

$$\frac{\gamma s}{q} + \frac{\gamma(1-s)}{p} = 1, \tag{6.23}$$

where $s \in [0, 1]$.

Proof. By using the Hölder inequality with $\frac{\gamma s}{q}+\frac{\gamma(1-s)}{p}=1$, we have

$$\begin{aligned}\int_a^b |u(x)|^\gamma dx &= \int_a^b |u(x)|^{\gamma s}|u(x)|^{\gamma(1-s)}dx \\ &\le \left(\int_a^b |u(x)|^q dx\right)^{\frac{\gamma s}{q}} \left(\int_a^b |u(x)|^p dx\right)^{\frac{\gamma(1-s)}{p}} \qquad (6.24) \\ &\overset{(6.9)}{\le} C\|\mathcal{D}^\alpha_{a+}u\|^{\gamma s}_{L^q(a,b)}\|u\|^{\gamma(1-s)}_{L^p(a,b)},\end{aligned}$$

showing (6.22). □

Remark 6.23. *Also, Theorem 6.22 holds for the Riemann-Liouville derivative.*

Let us consider the space $\dot{H}^\alpha_+(a,b)$ with $\alpha\in\left(\frac{1}{2},1\right]$ in the following form:

$$\dot{H}^\alpha_+(a,b) := \{u\in L^2(a,b),\ \mathcal{D}^\alpha_{a+}u\in L^2(a,b),\ u(a)=0\}.$$

A special case of Theorem 6.22 important for our further analysis is that of $q=2$ and $\alpha=1$, in which case we obtain a more classical Gagliardo-Nirenberg inequality:

Corollary 6.24. *We have the following Gagliardo-Nirenberg type inequality*

$$\|u\|_{L^\gamma(a,b)} \le C\|u\|^s_{\dot{H}^1_+(a,b)}\|u\|^{1-s}_{L^p(a,b)}, \qquad (6.25)$$

for $s\in[0,1]$.

We also record another more general special case of Theorem 6.22 with $q=2$:

Corollary 6.25. *Let* $\alpha\in\left(\frac{1}{2},1\right]$. *We have the following Gagliardo-Nirenberg type inequality,*

$$\|u\|_{L^\gamma(a,b)} \lesssim \|u\|^s_{\dot{H}^\alpha_+(a,b)}\|u\|^{1-s}_{L^p(a,b)}, \qquad (6.26)$$

for $\frac{1}{\gamma}=\frac{s}{2}+\frac{1-s}{p}$, *where* $s\in[0,1]$.

6.4. **Caffarelli-Kohn-Nirenberg type inequality for the Caputo fractional derivative.** In this subsection, we show the fractional Caffarelli-Kohn-Nirenberg type inequality.

Theorem 6.26. *Assume that* $a>0$, $\alpha\in\left(1-\frac{1}{q},1\right)$, $1<p,q<\infty$, $0<r<\infty$, *and* $p+q\ge r$. *Let* $\delta\in[0,1]\cap\left[\frac{r-q}{r},\frac{p}{r}\right]$ *and* $c,d,e\in\mathbb{R}$ *with the* $\frac{\delta}{p}+\frac{1-\delta}{q}=\frac{1}{r}$, $c=\delta(d-1)+e(1-\delta)$ *and* $u(a)=0$. *If* $1+(d-1)p>0$ *then we have*

$$\|x^c u\|_{L^r(a,b)} \le C\|x^d\mathcal{D}^\alpha_{a+}u\|^\delta_{L^p(a,b)}\|x^e u\|^{1-\delta}_{L^q(a,b)}. \qquad (6.27)$$

Proof. Case $\delta=0$.

If $\delta=0$, then $c=e$ and $q=r$. Then (6.27) is the inequality

$$\|x^c u\|_{L^r(a,b)} \le \|x^c u\|_{L^r(a,b)}.$$

Case $\delta=1$.

If $\delta=1$, then we have $c=d-1$ and $p=r$. Also, we have $1+cp=1+(d-1)p>0$. Then by using weighted fractional Hardy inequality (Theorem 6.20), we obtain

$$\begin{aligned}\|x^c u\|_{L^p(a,b)} &\le C\left\|x^{c+1}\mathcal{D}^\alpha_{a+}u\right\|_{L^p(a,b)} \\ &= C\left\|x^d\mathcal{D}^\alpha_{a+}u\right\|_{L^p(a,b)}.\end{aligned} \qquad (6.28)$$

Case $\delta \in [0,1] \cap [\frac{r-q}{r}, \frac{p}{r}]$.

By assumption $c = \delta(d-1) + e(1-\delta)$ and by using Hölder's inequality with $\frac{\delta}{p} + \frac{1-\delta}{q} = \frac{1}{r}$, we calculate

$$\begin{aligned} \|x^c u\|_{L^r(a,b)} &= \left(\int_a^b x^{cr} |u(x)|^r dx \right)^{\frac{1}{r}} \\ &= \left(\int_a^b \frac{|u(x)|^{\delta r}}{x^{\delta r(1-d)}} \frac{|u(x)|^{(1-\delta)r}}{x^{-er(1-\delta)}} dx \right)^{\frac{1}{r}} \\ &\leq \left\| \frac{u}{x^{1-d}} \right\|^{\delta}_{L^p(a,b)} \left\| \frac{u}{x^{-e}} \right\|^{1-\delta}_{L^q(a,b)}. \end{aligned} \tag{6.29}$$

By using weighted fractional Hardy inequality (Theorem 6.20) with $1 + (d-1)p > 0$, we obtain

$$\begin{aligned} \|x^c u\|_{L^r(a,b)} &\leq \left\| \frac{u}{x^{1-d}} \right\|^{\delta}_{L^p(a,b)} \left\| \frac{u}{x^{-e}} \right\|^{1-\delta}_{L^q(a,b)} \\ &\leq C \|x^d \mathcal{D}^{\alpha}_{a+} u\|^{\delta}_{L^p(a,b)} \|x^e u\|^{1-\delta}_{L^q(a,b)}, \end{aligned} \tag{6.30}$$

completing the proof. □

Remark 6.27. *Also, Theorem 6.26 holds for the Riemann-Liouville derivative.*

6.5. **Sequential Derivation Case.** In this subsection we collect results for the sequential derivatives. Indeed, it is important due to the non–commutativity and the absence of the semi–group property of fractional differential operators.

6.6. Fractional Poincaré–Sobolev type inequality for sequential fractional derivative.

Theorem 6.28. *Let* $\mathcal{D}^{\beta}_{a+} u(a) = 0$, $\mathcal{D}^{\alpha}_{a+} \mathcal{D}^{\beta}_{a+} u \in L^p(a,b)$ *with* $\alpha \in \left(\frac{1}{q}, 1\right)$ *and* $\beta \in (0,1)$. *Then the following inequality is true*

$$\|\mathcal{D}^{\beta}_{a+} u\|_{L^\infty(a,b)} \leq \frac{(b-a)^{\alpha - \frac{1}{p}}}{(\alpha q - q + 1)^{\frac{1}{q}} \Gamma(\alpha)} \left\| \mathcal{D}^{\alpha}_{a+} \mathcal{D}^{\beta}_{a+} u \right\|_{L^p(a,b)} \tag{6.31}$$

with $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. We consider the function

$$\mathcal{D}^{\beta}_{a+} u(t) = I^{\alpha}_{a+} \mathcal{D}^{\alpha}_{a+} \mathcal{D}^{\beta}_{a+} u(t). \tag{6.32}$$

Using the Hölder inequality

$$\begin{aligned}
\left|I^{\alpha}_{a+}\mathcal{D}^{\alpha}_{a+}\mathcal{D}^{\beta}_{a+}u(t)\right| &\le \frac{1}{\Gamma(\alpha)}\int\limits_a^t \left|(t-s)^{\alpha-1}\mathcal{D}^{\alpha}_{a+}\mathcal{D}^{\beta}_{a+}u(s)\right| ds \\
&\le \frac{1}{\Gamma(\alpha)}\left(\int\limits_a^t (t-s)^{\alpha q-q}ds\right)^{\frac{1}{q}}\left(\int\limits_a^t \left|\mathcal{D}^{\alpha}_{a+}\mathcal{D}^{\beta}_{a+}u(s)\right|^p ds\right)^{\frac{1}{p}} \\
&= \frac{(t-a)^{\alpha-1+\frac{1}{q}}}{(\alpha q-q+1)^{\frac{1}{q}}\,\Gamma(\alpha)}\left(\int\limits_a^t \left|\mathcal{D}^{\alpha}_{a+}\mathcal{D}^{\beta}_{a+}u(s)\right|^p ds\right)^{\frac{1}{p}} \\
&\le \frac{(b-a)^{\alpha-1+\frac{1}{q}}}{(\alpha q-q+1)^{\frac{1}{q}}\,\Gamma(\alpha)}\left\|\mathcal{D}^{\alpha}_{a+}\mathcal{D}^{\beta}_{a+}u\right\|_{L^p(a,b)}.
\end{aligned}$$

Then we have

$$\|\mathcal{D}^{\beta}_{a+}u\|_{L^\infty(a,b)} \le \frac{(b-a)^{\alpha-\frac{1}{p}}}{(\alpha q-q+1)^{\frac{1}{q}}\,\Gamma(\alpha)}\left\|\mathcal{D}^{\alpha}_{a+}\mathcal{D}^{\beta}_{a+}u\right\|_{L^p(a,b)},$$

completing proof. ☐

Remark 6.29. *In Theorem 6.28, if* $1<\theta<\infty$*, then we have*

$$\|\mathcal{D}^{\beta}_{a+}u\|_{L^\theta(a,b)} \le \frac{(b-a)^{\alpha-\frac{1}{p}+\frac{1}{\theta}}}{(\alpha q-q+1)^{\frac{1}{q}}\,\Gamma(\alpha)}\left\|\mathcal{D}^{\alpha}_{a+}\mathcal{D}^{\beta}_{a+}u\right\|_{L^p(a,b)}.$$

6.7. **Fractional Hardy type inequality for the sequential fractional derivative.**

Now we show the following sequential fractional Hardy inequality.

Theorem 6.30. *Let* $a>0$, $\gamma\in\mathbb{R}$ *and* $\mathcal{D}^{\beta}_{a+}u(a)=0$ *and* $\mathcal{D}^{\alpha}_{a+}\mathcal{D}^{\beta}_{a+}u\in L^p(a,b)$ *with* $\alpha\in\left(\frac{1}{q},1\right)$. *Then the following inequality is true*

$$\left\|\frac{\mathcal{D}^{\beta}_{a+}u}{x}\right\|_{L^p(a,b)} \le C\left\|\mathcal{D}^{\alpha}_{a+}\mathcal{D}^{\beta}_{a+}u\right\|_{L^p(a,b)} \tag{6.33}$$

with $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From $a<x<b$ we have $\frac{1}{b}<\frac{1}{x}<\frac{1}{a}$. By using Theorem 6.28, we calculate

$$\begin{aligned}
\left(\int_a^b \frac{|\mathcal{D}^{\beta}_{a+}u(x)|^p}{x^p}dx\right)^{\frac{1}{p}} &= \left(\int_a^b x^{-p}|\mathcal{D}^{\beta}_{a+}u(x)|^p dx\right)^{\frac{1}{p}} \\
&\le a^{-1}\|\mathcal{D}^{\beta}_{a+}u\|_{L^p(a,b)} \\
&\overset{(6.31)}{\le} \frac{a^{-1}(b-a)^{\alpha}}{(\alpha q-q+1)^{\frac{1}{q}}\,\Gamma(\alpha)}\left\|\mathcal{D}^{\alpha}_{a+}\mathcal{D}^{\beta}_{a+}u\right\|_{L^p(a,b)},
\end{aligned} \tag{6.34}$$

showing (6.33). ☐

6.8. **Fractional Gagliardo-Nirenberg type inequality for the sequential fractional derivative.** In the same way as Theorem 6.22 is proved, we can prove the following statement.

Theorem 6.31. *Assume that* $1 \leq p, q < \infty$, *and let* $\alpha \in (0,1)$ *be such that* $\beta \in \left(\frac{1}{q}, 1\right)$. *Suppose that* $\mathcal{D}^{\alpha}_{a+}\mathcal{D}^{\beta}_{a+}u \in L^q(a,b)$ *and* $\mathcal{D}^{\alpha}_{a}u \in L^p(a,b)$. *Then we have the following Gagliardo-Nirenberg type inequality,*

$$\int_a^b |\mathcal{D}^{\alpha}_{a+}u(x)|^{\gamma}dx \lesssim \left(\int_a^b |\mathcal{D}^{\beta}_{a+}\mathcal{D}^{\alpha}_{a+}u(x)|^q dx\right)^{\frac{s\gamma}{q}} \left(\int_a^b |\mathcal{D}^{\alpha}_{a+}u(x)|^p dx\right)^{\frac{(1-s)\gamma}{p}}, \tag{6.35}$$

with

$$\frac{s\gamma}{q} + \frac{(1-s)\gamma}{p} = 1, \tag{6.36}$$

where $s \in [0,1]$.

Proof. Let us calculate the following integral:

$$\begin{aligned}\int_a^b |\mathcal{D}^{\alpha}_{a+}u(x)|^{\gamma}dx &= \int_a^b |\mathcal{D}^{\alpha}_{a+}u(x)|^{s\gamma}|\mathcal{D}^{\alpha}_{a+}u(x)|^{(1-s)\gamma}dx \\ &\leq \left(\int_a^b |\mathcal{D}^{\alpha}_{a+}u(x)|^q dx\right)^{\frac{s\gamma}{q}} \left(\int_a^b |\mathcal{D}^{\alpha}_{a+}u(x)|^p dx\right)^{\frac{(1-s)\gamma}{p}},\end{aligned} \tag{6.37}$$

with

$$\frac{s\gamma}{q} + \frac{(1-s)\gamma}{p} = 1. \tag{6.38}$$

Then by using Theorem 6.28, we obtain

$$\begin{aligned}\int_a^b |\mathcal{D}^{\alpha}_{a+}u(x)|^{\gamma}dx &\leq \left(\int_a^b |\mathcal{D}^{\alpha}_{a+}u(x)|^q dx\right)^{\frac{s\gamma}{q}} \left(\int_a^b |\mathcal{D}^{\alpha}_{a+}u(x)|^p dx\right)^{\frac{(1-s)\gamma}{p}} \\ &\overset{(6.33)}{\leq} C\left(\int_a^b |\mathcal{D}^{\beta}_{a+}\mathcal{D}^{\alpha}_{a+}u(x)|^q dx\right)^{\frac{s\gamma}{q}} \left(\int_a^b |\mathcal{D}^{\alpha}_{a+}u(x)|^p dx\right)^{\frac{(1-s)\gamma}{p}},\end{aligned} \tag{6.39}$$

completing proof. ☐

6.9. **Poincaré–Sobolev type inequality for the Hadamard fractional derivative.** In this subsection, we show fractional order Poincaré–Sobolev type inequality.

Theorem 6.32. *Let* $a > 0$, $u \in L^p(a,b)$, $u(a) = 0$, $\mathfrak{D}^{\alpha}_{a+}u \in L^p_{\frac{1}{x}}(a,b)$ *and* $p > 1$. *Then for the* $\mathfrak{D}^{\alpha}_{a+}$ *Hadamard fractional derivative of order* $\alpha \in \left(\frac{1}{p}, 1\right]$ *we have the inequality*

$$\|u\|_{L^{\infty}(a,b)} \leq \frac{\left|\log\frac{b}{a}\right|^{\alpha-\frac{1}{p}}}{\left(\frac{\alpha p}{p-1} - \frac{1}{p-1}\right)^{\frac{p-1}{p}}\Gamma(\alpha)} \left\|\mathfrak{D}^{\alpha}_{a+}u\right\|_{L^p_{\frac{1}{x}}(a,b)}. \tag{6.40}$$

Proof. Let $u \in L^p_{\frac{1}{x}}(a,b)$, $u(a)=0$, $\mathfrak{D}^\alpha_{a+}u \in L^p(a,b)$ and consider the function

$$u(t) = \mathfrak{I}^\alpha_{a+}\mathfrak{D}^\alpha_{a+}u(t). \tag{6.41}$$

Using the Hölder inequality with $\frac{1}{p}+\frac{1}{q}=1$, we obtain

$$\begin{aligned}
\left|\mathfrak{I}^\alpha_{a+}\mathfrak{D}^\alpha_{a+}u(t)\right| &\le \frac{1}{\Gamma(\alpha)} \int_a^t \left|\left(\log\frac{t}{s}\right)^{\alpha-1} \mathfrak{D}^\alpha_{a+}u(s)\right| \frac{ds}{s^{\frac{1}{p}+\frac{1}{q}}} \\
&\le \frac{1}{\Gamma(\alpha)} \left(\int_a^t \left|\log\frac{t}{s}\right|^{\alpha q-q} \frac{ds}{s}\right)^{\frac{1}{q}} \left(\int_a^t \left|\mathfrak{D}^\alpha_{a+}u(s)\right|^p \frac{ds}{s}\right)^{\frac{1}{p}} \\
&\overset{\alpha>\frac{1}{p}}{=} \frac{\left|\log\frac{t}{a}\right|^{\alpha-1+\frac{1}{q}}}{(\alpha q-q+1)^{\frac{1}{q}}\Gamma(\alpha)} \left(\int_a^t \left|\mathfrak{D}^\alpha_{a+}u(s)\right|^p \frac{ds}{s}\right)^{\frac{1}{p}} \\
&\le \frac{\left|\log\frac{b}{a}\right|^{\alpha-1+\frac{1}{q}}}{(\alpha q-q+1)^{\frac{1}{q}}\Gamma(\alpha)} \left\|\mathfrak{D}^\alpha_{a+}u\right\|_{L^p_{\frac{1}{x}}(a,b)} \\
&= \frac{\left|\log\frac{b}{a}\right|^{\alpha-\frac{1}{p}}}{(\alpha q-q+1)^{\frac{1}{q}}\Gamma(\alpha)} \left\|\mathfrak{D}^\alpha_{a+}u\right\|_{L^p_{\frac{1}{x}}(a,b)} \\
&= \frac{\left|\log\frac{b}{a}\right|^{\alpha-\frac{1}{p}}}{\left(\frac{\alpha p}{p-1}-\frac{1}{p-1}\right)^{\frac{p-1}{p}}\Gamma(\alpha)} \left\|\mathfrak{D}^\alpha_{a+}u\right\|_{L^p_{\frac{1}{x}}(a,b)},
\end{aligned}$$

where $q=\frac{p}{p-1}>1$, showing (6.40).

∎

Remark 6.33. *In Theorem 6.32, by taking* $1<\theta<\infty$, *we have*

$$\|u\|_{L^\theta(a,b)} \le \frac{(b-a)^{\frac{1}{\theta}}\left|\log\frac{b}{a}\right|^{\alpha-\frac{1}{p}}}{\left(\frac{\alpha p}{p-1}-\frac{1}{p-1}\right)^{\frac{p-1}{p}}\Gamma(\alpha)} \left\|\mathfrak{D}^\alpha_{a+}u\right\|_{L^p_{\frac{1}{x}}(a,b)}. \tag{6.42}$$

6.10. Hardy type inequality for the Hadamard fractional derivative.

Let us show the Hardy type inequality.

Theorem 6.34. *Let* $a>0$, $u(a)=0$ *and* $\mathfrak{D}^\alpha_{a+}u \in L^p_{\frac{1}{x}}(a,b)$ *with* $p>1$. *Then for the* $\mathfrak{D}^\alpha_{a+}$ *Hadamard fractional derivative of order* $\alpha \in \left(\frac{1}{p},1\right]$ *we have the inequality*

$$\left\|\frac{u}{x}\right\|_{L^p(a,b)} \le \frac{a^{-1}(b-a)^{\frac{1}{p}}\left|\log\frac{b}{a}\right|^{\alpha-\frac{1}{p}}}{\left(\frac{\alpha p}{p-1}-\frac{1}{p-1}\right)^{\frac{p-1}{p}}\Gamma(\alpha)} \left\|\mathfrak{D}^\alpha_{a+}u\right\|_{L^p_{\frac{1}{x}}(a,b)}. \tag{6.43}$$

Proof. From $a < x < b$ we have $\frac{1}{b} < \frac{1}{x} < \frac{1}{a}$. By using Theorem 6.32, we calculate

$$\begin{aligned}\left(\int_a^b \frac{|u(x)|^p}{x^p}dx\right)^{\frac{1}{p}} &= \left(\int_a^b x^{-p}|u(x)|^p dx\right)^{\frac{1}{p}} \\ &\leq a^{-1}\|u\|_{L^p(a,b)} \\ &\overset{(6.40)}{\leq} \frac{a^{-1}(b-a)^{\frac{1}{p}}\left|\log\frac{b}{a}\right|^{\alpha-\frac{1}{p}}}{\left(\frac{\alpha p}{p-1}-\frac{1}{p-1}\right)^{\frac{p-1}{p}}\Gamma(\alpha)}\left\|\mathfrak{D}^{\alpha}_{a+}u\right\|_{L^p_{\frac{1}{x}}(a,b)},\end{aligned} \tag{6.44}$$

showing (6.43). □

Let us show the weighted Hardy inequality with Hadamard fractional derivative.

Theorem 6.35. *Let $a > 0$, $u(a) = 0$ and $\mathfrak{D}^{\alpha}_{a+}u \in L^p_{\frac{1}{x}}(a,b)$ with $p > 1$. Then for the $\mathfrak{D}^{\alpha}_{a+}$ Hadamard fractional derivative of order $\alpha \in \left(\frac{1}{p}, 1\right]$ and $\gamma \in \mathbb{R}$, we have*

$$\left\|\frac{u}{x^{\gamma+1}}\right\|_{L^p(a,b)} \leq C\left\|\frac{\mathfrak{D}^{\alpha}_{a+}u}{x^{\gamma}}\right\|_{L^p_{\frac{1}{x}}(a,b)}. \tag{6.45}$$

Proof. Let us split the proof in two cases $\gamma \geq 0$ and $\gamma < 0$. Firstly, let us prove the case $\gamma \geq 0$. From $a > 0$, we have $b^{-\gamma-1} < x^{-\gamma-1} < a^{-\gamma-1}$

$$\begin{aligned}\left(\int_a^b \frac{|u(x)|^p}{x^{(\gamma+1)p}}dx\right)^{\frac{1}{p}} &\leq a^{-\gamma-1}\left(\int_a^b |u(x)|^p dx\right)^{\frac{1}{p}} \\ &\overset{(6.40)}{\leq} \frac{a^{-\gamma-1}(b-a)^{\frac{1}{p}}\left|\log\frac{b}{a}\right|^{\alpha-\frac{1}{p}}}{\left(\frac{\alpha p}{p-1}-\frac{1}{p-1}\right)^{\frac{p-1}{p}}\Gamma(\alpha)}\left(\int_a^b |\mathfrak{D}^{\alpha}_{a+}u|^p\frac{dx}{x}\right)^{\frac{1}{p}} \\ &= \frac{a^{-\gamma-1}(b-a)^{\frac{1}{p}}\left|\log\frac{b}{a}\right|^{\alpha-\frac{1}{p}}}{\left(\frac{\alpha p}{p-1}-\frac{1}{p-1}\right)^{\frac{p-1}{p}}\Gamma(\alpha)}\left(\int_a^b \frac{x^{\gamma p}}{x^{\gamma p}}|\mathfrak{D}^{\alpha}_{a+}u|^p\frac{dx}{x}\right)^{\frac{1}{p}} \\ &\leq \frac{a^{-\gamma-1}(b-a)^{\frac{1}{p}}b^{\gamma}\left|\log\frac{b}{a}\right|^{\alpha-\frac{1}{p}}}{\left(\frac{\alpha p}{p-1}-\frac{1}{p-1}\right)^{\frac{p-1}{p}}\Gamma(\alpha)}\left(\int_a^b \frac{|\mathfrak{D}^{\alpha}_{a+}u|^p}{x^{\gamma p}}\frac{dx}{x}\right)^{\frac{1}{p}} \\ &= \frac{a^{-\gamma-1}(b-a)^{\frac{1}{p}}b^{\gamma}\left|\log\frac{b}{a}\right|^{\alpha-\frac{1}{p}}}{\left(\frac{\alpha p}{p-1}-\frac{1}{p-1}\right)^{\frac{p-1}{p}}\Gamma(\alpha)}\left\|\frac{\mathfrak{D}^{\alpha}_{a+}u}{x^{\gamma}}\right\|_{L^p_{\frac{1}{x}}(a,b)}.\end{aligned} \tag{6.46}$$

Let us show the case $\gamma < 0$,

$$
\begin{aligned}
\left(\int_a^b \frac{|u(x)|^p}{x^{(\gamma+1)p}}dx\right)^{\frac{1}{p}} &= \left(\int_a^b \frac{|u(x)|^p}{x^{(\gamma p+p)}}dx\right)^{\frac{1}{p}} \\
&\le b^{-\gamma}\left(\int_a^b \frac{|u(x)|^p}{x^p}dx\right)^{\frac{1}{p}} \\
&\overset{(6.43)}{\le} \frac{b^{-\gamma}a^{-1}(b-a)^{\frac{1}{p}}\left|\log\frac{b}{a}\right|^{\alpha-\frac{1}{p}}}{\left(\frac{\alpha p}{p-1}-\frac{1}{p-1}\right)^{\frac{p-1}{p}}\Gamma(\alpha)}\left\|\mathfrak{D}^{\alpha}_{a+}u\right\|_{L^p_{\frac{1}{x}}(a,b)} \\
&= \frac{b^{-\gamma}a^{-1}(b-a)^{\frac{1}{p}}\left|\log\frac{b}{a}\right|^{\alpha-\frac{1}{p}}}{\left(\frac{\alpha p}{p-1}-\frac{1}{p-1}\right)^{\frac{p-1}{p}}\Gamma(\alpha)}\left(\int_a^b |\mathfrak{D}^{\alpha}_{a+}u|^p\frac{dx}{x}\right)^{\frac{1}{p}} \\
&= \frac{b^{-\gamma}a^{-1}(b-a)^{\frac{1}{p}}\left|\log\frac{b}{a}\right|^{\alpha-\frac{1}{p}}}{\left(\frac{\alpha p}{p-1}-\frac{1}{p-1}\right)^{\frac{p-1}{p}}\Gamma(\alpha)}\left(\int_a^b \frac{x^{\gamma p}}{x^{\gamma p}}|\mathfrak{D}^{\alpha}_{a+}u|^p\frac{dx}{x}\right)^{\frac{1}{p}} \\
&\le \frac{b^{-\gamma}a^{\gamma-1}(b-a)^{\frac{1}{p}}\left|\log\frac{b}{a}\right|^{\alpha-\frac{1}{p}}}{\left(\frac{\alpha p}{p-1}-\frac{1}{p-1}\right)^{\frac{p-1}{p}}\Gamma(\alpha)}\left(\int_a^b \frac{|\mathfrak{D}^{\alpha}_{a+}u|^p}{x^{\gamma p}}\frac{dx}{x}\right)^{\frac{1}{p}} \\
&= \frac{b^{-\gamma}a^{\gamma-1}(b-a)^{\frac{1}{p}}\left|\log\frac{b}{a}\right|^{\alpha-\frac{1}{p}}}{\left(\frac{\alpha p}{p-1}-\frac{1}{p-1}\right)^{\frac{p-1}{p}}\Gamma(\alpha)}\left\|\frac{\mathfrak{D}^{\alpha}_{a+}u}{x^{\gamma}}\right\|_{L^p_{\frac{1}{x}}(a,b)},
\end{aligned}
\tag{6.47}
$$

showing (6.45). ☐

6.11. Fractional Gagliardo-Nirenberg type inequality with the Hadamard derivative.

Theorem 6.36. *Assume that* $\alpha \in \left(\frac{1}{q}, 1\right]$, $1 \le p, q < \infty$. *Then we have the following Gagliardo-Nirenberg type inequality,*

$$\|u\|_{L^{\gamma}(a,b)} \le C\|\mathfrak{D}^{\alpha}_{a+}u\|^{s}_{L^q_{\frac{1}{x}}(a,b)}\|u\|^{1-s}_{L^p(a,b)}, \tag{6.48}$$

with

$$\frac{\gamma s}{q} + \frac{\gamma(1-s)}{p} = 1, \tag{6.49}$$

where $s \in [0,1]$.

Proof. By using the Hölder inequality $\frac{\gamma s}{q} + \frac{\gamma(1-s)}{p} = 1$, we have

$$\begin{aligned} \int_a^b |u(x)|^\gamma dx &= \int_a^b |u(x)|^{\gamma s} |u|^{\gamma(1-s)} dx \\ &\leq \left(\int_a^b |u(x)|^q dx \right)^{\frac{\gamma s}{q}} \left(\int_a^b |u(x)|^p dx \right)^{\frac{\gamma(1-s)}{p}} \\ &\overset{(6.40)}{\leq} C \|\mathfrak{D}^\alpha_{a+} u\|^{\gamma s}_{L^q_{\frac{1}{x}}(a,b)} \|u\|^{\gamma(1-s)}_{L^p(a,b)}, \end{aligned} \tag{6.50}$$

completing proof. ☐

6.12. **Fractional Cafarrelli-Kohn-Nirenberg type inequality with Hadamard derivative.** In this subsection, we prove fractional Cafarrelli-Kohn-Nirenberg type inequality for Hadamard derivative.

Theorem 6.37. *Assume that* $a > 0$, $\alpha \in \left(1 - \frac{1}{q}, 1\right)$, $1 < p, q < \infty$, $0 < r < \infty$ *such that* $p + q \geq r$. *Let* $\delta \in [0,1] \cap [\frac{r-q}{r}, \frac{p}{r}]$ *and* $c, d, e \in \mathbb{R}$ *with the* $\frac{\delta}{p} + \frac{1-\delta}{q} = \frac{1}{r}$, $c = \delta(d-1) + e(1-\delta)$ *and* $u(a) = 0$. *If* $1 + (d-1)p > 0$ *then we have*

$$\|x^c u\|_{L^r(a,b)} \leq C \left\| x^d \mathfrak{D}^\alpha_{a+} u \right\|^\delta_{L^p_{\frac{1}{x}}(a,b)} \|x^e u\|^{1-\delta}_{L^q(a,b)}. \tag{6.51}$$

Proof. Case $\delta = 0$.

If $\delta = 0$, then $c = e$ and $q = r$. Then (6.27) is the inequality

$$\|x^c u\|_{L^r(a,b)} \leq \|x^c u\|_{L^r(a,b)}.$$

Case $\delta = 1$.

If $\delta = 1$, then we have $c = d - 1$ and $p = r$. Also, we have $1 + cp = 1 + (d-1)p > 0$. Then by using the weighted fractional Hardy inequality (Theorem 6.35) we obtain

$$\begin{aligned} \|x^c u\|_{L^p(a,b)} &\leq C \left\| x^{c+1} \mathfrak{D}^\alpha_{a+} u \right\|_{L^p_{\frac{1}{x}}(a,b)} \\ &= C \left\| x^d \mathfrak{D}^\alpha_{a+} u \right\|_{L^p_{\frac{1}{x}}(a,b)}. \end{aligned} \tag{6.52}$$

Case $\delta \in [0,1] \cap [\frac{r-q}{r}, \frac{p}{r}]$.

By assumption $c = \delta(d-1) + e(1-\delta)$ and by using Hölder's inequality with $\frac{\delta}{p} + \frac{1-\delta}{q} = \frac{1}{r}$, we calculate

$$\begin{aligned} \|x^c u\|_{L^r(a,b)} &= \left(\int_a^b x^{cr} |u(x)|^r dx \right)^{\frac{1}{r}} \\ &= \left(\int_a^b \frac{|u(x)|^{\delta r}}{x^{\delta r(1-d)}} \frac{|u(x)|^{(1-\delta)r}}{x^{-er(1-\delta)}} dx \right)^{\frac{1}{r}} \\ &\leq \left\| \frac{u}{x^{1-d}} \right\|^\delta_{L^p(a,b)} \left\| \frac{u}{x^{-e}} \right\|^{1-\delta}_{L^q(a,b)}. \end{aligned} \tag{6.53}$$

By using the weighted fractional Hardy inequality (Theorem 6.35) with $1+(d-1)p > 0$ we obtain

$$\begin{aligned} \|x^c u\|_{L^r(a,b)} &\leq \left\| \frac{u}{x^{1-d}} \right\|^{\delta}_{L^p(a,b)} \left\| \frac{u}{x^{-e}} \right\|^{1-\delta}_{L^q(a,b)} \\ &\leq C \|x^d \mathfrak{D}^{\alpha}_{a+} u\|^{\delta}_{L^p_{\frac{1}{x}}(a,b)} \|x^e u\|^{1-\delta}_{L^q(a,b)}, \end{aligned} \tag{6.54}$$

showing (6.51). ☐

6.13. **Lyapunov-type inequality.** Assume $\Omega \subset \mathbb{R}^N$, $N \geq 1$, be an open bounded domain, $-\infty < a < b < +\infty$ and $q(x)$ be real-valued, continuous function. Let us consider the following fractional differential equation:

$$\begin{cases} \mathcal{D}^{\alpha}_{a+,x} \mathcal{D}^{\beta}_{a+,x} u(x,y) - (-\Delta_y)^s_p u(x,y) + q(x) u(x,y) = 0, \text{ in } (a,b) \times \Omega, \\ u(a,y) = u(b,y) = 0, \quad y \in \mathbb{R}^N \setminus \Omega, \end{cases} \tag{6.55}$$

where $\mathcal{D}^{\mu}_{a+,x}$ is the Caputo fractional derivative in the variable x and $(-\Delta_y)^s_p$ is the fractional p-Laplacian in the variable y with $s \in (0,1)$ and $1 < p < \infty$.

By [125], we can choose the first eigenfunction of

$$\begin{cases} (-\Delta_y)^s_p \varphi_1(y) = \lambda_1(\Omega) \varphi_1(y), \quad y \in \Omega, \\ \varphi_1(y) = 0, \quad y \in \mathbb{R}^N \setminus \Omega, \end{cases} \tag{6.56}$$

corresponding to be positive and whose eigenvalue simple and positive, $\lambda_1(\Omega) > 0$.

In this section we obtain a Lyapunov-type inequality for (6.55).

Theorem 6.38. *Assume that $0 < \alpha, \beta \leq 1$ be such that $1 < \alpha + \beta \leq 2$, $s \in (0,1)$, $1 < p < \infty$ and $q(x) \in C([a,b])$. Then for (6.55), we have*

$$\int_a^b |q(x) - \lambda_1(\Omega)| dx \geq \frac{\Gamma(\alpha+\beta)(\alpha+2\beta-1)^{\alpha+2\beta-1}}{(b-a)^{\alpha+\beta-1}(\alpha+\beta-1)^{\alpha+\beta-1}\beta^{\beta}}, \tag{6.57}$$

where $\lambda_1(\Omega)$ is the first eigenvalue of (6.56).

Proof. By multiplying (6.55) with $\varphi_1(y)$ and integrating over Ω, we get

$$\begin{aligned}\int_\Omega \mathcal{D}^{\alpha}_{a+,x}\mathcal{D}^{\beta}_{a+,x}u(x,y)\varphi_1(y)dy - \int_\Omega((-\Delta_y)^s_p u(x,y))\varphi_1(y)dy&\\ + q(x)\int_\Omega u(x,y)\varphi_1(y)dy&\\ = \mathcal{D}^{\alpha}_{a+,x}\mathcal{D}^{\beta}_{a+,x}\int_\Omega u(x,y)\varphi_1(y)dy&\\ - \int_\Omega((-\Delta_y)^s_p u(x,y))\varphi_1(y)dy&\\ + q(x)\int_\Omega u(x,y)\varphi_1(y)dy&\\ = \mathcal{D}^{\alpha}_{a+,x}\mathcal{D}^{\beta}_{a+,x}\int_\Omega u(x,y)\varphi_1(y)dy&\\ - \int_\Omega((-\Delta_y)^s_p \varphi_1(y))u(x,y)dy&\\ + q(x)\int_\Omega u(x,y)\varphi_1(y)dy&\\ = \mathcal{D}^{\alpha}_{a+,x}\mathcal{D}^{\beta}_{a+,x}\int_\Omega u(x,y)\varphi_1(y)dy&\\ - \lambda_1(\Omega)\int_\Omega u(x,y)\varphi_1(y)dy&\\ + \int_\Omega u(x,y)\varphi_1(y)dy&\\ = \mathcal{D}^{\alpha}_{a+,x}\mathcal{D}^{\beta}_{a+,x}v(x) + q_1(x)v(x) = 0,&\end{aligned}$$

where $v(x) = \int_\Omega u(x,y)\varphi_1(y)dy$, $q_1(x) = q(x) - \lambda_1(\Omega)$, from boundary condition (6.55), we have

$$v(a) = 0,\ v(b) = 0.$$

Finally, we get

$$\begin{cases}\mathcal{D}^{\alpha}_{a+,x}\mathcal{D}^{\beta}_{a+,x}v(x) + q_1(x)v(x) = 0, \quad x \in (a,b),\\ v(a) = 0,\ v(b) = 0.\end{cases} \tag{6.58}$$

By [126], for the (6.58), we get

$$\begin{aligned}\int_a^b |q_1(x)|dx &= \int_a^b |q(x) - \lambda_1(\Omega)|dx\\ &\geq \frac{\Gamma(\alpha+\beta)(\alpha+2\beta-1)^{\alpha+2\beta-1}}{(b-a)^{\alpha+\beta-1}(\alpha+\beta-1)^{\alpha+\beta-1}\beta^{\beta}},\end{aligned} \tag{6.59}$$

completing the proof. ☐

Corollary 6.39. *By choosing $\alpha = \beta = 1$, $s = 1$ and $p = 2$, we have Theorem 2.2 in [121] with $\gamma = 0$.*

Let us consider the following eigenvalue problem in cylindrical domain:

$$\begin{cases} \mathcal{D}^{\alpha}_{a+,x}\mathcal{D}^{\beta}_{a+,x}u(x,y) - (-\Delta_y)^s u(x,y) + \nu u(x,y) = 0, \text{ in } (a,b)\times\Omega, \\ u(a,y) = u(b,y) = 0, \quad \forall y \in \Omega, \\ u(x,y) = 0, \quad y \in \mathbb{R}^N \setminus \Omega, \end{cases} \tag{6.60}$$

where $(-\Delta_y)^s$ is the fractional Laplacian. Denote that $|\cdot|$ is the Lebesgue measure. Then, we have the following two sides estimate of the first eigenvalue of (6.60) in the circular cylinder.

Theorem 6.40. *Let $0 < \alpha, \beta \leq 1$ be such that $1 < \alpha + \beta \leq 2$, $s \in (0,1)$ and $1 < p < \infty$. Then we have,*

$$\begin{aligned} (b-a)|\nu| + (b-a)\lambda_1(\Omega) &\geq (b-a)|\nu| + (b-a)\lambda_1(B) \\ &> \frac{\Gamma(\alpha+\beta)(\alpha+2\beta-1)^{\alpha+2\beta-1}}{(b-a)^{\alpha+\beta-1}(\alpha+\beta-1)^{\alpha+\beta-1}\beta^{\beta}}, \end{aligned} \tag{6.61}$$

where $\lambda_1(B)$ is the first eigenvalue of the eigenvalue problem (6.60) in a ball B with $|\Omega| = |B|$.

Proof. By using previous theorem, assume that B be a ball and $q(x) = \nu$ by using Theorem A.1 in [1], we have

$$\begin{aligned} (b-a)|\nu| + (b-a)\lambda_1(\Omega) &\geq (b-a)|\nu| + (b-a)\lambda_1(B) \\ &\geq \int_a^b |\nu - \lambda_1(B)| dx \\ &\geq \frac{\Gamma(\alpha+\beta)(\alpha+2\beta-1)^{\alpha+2\beta-1}}{(b-a)^{\alpha+\beta-1}(\alpha+\beta-1)^{\alpha+\beta-1}\beta^{\beta}}. \end{aligned} \tag{6.62}$$

□

Theorem 6.41. *Assume that $0 < \alpha, \beta \leq 1$ be such that $1 < \alpha + \beta \leq 2$, $s \in (0,1)$ and $1 < p < \infty$. Then we have,*

$$\begin{aligned} (b-a)|\nu| + (b-a)\lambda_1(\Omega) &\geq (b-a)|\nu| + (b-a)\lambda_1(B) \\ &> \frac{\Gamma(\alpha+\beta)(\alpha+2\beta-1)^{\alpha+2\beta-1}}{(b-a)^{\alpha+\beta-1}(\alpha+\beta-1)^{\alpha+\beta-1}\beta^{\beta}}, \end{aligned} \tag{6.63}$$

where $\lambda_1(B)$ is the first eigenvalue of the eigenvalue problem (6.60) in ball B with $|\Omega| = |B|$.

Let us consider the following fractional differential equation by the variable x:

$$\begin{cases} L_x u(x,y) - (-\Delta_y)^s u(x,y) + q(x)u(x,y) = 0, \quad (x,y) \in (a,b)\times\Omega, \\ u(a,y) = u(b,y) = 0, \quad \forall y \in \Omega, \\ u(x,y) = 0, \quad y \in \mathbb{R}^N \setminus \Omega, \end{cases} \tag{6.64}$$

where

$$L_x := \frac{D^{\alpha}_{b-,x} D^{\alpha}_{a+,x} + D^{\alpha}_{a+,x} D^{\alpha}_{b-,x}}{2}, \quad \frac{1}{2} < \alpha < 1.$$

Denote that the following functional spaces $AC^{\alpha,2}_a([a,b])$ and $AC^{\alpha,2}_b([a,b])$.

Definition 6.42. For all $\alpha \in (0,1)$ and every $1 \le p < \infty$, we denote by $AC_a^{\alpha,2}([a,b])$ the functional space defined by

$$AC_a^{\alpha,2}([a,b]) := \{f : f \in L^1([a,b]),\ D^\alpha_{a+,x} f \in L^2([a,b])\}. \tag{6.65}$$

$$AC_b^{\alpha,2}([a,b]) := \{f : f \in L^1([a,b]),\ D^\alpha_{b-,x} f \in L^2([a,b])\}. \tag{6.66}$$

Then, we have the following Lyapunov-type inequality:

Theorem 6.43. *Suppose that* $\frac{1}{2} < \alpha < 1$, $s \in (0,1)$, $1 < p < \infty$ *and* $q \in C([a,b])$. *Suppose that* $\int_\Omega u(x,y)\varphi_1(y)dy \in AC_a^{\alpha,2}([a,b]) \cap AC_b^{\alpha,2}([a,b]) \cap C([a,b])$. *Then for (6.64), we get*

$$\int_a^b |q(x) - \lambda_1(\Omega)| dx \ge \Gamma^2(\alpha) \left(\frac{2}{b-a}\right)^{2\alpha-1} (2\alpha - 1), \tag{6.67}$$

where $\lambda_1(\Omega)$ *is the first eigenvalue of (6.56).*

Proof. The proof is similar to that of Theorem 6.38. Shortly, we have

$$\begin{cases} L_x v(x) + q_1(x) v(x) = 0, & x \in (a,b), \\ v(a) = 0,\ v(b) = 0, \end{cases} \tag{6.68}$$

where $v(x) = \int_\Omega u(x,y)\varphi_1(y)dy$ and $q_1(x) = q(x) - \lambda_1(\Omega)$. By assumptions $v(x) \in AC_a^{\alpha,2}([a,b]) \cap AC_b^{\alpha,2}([a,b]) \cap C([a,b])$ and from [127], we get

$$\int_a^b |q(x) - \lambda_1(\Omega)| dx \ge \Gamma^2(\alpha) \left(\frac{2}{b-a}\right)^{2\alpha-1} (2\alpha - 1). \tag{6.69}$$

Theorem 6.43 is complete. ☐

6.14. Hartman-Wintner-type inequality.

In this section, we show Hartman-Wintner type inequality for problem (6.55).

Theorem 6.44. *Assume that* $0 < \alpha, \beta \le 1$ *be such that* $1 < \alpha + \beta \le 2$, $s \in (0,1)$, $1 < p < \infty$ *and* $q(x) \in C([a,b])$. *Assume that the fractional boundary value problem (6.55) has a nontrivial continuous solution. Then, we have*

$$\int_a^b (b-s)^{\alpha+\beta-1}(s-a)^\beta [q(x) - \lambda_1(\Omega)]^+ ds > \Gamma(\alpha+\beta)(b-a)^\beta, \tag{6.70}$$

where $[q(x) - \lambda_1(\Omega)]^+ = \max\{q(x) - \lambda_1(\Omega), 0\}$.

Proof. By multiplying (6.55) with $\varphi_1(y)$ and integrating over Ω, for the function $v(x) = \int_\Omega u(x,y)\varphi_1(y)dy$ we have problem (6.58). Problem (6.58) is equivalent to the integral equation (see. [126])

$$v(x) = \int_a^b G(x,s) q_1(s) v(s) ds,$$

where

$$G(x,s)=\frac{1}{\Gamma(\alpha+\beta)}\begin{cases}\frac{(b-s)^{\alpha+\beta-1}(x-a)^{\beta}}{(b-a)^{\beta}}-(x-s)^{\alpha+\beta-1}, & a\leq s\leq x\leq b,\\ \frac{(b-s)^{\alpha+\beta-1}(x-a)^{\beta}}{(b-a)^{\beta}}, & a\leq x\leq s\leq b,\end{cases} \tag{6.71}$$

$$G(x,s)\leq G(s,s),\ \text{ for } x,s\in[a,b]. \tag{6.72}$$

By using last fact with (6.72) for any $a\leq x\leq b$, we get

$$\begin{aligned}|v(x)| &\leq \int\limits_a^b |G(x,s)||q_1(s)||v(s)|ds\\ &\leq \int\limits_a^b G(s,s)|q_1(s)||v(s)|ds\\ &\leq \frac{(b-a)^{-\beta}}{\Gamma(\alpha+\beta)}\int\limits_a^b (b-s)^{\alpha+\beta-1}(s-a)^{\beta}q_1^{+}(s)|v(s)|ds.\end{aligned}$$

Theorem 6.70 is proved. ☐

Corollary 6.45. *By choosing $\alpha=\beta=1$ and $s=1$, $p=2$ in (6.70), we have the classical Hartman-Wintner inequality*

$$\int_a^b (b-s)(s-a)q_1^{+}(s)>b-a. \tag{6.73}$$

6.15. **De La Vallée Poussin-type inequality.** Let us consider in $(a,b)\times\Omega$ the following fractional differential Dirichlet problem:

$$\begin{cases}\frac{\partial^2}{\partial x^2}u(x,y)-(-\Delta_y)^s u(x,y)+f(x)\mathcal{D}^{\alpha}_{a+,x}u(x,y)+q(x)u(x,y)=0,\\ u(a,y)=u(b,y)=0, \quad \forall y\in\Omega,\\ u(x,y)=0, \quad y\in\mathbb{R}^N\setminus\Omega,\end{cases} \tag{6.74}$$

where $\alpha\in(0,1]$. Then, we show the de La Vallée Poussin-type inequality for (6.74).

Theorem 6.46. *Assume that $\alpha\in(0,1]$. Then, for (6.74), we have De La Vallée Poussin-type inequality in the following form:*

$$1<M_1(b-a)^{2-\alpha}+M_2\frac{(b-a)^2}{\Gamma(2-\alpha)},$$

where $M_1=\max_{a\leq x\leq b}|f(x)|$, $M_2=\max_{a\leq x\leq b}|q(x)-\lambda_1(\Omega)|$ and $\lambda_1(\Omega)$ is the first eigenvalue of the (6.56).

Proof. Similarly to Theorem 6.46, we get

$$v''(x)+f(x)\mathcal{D}^{\alpha}_{a+,x}v(x)+q_1(x)v(x)=0, \tag{6.75}$$

with

$$v(a)=v(b)=0,$$

where

$$v(x) = \int_{\Omega} u(x,y)\varphi_1(y)dy$$

and

$$q_1(x) = q(x) - \lambda_1(\Omega).$$

From [120, Theorem 3.1], we have

$$\begin{aligned} b-a &< \max\left\{\int_a^b \frac{(s-a)^{2-\alpha}}{\Gamma(2-\alpha)}|f(s)|ds, \int_a^b \frac{(s-a)^{1-\alpha}}{\Gamma(2-\alpha)}(b-s)|f(s)|ds\right\} \\ &\quad + \int_a^b (s-a)(b-s)|q(s)-\lambda_1(\Omega)|ds \\ &\quad \le M_1(b-a)^{3-\alpha} + M_2(b-a)^3, \end{aligned} \tag{6.76}$$

where $M_1 = \max_{a\le x\le b}|f(x)|$, $M_2 = \max_{a\le x\le b}|q(x)-\lambda_1(\Omega)|$.

Theorem 6.46 is complete. ☐

Corollary 6.47. *By choosing $\alpha = 1$, we get Theorem 2.2 in [122].*

Let us consider in $(a,b)\times\Omega$ the following fractional differential equation with Riemann-Liouville derivative and $1<\alpha\le 2$ and $0<\beta\le 1$:

$$\begin{cases} D^{\alpha}_{a+,x}u(x,y) - (-\Delta_y)^s u(x,y) + f(x)D^{\beta}_{a+,x}u(x,y) + q(x)u(x,y) = 0, \\ u(a,y) = u(b,y) = 0, \quad \forall y\in\Omega, \\ u(x,y) = 0, \quad y\in\mathbb{R}^N\setminus\Omega, \end{cases} \tag{6.77}$$

Then let us present de La Vallée Poussin-type inequality for (6.77),

Theorem 6.48. *Assume that $\alpha-\beta\ge 1$ with $1<\alpha\le 2$ and $0<\beta\le 1$. Then, we have*

$$\Gamma(\alpha-\beta) \le C_1M_1 + C_2M_2, \tag{6.78}$$

where $M_1 = \max\limits_{a\le x\le b}|f(x)|$, $M_2 = \max\limits_{a\le x\le b}|q(x)-\lambda_1(\Omega)|$,

$$C_1 = (b-a)^{\alpha-\beta}, \tag{6.79}$$

and

$$C_2 = \frac{(b-a)^{\alpha}}{\Gamma(1+\beta)}. \tag{6.80}$$

Proof. Similarly to Theorem 6.46, we get

$$D^{\alpha}_{a+}v(x) + f(x)D^{\beta}_{a+}v(x) + q_1(x)v(x) = 0, \tag{6.81}$$

with

$$v(a) = v(b) = 0,$$

where

$$v(x) = \int_{\Omega} u(x,y)\varphi_1(y)dy$$

and

$$q_1(x) = q(x) - \lambda_1(\Omega).$$

By using [120, Theorem 3.11], we have

$$\Gamma(\alpha-\beta) \le C_1'M_1 + C_2'M_2,$$

where

$$C_1' = \max\left\{\int_a^b F_1(s)ds,\ \int_a^b \frac{(s-a)^{\alpha-\beta-1}(b-s)^{\alpha-1}}{(b-a)^{\alpha-1}}ds\right\}, \tag{6.82}$$

and

$$C_2' = \max\left\{\int_a^b F_2(s)ds,\ \int_a^b \frac{(s-a)^{\alpha-\beta-1}(b-s)^{\alpha-1}}{(b-a)^{\alpha-1}}\frac{(s-a)^{\beta}}{\Gamma(\beta+1)}ds\right\}, \tag{6.83}$$

where

$$F_1(s) = \max\left\{\frac{(s-a)^{\alpha-\beta-1}(b-s)^{\alpha-1}}{(b-a)^{\alpha-1}} : \alpha-\beta-1>0,\ (b-s)^{\alpha-\beta-1} - \frac{(b-s)^{\alpha-1}}{(b-a)^{\beta}}\right\}$$

and

$$F_2(s) = F_1(s)\frac{(s-a)^{\beta}}{\Gamma(\beta+1)}.$$

Hence, we get

$$\begin{aligned}\Gamma(\alpha-\beta) &\le C_1'M_1 + C_2'M_2\\ &\le (b-a)^{\alpha-\beta}M_1 + \frac{(b-a)^{\alpha}}{\Gamma(1+\beta)}M_2.\end{aligned}$$

□

Corollary 6.49. *By choosing $\alpha = 2$ and $\beta = 1$ in Theorem 6.48, we get Theorem 2.2 in* [122].

Corollary 6.50. *By choosing $\alpha = 2$ in Theorem 6.48, we get Theorem 6.46.*

6.16. **Lyapunov-type inequality for a fractional differential system.** In this section we present Lyapunov-type inequality for fractional differential system. Let us consider in $(a,b)\times\Omega$ the following fractional differential systems:

$$\begin{cases} u_{xx}(x,y) - (-\Delta_y)^s v(x,y) + f(x)v(x,y) = 0,\\ v_{xx}(x,y) - (-\Delta_y)^s u(x,y) + g(x)u(x,y) = 0,\end{cases} \tag{6.84}$$

with homogeneous Dirichlet problem

$$u(a,y) = u(b,y) = v(a,y) = v(b,y) = 0, \quad y\in\Omega,$$

and

$$u(x,y) = v(x,y) = 0, \quad y \in \mathbb{R}^N \setminus \Omega.$$

Let us show one of the main result of this section:

Theorem 6.51. *Assume that that $f,g \ge 0$ and $f,g \in L^1([a,b])$. If (6.84) has not nontrivial solution, then we have*

$$4 \le (b-a)\left(\int_a^b |f(x)-\lambda_1(\Omega)|dx\right)^{\frac{1}{2}}\left(\int_a^b |g(x)-\lambda_1(\Omega)|dx\right)^{\frac{1}{2}}. \tag{6.85}$$

Proof. Suppose that

$$U(x)=\int_\Omega u(x,y)\varphi_1(y)dy,$$

and

$$V(x)=\int_\Omega v(x,y)\varphi_1(y)dy.$$

Similarly with the single equation case, we have

$$\begin{cases} U''(x)-f_1(x)V(x,y)=0,\\ V''(x)-g_1(x)U(x,y)=0, \end{cases} \tag{6.86}$$

with

$$\begin{gathered} U(a)=U(b)=0,\\ V(a)=V(b)=0,\\ f_1(x)=f(x)-\lambda_1(\Omega) \end{gathered}$$

and

$$g_1(x)=g(x)-\lambda_1(\Omega).$$

From [70], we have

$$4\le (b-a)\left(\int_a^b |f_1(x)|dx\right)^{\frac{1}{2}}\left(\int_a^b |g_1(x)|dx\right)^{\frac{1}{2}}.$$

Theorem 6.51 is proved. □

Let us consider in $(a,b)\times\Omega$ the following system:

$$\begin{cases} L_x^\alpha u(x,y)-(-\Delta_y)^s v(x,y)+f(x)v(x,y)=0,\\ L_x^\beta v(x,y)-(-\Delta_y)^s u(x,y)+g(x)u(x,y)=0, \end{cases} \tag{6.87}$$

with a homogeneous Dirichlet boundary condition

$$u(a,y)=u(b,y)=v(a,y)=v(b,y)=0,\quad y\in\Omega,$$

and

$$u(x,y)=v(x,y)=0,\quad y\in\mathbb{R}^N\setminus\Omega,$$

where

$$L_x^\alpha:=\frac{D_{b-,x}^\alpha D_{a+,x}^\alpha+D_{a+,x}^\alpha D_{b-,x}^\alpha}{2},\quad \frac{1}{2}<\alpha<1,$$

and

$$L_x^\beta:=\frac{D_{b-,x}^\beta D_{a+,x}^\beta+D_{a+,x}^\beta D_{b-,x}^\beta}{2},\quad \frac{1}{2}<\beta<1.$$

Theorem 6.52. *Suppose that $\frac{1}{2}<\alpha<1$, $\frac{1}{2}<\beta<1$ and $f,g\in L^1([a,b])$. Let u,v be a nontrivial solution of (6.87), then we have*

$$\begin{aligned} &\left(\frac{2}{b-a}\right)^{\alpha+\beta-1}(2\alpha-1)^{\frac{1}{2}}(2\beta-1)^{\frac{1}{2}}\Gamma(\alpha)\Gamma(\beta)\\ &\qquad\le\left(\int_a^b |f(x)-\lambda_1(\Omega)|dx\right)^{\frac{1}{2}}\left(\int_a^b |g(x)-\lambda_1(\Omega)|dx\right)^{\frac{1}{2}}. \end{aligned} \tag{6.88}$$

Proof. Proof of this Theorem is similar Theorem 6.51 we obtain

$$\begin{cases} L^{\alpha}U(x) - f_1(x)V(x,y) = 0, \\ L^{\beta}(x) - g_1(x)U(x,y) = 0, \end{cases} \tag{6.89}$$

where

$$U(x) = \int_{\Omega} u(x,y)\varphi_1(y)dy,$$

and

$$V(x) = \int_{\Omega} v(x,y)\varphi_1(y)dy.$$

By using Corollary 5.5 in [127], we get

$$\left(\frac{2}{b-a}\right)^{\alpha+\beta-1} (2\alpha-1)^{\frac{1}{2}}(2\beta-1)^{\frac{1}{2}}\Gamma(\alpha)\Gamma(\beta) \\ \leq \left(\int_a^b |f(x)-\lambda_1(\Omega)|dx\right)^{\frac{1}{2}} \left(\int_a^b |g(x)-\lambda_1(\Omega)|dx\right)^{\frac{1}{2}}. \tag{6.90}$$

Theorem 6.52 is proved. ☐

6.17. **Applications.** In this Section we show some applications of the obtained inequalities and we note that u is a real-valued function.

6.17.1. *Uncertainly principle.* The inequality (6.17) implies the following uncertainly principle:

Corollary 6.53. *Let $a > 0$, $u(a) = 0$ and $\mathcal{D}^{\alpha}_{a+}u \in L^p(a,b)$ with $p > 1$. Then for the Caputo fractional derivative $\mathcal{D}^{\alpha}_{a+}$ of order $\alpha \in \left(\frac{1}{p}, 1\right]$ we have following inequality*

$$\|u\|^2_{L^2(a,b)} \leq \frac{a^{-1}(b-a)^{\alpha}}{\left(\frac{\alpha p}{p-1} - \frac{1}{p-1}\right)^{\frac{p-1}{p}} \Gamma(\alpha)} \left\|\mathcal{D}^{\alpha}_{a+}u\right\|_{L^p(a,b)} \||x|^{\alpha}u\|_{L^q(a,b)}, \tag{6.91}$$

where $q = \frac{p}{p-1}$.

Proof. By using (6.17), we obtain

$$\frac{a^{-1}(b-a)^{\alpha}}{\left(\frac{\alpha p}{p-1} - \frac{1}{p-1}\right)^{\frac{p-1}{p}} \Gamma(\alpha)} \left\|\mathcal{D}^{\alpha}_{a+}u\right\|_{L^p(a,b)} \|xu\|_{L^q(a,b)} \overset{(6.17)}{\geq} \left\|\frac{u}{x}\right\|_{L^p(a,b)} \|xu\|_{L^q(a,b)} \\ \geq \|u\|^2_{L^2(a,b)}, \tag{6.92}$$

completing the proof. ☐

Remark 6.54. *Also, the uncertainly principle holds for the Riemann-Liouville derivative.*

6.17.2. *Embedding of spaces.* Let us consider the space $\dot{H}^\alpha_+(a,b)$ with $\alpha \in \left(\frac{1}{2}, 1\right]$ introduced in [128, 129] in the following form:

$$\dot{H}^\alpha_+(a,b) := \{u \in L^2(a,b),\ \mathcal{D}^\alpha_{a+}u \in L^2(a,b),\ u(a) = 0\}.$$

If $\alpha < \beta$, then by Poincaré–Sobolev-type inequality (6.9) we have $\dot{H}^\beta_+(a,b) \hookrightarrow \dot{H}^\alpha_+(a,b)$.

6.17.3. *A-priori estimate.* Here, we seek a real-valued solution to the following space-fractional diffusion problem

$$\begin{cases} u_t(x,t) + D^\alpha_{b-}\mathcal{D}^\alpha_{a+}u(x,t) = 0, & (x,t) \in (a,b) \times (0,T), \\ u(x,0) = u_0(x), \ \ \forall x \in (a,b), \end{cases} \tag{6.93}$$

where $\alpha \in \left(\frac{1}{2}, 1\right]$, $u \in L^\infty(0,T; \dot{H}^\alpha_+(a,b))$, $u_t \in L^2(0,T; \dot{H}^\alpha_+(a,b))$ and $u_0 \in L^2(a,b)$. We show an a-priori estimate for this problem. Let us define

$$I(t) = \|u(x,\cdot)\|^2_{L^2(a,b)} = \int_a^b |u(x,t)|^2 dx.$$

Then by multiplying (6.93) by u, integrating over (a,b), and by using integration by parts, we compute

$$\begin{aligned} \int_a^b u_t(x,t)u(x,t)dx &+ \int_a^b u(x,t)D^\alpha_{b-}\mathcal{D}^\alpha_{a+}u(x,t)dx \\ &= \frac{1}{2}\frac{d}{dt}\int_a^b |u(x,t)|^2 dx + \int_a^b |\mathcal{D}^\alpha_{a+}u(x,t)|^2 dx \\ &= \frac{1}{2}\frac{dI(t)}{dt} + \int_a^b |\mathcal{D}^\alpha_{a+}u(x,t)|^2 dx. \end{aligned} \tag{6.94}$$

By using (6.9) with $p = 2$ in (6.94), we get

$$0 = \frac{1}{2}\frac{dI(t)}{dt} + \int_a^b |\mathcal{D}^\alpha_{a+}u(x,t)|^2 dx \overset{(6.9)}{\geq} \frac{1}{2}\frac{dI(t)}{dt} + \frac{(2\alpha-1)\,\Gamma^2(\alpha)}{(b-a)^{2\alpha}}\int_a^b |u(x,t)|^2 dx, \tag{6.95}$$

it means $\frac{dI(t)}{dt} \leq 0$. That is, $I(t)$ is a non-decreasing function, then for $t > 0$, we have $I(t) \leq I(0)$. Finally,

$$\|u(x,\cdot)\|_{L^2(a,b)} \leq \|u_0\|_{L^2(a,b)}.$$

7. CONCLUSION

In this PhD thesis, we develop inequalities of fractional calculus on homogeneous Lie groups. More precisely, we develop the fractional calculus and non-commutative analysis, thereby combining two different directions in mathematics. This perspective turned out to be extremely useful on both a conceptual and a technical level. Let us review the obtained results in this dissertation:

In Chapter 3, we study fractional functional and geometric inequalities on homogeneous Lie groups. We obtain fractional Hardy, Sobolev, Gagliardo-Nirenberg, Caffarelli-Kohn-Nirenberg inequalities on homogeneous Lie groups and its logarithmic fractional analogues which are even new on Euclidean case. We prove the Hardy-Littlewood-Sobolev inequality on homogeneous Lie groups, which describes boundedness of the Riesz potential operator in $L^p - L^q$ spaces. Also, we obtain the Stein-Weiss inequality for the Riesz potential. In addition, we show the integer order logarithmic Sobolev-Folland-Stein inequality on stratified Lie groups.

In Chapter 4, we deal questions about reverse functional inequalities. We establish the reverse integral Hardy inequality on polarisable metric measure space with parameters $q < 0$ and $p \in (0,1)$. As a consequence, we obtain integral reverse Hardy inequality with parameters $q < 0$ and $p \in (0,1)$ on homogeneous Lie groups, hyperbolic space and Cartan-Hadamard manifolds. In addition, we obtain the integral reverse Hardy inequality on polarisable metric measure space with parameters $\infty < q \leq p < 0$, and consequently we show the reverse integral Hardy inequality on homogeneous Lie groups. Further, we obtain reverse Hardy-Littlewood-Sobolev, Stein-Weiss and improved Stein-Weiss inequalities on homogeneous Lie groups with parametres $q < 0$ and $p \in (0,1)$. Moreover, we obtain reverse Hardy-Littlewood-Sobolev, Stein-Weiss type and improved Stein-Weiss type inequalities with parameters $\infty < q \leq p < 0$, which are even new in Euclidean settings. In addition, we obtain reverse Hardy, L^p-Sobolev and L^p- Caffarelli-Kohn-Nirenberg inequalities with the radial derivative on homogeneous Lie groups.

In Chapter 5, we investigate nonlinear PDE on homogeneous groups by using the results in previous chapters. Firstly, we obtain Lyapunov inequalities for fractional p-sub-Laplacian equation and systems on homogeneous Lie groups. Then, we show existence of a weak solution for the nonlinear equation with the p-sub-Laplacian on Heisenberg and stratified groups and we show existence of weak solution for the nonlinear equation with the fractional p-sub-Laplacian and Hardy potential on homogeneous Lie groups. Moreover, we discuss blow-up results for the heat equation with fractional p-sub-Laplacian on homogeneous Lie groups, for the heat equation with fractional sub-Laplacian on stratified groups, viscoelastic equation, heat and wave Rockland equations on graded groups, respectively.

In Appendix, we study one-dimensional functional inequalities on Euclidean settings. Firstly, we obtain fractional Hardy, Poincaré type, Gagliardo-Nirenberg and Caffarelli-Kohn-Nirenberg inequalities for fractional order differential operators as Caputo, Riemann-Liouville and Hadamard fractional derivatives. Also, we show some applications of these inequalities. In addition, we show Lyapunov and Hartman-Wintner-type inequalities for a fractional partial differential equation with Dirichlet condition and we give some applications of these inequalities for the first eigenvalue and we show de La Vallée Poussin-type inequality for fractional elliptic boundary value problem.

Most of results in this thesis were published on peer-reviewed international journals.

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