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Model to accurately predict out-of-plane shear stiffness reduction in general cracked laminates

M. Hajikazemi[1,2♣], D. Garoz[1], W. Van Paepegem[1]

[1]Department of Materials, Textiles and Chemical Engineering, Faculty of Engineering and Architecture, Ghent University, Technologiepark Zwijnaarde 46, Ghent, Belgium

[2]Dutch Polymer Institute (DPI), P.O. Box 902, 5600 AX, Eindhoven, The Netherlands

Abstract

The practically important problem of the modification of laminate out-of-plane shear stiffness by ply cracks is hardly investigated in the literature. In this paper, out-of-plane shear stiffness reduction of laminates containing uniform and non-uniform distributions of ply cracks is studied. A novel variational model is developed to determine accurately stress transfer mechanisms and consequently out-of-plane shear stiffness reduction of general cracked laminates under applied out-of-plane shear loads. It is shown that the presence of ply cracks in a laminate under out-of-plane shear loads, perturbs the uniform distribution of shear stresses and induces high gradients of in-plane stresses leading to large shear stiffness reductions. The results are compared with those of the finite element method (FEM) implementing periodic boundary conditions. It is shown that there is excellent accordance between the results obtained from these approaches. The outcome of the paper provides necessary information for determination of damage-based constitutive laws for composites.

Keywords: Out-of-plane shear stiffness reduction; Ply cracking; Stress transfer; Periodic boundary conditions; non-uniform cracking

♣ Corresponding Author, Tel: +32479659875, E-mail address: Mohammad.hajikazemi@ugent.be, Postal address: Technologiepark 46, 9052 Zwijnaarde, Ghent, Belgium.

1. Introduction

Three types of behavior are usually considered when designing composite laminated structures [1]: (i) linear elastic behavior without the presence of any microscopic damage modes for relatively low applied loads, (ii) non-linear stress-strain behavior and deterioration of effective laminate properties due to the stable formation of microscopic damage mechanisms (ply cracking, delamination, fiber fracture, etc.) for larger applied loads, and (iii) finally catastrophic failure due to the unstable progressive formation of damage mechanisms. It has been observed in many experimental studies [2-4] that ply cracks (also known as matrix cracks, transverse cracks, intralaminar cracks) in laminates with off-axis plies are the first ply level damage mode under both applied thermo-mechanical service and environmental loads. The accumulation of ply cracks leads to the degradation of laminates' thermo-elastic constants. The current research work concerns the first two design stages with focus on obtaining the out-of-plane shear stiffness of intact and cracked laminates when there is a certain distribution of uniformly and non-uniformly spaced ply cracks.

Many analytical [5-9] and numerical [10, 11] models have been developed to determine stress transfer mechanisms between ply cracks in laminates under in-plane loads in order to predict the degradation of effective in-plane thermo-elastic properties of cracked symmetric laminates. Moreover, stress transfer in cracked un-symmetric laminates under flexural loads has also been addressed leading to determination of bending thermo-elastic properties [12-17] of general cracked laminates. In addition, cracked laminates under out-of-plane normal (through-thickness) applied loads are also investigated and, thus, the effective out-of-plane axial properties of cracked laminates can be determined [5]. Although ply cracking might largely degrade the effective out-of-plane shear stiffness of laminates, stress transfer analysis of cracked laminates under out-of-plane shear loads is sadly neglected in the literature. It is mainly due to the lack of physical boundary conditions and difficulty in making admissible stress fields in analytical models under out-of-plane shear loads. The use of finite element models is also tedious as complex three dimensional periodic boundary conditions are needed. However, an objective multiscale physics-based damage modeling [18, 19] of ply cracking should take into account the effects of ply cracking on all stiffness parameters. Even in continuum damage modeling of ply cracking, only reduction of the in-plane properties of a cracked ply like that of Ladevèze's meso-model [20] is not sufficient because a transversely isotropic ply after cracking and being homogenized is no longer transversely isotropic.

In the current paper, a novel variational model is developed to determine accurately the stress fields and subsequently out-of-plane shear stiffness of general laminates (possibly un-balanced and un-symmetric) under applied out-of-plane shear loads in the presence of a uniform distribution of ply cracks. *A completely different stress transfer mechanism than those under in-plane loads is observed. In a laminate under in-plane loads, the presence of cracks perturbs the uniform distribution of in-plane stresses and induces high gradients of out-of-plane shear stresses leading to the in-plane stiffness reductions. However, it is shown here that in a laminate under out-of-plane shear loads, the presence of cracks perturbs the uniform distribution of out-of-plane shear stresses and triggers high gradients of in-plane stresses leading to out-of-plane shear stiffness reductions. The study of this stress transfer mechanism in the cracked laminate is necessary for the modification of laminate out-of-plane shear stiffness and prediction of secondary damage modes, e.g. ply cracking in neighboring plies and delamination at the interfaces*. To verify the developed variational approach, the finite element method is applied on a repeated unit cell by implementing three dimensional periodic boundary conditions. A simple approximate methodology is also derived to obtain the out-of-plane shear stiffness of general cracked laminates containing non-uniformly spaced ply cracks. It is shown that the presence of cracks largely degrades the out-of-plane shear stiffness of laminates. *The single fundamental assumption of the developed method is that the out-of-plane shear stresses are linear through the thickness of each ply. This assumption is also relaxed by implementing a ply refinement technique by dividing each ply into several ply elements with the same material properties.*

2. Theoretical formulation of variational approach

2.1 Geometry and coordinate system

A multi-layered composite laminate made of N plies with general lay-up (possibly un-symmetric) under axial (τ_a) and transverse (τ_t) out-of-plane shear loads, is considered. A Cartesian coordinate system with its origin located at the center of the laminate will be considered as shown in Figs. 1a and b, where the x, y and z directions, respectively, define the in-plane axial, in-plane transverse and through-thickness directions. The N-1 interfaces between the layers are shown by $z=z_i$; i=1, 2... N-1. The lower and upper external surfaces are denoted by $z=z_0=-h$ and by $z=z_N=h$, where 2h is the total thickness of the laminate. The thickness of the i^{th} ply is specified by $h_i=z_i-z_{i-1}$. The laminate is assumed to have length 2L in the axial direction (x-axis) and width 2W in the transverse direction (y-axis).

2.2 Analysis of undamaged laminate

For an undamaged laminate subject to the input applied loading parameters τ_a and τ_t, the effective out-of-plane axial shear strain γ_a^0 and the effective out-of-plane transverse shear strain γ_t^0, control the deformation of the laminate. Moreover, the external edges of the undamaged laminate are subject to uniform shear stresses τ_a and τ_t, thus, for all layers in the laminate, we have

$$\tau_{xz}^{0(i)} = \tau_a, \quad \tau_{yz}^{0(i)} = \tau_t, \qquad \sigma_{xx}^{0(i)} = \sigma_{yy}^{0(i)} = \sigma_{zz}^{0(i)} = \tau_{xy}^{0(i)} \equiv 0, \quad i = 1 \ldots N. \tag{1}$$

where $\sigma_{xx}^{0(i)}, \tau_{xz}^{0(i)}$, etc., denote stress terms in the ith layer of the undamaged laminate. The material is monoclinic (no coupling between out-of-plane shear and axial directions) and thus, we have:

$$\gamma_{yz}^{0(i)} = s_{44}^i \tau_t + s_{45}^i \tau_a, \quad \gamma_{xz}^{0(i)} = s_{45}^i \tau_t + s_{55}^i \tau_a, \qquad \varepsilon_{xx}^{0(i)} = \varepsilon_{yy}^{0(i)} = \varepsilon_{zz}^{0(i)} = \gamma_{xy}^{0(i)} = 0. \tag{2}$$

where s_{44}^i, s_{45}^i and s_{55}^i terms are members of the compliance matrix $[s_{kl}^i]_{6\times 6}$ of the ith layer in the global (x, y, z) coordinate system and $\varepsilon_{xx}^{0(i)}$, $\gamma_{xz}^{0(i)}$, etc., denote strain terms in the undamaged laminate. The effective out-of-plane axial γ_a^0 and transverse γ_t^0 shear strains for the undamaged laminate are then given by

$$\begin{aligned} \gamma_t^0 &= \frac{1}{2h}\sum_{i=1}^{N} h_i \gamma_{yz}^{0(i)} \Rightarrow \gamma_t^0 = \frac{1}{G_t^0}\tau_t - \frac{\lambda_s{}^0}{G_a^0}\tau_a, \\ \gamma_a^0 &= \frac{1}{2h}\sum_{i=1}^{N} h_i \gamma_{xz}^{0(i)} \Rightarrow \gamma_a^0 = -\frac{\lambda_s{}^0}{G_a^0}\tau_t + \frac{1}{G_a^0}\tau_a, \end{aligned} \tag{3}$$

where the out-of-plane transverse G_t^0 and axial G_a^0 shear stiffness modules and out-of-plane shear coupling parameter $\lambda_s{}^0$ for the undamaged laminate, can be obtained as follows:

$$G_t^0 = \frac{1}{\frac{1}{2h}\sum_{i=1}^{N} h_i s_{44}^i}, \; G_a^0 = \frac{1}{\frac{1}{2h}\sum_{i=1}^{N} h_i s_{55}^i}, \; \lambda_s^0 = -\frac{\sum_{i=1}^{N} h_i s_{45}^i}{\sum_{i=1}^{N} h_i s_{55}^i}. \tag{4}$$

The analysis presented in this section might be regarded as an extension of the classical laminated plate theory where the effects of out-of-plane loads are included and out-of-plane shear stiffness terms are defined.

2.3 Analysis of the laminate containing uniformly spaced ply cracks

Imagine now that the laminate is damaged with uniformly spaced ply cracks, having a separation 2a, in some of its 90^0 plies (parallel to the y-axis, see Fig. 1a). As cracks are uniformly distributed, a unit cell of length 2a, thickness 2h and width 2W between two consecutive cracks with the (x, y, z) coordinate system located at the center of the unit cell, will be considered to derive the stress fields.

2.3.1 Stress transfer analysis

It is noted that the presence of cracks perturbs the undamaged stress fields and thus, the stresses in the i^{th} ply of the cracked laminate (σ^i_{mn}) will be written as follows:

$$\sigma^{(i)}_{mn}(x,z) = \sigma^{0(i)}_{mn}(z) + \sigma^{p(i)}_{mn}(x,z), \qquad m,n = x,y,\ z, \tag{5}$$

where $\sigma^{0(i)}_{mn}$ are the stresses in the i^{th} layer of the undamaged laminate (obtained in the previous section) and $\sigma^{p(i)}_{mn}$ are the unknown perturbation stresses in the i^{th} ply due to the presence of cracks.

We assume that out-of-plane perturbation shear stresses have piecewise linear forms, as follows:

$$\begin{aligned} \tau^{p(i)}_{xz} &= \frac{p_i^{'}(x)(z-z_{i-1})}{h_i} - \frac{p_{i-1}^{'}(x)(z-z_i)}{h_i}, \\ \tau^{p(i)}_{yz} &= \frac{q_i^{'}(x)(z-z_{i-1})}{h_i} - \frac{q_{i-1}^{'}(x)(z-z_i)}{h_i}, \quad i = 1...N, \end{aligned} \tag{6}$$

where $p_i(x)$ and $q_i(x)$ are 2(N+1) unknown perturbation stress terms which are functions of x only, and are zero when the laminate is undamaged. Also, the prime sign shows the derivative with respect to x. It should also be noted that the above assumed forms satisfy automatically the continuity of out-of-plane shear stresses at the interface between plies ($z=z_i$, i=1…N-1). The other stress terms can be found using the satisfaction of equilibrium equations $\sigma^i_{mn,n} = 0$ and traction continuity conditions between plies for out-of-plane normal stresses $\sigma^{p(i)}_{zz}$, as follows:

$$\sigma_{xx}^{p(i)} = \frac{-p_i(x) + p_{i-1}(x)}{h_i}, \; \tau_{xy}^{p(i)} = \frac{-q_i(x) + q_{i-1}(x)}{h_i}, \tag{7}$$
$$\sigma_{zz}^{p(i)} = -\frac{p_i^{''}(x)(z - z_{i-1})^2}{2h_i} + \frac{p_{i-1}^{''}(x)(z - z_i)^2}{2h_i} - \frac{p_0^{''}(x)h_1}{2} - \sum_{j=1}^{i-1} \frac{p_j^{''}(x)(h_j + h_{j+1})}{2},$$

The obtained perturbation stress fields should also balance the applied out-of-plane shear loads. The undamaged stress fields have already balanced the applied loads, thus, the following relations must be considered between the perturbation stresses:

$$\sum_{i=1}^{N} \int_{z_{i-1}}^{z_i} \tau_{xz}^{p(i)} dz = 0 \Rightarrow \sum_{i=1}^{N} \frac{h_i}{2}\left(p_i^{'}(x) + p_{i-1}^{'}(x)\right) = 0, \tag{8}$$
$$\sum_{i=1}^{N} \int_{z_{i-1}}^{z_i} \tau_{yz}^{p(i)} dz = 0 \Rightarrow \sum_{i=1}^{N} \frac{h_i}{2}\left(q_i^{'}(x) + q_{i-1}^{'}(x)\right) = 0,$$

where the satisfaction of the above equations ensure the normal traction free conditions (σ_{zz}=0) at z=±h. The in-plane perturbation transverse stresses $\sigma_{yy}^{p(i)}$ can be defined in terms of other stress terms with the assumption that under the assumed loading condition the in-plane transverse strains are zero in both cracked and undamaged laminates, as follows:

$$\varepsilon_{yy}^{p(i)} \equiv 0 = s_{12}^{i}\sigma_{xx}^{p(i)} + s_{22}^{i}\sigma_{yy}^{p(i)} + s_{23}^{i}\sigma_{zz}^{p(i)} + s_{26}^{i}\tau_{xy}^{p(i)} \Rightarrow \tag{9}$$
$$\sigma_{yy}^{p(i)} = -\frac{1}{s_{22}^{i}}\left(s_{12}^{i}\sigma_{xx}^{p(i)} + s_{23}^{i}\sigma_{zz}^{p(i)} + s_{26}^{i}\tau_{xy}^{p(i)}\right),$$

Finally, the Eqs. (6), (7) and (9) form an admissible stress field that satisfies exactly the equilibrium equations and all through-thickness traction continuity and boundary conditions for any 2(N+1) functions $p_i(x)$ and $q_i(x)$. In addition, Eqs. (8) and (9) assert that there are two relations between the perturbation functions, thus, the number of unknown perturbations function that must be found is 2N. These unknown functions will be obtained by minimization of the complementary energy.

It is shown [21] that in order to minimize the complementary energy, it is sufficient to minimize the perturbation complementary energy U^P, which can be written as follows:

$$U^{p}=\frac{1}{2}\int_{V}\sigma^{pT}s\,\sigma^{p}dV=\sum_{i=1}^{N}\left(\int_{-a}^{a}\int_{z_{i-1}}^{z_i}\frac{1}{2}\left\{\sigma^{p(i)}\right\}^{T}\left[s_{kl}^{i}\right]\left\{\sigma^{p(i)}\right\}dzdx\right). \tag{10}$$

Where V is the region occupied by the laminate.

The perturbation complementary energy can be simplified as Eq. (11) in terms of 2N unknown independent perturbation functions, by inserting the perturbation functions (Eqs. (6)-(9)) and the compliance matrices of each ply into Eq. (10) and integrating over z:

$$\begin{aligned} &U^{P}=\int_{-a}^{a}F\left(x,\{p\},\left\{p^{'}\right\},\left\{p^{''}\right\},\{q\},\left\{q^{'}\right\}\right)dx,\ \ where\\ &F=\{p\}^{T}[C_{11}^{00}]\{p\}+\{q\}^{T}[C_{22}^{00}]\{q\}+\{p\}^{T}[C_{12}^{00}]\{q\}\\ &\quad+\{p^{'}\}^{T}[C_{11}^{11}]\{p^{'}\}+\{q^{'}\}^{T}[C_{22}^{11}]\{q^{'}\}+\{p^{'}\}^{T}[C_{12}^{11}]\{q^{'}\}\\ &\quad+\{p^{''}\}^{T}[C_{11}^{22}]\{p^{''}\}+\{p^{''}\}^{T}[C_{11}^{20}]\{p\ \}+\{p^{''}\}^{T}[C_{12}^{20}]\{q\ \}, \end{aligned} \tag{11}$$

and where {p} and {q} represent independent unknown perturbation functions (written in a vector form) and the coefficient matrices $[C_{11}^{00}]$, etc., can be easily evaluated analytically in terms of ply properties. Finally, minimizing the functional in Eq. (11) leads to a set of ordinary differential equations with constant coefficients, as follows:

$$\begin{aligned} &[T_1]\{p^{''''}\}+[T_2]\{p^{''}\}+[T_3]\{p\}+[T_4]\{q^{''}\}+[T_5]\{q\}=0,\\ &[T_4]^{T}\{p^{''}\}+[T_5]^{T}\{p\}+[T_6]\{q^{''}\}+[T_7]\{q\}=0, \end{aligned} \tag{12}$$

where

$$\begin{aligned} &[T_1]=[C_{11}^{22}]+[C_{11}^{22}]^{T},\,[T_2]=[C_{11}^{20}]+[C_{11}^{20}]^{T}-[C_{11}^{11}]-[C_{11}^{11}]^{T},\ \ [T_3]=[C_{11}^{00}]+[C_{11}^{00}]^{T},\\ &[T_4]=-[C_{12}^{11}]+[C_{12}^{20}]^{T},[T_5]=[C_{12}^{00}],\,[T_6]=-[C_{22}^{11}]-[C_{22}^{11}]^{T},\,[T_7]=[C_{22}^{00}]+[C_{22}^{00}]^{T}. \end{aligned} \tag{13}$$

The readers can refer to many available mathematical text books or Ref. [6] to find the general solution of the differential equations similar to that of Eq. (12). However, after finding the general solution, boundary conditions are needed to determine the solution. The coupled system of differential equations in Eq. (12) requires, in total, 6N traction boundary conditions. *It is noted that for a laminate with N layers there are N+1 interface locations ($z=z_k$, $k=0\ldots N$) including the external lower and upper surfaces. Imagine there are N_c interface locations adjacent to cracked plies and N_u interface locations which are not adjacent to any cracked plies (there should be at least one interface of this type), thus, we have $N_c+N_u=N+1$. Suppose further that $I_p=\{0,1,2,\ldots,N\}$ is the set of all N+1 interface indices ($z=z_k$, $k=0\ldots N$). Moreover, $I_c \subset I_p$ is the set of indices of interface locations adjacent to cracked plies and $I_u \subset I_p$ is the set of indices of the uncracked interfaces. Therefore, we have clearly $I_c \cup I_u = I_p$ and $I_c \cap I_u \equiv 0$.*

The traction free conditions at the plane containing cracks $x=\pm a$ for N_c interface locations adjacent to cracked plies provide $6N_c$ boundary conditions, as follows:

$$\sigma_{xx}(\pm a)=0 \Rightarrow p_k(\pm a)=0, \quad k\in I_c, \tag{14}$$

$$\tau_{xy}(\pm a)=0 \Rightarrow q_k(\pm a)=0, \quad k\in I_c, \tag{15}$$

$$\tau_{xz}(\pm a, z_k)=0 \Rightarrow p_k'(\pm a)=-\tau_a, \quad k\in I_c. \tag{16}$$

For N_u interface locations which are not adjacent to any cracked plies, the periodic boundary conditions together with rotational anti-symmetry [22] (under assumed loading condition) about the vertical central axis assert the following 5(Nu-1) boundary conditions (see Eq. (8) by which one out of any N+1 unknowns functions were eliminated):

$$\sigma_{xx}(a)=\pm\sigma_{xx}(-a) \Rightarrow p_k(a)=\pm p_k(-a)=0, \quad k\in I_u, \tag{17}$$

$$\tau_{xy}(a)=\pm\tau_{xy}(-a) \Rightarrow q_k(a)=\pm q_k(-a)=0, \quad k\in I_u, \tag{18}$$

$$\tau_{xz}(a, z_k)=\tau_{xz}(-a, z_k) \Rightarrow p_k'(a)=p_k'(-a), \quad k\in I_u. \tag{19}$$

The physical construction of the problem does not offer any evident boundary condition even in terms of displacement. Here, we introduce the last traction boundary condition in terms of mathematical *natural boundary conditions [22]* for N_u-1 independent interface locations, which are not adjacent to any cracked plies. Indeed, to minimize the functional in Eq. (11) when boundary values are not fully defined, it is needed to satisfy the following equation in addition to the differential equations in Eq. (12),

$$\left(\frac{\partial F}{\partial\{p^{"}\}}\right)^T \{\partial p^{'}\}\Bigg|_{x=-a}^{x=a} = 0 \;\Rightarrow\; [T_1]\left(\{p^{"}(a)\} - \{p^{"}(-a)\}\right) = 0 \,. \tag{20}$$

The above equation clearly provides (N_u-1) boundary conditions, as required to find the solution.

2.3.2 The effective out-of-plane shear stiffness modules

The effective out-of-plane shear modules of the cracked laminate can be determined, provided that stress fields are already obtained as a function of applied shear loads. The principle of minimum complementary energy for the assumed loading condition can be simplified as follows:

$$U^P_{true} = \frac{1}{2}\{\tau_t \quad \tau_a\}\begin{bmatrix} \frac{1}{G_t} & \frac{\lambda_s}{G_a} \\ \frac{\lambda_s}{G_a} & \frac{1}{G_a} \end{bmatrix}\begin{Bmatrix} \tau_t \\ \tau_a \end{Bmatrix} V \le \frac{1}{2}\{\tau_t \quad \tau_a\}\begin{bmatrix} \frac{1}{G_t^0} & \frac{\lambda_s^0}{G_a^0} \\ \frac{\lambda_s^0}{G_a^0} & \frac{1}{G_a^0} \end{bmatrix}\begin{Bmatrix} \tau_t \\ \tau_a \end{Bmatrix} V + U^p = U^p_{adm}, \tag{21}$$

Where G_t , G_a and λ_s are, respectively, the out-of-plane transverse shear stiffness, axial shear stiffness and shear coupling parameter for the cracked laminate. Moreover, U^P_{true} denotes the true complementary energy while U^p_{adm} specifies the complementary energy computed with an admissible stress field. We have minimized the perturbation complementary energy U^p leading to an upper bound for the laminate stiffness parameters. Therefore, the effective shear modules of the cracked laminate can be obtained by applying three special loading cases and finding the perturbation complementary energy U_P using Eq. (11) , as follows:

$$\begin{aligned}
&\frac{1}{G_t} \le \frac{1}{G_t^0} + \frac{2}{(2h\times 2L)} U^p_{\{\tau_t,\tau_a\}=\{1,0\}}, \\
&\frac{1}{G_a} \le \frac{1}{G_a^0} + \frac{2}{(2h\times 2L)} U^p_{\{\tau_t,\tau_a\}=\{0,1\}}, \\
&2\frac{\lambda_s}{G_a} \le 2\frac{\lambda_s^0}{G_a^0} + \frac{1}{G_t^0} - \frac{1}{G_t} + \frac{1}{G_a^0} - \frac{1}{G_a} + \frac{2}{(2h\times 2L)} U^p_{\{\tau_t,\tau_a\}=\{1,1\}}.
\end{aligned} \tag{22}$$

2.4 Analysis of the laminate containing non-uniformly spaced ply cracks

Imagine now that the laminate is damaged with non-uniformly spaced ply cracks (see Fig. 1b). Unlike laminates with uniformly spaced cracks, it is not possible to select a repeating unit cell and the stress analysis must be conducted for the entire cracked laminate. Moreover, a progressive ply cracking simulation usually needs considering more than hundred non-uniformly distributed cracks. While it is still possible to perform an exact variational analysis (see Ref. [23] for in-plane loads) for laminates with non-uniform distribution of ply cracks under out-of-plane shear loads, the simplicity and computational efficiency of the approach might be negated by too many coupled differential equations and boundary conditions that will arise. It should be noted that performing such an analysis using FEM is also not feasible when there are too many cracks (>100). Therefore, an approximate approach based on an assumption is implemented to obtain out-of-plane shear modules of the laminate. We assume that for a non-uniformly cracked laminate under out-of-plane shear loads, crack tip opening and in-plane sliding displacements are negligible in comparison to crack tip out-of-plane sliding. In other words, we assume that for a laminate with non-uniform distribution of cracks $-L \le x \le L$ (see Fig. 1b), the stress distribution for each fragment J between the two neighboring ply cracks, where J=1…M, corresponds to that found in a uniformly cracked laminate having the same crack separation ($2a_J$). It is noted that all fragments are under the same effective average applied loads. Therefore, the total complementary energy of the cracked laminate can be determined by adding the complementary energy of each fragment which can be obtained separately, leading to simple expressions for laminate shear modules, as follows:

$$\frac{1}{G_t} = \frac{1}{L}\sum_{J=1}^{M}\frac{1}{G_t^{(J)}}\times a_J, \tag{23}$$

$$\frac{1}{G_a} = \frac{1}{L}\sum_{J=1}^{M}\frac{1}{G_a^{(J)}}\times a_J, \tag{24}$$

$$\frac{\lambda_s}{G_a} = \frac{1}{L}\sum_{J=1}^{M}\frac{\lambda_s^{(J)}}{G_a^{(J)}}\times a_J, \quad where \quad L = \sum_{J=1}^{M} a_J. \tag{25}$$

2.5 Finite element modeling of uniform ply cracking under out-of-plane shear loads

To verify the developed variational model for uniformly spaced ply cracks, a FEM simulation in ABAQUS/Standard general purpose software was performed using the 8-noded brick elements with full integration scheme. A unit cell similar to that considered in section (2.3), was considered and discretized into finite elements where traction free conditions on the crack surfaces were enforced. *In order to consider any state of applied far-field strain/stress as out-of-plane shear loads, periodic boundary conditions for all faces in the three dimensions were implemented [24]. The procedure of applying FEM and the formulation of periodic boundary conditions are already described in Ref. [24] and the reader should refer to this publication for more details.* The convergence study for the FEM models was performed and very refined meshes were used at locations close to the crack tips and ply interfaces. An example of such mesh is shown in Fig. 2.

3. Results and discussion

We first verify the accuracy of the developed model by comparing the stress fields obtained from the variational approach with those of FEM. To do so, two symmetric laminates of type $[0/90]_s$ and $[45/90]_s$, made of Carbon/Epoxy, containing uniformly distributed ply cracks with density of $\rho=1/2a=1/mm$ in the 90^0 ply, under an effective out-of-plane applied axial shear stress τ_a, are considered. The unidirectional material properties for these laminates are, E_{11}= 141.3 GPa, E_{22}=9.58 GPa, G_{12}=5 GPa, υ_{12}=0.3, υ_{23}=0.32 and t_{ply}=0.25 mm. For all results obtained from the developed variational model, a ply refinement technique [23] is implemented by which each layer is first divided into six elements of equal thickness to relax the effects of assumptions in Eq. (6). In addition, ply elements next to the interfaces which are adjacent to the cracks were subdivided in half, five times to ensure having converged results. The convergence study has also been performed for FEM to make sure that the results are converged and as can be seen in Fig. 2, very refined meshes are used specially close to the cracks. Figs.3a and b show, respectively, the through-thickness variations of the normalized out-of-plane axial shear τ_{xz}/τ_a and normal σ_{zz}/τ_a stresses at the plane containing cracks (x=a) for the $[0/90]_s$ laminate. Figs. 4a and b depict, respectively, the axial distribution of normalized interfacial out-of-plane shear τ_{xz}/τ_a and normal σ_{zz}/τ_a stresses at the upper 0/90 interface (z=0.25 mm). Figs. 5a and b also show the axial distribution of, respectively, the normalized out-of-plane axial shear stresses τ_{xz}/τ_a at the upper external surface (z=0.5 mm) and the normalized in-plane axial stresses σ_{xx}/τ_a at the upper 0/90 interface (z=0.25 mm in 90^0 ply). Moreover, Figs.6a and b show, respectively, the through-thickness variations of the normalized out-of-plane axial shear τ_{xz}/τ_a and normal σ_{zz}/τ_a stresses at the plane containing cracks (x=a) for $[45/90]_s$ laminate. Figs. 7a and b depict, respectively, the normalized axial distribution of in-plane axial σ_{xx}/τ_a and shear τ_{xy}/τ_a stresses at the upper 45/90 interface (z=0.25 mm in 90^0 ply). The general observation is that there is a perfect agreement between the two sets of results verifying the accuracy of both approaches and the implementation in each software. In order to show the capability of the current approach to deal with un-symmetric laminates, a symmetric $[90/45]_s$ laminate which is un-symmetrically cracked (containing cracks only in the upper 90^0 ply with density 1/mm) under an effective out-of-plane applied axial shear stress τ_a, is also considered. Through-thickness variations of the normalized out-of-plane axial τ_{xz}/τ_a and transverse τ_{yz}/τ_a shear stresses at the plane containing cracks (x=a) are shown, respectively, in Figs. 8a and 8b. The axial variations of the normalized out-of-plane axial τ_{xz}/τ_a and transverse τ_{yz}/τ_a shear stresses at the upper external surface (z=0.5mm) are also shown, respectively, in Figs. 9a and 9b.

In order to study the effects of ply cracking on the laminate out-of-plane shear stiffness parameters, two symmetric laminates of type $[0/90]_s$, $[60/90]_s$ and two un-symmetric laminates of type [0/90/30/45], [0/30/45/90], made of Glass/Epoxy, containing uniformly distributed ply cracks in 90^0 plies are considered. The unidirectional material properties for these laminates are, E_{11}= 45 GPa, E_{22}=12 GPa, G_{12}=5.8 GPa, υ_{12}=0.3, υ_{23}=0.42 and t_{ply}=0.25 mm. Fig. 10a depicts variation of the normalized axial $\frac{G_a}{G_a^0}$ and transverse $\frac{G_t}{G_t^0}$ out-of-plane shear stiffness terms versus crack density in 90^0 layer. Similarly, the variation of the out-of-plane shear coupling parameter λ_s versus crack density is shown in Fig. 10b. It can be seen that the out-of-plane axial shear stiffness G_a and coupling shear parameter λ_s are largely dependent to the crack density for different laminates. It can also be seen that that in a coordinate system where cracks are in 90^0 plies, the transverse shear stiffness parameters G_t remain constant and are independent of crack density.

Finally, in order to compare the out-of-plane shear stiffness parameters of uniformly and non-uniformly cracked laminates, a representative volume element of $[30/90]_s$ laminate, made of

Glass/Epoxy, containing two cracks (M=2) in the length 2L=2mm (see Fig. 1b) with uniform and non-uniform distribution of cracks in 90^0 ply, is considered. Table 1 compares the effective out-of-plane shear terms of laminates with different degree of non-uniformity.

Table. 1: Effective elastic constants of $[30/90]_s$ laminate with 2L=2mm and M=2 (see Fig. 1b) when the laminate is cracked with average crack density M/2L=1/mm.

Shear properties	$a_1/L=0.5$ (uniform cracking)	$a_1/L=0.4$	$a_1/L=0.3$	$a_1/L=0.2$	$a_1/L=0.1$
G_t (GPa)	5.089	5.089	5.089	5.089	5.089
G_a (GPa)	4.046	4.149	4.223	4.279	4.323
λ_s	0.0562	0.0577	0.0587	0.0595	0.0601

It can also be observed that assuming a uniform distribution of ply cracks ($a_1/L=0.5$) overestimates the out-of-plane shear stiffness reduction of the cracked laminates which is in agreement with previous observations on the in-plane laminate properties [23]. It also means that the effects of this non-uniformity should be taken into account specifically at the beginning of the cracking process where cracks are randomly distributed.

4. Conclusion

A new variational model is developed to predict accurately the stress transfer mechanisms in cracked general laminates under applied out-of-plane shear loads. The effective out-of-plane shear stiffness of laminates as a function of crack density is studied and it has been shown that ply cracking largely degrades the out-of-plane shear laminate properties. An approximate methodology is introduced to deal with non-uniformly spaced ply cracks. The comparison of the results with direct FE results implementing 3D periodic boundary conditions shows perfect accordance. The effects of ply cracking and its random or uniform distribution on the effective out-of-plane shear parameters of laminates should also be taken into account when modeling ply cracking at structural level.

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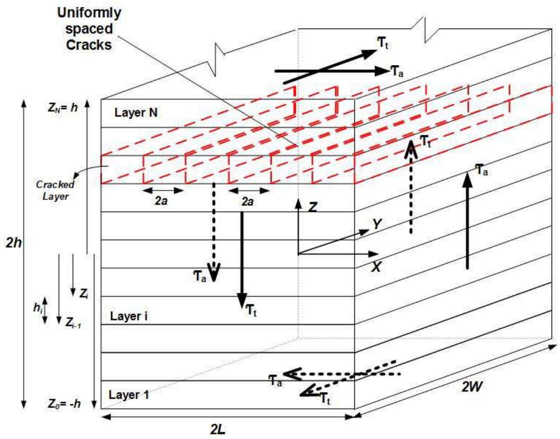
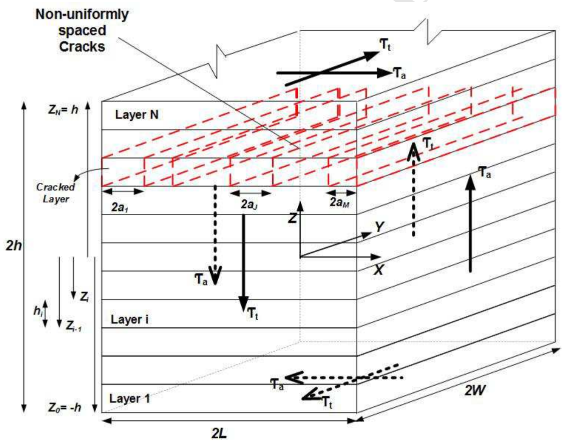


(a)



(b)

Fig. 1. Geometry of a laminate with arbitrary lay-up containing a) uniformly spaced and b) non-uniformly spaced ply cracks in one of 90-layers under out-of-plane shear loads.

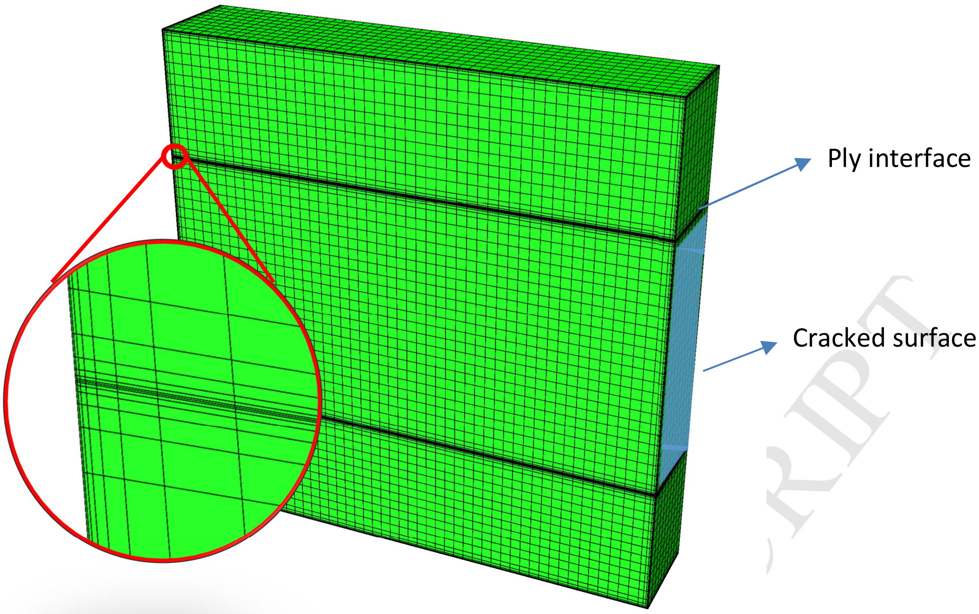
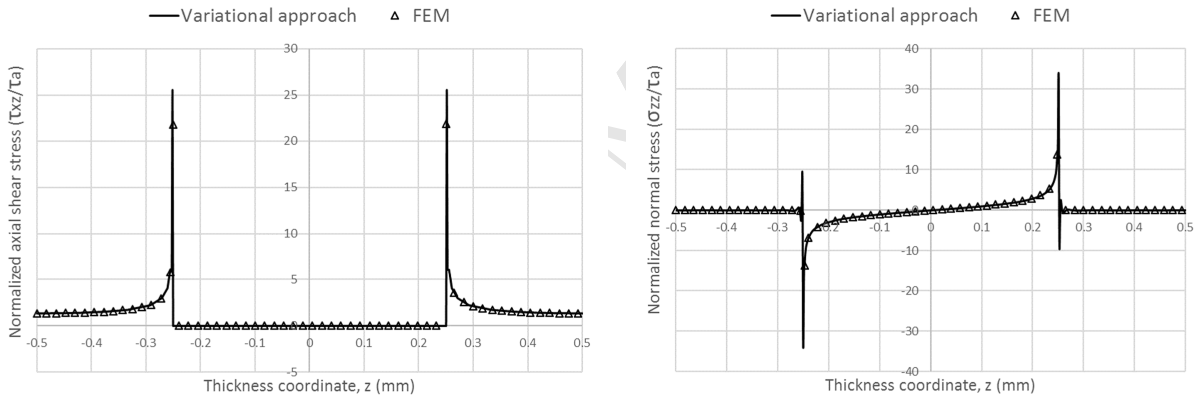
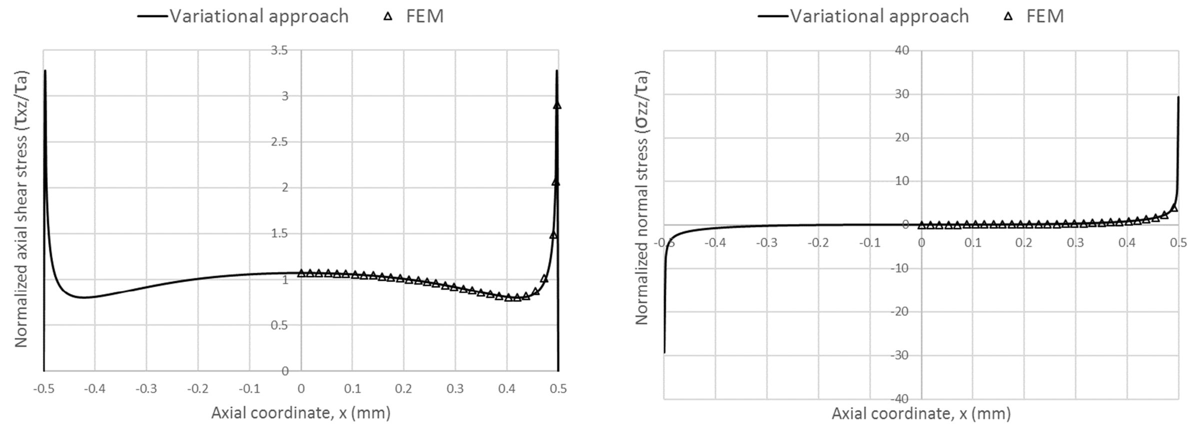


Fig. 2. An example of the finite element mesh implemented in this simulation.



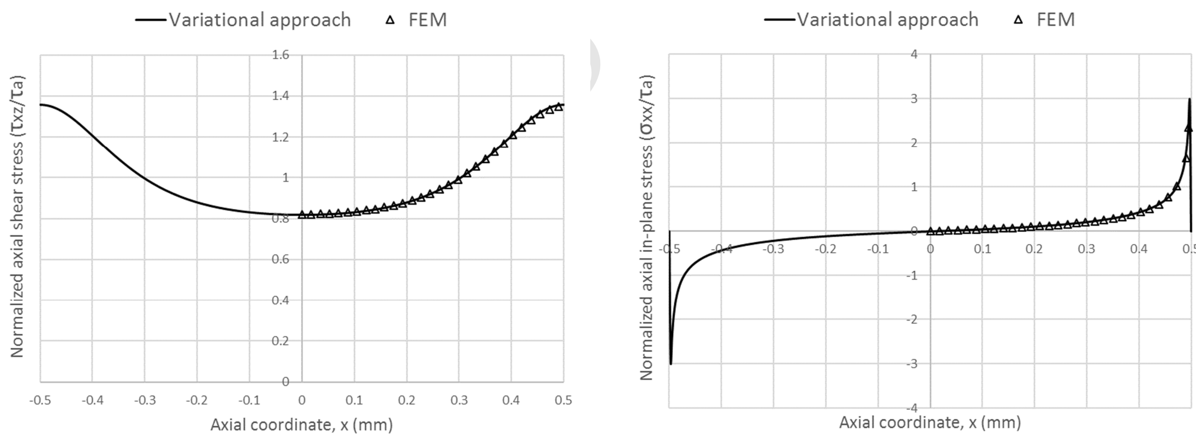
(a) (b)

Fig. 3. Through-thickness variation of a) the normalized out-of-plane axial shear τ_{xz}/τ_a and b) normal σ_{zz}/τ_a stresses at plane containing cracks (x=0.5mm) in the $[0/90]_s$ laminate.



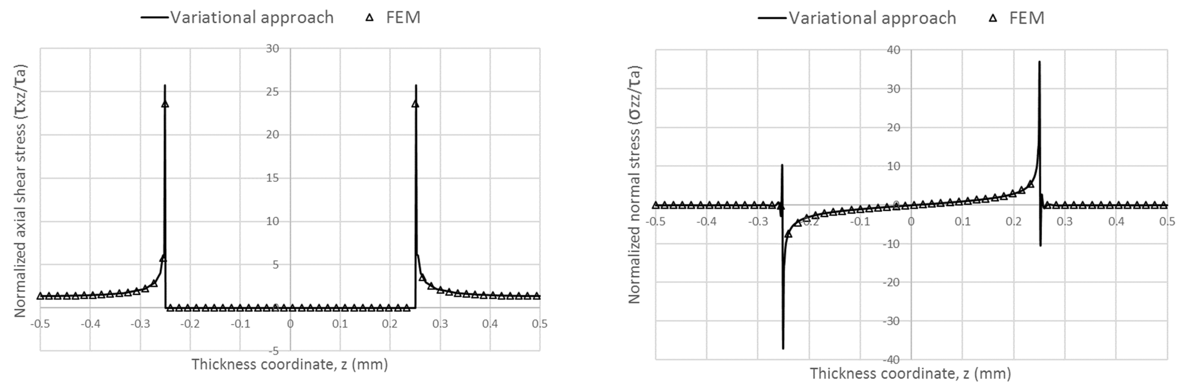
(a) (b)

Fig. 4. Axial distribution of a) the normalized out-of-plane axial shear τ_{xz}/τ_a and b) normal σ_{zz}/τ_a stresses at the upper 0/90 interface (z=0.25mm) in the $[0/90]_s$ laminate.



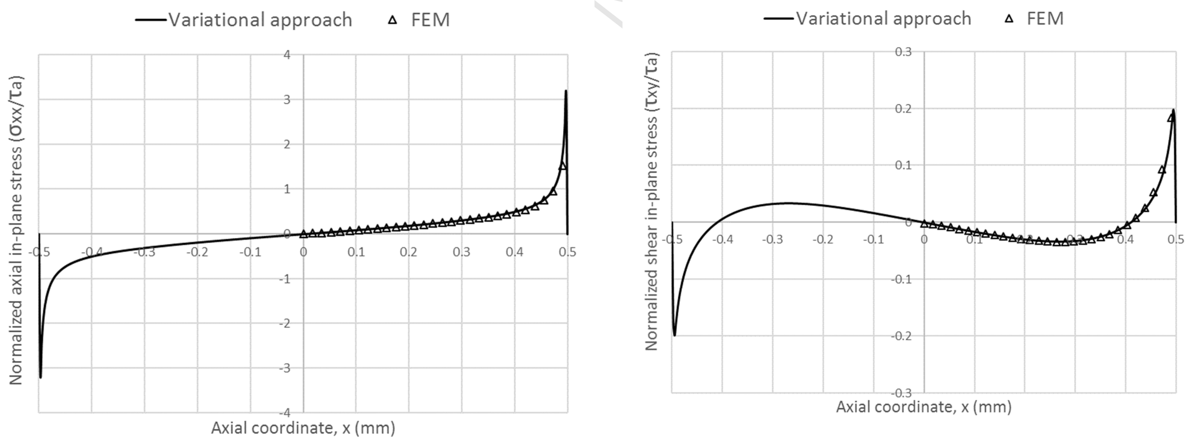
(a) (b)

Fig. 5. Axial distribution of a) the normalized out-of-plane axial shear stress τ_{xz}/τ_a at the upper external surface (z=0.5mm) and b) the normalized in-plane axial stresses σ_{xx}/τ_a at the upper 0/90 interface (z=0.25mm in 90^0 ply).



(a) (b)

Fig. 6. Through-thickness variation of a) the normalized out-of-plane axial shear τ_{xz}/τ_a and b) normal σ_{zz}/τ_a stresses at plane containing cracks (x=a) in the $[45/90]_s$ laminate.



(a) (b)

Fig. 7. Axial distribution of a) the normalized in-plane axial σ_{xx}/τ_a and b) shear τ_{xy}/τ_a stresses at the upper 45/90 interface (z=0.25mm in 90^0 ply) in the $[45/90]_s$ laminate.

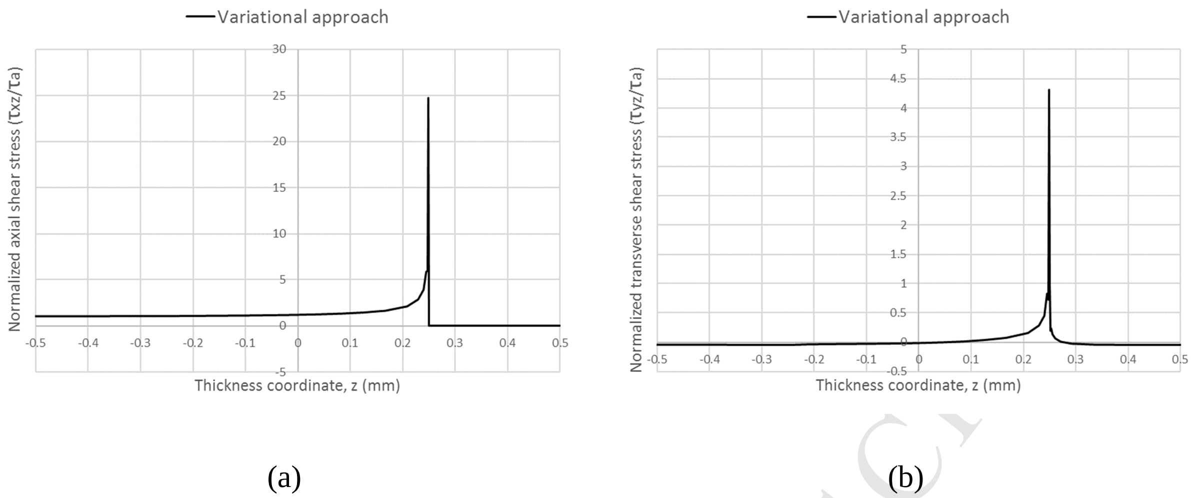


Fig. 8. Through-thickness variation of a) the normalized out-of-plane axial τ_{xz}/τ_a and b) transverse τ_{yz}/τ_a shear stresses at the plane containing cracks (x=a) in the unsymmetrically cracked $[90/45]_s$ laminate.

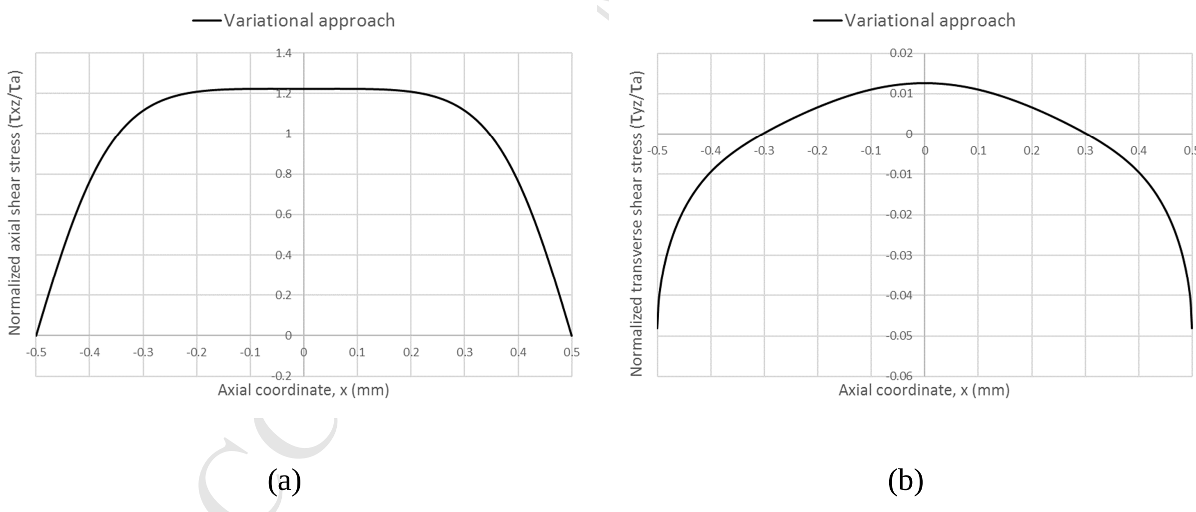
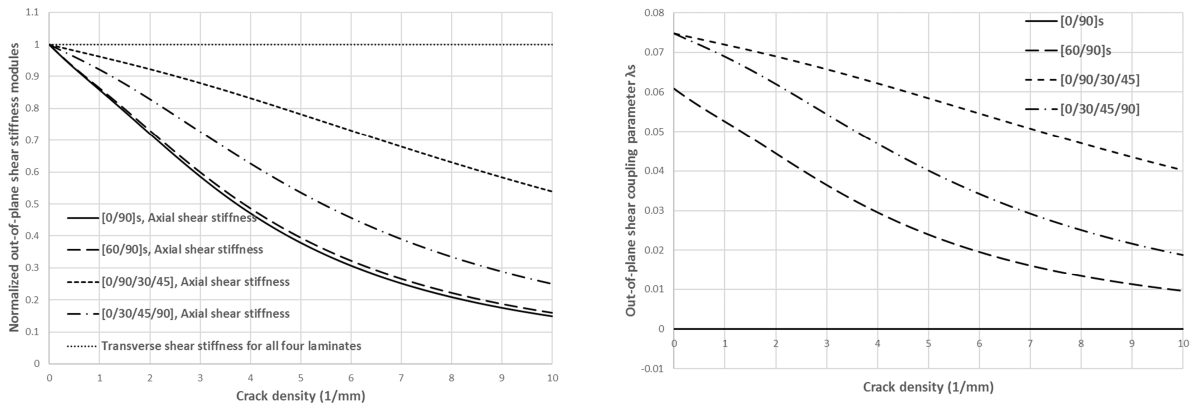


Fig. 9. Axial distribution of a) the normalized out-of-plane axial τ_{xz}/τ_a and b) transverse τ_{yz}/τ_a shear stresses at upper external surface (z=0.5mm) in the unsymmetrically cracked $[90/45]_s$ laminate.



(a) (b)

Fig. 10. Out-of-plane shear stiffness modules of laminates versus crack density in 90^0 ply.