On four codes with automorphism group $P\Sigma L(3,4)$ and pseudo-embeddings of the large Witt designs

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Abstract

A pseudo-embedding of a point-line geometry is a representation of the geometry into a projective space over the field $\mathbb{F}_2$ such that every line corresponds to a frame of a subspace. Such a representation is called homogeneous if every automorphism of the geometry lifts to an automorphism of the projective space. In this paper, we determine all homogeneous pseudo-embeddings of the three Witt designs that arise by extending the projective plane $\mathrm{PG}(2,4)$. Along our way, we come across some codes with automorphism group $P\Sigma L(3,4)$ and sets of points of $\mathrm{PG}(2,4)$ that have a particular intersection pattern with Baer subplanes or hyperovals.

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1 Introduction

This paper deals with a number of problems regarding the projective plane $\mathrm{PG}(2,4)$ and the Witt designs $W_{22} = S(3,6,22)$, $W_{23} = S(4,7,23)$ and $W_{24} = S(5,8,24)$.

It is known that the projective plane $\mathrm{PG}(2,4)$ has 360 Baer subplanes and 168 hyperovals. The set $\mathcal{B}$ of all Baer subplanes and the set $\mathcal{H}$ of all hyperovals form one orbit for the projective group $PGL(3,4)$. Under the action of $PSL(3,4)$, both sets split into three suborbits of equal size, say $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3$ and $\mathcal{H} = \mathcal{H}_1 \cup \mathcal{H}_2 \cup \mathcal{H}_3$, where the subindices can be chosen in such a way that the following two properties hold (see [11, 15, 17]):

- If $i \in \{1,2,3\}$, then any two Baer subplanes of $\mathcal{B}_i$ meet in an odd number of points, any two hyperovals of $\mathcal{H}_i$ meet in an even number of points and any Baer subplane of $\mathcal{B}_i$ meets every hyperoval of $\mathcal{H}_i$ in an even number of points.
- If $i, j \in \{1,2,3\}$ with $i \neq j$, then any Baer subplane of $\mathcal{B}_i$ meets every Baer subplane of $\mathcal{B}_j$ in an even number of points, any hyperoval of $\mathcal{H}_i$ intersects every hyperoval of $\mathcal{H}_j$ in an odd number of points, and any Baer subplane of $\mathcal{B}_i$ intersects every hyperoval of $\mathcal{H}_j$ in an odd number of points.

These facts lead to the following problems.

Problem 1. Which are the sets of points of $\mathrm{PG}(2,4)$ that meet a given Baer subplane of $\mathcal{B}_i$ in an even (respectively, odd) number of points?

Problem 2. Which are the sets of points of $\mathrm{PG}(2,4)$ that meet a given hyperoval of $\mathcal{H}_i$ in an even (respectively, odd) number of points?

In this paper, we solve both problems. While solving these problems, we come across four codes whose properties (weights of codewords, automorphisms) will be investigated.

Our results about these sets of points will have implications for the so-called pseudo-embeddings of the Witt designs W_{22}, W_{23} and W_{24}.

Suppose $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is a point-line geometry for which the number of points on each line is finite and at least three. A *pseudo-embedding* of $\mathcal{S}$ is a mapping ϵ from $\mathcal{P}$ to the set of points of a projective space $\mathrm{PG}(V)$, with V an $\mathbb{F}_2$-vector space, such that the following properties are satisfied:

(PS1) The image of ϵ generates $\mathrm{PG}(V)$.

(PS2) ϵ maps any line L of $\mathcal{S}$ to a frame of the subspace $\langle \epsilon(L) \rangle$ of $\mathrm{PG}(V)$.

Such a pseudo-embedding will shortly be denoted by $\epsilon : \mathcal{S} \to \mathrm{PG}(V)$. With a *frame* of a projective space of finite dimension d, we mean here a set of $d+2$ points no $d+1$ of which are contained in a hyperplane. In the above definition of pseudo-embedding, it is required that ϵ maps each line of $\mathcal{S}$ in an injective way to a frame of a subspace of $\mathrm{PG}(V)$. It is however not required that the map ϵ itself must also be injective. An injective pseudo-embedding is also called a *faithful* pseudo-embedding.

Two pseudo-embeddings $\epsilon_1 : \mathcal{S} \to \mathrm{PG}(V_1)$ and $\epsilon_2 : \mathcal{S} \to \mathrm{PG}(V_2)$ of the same point-line geometry $\mathcal{S}$ are called *isomorphic* if there exists a linear isomorphism θ from V_1 to V_2 such that $\epsilon_2 = \theta \circ \epsilon_1$.

If G acts as a group of automorphisms on $\mathcal{S}$, then we call a pseudo-embedding $\epsilon : \mathcal{S} \to \mathrm{PG}(V)$ *G-homogeneous* if for every $\theta \in G$, there exists a (necessarily unique) $\widetilde{\theta} \in GL(V)$ such that $\widetilde{\theta} \circ \epsilon = \epsilon \circ \theta$. If this is the case, then the map $G \to GL(V) : \theta \mapsto \widetilde{\theta}$ defines a modular representation of G, and V becomes a (possibly reducible) G-module. If G is the full group of automorphisms of $\mathcal{S}$, then a G-homogeneous pseudo-embedding is also called a *homogeneous pseudo-embedding*.

Suppose $\epsilon : \mathcal{S} \to \mathrm{PG}(V)$ is a pseudo-embedding of $\mathcal{S}$ and α is a subspace of $\mathrm{PG}(V)$ having no point in common with $\epsilon(\mathcal{P})$ nor with any of the $\langle \epsilon(L) \rangle$'s, where $L \in \mathcal{L}$. Then the map $x \mapsto \langle \alpha, \epsilon(x) \rangle$ defines a *pseudo-embedding* ϵ/α of $\mathcal{S}$ into the quotient projective space $\mathrm{PG}(V)/\alpha$ (whose points are the subspaces of $\mathrm{PG}(V)$ that contain α as a hyperplane). We call ϵ/α a *quotient* of ϵ. If ϵ_1, ϵ_2 are two pseudo-embeddings of $\mathcal{S}$, then we write $\epsilon_1 \geq \epsilon_2$ if ϵ_2 is isomorphic to a quotient of ϵ_1.

If $\widetilde{\epsilon}$ is a pseudo-embedding of $\mathcal{S}$ with the property that $\widetilde{\epsilon} \geq \epsilon$ for any other pseudo-embedding ϵ of $\mathcal{S}$, then we call $\widetilde{\epsilon}$ a *universal pseudo-embedding* of $\mathcal{S}$. It can be proved that if $\mathcal{S}$ has pseudo-embeddings, then it also has a universal pseudo-embedding which is moreover unique, up to isomorphism. If $\mathcal{S}$ has faithful pseudo-embeddings, then the universal embedding is faithful as well. The universal pseudo-embedding is always homogeneous. The vector dimension of the universal pseudo-embedding is called the *pseudo-embedding rank*. If $|\mathcal{P}| < \infty$, then the pseudo-embedding rank is equal to $|\mathcal{P}| - \dim(C)$, where C is the binary code of length $|\mathcal{P}|$ generated by the characteristic vectors of the lines of $\mathcal{S}$. So, besides the connection with modular representation theory alluded to above, there is also a connection between pseudo-embeddings and coding theory, and the study of certain problems in the area of coding theory can benefit from this connection.

Pseudo-embeddings have been introduced in [7] and further investigated in [6, 8]. We refer to these papers for proofs of the above-mentioned facts. In the present paper, we classify all homogeneous pseudo-embeddings of the Witt designs W_{22}, W_{23} and W_{24}. We show that W_{22} has, up to isomorphism, two homogeneous pseudo-embeddings, namely the universal one in $\mathrm{PG}(10, 2)$ and another one in $\mathrm{PG}(9, 2)$. We also show that W_{23} and W_{24} have up to isomorphism unique homogeneous pseudo-embeddings, namely their universal pseudo-embeddings in respectively $\mathrm{PG}(10, 2)$ and $\mathrm{PG}(11, 2)$. As a useful by-product we prove that the projective plane $\mathrm{PG}(2, 4)$ has up to isomorphism five $PSL(3, 4)$-homogeneous pseudo-embeddings. We also give explicit constructions (using coordinates) for each of these pseudo-embeddings.

Representations of point-line geometries in projective spaces where the lines correspond to certain nice configurations of points in subspaces have been intensively investigated in the literature. The standard representations are those where the lines of the geometry correspond to full lines of the projective space. Such representations are called *full projective embeddings*. Other situations have also been investigated: lines can correspond to subsets of lines (the so-called *lax projective embeddings*), conics [14, 18], ovals [12, 20, 23], ovoids [4, 9, 10, 16] and rational normal curves [19]. For pseudo-embeddings, the lines correspond to frames of subspaces and in this case the projective space should be defined over the field $\mathbb{F}_2$. In the case each line has exactly three points, such frames are projective lines and so the notions of pseudo-embeddings and full projective embeddings then coincide. For geometries with four points per line, such frames are hyperovals in planes, and for geometries with five points per line, such frames are ovoids of 3-spaces. It is therefore no surprise that pseudo-embeddings have showed up in some of the above literature. In [9] for instance, a connection was mentioned between the universal pseudo-embeddings of the projective space $\mathrm{PG}(2, 4)$ and the Witt design $S(5, 8, 24)$. We will meet this connection later in Section 7 of this paper.

Besides the "large" Witt designs W_{22}, W_{23} and W_{24}, there are two other Witt designs, namely $W_{11} = S(4, 5, 11)$ and $W_{12} = S(5, 6, 12)$. These arise by extending the affine plane $\mathrm{AG}(2, 3)$. In [7, Proposition 3.4(2)], it was shown that $\mathrm{AG}(2, 3)$ does not have pseudo-embeddings. The results of Section 7 (more precisely, Proposition 7.1) then imply that neither of the "small" Witt designs W_{11} and W_{12} can have pseudo-embeddings.

The homogeneous pseudo-embeddings of the large Witt designs we determine here realize models of these designs as sets of points in projective spaces where the blocks can be obtained as intersections with certain subspaces. This situation is very similar to the realization of the Witt design W_{12} in $\mathrm{PG}(5, 3)$ described by Coxeter in [5].

2 Preliminaries

In this section, we recall some known facts on pseudo-embeddings and Witt designs.

Suppose $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is a point-line geometry with the property that the number of points on each line is finite and at least 3.

An *even set* of $\mathcal{S}$ is a set of points meeting each line of $\mathcal{S}$ in an even number of points. The empty point set is the trivial example of an even set. The complements of the nontrivial even sets are also called *pseudo-hyperplanes*. We denote by $\mathcal{E}_\mathcal{S}$ the set of all even sets of $\mathcal{S}$. If $X_1, X_2 \in \mathcal{E}_\mathcal{S}$, then their symmetric difference $X_1 + X_2 := X_1 \Delta X_2$ also belongs to $\mathcal{E}_\mathcal{S}$. For every $X \in \mathcal{E}_\mathcal{S}$, we also define $0 \cdot X := \emptyset$ and $1 \cdot X := X$. With this definition, the set $\mathcal{E}_\mathcal{S}$ becomes an $\mathbb{F}_2$-vector space with zero vector $\emptyset \in \mathcal{E}_\mathcal{S}$. If $\mathcal{S}$ has pseudo-embeddings and $|\mathcal{E}_\mathcal{S}| < \infty$, then it is a consequence of Proposition 2.1 below that the dimension of the vector space $\mathcal{E}_\mathcal{S}$ coincides with the pseudo-embedding rank of $\mathcal{S}$.

If $\epsilon : \mathcal{S} \to \mathrm{PG}(V)$ is a pseudo-embedding of $\mathcal{S}$, then for every hyperplane Π of $\mathrm{PG}(V)$, the set $H_\Pi := \epsilon^{-1}(\epsilon(\mathcal{P}) \cap \Pi)$ is a pseudo-hyperplane of $\mathrm{PG}(V)$ by [7, Theorem 1.1]. The set of all pseudo-hyperplanes arising in this way will be denoted by $\mathcal{H}_\epsilon$. The set of all complements of the elements of $\mathcal{H}_\epsilon$ will be denoted by $\mathcal{E}_\epsilon \subseteq \mathcal{E}_\mathcal{S} \setminus \{\emptyset\}$. The following can be said about $\mathcal{H}_\epsilon$ if ϵ is the universal pseudo-embedding of $\mathcal{S}$, for a proof see Theorem 1.3 of [7].

Proposition 2.1 ([7]). *Suppose $\mathcal{S}$ has pseudo-embeddings and denote by $\widetilde{\epsilon} : \mathcal{S} \to \mathrm{PG}(\widetilde{V})$ the universal pseudo-embedding of $\mathcal{S}$. Then $\mathcal{H}_{\widetilde{\epsilon}}$ is the set of all pseudo-hyperplanes of $\mathcal{S}$. In fact, the map $\Pi \mapsto H_\Pi = \tilde{\epsilon}^{-1}(\tilde{\epsilon}(\mathcal{P}) \cap \Pi)$ defines a bijective correspondence between the set of hyperplanes of $\mathrm{PG}(\widetilde{V})$ and the set of pseudo-hyperplanes of $\mathcal{S}$.*

If $\epsilon : \mathcal{S} \to \mathrm{PG}(V)$ is a pseudo-embedding and the pseudo-hyperplane H of $\mathcal{S}$ is equal to $\epsilon^{-1}(\epsilon(\mathcal{P}) \cap \Pi)$ for some hyperplane Π of $\mathrm{PG}(V)$, then Π might not be the only hyperplane of $\mathrm{PG}(V)$ containing $\epsilon(\mathcal{P})$. If Π' is another such hyperplane, then the pseudo-hyperplane $H' = \epsilon^{-1}(\epsilon(\mathcal{P}) \cap \Pi')$ properly contains H. An example where such a situation occurs will be given in the paragraph following Lemma 8.6.

The following proposition was proved on page 79 of [7].

Proposition 2.2 ([7]). *Suppose ϵ_1 and ϵ_2 are two pseudo-embeddings of $\mathcal{S}$. Then ϵ_1 and ϵ_2 are isomorphic if and only if $\mathcal{H}_{\epsilon_1} = \mathcal{H}_{\epsilon_2}$.*

Not all point-line geometries with the property that the number of points on each line is finite and at least three have pseudo-embeddings. However, it is possible to give necessary and sufficient conditions for the existence of pseudo-embeddings in terms of pseudo-hyperplanes. The following proposition is precisely Theorem 1.4(1) of [7].

Proposition 2.3 ([7]). *$\mathcal{S}$ has a pseudo-embedding if and only if the set of all pseudo-hyperplanes of $\mathcal{S}$ satisfies the following conditions.*

- *If L is a line containing an odd number of points and x is a point of L, then there exists a pseudo-hyperplane intersecting L in the singleton $\{x\}$.*

- *If L is a line containing an even number of points and x_1, x_2 are two points of L, then there exists a pseudo-hyperplane intersecting L in the pair $\{x_1, x_2\}$.*

The following proposition was proved in Proposition 2.6 and Corollary 2.7 of [8].

Proposition 2.4 ([8])**.** *Suppose $G \leq Aut(\mathcal{S})$. A set $\mathcal{H}$ of pseudo-hyperplanes is of the form $\mathcal{H}_\epsilon$ for some G-homogeneous pseudo-embedding ϵ of $\mathcal{S}$ if and only the following properties hold:*

(a) *$\mathcal{H}$ is the union of G-orbits of pseudo-hyperplanes;*

(b) *if $H_1, H_2 \in \mathcal{H}$ with $H_1 \neq H_2$, then the complement of $H_1 \Delta H_2$ also belongs to $\mathcal{H}$;*

(c) *if L is a line of $\mathcal{S}$ containing an odd number of points and $x \in L$, then there exists an $H \in \mathcal{H}$ such that $H \cap L = \{x\}$;*

(d) *if L is a line of $\mathcal{S}$ containing an even number of points and $x_1, x_2 \in L$ with $x_1 \neq x_2$, then there exists an $H \in \mathcal{H}$ such that $H \cap L = \{x_1, x_2\}$;*

(e) *for every point x of $\mathcal{S}$, there exists an $H \in \mathcal{H}$ not containing x.*

The following proposition is precisely Theorem 3.1 of [6].

Proposition 2.5 ([6])**.** *Let V_1 and V_2 be two vector spaces over $\mathbb{F}_2$. For every $i \in \{1, 2\}$, let ϵ_i be a map from the point set $\mathcal{P}$ of $\mathcal{S}$ to the point set of $\mathrm{PG}(V_i)$ and let $\mathcal{H}_i$ be the set of all sets of the form $\epsilon_i^{-1}(\epsilon_i(\mathcal{P}) \cap \Pi)$, where Π is some hyperplane of $\mathrm{PG}(V_i)$. If ϵ_1 is a pseudo-embedding of $\mathcal{S}$ and $\mathcal{H}_1 = \mathcal{H}_2$, then also ϵ_2 is a pseudo-embedding of $\mathcal{S}$. Moreover, ϵ_2 is isomorphic to ϵ_1.*

The Witt designs W_{11}, W_{12}, W_{22}, W_{23} and W_{24} are examples of so-called Steiner systems. A *Steiner system of type* $S(t, k, v)$ with $t, k, v \in \mathbb{N}$ such that $2 \leq t \leq k \leq v$ is a block design $(\mathcal{P}, \mathcal{B})$ having $v = |\mathcal{P}|$ points such that every block $B \in \mathcal{B}$ contains exactly k points and every t distinct points are contained in a unique block. The Witt designs W_{11}, W_{12}, W_{22}, W_{23} and W_{24} are the unique Steiner systems of types $S(4, 5, 11)$, $S(5, 6, 12)$, $S(3, 6, 22)$, $S(4, 7, 23)$ and $S(5, 8, 24)$. Explicit constructions of these designs can be found in [11, 15, 17]), or in Sections 4, 5 and 6 of this paper.

Suppose $\mathcal{S} = (\mathcal{P}, \mathcal{B})$ is a Steiner system of type $S(t, k, v)$ with $t \geq 3$. Let $x \in \mathcal{P}$ and denote by $\mathcal{B}_x$ the set of all sets of the form $B \setminus \{x\}$ where $B \in \mathcal{B}$ contains x. Then $\mathcal{S}_x = (\mathcal{P}_x, \mathcal{B}_x)$ with $\mathcal{P}_x = \mathcal{P} \setminus \{x\}$ is a Steiner system of type $S(t-1, k-1, v-1)$ which is called a *derived design* of $\mathcal{S}$. Conversely, $\mathcal{S}$ is called an *extension* of $\mathcal{S}_x$. The Witt design W_{22} is an extension of the projective plane $\mathrm{PG}(2, 4)$ which is the unique Steiner system of type $S(2, 5, 21)$, W_{23} is an extension of W_{22} and W_{24} is an extension of W_{23}. The automorphism groups of W_{24} and W_{23} are the respective Mathieu groups M_{24} and M_{23}. The Mathieu group M_{22} is the derived subgroup of the automorphism group $Aut(W_{22})$ of W_{22} and has index 2 in $Aut(W_{22}) \cong M_{22} : 2$. The Mathieu group M_i, $i \in \{22, 23, 24\}$, acts $(i - 19)$-transitively on the points of W_i.

3 Four codes with automorphism group $P\Sigma L(3,4)$

During the classification of the homogeneous pseudo-embeddings of the large Witt designs, we will rely on some information about certain codes related to hyperovals and Baer subplanes of $\mathrm{PG}(2,4)$. Codes related to these point sets have already been investigated (as in [2]), but the novelty is that we will study some codes that originate from the $PSL(3,4)$-orbits of hyperovals and Baer subplanes.

Let V be a 3-dimensional vector space over $\mathbb{F}_4 = \{0,1,\omega,\omega^2\}$. Consider the projective plane $\mathrm{PG}(2,4) = \mathrm{PG}(V)$ with point set $\mathcal{P}$ and line set $\mathcal{L}$. For every set X of points of $\mathrm{PG}(2,4)$, we denote by $\overline{X} := \mathcal{P} \setminus X$ the complement of X. For every set $\mathcal{A}$ of sets of points of $\mathrm{PG}(2,4)$, we define $\overline{\mathcal{A}} := \{\overline{X} \,|\, X \in \mathcal{A}\}$. Recall that $\mathcal{B}$ denotes the set of all Baer subplanes of $\mathrm{PG}(2,4)$ and that $\mathcal{H}$ denotes the set of all hyperovals of $\mathrm{PG}(2,4)$. We consider the following additional sets of points in $\mathrm{PG}(2,4)$:

- $\mathcal{U}$ is the set of unitals of $\mathrm{PG}(2,4)$;
- $\mathcal{V}$ consists of all sets of the form $L_1 \Delta L_2$ where L_1 and L_2 are two distinct lines;
- $\mathcal{W}$ consists of all sets of the form $H \cup L$, where $H \in \mathcal{H}$ and $L \in \mathcal{L}$ with $H \cap L = \emptyset$.

The sizes of these sets are $|\mathcal{U}| = 280$, $|\mathcal{V}| = 210$ and $|\mathcal{W}| = 1008$. If $\mathcal{A}$ is a set of points of $\mathrm{PG}(2,4)$, then $|\overline{\mathcal{A}}| = |\mathcal{A}|$. Note that each element of $\overline{\mathcal{V}}$ is the set of points defined by a pencil of three lines.

The first claim of the following lemma is precisely Theorem 19.6.2 of Hirschfeld [13] and is a special case of results obtained by M. Tallini Scafati [21, 22].

Lemma 3.1. *The sets of points of* $\mathrm{PG}(2,4)$ *intersecting each line in an odd number of points are precisely the elements of* $\{\mathcal{P}\} \cup \mathcal{L} \cup \overline{\mathcal{V}} \cup \mathcal{B} \cup \mathcal{U} \cup \overline{\mathcal{H}} \cup \mathcal{W}$. *The sets of points of* $\mathrm{PG}(2,4)$ *intersecting each line in an even number of points are precisely the complements of these sets, i.e. the elements of* $\{\emptyset\} \cup \overline{\mathcal{L}} \cup \mathcal{V} \cup \overline{\mathcal{B}} \cup \overline{\mathcal{U}} \cup \mathcal{H} \cup \overline{\mathcal{W}}$.

Let $\mathcal{B}_1$, $\mathcal{B}_2$ and $\mathcal{B}_3$ be the three $PSL(3,4)$-orbits of Baer subplanes and $\mathcal{H}_1$, $\mathcal{H}_2$ and $\mathcal{H}_3$ the three $PSL(3,4)$-orbits of hyperovals (as in Section 1). For every $i \in \{1,2,3\}$, let $\mathcal{W}_i$ denote the set of all sets of the form $H \cup L$, where $H \in \mathcal{H}_i$ and L is a line disjoint from H. Then $|\mathcal{W}_1| = |\mathcal{W}_2| = |\mathcal{W}_3| = 336$.

Lemma 3.2. *A unital U intersects every Baer subplane in an odd number of points, and every hyperoval in an even number of points.*

Proof. The unital U can be written as $L_1 \Delta L_2 \Delta L_3$, where L_1, L_2 and L_3 are three nonconcurrent lines. Indeed, if we take a reference system such that U has equation $X_1^3 + X_2^3 + X_3^3 = 0$, then we take for L_i, $i \in \{1,2,3\}$, the line with equation $X_i = 0$. The claims then follow from the fact that each line intersects any Baer subplane in an odd number of points and any hyperoval in an even number of points. □

The following lemma is well-known and straightforward to prove.

Lemma 3.3. *The group $PSL(3,4)$ has one orbit on $\mathcal{U}$ (namely $\mathcal{U}$), one orbit on $\mathcal{V}$ (namely $\mathcal{V}$) and three orbits on $\mathcal{W}$ (namely $\mathcal{W}_1$, $\mathcal{W}_2$ and $\mathcal{W}_3$).*

If $(\bar{e}_1, \bar{e}_2, \bar{e}_3)$ is a basis of V, then the Baer subplanes

$$B = \{\langle \bar{e}_1 \rangle, \langle \bar{e}_2 \rangle, \langle \bar{e}_3 \rangle, \langle \bar{e}_1 + \bar{e}_2 \rangle, \langle \bar{e}_1 + \bar{e}_3 \rangle, \langle \bar{e}_2 + \bar{e}_3 \rangle, \langle \bar{e}_1 + \bar{e}_2 + \bar{e}_3 \rangle\},$$

$$B' = \{\langle \bar{e}_1 \rangle, \langle \bar{e}_2 \rangle, \langle \bar{e}_3 \rangle, \langle \bar{e}_1 + \bar{e}_2 \rangle, \langle \bar{e}_1 + \omega\bar{e}_3 \rangle, \langle \bar{e}_2 + \omega\bar{e}_3 \rangle, \langle \bar{e}_1 + \bar{e}_2 + \omega\bar{e}_3 \rangle\},$$

$$B'' = \{\langle \bar{e}_1 \rangle, \langle \bar{e}_2 \rangle, \langle \bar{e}_3 \rangle, \langle \bar{e}_1 + \bar{e}_2 \rangle, \langle \bar{e}_1 + \omega^2\bar{e}_3 \rangle, \langle \bar{e}_2 + \omega^2\bar{e}_3 \rangle, \langle \bar{e}_1 + \bar{e}_2 + \omega^2\bar{e}_3 \rangle\}$$

mutually intersect in precisely four points and so belong to distinct $\mathcal{B}_i$'s. Assume that we have chosen the $\mathcal{B}_i$'s in such a way that $B \in \mathcal{B}_1$, $B' \in \mathcal{B}_2$ and $B'' \in \mathcal{B}_3$.

We denote by $P\Sigma L(3,4)$ the subgroup of $P\Gamma L(3,4)$ generated by $PSL(3,4)$ and the additional field automorphism:

$$\theta : \langle x_1\bar{e}_1 + x_2\bar{e}_2 + x_3\bar{e}_3 \rangle \to \langle x_1^2\bar{e}_1 + x_2^2\bar{e}_2 + x_3^2\bar{e}_3 \rangle.$$

As θ fixes B, it stabilizes the set $\mathcal{B}_1$ (and hence also $\mathcal{H}_1$ and $\mathcal{W}_1$). As $P\Sigma L(3,4)$ has index 3 in $P\Gamma L(3,4)$, we thus have that $P\Sigma L(3,4)$ consists of those automorphisms of $\mathrm{PG}(2,4)$ that stabilizes the set $\mathcal{B}_1$ (or equivalently $\mathcal{H}_1$ or $\mathcal{W}_1$). As $(B')^\theta = B''$, we thus see that θ interchanges $\mathcal{B}_2$ and $\mathcal{B}_3$. Choosing another basis in V, we could similarly find an automorphism of $\mathrm{PG}(2,4)$ stabilizing $\mathcal{B}_2$ and interchanging $\mathcal{B}_1$ and $\mathcal{B}_3$. The group $P\Gamma L(3,4)$ thus induces the full symmetric group on the set $\{\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3\}$ and so $P\Gamma L(3,4)/PSL(3,4) \cong S_3$.

Lemma 3.4. *There exist two Baer subplanes $B, B' \in \mathcal{B}_1$ such that $B \Delta B' \in \mathcal{V}$.*

Proof. Let B be an arbitrary Baer subplane of $\mathcal{B}_1$, and choose a basis $\{\bar{e}_1, \bar{e}_2, \bar{e}_3\}$ in V such that

$$B = \{\langle \bar{e}_1 \rangle, \langle \bar{e}_2 \rangle, \langle \bar{e}_3 \rangle, \langle \bar{e}_1 + \bar{e}_2 \rangle, \langle \bar{e}_1 + \bar{e}_3 \rangle, \langle \bar{e}_2 + \bar{e}_3 \rangle, \langle \bar{e}_1 + \bar{e}_2 + \bar{e}_3 \rangle\}.$$

Now, let B' be the Baer subplane

$$B' = \{\langle \bar{e}_1 \rangle, \langle \bar{e}_2 \rangle, \langle \bar{e}_1+\bar{e}_2 \rangle, \langle \omega\bar{e}_1+\omega^2\bar{e}_2+\bar{e}_3 \rangle, \langle \omega^2\bar{e}_1+\omega^2\bar{e}_2+\bar{e}_3 \rangle, \langle \omega\bar{e}_1+\omega\bar{e}_2+\bar{e}_3 \rangle, \langle \omega^2\bar{e}_1+\omega\bar{e}_2+\bar{e}_3 \rangle\}.$$

Then $|B \cap B'| = 3$ and so $B' \in \mathcal{B}_1$. The set $B \Delta B'$ is equal to

$$\{\langle \bar{e}_3 \rangle, \langle \bar{e}_1+\bar{e}_3 \rangle, \langle \bar{e}_2+\bar{e}_3 \rangle, \langle \bar{e}_1+\bar{e}_2+\bar{e}_3 \rangle, \langle \omega\bar{e}_1+\omega^2\bar{e}_2+\bar{e}_3 \rangle, \langle \omega^2\bar{e}_1+\omega^2\bar{e}_2+\bar{e}_3 \rangle, \langle \omega\bar{e}_1+\omega\bar{e}_2+\bar{e}_3 \rangle,$$
$$\langle \omega^2\bar{e}_1 + \omega\bar{e}_2 + \bar{e}_3 \rangle\},$$

i.e. equal to $L_1 \Delta L_2$, where $L_1 = \langle \bar{e}_1 + \bar{e}_2, \bar{e}_3 \rangle$ and $L_2 = \langle \bar{e}_1 + \bar{e}_2, \bar{e}_2 + \bar{e}_3 \rangle$. □

Lemma 3.5. *Let X be a set of points of* $\mathrm{PG}(2,4)$ *intersecting each Baer subplane of* $\mathcal{B}_1$ *in an even number of points. Then either X meets every line of* $\mathrm{PG}(2,4)$ *in an even number of points or X meets every line of* $\mathrm{PG}(2,4)$ *in an odd number of points.*

Proof. Let L_1 and L_2 be two arbitrary distinct lines of $\mathrm{PG}(2,4)$. As $PSL(3,4)$ acts transitively on pairs of distinct lines of $\mathrm{PG}(2,4)$, there exist by Lemma 3.4 two Baer subplanes $B, B' \in \mathcal{B}_1$ such that $L_1 \Delta L_2 = B \Delta B'$. X intersects B, B' and hence also $L_1 \Delta L_2 = B \Delta B'$ in an even number of points. This implies that $|L_1 \cap X|$ and $|L_2 \cap X|$ have the same parity. □

Set	#	Set	#	Set	#
$\emptyset$	1	Elements of $\overline{\mathcal{B}_1}$	120	Elements of $\overline{\mathcal{H}_3}$	56
Elements of $\overline{\mathcal{L}}$	21	Elements of $\overline{\mathcal{U}}$	280	Elements of $\overline{\mathcal{W}_1}$	336
Elements of $\mathcal{V}$	210	Elements of $\mathcal{H}_1$	56	Elements of $\mathcal{W}_2$	336
Elements of $\mathcal{B}_2$	120	Elements of $\overline{\mathcal{H}_2}$	56	Elements of $\mathcal{W}_3$	336
Elements of $\mathcal{B}_3$	120				

Table 1: The sets meeting the elements of $\mathcal{B}_1$ in an even number of points

Set	#	Set	#	Set	#
$\emptyset$	1	Elements of $\mathcal{B}_1$	120	Elements of $\overline{\mathcal{H}_1}$	56
Elements of $\overline{\mathcal{L}}$	21	Elements of $\overline{\mathcal{U}}$	280	Elements of $\mathcal{W}_1$	336
Elements of $\mathcal{V}$	210				

Table 2: The sets meeting the elements of $\mathcal{B}_2 \cup \mathcal{B}_3$ in an even number of points

The following theorem is a consequence of Lemmas 3.1, 3.2, 3.3 and 3.5.

Theorem 3.6. *The sets of points of* PG(2, 4) *intersecting each Baer subplane of* $\mathcal{B}_1$ *in an even number of points are precisely the sets mentioned in Table 1. There are precisely* $2048 = 2^{11}$ *such sets and they fall into* 13 $PSL(3,4)$*-orbits.*

Corollary 3.7. *The binary code* C_b *generated by the characteristic vectors of the Baer subplanes of* $\mathcal{B}_1$ *has dimension* 10*, its dual code* $C_b^\perp$ *has dimension* 11 *and consists of all characteristic vectors of the sets mentioned in Table 1.*

Proof. The dual code $C_b^\perp$ consists of all characteristic vectors of the sets meeting all Baer subplanes of $\mathcal{B}_1$ in an even number of points. By Theorem 3.6, we then know that $C_b^\perp$ has dimension 11. The dimension of C_b therefore equals $|\mathcal{P}| - 11 = 10$. □

Lemma 3.8. *The sets of points meeting all Baer subplanes of* $\mathcal{B}_2 \cup \mathcal{B}_3$ *in an even number of points are precisely the sets mentioned in Table 2.*

Proof. This follows by applying Theorem 3.6 twice, once with $\mathcal{B}_1$ replaced by $\mathcal{B}_2$ and once with $\mathcal{B}_1$ replaced by $\mathcal{B}_3$. □

Theorem 3.9. *The codewords of* C_b *are precisely the characteristic vectors of the sets mentioned in Table 2. The dual code* $C_b^\perp$ *is the code generated by the characteristic vectors of the elements of* $\mathcal{B}_2 \cup \mathcal{B}_3$*.*

Proof. By Table 1, the characteristic vectors of the elements of $\mathcal{B}_2 \cup \mathcal{B}_3$ belong to $C_b^\perp$. So, all sets whose characteristic vectors belong to C_b can be found in Table 2. The first claim then follows from the fact that Table 2 lists $|C_b| = 2^{10}$ sets. Also, the binary code C

generated by the characteristic vectors of the elements of $\mathcal{B}_2 \cup \mathcal{B}_3$ is contained in $C_b^\perp$. So, it suffices to prove that $\dim(C) = \dim(C_b^\perp)$, or equivalently $\dim(C^\perp) = \dim(C_b) = 10$. From Table 2, it indeed follows that $\dim(C^\perp) = 10$. □

Theorem 3.10. *The nonzero codewords of C_b have weight 7, 8, 11, 12, 15 and 16. The nonzero codewords of $C_b^\perp$ have weight 6, 7, 8, 10, 11, 12, 14, 15, 16. The automorphism groups of C_b and $C_b^\perp$ are isomorphic to $P\Sigma L(3,4)$.*

Proof. The claims regarding the weights follow from Corollary 3.7 and Theorem 3.9. We can consider $Aut(C_b)$ as a subgroup of the symmetric group $S_{\mathcal{P}}$ acting naturally on the points of PG(2,4). Each element of $Aut(C_b)$ maps codewords of weight 16 to codewords of weight 16, and hence each line of PG(2,4) to a line of PG(2,4). So, $Aut(C_b)$ is a subgroup of $P\Gamma L(3,4)$. As also the codewords of weight 7 are mapped to the codewords of weight 7, also the elements of $\mathcal{B}_1$ should be fixed. So, $Aut(C_b)$ is a subgroup of $P\Sigma L(3,4)$, and necessarily isomorphic to the latter as $P\Sigma L(3,4)$ maps the sets mentioned in Table 2 to other sets mentioned in Table 2. Note also that the automorphism group of $C_b^\perp$ is the same as the one of C_b. □

Lemma 3.11. *There exist two hyperovals $H, H' \in \mathcal{H}_1$ such that $H \Delta H' \in \mathcal{V}$.*

Proof. Let $H = \{r, s, r_1, s_1, r_2, s_2\}$ be an arbitrary hyperoval of $\mathcal{H}_1$. Put $L_1 := r_1 s_1$, $L_2 := r_2 s_2$, $L := rs$ and $\{x\} = r_1 s_1 \cap r_2 s_2$. The third line through x meeting H (necessarily in two points) is the line $L = rs$. As H is a hyperoval, the lines rr_1, rs_1 intersect L_2 in points distinct from r_2, s_2 and x. Calling these two points t_2 and u_2, we have $L_2 = \{r_2, s_2, t_2, u_2, x\}$. The lines sr_1, ss_1 must intersect L_2 in the respective points u_2 and t_2 for a similar reason. By symmetry, if we put $L_1 = \{r_1, s_1, t_1, u_1, x\}$, then each of the lines rr_2, rs_2, sr_2, ss_2 intersects L_1 in one of the points t_1 and u_1. It follows that $H' = \{r, s, t_1, u_1, t_2, u_2\}$ is a collection of six points no three of which are collinear, i.e. a hyperoval. Obviously, $H \Delta H' = L_1 \Delta L_2$. As $|H \cap H'| = 2$, we also have $H' \in \mathcal{H}_1$. □

Similarly as in Lemma 3.5, one can prove the following.

Lemma 3.12. *Let X be a set of points of* PG(2,4) *intersecting each hyperoval of $\mathcal{H}_1$ in an even number of points. Then either X meets every line of* PG(2,4) *in an even number of points or X meets every line of* PG(2,4) *in an odd number of points.*

The following is a consequence of Lemmas 3.1, 3.2, 3.3 and 3.12.

Theorem 3.13. *The set of points of* PG(2,4) *intersecting each hyperoval of $\mathcal{H}_1$ in an even number of points are precisely the sets mentioned in Table 3. There are $2048 = 2^{11}$ such sets and they fall into 14 $PSL(3,4)$-orbits.*

Similarly as in Corollary 3.7, one can prove the following.

Corollary 3.14. *The binary code $\mathcal{C}_h$ generated by the characteristic vectors of the hyperovals of $\mathcal{H}_1$ has dimension 10, its dual code $\mathcal{C}_h^\perp$ has dimension 11 and consists of all characteristic vectors of the sets mentioned in Table 3.*

Set	#	Set	#	Set	#
$\emptyset$	1	Elements of $\overline{\mathcal{V}}$	210	Elements of $\mathcal{H}_1$	56
$\mathcal{P}$	1	Elements of $\mathcal{B}_1$	120	Elements of $\overline{\mathcal{H}_1}$	56
Elements of $\mathcal{L}$	21	Elements of $\overline{\mathcal{B}_1}$	120	Elements of $\mathcal{W}_1$	336
Elements of $\overline{\mathcal{L}}$	21	Elements of $\mathcal{U}$	280	Elements of $\overline{\mathcal{W}_1}$	336
Elements of $\mathcal{V}$	210	Elements of $\overline{\mathcal{U}}$	280		

Table 3: The sets meeting the elements of $\mathcal{H}_1$ in an even number of points

Set	#	Set	#	Set	#
$\emptyset$	1	Elements of $\overline{\mathcal{B}_1}$	120	Elements of $\mathcal{H}_1$	56
Elements of $\overline{\mathcal{L}}$	21	Elements of $\overline{\mathcal{U}}$	280	Elements of $\overline{\mathcal{W}_1}$	336
Elements of $\mathcal{V}$	210				

Table 4: The sets meeting the elements of $\mathcal{B}_1 \cup \mathcal{H}_1$ in an even number of points

Lemma 3.15. *The sets of points meeting all sets of $\mathcal{B}_1 \cup \mathcal{H}_1$ in an even number of points are precisely the sets mentioned in Table 4.*

Proof. These are precisely the sets occurring in Tables 1 and 3. □

Similarly as in Theorems 3.9 and 3.10, one can prove the following.

Theorem 3.16. *The codewords of C_h are precisely the characteristic vectors of the sets mentioned in Table 4. The dual code $C_h^\perp$ is the code generated by the characteristic vectors of the elements of $\mathcal{B}_1 \cup \mathcal{H}_1$.*

Theorem 3.17. *The nonzero codewords of C_h have weights* 6, 8, 10, 12, 14 *and* 16*. The nonzero codewords of $(C_h)^\perp$ have weight* 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16 *and* 21*. The automorphism groups of C_h and $(C_h)^\perp$ are isomorphic to $P\Sigma L(3,4)$.*

The following results accompany Theorems 3.6 and 3.13.

Theorem 3.18. (*a*) *The sets of points of* PG(2, 4) *intersecting each Baer subplane of $\mathcal{B}_1$ in an odd number of points are precisely the elements of*

$$\{\mathcal{P}\} \cup \mathcal{L} \cup \overline{\mathcal{V}} \cup \mathcal{B}_1 \cup \overline{\mathcal{B}_2} \cup \overline{\mathcal{B}_3} \cup \mathcal{U} \cup \overline{\mathcal{H}_1} \cup \mathcal{H}_2 \cup \mathcal{H}_3 \cup \mathcal{W}_1 \cup \overline{\mathcal{W}_2} \cup \overline{\mathcal{W}_3}.$$

(*b*) *The sets of points of* PG(2, 4) *intersecting each hyperoval of $\mathcal{H}_1$ in an odd number of points are precisely the elements of*

$$\mathcal{B}_2 \cup \overline{\mathcal{B}_2} \cup \mathcal{B}_3 \cup \overline{\mathcal{B}_3} \cup \mathcal{H}_2 \cup \overline{\mathcal{H}_2} \cup \mathcal{H}_3 \cup \overline{\mathcal{H}_3} \cup \mathcal{W}_2 \cup \overline{\mathcal{W}_2} \cup \mathcal{W}_3 \cup \overline{\mathcal{W}_3}.$$

Proof. (a) These are obviously the complements of the sets mentioned in Table 1.

(b) Again by relying on Lemma 3.11, one can show (similarly as in Lemma 3.5) that if X is such a set, then X intersects every line of PG(2, 4) in an even number of points or X intersects every line of PG(2, 4) in an odd number of points. The claim then follows from Lemmas 3.1 and 3.2. □

4 The even sets of W_{22}

Let $(\mathcal{P}_{22}, \mathcal{B}_{22})$ be the Witt design W_{22}. Then $\mathcal{P}_{22} = \mathcal{P} \cup \{\infty\}$, where ∞ is a symbol not contained in $\mathcal{P}$. There are two types of blocks in W_{22}:

- $\{\infty\} \cup L$, where $L \in \mathcal{L}$;
- the hyperovals of $\mathcal{H}_1$.

Note that the derived design of W_{22} with respect to ∞ is precisely PG(2, 4). Now, $Aut(W_{22}) \cong M_{22} : 2$ has a subgroup $G \cong M_{22}$ of index 2, $Aut(W_{22})_\infty \cong P\Sigma L(3,4)$ and $G_\infty \cong PSL(3,4)$. In fact, the group G_∞ stabilizes the sets $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3$, while each element of $Aut(W_{22})_\infty \setminus G_\infty$ stabilizes $\mathcal{H}_1$ and interchanges $\mathcal{H}_2$ and $\mathcal{H}_3$.

We classify all even sets X of points of W_{22}. Note that the pseudo-hyperplanes of W_{22} are precisely the even sets distinct from $\mathcal{P}_{22}$. We distinguish two cases.

(1) Suppose $\infty \in X$. Then $X = \{\infty\} \cup Y$ where Y is a set of points meeting each hyperoval of $\mathcal{H}_1$ in an even number of points and each line of PG(2, 4) in an odd number of points. By Lemma 3.1 and Theorem 3.13, we have the following possibilities for Y:

Y	#	$\lvert X\rvert = \lvert Y\rvert + 1$	Y	#	$\lvert X\rvert = \lvert Y\rvert + 1$
$\mathcal{P}$	1	22	Elements of $\mathcal{U}$	280	10
Elements of $\mathcal{L}$	21	6	Elements of $\overline{\mathcal{H}_1}$	56	16
Elements of $\overline{\mathcal{V}}$	210	14	Elements of $\mathcal{W}_1$	336	12
Elements of $\mathcal{B}_1$	120	8			

(2) Suppose $\infty \notin X$. Then X intersects every hyperoval of $\mathcal{H}_1$ and every line in an even number of points. By Lemma 3.1 and Theorem 3.13, we have the following possibilities for X:

X	#	$\lvert X\rvert$	X	#	$\lvert X\rvert$
$\emptyset$	1	0	Elements of $\overline{\mathcal{U}}$	280	12
Elements of $\overline{\mathcal{L}}$	21	16	Elements of $\mathcal{H}_1$	56	6
Elements of $\mathcal{V}$	210	8	Elements of $\overline{\mathcal{W}_1}$	336	10
Elements of $\overline{\mathcal{B}_1}$	120	14			

Theorem 4.1. *Up to isomorphism, W_{22} has 8 even sets. W_{22} has pseudo-embeddings and the universal pseudo-embedding of W_{22} has vector dimension 11.*

Proof. Since $Aut(W_{22}) \cong M_{22} : 2$ is 3-transitive on the points, every even set containing ∞ and distinct from $\mathcal{P}$ is isomorphic to an even set not containing ∞. By the above tables, the fact that $Aut(W_{22})_\infty \cong P\Sigma L(3,4)$ and Lemma 3.3, it then follows that there are up to isomorphism 8 even sets. By the above tables, it also follows that for every block B and every two points $x_1, x_2 \in B$, there exists a pseudo-hyperplane intersecting B in $\{x_1, x_2\}$. So, W_{22} has pseudo-embeddings by Proposition 2.3. By the above tables, there are $|\mathcal{E}_{W_{22}}| = 2048 = 2^{11}$ even sets. As explained in Section 2, the dimension 11 of the $\mathbb{F}_2$-vector space $\mathcal{E}_{W_{22}}$ is precisely the vector dimension of the universal pseudo-embedding of W_{22}. □

5 The even sets of W_{23}

Let $(\mathcal{P}_{23}, \mathcal{B}_{23})$ be the Witt design W_{23}. Then $\mathcal{P}_{23} = \mathcal{P} \cup \{\infty_1, \infty_2\}$, where ∞_1 and ∞_2 are two symbols not contained in $\mathcal{P}$. There are three types of blocks in W_{23}:

- $\{\infty_1, \infty_2\} \cup L$ where $L \in \mathcal{L}$;
- $\{\infty_i\} \cup H$ where $H \in \mathcal{H}_{3-i}$ and $i \in \{1,2\}$;
- $B \in \mathcal{B}_3$.

If we put $\infty := \infty_1$, then the derived design of W_{23} with respect to ∞_2 is precisely W_{22}. We have $Aut(W_{23}) \cong M_{23}$, $Aut(W_{23})_{\infty_2} \cong M_{22}$ and $Aut(W_{23})_{\infty_1,\infty_2} \cong PSL(3,4)$.

The pseudo-hyperplanes of W_{23} are precisely the complements of the nontrivial even sets of W_{23}. There are three possibilities for an even set X.

(a) $\infty_1, \infty_2 \notin X$. Then X intersects each element of $\mathcal{L} \cup \mathcal{H}_1 \cup \mathcal{H}_2 \cup \mathcal{B}_3$ in an even number of points. We have the following possibilities:

X	#	$\lvert X\rvert$	X	#	$\lvert X\rvert$
$\emptyset$	1	0	Elements of $\mathcal{V}$	210	8
Elements of $\overline{\mathcal{L}}$	21	16	Elements of $\overline{\mathcal{U}}$	280	12

(b) $\infty_1, \infty_2 \in X$. Then $X = Y \cup \{\infty_1, \infty_2\}$ where Y is a set of points of PG(2,4) meeting each set of $\mathcal{L} \cup \mathcal{B}_3$ is an even number of points and each set of $\mathcal{H}_1 \cup \mathcal{H}_2$ is an odd number of points. The possibilities for Y are as follows:

Y	#	$\lvert X\rvert = \lvert Y\rvert + 2$
Elements of $\overline{\mathcal{B}_3}$	120	16
Elements of $\mathcal{H}_3$	56	8
Elements of $\overline{\mathcal{W}_3}$	336	12

(c) $\infty_i \in X$ and $\infty_{3-i} \notin X$ for some $i \in \{1,2\}$. Then $X = Y \cup \{\infty_i\}$ where Y is a set of points of PG(2,4) intersecting each element of $\mathcal{L} \cup \mathcal{H}_{3-i}$ in an odd number of points and each element of $\mathcal{B}_3 \cup \mathcal{H}_i$ is an even number of points. We have the following possibilities:

Y	#	$\lvert X\rvert = \lvert Y\rvert + 1$
Elements of $\overline{\mathcal{H}_i}$	120	16
Elements of $\mathcal{B}_i$	56	8
Elements of $\mathcal{W}_i$	336	12

Theorem 5.1. *Up to isomorphism, W_{23} has four even sets. W_{23} has pseudo-embeddings and the universal pseudo-embedding of W_{23} has vector dimension 11.*

Proof. Since $Aut(W_{23}) \cong M_{23}$ is 4-transitive on the points, every even set of size at most 21 is isomorphic to an even set not containing ∞_1 nor ∞_2. By the above tables, the fact that $Aut(W_{23})_{\infty_1,\infty_2} \cong PSL(3,4)$ and Lemma 3.3, it then follows that there are up to isomorphism four even sets. By the above tables, it also follows that for every block B and every point $x \in B$, there exists a pseudo-hyperplane intersecting B in $\{x\}$. So, W_{23} has pseudo-embeddings by Proposition 2.3. By the above table, there are $|\mathcal{E}_{W_{23}}| = 2048 = 2^{11}$ even sets. The dimension 11 of the $\mathbb{F}_2$-vector space $\mathcal{E}_{W_{23}}$ is precisely the vector dimension of the universal pseudo-embedding of W_{23}. □

6 The even sets of W_{24}

Let $(\mathcal{P}_{24}, \mathcal{B}_{24})$ be the Witt design W_{24}. Then $\mathcal{P}_{24} = \mathcal{P} \cup \{\infty_1, \infty_2, \infty_3\}$, where ∞_1, ∞_2 and ∞_3 are three symbols not contained in $\mathcal{P}$. There are four types of lines:

- $\{\infty_1, \infty_2, \infty_3\} \cup L$ where $L \in \mathcal{L}$;
- $\{\infty_i, \infty_j\} \cup H$ where $H \in \mathcal{H}_k$ and $\{i,j,k\} = \{1,2,3\}$;
- $\{\infty_i\} \cup B$ where $B \in \mathcal{B}_i$ for some $i \in \{1,2,3\}$;
- elements of $\mathcal{V}$.

Note that the derived design of W_{24} with respect to ∞_3 is precisely W_{23}. We have $Aut(W_{24}) \cong M_{24}$, $Aut(W_{24})_{\infty_3} \cong M_{23}$ and $Aut(W_{24})_{\infty_1,\infty_2,\infty_3} \cong PSL(3,4)$. The pseudo-hyperplanes of W_{24} are precisely the even sets of W_{24} distinct from $\mathcal{P}_{24}$. We have the following possibilities for an even set X.

(a) Suppose $\infty_1, \infty_2, \infty_3 \notin X$. Then X intersects every set of $\mathcal{L} \cup \mathcal{B} \cup \mathcal{H} \cup \mathcal{V}$ in an even number of points. We then have the following possibilities for X:

X	#	$\lvert X\rvert$	X	#	$\lvert X\rvert$
$\emptyset$	1	0	Elements of $\mathcal{V}$	210	8
Elements of $\overline{\mathcal{L}}$	21	16	Elements of $\overline{\mathcal{U}}$	280	12

(b) Suppose $\infty_i \in X$ and $\{\infty_j, \infty_k\} \cap X = \emptyset$ where $\{i,j,k\} = \{1,2,3\}$. Then $X = \{\infty_i\} \cup Y$ where Y is a set of points of PG(2,4) intersecting each element of $\mathcal{L} \cup \mathcal{H}_j \cup \mathcal{H}_k \cup \mathcal{B}_i$ in an odd number of points and every set of $\mathcal{H}_i \cup \mathcal{B}_j \cup \mathcal{B}_k \cup \mathcal{V}$ in an even number of points. We then have the following possibilities for Y:

Y	#	$\lvert X\rvert = \lvert Y\rvert + 1$
Elements of $\mathcal{B}_i$	120	8
Elements of $\overline{\mathcal{H}_i}$	56	16
Elements of $\mathcal{W}_i$	336	12

(c) Suppose $\infty_j, \infty_k \in X$ and $\infty_i \notin X$ where $\{i,j,k\} = \{1,2,3\}$. Then $X = Y \cup \{\infty_j, \infty_k\}$ where Y is a certain set of points of PG(2, 4). The set Y intersects each element of $\mathcal{L} \cup \mathcal{H}_i \cup \mathcal{B}_i \cup \mathcal{V}$ in an even number of points and each element of $\mathcal{B}_j \cup \mathcal{B}_k \cup \mathcal{H}_j \cup \mathcal{H}_k$ in an odd number of points. We have the following possibilities:

Y	#	$\lvert X\rvert = \lvert Y\rvert + 2$
Elements of $\mathcal{H}_i$	56	8
Elements of $\overline{\mathcal{W}_i}$	336	12
Elements of $\overline{\mathcal{B}_i}$	120	16

(d) Suppose $\infty_1, \infty_2, \infty_3 \in X$. Then $X = \{\infty_1, \infty_2, \infty_3\} \cup Y$ where Y is a set of points of PG(2, 4) which intersect each element of $\mathcal{L} \cup \mathcal{B}$ in an odd number of points and each element of $\mathcal{H} \cup \mathcal{V}$ in an even number of points. We have the following possibilities:

Y	#	$\lvert X\rvert = \lvert Y\rvert + 3$	Y	#	$\lvert X\rvert = \lvert Y\rvert + 3$
$\mathcal{P}$	1	24	Elements of $\overline{\mathcal{V}}$	210	16
Elements of $\mathcal{L}$	21	8	Elements of $\mathcal{U}$	280	12

Theorem 6.1. *Up to isomorphism, W_{24} has five even sets. W_{24} has pseudo-embeddings and the universal pseudo-embedding has vector dimension* 12.

Proof. Since $Aut(W_{24}) \cong M_{24}$ is 5-transitive on the points, every even set of size at most 21 is isomorphic to an even set not containing ∞_1, ∞_2 nor ∞_3. By the above tables, $Aut(W_{24})_{\infty_1,\infty_2,\infty_3} \cong PSL(3,4)$ and Lemma 3.3, it then follows that there are up to isomorphism five even sets. By the above tables, it also follows that for every block B and every two points $x_1, x_2 \in B$, there exists a pseudo-hyperplane intersecting B in $\{x_1, x_2\}$. So, W_{24} has pseudo-hyperplanes by Proposition 2.3. By the above table, there are $|\mathcal{E}_{W_{24}}| = 4096 = 2^{12}$ even sets. The dimension 12 of the $\mathbb{F}_2$-vector space $\mathcal{E}_{W_{24}}$ is precisely the vector dimension of the universal pseudo-embedding of W_{24}. □

Theorem 6.1 is basically known. The code C^* generated by the characteristic vectors of the blocks of W_{24} is the so-called extended binary Golay code. This is a 12-dimensional code of length 24 for which the weights of the codewords are 0, 8, 12, 16 and 24. These codewords are the characteristic vectors of respectively the empty point set, the blocks, the dodecads, the complements of the blocks and the whole point set $\mathcal{P}_{24}$ (see e.g. Lemma 19.7(2) of [1]). We claim that these sets are precisely the even sets. Indeed, the fact that any two blocks of W_{24} meet in an even number of points implies that $C^* \subseteq (C^*)^\perp$ and hence that $C^* = (C^*)^\perp$ as $12 = \dim(C^*) \le \dim(C^*)^\perp = 24 - \dim(C^*) = 12$. So, the fact that W_{24} has up to isomorphism five even sets is basically a known result. However, for our classification of the homogeneous pseudo-embeddings of the large Witt designs and for our purpose to give explicit descriptions of these pseudo-embeddings, we need to describe the even set with respect to the three given points $\infty_1, \infty_2, \infty_3$, and the present section realizes this goal.

If we denote the ith coordinate vector of $\mathbb{F}_2^{24}$ by $\bar{e}_i$ ($i \in \{1, 2, \ldots, 24\}$), then the map from W_{24} to $\mathrm{PG}(\mathbb{F}_2^{24}/C^*)$ mapping the ith point of W_{24} to the 1-space $\langle \bar{e}_i + C^* \rangle$ of $\mathrm{PG}(\mathbb{F}_2^{24}/C^*)$ is isomorphic to the universal pseudo-embedding of W_{24}, see [7, Theorem 1.2(2)]. The vector space $\mathbb{F}_2^{24}/C^*$ can in a natural way be regarded as a module for the group $M_{24} = Aut(W_{24})$. This module is known as the 12-dimensional Todd module ([1, p. 92], [24]). The fact that $\mathbb{F}_2^{24}/C^*$ hosts a pseudo-embedding for W_{24} was already mentioned on page 414 of Todd [24] (although he did not use this terminology).

7 Derived pseudo-embeddings

Suppose $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is a point-line geometry with the property that the number of points on each line is finite and at least three. If $X \neq \mathcal{P}$ is a nonempty set of points of $\mathcal{S}$ and $\mathcal{L}_X$ denotes the set of lines of $\mathcal{S}$ containing X and having at least $|X| + 3$ points, then the subgeometry $\mathcal{S}_X$ of $\mathcal{S}$ determined by the point set $\mathcal{P} \setminus X$ and the line set $\mathcal{L}_X$ is called the *derivation of $\mathcal{S}$ with respect to X*. If X is a singleton $\{x\}$, then $\mathcal{S}_X$ will also be denoted by $\mathcal{S}_x$ and called *the derivation with respect to x*. If X_1 and X_2 are two disjoint subsets of $\mathcal{P}$, then $\mathcal{S}_{X_1 \cup X_2} = (\mathcal{S}_{X_1})_{X_2} = (\mathcal{S}_{X_2})_{X_1}$. The notion of derivation can be regarded as a generalization of the one given in Section 2 for Steiner systems.

Proposition 7.1. *Let $X \neq \mathcal{P}$ be a nonempty set of points of $\mathcal{S}$. Then the setwise stabilizer $Aut(\mathcal{S})_X$ can be regarded in a natural way as a group of automorphisms for $\mathcal{S}_X$.*

Suppose $\epsilon : \mathcal{S} \to \mathrm{PG}(W)$ is a pseudo-embedding of $\mathcal{S}$. For every $x \in \mathcal{P}$, let $\bar{w}_x \in W$ such that $\epsilon(x) = \langle \bar{w}_x \rangle$. Suppose the vectors $\bar{w}_x$, $x \in X$, are linearly independent and $\bar{w}_y \neq \sum_{x \in X} \bar{w}_x$ for every $y \in \mathcal{P} \setminus X$. Put $p^* := \langle \sum_{x \in X} \bar{w}_x \rangle$ and consider the quotient projective space $\mathrm{PG}(W)/p^*$ whose points are the lines of $\mathrm{PG}(W)$ through p^*. For every point y of $\mathcal{S}$ not contained in X, let $\epsilon'(y)$ be the line $p^*\epsilon(y)$ through p^*. Then the following hold:

(1) *ϵ' is a pseudo-embedding of $\mathcal{S}_X$ in $\mathrm{PG}(W')/p^*$, where W' is a suitable subspace of W through p^*;*

(2) *if ϵ is G-homogeneous for some $G \leq Aut(\mathcal{S})$, then ϵ' is G_X-homogeneous;*

(3) *$W' = W$ if and only if there exists no $Y \in \mathcal{E}_\epsilon$ with $Y \subseteq X$ and $|Y|$ even;*

(4) *the elements of $\mathcal{E}_{\epsilon'}$ are the nonempty sets of the form $Y \setminus X$, where $Y \in \mathcal{E}_\epsilon$ with $|Y \cap X|$ even.*

Proof. Put $X = \{x_1, x_2, \ldots, x_k\}$. Claim (1) follows from the fact that if $\{\epsilon(x_1), \epsilon(x_2), \ldots, \epsilon(x_k), y_1, y_2, \ldots, y_l\}$ is a frame in a subspace of $\mathrm{PG}(W)$ for some $l \in \mathbb{N} \setminus \{0, 1, 2\}$, then $\{p^*y_1, p^*y_2, \ldots, p^*y_l\}$ is a frame in a subspace of $\mathrm{PG}(W)/p^*$.

As for Claim (2). Suppose $\theta \in G_X$. Since ϵ is G-homogeneous, there exists an automorphism $\widetilde{\theta}$ of $\mathrm{PG}(W)$ such that $\epsilon \circ \theta = \widetilde{\theta} \circ \epsilon$. Since θ stabilizes X, $\widetilde{\theta}$ stabilizes $\{\epsilon(x_1), \epsilon(x_2), \ldots, \epsilon(x_k)\}$ and thus fixes p^*. Let $\widetilde{\theta}'$ be the automorphism of $\mathrm{PG}(W)/p^*$ induced by $\widetilde{\theta}$. Taking quotients with respect to p^* in both sides of the equality $\epsilon \circ \theta = \widetilde{\theta} \circ \epsilon$, we see that $\epsilon' \circ \theta = \widetilde{\theta}' \circ \epsilon'$. So, ϵ' is G_X-homogeneous.

For any hyperplane U of W, let $E_U \in \mathcal{E}_\epsilon$ denote the set of all points x of $\mathcal{S}$ for which $\bar{w}_x \notin U$. Now, $W' = W$ if and only if there exists no hyperplane U of W through $\sum_{x \in X} \bar{w}_x$ containing W', i.e. if and only if there exists no hyperplane U of W through $\sum_{x \in X} \bar{w}_x$ for which E_U is contained in X.

Now, any hyperplane U of W is an additive subgroup of index 2 and so U contains $\sum_{x \in X} \bar{w}_x$ if and only if E_U contains an even number of elements of X. Claim (3) follows.

As for Claim (4), the elements of $\mathcal{E}_{\epsilon'}$ are precisely the elements of the form $E_U \setminus X$, where U is a hyperplane of W through $\sum_{x \in X} \bar{w}_x$ not containing W', or equivalently the elements of the form $E_U \setminus X$, where U is a hyperplane of W through $\sum_{x \in X} \bar{w}_x$ for which $E_U \setminus X$ is nonempty. Also Claim (4) follows. □

We call the pseudo-embedding ϵ' in Proposition 7.1 a *derived pseudo-embedding*, or the *pseudo-embedding which arises from ϵ by derivation with respect to X* (or *by derivation with respect to x* if $X = \{x\}$ is a singleton).

Remarks. (1) Suppose that for every $y \in \mathcal{P} \setminus X$, there exists a line of size at least $|X| + 3$ containing $X \cup \{y\}$, then the condition $\bar{w}_y \neq \sum_{x \in X} \bar{w}_x$ is satisfied for any $y \in \mathcal{P} \setminus X$.

(2) In case, ϵ is the universal pseudo-embedding of $\mathcal{S} = W_{24}$ and $|X| = 3$, then it was observed in [9, p. 85] that ϵ' is isomorphic to the universal pseudo-embedding of $\mathcal{S}_X \cong \mathrm{PG}(2,4)$. This is a consequence of the fact that the vector dimension of the universal pseudo-embeddings of $\mathrm{PG}(2,4)$ is 11 ([7, Proposition 4.6], [9, p. 79]), which is one less than the vector dimension of the universal pseudo-embedding of W_{24}.

The following is a special case of Proposition 7.1.

Corollary 7.2. *Using the same notation as in Proposition 7.1, suppose that X is a singleton $\{x\}$. Then the pseudo-hyperplanes contained in $\mathcal{H}_{\epsilon'}$ are of the form $Y \setminus \{x\}$ with $Y \in \mathcal{H}_\epsilon$ such that $x \in Y$, and the even sets contained in $\mathcal{E}_{\epsilon'}$ are precisely the even sets of $\mathcal{E}_\epsilon$ not containing x.*

Corollary 7.3. (*a*) *The universal pseudo-embedding of W_{23} is isomorphic to the pseudo-embedding of W_{23} obtained by deriving the universal pseudo-embedding of W_{24} with respect to ∞_3.*

(*b*) *The universal pseudo-embedding of W_{22} is isomorphic to the pseudo-embedding of W_{22} obtained by deriving the universal pseudo-embedding of W_{24} with respect to $\{\infty_2, \infty_3\}$.*

Proof. This follows from the fact that in each case both pseudo-embeddings have the same dimension, see Theorems 4.1, 5.1 and 6.1. □

Recall that if ϵ is a universal pseudo-embedding, then $\mathcal{E}_\epsilon$ is the set of all nontrivial even sets. We now see that the nontrivial even sets of W_{22} and W_{23} (as described in Sections 4 and 5) are obtained from the nontrivial even sets of W_{24} (as described in Section 6) as stated in Proposition 7.1 and Corollary 7.2 (taking into account Corollary 7.3). Applying Proposition 7.1 and Corollary 7.2 another time, we find:

Corollary 7.4. *(a) W_{22} has a (nonuniversal) M_{22}-homogeneous embedding ϵ_{22} in* $\mathrm{PG}(9, 2)$ *which is obtained by deriving the universal pseudo-embedding of W_{23} with respect to ∞_2. The projective plane* $\mathrm{PG}(2,4)$ *has a $PSL(3,4)$-homogeneous pseudo-embedding ϵ_{21} in* $\mathrm{PG}(8,2)$ *which is obtained by deriving ϵ_{22} with respect to ∞_1.*

(b) $\mathcal{E}_{\epsilon_{22}} = \overline{\mathcal{L}} \cup \mathcal{V} \cup \overline{\mathcal{U}} \cup \{\{\infty\} \cup X \mid X \in \overline{\mathcal{H}_1} \cup \mathcal{B}_1 \cup \mathcal{W}_1\}$ and $\mathcal{E}_{\epsilon_{21}} = \overline{\mathcal{L}} \cup \mathcal{V} \cup \overline{\mathcal{U}}$.

In [6], we constructed a homogeneous pseudo-embedding ϵ_h of $\mathrm{PG}(2,4)$ that satisfies $\mathcal{E}_{\epsilon_h} = \overline{\mathcal{L}} \cup \mathcal{V} \cup \overline{\mathcal{U}}$ and which is called the *Hermitian Veronese pseudo-embedding* of $\mathrm{PG}(2,4)$. Combining this with Corollary 7.4(b) and Proposition 2.2, we find:

Corollary 7.5. *The pseudo-embedding ϵ_{21} of* $\mathrm{PG}(2,4)$ *constructed in Corollary 7.4(a) is isomorphic to the Hermitian Veronese pseudo-embedding ϵ_h of* $\mathrm{PG}(2,4)$.

8 Homogeneous pseudo-embeddings of the large Witt designs

In this section, we classify all homogeneous pseudo-embeddings of the Witt designs W_{22}, W_{23} and W_{24}. Certain knowledge on the $PSL(3,4)$-homogeneous pseudo–embeddings of $\mathrm{PG}(2,4)$ will be useful to that end. So, we start by studying these pseudo-embeddings.

8.1 $PSL(3,4)$-homogeneous pseudo-embeddings of $\mathrm{PG}(2,4)$

In the following table, we mention the even sets of $\mathcal{S} = \mathrm{PG}(2,4)$, see Lemma 3.1.

Set	E_1	E_2	E_3	E_4	E_5	E_6	E_7	E_8	E_9	E_{10}	E_{11}	E_{12}	E_{13}
Definition	$\{\emptyset\}$	$\overline{\mathcal{L}}$	$\mathcal{V}$	$\overline{\mathcal{B}_1}$	$\overline{\mathcal{B}_2}$	$\overline{\mathcal{B}_3}$	$\overline{\mathcal{U}}$	$\mathcal{H}_1$	$\mathcal{H}_2$	$\mathcal{H}_3$	$\overline{\mathcal{W}_1}$	$\overline{\mathcal{W}_2}$	$\overline{\mathcal{W}_3}$
Size	1	21	210	120	120	120	280	56	56	56	336	336	336

If $\epsilon : \mathrm{PG}(2,4) \to \mathrm{PG}(W)$ is a $PSL(3,4)$-homogeneous pseudo-embedding of $\mathrm{PG}(2,4)$, then by Proposition 2.4(a)+(b) and Lemma 3.3 $\mathcal{E}_\epsilon \cup \{\emptyset\}$ is a subspace of $\mathcal{E}_\mathcal{S}$ that is the union of some of the E_i's. We now determine all subspaces $\mathcal{E}$ that are the union of some of the E_i's. Obviously, $\mathcal{E}_\mathcal{S} = E_1 \cup E_2 \cup \cdots \cup E_{13}$ and $\{\emptyset\}$ are two such subspaces. So, we may suppose that $\{\emptyset\} \subsetneq \mathcal{E} \subsetneq \mathcal{E}_\mathcal{S}$. Since $|\mathcal{E}|$ is even, we have that $\mathcal{E}$ contains $E_1 \cup E_2$. By Corollary 7.4(b), we know that $\mathcal{E}' := E_1 \cup E_2 \cup E_3 \cup E_7$ is a subspace. Since $\mathcal{E} \cap \mathcal{E}'$ is a subspace (whose number of elements is a power of 2) we see that $\mathcal{E}$ contains $\mathcal{E}'$. Assume that $\mathcal{E}'$ is properly contained in $\mathcal{E}$. Since $|\mathcal{E}|$ is a power of 2, we see that the only possibility is that $\mathcal{E}$ contains besides E_1, E_2, E_3 and E_7, precisely one of $\overline{\mathcal{B}_1}, \overline{\mathcal{B}_2}, \overline{\mathcal{B}_3}$, precisely one of $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3$ and precisely one of $\overline{\mathcal{W}_1}, \overline{\mathcal{W}_2}, \overline{\mathcal{W}_3}$. If $\mathcal{E}$ contains $\mathcal{H}_i$ for some $i \in \{1, 2, 3\}$, then we know by Theorem 3.16 that $\mathcal{E}$ also contains $\overline{\mathcal{B}_i}$ and $\overline{\mathcal{W}_i}$. (Indeed, the sum of the characteristic vectors of two sets is the characteristic vector of their symmetric difference.) By Theorem 3.16, we also know that $\{\emptyset\} \cup \overline{\mathcal{L}} \cup \mathcal{V} \cup \overline{\mathcal{U}} \cup \overline{\mathcal{B}_i} \cup \mathcal{H}_i \cup \overline{\mathcal{W}_i}$ is a subspace. So, we have:

Lemma 8.1. *The following are the subspaces of $\mathcal{E}_\mathcal{S}$ that can be written as the union of a number of the E_i's:*

- $\{\emptyset\} = E_1$;
- $\{\emptyset\} \cup \overline{\mathcal{L}} \cup \mathcal{V} \cup \overline{\mathcal{U}} = E_1 \cup E_2 \cup E_3 \cup E_7$;
- $\{\emptyset\} \cup \overline{\mathcal{L}} \cup \mathcal{V} \cup \overline{\mathcal{U}} \cup \overline{\mathcal{B}_1} \cup \mathcal{H}_1 \cup \overline{\mathcal{W}_1} = E_1 \cup E_2 \cup E_3 \cup E_7 \cup E_4 \cup E_8 \cup E_{11}$;
- $\{\emptyset\} \cup \overline{\mathcal{L}} \cup \mathcal{V} \cup \overline{\mathcal{U}} \cup \overline{\mathcal{B}_2} \cup \mathcal{H}_2 \cup \overline{\mathcal{W}_2} = E_1 \cup E_2 \cup E_3 \cup E_7 \cup E_5 \cup E_9 \cup E_{12}$;
- $\{\emptyset\} \cup \overline{\mathcal{L}} \cup \mathcal{V} \cup \overline{\mathcal{U}} \cup \overline{\mathcal{B}_3} \cup \mathcal{H}_3 \cup \overline{\mathcal{W}_3} = E_1 \cup E_2 \cup E_3 \cup E_7 \cup E_6 \cup E_{10} \cup E_{13}$;
- $\mathcal{E}_\mathcal{S} = E_1 \cup E_2 \cup \cdots \cup E_{13}$.

By Propositions 2.2, 2.4 and Lemma 8.1, we then have:

Theorem 8.2. *Up to isomorphism,* PG(2, 4) *has five $PSL(3,4)$-homogeneous pseudo-embeddings: the Hermitean Veronese pseudo-embedding in* PG(8, 2)*, the universal pseudo-embedding in* PG(10, 2) *and three nonisomorphic pseudo-embeddings in* PG(9, 2).

We now define five "embeddings" of PG(2, 4) and show that they are the $PSL(3,4)$-homogeneous pseudo-embeddings of PG(2, 4) mentioned in Theorem 8.2.

(1) Let ϵ_h be the map from PG(2, 4) to PG(8, 2) mapping the point (X_1, X_2, X_3) to

$$(X_1^3, X_2^3, X_3^3, X_1X_2^2 + X_2X_1^2, \omega X_1X_2^2 + \omega^2 X_2X_1^2, X_1X_3^2 + X_3X_1^2, \omega X_1X_3^2 + \omega^2 X_3X_1^2,$$
$$X_2X_3^2 + X_3X_2^2, \omega X_2X_3^2 + \omega^2 X_3X_2^2).$$

(2) Let ϵ_u be the map from PG(2, 4) to PG(10, 2) mapping the point (X_1, X_2, X_3) to

$$(X_1^3, X_2^3, X_3^3, X_1X_2^2 + X_2X_1^2, \omega X_1X_2^2 + \omega^2 X_2X_1^2, X_1X_3^2 + X_3X_1^2, \omega X_1X_3^2 + \omega^2 X_3X_1^2,$$
$$X_2X_3^2 + X_3X_2^2, \omega X_2X_3^2 + \omega^2 X_3X_2^2, X_1X_2X_3 + X_1^2X_2^2X_3^2, \omega X_1X_2X_3 + \omega^2 X_1^2X_2^2X_3^2).$$

(3) For every $\lambda \in \mathbb{F}_4^*$, let ϵ_λ be the map from PG(2, 4) to PG(9, 2) mapping the point (X_1, X_2, X_3) to

$$(X_1^3, X_2^3, X_3^3, X_1X_2^2 + X_2X_1^2, \omega X_1X_2^2 + \omega^2 X_2X_1^2, X_1X_3^2 + X_3X_1^2, \omega X_1X_3^2 + \omega^2 X_3X_1^2,$$
$$X_2X_3^2 + X_3X_2^2, \omega X_2X_3^2 + \omega^2 X_3X_2^2, \lambda X_1X_2X_3 + \lambda^2 X_1^2X_2^2X_3^2).$$

By [6], we know that ϵ_h is the Hermitean Veronese pseudo-embedding of PG(2, 4) and that ϵ_u is isomorphic to the universal pseudo-embedding of PG(2, 4). We thus have $\mathcal{E}_{\epsilon_h} = \overline{\mathcal{L}} \cup \mathcal{V} \cup \overline{\mathcal{U}}$ and $\mathcal{E}_{\epsilon_u} = \overline{\mathcal{L}} \cup \mathcal{V} \cup \overline{\mathcal{U}} \cup \overline{\mathcal{B}} \cup \mathcal{H} \cup \overline{\mathcal{W}}$.

The following is a well-known property of the group $PSL(3,4)$.

Lemma 8.3. *The group $PSL(3,4)$ is generated by the following maps:*

(a) $\theta_\sigma : (X_1, X_2, X_3) \mapsto (X_{\sigma(1)}, X_{\sigma(2)}, X_{\sigma(3)})$, $\sigma \in S_3$;

(b) $\theta_\lambda : (X_1, X_2, X_3) \mapsto (X_1 + \lambda X_2, X_2, X_3)$, $\lambda \in \mathbb{F}_4$.

Lemma 8.4. *Let $\epsilon : \mathrm{PG}(V) \to \mathrm{PG}(W)$ be one of $\epsilon_h, \epsilon_u, \epsilon_1, \epsilon_\omega, \epsilon_{\omega^2}$. Then for every $\theta \in PSL(3,4)$, there exists a $\widetilde{\theta} \in GL(W)$ such that $\epsilon \circ \theta = \widetilde{\theta} \circ \epsilon$.*

Proof. It is straightforward to prove this if θ is one of the maps mentioned in Lemma 8.3. The claim then follows from the fact that these maps generate $PSL(3,4)$. □

Theorem 8.5. *Let $\epsilon : \mathrm{PG}(V) \to \mathrm{PG}(W)$ be one of $\epsilon_h, \epsilon_u, \epsilon_1, \epsilon_\omega, \epsilon_{\omega^2}$. Then ϵ is a $PSL(3,4)$-homogeneous pseudo-embedding of* $\mathrm{PG}(2,4)$.

Proof. One can easily verify that $\langle Im(\epsilon)\rangle = \mathrm{PG}(W)$ in each of the five cases. In view of Lemma 8.4, it remains to prove that ϵ maps every line L of $\mathrm{PG}(2,4)$ to a frame of a subspace of $\mathrm{PG}(W)$, and it suffices to prove the latter for the line $\langle (1,0,0), (0,1,0)\rangle$. But the verification for this line is straightforward. □

We already know $\mathcal{H}_\epsilon$ if ϵ is equal to ϵ_h or ϵ_u, and that there are three possibilities for $\mathcal{H}_\epsilon$ if ϵ is equal to ϵ_1, ϵ_ω or ϵ_{ω^2} (see Lemma 8.1). We now determine which of these three possibilities actually occurs. Define the following three Baer subplanes:

$$B_1 = \{\langle \bar{e}_1\rangle, \langle \bar{e}_2\rangle, \langle \bar{e}_3\rangle, \langle \bar{e}_1 + \bar{e}_2\rangle, \langle \bar{e}_1 + \bar{e}_3\rangle, \langle \bar{e}_2 + \bar{e}_3\rangle, \langle \bar{e}_1 + \bar{e}_2 + \bar{e}_3\rangle\},$$

$$B_2 = \{\langle \bar{e}_1\rangle, \langle \bar{e}_2\rangle, \langle \bar{e}_3\rangle, \langle \bar{e}_1 + \bar{e}_2\rangle, \langle \bar{e}_1 + \omega\bar{e}_3\rangle, \langle \bar{e}_2 + \omega\bar{e}_3\rangle, \langle \bar{e}_1 + \bar{e}_2 + \omega\bar{e}_3\rangle\},$$

$$B_3 = \{\langle \bar{e}_1\rangle, \langle \bar{e}_2\rangle, \langle \bar{e}_3\rangle, \langle \bar{e}_1 + \bar{e}_2\rangle, \langle \bar{e}_1 + \omega^2\bar{e}_3\rangle, \langle \bar{e}_2 + \omega^2\bar{e}_3\rangle, \langle \bar{e}_1 + \bar{e}_2 + \omega^2\bar{e}_3\rangle\}.$$

Any two of these intersect in four points and so they belong to distinct $\mathcal{B}_i$'s. Suppose that $B_i \in \mathcal{B}_i$ for every $i \in \{1,2,3\}$. The verification of the following lemma is straightforward.

Lemma 8.6.

- *B_1 has equation $X_1X_2X_3 + X_1^2X_2^2X_3^2 + X_1X_2^2 + X_2X_1^2 + X_1X_3^2 + X_3X_1^2 + X_2X_3^2 + X_3X_2^2 = 0$ and thus belongs to $\mathcal{H}_{\epsilon_1}$.*
- *B_2 has equation $\omega^2X_1X_2X_3 + \omega X_1^2X_2^2X_3^2 + X_1X_2^2 + X_2X_1^2 + \omega X_1X_3^2 + \omega^2X_3X_1^2 + \omega X_2X_3^2 + \omega^2X_3X_2^2 = 0$ and thus belongs to $\mathcal{H}_{\epsilon_{\omega^2}}$.*
- *B_3 has equation $\omega X_1X_2X_3 + \omega^2X_1^2X_2^2X_3^2 + X_1X_2^2 + X_2X_1^2 + \omega^2X_1X_3^2 + \omega X_3X_1^2 + \omega^2X_2X_3^2 + \omega X_3X_2^2 = 0$ and thus belongs to $\mathcal{H}_{\epsilon_\omega}$.*

Note that each Baer subplane B_i, $i \in \{1,2,3\}$, is contained in seven members of $\overline{\mathcal{V}}$ (one pencil of three lines for each of the seven points of B_i), but that none of these sets satisfy the equations mentioned in Lemma 8.6. If we consider the universal embedding ϵ_u of $\mathrm{PG}(2,4)$, then the image of B_i is thus contained in more than one hyperplane, although there is only one hyperplane whose intersection with the image of ϵ_u coincides with $\epsilon_u(B_i)$.

Lemmas 8.1, 8.6 and Theorem 8.5 now imply the following.

Corollary 8.7. *The following holds:*

- $\mathcal{E}_{\epsilon_1} = \overline{\mathcal{L}} \cup \mathcal{V} \cup \overline{\mathcal{U}} \cup \overline{\mathcal{B}_1} \cup \mathcal{H}_1 \cup \overline{\mathcal{W}_1}$,
- $\mathcal{E}_{\epsilon_{\omega^2}} = \overline{\mathcal{L}} \cup \mathcal{V} \cup \overline{\mathcal{U}} \cup \overline{\mathcal{B}_2} \cup \mathcal{H}_2 \cup \overline{\mathcal{W}_2}$,
- $\mathcal{E}_{\epsilon_\omega} = \overline{\mathcal{L}} \cup \mathcal{V} \cup \overline{\mathcal{U}} \cup \overline{\mathcal{B}_3} \cup \mathcal{H}_3 \cup \overline{\mathcal{W}_3}$.

The following is now a consequence of Theorems 8.2, 8.5 and Corollary 8.7.

Corollary 8.8. *The five nonisomorphic $PSL(3,4)$-homogeneous pseudo-embeddings of* $\mathrm{PG}(2,4)$ *are precisely the maps $\epsilon_h, \epsilon_u, \epsilon_1, \epsilon_\omega, \epsilon_{\omega^2}$.*

8.2 The (M_{24}-)homogeneous pseudo-embeddings of W_{24}

By Theorem 6.1, we know that $\mathcal{S} = W_{24}$ has up to isomorphism five even sets. We denote the isomorphism classes of these even sets by $E_0 = \{\emptyset\}$, E_8, E_{12}, E_{16} and $E_{24} = \{\mathcal{P}_{24}\}$, where E_i denotes the set of all even sets of size $i \in \{0, 8, 12, 16, 24\}$. By the tables of Section 6, we find:

$$|E_0| = 1, \quad |E_8| = 759, \quad |E_{12}| = 2576, \quad |E_{16}| = 759, \quad |E_{24}| = 1.$$

In order to find the (M_{24}-)homogeneous pseudo-embeddings of W_{24}, we must find by Proposition 2.4(a)+(b) the subspaces of $\mathcal{E}_\mathcal{S}$ that can be written as a union of some of the E_i's. As the number of elements in a subspace is a power of 2, there are only three subspaces that can be written like that, namely E_0, $E_0 \cup E_{24}$ and $\mathcal{E}_\mathcal{S} = E_0 \cup E_8 \cup E_{12} \cup E_{16} \cup E_{24}$. By Propositions 2.2 and 2.4, only the latter corresponds to an M_{24}-homogeneous pseudo-embedding. We thus have:

Theorem 8.9. *Up to isomorphism, W_{24} has a unique homogeneous pseudo-embedding, namely the universal pseudo-embedding in* $\mathrm{PG}(11,2)$*.*

Suppose now that $(\bar{e}_1, \bar{e}_2, \bar{e}_3)$ is a basis of V such that

$$B_1 = \{\langle \bar{e}_1 \rangle, \langle \bar{e}_2 \rangle, \langle \bar{e}_3 \rangle, \langle \bar{e}_1 + \bar{e}_2 \rangle, \langle \bar{e}_1 + \bar{e}_3 \rangle, \langle \bar{e}_2 + \bar{e}_3 \rangle, \langle \bar{e}_1 + \bar{e}_2 + \bar{e}_3 \rangle\} \in \mathcal{B}_1,$$

$$B_2 = \{\langle \bar{e}_1 \rangle, \langle \bar{e}_2 \rangle, \langle \bar{e}_3 \rangle, \langle \bar{e}_1 + \bar{e}_2 \rangle, \langle \bar{e}_1 + \omega\bar{e}_3 \rangle, \langle \bar{e}_2 + \omega\bar{e}_3 \rangle, \langle \bar{e}_1 + \bar{e}_2 + \omega\bar{e}_3 \rangle\} \in \mathcal{B}_2,$$

$$B_3 = \{\langle \bar{e}_1 \rangle, \langle \bar{e}_2 \rangle, \langle \bar{e}_3 \rangle, \langle \bar{e}_1 + \bar{e}_2 \rangle, \langle \bar{e}_1 + \omega^2\bar{e}_3 \rangle, \langle \bar{e}_2 + \omega^2\bar{e}_3 \rangle, \langle \bar{e}_1 + \bar{e}_2 + \omega^2\bar{e}_3 \rangle\} \in \mathcal{B}_3.$$

Let (X_1, X_2, X_3) denote the coordinates of a generic point of $\mathrm{PG}(V) = \mathrm{PG}(2,4)$ and let $(Y_1, Y_2, \ldots, Y_{12})$ denote the coordinates of a generic point of $\mathrm{PG}(11,2)$. The hyperplane of $\mathrm{PG}(11,2)$ with equation $a_1Y_1 + a_2Y_2 + \cdots + a_{12}Y_{12} = 0$ will be denoted by $[a_1, a_2, \ldots, a_{12}]$. Now, let $\widetilde{\epsilon}_{24}$ be the following map from $\mathcal{P}_{24} = \mathcal{P} \cup \{\infty_1, \infty_2, \infty_3\}$ to $\mathrm{PG}(11,2)$:

$(X_0, X_1, X_2) \mapsto (X_1^3, X_2^3, X_3^3, X_1X_2^2 + X_2X_1^2, \omega X_1X_2^2 + \omega^2 X_2X_1^2, X_1X_3^2 + X_3X_1^2, \omega X_1X_3^2 + \omega^2 X_3X_1^2,$

$$X_2X_3^2 + X_3X_2^2, \omega X_2X_3^2 + \omega^2 X_3X_2^2, X_1X_2X_3 + X_1^2X_2^2X_3^2 + 1,$$
$$\omega^2 X_1X_2X_3 + \omega X_1^2X_2^2X_3^2 + 1, \omega X_1X_2X_3 + \omega^2 X_1^2X_2^2X_3^2 + 1),$$
$$\infty_1 \mapsto (0, 0, \ldots, 0, 1, 0, 0),$$
$$\infty_2 \mapsto (0, 0, \ldots, 0, 0, 1, 0),$$
$$\infty_3 \mapsto (0, 0, \ldots, 0, 0, 0, 1).$$

Theorem 8.10. *$\widetilde{\epsilon}_{24}$ is isomorphic to the universal pseudo-embedding of W_{24}.*

Proof. By Propositions 2.1 and 2.5, it suffices to show that the pseudo-hyperplanes (or the even sets distinct from $\mathcal{P}_{24}$) are precisely the sets $A_\Pi := \widetilde{\epsilon}_{24}^{-1}(\widetilde{\epsilon}_{24}(\mathcal{P}_{24}) \cap \Pi)$, where Π is a hyperplane of $PG(11, 2)$. With ϵ_h, ϵ_1, ϵ_ω and ϵ_{ω^2} as defined in Section 8.1, we have:

If Π is of the form $[\ldots, 0, 0, 0]$, then A_Π is of the form $\{\infty_1, \infty_2, \infty_3\} \cup X$ where $X \in \mathcal{H}_{\epsilon_h}$.

If Π is of the form $[\ldots, 1, 0, 0]$, then A_Π is of the form $\{\infty_2, \infty_3\} \cup \overline{X}$ where $X \in \mathcal{H}_{\epsilon_1} \setminus \mathcal{H}_{\epsilon_h}$.

If Π is of the form $[\ldots, 0, 1, 0]$, then A_Π is of the form $\{\infty_1, \infty_3\} \cup \overline{X}$ where $X \in \mathcal{H}_{\epsilon_{\omega^2}} \setminus \mathcal{H}_{\epsilon_h}$.

If Π is of the form $[\ldots, 0, 0, 1]$, then A_Π is of the form $\{\infty_1, \infty_2\} \cup \overline{X}$ where $X \in \mathcal{H}_{\epsilon_\omega} \setminus \mathcal{H}_{\epsilon_h}$.

If Π is of the form $[\ldots, 1, 1, 0]$, then A_Π is of the form $\{\infty_3\} \cup X$ where $X \in \mathcal{H}_{\epsilon_\omega} \setminus \mathcal{H}_{\epsilon_h}$.

If Π is of the form $[\ldots, 1, 0, 1]$, then A_Π is of the form $\{\infty_2\} \cup X$ where $X \in \mathcal{H}_{\epsilon_{\omega^2}} \setminus \mathcal{H}_{\epsilon_h}$.

If Π is of the form $[\ldots, 0, 1, 1]$, then A_Π is of the form $\{\infty_1\} \cup X$ where $X \in \mathcal{H}_{\epsilon_1} \setminus \mathcal{H}_{\epsilon_h}$.

If Π is of the form $[\ldots, 1, 1, 1]$, then A_Π is of the form $\overline{X}$ where $X \in \mathcal{H}_{\epsilon_h} \cup \{\mathcal{P}\}$.

The claim then follows from Corollary 8.7 and the tables of Section 6, taking into account that for a pseudo-embedding ϵ, the elements of $\mathcal{H}_\epsilon$ are precisely the complements of the elements of $\mathcal{E}_\epsilon$. $\square$

Remark. Let $i \in \{1, 2, 3\}$. If we derive the universal pseudo-embedding of W_{24} with respect to ∞_i, and subsequently this derived pseudo-embedding with respect to $\{\infty_1, \infty_2, \infty_3\} \setminus \{\infty_i\}$, then we obtain a $PSL(3, 4)$-homogeneous pseudo-embedding of $PG(2, 4)$ in $PG(9, 2)$. In this way, we obtain the three nonisomorphic pseudo-embeddings of $PG(2, 4)$ described in Section 8.1.

8.3 The (M_{23}-)homogeneous pseudo-embeddings of W_{23}

By Theorem 5.1, we know that $\mathcal{S} = W_{23}$ has up to isomorphism four even sets. We denote the isomorphism classes of these even sets by $E_0 = \{\emptyset\}$, E_8, E_{12} and E_{16}, where E_i denotes the set of all even sets of size $i \in \{0, 8, 12, 16\}$. By the tables of Section 5, we find

$$|E_0| = 1, \quad |E_8| = 378, \quad |E_{12}| = 1288, \quad |E_{16}| = 381.$$

In order to find the (M_{23}-)homogeneous pseudo-embeddings of W_{23}, we must find by Proposition 2.4(a)+(b) the subspaces of $\mathcal{E}_\mathcal{S}$ that can be written as a union of some of the E_i's. As the number of elements in a subspace is a power of 2, there are only two subspaces that can be written like that, namely E_0 and $\mathcal{E}_\mathcal{S} = E_0 \cup E_8 \cup E_{12} \cup E_{16}$. By Propositions 2.2 and 2.4, we then know:

Theorem 8.11. *Up to isomorphism, W_{23} has a unique homogeneous pseudo-embedding, namely the universal pseudo-embedding in* PG(10, 2).

The following is a consequence of Corollary 7.3(a) and Theorem 8.10.

Corollary 8.12. *Let $\widetilde{\epsilon}_{23}$ be the following map from $\mathcal{P}_{23} = \mathcal{P} \cup \{\infty_1, \infty_2\}$ to* PG(10, 2)*:*

$$(X_0, X_1, X_2) \mapsto (X_1^3, X_2^3, X_3^3, X_1X_2^2+X_2X_1^2, \omega X_1X_2^2+\omega^2 X_2X_1^2, X_1X_3^2+X_3X_1^2, \omega X_1X_3^2+\omega^2 X_3X_1^2,$$

$$X_2X_3^2 + X_3X_2^2, \omega X_2X_3^2 + \omega^2 X_3X_2^2, X_1X_2X_3 + X_1^2X_2^2X_3^2 + 1, \omega^2 X_1X_2X_3 + \omega X_1^2X_2^2X_3^2 + 1),$$

$$\infty_1 \mapsto (0, 0, \ldots, 0, 1, 0),$$

$$\infty_2 \mapsto (0, 0, \ldots, 0, 0, 1).$$

Then $\widetilde{\epsilon}_{23}$ is isomorphic to the universal pseudo-embedding of W_{23}.

8.4 The M_{22}- and $(M_{22} : 2)$-homogeneous pseudo-embeddings of W_{22}

By Theorem 4.1, we know that $\mathcal{S} = W_{22}$ has up to isomorphism eight even sets. We denote the isomorphism classes of these even sets by $E_0 = \{\emptyset\}$, E_6, E_8, E_{10}, E_{12}, E_{14}, E_{16} and $E_{22} = \{\mathcal{P}_{22}\}$, where E_i denotes the set of all even sets of size $i \in \{0, 6, 8, 10, 12, 14, 16, 22\}$. By the tables of Section 4, we find

$$|E_0| = 1, |E_6| = 77, |E_8| = 330, |E_{10}| = 616, |E_{12}| = 616, |E_{14}| = 330, |E_{16}| = 77, |E_{22}| = 1.$$

The E_i's are orbits for both the groups M_{22} and $M_{22} : 2$. In order to find the M_{22}- and $(M_{22} : 2)$-homogeneous pseudo-embeddings, we must find by Proposition 2.4(a)+(b) the subspaces $\mathcal{E}$ of $\mathcal{E}_\mathcal{S}$ that can be written as a union of some of the E_i's. As E_0 and $\mathcal{E}_\mathcal{S} = E_0 \cup E_6 \cup \cdots \cup E_{22}$ are two such subspaces, we may assume that $E_0 \subsetneq \mathcal{E} \subsetneq \mathcal{E}_\mathcal{S}$. By Corollary 7.4(b), we also know that $\mathcal{E}' = E_0 \cup E_8 \cup E_{12} \cup E_{16}$ is such a subspace. As $\mathcal{E}' \cap \mathcal{E}$ is a subspace whose number of elements is a power of 2, we necessarily have $\mathcal{E} \cap \mathcal{E}' = \mathcal{E}'$ or $\mathcal{E} \cap \mathcal{E}' = E_0$. The fact that $|\mathcal{E}|$ is a power of 2 then implies that either $\mathcal{E} = \mathcal{E}'$, $\mathcal{E} = E_0 \cup E_{22}$ or $\mathcal{E} = E_0 \cup E_6 \cup E_{10} \cup E_{14}$. The latter case can however not occur. Since $\mathcal{E}$ then contains all subsets of the form $\{\infty\} \cup L$ with L a line of PG(2, 4), it would also contain all subsets of the form $(\{\infty\} \cup L_1)\Delta(\{\infty\} \cup L_2) = L_1 \Delta L_2$, where L_1 and L_2 are two distinct lines of PG(2, 4). This is impossible as any set of the form $L_1 \Delta L_2$ belongs to E_8. We thus conclude:

Lemma 8.13. *The following are the subspaces of $\mathcal{E}_\mathcal{S}$ that can be written as the union of a number of the E_i's:*

- $E_0 = \{\emptyset\}$;
- $E_0 \cup E_{22} = \{\emptyset, \mathcal{P}_{22}\}$;
- $E_0 \cup E_8 \cup E_{12} \cup E_{16}$;
- $\mathcal{E}_\mathcal{S} = E_0 \cup E_6 \cup \cdots \cup E_{22}$.

By Corollary 7.4, Propositions 2.2, 2.4 and Lemma 8.13, we then have:

Theorem 8.14. *Up to isomorphism, W_{22} has two M_{22}-homogeneous pseudo-embeddings: the universal pseudo-embedding in* PG(10, 2) *and the pseudo-embedding in* PG(9, 2) *that arises by deriving the universal pseudo-embedding of W_{23} with respect to ∞_2. These M_{22}-homogeneous pseudo-embeddings are also $(M_{22} : 2)$-homogeneous.*

The following is a consequence of Corollaries 7.4(a) and 8.12.

Corollary 8.15. *Let ϵ_{22} be the following map from $\mathcal{P}_{22} = \mathcal{P} \cup \{\infty\}$ to* PG(9, 2)*:*

$$(X_0, X_1, X_2) \mapsto (X_1^3, X_2^3, X_3^3, X_1X_2^2 + X_2X_1^2, \omega X_1X_2^2 + \omega^2X_2X_1^2, X_1X_3^2 + X_3X_1^2,$$

$$\omega X_1X_3^2 + \omega^2X_3X_1^2, X_2X_3^2 + X_3X_2^2, \omega X_2X_3^2 + \omega^2X_3X_2^2, X_1X_2X_3 + X_1^2X_2^2X_3^2 + 1),$$

$$\infty \mapsto (0, 0, \ldots, 0, 1).$$

Then ϵ_{22} is isomorphic to the M_{22}-homogeneous pseudo-embedding of W_{22} in PG(9, 2).

The following is a consequence of Corollary 7.3(b) and Theorem 8.10.

Corollary 8.16. *Let $\widetilde{\epsilon}_{22}$ be the following map from $\mathcal{P}_{22} = \mathcal{P} \cup \{\infty\}$ to* PG(10, 2)*:*

$$(X_0, X_1, X_2) \mapsto (X_1^3, X_2^3, X_3^3, X_1X_2^2 + X_2X_1^2, \omega X_1X_2^2 + \omega^2X_2X_1^2, X_1X_3^2 + X_3X_1^2, \omega X_1X_3^2 + \omega^2X_3X_1^2,$$

$$X_2X_3^2 + X_3X_2^2, \omega X_2X_3^2 + \omega^2X_3X_2^2, X_1X_2X_3 + X_1^2X_2^2X_3^2, 1),$$

$$\infty \mapsto (0, 0, \ldots, 0, 1).$$

Then $\widetilde{\epsilon}_{22}$ is isomorphic to the universal pseudo-embedding of W_{22}.

Remark. It is also possible to prove Corollaries 8.15 and 8.16 in a similar way as Theorem 8.10.

9 The pseudo-generating ranks of W_{22}, W_{23} and W_{24}

For point-line geometries with three points per line, the notion of a pseudo-embedding coincides with the notion of an ordinary projective embedding. The vector dimension of the universal (pseudo-)embedding is in this case also called the *embedding rank*. One of the main tools in determining the embedding rank is finding a generating set of smallest possible size. The latter number is called the *generating rank* of the geometry. The determination of generating ranks of geometries has been the subject of active research, see e.g. [3].

In the theory of pseudo-embeddings, there is a similar tool for determining pseudo-embedding ranks, namely the one of pseudo-generating sets (of smallest possible size). Although we determined the pseudo-embedding ranks of the large Witt designs without this tool, we will for reasons of completeness now also determine the smallest sizes of pseudo-generating sets for these designs.

As before, let $\mathcal{S}$ be a point-line geometry for which the number of points on each line is finite an at least three. A *pseudo-subspace* of $\mathcal{S}$ is a set S of points such that every line that has at most one point outside S has all its points in S. Every pseudo-hyperplane is a pseudo-subspace. The whole point set $\mathcal{P}$ is an example of a pseudo-subspace and the intersection of any number of pseudo-subspaces is again a pseudo-subspace. This implies that every nonempty set X of points is contained in a smallest pseudo-subspace, namely the intersection of all pseudo-subspaces containing X. If this smallest pseudo-subspace coincides with $\mathcal{P}$, then X is called a *pseudo-generating set*. The smallest size of a pseudo-generating set is called the *pseudo-embedding rank*. The following result which was proved in [7, Theorem 1.5] explains why pseudo-generating sets can be important for determining universal pseudo-embeddings of geometries.

Proposition 9.1 ([7]). *Suppose $\mathcal{S}$ has pseudo-embeddings. Then the following hold.*

(1) *The pseudo-embedding rank of $\mathcal{S}$ is bounded above by the pseudo-generating rank of $\mathcal{S}$.*

(2) *If there exists a pseudo-embedding $\epsilon : \mathcal{S} \to \mathrm{PG}(V)$ and a pseudo-generating set X of $\mathcal{S}$ such that $|X| = \dim(V) < \infty$, then the embedding and generating ranks of $\mathcal{S}$ are equal to $\dim(V)$ and ϵ is isomorphic to the universal pseudo-embedding of $\mathcal{S}$.*

In fact, we could say more in Proposition 9.1: the image $\epsilon(X)$ of X then forms a basis of the universal embedding space $\mathrm{PG}(V)$. Indeed, if $\epsilon(X)$ is not a basis, then it is contained in a hyperplane Π of $\mathrm{PG}(V)$ and the pseudo-hyperplane $\epsilon^{-1}(\epsilon(\mathcal{P}) \cap \Pi)$ would then be a pseudo-subspace containing X, in contradiction with the fact that X is a pseudo-generating set.

In the following lemma, we determine a pseudo-generating set of size 11 in $\mathrm{PG}(2,4)$. That such a pseudo-generating set exists was already known, see [7, Lemma 4.5(1)] and [9, p. 79], but the particular construction of such a set given here will be used in the construction of a pseudo-generating set of size 11 in W_{22}.

Lemma 9.2. PG$(2,4)$ *has a pseudo-generating set of size* 11.

Proof. Let x be a point and L_1, L_2, L_3, L_4, L_5 the five lines through x. Put $X_1 := L_1 \setminus \{x\}$, let X_2 be a subset of size 3 of $L_2 \setminus \{x\}$, X_3 a subset of size 3 of $L_3 \setminus \{x\}$ and $L_4 = \{x_4\}$ a singleton contained in $L_4 \setminus \{x\}$. We show that the set $X_1 \cup X_2 \cup X_3 \cup X_4$, which has size 11, is a pseudo-generating set. Denote by S the smallest pseudo-subspace containing $X_1 \cup X_2 \cup X_3 \cup X_4$. Since $X_1 \subseteq S$, also the remaining point x of L_1 is contained in S. The latter implies that also the unique points on $L_2 \setminus (X_2 \cup \{x\})$ and $L_3 \setminus (X_3 \cup \{x\})$ are contained in S. Hence, $L_1 \cup L_2 \cup L_3 \subseteq S$. By considering lines through $x_4 \in S$ not containing x, we then see that all points of L_5 are contained in S. By considering lines not containing x, we subsequently see that also all points of L_4 are contained in S. Hence, S coincides with the whole point set of PG$(2,4)$. □

Lemma 9.3. *The Witt design* W_{22} *has a pseudo-generating set of size* 11.

Proof. Let $H \in \mathcal{H}_1$. Let $x \in \mathrm{PG}(2,4) \setminus H$, let L_1, L_2 and L_3 be the three lines through x intersecting H in two points, and let L_4 be an additional line through x. Put $X_1 := L_1 \setminus \{x\}$, let X_i with $i \in \{2,3\}$ be a subset of size 3 of $L_i \setminus \{x\}$ containing $L_i \cap \mathcal{H}$, and let X_4 be a singleton contained in $L_4 \setminus \{x\}$. Then $X_1 \cup X_2 \cup X_3 \cup X_4$ is a pseudo-generating set of PG$(2,4)$ (see the proof of Lemma 9.2) and hence $X_1 \cup X_2 \cup X_3 \cup X_4 \cup \{\infty\}$ is a pseudo-generating set of W_{22}.

Now, let $h \in H$ be arbitrary. Then obviously, every pseudo-subspace containing $(X_1 \cup X_2 \cup X_3 \cup X_4 \cup \{\infty\}) \setminus \{h\}$ also contains h (by considering the line H of W_{22}) and hence coincides with W_{22}. So, $(X_1 \cup X_2 \cup X_3 \cup X_4 \cup \{\infty\}) \setminus h$ is a pseudo-generating set of size 11 of W_{22}. □

A *quadrangle* of PG$(2,4)$ is a set of four points, no three of which are collinear. Any quadrangle of PG$(2,4)$ is contained in a unique Baer subplane and a unique hyperoval. The following lemma will be useful in the reasoning that allows to determine the pseudo-generating rank of W_{23}.

Lemma 9.4. *Let* B *be a Baer subplane of* PG$(2,4)$*, let* $x \in B$*, and let* L_1, L_2, L_3 *be the three lines through* x *meeting* B *in exactly three points. Then* $((L_1 \setminus \{x\}) \cap B) \cup ((L_2 \setminus \{x\}) \cap B) \cup (L_3 \setminus B)$ *is the unique hyperoval containing the quadrangle* $((L_1 \setminus \{x\}) \cap B) \cup ((L_2 \setminus \{x\}) \cap B)$.

Proof. If this set would not be a hyperoval, then there would exist a line meeting $(L_1 \setminus \{x\}) \cap B$, $(L_2 \setminus \{x\}) \cap B$ and $L_3 \setminus B$. But that is impossible as any line meeting $(L_1 \setminus \{x\}) \cap B$ and $(L_2 \setminus \{x\}) \cap B$ also meets $(L_3 \setminus \{x\}) \cap B$. □

Lemma 9.5. *The Witt design* W_{23} *has a pseudo-generating set of size* 11.

Proof. Let x be a point of PG$(2,4)$ and B a Baer subplane of $\mathcal{B}_3$ through x. Let L_1, L_2, L_3, L_4 and L_5 be the five lines through x such that $|L_1 \cap B| = |L_2 \cap B| = |L_3 \cap B| = 3$. Put $L_i \cap B = \{x, y_i, z_i\}$ and $L_i = \{x, y_i, z_i, u_i, v_i\}$ for every $i \in \{1,2,3\}$. By Lemma 9.4, the set $\{y_1, z_1, y_2, z_2, u_3, v_3\}$ is the unique hyperoval containing $\{y_1, z_1, y_2, z_2\}$. Let

H be the unique hyperoval containing $\{y_1, z_1, y_2, u_2\}$. If $H \cap \{y_3, z_3\} \neq \emptyset$, then there exists a line through y_2 containing a point of $\{y_1, z_1\} \subseteq H$ and a point of $\{y_3, z_3\} \cap H$, which is impossible. If $H \cap \{u_3, v_3\} \neq \emptyset$, then H has at least four points in common with the hyperoval $\{y_1, z_1, y_2, z_2, u_3, v_3\}$ and hence coincides with it. This is impossible as u_2 belongs to precisely one of these hyperovals. We thus have $|L_1 \cap H| = |L_2 \cap H| = 2$ and $|L_3 \cap H| = 0$. Without loss of generality, we may suppose that $|L_4 \cap H| = 2$. As $|H \cap B| = 3$, we know that $H \notin \mathcal{H}_3$. Hence, $H \cup \{\infty\}$ is a line of W_{23} for some $\infty \in \{\infty_1, \infty_2\}$.

Now, put $Y_1 := L_1 \setminus \{x\}$, $Y_2 := \{y_2, z_2\}$, $Y_3 := \{y_3\}$ and $Y_4 := H \cap L_4$. Then $X := Y_1 \cup Y_2 \cup Y_3 \cup Y_4 \cup \{\infty_1, \infty_2\}$ is a set of 11 points of W_{23}. We show that X is a pseudo-generating set. To that end, consider the smallest pseudo-subspace S containing X.

As the line $L_1 \cup \{\infty_1, \infty_2\}$ contains 6 points of X (namely the six points of $(L_1 \cup \{\infty_1, \infty_2\}) \setminus \{x\})$ we see that also $x \in S$. As the line B of W_{23} contains 6 points of S (namely $y_1, z_1, x, y_2, z_2, y_3$), also the seventh point z_3 belongs to S. As the line $\{\infty\} \cup H$ contains 6 points of S (namely ∞, y_1, z_1, y_2 and the two points of $L_4 \cap H$), we see that also the seventh point u_2 belongs to S. Now, the line $\{\infty_1, \infty_2\} \cup L_2$ has six points in common with S (namely the points of $(\{\infty_1, \infty_2\} \cup L_2) \setminus \{v_2\})$ and hence also the seventh point v_2 belongs to S. At this stage, we thus already know that $L_1 \cup L_2 \cup \{y_3, z_3\} \cup (L_4 \cap H) \subseteq S$.

Let B' be the unique Baer subplane containing $\{x, y_1, z_1\} \cup (L_4 \cap H)$. As H is the unique hyperoval containing $\{y_1, z_1\} \cup (L_4 \cap H)$, we know by Lemma 9.4 that the line connecting the two remaining points y_2 and u_2 of H must intersect B' in three points. So, $\{x, z_2, v_2\} \subseteq B'$.

Now, the lines $z_2 y_1$ and $z_2 z_1$ intersect L_3 in points of S (namely y_3 and z_3) and L_4 in points of S (namely the points of $L_4 \cap H$). So, these lines intersect L_5 in points a_1 and a_2 belonging to S.

As the lines $z_2 y_1$ and $z_2 z_1$ intersect $U := \{a_1, a_2\} \cup (L_4 \cap H) \cup \{y_3, z_3\}$ in two disjoint sets of size 3, $U \cup \{x\}$ cannot be a Baer subplane. So, there exists a line M not containing x meeting U in precisely two points. Let u be the point of $(L_3 \cup L_4 \cup L_5) \cap M$ not contained in U and suppose $u \in L_i$ with $i \in \{3, 4, 5\}$. Then the line $\{\infty_1, \infty_2\} \cup M$ contains six points of S (namely $\infty_1, \infty_2, M \cap L_1, M \cap L_2$ and the two points of $M \cap U$) and so also $u \in S$. But as the line $L_i \cup \{\infty_1, \infty_2\}$ of W_{23} now contains six points of S (namely ∞_1, ∞_2, x, u and the two points of $L_i \cap U$), we also see that $L_i \subset S$. Now, put $\{3, 4, 5\} \setminus \{i\} = \{j, k\}$. By considering lines of $PG(3, 4)$ through a point of $L_j \cap U$ (respectively $L_k \cap U$) not containing x and extending them to lines of W_{23}, we see that all points of L_k (respectively L_j) must belong to S. Hence, S consists of all points of W_{23}. □

Lemma 9.6. *The Witt design W_{24} has a pseudo-generating set of size* 12.

Proof. Recall that W_{23} is obtained from W_{24} by derivation with respect to ∞_3. If Y is a pseudo-generating set of size 11 of W_{23} (see Lemma 9.5), then $X := Y \cup \{\infty_3\}$ is a pseudo-generating set of size 12 of W_{24}. □

By Theorems 4.1, 5.1, 6.1, 8.2, Proposition 9.1 and Lemmas 9.2, 9.3, 9.5, 9.6, we have:

Corollary 9.7. *The pseudo-generating ranks of* PG(2, 4)*,* W_{22} *and* W_{23} *are equal to* 11*. The pseudo-generating rank of* W_{24} *is equal to* 12*.*

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