Reasoning with Rules and Rights: Term-Modal Deontic Logic*

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Abstract

General obligations such as 'every driver has to give way to a driver coming from the right' are central in legal reasoning, but have been mostly overlooked in deontic logic. We claim that a simple extension of Standard Deontic Logic to the predicative level is insufficient to capture general obligations. Instead, we argue for an explicit representation of bearers (and counterparties) of obligations as terms in a quantified deontic logic. To achieve this we develop a term-modal counterpart of Standard Deontic Logic and give a sound and strongly complete axiomatization for it. We go on to show that this logic is not only suitable for capturing reasoning with general obligations, but also with (multital and paucital) Hohfeldian rights relations and rules of rights.

Keywords: term-modal logic, multital rights, directed obligations, personal obligations, quantified deontic logic, Hohfeld

1 Introduction

Deriving specific obligations and permissions from general deontic statements is at the core of normative reasoning. When shopping with her older sister,

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little Mary relies on the rule "taking things from other people without asking is forbidden" to infer that she ought not to steal candy. Visiting her grandma Sophie, Mary learns that she may take the apples that fell from the neighbour's tree into grandma's garden: "one is not allowed to pick apples from one's neighbour's tree, even if they are on a branch that is bending over the fence between the two gardens, but one is allowed to take them once they have fallen into the grass on one's own side of the fence". And when Mary asks if she can have some of the delicious-smelling apple cider that her grandma made, she is told that "children are not allowed to drink apple cider". Although Mary is only five, she immediately infers from this that she is forbidden to drink cider.

We will argue in this paper that deontic logicians have so far failed to adequately capture basic inferences such as these. As deontic logic is concerned with the formal analysis of normative concepts and normative reasoning, this may seem surprising. Importantly, however, the reason is not that the structure of normative discourse escapes logical analysis. The vast literature of the past sixty years proves otherwise, and, in our opinion, legal philosophers are right to consider deontic logic as "an essential tool to understand both the systematic structure of law and its dynamic nature" [31, p. xix]. The reason is rather that deontic logic, even more so than other branches of modal logic, is very much stuck at the propositional level, and that straightforward extensions to the predicative level are insufficient for the representation of general deontic statements. The results we currently have on the logical analysis of normative concepts (obligation, permission, prohibition, rights, duties, responsibility, ...), and of the relations between them, are impressive, but are (with very few exceptions) restricted to what can be analysed at the propositional level.

As far as the logical analysis of legal reasoning is concerned, staying at the propositional level constitutes a serious limitation. Legal arguments involve general rules and principles. The conclusion that Mister Smith ought to appear in court follows from the fact that he was summoned to court as a witness and the legal rule "Everyone who is summoned to court as a witness is obliged to appear in court". When arguing that Mister Jones is not entitled to his share of his late wife's matrimonial estate, a court may refer to the fact that Mister Jones killed his wife and the legal principle "no one shall benefit from his own wrongdoing". Arguments based on such general statements cannot be analysed at the propositional level.[1]

The goal of this paper is to present a new deontic logic, **TMDL**, that allows for the representation of general deontic statements. As the logical analysis of such statements requires quantification, **TMDL** is a first-order logic, but it is not a straightforward extension of some existing propositional deontic logic to the predicative level. In **TMDL**, deontic operators are indexed with terms of the language, and quantification is allowed over these indices. It is thus possible to quantify not only over objects in the domain, but also over the

[1]In legal reasoning, deriving consequences from general statements is not restricted to simple applications of so-called *legal syllogisms*. In view of the legal principle "like cases should be treated alike", it is commonly accepted that *all* legal decisions should be founded on general rules (see, for instance, [7, p. 25]).

deontic operators associated with these objects, which gives the logic a kind of second-order flavour (see also [25]). As we will show, it is precisely this property that enables one to represent legal rules, and thus to reconstruct the arguments underlying legal justification as logically valid arguments.[2]

The logic **TMDL** belongs to the family of term-modal logics as developed by Fitting et al. [9], but, to the best of our knowledge, this is the first application of term-modal logics to deontic reasoning. There are some technical differences as well: the language of **TMDL** contains identity and two kinds of modal operators (with one index and with two indices), the semantics is different (a constant-domain semantics instead of an increasing-domain semantics), and we present a Hilbert-style axiomatisation for **TMDL**, whereas Fitting et al. present sequent and tableau calculi.[3]

The relevance of **TMDL** for legal reasoning is not restricted to the logical reconstruction of legal arguments. General deontic statements also figure in what Hohfeld called *multital rights*—rights of a single person that avail against an indefinite class of others. Sophie's neighbour does not only have a right against Sophie that her apples are not picked from the tree, but she has a similar right against all others that were not given explicit permission to pick the apples. Hohfeldian rights have been analysed within the framework of deontic logic (see, for instance, [22, 34, 29, 6]), but only at the propositional level, and restricted to what Hohfeld called *paucital rights*—rights of a single person availing against a single other person.[4] We will show that **TMDL** leads to a natural treatment of all first-order legal rights that Hohfeld distinguished (claims, privileges, duties and no-claims), both in their paucital and their multital version.

The paper is structured as follows. After providing some background on different kinds of normative statements (Section 2) and on Hohfeldian rights (Section 3), we discuss possible problems of translating these observations to a logic (Section 4). In Section 5, we discuss the term-modal logics of Fitting et al. and present both the syntax and the semantics of **TMDL**. Soundness and strong completeness is proven in Section 6. In Section 7, we show that **TMDL** succeeds in capturing the aspects of legal reasoning laid out in Sections 2 and 3. Section 8 concludes with a short summary of the paper and some suggestions for future work.

2 Legal Statements: Some Distinctions

In the introduction we mentioned that simple extensions to the predicative level of existing deontic logics are not adequate for the representation of general deontic statements. To understand why, we first need to distinguish two kinds of normative statements.

[2] Within a logical approach to legal reasoning, logical validity is seen as a necessary requirement for the acceptability of legal arguments, see [7, p. 25]).

[3] [1] also uses a Hilbert-style axiomatisation.

[4] A notable exception to the limitation to paucital rights is [29].

2.1 Situational versus Personal Normative Statements

Some normative statements are *situational*, in the sense that a judgement is passed on a situation, and not on an individual. The humanitarian outcry "Children ought not to die like this!" forms an example of a situational obligation—what is meant is not necessarily that the children or others are at fault if the children die, but that the situation ought not to be such that they die [21, p. 8].

The obligations that little Mary derives are of a different kind. The statements "I, Mary, ought not to steal" and "I, Mary, ought not to drink apple cider" hold *for Mary*, and Mary is at fault when she (knowingly) violates them. The person for whom an obligation or permission holds is called the *bearer*, and obligations (permissions) that are relative to some bearer are called bearer-relative, or *personal*.[5]

An important subcategory of personal obligations are so-called *directed obligations*. These are not only tied to the bearer of the obligation, but also to the *counterparty*—the person to whom the bearer has the obligation. If Sophie and her neighbour Julie come to an agreement that the former will deliver 50 kilograms of apples to the latter in exchange for 5 liters of apple cider, then Sophie has a directed obligation *towards Julie* to deliver 50 kilograms of apples to her by the agreed time. In turn, Julie has a directed obligation towards Sophie to give her 5 liters of apple cider in exchange.

Permissions too can be directed. If Sophie and Julie agree that the former may put a 3 meter high fence around her orchard, then Sophie has a directed permission towards Julie to build this fence. This does not exclude that Sophie may be forbidden towards Emilie, another of Sophie's neighbours, to erect this fence. Indeed, it may in general be forbidden to build a fence higher than 1 meter without the consent of each of the neighbours, and Emilie may object to such a fence.

In this paper, permissions and obligations will be treated as interdefinable concepts. In standard deontic logic, situational permission is the dual of situational obligation. Therefore, some proposition φ is permitted iff it is not obligatory that not φ.[6] We follow this pattern for undirected bearer-relative permission and obligation [15]. Thus, φ is permitted for a iff it is not obligatory for a that not φ. We will also treat directed permission as the dual of directed obligation, meaning that a has a permission towards b that φ iff a does not have an obligation towards b that not φ.[7]

[5] There is no consensus in the literature on the terminology. Therefore our use of terminology might differ slightly from that found in other work.

[6] We have in mind what is sometimes called 'weak' permission [15, p. 464]. Logics of other notions of permission have been developed in recent work, cf. [14]. We leave the investigation of such notions in combination with term-modal logic for future work.

[7] Herrestad and Krogh claim that they have "few clear and convincing intuitions" regarding directed permissions and argue that directed permissions are not the duals of directed obligations [15, p. 504].

2.2 General Deontic Statements

General deontic statements involve quantifying over the bearers (and counterparties) of personal obligations and permissions. Because of this, simple extensions to the predicative level are insufficient for their representation.

Almost none of the available systems allow for the representation of general deontic statements. If they do, they can at best handle those that we will call *categorical*—deontic statements that apply, without any conditions or restrictions, to all individuals. An example of such a categorical statement is the biblical commandment 'Thou shalt not kill'. This commandment could be rephrased as 'Everyone has an obligation not to kill' or 'Everybody is the bearer of an obligation not to kill'. Other examples are "everybody is forbidden to take things from another person without asking", "nobody is permitted to smoke in public places", and "everybody is permitted to walk in Central Park".

Although some legal rules are categorical in this sense, most contain some condition or restriction. Take for example this article of the Belgian traffic regulations: 'Every driver has to give way to the driver coming from the right, unless he is driving on a roundabout or the driver from the right is coming from a forbidden direction.'[8] This is not a categorical deontic statement, since not everyone has to give way, but only those people who satisfy certain conditions (i.e. they are drivers, they encounter another driver coming from their right, etc.). Other examples are "Only those belonging to the emergency services are allowed to drive with blue lights", "One cannot buy alcohol if one is younger than 16, and one cannot buy spirits if one is younger than 18", and "When sitting in a car, smoking is forbidden, if children younger than 16 are present in the car".

The general deontic statements mentioned above are all examples of quantification over the bearers of undirected personal obligations and permissions. When it comes to directed obligations, we also have quantification over the counterparties and even over both the bearers and counterparties at the same time. An example of the first kind is 'a has an obligation towards all of her employees to pay them their wages'. An example of the second kind is 'all employers have an obligation towards their employees to pay them their wages'. Note that both of these are also examples of general deontic statements that are not categorical.

Several authors have pointed out that reasoning from general rules to specific obligations is often defeasible (see, for instance, [38, 33]). There might be exceptions to a general rule. If the case under consideration is such an exception, then the derivation of the specific obligation must be blocked. In this paper we will bracket this observation. The monotonic logic we develop here may, however, serve as a basis for a logic that allows us to handle the defeasibility of

[8] "Elke bestuurder moet voorrang verlenen aan de bestuurder die van rechts komt, behalve indien hij op een rotonde rijdt of indien de bestuurder die van rechts komt uit een verboden rijrichting komt." Artikel 12.3.1. van het Koninklijk besluit houdende algemeen reglement op de politie van het wegverkeer en van het gebruik van de openbare weg.

legal rules.[9]

3 Hohfeldian rights

There is a strong connection between directed obligations and permissions and the Hohfeldian theory of rights. Hohfeld published two papers on rights relations [17, 18]. In these papers, he sets out an analytical theory of legal rights. The starting point of this theory is the observation that the term 'right' is used to refer to different, and conflicting, legal relations between persons [17, p. 30]. Hohfeld makes these different legal relations explicit and explores the (logical) connections between them. These relations and connections are illustrated in Figure 1.

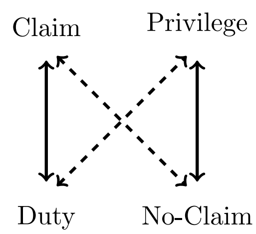

Figure 1: Hohfeldian rights relations

3.1 Hohfeldian rights relations

The first legal relation that Hohfeld identifies is a *claim*. Hohfeld uses both the terms 'claim' and 'right' to refer to this legal relation, but to avoid confusion we will use 'claim' for this specific legal relation, whereas we will use 'right' as a collective noun to refer to all the different legal relations we discuss. Hohfeld does not give a definition of claims, but he does provide the following example as an illustration of a claim.

> If a has contracted to work for b during the ensuing six months, b has an *affirmative* right *in personam* that a shall render such service, as agreed [18, p. 719].[10]

This example shows that claims are *relational*. If a person has a claim, then she will have that claim on someone else.

A claim of a on b always is correlative with, and equivalent to, a duty of b towards a [17]. For example, if a has a claim on b that b shall work for a, then

[9]See [32, 19, 3] for combinations of non-monotonic logic and deontic logic.

[10]In all of our citations we will change the notation used for constants and variables (ranging over persons) to ensure a uniform notation and easy reading experience. We will use $a, b, \ldots$ for individual persons and $x, y, \ldots$ for variables ranging over persons.

b has a corresponding duty toward a that b shall work for a. The correlativity of claims and duties is illustrated by the undashed arrow in Figure 1.

Several authors have identified the Hohfeldian concept of a duty with a directed obligation [15, 13, 28]. The bearer of a directed obligation is identified with the person who has the Hohfeldian duty and the counterparty of a directed obligation is identified with the person towards whom the first person has a duty. We follow this identification of Hohfeldian duties and directed obligations.

Hohfeld identifies *privilege* as a second legal relation that is often called a right, but Hohfeld does not give a definition. An example can illustrate the difference between a claim and a privilege. Suppose that a owns a piece of land. Then a has a claim on b that b does not walk on a's land. In contrast, a is allowed to walk on her own land, meaning that a has a privilege towards b to walk on a's land. In the same way that a duty can be identified with a directed obligation, so can a privilege be identified with the notion of a directed permission.

An important difference between the notion of a privilege and the notion of a claim is that the privilege of a (towards b) does not correspond to a duty on b that a walks on this land. Instead of such a duty, the privilege of a to walk on their own land corresponds to a *no-claim* on b: b does not have a claim on a that a does not walk on her (a's) own land. Hohfeld points out that such a no-claim is the opposite of a claim. This relation of opposites is illustrated in Figure 1 by a dashed arrow.

In the same way that a claim and a no-claim are opposites, a duty is the opposite of a privilege. However, this notion of opposites has a complication: "when it is said that a given privilege is the mere negation of a *duty*, what is meant, of course, is a duty having a content or tenor precisely *opposite* to that of the privilege in question" [17, p. 32]. Thus, if a has a privilege towards b that a walks on a's land, then this is equivalent to a *negation* of a duty of a towards b that a does *not* walk on a's land.

Claims and privileges are not the only legal relations that Hohfeld identifies as being meant when the word 'right' is used. Other such relations are those that Hohfeld calls 'power' and 'immunity'. These rights relations are sometimes called higher order rights, because they concern changing the first order rights relations (claims, privileges, duties and no-claims) [27, pp. 203-204]. We limit the scope of this paper to first order rights. For a recent formal account of higher order rights we refer to Dong and Roy [6] or Markovich [29].

3.2 Paucital and multital rights

Hohfeld also discusses the often made distinction between rights *in rem* and rights *in personam*. He identifies the latter with *paucital* rights, described as follows.

> A paucital right, or claim, (right *in personam*) is either a unique right residing in a person (or group of persons) and availing against a single person (or single group of persons); or else it is one of a

> *few* fundamentally similar, yet separate, rights availing respectively against a few definite persons[11] [18, p. 718].

An example of such a paucital right is the claim (discussed above) that a has on b if a and b have signed a contract stating that b will work for a.

Paucital rights are contrasted with *multital* rights. These multital rights are described as follows.

> A multital right, or claim, (right *in rem*) is always *one* of a large *class* of *fundamentally similar* yet separate rights, actual and potential, residing in a *single* person (or single group of persons) but availing *respectively* against persons constituting a very large and indefinite class of people [18, p. 718].

Examples of such multital rights are a's right that b shall not commit a battery on him or a's right that b shall not enter property owned by a [18, p. 719]. What makes these multital rights (as opposed to paucital rights) is that a does not only have these rights towards b, but towards all, or a very large (and indefinite) class of people.

So far we have only discussed the paucital-multital distinction when applied to claims. However, the distinction also applies to the other rights relations that we have discussed. There are paucital claims, duties, privileges and no-claims and there are multital claims, duties, privileges and no-claims (see for example page 747 of Hohfeld's second paper for multital privileges [18]). Both the paucital and the multital versions of these legal relations stand in the discussed logical relations of opposites and correlatives to each other. So, for example, a multital claim corresponds to a multital duty and a paucital privilege corresponds to a paucital no-claim.

It is important to distinguish multital rights from the class of multital rights to which they belong. Consider the following example.

> If a owns and occupies Whiteacre, not only b but also a great many other persons – not necessarily all persons – are under a duty, e.g., not to enter on a's land. a's right against b is a multital right, or right in rem, for it is simply one of a's class of similar, though separate, rights, actual and potential, against very many persons [18, p. 719].

This example shows that Hohfeld identifies the right in rem with the single claim that a has against b.

However, it is tempting to use the phrase 'right in rem' to refer to what Hohfeld calls the class of similar rights that a has against 'very many persons'. Such an identification is also suggested by the examples of multital rights that Hohfeld gives when he classifies different kinds of multital rights, for example:

> his right that any ordinary person shall not strike him, or that any ordinary person shall not restrain his physical liberty, i.e., "falsely imprison" him; [18, p. 733]

[11] The emphasis in this and further quotations is Hohfeld's own.

In this example, Hohfeld seems to use the word 'right' to refer to a class of multital rights instead of a single multital right. Hohfeld also discusses several authors who seem to use the words 'right in rem' for what Hohfeld calls a class of multital rights [18, pp. 720-733].

We will not answer the question whether 'right in rem' is (or should be) used to refer to multital rights or to classes of multital rights. Nevertheless, we do think that this confusion shows that classes of multital rights play a prominent role in the conceptualisation of rights relations. When one talks of a's right not to be beaten up, one will usually refer to the class of multital rights relations in which a stands to very many persons and not to the specific multital right that a has against some specific b not to be beaten by b. That we interpret a's right not to be beaten up in this way is illustrated by the fact that if b and c both beat up a, we say that both have violated a's right not to be beaten up. If we interpreted a's right not to be beaten up as a specific claim of a towards one specific other person, then we would say that b and c had violated two different rights of a. Thus, what Hohfeld calls classes of multital rights play a prominent role in our conceptualisation of and reasoning with rights.

3.3 Classes of multital rights and rules of rights

We saw above that some rights can be identified with directed obligations and permissions. A natural follow-up question would be whether classes of multital rights can also be framed in terms of directed obligations and permissions. This is possible, but in order to do so we must again quantify over the bearers and counterparties of directed obligations and permissions.

We will illustrate this with two examples. For the first example, we will look at classes of multital privileges. Hohfeld states that a, the owner of a piece of land, "has an indefinite number of legal privileges of entering on the land" [18, p. 746].[12] Thus a has a privilege against others to enter on her own land and those others have corresponding no-claims [18, p. 746]. As we identified directed permission with privilege, we can represent the class of multital privileges of a towards others in a semi-formal language as: 'for all x, if x is not a, then a has the permission towards x that a walks on the land owned by a.' In this first example, we have quantified over the counterparties of a permission to represent a class of multital privileges.

For our second example, we consider a class of multital claims. Take Hohfeld's assertion that somebody, let us call him a, has a "right that any ordinary person shall not strike him" [18, p. 733]. In the Hohfeldian analysis, this is understood as a class of multital claims that a has on any other ordinary person that that person shall not strike a. Since claims correspond to duties and we have identified duties with directed obligations, we can represent this class of

[12]The full quote is "A has an indefinite number of legal privileges of entering on the land, using the land, harming the land, etc" [18, p. 746]. The phrase 'indefinite number of privileges' refers to an indefinite number of privileges to enter on the land, an indefinite number of privileges of using the land, etc. This is illustrated by the fact that Hohfeld later refers to these privileges as "multital, or in rem, 'privilege-no-right' relations" [18, p. 747].

multital claims in a semi-formal language as: 'for all x, if x is not a and x is an ordinary person, then x has an obligation towards a that x does not strike a.' To represent this class of multital claims, we have quantified over the bearers of a directed obligation.

Another way in which people quantify over the bearers and counterparties of the directed obligations and permissions that are used to represent rights is in what Kanger calls 'rules of rights' [24, p. 131]. Here we will not discuss Kanger's formal theory of rights, but only his observation that there are rules that grant rights relation to persons who satisfy certain conditions. These 'rules of rights' are especially prevalent in legal texts. Kanger gives the following example of a rule of rights:

> For every x and y such that x is a pedestrian and y is a motorist who encounters x, it is the case that x has versus y a right of atomic type: claim (...) to the effect that y does not run into x [24, p. 131].

As this example shows, in rules of rights we quantify over both the owners of the right and over the people with whom they stand in a right relation.[13] If we interpret claims as directed obligations, then rules of rights involve quantification over both the bearer and the counterparty of the associated directed obligation.

We end with a small terminological note. From here on we will also include statements expressing rules of rights or classes of multital rights in the category of general deontic statements.

3.4 The parking spot example

Let us illustrate the distinctions set out in this section with one example, loosely inspired by [34, pp. 362-363]. Suppose that a has contracted with b that a will park his van in a certain parking spot (so b can load the van). As a result of this contract, b has a claim on a that a will park his van in this parking spot and a has a corresponding obligation towards b that a will park his van in the parking spot.

Now suppose furthermore that this parking spot is reserved for disabled people and that a is disabled, but b is not. Then a has both a privilege towards b to park in this spot, and a claim on b that b will not park in this spot. Both this privilege and this claim are multital rights of a. However, there is a difference between the two classes of rights of which they are a part. Person a has a class of multital privileges to park in this spot against *all* others, but the class of claims that others do not park in this spot is *limited* to those who are not disabled. Against another disabled person, c, a does not have the claim that c does not park in this parking spot.

These multital rights are instances of different rules of rights. The first of these is the rule that all disabled people have a privilege against all others to

[13] Kanger's theory of 'atomic types of rights' differs from the Hohfeldian theory of rights, but the point about the existence of 'rules of rights' holds regardless of the specific theory of rights one accepts.

park in this spot. Secondly, there is a rule that all disabled people have a claim on all non-disabled people that those non-disabled people do not park in this parking space.

4 The translation to logic

In this section we discuss how we can translate the observations in sections 2 and 3 to a logic. It will be clear that standard deontic logic (SDL) is not suited for this task. In this section we will argue for two modifications to SDL, while describing some problems of trying to capture general deontic statements in a propositional or semi-propositional framework.

In order to represent undirected personal obligations, our formalism must explicitly mention who the bearer of an obligation is. In the literature there are two proposals for this. The first way of doing this is by indexing an O-operator with a symbol denoting the bearer of the obligation (see for example [15, 13]). Thus, O_a is to be read as 'it is obligatory for a that' or 'a has an obligation that'. There is, however, a competing proposal in the literature.

Hilpinen has argued that what we call personal obligations can be represented by a combination of an obligation operator that is not indexed and an action operator [16]. According to him, 'φ is obligatory for a' can be reduced to 'It is obligatory that a sees to it that φ' [16, p. 167]. Thus, according to this reduction, the bearer of a personal obligation that φ is merely the agent of the action described by φ. However, by using this reduction we are unable to capture the full range of personal obligations that are under consideration in this paper.

Examples that cannot be captured by this reduction are obligations that are personal, but where what is obligatory is not an action performed by the bearer of the obligation. McNamara calls these personal, non-agential obligations [30, p. 121]. That such personal, non-agential obligations exist (and that therefore the reduction is insufficient) has been argued by Krogh and Herrestad [26, pp. 150-151] and by McNamara [30, pp. 120-123]. They discuss the following example.

> Consider the case where the manager of a firm is under an obligation that the company's financial status is reported to the company board once a month. Let us assume that this manager has a particularly useful assistant. Without the manager's consent this assistant sends the financial status to the board each month, thus seeing to it that the manager's obligation is fulfilled. (...) As far as we are concerned, the managers obligations are personal, but may be fulfilled by someone else. [26, p. 151]

Examples of personal, non-agential obligations can also be found in the legal domain, for example in laws prohibiting the possession of illegal narcotics or weapons. Possession of such items is considered to be neither an act, nor the omission of an act [39, p. 753].[14] Thus, a can have the personal obligation that it is not the case that a has drugs in her possession. There does not need to be any action on the part of a for her to violate this obligation (especially not given the sometimes very wide definition of 'constructive possession' [39]).[15] Such a personal, non-agential obligation cannot be expressed by the combination of an obligation operator that is not indexed and an action operator.

Another counterexample to the reduction advocated by Hilpinen can be found in personal obligations where the agent of the obligatory action is specified, but is not the bearer of the obligation. An example is the concept of 'ministerial responsibility'. In some constitutional monarchies, the king is inviolable (*onschendbaar*), and the ministers of the cabinet are responsible for his wrongdoing [2, p. 146]. Thus, the ministers have an obligation that the king does not, for example, sign unconstitutional laws. These examples cannot be represented with the reduction advocated by Hilpinen, since there is no way to differentiate the bearer of the obligation from the agent of the obligatory action. Thus, indexing to represent bearers is preferable over using the reduction advocated by Hilpinen, if we also want to capture these examples.

Analogously, we propose to index our O operator with both the counterparty and the bearer when it comes to representing directed obligations (again, this follows previous work [15, 28, 35]). This explicit mention of the counterparty of a directed obligation is necessary, since who the counterparty is, cannot always be inferred from what is obligatory. Consider for example the case where a has contracted b to tutor her son c. In this case b has an obligation towards a that b tutors c. Since there is no mention of a (the counterparty) in what is obligatory, a formalisation of this sentence that does not explicitly mention the counterparty as counterparty would not represent who the counterparty is. Similar examples can be found for directed permissions (for example, Sophie's directed permission towards Julie to build a fence that was discussed in Section 2).

Turning away from personal permissions and obligations for a moment, we can now look at the representation of general deontic statements. We argue that we need the machinery of first-order logic for this representation. Some have proposed to model (subclasses of) general deontic statements as (finite) conjunctions of bearer-relative obligations or permissions (see for example [15]). Seeing general deontic statements as such finite conjunctions is problematic, given that we want to model laws and reasoning with general deontic statements.

To see this we must distinguish *definite* from *indefinite* general deontic statements. When a teacher ends a class by telling the room of students "everybody must finish the homework before Monday", then this is a definite general deontic statement: there is a definite number of individuals that is subject to the statement (the students in the class). Such a definite general deontic statement can be represented by a conjunction of bearer-relative obligations. However, there are also indefinite general deontic statements. Laws usually have this indefinite

[14] Going even further: "A few states have held that in narcotics cases knowledge is not an essential element of possession" [39, p. 753].

[15] Whether such a law is just is a different discussion.

character.[16] When one issues a law, it is not possible to give a definite list of the people who are subject to the law. Instead, a law holds for everyone, or everyone satisfying certain conditions. As a result, laws, and indefinite general deontic statements in general, should not be represented as finite conjunctions of bearer-relative obligations.

Thus we are in need of proper quantification. If we combine this with the previous idea of representing bearers and counterparties as indexes, we conclude that we must quantify over the indexes of our operators, to represent the quantification over bearers and counterparties. This would seem to point to a semi-propositional representation. Where p represents 'the books are returned to the library', we can represent the general deontic statement 'everyone has the obligation that the books are returned to the library' as $(\forall x)(\mathsf{O}_x p)$.

However, such a semi-propositional representation still falls short. Take for example the general obligation 'everyone must pay taxes'. If $\mathsf{O}_a p$ represents the undirected bearer-relative obligation 'a must pay taxes', then $(\forall x)(\mathsf{O}_x p)$ does not represent 'everyone must pay taxes', but 'it is obligatory for everyone that a pays taxes'. What this shows is that, as long as we stay at such a semi-propositional level, we cannot explicitly represent the link between the bearer of an obligation, and the occurrence of the bearer within the scope of the obligation-operator (for example when it is an action of that bearer that is required). We will call this the *bearer-in-scope problem*.[17]

It is often not only the bearer of the obligation that will occur within the scope of the operator, but also the counterparty. If one wants to quantify over the counterparties, then one must also quantify over these occurrences within the scope of the operator. Consider the statement 'a has an obligation towards b not to hit b' and the statement 'for all x, a has an obligation towards x not to hit x'. The first statement is an instance of (and thus follows from) the second statement. If a logic is unable to represent this logical relation between the first and the second statement, then we say that it suffers from the *counterparty-in-scope problem*.

Finally, in general deontic statements the bearers and counterparties do not only appear within the scope of the deontic operators, but also outside. This is especially true in non-categorical statements. Consider again 'Every driver has to give way to the driver coming from the right, unless he is driving on a roundabout or the driver from the right is coming from a forbidden direction.' We shall say that logics are *limited to categorical statements* if they can only represent categorical and not other general deontic statements.

There have been earlier attempts to capture subcategories of what we have called general deontic statements (most notably [13, 15, 23]). However, these attempts have mostly stayed at the propositional level. As a result they suffer from the problems described above (though some have found solutions to some of these problems within a propositional framework). The only exception is Kanger, who does try to capture reasoning with rights in a first-order frame-

[16] Cfr. also Hohfelds *indefinite* classes of multital rights [18, p. 718]

[17] This problem was already noted by Hansson [13].

work. However, he can only represent what we have called undirected personal obligation by a reduction akin to the one proposed by Hilpinen [23]. We have argued that this is insufficient for our purposes.

5 Term-modal deontic logic

In this section we first discuss Fitting et al.'s term-modal logics [9, 36]. We show that these logics give us a good starting point for formalising quantification over bearers and counterparties. Nevertheless, we point out some shortcomings in the approach of Fitting et al. when it is to be used for deontic reasoning (an application context that they did not have in mind). We then present **TMDL** that overcomes these shortcomings.

5.1 From term-modal logic to term-modal deontic logic

Term-modal logics were developed for an epistemic application context [36, p. 1]. The starting point of Fitting et al. is knowledge representation in a multi-agent context. Some earlier approaches to this topic used multimodal epistemic logics where there are accessibility relations and associated modalities for every agent. Fitting et al. point out that these logics cannot express sentences like 'every Christian believes in the existence of God'. To accommodate these sentences Fitting et al. propose to index the modal operators with terms of the language (i.e. with names of agents) instead of with agents. This then allows them to quantify over the terms that index the modal operators.

Semantically, Fitting et al. introduce a ternary accessibility relation between the set of worlds, the set of agents, and the set of worlds. This ternary accessibility relation replaces the binary accessibility relation of normal modal logic. The new ternary accessibility relation R^f satisfies a condition that corresponds to increasing domains. Where w_1 and w_2 are worlds, D_{w_1} and D_{w_2} are the domains associated with w_1 and w_2 respectively and $p \in D_{w_1}$, the following holds: if $\langle w_1, p, w_2 \rangle \in R^f$, then $D_{w_1} \subseteq D_{w_2}$. Fitting et al. do not give a reason for the increasing domains condition, but they do provide an example of an application context where this condition would seem to be applicable: computer processes that spawn new processes and in doing so increase the number of computer processes in the system.

In our legal application context such a reason for increasing domains is missing. New persons do get born, but other persons also die. This would seem to point to varying domain semantics. In varying domain semantics the domains of different worlds are independent of each other. Such varying domain semantics for term-modal logic have been developed by Corsi and Orlandelli [4].

There is however another option: constant domain semantics. With constant domain semantics there is just one domain for the whole model.[18] Employing constant domain semantics greatly simplifies the meta-theory. On top of that

[18]Equivalently, one can obtain constant domain semantics by assigning the same domain to every world.

we can still model the fact that different persons are alive and dead in different worlds by simply introducing predicates for 'x is alive' and 'x is dead'. Another argument in favour of constant domain semantics is that it allow us to represent the rights of unborn and deceased persons.

A shortcoming of the term-modal logics of Fitting et al. for our desired applications is that their language does not contain identity or double indexed modal operators. As we will show in Section 7, both are useful for the representation of rights and directed obligations. We will use identity to represent the fact that a person has a claim against all persons other than the claim-holder and we will use the double-indexed modal operators to indicate the directedness of obligations.

TMDL will solve these shortcomings for our application context. **TMDL** has constant domain semantics and its language contains identity and double indexed modal operators. In addition, we present a Hilbert-style axiomatization for **TMDL**, whereas Fitting et al. present tableau and sequent calculi.

5.2 The language of TMDL

Let $C = \{a, b, \ldots\}$ be the set of constants denoting persons and $V = \{x, y, \ldots\}$ be the set of variables. We let $\alpha, \beta, \ldots$ range over C and $\nu, \xi, \ldots$ over V. Let $T = C \cup V$ be the set of terms and $\theta, \kappa, \ldots$ be the metavariables ranging over it. For each natural number $n \geq 1$ we let $\mathcal{P}^n$ be a set of n-ary predicate symbols and we let $\mathcal{P}$ be the union of all $\mathcal{P}^n$. We let P range over $\mathcal{P}$. Lastly, we let φ, ψ, χ be metavariables for formulas and we use $\Gamma, \Delta, \Theta, \ldots$ as metavariables for sets of formulas. Our language $\mathcal{L}$ is defined with the following Backus-Naur form:

$$\varphi ::= P\theta_1 \ldots \theta_n \mid \theta = \kappa \mid \neg\varphi \mid \varphi \vee \varphi \mid \mathsf{O}_\theta\varphi \mid \mathsf{O}^\theta_\kappa\varphi \mid (\forall\nu)\varphi$$

The other Boolean connectives are defined in the standard way. Additionally, $(\exists\nu)\varphi =_{\mathsf{df}} \neg(\forall\nu)\neg\varphi$, $\mathsf{P}_\theta\varphi =_{\mathsf{df}} \neg\mathsf{O}_\theta\neg\varphi$ and $\mathsf{P}^\theta_\kappa\varphi =_{\mathsf{df}} \neg\mathsf{O}^\theta_\kappa\neg\varphi$. We will often write $\theta \neq \kappa$ instead of $\neg\theta = \kappa$.

The notion of free and bound variables are as usual, with two additions (cfr. Fitting et al. [9]): (1) The free occurrences of variables in $\mathsf{O}_\theta\varphi$ are all free occurrences of variables in φ and in addition θ if θ is a variable, and (2) the free occurrences of variables in $\mathsf{O}^\kappa_\theta\varphi$ are θ, if θ is a variable, κ, if κ is a variable, and all free occurrences of variables in φ. A wff φ is a sentence iff all the variables in φ are bound. We define the set $\mathcal{S}$ of sentences of $\mathcal{L}$: $\mathcal{S} =_{\mathsf{df}} \{\varphi \in \mathcal{L} \mid \varphi \text{ is a sentence}\}$.

5.3 Semantics of TMDL

We define **TMDL**-models as follows:

Definition 1. *A* **TMDL**-*model is a tuple* $M = \langle W, \mathcal{A}, R, R^D, I\rangle$, *where:*

1. $W \neq \emptyset$

2. $\mathcal{A} \neq \emptyset$

3. $R \subseteq W \times \mathcal{A} \times W$ *(where* $w \in W$ *and* $p \in \mathcal{A}$*, we write* $R(w,p)$ *for* $\{w' \in W \mid \langle w,p,w' \rangle \in R\}$*)*
3.1. *For all* $p \in \mathcal{A}$ *and* $w \in W$*,* $R(w,p) \neq \emptyset$
4. $R^D \subseteq W \times \mathcal{A} \times \mathcal{A} \times W$ *(where* $w \in W$ *and* $p_1, p_2 \in \mathcal{A}$*, we write* $R^D(w,p_1,p_2)$ *for* $\{w' \in W \mid \langle w,p_1,p_2,w' \rangle \in R^D\}$*)*
4.1. *For all* $w \in W$*,* $p_1, p_2 \in \mathcal{A}$*:* $R^D(w,p_1,p_2) \neq \emptyset$*.*
4.2. *For all* $w \in W$*,* $p_1, p_2 \in \mathcal{A}$*:* $R(w,p_1) \subseteq R(w,p_1,p_2)$*.*
5. I *is an* interpretation *function such that:*
5.1. $I : T \to \mathcal{A}$
5.2. $I : \mathcal{P}^n \times W \to \wp(\mathcal{A}^n)$ *for every natural number* $n \geq 1$*.*

We interpret W as a set of possible worlds and $\mathcal{A}$, the domain, as a set of agents.[19] R is an accessibility relation. $R(w,p_1)$ is interpreted as the set of worlds where all the obligations that agent p_1 has in world w are fulfilled. $R(w,p_1,p_2)$ is interpreted as the set of worlds where all obligations that agent p_1 has towards p_2 in world w are fulfilled. Conditions 3.1. and 4.1. ensure seriality, whereas condition 4.2. ensures that all worlds where the obligations of agent p_1 are fulfilled, are also worlds where the obligations p_1 has towards others are fulfilled. This models the intuition that if p_1 has a directed obligation that φ, then she also has an undirected bearer-relative obligation that φ.

Definition 2. *For any* $\nu \in V$*,* $M' = \langle W, \mathcal{A}, R, R^D, I' \rangle$ *is a* ν*-alternative to* $M = \langle W, \mathcal{A}, R, R^D I \rangle$ *iff* I' *differs at most from* I *in the member of* $\mathcal{A}$ *that* I' *assigns to* ν*.*

Definition 3 (Semantic Clauses)**.** *For any* **TMDL***-model* $M = \langle W, \mathcal{A}, R, R^D, I \rangle$*:*
SC1 $M, w \models P\theta_1 \ldots \theta_n$ *iff* $\langle I(\theta_1), \ldots, I(\theta_n) \rangle \in I(P,w)$
SC2 $M, w \models \neg\varphi$ *iff* $M, w \not\models \varphi$
SC3 $M, w \models \varphi \vee \psi$ *iff* $M, w \models \varphi$ *or* $M, w \models \psi$
SC4 $M, w \models \theta = \kappa$ *iff* $I(\theta) = I(\kappa)$
SC5 $M, w \models \mathsf{O}_\theta \varphi$ *iff* $M, w' \models \varphi$ *for all* $w' \in R(w, I(\theta))$
SC6 $M, w \models (\forall\nu)\varphi$ *iff for every* ν*-alternative* M'*:* $M', w \models \varphi$
SC7 $M, w \models \mathsf{O}^\kappa_\theta \varphi$ *iff* $M, w' \models \varphi$ *for all* $w' \in R^D(w, I(\theta), I(\kappa))$

Let $\Gamma \subseteq \mathcal{S}$ and $\varphi \in \mathcal{S}$, then we can define semantic consequence and validity as follows.

Definition 4 (Semantic consequence)**.** φ *is a semantic consequence of* Γ*,* $\Gamma \Vdash \varphi$ *iff for every* **TMDL***-model* $M = \langle W, \mathcal{A}, R, R^D, I \rangle$ *and* $w \in W$*: if* $M, w \models \psi$ *for all* $\psi \in \Gamma$*, then* $M, w \models \varphi$*.*

Definition 5 (Validity)**.** φ *is valid,* $\Vdash \varphi$ *iff for every* **TMDL***-model* $M = \langle W, \mathcal{A}, R, R^D, I \rangle$ *and* $w \in W$*:* $M, w \models \varphi$*.*

[19] It is also possible to expand the domain to also include other objects, like the apples from the introduction. See for example [1].

5.4 Axiomatisation of TMDL

A sound and strongly complete axiomatisation of **TMDL** is obtained by closing a complete axiomatisation of classical propositional logic (CL) with all instances of the axiom schemata in Table 1 under the rules of Table 2. $\varphi(\theta/\kappa)$ is the result of replacing all free occurrences of κ in φ by θ, relettering bound variables if necessary to avoid rendering new occurrences of θ bound in $\varphi(\theta/\kappa)$. $\varphi(\theta//\kappa)$ is the result of replacing various (not necessarily all or even any) free occurrences of θ in φ by occurrences of κ, again relettering if necessary [37, p. 57].

φ is a **TMDL**-theorem (denoted $\vdash \varphi$) iff φ can be derived from the **TMDL**-axioms and rules. $\varphi \in \mathcal{S}$ is **TMDL**-derivable from $\Gamma \subseteq \mathcal{S}$ (denoted $\Gamma \vdash \varphi$) iff there are $\psi_1, \ldots, \psi_n \in \Gamma$ such that $\vdash (\psi_1 \wedge \ldots \wedge \psi_n) \rightarrow \varphi$. From this it follows immediately that $\vdash$ is compact.

(K)	$\mathsf{O}_\alpha(\varphi \rightarrow \psi) \rightarrow (\mathsf{O}_\alpha\varphi \rightarrow \mathsf{O}_\alpha\psi)$
(D)	$\mathsf{O}_\alpha\varphi \rightarrow \neg\mathsf{O}_\alpha\neg\varphi$
(BF)	$(\forall\nu)\mathsf{O}_\alpha\varphi \rightarrow \mathsf{O}_\alpha(\forall\nu)\varphi$
(DK)	$\mathsf{O}^\beta_\alpha(\varphi \rightarrow \psi) \rightarrow (\mathsf{O}^\beta_\alpha\varphi \rightarrow \mathsf{O}^\beta_\alpha\psi)$
(DD)	$\mathsf{O}^\beta_\alpha\varphi \rightarrow \neg\mathsf{O}^\beta_\alpha\neg\varphi$
(DBF)	$(\forall\nu)\mathsf{O}^\beta_\alpha\varphi \rightarrow \mathsf{O}^\beta_\alpha(\forall\nu)\varphi$
(DIU)	$\mathsf{O}^\beta_\alpha\varphi \rightarrow \mathsf{O}_\alpha\varphi$
(UI)	$(\forall\nu)\varphi \rightarrow \varphi(\alpha/\nu)$
(REF)	$\alpha = \alpha$
(SUB)	$(\alpha = \beta) \rightarrow (\varphi \rightarrow \varphi(\alpha//\beta))$
(ND)	$(\alpha \neq \beta) \rightarrow \mathsf{O}_\gamma(\alpha \neq \beta)$
(DND)	$(\alpha \neq \beta) \rightarrow \mathsf{O}^\delta_\gamma(\alpha \neq \beta)$

Table 1: Axiom schemata

(MP)	if $\varphi \rightarrow \psi$ and φ, then ψ
(NEC)	if $\vdash \varphi$, then $\mathsf{O}_\alpha\varphi$
(DNEC)	if $\vdash \varphi$, then $\vdash \mathsf{O}^\beta_\alpha\varphi$.
(UG)	if $\vdash \varphi \rightarrow \psi(\alpha/\nu)$ and α not in φ or ψ, then $\vdash \varphi \rightarrow (\forall\nu)\psi$.

Table 2: Rules

The axioms and rules in Table 1 and Table 2 require some explanation. (K) and (DK) are the bearer-relative and directed analogues of the K-axiom of **SDL**. Similarly (D) and (DD) are the analogues of the D-axiom and (NEC) and (DNEC) of the necessitation rule of **SDL**. Choosing for constant domain semantics entails accepting the Barcan Formula $((\forall x)(\Box\varphi) \rightarrow \Box(\forall x)\varphi)$. (BF) and (DBF) are the analogues of this Barcan Formula for **TMDL**. The axiom (DIU) formalises the intuition that whenever a person has a directed obligation that φ, then this person also has a bearer-relative obligation that φ (cfr. Section 2). (UI), (REF), (SUB) and (UG) are inherited from first order logic. Finally, a first order modal logic with identity that treats terms as rigid designators also validates a principle of necessary distinctness ($\alpha \neq \beta \to \Box\alpha \neq \beta$). (ND) and (DND) are the analogous principles in **TMDL**.

The reader will have noticed that some of the axiom schemata in Table 1 are superfluous. For example, (ND) follows from (DND) and (DIU). The reason that we have chosen for this presentation is that it shows the modularity of the logic. For example, one can get an axiomatisation of just the bearer-relative part of **TMDL** by simply leaving out (DK), (DD), (DBF), (DIU), (DND) and (DNEC).

We prove the following theorems. The first two theorems are useful for the completeness proof and the final two are desirable and therefore deserve to be mentioned.

Theorem 1. $\vdash \alpha = \beta \leftrightarrow \mathsf{O}_\gamma(\alpha = \beta)$

Proof. For left to right, since $\alpha = \alpha$ is a theorem we can use (NEC) to derive $\mathsf{O}_\gamma(\alpha = \alpha)$. Using (SUB) and the fact that in CL from $\varphi \to (\psi \to \chi)$ and ψ it follows that $\varphi \to \chi$, we can derive $\alpha = \beta \to \mathsf{O}_\gamma(\alpha = \beta)$. For right to left, suppose $\mathsf{O}_\gamma(\alpha = \beta)$. By (D), we have $\neg\mathsf{O}_\gamma(\alpha \neq \beta)$. By (ND) and contraposition, $\neg\alpha \neq \beta$, hence $\alpha = \beta$. □

Theorem 2. $\vdash \alpha = \beta \leftrightarrow \mathsf{O}^\delta_\gamma(\alpha = \beta)$

Proof. The proof is analogous to the proof for Theorem 1: For left to right, we can derive $\mathsf{O}^\delta_\gamma(\alpha = \beta)$ from $\alpha = \beta$ by (DNEC) and (SUB). For the right to left direction, we can derive $\alpha = \beta$ from $\mathsf{O}^\delta_\gamma(\alpha = \beta)$ by (DD), (DND) and contraposition. □

Theorem 3. $\vdash (\mathsf{O}^\alpha_\gamma\varphi \wedge \mathsf{O}^\beta_\gamma\psi) \to \mathsf{O}_\gamma(\varphi \wedge \psi)$

Proof. $\mathsf{O}_\gamma(\varphi \wedge \psi)$ follows from $\mathsf{O}^\alpha_\gamma\varphi \wedge \mathsf{O}^\beta_\gamma\psi$ by (DIU) and (K). □

Theorem 4. $\vdash (\mathsf{O}^{\alpha_1}_\beta\varphi_1 \wedge \ldots \wedge \mathsf{O}^{\alpha_n}_\beta\varphi_n) \to \neg\mathsf{O}^\gamma_\beta\neg(\varphi_1 \wedge \ldots \wedge \varphi_n)$

Proof. Suppose $\mathsf{O}^{\alpha_1}_\beta\varphi_1 \wedge \ldots \wedge \mathsf{O}^{\alpha_n}_\beta\varphi_n$. By (DIU), $\mathsf{O}_\beta\varphi_1 \wedge \ldots \wedge \mathsf{O}_\beta\varphi_n$ follows. Then by (K), $\mathsf{O}_\beta(\varphi_1 \wedge \ldots \wedge \varphi_n)$. By (D), $\neg\mathsf{O}_\beta\neg(\varphi_1 \wedge \ldots \wedge \varphi_n)$ and by (DIU) and contraposition, $\neg\mathsf{O}^\gamma_\beta\neg(\varphi_1 \wedge \ldots \wedge \varphi_n)$. □

6 Completeness

In this section, we prove strong completeness for **TMDL** (Theorem 5). The proof is very similar to that for first-order modal logics, see for example [8] and [20], but is significantly simpler than the proof for the completenes in [9]. The proofs of Lemma 2 and Lemma 3 (cfr. infra) are largely analogous to the proofs in [20, Chap. 14].

In line with the literature, we construct a canonical model, prove that it is a **TMDL**-model (Lemma 7) and that the truth lemma holds for it (Lemma 8). Note that for every α and β, $R(w, I(\alpha))$ and $R^D(w, I(\alpha), I(\beta))$ are analogous to the set of accessible worlds in normal modal logic and that O_α and O_α^β are the analogues of the normal modal $\Box$-operator.

As we are dealing with a first-order logic, the worlds in the canonical model cannot simply be maximal consistent sets. A maximal consistent set Θ might contain $\neg(\forall\nu)\varphi$, but there might not be an α such that $\neg\varphi(\alpha/\nu) \in \Theta$, i.e. there might not be a 'witness' for $\neg(\forall\nu)\varphi$. This would cause a problem for the truth lemma (the case where φ is of the form $(\forall\nu)\psi$). To solve this problem, we work with an extended language $\mathcal{L}^+$ that contains an infinite number of constants that do not occur in $\mathcal{L}$. Following [20], we prove that any consistent set $\Gamma \subseteq \mathcal{S}$ has a maximal consistent and ω-complete extension in this new language $\mathcal{L}^+$ (Lemma 2). Sets that are maximal consistent and ω-complete in $\mathcal{L}^+$ we call $\mathcal{L}^+$-saturated sets (see also [37]).

We define $\mathcal{L}^+$ in a way analogous to $\mathcal{L}$, but with C replaced by $C^+ = C \cup \{\alpha_1^+, \alpha_2^+, \ldots\}$. We stipulate that $\alpha_1^+, \alpha_2^+, \ldots \notin C$. $\mathcal{S}^+ =_{\mathsf{df}} \{\varphi \in \mathcal{L}^+ \mid \varphi \text{ is a sentence}\}$.

Definition 6. *$\Gamma \subseteq \mathcal{S}^+$ is $\mathcal{L}^+$-saturated iff*
1. *Γ is consistent.*
2. *for all $\varphi \in \mathcal{S}^+$: $\varphi \in \Gamma$ or $\neg\varphi \in \Gamma$.*
3. *for all $\varphi \in \mathcal{S}^+$ and $\nu \in V$: there is an $\alpha \in C^+$ such that $\varphi(\alpha/\nu) \to (\forall\nu)\varphi \in \Gamma$.*

If a consistent set of sentences Γ satifies the second condition of Definition 6, we call it *maximal consistent.* If some Γ satisfies its third condition, we call it *ω-complete.*

The proof of the following lemma is obvious in view of the Lindenbaum construction, and is left to the reader (see [8, 20] for examples).

Lemma 1. *For any set $\Gamma \subseteq \mathcal{S}$ (respectively, $\Gamma \subseteq \mathcal{S}^+$), if Γ is consistent, there is a maximal consistent $\Gamma' \subseteq \mathcal{S}$ (respectively, $\Gamma' \subseteq \mathcal{S}^+$) such that Γ' is maximal consistent and $\Gamma \subseteq \Gamma'$.*

For the proof of the following two lemmas, we assume that all sentences of the form $(\forall\nu)\varphi$ are enumerated in a fixed order $(\forall\nu)\chi_1$, $(\forall\nu)\chi_2, \ldots$

Lemma 2. *For any set $\Gamma \subseteq \mathcal{S}$: if Γ is consistent then there is an $\mathcal{L}^+$-saturated set $\Gamma' \subseteq \mathcal{S}^+$ such that $\Gamma \subseteq \Gamma'$.*

Proof. Suppose $\Gamma \subseteq \mathcal{S}$ is consistent, and define a sequence of sets as follows:

$$\Gamma_0 = \Gamma$$

$$\Gamma_{n+1} = \Gamma_n \cup \{\chi_{n+1}(\alpha_{n+1}^+/\nu) \to (\forall\nu)\chi_{n+1}\}$$

We prove by induction that Γ_n is consistent, for all n. The base case is obvious, as Γ is consistent. For the induction step, suppose that Γ_n is consistent and Γ_{n+1} is not. It follows that there are $\psi_1, \ldots, \psi_k \in \Gamma_n$ such that $\vdash (\psi_1 \wedge \ldots \wedge \psi_k) \rightarrow \chi_{n+1}(\alpha^+_{n+1}/\nu)$ and $\vdash (\psi_1 \wedge \ldots \wedge \psi_k) \rightarrow \neg(\forall\nu)\chi_{n+1}$. By the construction, α^+_{n+1} does not occur in $\psi_1 \wedge \ldots \wedge \psi_k$ and not in χ_{n+1}. By (UG), $\vdash (\psi_1 \wedge \ldots \wedge \psi_k) \rightarrow (\forall\nu)\chi_{n+1}$. Thus Γ_n is inconsistent. This contradicts the supposition.

Let Γ^* be the union of all Γ_n. Γ^* is consistent (by the above and the fact that the logic is compact) and ω-complete (it satisfies condition 3 of Definition 6). In view of Lemma 1, Γ^* can be extended to an $\mathcal{L}^+$-saturated set Γ'. □

Lemma 3. *For any set $\Gamma \subseteq \mathcal{S}^+$, $\alpha \in C^+$ and $\varphi \in \mathcal{S}^+$: If Γ is $\mathcal{L}^+$-saturated and $\mathsf{O}_\alpha\varphi \notin \Gamma$, then there is an $\mathcal{L}^+$-saturated set Θ such that $\{\psi \mid \mathsf{O}_\alpha\psi \in \Gamma\} \cup \{\neg\varphi\} \subseteq \Theta$.*

Proof. In view of Lemma 1, it is sufficient to show that, if Γ is $\mathcal{L}^+$-saturated and $\mathsf{O}_\alpha\varphi \notin \Gamma$, the set $\{\psi \mid \mathsf{O}_\alpha\psi \in \Gamma\} \cup \{\neg\varphi\}$ is consistent and moreover can be extended to a set that is ω-complete. To prove that such an ω-complete extension exists, a different construction is needed than for Lemma 2. (Since this lemma concerns $\Gamma \subseteq \mathcal{S}^+$ instead of $\Gamma \subseteq \mathcal{S}$.)

So, suppose that Γ is $\mathcal{L}^+$-saturated and $\mathsf{O}_\alpha\varphi \notin \Gamma$. Suppose further that $\{\psi \mid \mathsf{O}_\alpha\psi \in \Gamma\} \cup \{\neg\varphi\}$ is not consistent. It follows that there is a finite subset $\{\psi_1, \ldots, \psi_n\}$ of $\{\psi \mid \mathsf{O}_\alpha\psi \in \Gamma\}$ such that $\vdash (\psi_1 \wedge \ldots \wedge \psi_n) \rightarrow \varphi$. By (NEC), $\vdash \mathsf{O}_\alpha((\psi_1 \wedge \ldots \wedge \psi_n) \rightarrow \varphi)$, and by (K), $\vdash \mathsf{O}_\alpha(\psi_1 \wedge \ldots \wedge \psi_n) \rightarrow \mathsf{O}_\alpha\varphi$. But then, in view of the main supposition, Γ would be inconsistent (as $\mathsf{O}_\alpha\varphi \notin \Gamma$, and hence, $\neg\mathsf{O}_\alpha\varphi \in \Gamma$).

To prove that $\{\psi \mid \mathsf{O}_\alpha\psi \in \Gamma\} \cup \{\neg\varphi\}$ can be extended to a set that is ω-complete, we define a sequence of sentences $\varphi_0, \varphi_1, \varphi_2, \ldots$. We let φ_0 be $\neg\varphi$, and define φ_{n+1} as $\varphi_n \wedge (\chi_{n+1}(\alpha_k/\nu) \rightarrow (\forall\nu)\chi_{n+1})$, where α_k is the first constant in C^+ for which $\{\psi \mid \mathsf{O}_\alpha\psi \in \Gamma\} \cup \{\varphi_n \wedge (\chi_{n+1}(\alpha_k/\nu) \rightarrow (\forall\nu)\chi_{n+1})\}$ is consistent.

The set $\Theta^* = \{\psi \mid \mathsf{O}_\alpha\psi \in \Gamma\} \cup \{\varphi_0, \varphi_1, \varphi_2, \ldots\}$ is ω-complete and in view of Lemma 1 it can be extended to an $\mathcal{L}^+$-saturated set Θ such that $\{\psi \mid \mathsf{O}_\alpha\psi \in \Gamma\} \cup \{\neg\varphi\} \subseteq \Theta$. We do have to prove, however, that the construction is well-defined, namely that there is an $\alpha_k \in C^+$ such that $\{\psi \mid \mathsf{O}_\alpha\psi \in \Gamma\} \cup \{\varphi_n \wedge (\chi_{n+1}(\alpha_k/\nu) \rightarrow (\forall\nu)\chi_{n+1})\}$ is consistent.

To prove that there is always such an $\alpha_k \in C^+$, suppose $\{\psi \mid \mathsf{O}_\alpha\psi \in \Gamma\} \cup \{\varphi_n\}$ is consistent and suppose there is no such α_k. Then for every $\alpha_k \in C^+$ there is a set of sentences $\{\psi_1, \ldots, \psi_m\} \subseteq \{\psi \mid \mathsf{O}_\alpha\psi \in \Gamma\}$ such that

$$\vdash (\psi_1 \wedge \ldots \wedge \psi_m) \rightarrow (\varphi_n \rightarrow \neg(\chi_{n+1}(\alpha_k/\nu) \rightarrow (\forall\nu)\chi_{n+1})) \qquad (1)$$

Hence, by (K) and CL,

$$\vdash (\mathsf{O}_\alpha\psi_1 \wedge \ldots \wedge \mathsf{O}_\alpha\psi_m) \rightarrow \mathsf{O}_\alpha(\varphi_n \rightarrow \neg(\chi_{n+1}(\alpha_k/\nu) \rightarrow (\forall\nu)\chi_{n+1})) \qquad (2)$$

holds. Since Γ is $\mathcal{L}^+$-saturated,

$$\mathsf{O}_\alpha(\varphi_n \rightarrow \neg(\chi_{n+1}(\alpha_k/\nu) \rightarrow (\forall\nu)\chi_{n+1})) \qquad (3)$$

is a member of Γ for every $\alpha_k \in C^+$.

Let $\nu_1 \in V$ be such that ν_1 does not occur in φ_n or χ_{n+1}. Since Γ is $\mathcal{L}^+$-saturated, there is an $\alpha_l \in C^+$ such that

$$\begin{aligned}\mathsf{O}_\alpha(\varphi_n \to (\neg\chi_{n+1}(\alpha_l/\nu) \to (\forall\nu)\chi_{n+1})) \to \\ (\forall\nu_1)(\mathsf{O}_\alpha(\varphi_n \to \neg(\chi_{n+1}(\nu_1/\nu) \to (\forall\nu)\chi_{n+1})))\end{aligned} \tag{4}$$

is a member of Γ.

In view of (3) and (MP),

$$(\forall\nu_1)(\mathsf{O}_\alpha(\varphi_n \to \neg(\chi_{n+1}(\nu_1/\nu) \to (\forall\nu)\chi_{n+1}))) \tag{5}$$

is a member of Γ, and by (BF), so is

$$\mathsf{O}_\alpha((\forall\nu_1)(\varphi_n \to \neg(\chi_{n+1}(\nu_1/\nu) \to (\forall\nu)\chi_{n+1}))). \tag{6}$$

Since ν_1 does not occur in φ_n, also

$$\mathsf{O}_\alpha(\varphi_n \to (\forall\nu_1)(\neg(\chi_{n+1}(\nu_1/\nu) \to (\forall\nu)\chi_{n+1}))) \tag{7}$$

is a member of Γ. However, $(\forall\nu_1)(\neg(\chi_{n+1}(\nu_1/\nu) \to (\forall\nu)\chi_{n+1}))$ is a contradiction, hence $\mathsf{O}_\alpha\neg\varphi_n \in \Gamma$, and $\neg\varphi_n \in \{\psi \mid \mathsf{O}_\alpha\psi \in \Gamma\}$. But then, $\{\psi \mid \mathsf{O}_\alpha\psi \in \Gamma\} \cup \{\varphi_n\}$ is inconsistent, which contradicts the supposition. $\square$

Corollary 1. *For any set $\Gamma \subseteq \mathcal{S}^+$ and $\alpha \in C^+$: If Γ is $\mathcal{L}^+$-saturated, then there is an $\mathcal{L}^+$-saturated set Θ s.t. $\{\psi \mid \mathsf{O}_\alpha\psi \in \Gamma\} \subseteq \Theta$.*

Lemma 4. *For any set $\Gamma \subseteq \mathcal{S}^+$, $\alpha, \beta \in C^+$ and $\varphi \in \mathcal{S}^+$: If Γ is $\mathcal{L}^+$-saturated and $\mathsf{O}^\beta_\alpha\varphi \notin \Gamma$, then there is an $\mathcal{L}^+$-saturated set Θ such that $\{\psi \mid \mathsf{O}^\beta_\alpha\psi \in \Gamma\} \cup \{\neg\varphi\} \subseteq \Theta$.*

Proof. The proof is analogous to the proof of Lemma 3, except that every occurrence of O_α is replaced by O^β_α, (NEC) by (DNEC), (K) by (DK), and (BF) by (DBF). $\square$

Corollary 2. *For any set $\Gamma \subseteq \mathcal{S}^+$ and $\alpha, \beta \in C^+$: If Γ is $\mathcal{L}^+$-saturated, then there is an $\mathcal{L}^+$-saturated set Θ such that $\{\psi \mid \mathsf{O}^\beta_\alpha\psi \in \Gamma\} \subseteq \Theta$.*

Where $\Gamma \subseteq \mathcal{S}^+$ is an $\mathcal{L}^+$-saturated set and $\alpha \in C^+$, $[[\alpha]]_\Gamma$ stands for $\{\beta \in C^+ \mid \alpha = \beta \in \Gamma\}$.

Definition 7. *Where Γ is an $\mathcal{L}^+$-saturated set, the canonical model $M_\Gamma = \langle W_\Gamma, \mathcal{A}_\Gamma, R_\Gamma, R^D_\Gamma, I_\Gamma\rangle$ is defined as follows:*

i. *W_Γ is the set of all $\mathcal{L}^+$-saturated sets Δ s.t. $\alpha = \beta \in \Gamma$ iff $\alpha = \beta \in \Delta$.*

ii. *$\mathcal{A}_\Gamma = \{[[\alpha]]_\Gamma \mid \alpha \in C^+\}$*

iii. *For all $\Delta \in W_\Gamma$ and $[[\alpha]]_\Gamma \in \mathcal{A}_\Gamma$: $R_\Gamma(\Delta, [[\alpha]]_\Gamma) = \{\Theta \in W_\Gamma \mid \{\varphi \mid \mathsf{O}_\alpha\varphi \in \Delta\} \subseteq \Theta\}$*

iv. *For all $\Delta \in W_\Gamma$ and $[[\alpha]]_\Gamma, [[\beta]]_\Gamma \in \mathcal{A}_\Gamma$: $R_\Gamma(\Delta, [[\alpha]]_\Gamma, [[\beta]]_\Gamma) = \{\Theta \in W_\Gamma \mid \{\varphi \mid \mathsf{O}^\beta_\alpha\varphi \in \Delta\} \subseteq \Theta\}$*

v. *I_Γ is defined as follows:*

v.1. *v.1.1.* *for all $\alpha \in C^+$: $I(\alpha) = [[\alpha]]_\Gamma$*

v.1.2. *for all* $\nu \in V$*:* $I(\nu) = [[\alpha_1^+]]_\Gamma$

v.2. *for all* $\Delta \in W_\Gamma$ *and n-ary* $P \in \mathcal{P}^n$*:* $I(P, \Delta) = \{\langle [[\alpha_1]]_\Gamma, \ldots, [[\alpha_n]]_\Gamma \rangle \mid P(\alpha_1, \ldots, \alpha_n) \in \Delta\}$

Lemma 5. *If* $\Delta \in W_\Gamma$ *and* Θ *is an* $\mathcal{L}^+$*-saturated set such that* $\{\varphi \mid \mathsf{O}_\alpha\varphi \in \Delta\} \subseteq \Theta$*, then* $\Theta \in W_\Gamma$*.*

Proof. Suppose that $\Delta \in W_\Gamma$ and Θ is an $\mathcal{L}^+$-saturated set such that $\{\varphi \mid \mathsf{O}_\alpha\varphi \in \Delta\} \subseteq \Theta$. By Theorem 1, and the fact that $\Delta \in W_\Gamma$: $\gamma = \delta \in \{\varphi \mid \mathsf{O}_\alpha\varphi \in \Delta\}$ iff $\gamma = \delta \in \Gamma$. Hence $\gamma = \delta \in \Theta$ iff $\gamma = \delta \in \Gamma$. Hence, by Definition 7.i., $\Theta \in W_\Gamma$. □

Lemma 6. *If* $\Delta \in W_\Gamma$ *and* Θ *is an* $\mathcal{L}^+$*-saturated set such that* $\{\varphi \mid \mathsf{O}_\alpha^\beta\varphi \in \Delta\} \subseteq \Theta$*, then* $\Theta \in W_\Gamma$*.*

Proof. The proof is analogous to the proof of Lemma 5, except that O_α is replaced by O_α^β and Theorem 1 is replaced by Theorem 2. □

Lemma 7. *For all* $\mathcal{L}^+$*-saturated sets* Γ*,* M_Γ *is a* **TMDL***-model.*

Proof. We prove that each of the conditions in Definition 1 is fulfilled.

1. Since $\Gamma \in W_\Gamma$, $W_\Gamma \neq \emptyset$.
2. Since $C^+ \neq \emptyset$, $\mathcal{A}_\Gamma \neq \emptyset$.
3. For any $[[\alpha]]_\Gamma \in \mathcal{A}_\Gamma$ and $\Delta \in W_\Gamma$: $\{\varphi \mid \mathsf{O}_\alpha\varphi \in \Delta\}$ is consistent by (D) and non-empty by (NEC). By Corollary 1, there is an $\mathcal{L}^+$-saturated extension Θ of $\{\varphi \mid \mathsf{O}_\alpha\varphi \in \Delta\}$. By Lemma 5, $\Theta \in W_\Gamma$. By Definition 7.iii. $\Theta \in R_\Gamma(\Delta, [[\alpha]]_\Gamma)$ and thus $R(\Delta, [[\alpha]]_\Gamma) \neq \emptyset$.

4.1. For any $[[\alpha]]_\Gamma, [[\beta]]_\Gamma \in \mathcal{A}_\Gamma$ and $\Delta \in W_\Gamma$: $\{\varphi \mid \mathsf{O}_\alpha^\beta\varphi \in \Delta\}$ is consistent by (DD) and non-empty by (DNEC). By Corollary 2, there is an $\mathcal{L}^+$-saturated extension Θ of $\{\varphi \mid \mathsf{O}_\alpha^\beta\varphi \in \Delta\}$. By Lemma 6, $\Theta \in W_\Gamma$. By Definition 7.iv. $\Theta \in R_\Gamma(\Delta, [[\alpha]]_\Gamma, [[\beta]]_\Gamma)$ and thus $R(\Delta, [[\alpha]]_\Gamma, [[\beta]]_\Gamma) \neq \emptyset$.

4.2. For any $[[\alpha]]_\Gamma, [[\beta]]_\Gamma \in \mathcal{A}_\Gamma$ and $\Delta \in W_\Gamma$: $\{\varphi \mid \mathsf{O}_\alpha^\beta\varphi \in \Delta\} \subseteq \{\varphi \mid \mathsf{O}_\alpha\varphi \in \Delta\}$ by (DIU). Hence for any $[[\alpha]]_\Gamma, [[\beta]]_\Gamma \in \mathcal{A}_\Gamma$ and $\Delta \in W_\Gamma$, $\{\Theta \in W_\Gamma \mid \{\varphi \mid \mathsf{O}_\alpha\varphi \in \Delta\} \subseteq \Theta\} \subseteq \{\Theta \in W_\Gamma \mid \{\varphi \mid \mathsf{O}_\alpha^\beta\varphi \in \Delta\} \subseteq \Theta\}$. Thus for all $[[\alpha]]_\Gamma, [[\beta]]_\Gamma \in \mathcal{A}_\Gamma$ and $\Delta \in W_\Gamma$, $R_\Gamma(\Delta, \alpha) \subseteq R_\Gamma^D(\Delta, \alpha, \beta)$ by Definition 7.iii. and Definition 7.iv.

5. By Definition 7.v.1., I_Γ satisfies condition 5.1 of Definition 1 and by Definition 7.v.2., I_Γ satisfies condition 5.2 of Definition 1.

□

Lemma 8 (Truth Lemma)**.** *For all* $\Gamma \subseteq \mathcal{S}^+$*,* $\varphi \in \mathcal{S}^+$ *and* $\Delta \in W_\Gamma$*:* $M_\Gamma, \Delta \models \varphi$ *iff* $\varphi \in \Delta$*.*

Proof. The proof proceeds by induction on the complexity of φ. The lemma obviously holds for sentences of the form $P(\theta_1, \ldots, \theta_n)$, which forms the base case. The cases where φ is of the form $\neg\psi$ or $\chi \vee \psi$ are standard. This leaves us with four cases.

Case 1: φ is of the form $(\forall\nu)\psi$. For the right to left direction, suppose $(\forall\nu)\psi \in \Delta$. Consider an arbitrary ν-alternative M'_Γ, with interpretation function I'_Γ. $I'_\Gamma(\nu) = [[\alpha]]_\Gamma$ for some constant $\alpha \in C^+$. By (UI) and the fact that Δ is maximally consistent, $\psi(\alpha/\nu) \in \Delta$. By the induction hypothesis, $M_\Gamma, \Delta \models \psi(\alpha/\nu)$. Since M'_Γ is a ν-alternative of M_Γ, $M'_\Gamma, \Delta \models \psi(\alpha/\nu)$. Since $I'_\Gamma(\alpha) = I'_\Gamma(\nu)$, $M'_\Gamma, \Delta \models \psi$. Since M'_Γ was an arbitrary ν-alternative, $M_\Gamma, \Delta \models (\forall\nu)\psi$.

For the other direction, suppose $(\forall\nu)\psi \notin \Delta$. As Δ is maximally consistent, it follows that $\neg(\forall\nu)\psi \in \Delta$. Hence, as Δ is $\mathcal{L}^+$-saturated, there is some $\alpha \in C^+$ such that $\neg\psi(\alpha/\nu) \in \Delta$. But then, $\psi(\alpha/\nu) \notin \Delta$, and so, in view of the induction hypothesis, $M_\Gamma, \Delta \not\models \psi(\alpha/\nu)$. Hence, $M_\Gamma, \Delta \not\models (\forall\nu)\psi$.

Case 2: φ is of the form $\alpha = \beta$. $M_\Gamma, \Delta \models \alpha = \beta$ iff $I_\Gamma(\alpha) = I_\Gamma(\beta)$ iff $[[\alpha]]_\Gamma = [[\beta]]_\Gamma$ (by Definition 7v.1.2.) iff $\alpha = \beta \in \Gamma$ (by the definition of $[[\alpha]]_\Gamma$) iff $\alpha = \beta \in \Delta$ (since $\Delta \in W_\Gamma$).

Case 3: φ is of the form $\mathsf{O}_\alpha\psi$. For right to left, suppose $\mathsf{O}_\alpha\psi \in \Delta$. So for every $\Theta \in R_\Gamma(\Delta, I_\Gamma(\alpha))$, $\psi \in \Theta$. By the induction hypothesis, $M_\Gamma, \Theta \models \psi$ for every $\Theta \in R_\Gamma(\Delta, I_\Gamma(\alpha))$. Hence $M_\Gamma, \Delta \models \mathsf{O}_\alpha\psi$.

For left to right, suppose $\mathsf{O}_\alpha\psi \notin \Delta$. By Lemma 3, there is an $\mathcal{L}^+$-saturated set Θ such that $\{\chi \mid \mathsf{O}_\alpha\chi \in \Delta\} \cup \{\neg\psi\} \subseteq \Theta$. By Lemma 5 and Definition 7.iii., $\Theta \in R_\Gamma(\Delta, [[\alpha]]_\Gamma)$. By the induction hypothesis, $M_\Gamma, \Theta \not\models \psi$ and hence $M_\Gamma, \Delta \not\models \mathsf{O}_\alpha\psi$.

Case 4: The proof for this case (where φ is of the form $\mathsf{O}^\beta_\alpha\psi$) is analogous to that of the previous case, except for some obvious replacements in the notation and the justification. $\square$

Theorem 5 (Soundness and Strong Completeness for **TMDL**)**.** $\Gamma \vdash \varphi$ *iff* $\Gamma \Vdash \varphi$

Proof. Soundness is a matter of routine, each of the axioms and rules is valid. For completeness, suppose $\Gamma \nvdash \varphi$. Hence, $\Gamma \cup \{\neg\varphi\}$ is consistent. By Lemma 2, we can construct an $\mathcal{L}^+$-saturated set Θ such that $\Gamma \cup \{\neg\varphi\} \subseteq \Theta$. By Lemma 7, we know that the canonical model M_Θ is a **TMDL**-model. By Lemma 8, $M_\Theta, \Theta \models \varphi$ iff $\varphi \in \Theta$. Since $\varphi \notin \Theta$, $M_\Theta, \Theta \not\models \varphi$. Hence, $\Gamma \nVdash \varphi$. $\square$

7 Using TMDL to represent general deontic statements

In this section we show how **TMDL** succeeds in capturing the general deontic statements that we identified in sections 2 and 3. We first discuss the interaction principles between directed and undirected personal obligations, before turning to quantification over bearers and counterparties of directed obligations and permissions in **TMDL**. After that we turn to the formal account of the Hohfeldian theory of rights that we obtain from **TMDL**.

7.1 The principles (DIU) and (DPIU)

It is possible to argue that a principle like (DIU) is counter-intuitive. One can then read the following example of Hage as a counterexample to (DIU).

> For instance, if Antony contracts with Giovanni to transfer his car to Giovanni, and if he also contracts with Guido to transfer his car to him, then Antony both has an obligation toward Giovanni and toward Guido. It is impossible for Antony to comply with both obligations and therefore it is not the case that Antony both ought to transfer the car to Giovanni and to Guido [12, pp. 126 - 127].

If one were to follow this line of argument, then one can reject the axiom (DIU). To obtain a fragment of **TMDL** without (DIU), it suffices to reject (DIU) and Condition 4.2 of Definition 1.[20]

However, we disagree with this analysis of the example. In our view this is an example of a deontic conflict. Antony has two conflicting obligations and this leads to a problem not because we accept (DIU), but because our logic is not conflict-tolerant. The development of conflict-tolerant variants of **TMDL** is left for future work.[21]

A different principle that might seem intuitive is (DPIU): $(\forall x)\mathsf{P}_a^x\varphi \to \mathsf{P}_a\varphi$. This principle states that if a has a directed permission towards everyone that φ, then a has a bearer-relative permission that φ. This principle is not valid in **TMDL**. We give two reasons why we think that this principle should indeed be invalid.

First, $\mathsf{O}_a\varphi \to (\exists x)\mathsf{O}_a^x\varphi$ follows from (DPIU), by contraposition and the definitions of $(\exists\nu)\varphi$, $\mathsf{P}_\theta\varphi$ and $\mathsf{P}_\kappa^\theta\varphi$.[22] However, this is a counter-intuitive principle. An agent a can have an obligation that φ towards b and an obligation that ψ towards a c such that $c \neq b$. From this it follows that a has an obligation that $\varphi \wedge \psi$, but this does not imply that there is a person towards whom a has the obligation that $\varphi \wedge \psi$.

Secondly, there are intuitive counterexamples to (DPIU). Consider Hansson's Petaluma example:

> every landowner in Petaluma, Calif., has forbidden a to walk on his (the landowner's) land. It is obvious that a then has no permission to walk on private land in Petaluma. But it is still the case that he has such permission with respect to every individual y, for even if y is a landowner in Petaluma he cannot forbid a to walk on land owned by others in Petaluma. (I have assumed that there is more than one landowner in Petaluma.) [13, 245-256]

This example is set up such that a has against all persons a directed permission to walk on private land, but he does not have the bearer-relative permission to do so. Thus, this is a counterexample to (DPIU).

[20] Note that $\{\mathsf{O}_a^b\varphi, \mathsf{O}_a^c\neg\varphi\}$ is inconsistent in **TMDL**, but not in this new logic.

[21] For conflict-tolerant deontic logic, see [32, 11].

[22] The converse is not valid.

7.2 General deontic statements in TMDL

We can represent categorical deontic statements in **TMDL**. Take for example 'Everyone has an obligation not to kill'. Let Kxy stand for 'x kills y', then we can formalise this as $(\forall x)(\forall y)\mathsf{O}_x\neg Kxy$. Note that this example shows that **TMDL** does not suffer from the bearer-in-scope problem (and that **TMDL** does not rely on finite conjunctions for the formalisation of general deontic statements).

We can also easily represent general deontic statements that are not categorical. Let us return to the example from the Belgian traffic regulations: 'Every driver has to give way to the driver coming from the right, unless he is driving on a roundabout or the driver from the right is coming from a forbidden direction.' Let Dx be interpreted as 'x is a driver', Cx as 'x is on a roundabout', Fx as 'x is coming from a forbidden direction', Rxy as 'x comes from y's right' and Gxy as 'x gives way to y'. Then we can formalise this rule as $(\forall x)(\forall y)((Dx \wedge Dy \wedge \neg Cx \wedge \neg Fy \wedge Ryx) \rightarrow \mathsf{O}_x(Gxy))$. This example shows that **TMDL** is not limited to categorical statements .

In addition to representing general deontic statements in **TMDL**, we can also capture reasoning with these statements. Let us take the example in Section 2 about appearing in court to illustrate this. The first premise is "Everyone who is summoned to court as a witness, is obliged to appear in court" [7, p. 26]. Let Sx stand for 'x is summoned to court as a witness' and Ax for 'x appears in court'. Then we can formalise this first premise as $(\forall x)(Sx \rightarrow \mathsf{O}_x Ax)$. The second premise is: "a is summoned to court as a witness" [7, p. 26]. This can be formalised as Sa. By (UI) and CL it follows that $\mathsf{O}_a Aa$, which is to be interpreted as the intended conclusion: "a is obliged to appear in court" [7, p. 26].

When it comes to directed obligations and permissions, **TMDL** can also capture quantification over the counterparties. Consider again the sentence 'a has an obligation towards all of her employees to pay them their wages'. Let Exy be interpreted as 'x is an employee of y' and Wxy as 'x pays the wage of y'. **TMDL** allows us to formalise the sentence as $(\forall x)(Exa \rightarrow \mathsf{O}_a^x Wax)$.

Naturally, it follows that we can quantify over both the bearers and counterparties of directed obligations at the same time. For example, a formalisation of 'all employers have an obligation towards their employees to pay them their wages' is $(\forall x)(\forall y)(Exy \rightarrow \mathsf{O}_y^x Wyx)$. These two examples illustrate that **TMDL** does not suffer from the counterparty-in-scope problem.

Finally, we can also represent the personal obligations that those using a Hilpinen-inspired account cannot. As an example we can look at the obligation of minister a that the king b does not sign an unconstitutional law. Let Sx stand for 'x signs an unconstitutional law', then this obligation can be represented as $\mathsf{O}_a\neg Sb$.

7.3 The Hohfeldian rights relations in TMDL

In Section 3 we discussed the Hohfeldian rights relations. In this section we will show how we can use **TMDL** to formalise these relations. We then move on to classes of multital rights and to rules of rights, before bringing everything together with the parking example of Section 3.

First we define $\mathsf{CL}^x_y\varphi$, to be read as 'y has a claim on x that φ'. According to the Hohfeldian analysis a claim of y on x is equivalent to a duty of x towards y. Since we identified duties with directed obligations, Definition 8 follows naturally.

Definition 8 (Claim). $\mathsf{CL}^x_y\varphi =_{\mathsf{df}} \mathsf{O}^y_x\varphi$

The formal definition of a no-claim is slightly more complex. Recall that we identified privileges with directed permissions. Hohfeld points out that a privilege is the opposite of a duty, i.e. the opposite of a claim and thus a no-claim, "having a content or tenor precisely opposite to that of the privilege in question" [17, p. 32]. The language of **TMDL** allows us to express this 'opposite content or tenor' more precisely with Definition 9, where $\mathsf{NC}^x_y\varphi$ is to be interpreted as 'y has a no-claim on x that φ'.

Definition 9 (No-claim). $\mathsf{NC}^x_y\varphi =_{\mathsf{df}} \mathsf{P}^y_x\neg\varphi$

Given Definition 8 and Definition 9 we obtain Figure 2, the formal counterpart of Figure 1. Again the dashed arrows denote a relation of opposites, whereas the normal arrows denote a relation of correspondence. These correspondences follow immediately from Definition 8 and Definition 9. The relation of opposites follows from the definition of $\mathsf{P}^x_y\varphi$: $\mathsf{O}^x_y\varphi \leftrightarrow \neg\mathsf{P}^x_y\neg\varphi$ and hence $\mathsf{O}^x_y\varphi$ is the opposite of $\mathsf{P}^x_y\neg\varphi$. Similarly, from $\mathsf{NC}^x_y\varphi \leftrightarrow \mathsf{P}^y_x\neg\varphi$, $\mathsf{P}^y_x\neg\varphi \leftrightarrow \neg\mathsf{O}^y_x\varphi$ and $\neg\mathsf{O}^y_x\neg\varphi \leftrightarrow \neg\mathsf{CL}^x_y\varphi$, it follows that $\mathsf{NC}^x_y\varphi \leftrightarrow \neg\mathsf{CL}^x_y\varphi$ and thus that $\mathsf{NC}^x_y\varphi$ is the opposite of $\mathsf{CL}^x_y\varphi$.

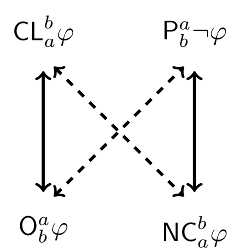

Figure 2: Hohfeldian rights relations formalised

It has been pointed out in the literature that there is some asymmetry in the Hohfeldian account [24, 27, 34]. A duty that φ is the opposite of a privilege that the *negation* of φ, whereas the opposite of a claim that φ is a no-claim that φ. Similarly, a claim that φ is equivalent to a duty that φ, but a privilege that φ is equivalent to a no-claim that *not* φ.[23] From this asymmetry Sergot concludes that Hohfeld's account is "not precise enough to give a formal theory" and that we instead need a more precise theory such as that proposed by Kanger [34, p. 357]. We disagree with this conclusion.

Instead we propose that this asymmetry is due to the fact that there is no term in English that is the dual of a claim in the same way that a directed permission is the dual of a directed obligation.[24] That the asymmetry is not an indication of the impreciseness of the Hohfeldian framework is further illustrated by the fact that the symmetry returns when we substitute the Hohfeldian notions with equivalent expressions employing only a directed obligation operator (see Figure 3).

$$
\begin{tikzcd}
\mathsf{O}^a_b\varphi \arrow[d, leftrightarrow] \arrow[dr, leftrightarrow, dashed] & \neg\mathsf{O}^a_b\neg\neg\varphi \arrow[d, leftrightarrow] \arrow[dl, leftrightarrow, dashed] \\
\mathsf{O}^a_b\varphi & \neg\mathsf{O}^a_b\neg\neg\varphi
\end{tikzcd}
$$

Figure 3: Hohfeldian rights relations as directed obligations

All of this is not to say that the theories of legal relations developed by Kanger and others are without merit. On the contrary, we think that these theories offer valuable insights and would benefit from term-modal versions. However, since Kanger and those inspired by him employ an agency operator, the term-modal versions of their logics fall outside the scope of the present paper. We hence leave a full integration of those theories for future work.

7.4 Multital rights relations in TMDL

In Section 3 we noted that classes of multital rights and rules of rights also play a prominent role in legal reasoning. We now turn to a formal account of these notions, starting with classes of multital rights. To simplify our account, we will use some terms from our discussion on directed obligations and permissions when talking about rights. We will call the person who has a right the bearer of that right and we will call the person against whom someone has a right the counterparty of that right. Thus, if a has a claim on b, then we call a the bearer of that claim and b the counterparty.

We can formalise classes of multital rights by quantifying over the indexes of the operators. In Section 3 we discussed the class of permissions that a has towards all others that a walks on a's land. If we interpret Wxy as 'x

[23] This asymmetry was also noted by Hohfeld, see Section 3.

[24] Note that Hohfeld himself also laments the fact that there is no single term available to express the correlative of a permission [17, p. 33].

walks on land owned by y', then we can formalise this class of permissions as: $(\forall x)(x \neq a \rightarrow \mathsf{P}^x_a Waa)$. Another example from Section 3 was the class of multital claims that a has towards all other ordinary persons that those persons shall not strike a. Let Sxy be 'x strikes y' and let Nx be 'x is an ordinary person', then we can formalise this class of claims as $(\forall x)((x \neq a \wedge Nx) \rightarrow \mathsf{CL}^x_a \neg Sxa)$.

Hohfeld does point out that some classes of multital rights have exceptions [18]. Given that a is the owner of a property, a has multital claims against others that they do not enter his property. However, a can give his friends c and d "leave and license" to enter the property [18, p. 743]. In this case a has no claims against c and d that they shall not enter his property, but a does still have such claims against all others. Let Exy be interpreted as 'x enters the property of y'. Then we can formalise these exceptions in two ways. We can simply list the exceptions: $(\forall x)((x \neq a \wedge x \neq c \wedge x \neq d) \rightarrow \mathsf{CL}^x_a \neg Exa)$. Alternatively, we can make explicit the reason for the exception. Let Lxy mean 'x has given leave and license to y to enter x's property', then we can use $(\forall x)((x \neq a \wedge \neg Lax) \rightarrow \mathsf{CL}^x_a \neg Exa)$ as a formalisation of the class of multital claims that a has.

In Section 3 it was shown that we can represent rules of rights by quantifying over both the counterparty and the bearer of the right. **TMDL** allows for this. Let us take Kanger's example discussed in Section 3:

> For every x and y such that x is a pedestrian and y is a motorist who encounters x, it is the case that x has versus y a right of atomic type: claim (...) to the effect that y does not run into x [24, p. 131].

Let Fx mean 'x is a pedestrian', Mx 'x is a motorist', Exy 'x encounters y' and Rxy 'x runs into y'. Then we can formalise this rule of rights as: $(\forall x)(\forall y)((Fx \wedge My \wedge Eyx) \rightarrow \mathsf{CL}^y_x \neg Ryx)$

To end this section we will return to the parking example. This example illustrates the different logical relations between rules of rights, classes of multital rights and specific multital rights. In this example, there were two rules of rights. The first of these was the rule that all disabled people have a privilege against all others to park in the parking spot. Let Sx be 'x parks in the parking spot' and Dx 'x is disabled', then we can formalise this first rule as $(\forall x)(Dx \rightarrow (\forall y)\mathsf{P}^y_x Px)$. By (UI), CL and the fact that a is disabled (Da), it follows that $(\forall y)\mathsf{P}^y_a Pa$, which represents the class of multital privileges that a has to park in the spot. By applying (UI) again we get to the specific multital privilege that a has against b to park in the spot: $\mathsf{P}^b_a Pa$. A similar account can be given of the second rule of rights.

8 Conclusion

In this article we argued that previous deontic logics are inadequate in representing general deontic statements. We showed that using these logics to formalise reasoning with general deontic statements leads to a number of problems. We then presented **TMDL**, which allows for quantification over the indexes of obligation operators and as a result is able to capture reasoning with those statements and avoids the problems associated with previous logics.

TMDL is a conservative extension of **SDL**. As a result many of the paradoxes and deficiencies of **SDL** are inherited by **TMDL**. To avoid this we should look at term-modal versions of other (for example non-normal or conditional) deontic logics that avoid these paradoxes.[25] Indeed, **TMDL** opens many avenues for future research.

In view of the interest in deontic logic from computer scientists it might be interesting to look into decidable fragments of **TMDL**. There is already work on decidable fragments of term-modal logic, for example [5]. In light of this work it is plausible that decidable fragments of **TMDL** can be found.

Another promising avenue of research would be to expand **TMDL** with an agency operator that is indexed by terms of the language. This will help to capture the more extensive classification of rights relations proposed by Kanger, Lindahl and others [24, 27, 34]. This could then be used to formalise multital versions of their rights relations and rules of rights involving their rights relations. Related to this, one could develop an extension of **TMDL** that also captures higher order rights.

There are more possibilities for future research. First among these is the application of **TMDL** to ethical reasoning. General and restricted obligations as well as moral rights play a role there. In this context it is also interesting to look at the interaction of situational obligations with the bearer-relative and directed obligations presented in this paper. Secondly, the domains of **TMDL** only consist of persons. It might be interesting to expand the domains with other objects. For example, an interesting problem is posed by the observation that animals can (arguably) be bearers of rights, but not of obligations. Thirdly, it was already pointed out that a defeasible version of **TMDL** might be better suited to explicate the defeasible nature of legal reasoning or the reasoning with deontic conflicts.

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