12

$9j$-Coefficients and higher

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12.1 Introduction

$3j$-Coefficients (or $3j$-symbols), $6j$-coefficients, $9j$-coefficients and higher (referred to as $3nj$-coefficients) play a crucial role in various physical applications dealing with the quantization of angular momentum. This is because the quantum operators of angular momentum satisfy the $\mathfrak{su}(2)$ commutation relations. So the $3nj$-coefficients in this chapter are $3nj$-coefficients of the Lie algebra $\mathfrak{su}(2)$. For these coefficients, we shall emphasize their hypergeometric expressions and their relations to discrete orthogonal polynomials. Note that $3nj$-coefficients can also be considered for other Lie algebras. For positive discrete series representations of $\mathfrak{su}(1,1)$, the $3nj$-coefficients carry different labels but have the same structure as those of $\mathfrak{su}(2)$ [29, 37], and the related orthogonal polynomials are the same. For other Lie algebras, the definition of $3j$-coefficients (i.e., coupling coefficients or Clebsch–Gordan coefficients related to the decomposition of tensor products of irreducible representations) is more involved, since in general multiplicities appear in the decomposition of tensor products [35, 11]. Note that there is also a vast literature on the q-analogues of $3nj$-coefficients in the context of quantum groups or quantized enveloping algebras: for the quantum universal enveloping algebra $U_q(\mathfrak{su}(2))$, the $3j$- and $6j$-coefficients are straightforward q-analogues of those of $\mathfrak{su}(2)$, and the related discrete orthogonal polynomials are the corresponding q-orthogonal polynomials in terms of basic hypergeometric series [21, 23, 1, 2].

Here, we shall be dealing only with $3nj$-coefficients for $\mathfrak{su}(2)$. First, we give a short summary of the relevant class of representations of the Lie algebra $\mathfrak{su}(2)$. An important notion is the tensor product of such representations. In the tensor product decomposition, the important Clebsch–Gordan coefficients appear. $3j$-Coefficients are proportional to these Clebsch–Gordan coefficients. We give some useful expressions (as hypergeometric series) and their relation to Hahn polynomials. Next, the tensor product of three representations is considered, and the relevant Racah coefficients (or $6j$-coefficients) are defined. The explicit expression of a Racah coefficient as a hypergeometric series of ${}_4F_3$-type and the connection with Racah polynomials and their orthogonality is given. $9j$-Coefficients are defined in the context of the tensor product of four representations. They are related to a discrete orthogonal polynomial in two variables (but no expression as a hypergeometric double sum is known). Finally, we consider the general tensor product of $(n+1)$ representations and "generalized recoupling coefficients" or $3nj$-coefficients.

There are several standard books on quantum theory of angular momentum and $3nj$-coefficients. A classical reference is [13], and an interesting collection of historical papers on the subject is [10]. The books by Biedenharn & Louck [8, 9] treat the subject thoroughly. An excellent collection of formulas is found in [38]. Srinivasa Rao & Rajeswari [33] emphasize the connection with hypergeometric series and some special topics such as zeros and the numerical computation. In [37], a self-contained mathematical introduction is given for $3nj$-coefficients of both the Lie algebras $\mathfrak{su}(2)$ and $\mathfrak{su}(1,1)$.

12.2 Representations of the Lie algebra $\mathfrak{su}(2)$

An introduction to Lie algebras and their representations is not given here; it can be found e.g. in [18]. We shall just recall some basic notions related to $\mathfrak{sl}(2,\mathbb{C})$ and $\mathfrak{su}(2)$. As a vector space, the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ consists of all traceless complex (2×2) matrices. In this matrix form, the Lie algebra bracket $[x,y]$ is the commutator $xy-yx$. We consider the following standard basis of $\mathfrak{sl}(2,\mathbb{C})$:

$$J_0 := \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}, \quad J_+ := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad J_- := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

The basic commutation relations then read: $[J_0, J_\pm] = \pm J_\pm$, $[J_+, J_-] = 2J_0$. In the universal enveloping algebra of $\mathfrak{sl}(2,\mathbb{C})$, the following element (called the *Casimir operator*) is *central* (i.e., it commutes with every element):

$$C := J_+J_- + J_0^2 - J_0 = J_-J_+ + J_0^2 + J_0.$$

A $*$*-operation* on a complex Lie algebra is a conjugate-linear anti-automorphic involution. With a $*$-operation there is associated a real subalgebra (real form) consisting of all elements x in the complex Lie algebra for which $x^* = -x$. For $\mathfrak{sl}(2,\mathbb{C})$, there exist two non-equivalent $*$-operations, one corresponding to $\mathfrak{su}(2)$ and one to $\mathfrak{su}(1,1)$. For the real form $\mathfrak{su}(2)$, this is explicitly given by $J_0^* = J_0$, $J_\pm^* = J_\mp$. It consists of the matrices $\begin{pmatrix} ia & b \\ -\bar{b} & -ia \end{pmatrix}$ $(a \in \mathbb{R},\ b \in \mathbb{C})$.

A *representation* of the Lie algebra $\mathfrak{g}$ in a finite dimensional complex vector space V is a linear map $\phi\colon \mathfrak{g} \to \mathrm{End}(V)$ such that $\phi([x,y]) = \phi(x)\phi(y) - \phi(y)\phi(x)$ for all elements $x, y \in \mathfrak{g}$. V is then called the *representation space*. It is convenient to use the language of modules, thus to refer to V as a $\mathfrak{g}$-module and to the action $\phi(x)(v)$ as $x \cdot v$ $(v \in V)$. The representation ϕ or the representation space V is *irreducible* if V has no non-trivial invariant subspaces under the action of $\mathfrak{g}$. It is *completely reducible* if V is a direct sum of irreducible representation subspaces. A representation ϕ of a real Lie algebra $\mathfrak{g}$ is *unitary* if V is a vector space with hermitian inner product $\langle\,.\,,\,.\,\rangle$ and $\langle x\cdot v, w\rangle = -\langle v, x\cdot w\rangle$ for all $x \in \mathfrak{g}$ and all $v, w \in V$. In this case, one also refers to V as a unitary representation. Unitary representations are completely reducible.

The following lists all irreducible unitary representations of $\mathfrak{su}(2)$.

Theorem 12.2.1 *For every $j \in \frac{1}{2}\mathbb{N} = \{0, \frac{1}{2}, 1, \frac{3}{2}, 2, \dots\}$, there is a unique (up to equivalence) irreducible unitary representation of $\mathfrak{su}(2)$ of dimension $2j+1$. An orthonormal basis for the corresponding representation space D_j is denoted by $\{e_m^{(j)} \mid m = -j, -j+1, \dots, j\}$. The action of $J_0, J_\pm$ is given by:*

$$J_0\, e_m^{(j)} = m\, e_m^{(j)}, \quad J_+\, e_m^{(j)} = \sqrt{(j-m)(j+m+1)}\, e_{m+1}^{(j)}, \quad J_- e_m^{(j)} = \sqrt{(j+m)(j-m+1)}\, e_{m-1}^{(j)}. \tag{12.2.1}$$

For the Casimir operator, one has $C^* = C$ and $C\, e_m^{(j)} = j(j+1)\, e_m^{(j)}$.

In the following, D_j will denote both the representation space and the representation on that vector space.

12.3 Clebsch–Gordan coefficients and $3j$-coefficients

An important notion to introduce is the concept of *tensor product* of two unitary representations of a real Lie algebra $\mathfrak{g}$. Let V and W be $\mathfrak{g}$-modules, and let $V \otimes W$ be the tensor product of the underlying vector spaces. Recall that if V and W have respective bases $v_1, v_2, \dots$ and $w_1, w_2, \dots$, then $V \otimes W$ has a basis consisting of the vectors $v_i \otimes w_j$. Now $V \otimes W$ has the structure of a $\mathfrak{g}$-module by defining:

$$x \cdot (v \otimes w) := x \cdot v \otimes w + v \otimes x \cdot w. \tag{12.3.1}$$

This tensor product space is naturally equipped with an inner product by $\langle v \otimes w, v' \otimes w' \rangle := \langle v, v' \rangle \langle w, w' \rangle$. If V and W are unitary representations, then the tensor product representation is also unitary with respect to this inner product.

Turning to the case of $\mathfrak{su}(2)$, let $j_1, j_2 \in \frac{1}{2}\mathbb{N}$, and consider the tensor product $D_{j_1} \otimes D_{j_2}$, sometimes denoted by $(j_1) \otimes (j_2)$. A set of basis vectors of $D_{j_1} \otimes D_{j_2}$ is given by

$$e_{m_1}^{(j_1)} \otimes e_{m_2}^{(j_2)}, \qquad m_1 = -j_1, -j_1+1, \dots, j_1; \;\; m_2 = -j_2, -j_2+1, \dots, j_2.$$

This basis is often referred to as the *uncoupled basis*. The total dimension of $D_{j_1} \otimes D_{j_2}$ is $(2j_1+1)(2j_2+1)$. The action of the $\mathfrak{su}(2)$ basis elements on these vectors is determined by (12.3.1).

In general, the module $D_{j_1} \otimes D_{j_2}$ is not irreducible, but it is completely reducible. Its irreducible components are again representations D_j of the form given in Theorem 12.2.1: $D_{j_1} \otimes D_{j_2} = \oplus_j D_j$. Herein, j takes the values $|j_1 - j_2|, |j_1 - j_2| + 1, \dots, j_1 + j_2$ (each value once, i.e. there is no multiplicity in the decomposition). So it must be possible to write a basis of $D_{j_1} \otimes D_{j_2}$ in terms of the standard basis vectors of the representations D_j appearing in the decomposition. This basis is referred to as the *coupled basis*. The coefficients expressing the coupled basis vectors in terms of the uncoupled basis vectors are known as the *Clebsch–Gordan coefficients* of $\mathfrak{su}(2)$. They appeared already in the work of Clebsch and Gordan on invariant theory of algebraic forms. But it was Wigner [26, 42, 43] who studied these coefficients systematically and who introduced the related $3j$-coefficients.

Theorem 12.3.1 *The tensor product $D_{j_1} \otimes D_{j_2}$ decomposes into irreducible unitary representations D_j of $\mathfrak{su}(2)$, $D_{j_1} \otimes D_{j_2} = \bigoplus_{j=|j_1-j_2|}^{j_1+j_2} D_j$. An orthonormal basis of $D_{j_1} \otimes D_{j_2}$ is given by the vectors*

$$e_m^{(j_1 j_2)j} = \sum_{m_1} C_{m_1,m-m_1,m}^{j_1,j_2,j}\, e_{m_1}^{(j_1)} \otimes e_{m-m_1}^{(j_2)} \quad (|j_1 - j_2| \le j \le j_1 + j_2, \ \ -j \le m \le j), \tag{12.3.2}$$

where the coefficients $C_{m_1,m-m_1,m}^{j_1,j_2,j}$ are Clebsch–Gordan coefficients *of $\mathfrak{su}(2)$, for which an expression is given below by (12.3.4). The action of J_0, $J_\pm$ on the basis vectors $e_m^{(j_1 j_2)j}$ is the standard action (12.2.1) of the representation D_j.*

In (12.3.2), the summation index m_1 runs from $-j_1$ to j_1 in steps of 1, and such that $-j_2 \le m - m_1 \le j_2$: thus $\max(-j_1, m - j_2) \le m_1 \le \min(j_1, m + j_2)$. The Clebsch–Gordan coefficient $C_{m_1,m_2,m}^{j_1,j_2,j}$ can be considered as a real function of six arguments from $\frac{1}{2}\mathbb{N}$. Following Theorem 12.3.1, these arguments satisfy the following conditions:

(c1) (j_1, j_2, j) forms a *triad*, i.e., $-j_1 + j_2 + j$, $j_1 - j_2 + j$ and $j_1 + j_2 - j$ are nonnegative integers;
(c2) m_1 is a *projection* of j_1, i.e., $m_1 \in \{-j_1, -j_1 + 1, \ldots, j_1\}$ (and similarly, m_2 is a projection of j_2 and m is a projection of j);
(c3) $m = m_1 + m_2$.

Usually, one extends the definition by saying that $C_{m_1,m_2,m}^{j_1,j_2,j} = 0$ if one of the conditions (c1), (c2), (c3) is not satisfied. Then one can write

$$e_m^{(j_1 j_2)j} = \sum_{m_1,m_2} C_{m_1,m_2,m}^{j_1,j_2,j}\, e_{m_1}^{(j_1)} \otimes e_{m_2}^{(j_2)}. \tag{12.3.3}$$

An explicit formula for $C_{m_1,m_2,m}^{j_1,j_2,j}$ can, for instance, be obtained by using a differential operator realization of $\mathfrak{su}(2)$ and a polynomial realization of the basis vectors of the representations. One finds:

$$\begin{aligned} C_{m_1,m_2,m}^{j_1,j_2,j} = {} & \sqrt{(2j+1)}\,\Delta(j_1, j_2, j)\,\delta(j_1, m_1, j_2, m_2, j, m) \\ \times \sum_k & \frac{(-1)^k}{k!\,(j_1 - m_1 - k)!\,(j_1 + j_2 - j - k)!\,(j_2 + m_2 - k)!\,(j - j_2 + m_1 + k)!\,(j - j_1 - m_2 + k)!}, \end{aligned} \tag{12.3.4}$$

where

$$\Delta(j_1, j_2, j) = \sqrt{\frac{(-j_1 + j_2 + j)!\,(j_1 - j_2 + j)!(j_1 + j_2 - j)!}{(j_1 + j_2 + j + 1)!}}, \tag{12.3.5}$$

$$\delta(j_1, m_1, j_2, m_2, j, m) = \sqrt{(j_1 - m_1)!\,(j_1 + m_1)!\,(j_2 - m_2)!\,(j_2 + m_2)!\,(j - m)!\,(j + m)!}\,.$$

This rather symmetrical form is due to Van der Waerden [40] and Racah [27]. The expression is generally valid (that is, for all arguments satisfying (c1)–(c3)). The summation is over all integer k-values such that the factorials in the denominator of (12.3.4) are nonnegative.

It is clear that the summation in (12.3.4) can be rewritten in terms of a terminating ${}_3F_2$

series of unit argument; indeed, assume that $j - j_2 + m_1 \geq 0$ and $j - j_1 - m_2 \geq 0$, then this sum equals

$$\begin{aligned}
&\left((j_1 - m_1)!\,(j_1 + j_2 - j)!\,(j_2 + m_2)!\,(j - j_2 + m_1)!\,(j - j_1 - m_2)!\right)^{-1} \\
&\qquad\qquad \times\ {}_3F_2\left({-j_1 + m_1, -j_1 - j_2 + j, -j_2 - m_2 \atop j - j_2 + m_1 + 1, j - j_1 - m_2 + 1} ; 1 \right). \qquad (12.3.6)
\end{aligned}$$

Once the Clebsch–Gordan coefficient is rewritten as a terminating ${}_3F_2(1)$, one can use a transformation [4, Corollary 3.3.4] (known as *Sheppard's transformation*, and sometimes referred to as *Thomae's transformation*) to find yet other formulas. Applyication of Sheppard's transformation to (12.3.6), while keeping $-j_1 + m_1$ as negative numerator parameter, yields

$$C^{j_1,j_2,j}_{m_1,m_2,m} = C'\ {}_3F_2\left({-j_1 + m_1, -j_1 - j_2 + j, -j_1 - j_2 - j - 1 \atop -2j_1, -j_1 - j_2 + m} ; 1 \right); \qquad (12.3.7)$$

herein C' is some constant determined in (12.3.9). This expression is generally valid under conditions (c1)–(c3).

Various symmetries can be deduced for Clebsch–Gordan coefficients. Some of these symmetries follow by replacing the summation index in (12.3.4). Others can be deduced by performing permutations of the numerator and/or denominator parameters in a ${}_3F_2(1)$ expression as in (12.3.7). In order to express these symmetries, one often introduces the so-called $3j$-*coefficient* (due to Wigner)

$$\begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & -m \end{pmatrix} := \frac{(-1)^{j_1 - j_2 + m}}{\sqrt{2j+1}}\, C^{j_1,j_2,j}_{m_1,m_2,m}$$

From (12.3.4) one finds the classical expression for the $3j$-coefficient:

Proposition 12.3.2 *Let $j_1, j_2, j_3, m_1, m_2, m_3 \in \frac{1}{2}\mathbb{N}$. If (j_1, j_2, j_3) forms a triad, m_i is a projection of j_i $(i = 1, 2, 3)$ and $m_1 + m_2 + m_3 = 0$, then the $3j$-coefficient is determined by*

$$\begin{aligned}
&\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = (-1)^{j_1 - j_2 - m_3}\, \Delta(j_1, j_2, j_3)\, \delta(j_1, m_1, j_2, m_2, j_3, m) \\
&\times \sum_k \frac{(-1)^k}{k!\,(j_1 - m_1 - k)!\,(j_1 + j_2 - j_3 - k)!\,(j_2 + m_2 - k)!\,(j_3 - j_2 + m_1 + k)!\,(j_3 - j_1 - m_2 + k)!}\,.
\end{aligned} \qquad (12.3.8)$$

In all other cases, the $3j$-coefficient is zero. In (12.3.8), the summation is over all integers such that the arguments in the factorials are nonnegative. Alternatively, one can write:

$$\begin{aligned}
&\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = (-1)^{j_1 - j_2 - m_3}\, \frac{(2j_1)!\,(j_1 + j_2 + m_3)!\,(j_3 - m_3)!}{(j_1 - j_2 + j_3)\,(j_1 + j_2 - j_3)!} \\
&\qquad \times \frac{\Delta(j_1, j_2, j_3)}{\delta(j_1, m_1, j_2, m_2, j_3, m_3)}\ {}_3F_2\left({m_1 - j_1, j_3 - j_1 - j_2, -j_1 - j_2 - j_3 - 1 \atop -2j_1, -j_1 - j_2 - m_3} ; 1 \right). \qquad (12.3.9)
\end{aligned}$$

The symmetries of the $3j$-coefficient can be described through the corresponding *Regge array* [30]:

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = R_{3j} \begin{bmatrix} -j_1+j_2+j_3 & j_1-j_2+j_3 & j_1+j_2-j_3 \\ j_1-m_1 & j_2-m_2 & j_3-m_3 \\ j_1+m_1 & j_2+m_2 & j_3+m_3 \end{bmatrix}.$$

In this 3×3 array, all entries are nonnegative integers such that for each row and each column the sum of the entries equals $J = j_1 + j_2 + j_3$; conversely, every 3×3 array with nonnegative integers such that all row and column sums are the same, corresponds to a Regge array or a $3j$-coefficient. The symmetries are easy to describe in terms of the Regge array. They generate a group of 72 symmetries, described as follows:

- The Regge array is invariant under transposition.
- Under permutation of the rows (resp. columns), the Regge array remains invariant up to a sign. For cyclic permutations this sign is $+1$; for non-cyclic permutations this sign is $(-1)^J$.

Observe that for certain special values of the arguments, the single sum expression in equation (12.3.8) reduces to a single term. This is the case, e.g., when $m_1 = j_1$. Many such *closed form expressions* (i.e. without a summation expression) are listed in [38].

The coupled and uncoupled basis vectors in (12.3.3) are orthonormal bases for $D_{j_1} \otimes D_{j_2}$. So the matrix relating these two bases in (12.3.3) is orthogonal. This implies that the Clebsch–Gordan coefficients satisfy the following orthogonality relations:

$$\sum_{m_1,m_2} C^{j_1,j_2,j}_{m_1,m_2,m} C^{j_1,j_2,j'}_{m_1,m_2,m'} = \delta_{j,j'}\,\delta_{m,m'}\,, \tag{12.3.10a}$$

$$\sum_{j,m} C^{j_1,j_2,j}_{m_1,m_2,m} C^{j_1,j_2,j}_{m'_1,m'_2,m} = \delta_{m_1,m'_1}\,\delta_{m_2,m'_2}\,. \tag{12.3.10b}$$

These relations can also be expressed by means of $3j$-coefficients, e.g.,

$$\sum_{m_1,m_2} (2j_3+1) \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j'_3 \\ m_1 & m_2 & m'_3 \end{pmatrix} = \delta_{j_3,j'_3}\,\delta_{m_3,m'_3}\,.$$

The orthogonality relations for $\mathfrak{su}(2)$ Clebsch–Gordan coefficients or $3j$-coefficients are actually related to the (discrete) orthogonality of Hahn polynomials. Consider the expression (12.3.7), and let us write

$$N = 2j_1,\ x = j_1 - m_1,\ n = j_1 + j_2 - j,\ \alpha = m - j_1 - j_2 - 1,\ \beta = -j_1 - j_2 - m - 1.$$

Suppose j_1, j_2 and m are fixed numbers, with $j_2 - j_1 \geq |m|$. Then m_1 can vary between $-j_1$ and j_1, and j can vary between $j_2 - j_1$ and $j_2 + j_1$. In terms of the new variables, this means: N is a fixed nonnegative integer, α and β are fixed (with $\alpha,\beta \leq -N-1$); the quantities x and n are nonnegative integers with $0 \leq x \leq N$ and $0 \leq n \leq N$. The ${}_3F_2$ series appearing in (12.3.7) is then of the following form:

$$Q_n(x;\alpha,\beta,N) = {}_3F_2\left(\begin{matrix} -x, -n, n+\alpha+\beta+1 \\ -N, \alpha+1 \end{matrix}; 1\right).$$

Herein, $Q_n(x;\alpha,\beta,N)$ is the *Hahn polynomial* of degree n in the variable x, see [22, §9.5].

Interestingly, the orthogonality (12.3.10a) of $\mathfrak{su}(2)$ Clebsch–Gordan coefficients (or $3j$-coefficients) is equivalent to the orthogonality [22, (9.5.2)] of Hahn polynomials. In a similar way, one can verify that the orthogonality relation (12.3.10b) is equivalent to the orthogonality relation [22, (9.6.2)] of dual Hahn polynomials. The relation between Hahn polynomials and $\mathfrak{su}(2)$ $3j$-coefficients was known to some people, but appeared explicitly only in 1981 [24]. There, the relationship is established in the context of representations of the Lie group SU(2) rather than in terms of representations of the Lie algebra $\mathfrak{su}(2)$, as here.

To conclude this section, let us mention that the action of J_+ or J_- on (12.3.3) yields certain recurrence relations for Clebsch–Gordan coefficients or $3j$-coefficients. Appropriately combined recurrence relations then lead to the classical 3-term recurrence relation of Hahn or dual Hahn polynomials [22, Sections 9.5, 9.6].

12.4 Racah coefficients and $6j$-coefficients

Consider the tensor product of three irreducible unitary representations of $\mathfrak{su}(2)$,

$$D_{j_1} \otimes D_{j_2} \otimes D_{j_3} = (D_{j_1} \otimes D_{j_2}) \otimes D_{j_3} = D_{j_1} \otimes (D_{j_2} \otimes D_{j_3}). \qquad (12.4.1)$$

Clearly, a basis for this tensor product is given by $e^{(j_1)}_{m_1} \otimes e^{(j_2)}_{m_2} \otimes e^{(j_3)}_{m_3}$, where m_i is a projection of j_i. This is the *uncoupled basis*. In order to decompose the actual tensor product $D_{j_1} \otimes D_{j_2} \otimes D_{j_3}$ into irreducible $\mathfrak{su}(2)$ representations, one can proceed in two ways. First, decompose $D_{j_1} \otimes D_{j_2}$ into irreducibles, say $\oplus D_{j_{12}}$, and then decompose each tensor product $D_{j_{12}} \otimes D_{j_3}$ into irreducibles D_j. Secondly, decompose $D_{j_2} \otimes D_{j_3}$ into irreducibles, say $\oplus D_{j_{23}}$, and then decompose each tensor product $D_{j_1} \otimes D_{j_{23}}$ into irreducibles D_j. So one can immediately define two sets of orthonormal basis vectors for the irreducible components of (12.4.1), corresponding to these two coupling schemes:

$$\begin{aligned} e_m^{((j_1j_2)j_{12}j_3)j} &= \sum_{m_{12},m_3} C^{j_{12},j_3,j}_{m_{12},m_3,m} e^{(j_1j_2)j_{12}}_{m_{12}} \otimes e^{(j_3)}_{m_3} \qquad (12.4.2a) \\ &= \sum_{\substack{m_1,m_2,m_3 \\ m_1+m_2+m_3=m}} C^{j_1,j_2,j_{12}}_{m_1,m_2,m_{12}} C^{j_{12},j_3,j}_{m_{12},m_3,m}\, e^{(j_1)}_{m_1} \otimes e^{(j_2)}_{m_2} \otimes e^{(j_3)}_{m_3}, \end{aligned}$$

and

$$\begin{aligned} e_m^{(j_1(j_2j_3)j_{23})j} &= \sum_{m_1,m_{23}} C^{j_1,j_{23},j}_{m_1,m_{23},m} e^{(j_1)}_{m_1} \otimes e^{(j_2j_3)j_{23}}_{m_{23}} \qquad (12.4.2b) \\ &= \sum_{\substack{m_1,m_2,m_3 \\ m_1+m_2+m_3=m}} C^{j_2,j_3,j_{23}}_{m_2,m_3,m_{23}} C^{j_1,j_{23},j}_{m_1,m_{23},m}\, e^{(j_1)}_{m_1} \otimes e^{(j_2)}_{m_2} \otimes e^{(j_3)}_{m_3}. \end{aligned}$$

Let us denote the matrix transforming the basis (12.4.2a) into (12.4.2b) by U. Its matrix elements are given by $\langle e_m^{(j_1(j_2j_3)j_{23})j}, e_{m'}^{((j_1j_2)j_{12}j_3)j'}\rangle$. From the action of the $\mathfrak{su}(2)$ Casimir operator C and of the $\mathfrak{su}(2)$ diagonal operator J_0, it is easy to see that this element is zero if $j' \neq j$ and if $m' \neq m$. Furthermore, by the action of J_+ one verifies that this element is independent of m.

So one can write

$$\langle e_m^{(j_1(j_2j_3)j_{23})j}, e_{m'}^{((j_1j_2)j_{12}j_3)j'} \rangle = \delta_{j,j'}\,\delta_{m,m'}\,U_{j_3,j,j_{23}}^{j_1,j_2,j_{12}}. \tag{12.4.3}$$

The coefficients $U_{j_3,j,j_{23}}^{j_1,j_2,j_{12}}$ are called the *Racah coefficients*. So, we can write

$$e_m^{((j_1j_2)j_{12}j_3)j} = \sum_{j_{23}} U_{j_3,j,j_{23}}^{j_1,j_2,j_{12}}\, e_m^{(j_1(j_2j_3)j_{23})j}, \tag{12.4.4a}$$

and vice versa, since U is an orthogonal matrix,

$$e_m^{(j_1(j_2j_3)j_{23})j} = \sum_{j_{12}} U_{j_3,j,j_{23}}^{j_1,j_2,j_{12}}\, e_m^{((j_1j_2)j_{12}j_3)j}. \tag{12.4.4b}$$

The orthogonality of the matrix is also expressed by:

$$\sum_{j_{12}} U_{j_3,j,j_{23}}^{j_1,j_2,j_{12}}\, U_{j_3,j,j'_{23}}^{j_1,j_2,j_{12}} = \delta_{j_{23},j'_{23}}, \qquad \sum_{j_{23}} U_{j_3,j,j_{23}}^{j_1,j_2,j_{12}}\, U_{j_3,j,j_{23}}^{j_1,j_2,j'_{12}} = \delta_{j_{12},j'_{12}}.$$

An expression for the Racah coefficient follows from (12.4.3), (12.4.2a) and (12.4.2b):

$$U_{j_3,j,j_{23}}^{j_1,j_2,j_{12}} = \sum_{\substack{m_1,m_2,m_3\\ m_1+m_2+m_3=m}} C_{m_1,m_2,m_{12}}^{j_1,j_2,j_{12}}\, C_{m_{12},m_3,m}^{j_{12},j_3,j}\, C_{m_2,m_3,m_{23}}^{j_2,j_3,j_{23}}\, C_{m_1,m_{23},m}^{j_1,j_{23},j}. \tag{12.4.5}$$

Herein, m is an arbitrary but fixed projection of j. The sum is over m_1, m_2 and m_3 such that $m_1 + m_2 + m_3 = m$; m_{12} stands for $m_1 + m_2$ and m_{23} for $m_2 + m_3$. So this is a double sum over the product of four Clebsch–Gordan coefficients. This is clearly a rather complicated object; Racah was the first to simplify this expression by various summation manipulations and to rewrite it as a single sum [27].

The Racah coefficient has a number of symmetries that can be deduced from symmetry properties of Clebsch–Gordan coefficients. In this context, it is appropriate to introduce the so-called $6j$*-coefficient*. Wigner was the first to introduce the $6j$-coefficient in his Princeton Lectures (1940), published much later in [43]. They take the form

$$\begin{Bmatrix} a & b & c\\ d & e & f\end{Bmatrix} := (-1)^{a+b+d+e}\,\frac{U_{d,e,f}^{a,b,c}}{\sqrt{(2c+1)(2f+1)}},$$

where (a, b, c), (d, e, c), (d, b, f) and (a, e, f) are triads. Then the $6j$-coefficient is invariant under any permutation of its columns, or under the interchange of the upper and lower arguments in each of any two columns.

Even more, one can also use the Regge symmetries of the Clebsch–Gordan coefficients, and obtain similar symmetries for the $6j$-coefficient. In order to describe these, let the *Regge array* for the $6j$-coefficient be defined as the 3×4 array [31]

$$\begin{Bmatrix} a & b & c\\ d & e & f\end{Bmatrix} = R_{6j}\begin{bmatrix} d+e-c & b+d-f & a+e-f & a+b-c\\ d+f-b & c+d-e & a+c-b & a+f-e\\ e+f-a & b+c-a & c+e-d & b+f-d\end{bmatrix}.$$

Then the value of the Regge array is invariant under any permutation of its rows or columns. Note that the arguments of the Regge array are such that all entries are nonnegative integers, and the differences between corresponding elements of rows (resp. columns) are constant. Conversely, every 3×4 array of nonnegative integers with this property corresponds to a Regge array, or a $6j$-coefficient.

As mentioned, Racah managed to obtain a single sum expression for the Racah coefficient (or $6j$-coefficient). A simple method to obtain this single sum expression from (12.4.5) is outlined in [39] or in [37]. The final result is:

Proposition 12.4.1 *Let $a, b, c, d, e, f \in \frac{1}{2}\mathbb{N}$, where (a, b, c), (d, e, c), (d, b, f) and (a, e, f) are triads. Then the $6j$-coefficient is given by*

$$\begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix} = \Delta(a,b,c)\,\Delta(c,d,e)\,\Delta(a,e,f)\,\Delta(b,d,f) \times \sum_k \frac{(-1)^k\,(1+k)!}{(k-t_1)!\,(k-t_2)!\,(k-t_3)!\,(k-t_4)!\,(c_1-k)!\,(c_2-k)!\,(c_3-k)!}. \quad (12.4.6)$$

Herein, the t_i correspond to the four triad sums ($t_1 = a+b+c$, $t_2 = d+e+c$, $t_3 = d+b+f$, $t_4 = a+e+f$) and the c_i correspond to the sums of two columns ($c_1 = a+d+b+e$, $c_2 = a+d+c+f$, $c_3 = b+e+c+f$). The sum is over all integer k-values such that all factorials assume nonnegative arguments, and Δ is defined by (12.3.5).

By replacement of the summation variable in (12.4.6), one can (under certain assumptions) rewrite the sum in terms of a ${}_4F_3$ series:

$$\begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix} = \frac{(-1)^{b+c+e+f}\,\Delta(a,b,c)\,\Delta(c,d,e)\,\Delta(a,e,f)\,\Delta(b,d,f)}{(b+c-a)!\,(c-d+e)!\,(e+f-a)!\,(b-d+f)!} \times \frac{(1+b+c+e+f)!}{(a+d-b-e)!\,(a+d-c-f)!} \times {}_4F_3\left({a-b-c, d-b-f, a-e-f, d-c-e \atop -b-e-c-f-1, a+d-b-e+1, a+d-c-f+1} ; 1 \right). \quad (12.4.7)$$

This expression is valid for $a+d \geq b+e$ and $a+d \geq c+f$ (which can always be assumed after applying a symmetry corresponding to a permutation of columns of the $6j$-coefficient). The ${}_4F_3$ in (12.4.7) is a terminating balanced ${}_4F_3$ of unit argument; for such series there exist transformation formulas due to Whipple [41], [4, Theorem 3.3.3]. This allows the relation between $6j$-coefficients and ${}_4F_3$ series to be written in various forms, see [38]. One of these forms is

$$\begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix} = (-1)^{b+c+e+f}\,\frac{(2b)!\,(b+c-e+f)!\,(b+c+e+f+1)!}{\nabla(b,a,c)\,\nabla(c,d,e)\,\nabla(f,a,e)\,\nabla(b,d,f)} \times {}_4F_3\left({a-b-c, d-b-f, -a-b-c-1, -b-d-f-1 \atop -2b, -b-c+e-f, -b-c-e-f-1} ; 1 \right), \quad (12.4.8)$$

where

$$\nabla(a,b,c) = \sqrt{(a+b-c)!\,(a-b+c)!\,(a+b+c+1)!\,/\,(-a+b+c)!}\,. \quad (12.4.9)$$

Expression (12.4.8) is valid for all possible arguments of the $6j$-coefficient, provided of course that (a,b,c), (d,e,c), (d,b,f) and (a,e,f) are triads.

Let us rewrite:

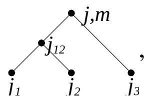
$$n = -a + b + c, \quad x = b - d + f, \quad \alpha \equiv -N - 1 = -1 - b - c + e - f,$$
$$\beta = -1 + f - b - c - e, \quad \gamma = -2b - 1, \quad \delta = -2f - 1.$$

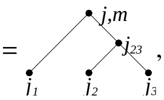
Let b, c, e and f be fixed numbers (parameters), with $e - f \geq |b - c|$ and $e - c \geq |b - f|$. Then a and d can be thought of as variables, with a varying between $e - f$ and $b + c$, and d running from $e - c$ to $b + f$. In terms of the new parameters/variables, this means that N is a fixed nonnegative integer parameter, and x and n are nonnegative integer variables with $0 \leq x \leq N$ and $0 \leq n \leq N$. The ${}_4F_3$-series of (12.4.8) is of the following form:

$$R_n(\lambda(x);\alpha,\beta,\gamma,\delta) := {}_4F_3\left(\begin{matrix}-n, n+\alpha+\beta+1, -x, x+\gamma+\delta+1\\ \alpha+1, \beta+\delta+1, \gamma+1\end{matrix}; 1\right), \quad \lambda(x) := x(x+\gamma+\delta+1).$$

This is the *Racah polynomial* $R_n(\lambda(x)) \equiv R_n(\lambda(x);\alpha,\beta,\gamma,\delta)$, see [22, (9.2.1)]. The orthogonality of the Racah coefficients – appropriately rewritten – is equivalent to the orthogonality relation [22, (9.2.2)] of Racah polynomials. The orthogonality of Racah coefficients is of historical importance: it motivated J. A. Wilson to introduce Racah polynomials in his Ph.D. thesis [44] (1978). Soon afterwards, this led Askey and Wilson to q-Racah polynomials and Askey–Wilson polynomials [5, 6].

The coupled vectors (12.4.2a), (12.4.2b) in $D_{j_1} \otimes D_{j_2} \otimes D_{j_3}$, related by Racah coefficients, are examples of *binary coupling schemes* [9, 37]. In an obvious notation, this reads:

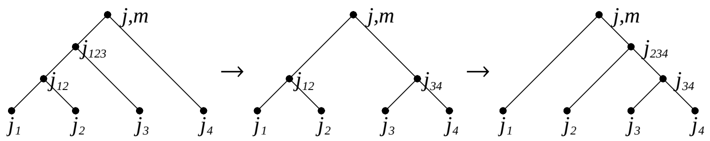
$e_m^{((j_1j_2)j_{12}j_3)j} =$ 

, $\quad e_m^{(j_1(j_2j_3)j_{23})j} =$ 

, (12.4.10)

and to express one type in terms of the other, one uses Racah coefficients, see (12.4.4a) or (12.4.4b).

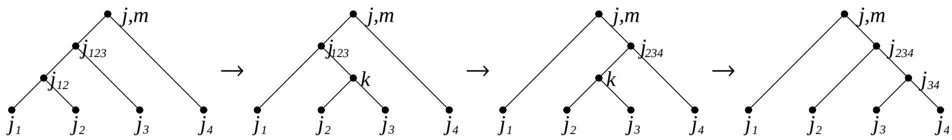
Consider now the tensor product of four $\mathfrak{su}(2)$ representations $D_{j_1} \otimes D_{j_2} \otimes D_{j_3} \otimes D_{j_4}$. Using (12.4.4a) twice, according to the order



one has

$$e_m^{(((j_1j_2)j_{12}j_3)j_{123}j_4)j} = \sum_{j_{34},j_{234}} U_{j_4,j,j_{34}}^{j_{12},j_3,j_{123}}\, U_{j_{34},j,j_{234}}^{j_1,j_2,j_{12}}\, e_m^{(j_1(j_2(j_3j_4)j_{34})j_{234})j}. \tag{12.4.11}$$

Alternatively, one can use (12.4.4a) three times according to a different order:



Then

$$e_m^{(((j_1 j_2) j_{12} j_3) j_{123} j_4) j} = \sum_{k, j_{234}, j_{34}} U^{j_1, j_2, j_{12}}_{j_3, j_{123}, k} \, U^{j_1, k, j_{123}}_{j_4, j, j_{234}} \, U^{j_2, j_3, k}_{j_4, j_{234}, j_{34}} \, e_m^{(j_1 (j_2 (j_3 j_4) j_{34}) j_{234}) j}. \qquad (12.4.12)$$

Comparison of (12.4.11) with (12.4.12) yields [7, 14]:

Theorem 12.4.2 *The Racah coefficients of* $\mathfrak{su}(2)$ *satisfy the following identity, known as the* Biedenharn–Elliott identity*:*

$$U^{j_1, j_2, j_{12}}_{j_{34}, j, j_{234}} \, U^{j_{12}, j_3, j_{123}}_{j_4, j, j_{34}} = \sum_k U^{j_1, j_2, j_{12}}_{j_3, j_{123}, k} \, U^{j_1, k, j_{123}}_{j_4, j, j_{234}} \, U^{j_2, j_3, k}_{j_4, j_{234}, j_{34}}.$$

In terms of $6j$*-coefficients, this can be rewritten as:*

$$\sum_x (-1)^{J+x}(2x+1) \begin{Bmatrix} a & b & x \\ c & d & p \end{Bmatrix} \begin{Bmatrix} c & d & x \\ e & f & q \end{Bmatrix} \begin{Bmatrix} e & f & x \\ b & a & r \end{Bmatrix} = \begin{Bmatrix} p & q & r \\ e & a & d \end{Bmatrix} \begin{Bmatrix} p & q & r \\ f & b & c \end{Bmatrix},$$

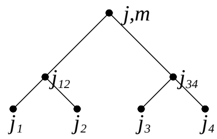
where $J = a+b+c+d+e+f+p+q+r$*, and all labels are representation labels (elements of* $\frac{1}{2}\mathbb{N}$*). The sum is over all* x *(in steps of 1) with* $\max(|a-b|, |c-d|, |e-f|) \leq x \leq \min(a+b, c+d, e+f)$.

The Biedenharn–Elliott identity can be considered as a master identity for special functions, since it gives rise to many known identities for orthogonal polynomials as limit cases (sometimes after analytic continuation); see e.g. [37, §5.5].

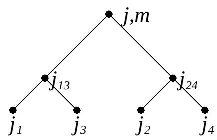
12.5 The $9j$-coefficient

Consider again the tensor product of four $\mathfrak{su}(2)$ representations $D_{j_1} \otimes D_{j_2} \otimes D_{j_3} \otimes D_{j_4}$, and the vectors corresponding to the following couplings:

$$v = e_m^{((j_1 j_2) j_{12} (j_3 j_4) j_{34}) j} = \qquad (12.5.1a)$$



$$v' = e_m^{((j_1 j_3) j_{13} (j_2 j_4) j_{24}) j} = \qquad (12.5.1b)$$



In terms of Clebsch–Gordan coefficients, these read:

$$v = \sum_{\substack{m_1, m_2, m_3, m_4 \\ j_{12}, j_{34}}} C^{j_1, j_2, j_{12}}_{m_1, m_2, m_{12}} C^{j_3, j_4, j_{34}}_{m_3, m_4, m_{34}} C^{j_{12}, j_{34}, j}_{m_{12}, m_{34}, m} \, e^{j_1}_{m_1} \otimes e^{j_2}_{m_2} \otimes e^{j_3}_{m_3} \otimes e^{j_4}_{m_4}, \qquad (12.5.2a)$$

$$v' = \sum_{\substack{m_1, m_2, m_3, m_4 \\ j_{13}, j_{24}}} C^{j_1, j_3, j_{13}}_{m_1, m_3, m_{13}} C^{j_2, j_4, j_{24}}_{m_2, m_4, m_{24}} C^{j_{13}, j_{24}, j}_{m_{13}, m_{24}, m} \, e^{j_1}_{m_1} \otimes e^{j_2}_{m_2} \otimes e^{j_3}_{m_3} \otimes e^{j_4}_{m_4}. \qquad (12.5.2b)$$

The matrix relating the basis (12.5.1a) and (12.5.1b) consists of the $9j$*-coefficients*. More precisely, one defines [43]

$$\begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j_4 & j_{34} \\ j_{13} & j_{24} & j \end{Bmatrix} := \frac{\langle e_m^{((j_1 j_2) j_{12} (j_3 j_4) j_{34}) j}, e_m^{((j_1 j_3) j_{13} (j_2 j_4) j_{24}) j} \rangle}{\sqrt{(2j_{12}+1)(2j_{34}+1)(2j_{13}+1)(2j_{24}+1)}}\,. \tag{12.5.3}$$

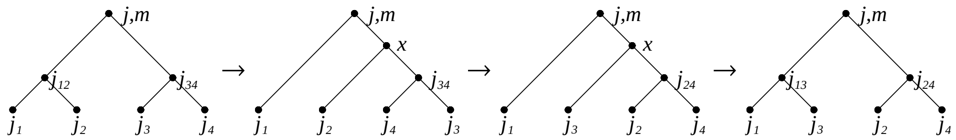
Since the two bases (12.5.2a), (12.5.2b) are orthonormal, the corresponding transformation matrix is orthogonal, and this leads to orthogonality relations of $9j$-coefficients:

$$\sum_{j_{12},j_{34}} (2j_{12}+1)(2j_{34}+1) \begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j_4 & j_{34} \\ j_{13} & j_{24} & j \end{Bmatrix} \begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j_4 & j_{34} \\ j'_{13} & j'_{24} & j \end{Bmatrix} = \frac{\delta_{j_{13},j'_{13}} \delta_{j_{24},j'_{24}}}{(2j_{13}+1)(2j_{24}+1)}\,. \tag{12.5.4}$$

From (12.5.2a), (12.5.2b) it is clear that the $9j$-coefficient can be written as a multiple sum over the product of six Clebsch–Gordan coefficients. Rewriting this in terms of $3j$-coefficients yields, in an appropriate notation:

$$\begin{Bmatrix} a & b & c \\ d & e & f \\ g & h & j \end{Bmatrix} = \sum_{\text{all } m} \begin{pmatrix} a & b & c \\ m_a & m_b & m_c \end{pmatrix} \begin{pmatrix} d & e & f \\ m_d & m_e & m_f \end{pmatrix} \begin{pmatrix} g & h & j \\ m_g & m_h & m_j \end{pmatrix} \begin{pmatrix} a & d & g \\ m_a & m_d & m_g \end{pmatrix} \begin{pmatrix} b & e & h \\ m_b & m_e & m_h \end{pmatrix} \begin{pmatrix} c & f & j \\ m_c & m_f & m_j \end{pmatrix}. \tag{12.5.5}$$

Alternatively, just as in (12.4.12), the two vectors (12.5.1a) and (12.5.1b) can be related through the product of Racah coefficients, according to



This leads to a single sum over the product of three Racah coefficients for the $9j$-coefficient. When rewritten in terms of $6j$-coefficients, this gives:

$$\begin{Bmatrix} a & b & c \\ d & e & f \\ g & h & j \end{Bmatrix} = \sum_x (-1)^{2x}(2x+1) \begin{Bmatrix} a & b & c \\ f & j & x \end{Bmatrix} \begin{Bmatrix} d & e & f \\ b & x & h \end{Bmatrix} \begin{Bmatrix} g & h & j \\ x & a & d \end{Bmatrix}. \tag{12.5.6}$$

From symmetry properties of $3j$ and $6j$-coefficients and by (12.5.5) and (12.5.6), one obtains the symmetries of $9j$-coefficients. The $9j$-coefficient is, up to a sign, invariant with respect to permutations of its columns, permutations of its rows, and under transposition. Even permutations (and transposition) leave the $9j$-coefficient unchanged, whereas odd permutations introduce a factor $(-1)^J$, where J is the sum of all nine arguments of the $9j$-coefficient.

Just as the orthogonality of $3j$ and $6j$-coefficients can be related to the orthogonality of a discrete polynomial, this can be done for the $9j$-coefficient. For this purpose, write the arguments as

$$\begin{Bmatrix} a & b & a+b-x \\ c & d & c+d-y \\ a+c-m & b+d-n & a+b+c+d-N \end{Bmatrix},$$

where m, n, x and y take nonnegative integer values with $m+n \le N$ and $x+y \le N$. This expression can be written as a factor times $R_{m,n}(x,y) \equiv R_{m,n}(x,y;\alpha,\beta,\gamma,\delta,N)$, where $R_{m,n}(x,y)$ is a polynomial of degree $N-n$ in $\lambda(x) = x(x+\alpha+\beta+1)$ and of degree $N-m$ in $\mu(y) = y(y+\gamma+\delta+1)$, and where $(\alpha,\beta,\gamma,\delta)$ is equal to $(-2a-1,-2b-1,-2c-1,-2d-1)$ [36] (see also [28] for a different approach). The orthogonality relation (12.5.4) then reads:

$$\sum_{x=0}^{N}\sum_{y=0}^{N-x} w(x,y)R_{m,n}(x,y)R_{m',n'}(x,y) = \delta_{m,m'}\delta_{n,n'}h_{m,n}, \tag{12.5.7}$$

where $w(x,y)$ is some expression in x, y and the five parameters $\alpha,\beta,\gamma,\delta,N$, and $h_{m,n}$ is an expression involving m, n and the same five parameters. So the $R_{m,n}(x,y)$ are a discrete version of orthogonal polynomials on the triangle (see §2.3.3 in Chapter 2 for the continuous case). In terms of the common convention for Pochhammer symbols, the weight function is

$$\begin{aligned} w(x,y) = {} & \frac{(-1)^{x+y}(-N)_{x+y}}{x!y!(\alpha+\beta+\gamma+\delta+N+3)_{x+y}} \, \frac{(\alpha+1)_x(\delta+1)_y}{(\beta+1)_x(\gamma+1)_y} \\ & \times \frac{(\alpha+\beta+1)_x(\alpha+\beta+2)_{2x}}{(\alpha+\beta+1)_{2x}(\alpha+\beta+N+2)_{x-y}} \, \frac{(\gamma+\delta+1)_y(\gamma+\delta+2)_{2y}}{(\gamma+\delta+1)_{2y}(\gamma+\delta+N+2)_{y-x}}; \end{aligned}$$

the expression for $h_{m,n}$ is more complicated [36].

Although (12.5.7) looks like a neat two-variable extension of the orthogonality of Hahn and Racah polynomials, the setback is that the known forms of the expression $R_{m,n}(x,y)$ are complicated. Although the transposition symmetry for the $9j$-coefficient implies a duality between (m,n) and (x,y), none of the known expressions for $R_{m,n}(x,y)$ [36] displays this duality explicitly (see also [17]). Furthermore, no difference operators are known for which $R_{m,n}(x,y)$ are eigenfunctions. Ideally, one would expect a double sum expression of hypergeometric type for $R_{m,n}(x,y)$. So far, such an expression is not available. This is because all known expressions of $9j$-coefficients are rather involved. One such form is obtained as follows: one starts from a formula similar to (12.5.5) but expressing the product of one $3j$-coefficient with the $9j$-coefficient as an essentially double sum over the product of five $3j$-coefficients. Then, making appropriate choices for the projection numbers appearing in these $3j$-coefficients (i.e. choices that reduce some $3j$'s to closed forms), this gives for the $9j$-coefficient an expression as a double sum over the product of three $3j$-coefficients (times factors). So using (12.3.8), this reduces to a fivefold summation expression. It is not too difficult to see that one of these summations can be performed due to Vandermonde's theorem [4, Corollary 2.2.3], leaving a complicated fourfold summation expression for the $9j$-coefficient. This fourfold expression can be found in some books, e.g. in [38] and in [19].

Ališauskas & Jucys [3] went on manipulating this fourfold sum expression, changing summation variables in several ways, and by this tour de force they managed to perform one further sum (again using Vandermonde's theorem) and finally ended up with a *triple sum series* for the $9j$-coefficient [3]. Their method was later reproduced in the book by Jucys & Bandzaĭtis [19]. Much later, Rosengren [32] deduced Ališauskas's triple sum series in a simpler way: starting from the single sum over the product of three $6j$-coefficients (12.5.6), expressing these coefficients as single sums through (12.4.6) or an alternative single sum, then manipulating the summation variables such that one summation can be performed using a summation formula for a very well poised ${}_4F_3(-1)$ series [4, Corollary 3.5.3] following from a limit of Dougall's formula [4, Theorem 3.5.1], and thus ending up with a triple sum series.

Ališauskas [1, 2] later derived several triple sum series, also for the q-case. We present one form here:

$$\begin{Bmatrix} a & b & c \\ d & e & f \\ g & h & j \end{Bmatrix} = (-1)^{c+f-j} \frac{\nabla(d,a,g)\nabla(b,e,h)\nabla(j,g,h)}{\nabla(d,e,f)\nabla(b,a,c)\nabla(j,c,f)} \times \sum_{x,y,z} X_x Y_y Z_z \frac{(a+d-h+j-y-z)!}{(-b+d-f+h+x+y)!(-a+b-f+j+x+z)!} \tag{12.5.8}$$

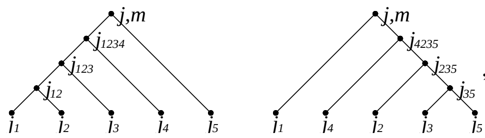
with

$$X_x = (-1)^x \frac{(2f-x)!\,(d+e-f+x)!\,(c-f+j+x)!}{x!\,(e+f-d-x)!\,(c+f-j-x)!},$$
$$Y_y = (-1)^y \frac{(-b+e+h+y)!\,g+h-j+y)!}{y!\,(2h+1+y)!\,(b+e-h-y)!\,(g-h+j-y)!},$$
$$Z_z = (-1)^z \frac{(2a-z)!\,(-a+b+c+z)!}{z!\,(a+d+g+1-z)!\,(a+d-g-z)!\,(a-b+c-z)!}.$$

Herein $\nabla(a,b,c)$ has been defined in (12.4.9) and the sum in (12.5.8) is over all integer values of x, y and z such that all factorials in the summation are nonnegative. Srinivasa Rao [33] rewrote this expression as a triple hypergeometric series (generalizations of Appell's series, see Chapter 3), and deduced some identities from this for so-called stretched $9j$-coefficients [34].

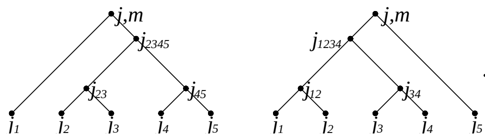
12.6 Beyond $9j$: graphical methods

More generally, one can consider the tensor product of $n+1$ irreducible unitary representations of $\mathfrak{su}(2)$ and their related $3nj$-coefficients. In the tensor product $V = D_{j_1} \otimes D_{j_2} \otimes \cdots \otimes D_{j_{n+1}}$, a basis (the uncoupled basis) is given by $e^{(j_1)}_{m_1} \otimes e^{(j_2)}_{m_2} \otimes \cdots \otimes e^{(j_{n+1})}_{m_{n+1}}$, where m_i is a projection of j_i. Just as in (12.5.1a), coupled basis vectors can be defined by means of *binary coupling schemes*. The idea is a simple extension of the two ways in which the tensor product of three representations can be "coupled", see (12.4.1). Of course, as n increases, the number of ways that representations can be coupled also increases. A *$3nj$-coefficient* is then, as in (12.5.3), proportional to the inner product of two vectors corresponding to different couplings. E.g., when $n = 4$, there are essentially two distinct $12j$-coefficients (that do not reduce to products of $9j$ and/or $6j$-coefficients). The $12j$-coefficient of the first type corresponds to the inner product of the vectors described by



,

and the $12j$-coefficient of the second type to



.

Since $3nj$-coefficients relate two sets of orthonormal basis vectors, they satisfy an orthogonality relation like (12.5.4). For $12j$-coefficients, this orthogonality relation involves a triple sum, see e.g. [38, §10.13]. As far as we know, orthogonality relations for $12j$-coefficients or higher have not explicitly been related to the orthogonality of discrete multivariable polynomials. But there exist other interesting interpretations: for example, $3nj$-coefficients have been identified as connection coefficients between orthogonal polynomials in n variables [25].

From the above example of $12j$-coefficients, it is clear that in general a $3nj$-coefficient is determined by two binary coupling schemes T_1 and T_2 on $n+1$ elements. Just as in the case of the $9j$-coefficient (see the sequence of binary coupling schemes preceding (12.5.6)), one can find a sequence of binary coupling schemes starting with T_1 and ending with T_2, such that two consecutive elements in the sequence are related through an elementary transformation (i.e., a transformation turning the left-hand side of (12.4.10) into its right-hand side). Consequently, as in (12.5.6), this yields an expression of the $3nj$-coefficient as a (multiple) sum over products of $6j$-coefficients. This method is usually referred to as the *method of trees*, see [9, Topic 12]. It involves combinatorial problems (enumeration of all binary coupling schemes), and interesting graph theoretical problems (e.g. finding the shortest sequence to go from T_1 to T_2 by means of elementary transformations), see [16].

The method of trees, as described here briefly, is still quite general: e.g., it could also be applied to $3nj$-coefficients of $\mathfrak{su}(1,1)$ [37]. For $3nj$-coefficients of $\mathfrak{su}(2)$, there exists however a more powerful method, namely that of Jucys graphs. The *Jucys graph* of a $3nj$-coefficient is obtained by "gluing" the $n+1$ leaves of the binary coupling trees T_1 and T_2 together (thereby deleting these leaves as vertices of the resulting graph) and connecting their two roots by an extra edge, thus yielding a cubic graph. On such a cubic graph, various transformations or rules can be applied, which reduce the cubic graph (four basic reduction rules are sufficient, see [15]). Each reduction rule yields a certain contribution to a formula, and in this way new expressions can be obtained for $3nj$-coefficients. This graphical method of Jucys has become an art of its own, leading to magnificent formulas relating sums over products of $3nj$-coefficients; see [20, 38, 12].

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