

REGULAR PSEUDO-HYPEROVALS AND REGULAR PSEUDO-OVALS IN EVEN CHARACTERISTIC

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ABSTRACT. S. Rottey and G. Van de Voorde characterized regular pseudo-ovals of $\mathbf{PG}(3n-1, q)$, $q = 2^h$, $h > 1$ and n prime. Here an alternative proof is given and slightly stronger results are obtained.

1. INTRODUCTION

Pseudo-ovals and pseudo-hyperovals were introduced in [10]; see also [12]. These objects play a key role in the theory of translation generalized quadrangles [6, 12]. Pseudo-hyperovals only exist in even characteristic. A characterization of regular pseudo-ovals in odd characteristic was given in [2]; see also [12]. In [8] a characterization of regular pseudo-ovals and regular pseudo-hyperovals in $\mathbf{PG}(3n-1, q)$, q even, $q \neq 2$ and n prime, is obtained. Here a shorter proof is given and slightly stronger results are obtained.

2. OVALS AND HYPEROVALS

A k -arc in $\mathbf{PG}(2, q)$ is a set of k points, $k \geq 3$, no three of which are collinear. Any non-singular conic of $\mathbf{PG}(2, q)$ is a $(q+1)$ -arc. If \mathcal{K} is any k -arc of $\mathbf{PG}(2, q)$, then $k \leq q+2$. For q odd $k \leq q+1$ and for q even a $(q+1)$ -arc extends to a $(q+2)$ -arc; see [3]. A $(q+1)$ -arc is an *oval*; a $(q+2)$ -arc, q even, is a *complete oval* or *hyperoval*.

A famous theorem of B. Segre [9] tells us that for q odd every oval of $\mathbf{PG}(2, q)$ is a non-singular conic. For q even, there are many ovals that are not conics [3]; also, there are many hyperovals that do not contain a conic [3].

3. GENERALIZED OVALS AND HYPEROVALS

Arcs, ovals and hyperovals can be generalized by replacing their points with m -dimensional subspaces to obtain generalized k -arcs, generalized ovals and generalized hyperovals. These have strong connections to generalized quadrangles, projective planes, circle geometries, flocks and other structures. See [6, 12, 10, 11, 2, 7]. Below, some basic definitions and results are formulated; for an extensive study, many applications and open problems, see [12].

A *generalized k -arc* of $\mathbf{PG}(3n-1, q)$, $n \geq 1$, is a set of k $(n-1)$ -dimensional subspaces of $\mathbf{PG}(3n-1, q)$ every three of which generate $\mathbf{PG}(3n-1, q)$. If q is odd then $k \leq q^n + 1$, if q is even then $k \leq q^n + 2$. Every generalized $(q^n + 1)$ -arc of $\mathbf{PG}(3n-1, q)$, q even, can be extended to a generalized $(q^n + 2)$ -arc.

If \mathcal{O} is a generalized $(q^n + 1)$ -arc in $\mathbf{PG}(3n - 1, q)$, then it is a *pseudo-oval* or *generalized oval* or $[n - 1]$ -*oval* of $\mathbf{PG}(3n - 1, q)$. For $n = 1$, a $[0]$ -oval is just an oval of $\mathbf{PG}(2, q)$. If \mathcal{O} is a generalized $(q^n + 2)$ -arc in $\mathbf{PG}(3n - 1, q)$, q even, then it is a *pseudo-hyperoval* or *generalized hyperoval* or $[n - 1]$ -*hyperoval* of $\mathbf{PG}(3n - 1, q)$. For $n = 1$, a $[0]$ -hyperoval is just a hyperoval of $\mathbf{PG}(2, q)$.

If $\mathcal{O} = \{\pi_0, \pi_1, \dots, \pi_{q^n}\}$ is a pseudo-oval of $\mathbf{PG}(3n - 1, q)$, then π_i is contained in exactly one $(2n - 1)$ -dimensional subspace τ_i of $\mathbf{PG}(3n - 1, q)$ which has no point in common with $(\pi_0 \cup \pi_1 \cup \dots \cup \pi_{q^n}) \setminus \pi_i$, with $i = 0, 1, \dots, q^n$; the space τ_i is the *tangent space* of \mathcal{O} at π_i . For q even the $q^n + 1$ tangent spaces of \mathcal{O} contain a common $(n - 1)$ -dimensional space π_{q^n+1} , the *nucleus* of \mathcal{O} ; also, $\mathcal{O} \cup \{\pi_{q^n+1}\}$ is a pseudo-hyperoval of $\mathbf{PG}(3n - 1, q)$. For q odd, the tangent spaces of a pseudo-oval \mathcal{O} are the elements of a pseudo-oval \mathcal{O}^* in the dual space of $\mathbf{PG}(3n - 1, q)$.

4. REGULAR PSEUDO-OVALS AND PSEUDO-HYPEROVALS

In the extension $\mathbf{PG}(3n-1, q^n)$ of $\mathbf{PG}(3n-1, q)$, consider n planes ξ_i , $i = 1, 2, \dots, n$, that are conjugate in the extension \mathbb{F}_{q^n} of \mathbb{F}_q and which span $\mathbf{PG}(3n - 1, q^n)$. This means that they form an orbit of the Galois group corresponding to this extension and span $\mathbf{PG}(3n - 1, q^n)$.

In ξ_1 consider an oval $\mathcal{O}_1 = \{x_0^{(1)}, x_1^{(1)}, \dots, x_{q^n}^{(1)}\}$. Further, let $x_i^{(1)}, x_i^{(2)}, \dots, x_i^{(n)}$, with $i = 0, 1, \dots, q^n$, be conjugate in \mathbb{F}_{q^n} over \mathbb{F}_q . The points $x_i^{(1)}, x_i^{(2)}, \dots, x_i^{(n)}$ define an $(n - 1)$ -dimensional subspace π_i over \mathbb{F}_q for $i = 0, 1, \dots, q^n$. Then, $\mathcal{O} = \{\pi_0, \pi_1, \dots, \pi_{q^n}\}$ is a generalized oval of $\mathbf{PG}(3n - 1, q)$. These objects are the *regular* or *elementary pseudo-ovals*. If \mathcal{O}_1 is replaced by a hyperoval, and so q is even, then the corresponding \mathcal{O} is a *regular* or *elementary pseudo-hyperoval*.

All known pseudo-ovals and pseudo-hyperovals are regular.

5. CHARACTERIZATIONS

Let $\mathcal{O} = \{\pi_0, \pi_1, \dots, \pi_{q^n}\}$ be a pseudo-oval in $\mathbf{PG}(3n-1, q)$. The tangent space of \mathcal{O} at π_i will be denoted by τ_i , with $i = 0, 1, \dots, q^n$. Choose π_i , $i \in \{0, 1, \dots, q^n\}$, and let $\mathbf{PG}(2n-1, q) \subseteq \mathbf{PG}(3n-1, q)$ be skew to π_i . Further, let $\tau_i \cap \mathbf{PG}(2n-1, q) = \eta_i$ and $\langle \pi_i, \pi_j \rangle \cap \mathbf{PG}(2n-1, q) = \eta_j$, with $j \neq i$. Then $\{\eta_0, \eta_1, \dots, \eta_{q^n}\} = \Delta_i$ is an $(n - 1)$ -spread of $\mathbf{PG}(2n-1, q)$.

Now, let q be even and let π be the nucleus of \mathcal{O} . Let $\mathbf{PG}(2n-1, q) \subseteq \mathbf{PG}(3n-1, q)$ be skew to π . If $\zeta_j = \mathbf{PG}(2n-1, q) \cap \langle \pi, \pi_j \rangle$, then $\{\zeta_0, \zeta_1, \dots, \zeta_{q^n}\} = \Delta$ is an $(n - 1)$ -spread of $\mathbf{PG}(2n-1, q)$.

Next, let q be odd. Choose τ_i , with $i \in \{0, 1, \dots, q^n\}$. If $\tau_i \cap \tau_j = \delta_j$, with $j \neq i$, then $\{\delta_0, \delta_1, \dots, \delta_{i-1}, \pi_i, \delta_{i+1}, \dots, \delta_{q^n}\} = \Delta_i^*$ is an $(n - 1)$ -spread of τ_i .

Finally, let q be even and let $\mathcal{O} = \{\pi_0, \pi_1, \dots, \pi_{q^n+1}\}$ be a pseudo-hyperoval in $\mathbf{PG}(3n-1, q)$. Choose π_i , with $i \in \{0, 1, \dots, q^n+1\}$, and let $\mathbf{PG}(2n-1, q) \subseteq \mathbf{PG}(3n-1, q)$ be skew to π_i . Let $\langle \pi_i, \pi_j \rangle \cap \mathbf{PG}(2n-1, q) = \eta_j$, with $j \neq i$. Then $\{\eta_0, \eta_1, \dots, \eta_{i-1}, \eta_{i+1}, \dots, \eta_{q^n+1}\} = \Delta_i$ is an $(n - 1)$ -spread of $\mathbf{PG}(2n-1, q)$.

Theorem 5.1 (Casse, Thas and Wild [2]). *Consider a pseudo-oval \mathcal{O} with q odd. Then at least one of the $(n - 1)$ -spreads $\Delta_0, \Delta_1, \dots, \Delta_{q^n}, \Delta_0^*, \Delta_1^*, \dots, \Delta_{q^n}^*$ is regular*

if and only if they all are regular if and only if the pseudo-oval \mathcal{O} is regular. In such a case \mathcal{O} is essentially a conic over \mathbb{F}_{q^n} .

Theorem 5.2 (Rottey and Van de Voorde [8]). *Consider a pseudo-oval \mathcal{O} in $\mathbf{PG}(3n-1, q)$ with $q = 2^h$, $h > 1$, n prime. Then \mathcal{O} is regular if and only if all $(n-1)$ -spreads $\Delta_0, \Delta_1, \dots, \Delta_{q^n}$ are regular.*

6. ALTERNATIVE PROOF AND IMPROVEMENTS

Theorem 6.1. *Consider a pseudo-hyperoval \mathcal{O} in $\mathbf{PG}(3n-1, q)$, $q = 2^h$, $h > 1$ and n prime. Then \mathcal{O} is regular if and only if all $(n-1)$ -spreads Δ_i , with $i = 0, 1, \dots, q^n+1$, are regular.*

Proof. If \mathcal{O} is regular, then clearly all $(n-1)$ -spreads Δ_i , with $i = 0, 1, \dots, q^n+1$, are regular.

Conversely, assume that the $(n-1)$ -spreads $\Delta_0, \Delta_1, \dots, \Delta_{q^n+1}$ are regular. Let $\mathcal{O} = \{\pi_0, \pi_1, \dots, \pi_{q^n+1}\}$ and let $\hat{\mathcal{O}} = \{\beta_0, \beta_1, \dots, \beta_{q^n+1}\}$ be the dual of \mathcal{O} , with β_i being the dual of π_i .

Choose $\beta_i, i \in \{0, 1, \dots, q^n+1\}$, and let $\beta_i \cap \beta_j = \alpha_{ij}, j \neq i$. Then

$$(1) \quad \{\alpha_{i0}, \alpha_{i1}, \dots, \alpha_{i,i-1}, \alpha_{i,i+1}, \dots, \alpha_{i,q^n+1}\} = \Gamma_i$$

is an $(n-1)$ -spread of β_i .

Now consider $\beta_i, \beta_j, \Gamma_i, \Gamma_j, \alpha_{ij}, j \neq i$. In Γ_j we next consider a $(n-1)$ -regulus γ_j containing α_{ij} . The $(n-1)$ -regulus γ_j is a set of maximal spaces of a Segre variety $\mathcal{S}_{1;n-1}$; see Section 4.5 in [4]. The $(n-1)$ -regulus γ_j and the $(n-1)$ -spread Γ_i of β_i generate a regular $(n-1)$ -spread $\Sigma(\gamma_j, \Gamma_i)$ of $\mathbf{PG}(3n-1, q)$. This can be seen as follows. The elements of Γ_i intersect n lines U_1, U_2, \dots, U_n which are conjugate in \mathbb{F}_{q^n} over \mathbb{F}_q , that is, they form an orbit of the Galois group corresponding to this extension. Let $\alpha_{ij} \cap U_l = \{u_l\}$, with $l = 1, 2, \dots, n$. Now consider the transversals T_1, T_2, \dots, T_n of the elements of γ_j , with T_l containing u_l . The n planes $T_l U_l = \theta_l$ intersect all elements of γ_j and Γ_i . The $(n-1)$ -dimensional subspaces of $\mathbf{PG}(3n-1, q)$ intersecting $\theta_1, \theta_2, \dots, \theta_n$ are the elements of the regular $(n-1)$ -spread $\Sigma(\gamma_j, \Gamma_i)$. The elements of this spread correspond to the points of a plane $\mathbf{PG}(2, q^n)$, with its lines corresponding to the $(2n-1)$ -dimensional spaces containing at least two (and then q^n+1) elements of the spread. Hence the $q+2$ elements of $\hat{\mathcal{O}}$ containing an element of γ_j , say $\beta_i = \beta_{i_1}, \beta_{i_2}, \dots, \beta_{i_{q+1}}, \beta_{i_{q+2}} = \beta_j$, correspond to lines of $\mathbf{PG}(2, q^n)$. Dualizing, the elements $\pi_{i_1}, \pi_{i_2}, \dots, \pi_{i_{q+2}}$ correspond to points of $\mathbf{PG}(2, q^n)$.

Now consider β_{i_2} and γ_j , and repeat the argument. Then there arise n planes θ'_l intersecting all elements of γ_j and Γ_{i_2} . The $(n-1)$ -dimensional subspaces of $\mathbf{PG}(3n-1, q)$ intersecting $\theta'_1, \theta'_2, \dots, \theta'_n$ are the elements of the regular $(n-1)$ -spread $\Sigma(\gamma_j, \Gamma_{i_2})$. The elements of this spread correspond to the points of a plane $\mathbf{PG}'(2, q^n)$, and the lines of this plane correspond to the $(2n-1)$ -dimensional spaces containing q^n+1 elements of the spread. Hence $\beta_{i_1}, \beta_{i_2}, \dots, \beta_{i_{q+2}}$ correspond to lines of $\mathbf{PG}'(2, q^n)$. Dualizing, the elements $\pi_{i_1}, \pi_{i_2}, \dots, \pi_{i_{q+2}}$ correspond to points of $\mathbf{PG}'(2, q^n)$.

First, assume that $\{\theta_1, \theta_2, \dots, \theta_n\} \cap \{\theta'_1, \theta'_2, \dots, \theta'_n\} = \emptyset$. Consider $\pi_{i_1}, \pi_{i_2}, \pi_{i_3}, \pi_{i_4}$. The planes of $\mathbf{PG}(3n-1, q^n)$ intersecting these four spaces constitute a set \mathcal{M} of

maximal spaces of a Segre variety $\mathcal{S}_{2;n-1}$ [1]. The planes $\theta_1, \theta_2, \dots, \theta_n, \theta'_1, \theta'_2, \dots, \theta'_n$ are elements of \mathcal{M} . It follows that $(\theta_1 \cup \theta_2 \cup \dots \cup \theta_n) \cap (\theta'_1 \cup \theta'_2 \cup \dots \cup \theta'_n) = \emptyset$.

Consider any $(n-1)$ -dimensional subspace $\pi \in \{\pi_{i_5}, \pi_{i_6}, \dots, \pi_{i_{q+2}}\}$ of $\mathbf{PG}(3n-1, q)$. We will show that π is a maximal subspace of $\mathcal{S}_{2;n-1}$. Let $\theta_i \cap \pi_j = \{t_{ij}\}, \theta'_i \cap \pi_j = \{t'_{ij}\}, i = 1, 2, \dots, n, j = i_1, i_2, \dots, i_{q+2}$. If $t_{ij_1} t_{ij_2} \cap t_{ij_3} t_{ij_4} = \{v_i\}, t'_{ij_1} t'_{ij_2} \cap t'_{ij_3} t'_{ij_4} = \{v'_i\}$, with j_1, j_2, j_3, j_4 distinct, then v_1, v_2, \dots, v_n are conjugate and similarly v'_1, v'_2, \dots, v'_n are conjugate. Hence $\langle v_1, v_2, \dots, v_n \rangle = \langle v'_1, v'_2, \dots, v'_n \rangle$ defines a $(n-1)$ -dimensional space over \mathbb{F}_q which intersects $\theta_1, \theta_2, \dots, \theta'_n$ (over \mathbb{F}_{q^n}). The points t_{ij} , with $j = i_1, i_2, \dots, i_{q+2}$, generate a subplane of θ_i , and the points t'_{ij} , with $j = i_1, i_2, \dots, i_{q+2}$, generate a subplane of θ'_i , with $i = 1, 2, \dots, n$. Let $q = 2^h$ and let \mathbb{F}_{2^v} be the subfield of $\mathbb{F}_{q^n} = \mathbb{F}_{2^{hn}}$ over which these subplanes are defined; so $v|hn$. Then $v < hn$ as otherwise the spreads of $\mathbf{PG}(3n-1, q)$ defined by $\theta_1, \theta_2, \dots, \theta_n$ and $\theta'_1, \theta'_2, \dots, \theta'_n$ coincide, clearly not possible. The $(n-1)$ -regulus γ_j implies that the subplanes contain a line over \mathbb{F}_q , so $h|v$. As n is prime we have $v = h$, so $2^v = q$. Hence the $2n$ subplanes are defined over \mathbb{F}_q . It follows that the $q+2$ elements $\pi_{i_1}, \pi_{i_2}, \dots, \pi_{i_{q+2}}$ are maximal subspaces of the Segre variety $\mathcal{S}_{2;n-1}$. Hence π is a maximal subspace of $\mathcal{S}_{2;n-1}$. It follows that $\pi_1, \pi_2, \dots, \pi_{q+2}$ are maximal subspaces of $\mathcal{S}_{2;n-1}$.

Now consider a $\mathbf{PG}(2, q)$ which intersects $\pi_{i_1}, \pi_{i_2}, \pi_{i_3}, \pi_{i_4}$. The $(n-1)$ -dimensional spaces $\pi_{i_1}, \pi_{i_2}, \dots, \pi_{i_{q+2}}$ are maximal spaces of $\mathcal{S}_{2;n-1}$ which intersect $\mathbf{PG}(2, q)$; they are maximal spaces of the Segre variety $\mathcal{S}_{2;n-1} \cap \mathbf{PG}(3n-1, q)$ of $\mathbf{PG}(3n-1, q)$.

Consider π_{i_1} and also a $\mathbf{PG}(2n-1, q)$ skew to π_{i_1} . If we project $\pi_{i_2}, \pi_{i_3}, \dots, \pi_{i_{q+2}}$ from π_{i_1} onto $\mathbf{PG}(2n-1, q)$, then by the foregoing paragraph the $q+1$ projections constitute a $(n-1)$ -regulus of $\mathbf{PG}(2n-1, q)$. Similarly, if we project from π_{i_s} , s any element of $\{1, 2, \dots, q+2\}$. Equivalently, if $s \in \{1, 2, \dots, q+2\}$ then the spaces $\beta_{i_s} \cap \beta_{i_t}$, with $t = 1, 2, \dots, s-1, s+1, \dots, q+2$, form a $(n-1)$ -regulus of β_{i_s} .

Now assume that the condition $\{\theta_1, \theta_2, \dots, \theta_n\} \cap \{\theta'_1, \theta'_2, \dots, \theta'_n\} = \emptyset$ is satisfied for any choice of $\beta_i, \beta_j, \gamma_j, \beta_{i_2}$. In such a case every $(n-1)$ -regulus contained in a spread Γ_s defines a Segre variety $\mathcal{S}_{2;n-1}$ over \mathbb{F}_q . Let us define the following design \mathcal{D} . Points of \mathcal{D} are the elements of $\hat{\mathcal{O}}$, a block of \mathcal{D} is a set of $q+2$ elements of $\hat{\mathcal{O}}$, containing at least one space of a $(n-1)$ -regulus contained in some regular spread Γ_s , and incidence is containment. Then \mathcal{D} is a $4 - (q^n + 2, q+2, 1)$ design. By Kantor [5] this implies that $q = 2$, a contradiction.

Consequently, we may assume that for at least one quadruple $\beta_i, \beta_j, \gamma_j, \beta_{i_2}$ we have

$$(2) \quad \{\theta_1, \theta_2, \dots, \theta_n\} = \{\theta'_1, \theta'_2, \dots, \theta'_n\}.$$

In such a case the $q^n + 2$ elements of $\hat{\mathcal{O}}$ correspond to lines of the plane $\mathbf{PG}(2, q^n)$. It follows that \mathcal{O} is regular. \blacksquare

Theorem 6.2. *Consider a pseudo-oval \mathcal{O} in $\mathbf{PG}(3n-1, q)$, with $q = 2^h, h > 1$ and n prime. Then \mathcal{O} is regular if and only if all $(n-1)$ -spreads $\Delta_0, \Delta_1, \dots, \Delta_{q^n}$ are regular.*

Proof. If \mathcal{O} is regular, then clearly all $(n-1)$ -spreads $\Delta_0, \Delta_1, \dots, \Delta_{q^n}$ are regular. Conversely, assume that the $(n-1)$ -spreads $\Delta_0, \Delta_1, \dots, \Delta_{q^n}$ are regular. Let $\mathcal{O} = \{\pi_0, \pi_1, \dots, \pi_{q^n}\}$, let π_{q^n+1} be the nucleus of \mathcal{O} , let $\bar{\mathcal{O}} = \mathcal{O} \cup \{\pi_{q^n+1}\}$, let $\hat{\mathcal{O}}$ be the dual of \mathcal{O} , let $\hat{\bar{\mathcal{O}}}$ be the dual of $\bar{\mathcal{O}}$, and let β_i be the dual of π_i .

Choose $\beta_i, i \in \{0, 1, \dots, q^n + 1\}$, and let $\beta_i \cap \beta_j = \alpha_{ij}, j \neq i$. Then

$$(3) \quad \{\alpha_{i0}, \alpha_{i1}, \dots, \alpha_{i,i-1}, \alpha_{i,i+1}, \dots, \alpha_{i,q^n+1}\} = \Gamma_i$$

is an $(n-1)$ -spread of β_i .

Now consider $\beta_i, \beta_j, \Gamma_i, \Gamma_j, \alpha_{ij}$, with $j \neq i$ and $i, j \in \{0, 1, \dots, q^n\}$. In Γ_j we next consider a $(n-1)$ -regulus γ_j containing α_{ij} and α_{j,q^n+1} . The $(n-1)$ -regulus γ_j is a set of maximal spaces of a Segre variety $\mathcal{S}_{1;n-1}$. The $(n-1)$ -regulus γ_j and the $(n-1)$ -spread Γ_i of β_i generate a regular $(n-1)$ -spread $\Sigma(\gamma_j, \Gamma_i)$ of $\mathbf{PG}(3n-1, q)$. Such as in the proof of Theorem 6.1 we introduce the elements $U_l, u_l, T_l, \theta_l, l = 1, 2, \dots, n$, and the plane $\mathbf{PG}(2, q^n)$. The $q+2$ elements of $\hat{\mathcal{O}}$ containing an element of γ_j , say $\beta_i = \beta_{i_1}, \beta_{i_2}, \dots, \beta_{i_q}, \beta_j = \beta_{i_{q+1}}, \beta_{q^n+1}$, correspond to lines of $\mathbf{PG}(2, q^n)$. Dualizing, the elements $\pi_{i_1}, \pi_{i_2}, \dots, \pi_{i_{q+1}}, \pi_{q^n+1}$ correspond to points of $\mathbf{PG}(2, q^n)$.

Now consider β_{i_2} and γ_j , and repeat the argument. Then there arise n planes θ'_l of $\mathbf{PG}(3n-1, q^n)$ intersecting all elements of γ_j and Γ_{i_2} , and a $(n-1)$ -spread $\Sigma(\gamma_j, \Gamma_{i_2})$ of $\mathbf{PG}(3n-1, q)$. The elements of this spread correspond to the points of a plane $\mathbf{PG}'(2, q^n)$. The spaces $\beta_{i_1}, \beta_{i_2}, \dots, \beta_{i_{q+1}}, \beta_{q^n+1}$ correspond to lines of $\mathbf{PG}'(2, q^n)$. Dualizing, the elements $\pi_{i_1}, \pi_{i_2}, \dots, \pi_{i_{q+1}}, \pi_{q^n+1}$ correspond to points of $\mathbf{PG}'(2, q^n)$.

First, assume that $\{\theta_1, \theta_2, \dots, \theta_n\} \cap \{\theta'_1, \theta'_2, \dots, \theta'_n\} = \emptyset$. Consider $\pi_{i_1}, \pi_{i_2}, \pi_{i_3}, \pi_{i_4}$. The planes of $\mathbf{PG}(3n-1, q^n)$ intersecting these four spaces constitute a set \mathcal{M} of maximal spaces of a Segre variety $\mathcal{S}_{2;n-1}$. The planes $\theta_1, \theta_2, \dots, \theta_n, \theta'_1, \theta'_2, \dots, \theta'_n$ are elements of \mathcal{M} . It follows that $(\theta_1 \cup \theta_2 \cup \dots \cup \theta_n) \cap (\theta'_1 \cup \theta'_2 \cup \dots \cup \theta'_n) = \emptyset$. Let $\pi \in \{\pi_{i_5}, \pi_{i_6}, \dots, \pi_{i_{q+1}}, \pi_{q^n+1}\}$. As in the proof of Theorem 6.1 one shows that π is a maximal subspace of $\mathcal{S}_{2;n-1}$. It follows that $\pi_{i_1}, \pi_{i_2}, \dots, \pi_{i_{q+1}}, \pi_{q^n+1}$ are maximal subspaces of $\mathcal{S}_{2;n-1}$.

Next consider a $\mathbf{PG}(2, q)$ which intersects $\pi_{i_1}, \pi_{i_2}, \pi_{i_3}, \pi_{i_4}$. The $(n-1)$ -dimensional spaces $\pi_{i_1}, \pi_{i_2}, \dots, \pi_{i_{q+1}}, \pi_{q^n+1}$ are maximal spaces of $\mathcal{S}_{2;n-1}$ which intersect the plane $\mathbf{PG}(2, q)$; they are maximal spaces of the Segre variety $\mathcal{S}_{2;n-1} \cap \mathbf{PG}(3n-1, q)$ of $\mathbf{PG}(3n-1, q)$. As in the proof of Theorem 6.1 it follows that the spaces $\beta_{q^n+1} \cap \beta_{i_t}$, with $t = 1, 2, \dots, q+1$, form a $(n-1)$ -regulus of β_{q^n+1} .

Now assume that the condition $\{\theta_1, \theta_2, \dots, \theta_n\} \cap \{\theta'_1, \theta'_2, \dots, \theta'_n\} = \emptyset$ is satisfied for any choice of $\beta_i, \beta_j, \gamma_j, \beta_{i_2}$, $j \neq i$ and $i, j \in \{0, 1, \dots, q^n\}$. Let $\alpha_1, \alpha_2, \alpha_3$ be distinct elements of Γ_{q^n+1} . Then $\beta_i, \beta_j, \gamma_j, \beta_{i_2}$ can be chosen in such a way that $\alpha_1 \in \beta_i, \alpha_2 \in \beta_j, \alpha_3 \in \gamma_j, \beta_{i_2} \cap \beta_j \in \gamma_j$ with $\alpha_3 \in \beta_{i_2}$. Hence the $(n-1)$ -regulus in β_{q^n+1} defined by $\alpha_1, \alpha_2, \alpha_3$ is subset of Γ_{q^n+1} . From [4], Theorem 4.123, now follows that the $(n-1)$ -spread Γ_{q^n+1} of β_{q^n+1} is regular. By Theorem 6.1 the pseudo-hyperoval $\bar{\mathcal{O}}$ is regular, and so \mathcal{O} is regular. But in such a case the condition $\{\theta_1, \theta_2, \dots, \theta_n\} \cap \{\theta'_1, \theta'_2, \dots, \theta'_n\} = \emptyset$ is never satisfied, a contradiction.

Consequently, we may assume that for at least one quadruple $\beta_i, \beta_j, \gamma_j, \beta_{i_2}$ we have $\{\theta_1, \theta_2, \dots, \theta_n\} = \{\theta'_1, \theta'_2, \dots, \theta'_n\}$. In such a case the $q^n + 2$ elements of $\hat{\mathcal{O}}$ correspond to lines of the plane $\mathbf{PG}(2, q^n)$. It follows that $\bar{\mathcal{O}}$, and hence also \mathcal{O} , is regular. \blacksquare

Theorem 6.3. Consider a pseudo-hyperoval \mathcal{O} in $\mathbf{PG}(3n-1, q)$, $q = 2^h, h > 1$ and n prime. Then \mathcal{O} is regular if and only if at least $q^n - 1$ elements of $\{\Delta_0, \Delta_1, \dots, \Delta_{q^n+1}\}$ are regular.

Proof. If \mathcal{O} is regular, then clearly all $(n-1)$ -spreads Δ_i , with $i = 0, 1, \dots, q^n + 1$, are regular.

Conversely, assume that ρ , with $\rho \geq q^n - 1$, elements of $\{\Delta_0, \Delta_1, \dots, \Delta_{q^n+1}\}$ are regular.

If $\rho = q^n + 2$, then \mathcal{O} is regular by Theorem 6.1; if $\rho = q^n + 1$, then \mathcal{O} is regular by Theorem 6.2.

Now assume that $\rho = q^n$ and that $\Delta_2, \Delta_3, \dots, \Delta_{q^n+1}$ are regular. We have to prove that Δ_0 is regular. We use the arguments in the proof of Theorem 6.2. If one of the elements $\alpha_1, \alpha_2, \alpha_3$, say α_1 , in the proof of Theorem 6.2 is $\beta_0 \cap \beta_1$, then let γ_j contain $\beta_j \cap \beta_i, \beta_j \cap \beta_0, \beta_j \cap \beta_1$ and let $\beta_{i_2} \neq \beta_1$, with $i, j \in \{2, 3, \dots, q^n + 1\}$. Now see the proof of the preceding theorem.

Finally, assume that $\rho = q^n - 1$ and that $\Delta_3, \Delta_4, \dots, \Delta_{q^n+1}$ are regular. We have to prove that Δ_0 is regular. We use the arguments in the proof of Theorem 6.2. If exactly one of the elements $\alpha_1, \alpha_2, \alpha_3$, say α_1 , in the proof of Theorem 6.2 is $\beta_0 \cap \beta_1$ or $\beta_0 \cap \beta_2$, then proceed as in the preceding paragraph with $\beta_{i_2} \neq \beta_1, \beta_2$. Now assume that two of the elements $\alpha_1, \alpha_2, \alpha_3$, say α_1 and α_2 , are $\beta_0 \cap \beta_1$ and $\beta_0 \cap \beta_2$. Now consider all $(n-1)$ -reguli in Δ_0 containing α_1 and α_3 , and assume, by way of contradiction, that no one of these $(n-1)$ -reguli contains α_2 . The number of these $(n-1)$ -reguli is $\frac{q^n-2}{q-1}$, and so $q = 2$, a contradiction. It follows that the $(n-1)$ -regulus in β_0 defined by $\alpha_1, \alpha_2, \alpha_3$ is contained in Δ_0 . Now we proceed as in the proof of Theorem 6.2. ■

7. FINAL REMARKS

7.1. The cases $q = 2$ and n not prime

For $q = 2$ or n not prime other arguments have to be developed.

7.2. Improvement of Theorem 6.3

Let $\mathcal{D} = (P, B, \in)$ be an incidence structure satisfying the following conditions.

- (i) $|P| = q^n + 1$, q even, $q \neq 2$;
- (ii) the elements of B are subsets of size $q + 1$ of P and every three distinct elements of P are contained in at most one element of B ;
- (iii) Q is a subset of size δ of P such that any triple of elements in P with at most one element in Q , is contained in exactly one element of B .

Assumption : Any such \mathcal{D} is a $3 - (q^n + 1, q + 1, 1)$ design whenever $\delta \leq \delta_0$ with $\delta_0 \leq q - 2$.

Theorem 7.1. Consider a pseudo-hyperoval \mathcal{O} in $\text{PG}(3n - 1, q)$, $q = 2^h, h > 1$ and n prime. Then \mathcal{O} is regular if and only if at least $q^n + 1 - \delta_0$ elements of $\{\Delta_0, \Delta_1, \dots, \Delta_{q^n+1}\}$ are regular.

Proof. Similar to the proof of Theorem 6.3. ■

7.3. Acknowledgement

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