

New Strongly Regular Graphs from Finite Geometries via Switching

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Abstract

We show that the strongly regular graph on non-isotropic points of one type of the polar spaces of type $U(n, 2)$, $O(n, 3)$, $O(n, 5)$, $O^+(n, 3)$, and $O^-(n, 3)$ are not determined by its parameters for $n \geq 6$. We prove this by using a variation of Godsil-McKay switching recently described by Wang, Qiu, and Hu. This also results in a new, shorter proof of a previous result of the first author which showed that the collinearity graph of a polar space is not determined by its spectrum. The same switching gives a linear algebra explanation for the construction of a large number of non-isomorphic designs.

1 Introduction

Two graphs Γ and $\bar{\Gamma}$ are cospectral if their adjacency matrices A and \bar{A} are cospectral, that is there exists an orthogonal matrix Q with $Q^T A Q = \bar{A}$. In 1982 Godsil and McKay described a possible choice for Q which has an easy combinatorial description [10] – nowadays known as Godsil–McKay switching – and proved to be very useful in constructing cospectral graphs [2, 7, 8, 9, 11, 20, 21]. Godsil–McKay switching can be described as follows. Here we denote the neighborhood of a vertex x in a graph Γ by $\Gamma(x)$.

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Theorem 1 (GM switching). *Let Γ be a graph whose vertex set is partitioned as $C_1 \cup \dots \cup C_t \cup D$. Assume that $C := C_1 \cup \dots \cup C_t$ is an equitable partition of the induced subgraph on C , and that $|\Gamma(x) \cap C_i|/|C_i| \in \{0, \frac{1}{2}, 1\}$ for all $x \in D$ and all $i \in \{1, \dots, t\}$. Construct a graph $\bar{\Gamma}$ from Γ by modifying edges between C_i and D as follows:*

$$\bar{\Gamma}(x) \cap C_i = \begin{cases} C_i \setminus \Gamma(x) & \text{if } |\Gamma(x) \cap C_i|/|C_i| = \frac{1}{2}, \\ \Gamma(x) \cap C_i & \text{otherwise,} \end{cases}$$

for $x \in D$. Then $\bar{\Gamma}$ is cospectral with Γ .

Abiad and Haemers used Godsil–McKay switching [10] in [1] to obtain graphs cospectral to the collinearity graphs of the symplectic polar space of rank at least 3 over \mathbb{F}_2 . Barwick et al. generalized this, also using Godsil–McKay switching, to quadrics of rank at least 3 over \mathbb{F}_2 [4]. Using different techniques, Kantor showed in [18] that if a polar space has a spread, then one can define a new (not necessarily non-isomorphic) strongly regular graph with the parameters as a finite classical polar space \mathbb{P} over \mathbb{F}_q if \mathbb{P} possesses a spread of maximals. For instance this implies nowadays that the strongly regular collinearity graphs of symplectic polar spaces are not uniquely defined by their spectrum for sufficiently large q . In [15] the first author generalized all of the results obtained by switching to all finite classical polar spaces of rank at least 3 over \mathbb{F}_q by using purely geometrical arguments.

This note provides an algebraic explanation for the construction given in [15] by providing a specific family of orthogonal matrices Q . This gives rise to the following new switching which was recently discovered – in a slightly more general form – by Wang, Qiu, and Hu [25]. We discovered the same switching independently while looking for a linear algebra argument for the results by the first author in [15].

Theorem 2 (WQH switching). *Let Γ be a graph whose vertex set is partitioned as $C_1 \cup C_2 \cup D$. Assume that $|C_1| = |C_2|$ and that the induced subgraphs on C_1 , C_2 , and $C_1 \cup C_2$ are regular, where the degrees in the induced subgraphs on C_1 and C_2 are the same. Suppose that all $x \in D$ satisfy one of the following:*

1. $|\Gamma(x) \cap C_1| = |\Gamma(x) \cap C_2|$, or
2. $\Gamma(x) \cap (C_1 \cup C_2) \in \{C_1, C_2\}$.

Construct a graph $\bar{\Gamma}$ from Γ by modifying the edges between $C_1 \cup C_2$ and D as follows:

$$\bar{\Gamma}(x) \cap (C_1 \cup C_2) = \begin{cases} C_1 & \text{if } \Gamma(x) \cap (C_1 \cup C_2) = C_2, \\ C_2 & \text{if } \Gamma(x) \cap (C_1 \cup C_2) = C_1, \\ \Gamma(x) \cap (C_1 \cup C_2) & \text{otherwise,} \end{cases}$$

for $x \in D$. Then $\bar{\Gamma}$ is cospectral with Γ .

Notice that we simplified the statement from [25] where the stated conditions are also necessary. In our case, the conditions are just sufficient. One can show that if $t = 1$ and $|C_1| = 4$ in Theorem 1, then it is equivalent to $|C_1| = |C_2| = 2$ in Theorem 2.

In the later sections we show that WQH switching indeed yields strongly regular graphs with the same parameters as the collinearity graphs of polar spaces, thus reproducing the results from [15] with a much shorter proof. We start by applying WQH switching to graphs coming from designs to reproduce a classical construction for non-isomorphic 2-designs due to Jungnickel and Tonchev [16, 17] (see also Shrikhande [23]), using the linear algebra method. Furthermore, we use the switching to show that all the strongly regular collinearity graphs coming from one type of non-isotropic points in the ambient space of the polar spaces $U(n, 2)$, $O(n, 3)$, $O(n, 5)$, $O^+(n, 3)$, and $O^-(n, 3)$ are not uniquely determined by their parameters for $n \geq 6$. This extends work by Hui and Rodrigues who showed that the strongly regular collinearity graph on non-isotropic points of $O^+(n, 2)$ and $O^-(n, 2)$ are not uniquely determined by their parameters for $n \geq 6$ [14].

Definition 3. If we can apply Theorem 2 to a graph Γ for a pair $\{C_1, C_2\}$, then we call $\{C_1, C_2\}$ a *switching set* of Γ . We refer to $\bar{\Gamma}$ (as in Theorem 2) as the graph *obtained by switching*.

2 Block Designs and Grassmann Graphs

A 2 - (v, b, λ) design $(\mathcal{P}, \mathcal{B})$ consists of a family \mathcal{B} (blocks) of b -sets of an v -element set \mathcal{P} (points) such that every pair of points lies in exactly λ blocks. All k -spaces of an n -dimensional vector space over \mathbb{F}_q^n form a 2-design. Jungnickel [16] constructed many non-isomorphic 2-designs with the same parameters. We reproduce a special case of his construction by applying

WQH switching. For $\lambda = 1$, the *block graph* of $(\mathcal{P}, \mathcal{B})$ is the graph with \mathcal{B} as vertices where two vertices are adjacent if the blocks intersect.

Let $(\mathcal{P}, \mathcal{B})$ be a $2-(v, b, 1)$ design with a $2-(v', b, 1)$ subdesign $(\mathcal{P}', \mathcal{B}')$. Let $p_1, p_2 \in \mathcal{P}'$. Let C_1 be the set of all blocks of \mathcal{B}' containing p_1 and not p_2 , and let C_2 be the set of all blocks of \mathcal{B}' containing p_2 and not p_1 .

Theorem 4. *The pair $\{C_1, C_2\}$ is a switching set of the block graph Γ of $(\mathcal{P}, \mathcal{B})$.*

Proof. The points p_1 and p_2 lie in exactly one block together in \mathcal{B}' , so $|C_1| = \frac{v'-1}{b-1} - 1 = |C_2|$.

First we show that $C_1 \cup C_2$ is an equitable partition. Each block in C_i is adjacent to all blocks in C_i . Each block in C_1 is adjacent to $b - 1$ blocks in C_2 . As $|C_1| = |C_2|$, this suffices.

We have to verify that every block B not in $C_1 \cup C_2$ satisfies $|\Gamma(B) \cap C_1| = |\Gamma(B) \cap C_2|$ or $\Gamma(B) \cap (C_1 \cup C_2) \in \{C_1, C_2\}$. The block B through p_1 and p_2 clearly has $\Gamma(B) \cap (C_1 \cup C_2) = C_1 \cup C_2$. A block $B \in \mathcal{B}'$, which does not contain p_1 or p_2 , meets b elements of C_i for $i \in \{1, 2\}$ as $\lambda = 1$. A block in $\mathcal{B} \setminus \mathcal{B}'$ contains 0 or 1 point of \mathcal{P}' as $\lambda = 1$. A block $B \in \mathcal{B} \setminus \mathcal{B}'$ which contains exactly one p_i for $i \in \{1, 2\}$, satisfies $\Gamma(B) \cap (C_1 \cup C_2) = C_i$. A block $B \in \mathcal{B}$ which does not contain p_1 or p_2 meets 0 elements of C_i if $B \cap \mathcal{P}' = \emptyset$ and 1 element of C_i if $|B \cap \mathcal{P}'| = 1$ for all $i \in \{1, 2\}$. \square

The graph $\bar{\Gamma}$ obtained by switching with respect to the switching set given in Theorem 4 is also a block graph. Indeed, it is the block graph of the modified design obtained from $(\mathcal{P}, \mathcal{B})$ by replacing C_i by

$$\{L \cup \{p_{i'}\} \setminus \{p_i\} \mid L \in C_i\},$$

where $\{i, i'\} = \{1, 2\}$. This is a special case of the construction given in [16].

As an example let \mathcal{P} be the set of all 1-spaces of \mathbb{F}_q^n and let \mathcal{B} be the set of all 2-spaces of \mathbb{F}_q^n . For $(\mathcal{P}', \mathcal{B}')$ we choose an s -space S of \mathbb{F}_q^n with $2 < s < n$.

Lemma 5. *The graphs obtained from the block graph Γ of the above design $(\mathcal{P}, \mathcal{B})$ by switching are non-isomorphic to Γ .*

Proof. Let ℓ be the 2-space spanned by p_1 and p_2 . Let T be a 3-space which meets S in a line $\ell' \neq \ell$ with $p_1 \in \ell'$. After switching, all blocks in $\{B \in \mathcal{B} \setminus \mathcal{B}' : p_1 \in B\}$ form a maximal clique of size $q^2 + q$. It is well-known that all maximal cliques of Γ have size $q^2 + q + 1$ (cf. [22, Proposition 2.20]). \square

There are many similar constructions for designs in the literature. For instance, we can replace the 2-space by a k -space with $1 < k < n$. Our emphasis here is that we provide an explanation for this construction by considering the block graph and using the linear algebra method.

3 Polar Spaces

We apply Theorem 2 to several strongly regular graphs related to finite classical polar spaces, so that we obtain new, non-isomorphic strongly regular graphs with the same parameters. We refer the reader to [5, Section 9.4], [12], [15] and [24, Chapter 8] for more detailed descriptions of finite classical polar spaces. In the following we give a brief overview. We use projective notation, so we denote 1-spaces of \mathbb{F}_q^n as *points*, 2-spaces as *lines*, and $(n-1)$ -spaces as *hyperplanes*. There are six finite classical polar spaces: $O^+(2d, q)$, $U(2d, \sqrt{q})$, $U(2d+1, \sqrt{q})$, $Sp(2d, q)$, $O(2d+1, q)$, $O^-(2d+2, q)$. All polar spaces can be defined by a polarity \perp of \mathbb{F}_q^n , except for q even in the case of the quadrics $O^+(2d, q)$, $O(2d+1, q)$ and $O^-(2d+2, q)$. As we are not showing any results for quadrics in even characteristic, we assume in the following that either q is odd or that the polar space is not a quadric. A polarity is an incidence preserving bijection between k -spaces and $(n-k)$ -spaces of order two. Call a subspace S *isotropic* if $S \subseteq S^\perp$. A polar space over \mathbb{F}_q^n consists of all isotropic subspaces. We call a subspace T of \mathbb{F}_q^n *degenerate* if $T^\perp \cap T$ is non-empty. If T is degenerate, then $T^\perp / (T \cap T^\perp)$ is a non-degenerate finite classical polar space.

Recall the following property of polar spaces: Let $\{x, y, z\}$ be non-isotropic points of the ambient space of a finite classical polar space such that $x, y \in z^\perp$ and $x \in y^\perp$. Then $\langle x, y, z \rangle$ is isomorphic to $U(3, \sqrt{q})$ or $O(3, q)$.

Throughout the following let \mathbb{P} denote a polar space of rank $d \geq 3$ over a finite field of order q with an associated polarity \perp such as each isotropic $(d-1)$ -space lies in $q^e + 1$ maximal isotropic subspaces.

4 Collinearity Graphs of Polar Spaces

In this section we recover a special case of the result of [15] using Theorem 2. The result in [15] was the original motivation for the research presented here. The collinearity graph Γ of \mathbb{P} has the isotropic points of \mathbb{P} as vertices and

two points are adjacent if they are collinear. The graph Γ is well-known to be strongly regular. Let P be an isotropic m -space, where $2 \leq m \leq d$. Let L_1 and L_2 be hyperplanes of P with $L_1 \neq L_2$. Set

$$C_1 = L_1 \setminus L_2, \quad C_2 = L_2 \setminus L_1.$$

Theorem 6. *The pair $\{C_1, C_2\}$ is a switching set of Γ . If $d = m \geq 3$, then the graph $\bar{\Gamma}$ obtained by switching is non-isomorphic to Γ .*

Proof. First we show that $\{C_1, C_2\}$ is a switching set. Clearly, the induced subgraph on $C_1 \cup C_2$ is a complete graph, hence we only have to investigate the size of $x^\perp \cap C_1$ and $x^\perp \cap C_2$ for $x \notin C_1 \cup C_2$. If $x \in P^\perp$, then $x^\perp \cap (C_1 \cup C_2) = C_1 \cup C_2$. For $x \notin P^\perp$, let $L = x^\perp \cap P$. If $L = L_i$ for $i \in \{1, 2\}$, then $x^\perp \cap (C_1 \cup C_2) = C_i$. Otherwise, $\dim(x^\perp \cap L_1) = \dim(x^\perp \cap L_2)$, so $|x^\perp \cap C_1| = |x^\perp \cap C_2|$. Hence, $\{C_1, C_2\}$ is a switching set.

It is easy to see that the graph Γ_0 considered in Section 4 of [15] corresponds to Γ (and Γ_1 to $\bar{\Gamma}$). Non-isomorphy is shown there and it requires $d \geq 3$. \square

Recall that Γ is a strongly regular graph, so $\bar{\Gamma}$ is a strongly regular graph too. We do not know if all graphs constructed in [15] can be reconstructed using Theorem 2.

5 Non-isotropic Points of Unitary Polar Spaces

Let \perp be a polarity on \mathbb{F}_q^n such that there are non-isotropic points, so the associated polar space \mathbb{P} is one of $O^+(2d, q)$, $U(2d, \sqrt{q})$, $U(2d+1, \sqrt{q})$, $O(2d+1, q)$, $O^-(2d+2, q)$. In the case of the quadrics, q is odd. Let Γ be the graph with the non-isotropic points of \mathbb{P} as vertices and points x, y are adjacent if $x \in y^\perp$. For this paper we call Γ a *polarity graph*.¹ As a motivation for Section 6, we show that Γ is not determined by their spectrum for $n \geq 4$. Note that Γ is in general not strongly regular, but has several nice properties. For instance it is K_{n+2} -free and therefore for many parameters, it gives the best known constructive lower bound on certain Ramsey numbers [3]. For $q = 4$, in the case of $U(n, 2)$, the graph Γ is indeed strongly regular such that

¹This is slightly non-standard as the polarity graph usually includes isotropic points as well.

the complement $\bar{\Gamma}$ of Γ has the parameters (with $\epsilon = (-1)^n$):

$$\begin{aligned} v &= 2^{n-1}(2^n - \epsilon)/3, & k &= (2^{n-1} + \epsilon)(2^{n-2} - \epsilon), \\ \lambda &= 3 \cdot 2^{2n-5} - \epsilon 2^{n-2} - 2, & \mu &= 3 \cdot 2^{n-3}(2^{n-2} - \epsilon). \end{aligned}$$

One can see as follows that Γ is strongly regular (as we are not aware of a reference): In [13, C14] a strongly regular graph Γ' on the non-isotropic points of $U(n, 2)$ is defined, where two points are adjacent if they span a degenerate subspace. As a non-degenerate lines isomorphic to $U(2, 2)$ contains exactly two non-isotropic points, which are pairwise orthogonal, $\Gamma' = \bar{\Gamma}$.

Let $3 \leq m \leq d$. Let L_1 and L_2 be non-isotropic subspaces of dimension $m-1$ such that $L_1 \cap L_2$ is the radical of L_1 and L_2 and has dimension $m-2$, that is $L_1 \cap L_2 = (L_1 \cap L_1^\perp) \cap (L_2 \cap L_2^\perp)$. Set $P = \langle L_1, L_2 \rangle$, $p = L_1 \cap L_2$, $C_1 = L_1 \setminus L_2$, and $C_2 = L_2 \setminus L_1$.

Theorem 7. *The pair $\{C_1, C_2\}$ is a switching set of Γ .*

Proof. Let x be a non-isotropic point in L_i for some $i \in \{1, 2\}$. As $x \notin x^\perp$, x^\perp meets P in a hyperplane. Clearly, $p \subseteq x^\perp$ by construction. As $x \in L_i$ is non-isotropic, $x^\perp \cap L_i$ contains no non-isotropic points. For $\{i, j\} = \{1, 2\}$, $x^\perp \cap L_j$ contains all or no non-isotropic points of L_j , depending on whether $L_j = L_i^\perp \cap P$ or not. Thus, the induced subgraph on $C_1 \cup C_2$ is a complete bipartite graph or an empty graph. Hence, the regularity condition of Theorem 2 is satisfied.

Next we have to consider $x \notin C_1 \cup C_2$. For x a non-isotropic point not in $L_1 \cup L_2$, we have that either $P \subseteq x^\perp$ or $x^\perp \cap P$ is a hyperplane of P . If $P \subseteq x^\perp$, then clearly $|x^\perp \cap C_1| = |x^\perp \cap C_2|$. If $P \cap x^\perp$ is a hyperplane of P and $p \not\subseteq P \cap x^\perp$, then $|x^\perp \cap L_1| = |x^\perp \cap L_2|$. If $P \cap x^\perp$ is a hyperplane of P and $p \subseteq P \cap x^\perp$, then $P \cap x^\perp$ is in $\{L_1, L_2\}$, so $x^\perp \cap (C_1 \cup C_2) \in \{C_1, C_2\}$, or $x^\perp \cap L_1 = p = x^\perp \cap L_2$, so $x^\perp \cap (C_1 \cup C_2)$ is empty.

Hence, $\{C_1, C_2\}$ is a switching set. □

We show non-isomorphy for $U(n, \sqrt{q})$ for $m = 3$. We believe that the same is true for $m > 3$, but do not intend to show it. The argument for $O^+(n, q)$, $O^-(n, q)$, and $O(n, q)$ is analogous, but slightly more tedious as the perp of $U(3, \sqrt{q})$ in $U(n, \sqrt{q})$ is always isomorphic to $U(n-3, \sqrt{q})$, while the perp of $O(3, q)$ is either $O^+(n-3, q)$, $O^-(n-3, q)$, or $O(n-3, q)$.

Lemma 8. *For a clique $\{x, y, z\}$ of Γ the value $|\Gamma(x) \cap \Gamma(y) \cap \Gamma(z)|$ is independent of our choice of $\{x, y, z\}$.*

Proof. Since $\Gamma(x) \cap \Gamma(y) \cap \Gamma(z)$ is contained in $\langle x, y, z \rangle^\perp \cong U(n-3, \sqrt{q})$, $|\Gamma(x) \cap \Gamma(y) \cap \Gamma(z)|$ is the number of non-isotropic points in $U(n-3, \sqrt{q})$ which is independent of our choice of $\{x, y, z\}$. \square

Theorem 9. *Suppose that $m = 3$ and P/p isomorphic to $U(2, \sqrt{q})$. If $n \geq 6$, then the graph $\bar{\Gamma}$ by switching is non-isomorphic to Γ .*

Proof. Let x, y, z be pairwise adjacent vertices not in P with $L_x = x^\perp \cap P$, $L_y = y^\perp \cap P$, $L_z = z^\perp \cap P$ such that $L_x = L_1$, L_x, L_y and L_z are non-isotropic lines in P with $L_y \neq L_z$ and $L_y \cap L_x = L_z \cap L_x \neq p$.

Claim. There exist x, y , and z as described.

Assume for now that the claim is true. Then $\Gamma(x) \cap \Gamma(y) \cap \Gamma(z)$ contains the unique non-isotropic point of $L_1 \cap L_y \cap L_z$, while $L_2 \cap L_y \cap L_z$ contains no non-isotropic points, so

$$|\bar{\Gamma}(x) \cap \bar{\Gamma}(y) \cap \bar{\Gamma}(z)| = |\Gamma(x) \cap \Gamma(y) \cap \Gamma(z)| - 1$$

as no other adjacencies change. By Corollary 8, $\bar{\Gamma}$ is non-isomorphic to Γ .

We still have to show the claim.

Take a $U(6, \sqrt{q})$ space containing P with the decomposition

$$\langle p_0, q_0 \rangle \perp \langle p_1, q_1 \rangle \perp \langle p_2, q_2 \rangle,$$

where the p_i are isotropic points and $\langle p_i, q_i \rangle$ are non-degenerate lines. Let $\{r_i, s_i\}$ be an orthogonal basis of $\langle p_i, q_i \rangle$, that is in particular $s_i \in r_i^\perp$. Without loss of generality $P = \langle p_0, r_1, r_2 \rangle$, where $p = p_0$ and $L_1 = \langle p_0, r_1 \rangle$. Set

$$x = s_1 + p_2, \quad y = r_0, \quad z = s_0 + p_2.$$

We now have the desired properties: The line $\langle r_1, r_2 \rangle$ is isomorphic to $U(2, \sqrt{q})$ and p is orthogonal to $\langle r_1, r_2 \rangle$, so P/p is isomorphic to $U(2, \sqrt{q})$. The points x, y , and z are pairwise orthogonal, so $\{x, y, z\}$ is a clique. We have that $x^\perp \cap P = \langle p_0, r_1 \rangle$. Furthermore, $L_y = y^\perp \cap P = \langle r_1, r_2 \rangle$, but $r_2 \notin z^\perp \cap P = L_z$, so $L_y \neq L_z$. Additionally, $L_y \cap L_x = \langle r_1 \rangle = L_z \cap L_x$, so in particular p is not in L_y or L_z . \square

We conclude that there are graphs cospectral, but non-isomorphic, to Γ for $n \geq 6$. In particular, for $q = 4$ we obtain new strongly regular graphs with parameters as above.

6 Non-isotropic Points of Quadrics

Let \mathbb{P} be a polar space of type $O(2d+1, q)$, $O^+(2d, q)$, or $O^-(2d, q)$, where $q \geq 3$ odd and $d \geq 2$, induced by some quadratic form σ on \mathbb{F}_q^n . Here $n = 2d+1$ if $\mathbb{P} \cong O(2d+1, q)$ and $n = 2d$ otherwise. For a non-isotropic point p we have that $\sigma(p)$ is a square or a non-square of \mathbb{F}_q . For $O(2d+1, q)$, we may choose the quadratic form σ in such a way that this is equivalent to p^\perp isomorphic to $O^+(2d, q)$, respectively, $O^-(2d, q)$. Set $V_+ = \{p : \sigma(p) \text{ square}\}$ and $V_- = \{p : \sigma(p) \text{ non-square}\}$. Let Γ_ϵ be the graph with the points in V_ϵ as vertices where two points x, y are adjacent if $x \in y^\perp$. It is well known that this is a strongly regular graph with parameters

$$\begin{aligned} v &= 3^d(3^d + \epsilon)/2, & k &= 3^{d-1}(3^d - \epsilon)/2, \\ \lambda &= \mu = 3^{d-1}(3^{d-1} - \epsilon)/2, \end{aligned}$$

if $\mathbb{P} \cong O(2d+1, 3)$ [13, §8.10], with parameters

$$\begin{aligned} v &= 3^{d-1}(3^d - \zeta)/2, & k &= 3^{d-1}(3^{d-1} - \zeta)/2, \\ \lambda &= 3^{d-2}(3^{d-1} + \zeta)/2, & \mu &= 3^{d-1}(3^{d-2} - \zeta)/2, \end{aligned}$$

if $\mathbb{P} \cong O^\zeta(2d, 3)$ [13, p. 377],² and with parameters

$$\begin{aligned} v &= 5^d(5^d + \epsilon)/2, & k &= 5^{d-1}(5^d - \epsilon)/2, \\ \lambda &= 5^{d-1}(5^{d-1} + \epsilon)/2, & \mu &= 5^{d-1}(5^{d-1} - \epsilon)/2, \end{aligned}$$

if $\mathbb{P} \cong O(2d+1, 5)$ [6, §7.D]. To our knowledge it is not known if these strongly regular graphs are determined by their parameters except for a finite number of small cases.

Let $3 \leq m \leq d$. Following the construction of the previous section, let L_1 and L_2 be non-isotropic subspaces of dimension $m-1$ such that $L_1 \cap L_2$ is the radical of L_1 and L_2 and has dimension $m-2$, that is $L_1 \cap L_2 = (L_1 \cap L_1^\perp) \cap (L_2 \cap L_2^\perp)$. Set $P = \langle L_1, L_2 \rangle$, $p = L_1 \cap L_2$, $C_1 = L_1 \setminus L_2$, and $C_2 = L_2 \setminus L_1$. Furthermore, suppose that the points of C_1 and C_2 are of the same type, so both are in V_ϵ . Such L_1 and L_2 exist unless $P/p = O^+(2, 3)$.

The arguments in this section are almost identical to the ones for non-isotropic points in $U(n, \sqrt{q})$, so we only elaborate on the parts of the argument which are slightly different.

²Notice that μ is given incorrectly in [13]. One can find the correct parameters on <https://www.win.tue.nl/~aeb/graphs/srghub.html>.

Theorem 10. *Let $\delta = -$ if $q \equiv 3 \pmod{4}$ and $\delta = +$ if $q \equiv 1 \pmod{4}$. The pair $\{C_1, C_2\}$ is a switching set of Γ_ϵ . Suppose that $m = 3$ and P/p is isomorphic to $O^\delta(2, q)$. If $n \geq 7$, then the graph $\bar{\Gamma}_\epsilon$ obtained by switching is non-isomorphic to Γ_ϵ .*

Proof. Recall that all non-isotropic points of L_i are in V_ϵ . Then the argument for $\{C_1, C_2\}$ being a switching set is identical to the proof of Theorem 7.

Write $\Gamma = \Gamma_\epsilon$ for brevity. In Γ the number $|\Gamma(x) \cap \Gamma(y) \cap \Gamma(z)|$ is independent of the choice of x, y, z for a clique $\{x, y, z\}$ of size 3. As in the proof of Theorem 9, it suffices to show that $|\Gamma(x) \cap \Gamma(y) \cap \Gamma(z)|$ is one larger than $|\bar{\Gamma}(x) \cap \bar{\Gamma}(y) \cap \bar{\Gamma}(z)|$ for at least one choice of $\{x, y, z\}$. A sufficient condition for such $\{x, y, z\}$ to satisfy this conclusion is the one stated in the beginning of the proof of Theorem 9.

All that is left to do is to show that $\{x, y, z\}$ exists. It is well known that a point of $O^\delta(2, q) \cap V_\epsilon$ is orthogonal to a point of $O^\delta(2, q) \cap V_\epsilon$.

Write $P = \langle p \rangle \perp \langle r_1, r_2 \rangle$, $\langle r_1, r_2 \rangle \cong O^\delta(2, q)$, $L_1 = \langle p, r_1 \rangle$, $r_1 \in r_2^\perp$. Since p^\perp/p contains an $O(5, q)$ space which contains $O^\delta(4, q) = O^\delta(2, q) \perp O^+(2, q)$ space, there exist $s_1 \in V_\epsilon$ and a line $L \ni r_2$ with $L \cong O^+(2, q)$ such that

$$p^\perp \supseteq \langle p \rangle \perp \langle r_1, s_1 \rangle \perp L.$$

Since $n \geq 7$, there exists an $O(7, q)$ space

$$Q \perp \langle r_1, s_1 \rangle \perp L \text{ with } p \in Q \cong O(3, q).$$

Let $r_0, s_0 \in V_\epsilon \cap Q$ such that $\langle r_0, s_0, p_0 \rangle$ is a basis of Q with $s_0 \in r_0^\perp$ and $p \notin r_0^\perp, s_0^\perp$. Let p_2 be an isotropic point of L , and set

$$x = s_1 + p_2, \quad y = r_0, \quad z = s_0 + p_2.$$

We have that $x^\perp \cap P = \langle p_0, r_1 \rangle$. Furthermore, $L_y = y^\perp \cap P = \langle r_1, r_2 \rangle$, but $r_2 \notin z^\perp \cap P = L_z$, so $L_y \neq L_z$. Additionally, $L_y \cap L_x = \langle r_1 \rangle = L_z \cap L_x$, so in particular p is not in L_y or L_z . \square

We want to emphasize that the construction and the non-isomorphy result are independent of q , but that in the general case Γ_ϵ and $\bar{\Gamma}_\epsilon$ are not strongly regular. For $O^-(6, 3)$, the general non-isomorphy result of Theorem 10 does not apply. However, we have verified by computer that the above construction actually yields a strongly regular graph not isomorphic to Γ_- . For $O^+(6, 3)$, the graph remains unchanged by the switching, but it is non-isomorphic to the block graph of $\text{AG}(3, 3)$, another strongly regular graph with the same parameters.

7 Future Work

There are infinite families of strongly regular graphs on the non-isotropic points of $U(n, \sqrt{q})$ [13, C14] and one type of non-isotropic points of $O(2d + 1, q)$ [6, §7C], where adjacency is not defined by orthogonality, but lying on a tangent. We wonder if these strongly regular graphs are determined by their spectrum or if an argument similar to the arguments in this paper can construct new graphs of the same type. The smallest case of $U(n, \sqrt{q})$, that is $q = 4$, was covered in Section 5.

In a celebrated result Keevash did not only show that designs exist, but that many designs exist [19]. We are wondering if the probability of the condition in Theorem 4 is bounded away from 0 for these designs.

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