Three homogeneous embeddings of $DW(2n-1,2)$

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Abstract

We construct a homogeneous full projective embedding of the dual polar space $DW(2n-1,2)$ from the hyperplane intersections of hyperbolic type of the parabolic quadric $Q(2n,2)$. We believe that this embedding is universal, but have not succeeded in proving that. As a by-product of our investigations, we have obtained necessary and sufficient conditions for this to be the case and came across two other homogeneous full projective embeddings of $DW(2n-1,2)$, one with vector dimension $\frac{2^{2n-1}+3\cdot 2^{n-1}-2}{3}$ and another one with vector dimension $\frac{2^{2n-1}+3\cdot 2^{n-1}-2-6n}{3}$.

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1 Introduction

The study of projective embeddings of point-line geometries is motivated by the fact that such embeddings often offer extra insight in certain aspects or properties of the geometries under consideration. Projective spaces are the natural habitat for certain embeddable geometries and for certain applications (like Tits' classification of polar spaces), it is often necessary as a first step to show that the studied geometries admit projective embeddings. Among all projective embeddings, the most interesting and symmetric ones are those that are homogeneous. An interesting feature of these embeddings is that they give rise to (ordinary or projective) representations of the automorphism groups of the geometries. In this paper, we construct and investigate three homogeneous full projective embeddings of the dual polar space $DW(2n-1,2)$.

Let ζ be a symplectic polarity of the projective space $\mathrm{PG}(2n-1,2)$, $n \geq 2$. A subspace π of $\mathrm{PG}(2n-1,2)$ is said to be *totally isotropic (with respect to ζ)* if $\pi \subseteq \pi^\zeta$. Associated with ζ, there is a *symplectic dual polar space $DW(2n-1,2)$ of rank n*. This is the point-line geometry whose points are the $(n-1)$-dimensional totally isotropic subspaces and whose lines are the $(n-2)$-dimensional totally isotropic subspaces, with incidence being reverse containment.

A *full projective embedding* of $DW(2n-1,2)$ is an injective mapping ϵ from the point set $\mathcal{P}$ of $DW(2n-1,2)$ to the set of points of a projective space $\mathrm{PG}(V)$ mapping every line of $DW(2n-1,2)$ to a full line of $\mathrm{PG}(V)$ such that the image $\mathrm{Im}(\epsilon)$ of ϵ generates the whole projective space $\mathrm{PG}(V)$. As every line of $DW(2n-1,2)$ is incident with precisely three points, the vector space V should then be defined over the field $\mathbb{F}_2$. A full projective embedding ϵ of $DW(2n-1,2)$ into a projective space $\mathrm{PG}(V)$ will shortly be denoted by $\epsilon : DW(2n-1,2) \to \mathrm{PG}(V)$. The dimension $\dim(V)$ of V is called the *vector dimension* of ϵ.

Suppose $\epsilon : DW(2n-1,2) \to \mathrm{PG}(V)$ is a full projective embedding and α is a subspace of $\mathrm{PG}(V)$ disjoint from any line containing at least two points of $\mathrm{Im}(\epsilon)$. Then a new projective embedding $\epsilon/\alpha : DW(2n-1,2) \to \mathrm{PG}(V)/\alpha$ can be defined which maps each point x of $DW(2n-1,2)$ to the point $\langle \alpha, \epsilon(x)\rangle$ of the quotient space $\mathrm{PG}(V)/\alpha$. The embedding ϵ/α is called a *quotient* of ϵ.

Two full projective embeddings $\epsilon_1 : DW(2n-1,2) \to \mathrm{PG}(V_1)$ and $\epsilon_2 : DW(2n-1,2) \to \mathrm{PG}(V_2)$ are called *isomorphic*, denoted by $\epsilon_1 \cong \epsilon_2$, if there exists an isomorphism $\phi : \mathrm{PG}(V_1) \to \mathrm{PG}(V_2)$ such that $\epsilon_2 = \phi \circ \epsilon_1$. We say that $\epsilon_1 \geq \epsilon_2$ if ϵ_2 is isomorphic to a quotient of ϵ_1. A full projective embedding ϵ of $DW(2n-1,2)$ is called *universal* if $\epsilon \geq \epsilon'$ for any full projective embedding ϵ' of $DW(2n-1,2)$. Up to isomorphism, $DW(2n-1,2)$ has a unique universal embedding which we will denote by $\widetilde{\epsilon}_0 : DW(2n-1,2) \to \mathrm{PG}(\widetilde{V}_0)$. The vector dimension $\dim(\widetilde{V}_0)$ of $\widetilde{\epsilon}_0$ is also called the *embedding rank* of $DW(2n-1,2)$. The structure of the universal embedding of $DW(2n-1,2)$ is not so well understood, but its vector dimension is known to be equal to $\frac{(2^n+1)(2^{n-1}+1)}{3}$, as was proved by Yoshiara [21, Proposition 6.4] for $n=3$, by Cooperstein [8] for $n \in \{4,5\}$ and for general n independently by Blokhuis & Brouwer [2] and Li [15].

A full projective embedding $\epsilon : DW(2n-1,2) \to \mathrm{PG}(V)$ is called *homogeneous* if for every automorphism θ of $DW(2n-1,2)$, there exists a (necessarily unique) automorphism $\overline{\theta}$ of $\mathrm{PG}(V)$ such that $\epsilon \circ \theta = \overline{\theta} \circ \epsilon$. The universal embedding is homogeneous. In this case, we denote $\overline{\theta}$ also by $\widetilde{\theta}$ and we put $\widetilde{G} := \{\widetilde{\theta} \,|\, \theta \in G\}$, where G is the full automorphism group of $DW(2n-1,2)$.

Suppose $\widetilde{\epsilon}_0 : DW(2n-1,2) \to \mathrm{PG}(\widetilde{V}_0)$ is the universal embedding of $DW(2n-1,2)$, $n \geq 3$. Then we will show that there exist $\widetilde{G}$-invariant subspaces $\widetilde{\alpha}_1$ and $\widetilde{\alpha}_2$ of $\mathrm{PG}(\widetilde{V}_0)$ of respective projective dimensions 0 and $2n$. This implies (see Lemma 2.3) that the quotient embeddings $\widetilde{\epsilon}_1 \cong \widetilde{\epsilon}_0/\widetilde{\alpha}_1$ and $\widetilde{\epsilon}_2 \cong \widetilde{\epsilon}_0/\widetilde{\alpha}_2$ are homogeneous projective embeddings. So, we have:

Theorem 1.1 *There exist homogeneous full projective embeddings $\widetilde{\epsilon}_1$ and $\widetilde{\epsilon}_2$ of $DW(2n-1,2)$, $n \geq 3$, whose vector dimensions are respectively equal to $\frac{2^{2n-1}+3\cdot 2^{n-1}-2}{3}$ and $\frac{2^{2n-1}+3\cdot 2^{n-1}-2-6n}{3}$.*

The dual polar space $DW(2n-1,2)$ is isomorphic to the dual polar space $DQ(2n,2)$ whose points are the $(n-1)$-dimensional subspaces (or generators) of a parabolic quadric $Q(2n,2)$ of $\mathrm{PG}(2n,2)$ and whose lines are the $(n-2)$-dimensional subspaces of $Q(2n,2)$, with incidence being reverse containment. Let $\mathcal{Q}^+$ denote the set of all hyperplanes of $\mathrm{PG}(2n,2)$ that intersect $Q(2n,2)$ in a hyperbolic quadric of type $Q^+(2n-1,2)$. Consider then a vector space W_h of dimension $|\mathcal{Q}^+|$ having a basis $\{\bar{v}_q \,|\, q \in \mathcal{Q}^+\}$ whose elements are indexed by the elements of $\mathcal{Q}^+$. For every point α of $DQ(2n,2)$, put

$$\epsilon_h(\alpha) := \langle \sum_{\alpha \subset q \in \mathcal{Q}^+} \bar{v}_q \rangle \in \mathrm{PG}(W_h).$$

We will show the following.

Theorem 1.2 *The map ϵ_h defines a homogeneous full projective embedding of $DQ(2n,2)$ into the subspace $\mathrm{PG}(V_h)$ of $\mathrm{PG}(W_h)$ generated by the image of ϵ_h.*

We refer to the embedding ϵ_h as the *hyperbolic embedding* of $DQ(2n, 2) \cong DW(2n-1, 2)$. We conjecture that this embedding is universal, but were not successful in proving that. Instead, we have determined necessary and sufficient conditions for this to be the case, see Propositions 3.5, 7.7(d), 7.9 and Corollary 7.10. The conjecture that the embedding ϵ_h is universal is supported by some computer computations of Peter Vandendriessche which show the validity of this conjecture for $n \leq 7$.

2 Preliminaries

With the symplectic polarity ζ of $\mathrm{PG}(2n-1, 2)$, $n \geq 2$, there is also associated a polar space $W(2n-1, 2)$ in the sense of Tits [20, Chapter 7]. The points of $W(2n-1, 2)$ are the points of $\mathrm{PG}(2n-1, 2)$ and the singular subspaces of $W(2n-1, 2)$ are the subspaces of $\mathrm{PG}(2n-1, 2)$ that are totally isotropic with respect to ζ.

Let $Q(2n, 2)$ with $n \geq 2$ be a parabolic quadric of $\mathrm{PG}(2n, 2)$, i.e. a set of points having equation $X_0^2 + X_1X_2 + \cdots + X_{2n-1}X_{2n} = 0$ with respect to a suitable reference system. Denote by k the *kernel* of $Q(2n, 2)$, i.e. the intersection of all tangent hyperplanes. Note that a hyperplane of $\mathrm{PG}(2n, 2)$ is tangent to $Q(2n, 2)$ if and only if it contains k. With $Q(2n, 2)$, there is associated a polar space, which we also denote by $Q(2n, 2)$. The points of this polar space are the points of $Q(2n, 2)$ and the singular subspaces are the subspaces of $\mathrm{PG}(n, 2)$ contained in $Q(2n, 2)$. The polar spaces $Q(2n, 2)$ and $W(2n-1, 2)$ are isomorphic: the projection from the kernel k on a hyperplane not containing k defines an isomorphism. With $Q(2n, 2)$, there is also associated a dual polar space $DQ(2n, 2)$, whose points and lines are the $(n-1)$-dimensional and $(n-2)$-dimensional subspaces of $Q(2n, 2)$, with incidence being reverse containment. As $Q(2n, 2) \cong W(2n-1, 2)$, we also have $DQ(2n, 2) \cong DW(2n-1, 2)$.

Consider again the symplectic dual polar space $DW(2n-1, 2)$, $n \geq 2$, and denote by $\mathcal{P}$ its point set. If x and y are two points of $DW(2n-1, 2)$, then $\mathrm{d}(x, y)$ denotes the distance between x and y in the collinearity graph Γ of $DW(2n-1, 2)$. If $x \in \mathcal{P}$, then $\Gamma_i(x)$ with $i \in \mathbb{N}$ denotes the set of points at distance i from x and $\Gamma_{\leq i}(x)$ denotes the set of points at distance at most i from x. For every point x of $DW(2n-1, 2)$, we define $x^\perp := \{x\} \cup \Gamma_1(x)$. The graph Γ is distance-regular [3] with diameter n and valency $2^{n+1} - 2$. If x and y are two vertices of Γ at distance i from each other, then there are $c_i = 2^i - 1$ points collinear with x at distance $i - 1$ from y.

If X_1 and X_2 are two sets of points of $DW(2n-1, 2)$, then $X_1 * X_2$ denotes the complement $\mathcal{P} \setminus (X_1 \Delta X_2)$ of the symmetric difference $X_1 \Delta X_2$ of X_1 and X_2. The operator $*$ is commutative and associative. We moreover have that $X * \mathcal{P} = X$ and $X * X = \mathcal{P}$ for every set X of points.

A *hyperplane* of $DW(2n-1, 2)$ is a proper subset of $\mathcal{P}$ that meets each line of $DW(2n-1, 2)$ in either a singleton or the whole line. If H_1 and H_2 are two distinct hyperplanes of $DW(2n-1, 2)$, then $H_1 * H_2$ is again a hyperplane. We denote by $\widetilde{V_0^*}$ the set of all hyperplanes of $DW(2n-1, 2)$ together with the whole point set $\mathcal{P}$. The set $\widetilde{V_0^*}$ can be given the structure of an $\mathbb{F}_2$-vector space by defining $X_1 + X_2 := X_1 * X_2$, $0 \cdot X := \mathcal{P}$ and $1 \cdot X := X$ for all $X, X_1, X_2 \in \widetilde{V_0^*}$. If $\mathcal{H}$ is a nonempty set of hyperplanes of $DW(2n-1, 2)$, then $\overline{\mathcal{H}}$ denotes the set of all hyperplanes of the form $H_1 * H_2 * \cdots * H_k$, where all H_i's belong to $\mathcal{H}$, i.e. $\overline{\mathcal{H}} = \langle \mathcal{H} \rangle \setminus \{\mathcal{P}\}$, where $\langle \mathcal{H} \rangle$ is the subspace of $\widetilde{V_0^*}$ generated by $\mathcal{H}$.

An isomorphism class $\mathcal{H}$ of hyperplanes of $DW(2n-1, 2)$ is called *universal* if every hyperplane belongs to $\overline{\mathcal{H}}$. A hyperplane of $DW(2n-1, 2)$ is called *universal* if the isomorphism class of hyperplanes to which it belongs is universal. Universal hyperplanes are interesting from a computational point of view. Such hyperplanes have already successfully been used to enumerate all hyperplanes of certain point-line geometries by means of successively applying the $*$-operator.

If x is a point of $DW(2n-1,2)$, then the set $\Gamma_{\leq n-1}(x)$ is a hyperplane of $DW(2n-1,2)$, the so-called *singular hyperplane with center* x. If Π is a hyperplane of $\mathrm{PG}(2n,2)$ intersecting $Q(2n,2)$ in a hyperbolic quadric $Q^+(2n-1,2)$, then by Pasini and Shpectorov [17, Section 1.2] the set of generators of $Q(2n,2)$ not contained in $Q^+(2n-1,2)$ is a hyperplane of $DQ(2n,2)$. Any such hyperplane will be called a *hyperbolic hyperplane* of $DQ(2n,2) \cong DW(2n-1,2)$.

Suppose $\epsilon : DW(2n-1,2) \to \mathrm{PG}(V)$ is a full projective embedding of $DW(2n-1,2)$. Then $\overline{\mathrm{Im}(\epsilon)}$ denotes the set of all points of $\mathrm{PG}(V)$ that are on a line containing two points of $\mathrm{Im}(\epsilon)$. We denote by $\mathcal{H}_\epsilon$ the set of all sets of the form $\epsilon^{-1}(\epsilon(\mathcal{P}) \cap \Pi)$, where Π is some hyperplane of $\mathrm{PG}(V)$. Then every element of $\mathcal{H}_\epsilon$ is a hyperplane of $DW(2n-1,2)$. By Ronan [19, Corollary 2, page 180], we know the following.

Lemma 2.1 ([19]) *A full projective embedding ϵ of $DW(2n-1,2)$ is universal if and only if $\mathcal{H}_\epsilon$ coincides with the set of all hyperplanes of $DW(2n-1,2)$.*

So, we have $\widetilde{V}_0^* = \mathcal{H}_{\widetilde{\epsilon}_0} \cup \{\mathcal{P}\}$ and the map $\Pi \mapsto \widetilde{\epsilon}_0^{-1}(\widetilde{\epsilon}_0(\mathcal{P}) \cap \Pi)$ defines a bijection between the hyperplanes of $\mathrm{PG}(\widetilde{V}_0)$ and the elements of $\mathcal{H}_{\widetilde{\epsilon}_0} = \widetilde{V}_0^* \setminus \{\mathcal{P}\}$.

The following two lemmas will be useful. For a proof of the first lemma, see e.g. [12, Corollary 2.7]. For the second lemma, see Proposition 2.4 and Lemma 4.3 of [1] (see also [18, Lemma 12]).

Lemma 2.2 ([12]) *A full projective embedding ϵ of $DW(2n-1,2)$ is homogeneous if and only if $H^\theta \in \mathcal{H}_\epsilon$ for every $H \in \mathcal{H}_\epsilon$ and every automorphism θ of $DW(2n-1,2)$.*

Lemma 2.3 ([1]) *Suppose $\epsilon : DW(2n-1,2) \to \mathrm{PG}(V)$ is a homogeneous full projective embedding of $DW(2n-1,2)$ and α is a proper subspace of $\mathrm{PG}(V)$. Let $\overline{G}$ denote the induced action of $Aut(DW(2n-1,2))$ on $\mathrm{PG}(V)$. Then:*

(1) *If α is stabilized by $\overline{G}$, then α is disjoint from $\overline{\mathrm{Im}(\epsilon)}$.*

(2) *If α is disjoint from $\overline{\mathrm{Im}(\epsilon)}$, then ϵ/α is homogeneous if and only if α is stabilized by $\overline{G}$.*

The dual polar space $DW(2n-1,2)$ has two nice homogeneous full projective embeddings: the *Grassmann embedding* as described in Cooperstein [9, Proposition 5.1] has vector dimension $\binom{2n}{n} - \binom{2n}{n-2}$, and the *spin-embedding* of $DQ(2n,2) \cong DW(2n-1,2)$ as described in Buekenhout and Cameron [5, Section 7] has vector dimension 2^n.

Suppose $\widetilde{\epsilon}_0 : DW(2n-1,2) \to \mathrm{PG}(\widetilde{V}_0)$ is the universal embedding of $DW(2n-1,2)$. Let A denote the set of all subspaces α of $\mathrm{PG}(\widetilde{V}_0)$ which are disjoint from $\overline{\mathrm{Im}(\widetilde{\epsilon}_0)}$. With respect to set inclusion, A is a poset. We define $\mathcal{E} := \{\widetilde{\epsilon}_0/\alpha \,|\, \alpha \in A\}$. As $\widetilde{\epsilon}_0$ is universal, every full projective embedding of $DW(2n-1,2)$ is isomorphic to an element of $\mathcal{E}$. This element is unique: if $\alpha, \beta \in A$, then $\widetilde{\epsilon}_0/\alpha$ and $\widetilde{\epsilon}_0/\beta$ are isomorphic if and only if $\alpha = \beta$. The pair $(\mathcal{E}, \geq)$ defines a poset isomorphic to $(A, \subseteq)$: if $\alpha, \beta \in A$, then $\alpha \subseteq \beta$ if and only if $\widetilde{\epsilon}_0/\alpha \geq \widetilde{\epsilon}_0/\beta$.

Lemma 2.4 *If ϵ_1 and ϵ_2 are two full projective embeddings of $DW(2n-1,2)$, then $\mathcal{H}_{\epsilon_1} \subseteq \mathcal{H}_{\epsilon_2}$ if and only if $\epsilon_1 \leq \epsilon_2$. As a consequence, the embeddings ϵ_1 and ϵ_2 are isomorphic if and only if $\mathcal{H}_{\epsilon_1} = \mathcal{H}_{\epsilon_2}$.*

Proof. Consider the universal embedding $\widetilde{\epsilon}_0 : DW(2n-1,2) \to \mathrm{PG}(\widetilde{V_0})$. For every hyperplane H of $DW(2n-1,2)$, let Π_H denote the unique hyperplane of $\mathrm{PG}(\widetilde{V_0})$ such that $H = \widetilde{\epsilon}_0^{-1}(\widetilde{\epsilon}_0(\mathcal{P}) \cap \Pi_H)$. If α_i with $i \in \{1,2\}$ is the unique element of A for which $\epsilon_i \cong \widetilde{\epsilon}_0/\alpha_i$, then α_i is the intersection of all hyperplanes Π_H, where $H \in \mathcal{H}_{\epsilon_i}$. So, $\mathcal{H}_{\epsilon_1} \subseteq \mathcal{H}_{\epsilon_2}$ if and only if $\alpha_2 \subseteq \alpha_1$. The latter happens precisely when $\widetilde{\epsilon}_0/\alpha_1 \leq \widetilde{\epsilon}_0/\alpha_2$, i.e. $\epsilon_1 \leq \epsilon_2$. ■

Lemma 2.5 *If ϵ is a full projective embedding of $DW(2n-1,2)$, then $H_1 * H_2 \in \mathcal{H}_\epsilon$ for any two distinct elements $H_1, H_2 \in \mathcal{H}_\epsilon$.*

Proof. Put $\epsilon : DW(2n-1,2) \to \mathrm{PG}(V)$. Let Π_i with $i \in \{1,2\}$ be the unique hyperplane of $\mathrm{PG}(V)$ for which $\epsilon^{-1}(\epsilon(\mathcal{P}) \cap \Pi_i) = H_i$, and let Π_3 denote the third hyperplane through $\Pi_1 \cap \Pi_2$. Then $\epsilon^{-1}(\epsilon(\mathcal{P}) \cap \Pi_3) = H_1 * H_2$ and so $H_1 * H_2 \in \mathcal{H}_\epsilon$. ■

A *subspace* of $DW(2n-1,2)$ is a *set of points* having the property that every line containing at least two points of X has all its points in X. If X is a nonempty subspace, then we denote by $\widetilde{X}$ the subgeometry of $DW(2n-1,2)$ determined by the points of X and the lines of $DW(2n-1,2)$ that have all their points in X. A subspace X is called *convex* if every shortest path between two points of X is again a point of X. If X is a nonempty convex subspace of diameter $m \geq 2$, then $\widetilde{X} \cong DW(2m-1,2)$.

If F is a nonempty convex subspace of $DW(2n-1,2)$, then every full projective embedding $\epsilon : DW(2n-1,2) \to \mathrm{PG}(V)$ of $DW(2n-1,2)$ will induce a full projective embedding ϵ_F of $\widetilde{F}$. The following is a special case of [11, Theorem 1.4].

Lemma 2.6 ([11]) *Suppose F is a convex subspace of diameter at least 2 of $DW(2n-1,2)$. Then the full projective embedding of $\widetilde{F}$ induced by the universal embedding of $DW(2n-1,2)$ is isomorphic to the universal embedding of $\widetilde{F}$.*

For every singular subspace α of dimension $n-1-\delta$, $\delta \in \{0,1,\ldots,n\}$, of $W(2n-1,2)$, we denote by F_α the set of all $(n-1)$-dimensional singular subspaces of $W(2n-1,2)$ containing α. Then F_α is a convex subspace of diameter δ of $DW(2n-1,2)$, and every convex subspace of diameter δ can be obtained this way. This correspondence between singular subspaces of $W(2n-1,2)$ and convex subspaces of $DW(2n-1,2)$ is bijective, and we will say that F_α is the convex subspace of $DW(2n-1,2)$ corresponding to α, or that α is the singular subspace of $W(2n-1,2)$ corresponding to F_α.

Every two points of $DW(2n-1,2)$ at distance δ from each other are contained in a unique convex subspace of diameter δ. Such a convex subspace is a singleton if $\delta = 0$, a line if $\delta = 1$ and the whole point set $\mathcal{P}$ if δ is equal to n. Convex subspaces of diameter 2 are called *quads* and those of diameter $n-1$ are called *maxes*. The maxes are thus the convex subspaces that correspond to the points of $W(2n-1,2)$. Every two distinct intersecting lines are contained in a unique quad.

If x is a point and F a nonempty convex subspace, then F contains a unique point x' nearest to x and $\mathrm{d}(x,y) = \mathrm{d}(x,x') + \mathrm{d}(x',y)$ for every $y \in F$. The point x' is also denoted by $\pi_F(x)$ and called the *projection of x on F*. In the particular case that F is a line L, we thus have that for every point x, there exists a unique point on L nearest to x. That means that $DW(2n-1,2)$

is a so-called *near polygon.* The maximal distance from a point of $DW(2n-1,2)$ to a convex subspace of diameter δ is equal to $n-\delta$. In particular, every point has distance at most 1 from a max.

Suppose F is a convex subspace of diameter δ and let G be a hyperplane of $\widetilde{F}$. Then the points of $DW(2n-1,2)$ at distance at most $n-\delta-1$ from F together with the points x at distance $n-\delta$ from F for which $\pi_F(x) \in G$ is a hyperplane of $DW(2n-1,2)$, called the *extension* of G.

If M_1 and M_2 are two disjoint maxes, then every point $x_1 \in M_1$ is collinear with a unique point of M_2, namely $\pi_{M_2}(x_1)$, and the map $x_1 \mapsto \pi_{M_2}(x_1)$ defines an isomorphism between $\widetilde{M_1}$ and $\widetilde{M_2}$. If two maxes M_1 and M_2 meet, then $M_1 \cap M_2$ is a convex subspace of diameter $n-2$. Suppose $\{x_1, x_2, x_3\}$ is a line of the ambient projective space $\mathrm{PG}(2n-1,2)$ of $W(2n-1,2)$, and denote by M_{x_1}, M_{x_2} and M_{x_3} the maxes corresponding to respectively x_1, x_2 and x_3. If $\{x_1, x_2, x_3\}$ is a singular line of $W(2n-1,2)$, then M_{x_1}, M_{x_2} and M_{x_3} are the three maxes through the convex subspace of diameter $n-2$ of $DW(2n-1,2)$ corresponding to the singular line $\{x_1, x_2, x_3\}$. If $\{x_1, x_2, x_3\}$ is not a singular line, then it is called a *hyperbolic line.* In this case, we call $\{M_{x_1}, M_{x_2}, M_{x_3}\}$ a *hyperbolic set of maxes.* The maxes M_{x_1}, M_{x_2} and M_{x_3} are mutually disjoint and every line meeting two of them also meets the third.

We denote by $\widetilde{W}(2n-1,2)$ the point-line geometry whose points and lines are the points and hyperbolic lines of $W(2n-1,2)$, with incidence being containment. $\widetilde{W}(2n-1,2)$ is called the *geometry of the hyperbolic lines of* $W(2n-1,2)$. Up to isomorphism, there are two full projective embeddings of $\widetilde{W}(2n-1,2)$, see Hall [14, Theorem 3].

(1) Let $Q(2n,2)$ be a nonsingular parabolic quadric of $\mathrm{PG}(2n,2)$ with kernel k. Then the points of $\mathrm{PG}(2n,2)$ not contained in $Q(2n,2) \cup \{k\}$ together with the lines of $\mathrm{PG}(2n,2)$ disjoint from $Q(2n,2)$ define a point-line geometry isomorphic to $\widetilde{W}(2n-1,2)$. This defines the *universal embedding* of $\widetilde{W}(2n-1,2)$.

(2) The projection from the kernel k on a hyperplane of $\mathrm{PG}(2n,2)$ disjoint from k gives rise to the *natural embedding* of $\widetilde{W}(2n-1,2)$, which is the embedding induced by the natural embedding of $W(2n-1,2)$ in $\mathrm{PG}(2n-1,2)$.

Let f be a nondegenerate alternating bilinear form on $V(2n,2)$ defining the symplectic polar space $W(2n-1,2)$. For every nonzero vector $\bar{v} \in V(2n,2)$, we can define the following element of $GL(V(2n,2))$: $\bar{w} \mapsto \bar{w} + f(\bar{v},\bar{w}) \cdot \bar{v}$. Such a linear transformation is called a *symplectic transvection* and leaves the form f invariant. It thus determines an automorphism of $W(2n-1,2)$ and an automorphism $\theta_{\bar{v}}$ of $DW(2n-1,2)$. Such an automorphism of $DW(2n-1,2)$ is a *reflection*: if M denotes the max of $DW(2n-1,2)$ corresponding to the point $\langle \bar{v} \rangle$, then $\theta_{\bar{v}} = \mathcal{R}_M$, with $\mathcal{R}_M$ the *reflection about* M which fixes each point $x \in M$ and maps each point $y \notin M$ to the unique point of the line $y\pi_M(y)$ distinct from y and $\pi_M(y)$. The symplectic group $Sp(V(2n,2),f)$ is generated by all symplectic transvections, see e.g. [7, Theorem 3.6.3]. This implies that the reflections about the maxes generate the full automorphism group of $DW(2n-1,2)$.

3 The hyperbolic embedding ϵ_h of $DW(2n-1,2)$

The following proposition allows to construct (homogeneous) full projective embeddings of $DW(2n-1,2)$. It is basically contained in the unpublished manuscript [4].

Proposition 3.1 *Let $\mathcal{H}$ be a set of hyperplanes of $DW(2n-1,2)$ satisfying the property that for every two distinct points x and y, there exists a hyperplane of $\mathcal{H}$ containing x, but not y. Let $W_{\mathcal{H}}$ denote the vector space of dimension $|\mathcal{H}|$ having a basis $\{\bar{e}_h \,|\, h \in \mathcal{H}\}$ indexed by the elements of $\mathcal{H}$. For every point x of $DW(2n-1,2)$, we define*

$$\epsilon_{\mathcal{H}}(x) := \langle \sum_{x \notin h \in \mathcal{H}} \bar{e}_h \rangle.$$

Then $\epsilon_{\mathcal{H}}$ defines a full projective embedding of $DW(2n-1,2)$ into a subspace $\mathrm{PG}(V_{\mathcal{H}})$ *of* $\mathrm{PG}(W_{\mathcal{H}})$.

Proof. The condition on the set $\mathcal{H}$ of hyperplanes implies that $\epsilon_{\mathcal{H}}$ is injective. We thus only need to show that $\epsilon_{\mathcal{H}}$ maps lines of $DW(2n-1,2)$ to lines of $\mathrm{PG}(W_{\mathcal{H}})$. Let $\{x_1,x_2,x_3\}$ be a line of $DW(2n-1,2)$. Put

$$\epsilon_{\mathcal{H}}(x_i) := \langle \sum_{h \in \mathcal{H}} a_h^{(i)} \bar{e}_h \rangle, \qquad i \in \{1,2,3\},$$

where each $a_h^{(i)}$ is either 0 or 1. If $\{x_1,x_2,x_3\} \subseteq h \in \mathcal{H}$, then $a_h^{(1)} = a_h^{(2)} = a_h^{(3)} = 0$. If $\{x_1,x_2,x_3\} \cap h$ is a singleton, then two of $a_h^{(1)}, a_h^{(2)}, a_h^{(3)}$ are equal to 1 and the third is equal to 0. In any case, we thus have $a_h^{(1)} + a_h^{(2)} + a_h^{(3)} = 0$, implying that $\{\epsilon_{\mathcal{H}}(x_1), \epsilon_{\mathcal{H}}(x_2), \epsilon_{\mathcal{H}}(x_3)\}$ is a line of $\mathrm{PG}(W_{\mathcal{H}})$. ■

Suitable sets of hyperplanes for which Proposition 3.1 can be applied are the isomorphism classes of hyperplanes as the following proposition shows.

Proposition 3.2 *If $\mathcal{H}$ is an isomorphism class of hyperplanes of $DW(2n-1,2)$, then for every two distinct points x and y, there exists a hyperplane of $\mathcal{H}$ containing x, but not y. As a consequence, the full projective embedding $\epsilon_{\mathcal{H}}$ is well-defined.*

Proof. Suppose x and y are two distinct points at distance $\delta \geq 1$ from each other such that any hyperplane of $\mathcal{H}$ containing x also contains y. As the collinearity graph Γ of $DW(2n-1,2)$ is distance-transitive, this implies that every hyperplane containing x also contains $\Gamma_\delta(x)$ and hence also the smallest subspace $\Gamma_{\leq\delta}(x)$ of $DW(2n-1,2)$ containing $\Gamma_\delta(x)$. In particular, every hyperplane of $\mathcal{H}$ containing x also contains $\Gamma_1(x)$. By the vertex-transitivity of Γ, we then see that if a hyperplane of $\mathcal{H}$ contains a point z, then it also contains every point of $\Gamma_1(z)$, in contradiction with the fact that every hyperplane is a proper subset of the point set $\mathcal{P}$ of $DW(2n-1,2)$. ■

The importance of Proposition 3.1 increases when the description of the hyperplanes of (the isomorphism class) $\mathcal{H}$ is more simple or when $|\mathcal{H}|$ is small, for instance close to the embedding rank $\frac{(2^n+1)(2^{n-1}+1)}{3}$ of $DW(2n-1,2)$. If $\mathcal{H}$ is the class of singular hyperplanes, then $\epsilon_{\mathcal{H}}$ is the so-called *minimal full polarized embedding* of $DW(2n-1,2)$ [6] (also called the *near polygon embedding* in [4]), which is isomorphic to the *spin-embedding* of $DW(2n-1,2)$. In this case, we have $|\mathcal{H}| = |\mathcal{P}| = (2+1)(2^2+1)\cdots(2^n+1)$. The smallest isomorphism class of hyperplanes of $DW(2n-1,2)$ that the author is aware of is the class of the hyperbolic hyperplanes. This class only contains $2^{n-1}(2^n+1)$ hyperplanes, which is smaller than three times the embedding rank! Note that if $\mathcal{H}$ is the set of hyperbolic hyperplanes of $DW(2n-1,2)$, then $\epsilon_{\mathcal{H}}$ is precisely the hyperbolic embedding of $DW(2n-1,2)$ defined in Section 1.

Proposition 3.3 *Let $\mathcal{H}$ be a set of hyperplanes of $DW(2n-1,2)$ satisfying the property that for every two distinct points x and y, there exists a hyperplane of $\mathcal{H}$ containing x, but not y. If $\epsilon := \epsilon_{\mathcal{H}}$, then $\mathcal{H}_\epsilon = \overline{\mathcal{H}}$.*

Proof. We continue with the notation of Proposition 3.1. A hyperplane of $\mathrm{PG}(W_{\mathcal{H}})$ is described by an equation of the form $\sum_{h\in\mathcal{H}} a_h X_h = 0$, where $a_h \in \{0,1\}$ for all $h \in \mathcal{H}$ and $\langle \sum_{h\in\mathcal{H}} X_h \bar{e}_h \rangle$ denotes a generic point of $\mathrm{PG}(W_{\mathcal{H}})$. For every $h \in \mathcal{H}$, consider the hyperplane Π_h of $\mathrm{PG}(W_{\mathcal{H}})$ with equation $X_h = 0$. The set of all points of $DW(2n-1,2)$ that are mapped by ϵ into Π_h is precisely h.

Now, consider a generic hyperplane of $\mathrm{PG}(W_{\mathcal{H}})$ described by an equation of the form

$$\sum_{h\in\mathcal{H}} a_h X_h = 0,$$

and denote by $\mathcal{H}'$ the set of all elements $h \in \mathcal{H}$ for which $a_h = 1$. If $\mathcal{H}' = \{h_1, h_2, \ldots, h_k\}$ with $h_1, h_2, \ldots, h_k$ mutually distinct, then the set of all points of $DW(2n-1,2)$ that are mapped into Π is precisely $h_1 * h_2 * \cdots * h_k$. This set is either the whole point set or a hyperplane of $DW(2n-1,2)$. The validity of the proposition now follows from the fact that for every hyperplane π of $\mathrm{PG}(V_{\mathcal{H}})$, there exists a hyperplane of $\mathrm{PG}(W_{\mathcal{H}})$ intersecting $\mathrm{PG}(V_{\mathcal{H}})$ in π. ■

Proposition 3.4 *If $\mathcal{H}$ is an isomorphism class of hyperplanes of $DW(2n-1,2)$, then $\epsilon_{\mathcal{H}}$ is a homogeneous embedding. In particular, the hyperbolic embedding of $DW(2n-1,2)$ is homogeneous.*

Proof. Put $\epsilon := \epsilon_{\mathcal{H}}$ and let H be an arbitrary hyperplane of $\mathcal{H}_\epsilon$. Then by Proposition 3.3, there exist hyperplanes $H_1, H_2, \ldots, H_k \in \mathcal{H}$ such that $H = H_1 * H_2 * \cdots * H_k$. For every automorphism θ of $DW(2n-1,2)$, we then have $H^\theta = H_1^\theta * H_2^\theta * \cdots * H_k^\theta \in \overline{\mathcal{H}} = \mathcal{H}_\epsilon$. By Lemma 2.2, we then know that ϵ is homogeneous. ■

We can now show the following.

Proposition 3.5 *The following are equivalent:*

(1) *The hyperbolic embedding of $DW(2n-1,2)$ is universal.*

(2) *Every hyperbolic hyperplane of $DW(2n-1,2)$ is universal.*

(3) *Let M be a 0-1 matrix over $\mathbb{F}_2$ whose rows are indexed by the generators α of $Q(2n,2)$ and whose columns are indexed by the hyperbolic quadrics h of type $Q^+(2n-1,2)$ on $Q(2n,2)$, where $M_{\alpha,h} = 1$ if and only if α is contained in h. Then M has rank $\frac{(2^n+1)(2^{n-1}+1)}{3}$.*

Proof. The equivalence of (1) and (2) is a consequence of Lemma 2.1, Proposition 3.1 and Proposition 3.3. The vector dimension of the universal embedding of $DW(2n-1,2)$ is equal to $\frac{(2^n+1)(2^{n-1}+1)}{3}$, and every full projective embedding of $DW(2n-1,2)$ having this vector dimension necessarily is universal. By definition of the hyperbolic embedding ϵ_h, the rank of M is equal to the vector dimension of ϵ_h. We thus also see that also (1) and (3) are equivalent. ■

We conjecture that the hyperbolic embedding of $DW(2n-1,2)$ is universal, but were not able to prove that. We were however able to prove the following (see Proposition 7.9):

Suppose $n \geq 3$. Then the hyperbolic embedding of $DW(2n-1,2)$ is universal if and only if the hyperbolic embedding of $DW(2n-3,2)$ is universal and $DW(2n-1,2)$ has universal hyperplanes.

So, if we are able to show that $DW(2n-1,2)$ has universal hyperplanes for every $n \geq 2$, then the hyperbolic embedding of $DW(2n-1,2)$ is universal for every $n \geq 2$ (Corollary 7.10).

As already mentioned in the introduction, the conjecture that the embedding ϵ_h is universal is supported by computer computations of Peter Vandendriessche which showed the validity of this conjecture for $n \leq 7$.

4 The homogeneous embedding $\widetilde{\epsilon_1}$ of $DW(2n-1,2)$

The following proposition will allow us to construct two additional homogeneous full projective embeddings of $\Delta = DW(2n-1,2)$.

Proposition 4.1 *Let $\epsilon : DW(2n-1,2) \to \mathrm{PG}(V)$ be a full projective embedding of $DW(2n-1,2)$, $n \geq 2$. Let x and y be two opposite points of $DW(2n-1,2)$. Let $z_1, z_2, \ldots, z_{2^n-1}$ be the points contained in $\Gamma_1(x) \cap \Gamma_{n-1}(y)$, and for every $i \in \{1, 2, \ldots, 2^n-1\}$, let $\bar{v}_i \in V$ such that $\epsilon(z_i) = \langle \bar{v}_i \rangle$. Put $\Omega_{x,y} := \sum_{i=1}^{2^n-1} \bar{v}_i$. Then $\Omega_{x,y}$ is independent of the pair (x,y) of opposite points of $DW(2n-1,2)$.*

Proof. We first show that $\Omega_{x,y_1} = \Omega_{x,y_2}$ if x, y_1, y_2 are points of $DW(2n-1,2)$ such that $\mathrm{d}(x,y_1) = \mathrm{d}(x,y_2) = n$ and $\mathrm{d}(y_1,y_2) = 1$. Let y_3 be the third point on the line y_1y_2. Then $\mathrm{d}(x,y_3) = n-1$, $|\Gamma_1(x) \cap \Gamma_{n-2}(y_3)| = 2^{n-1}-1$ and $\Gamma_1(x) \cap \Gamma_{n-2}(y_3) = \Big(\Gamma_1(x) \cap \Gamma_{n-1}(y_1)\Big) \cap \Big(\Gamma_1(x) \cap \Gamma_{n-1}(y_2)\Big)$. Put

$$\begin{aligned} \Gamma_1(x) \cap \Gamma_{n-2}(y_3) &= \{z_1, z_2, \ldots, z_{2^{n-1}-1}\}, \\ \Gamma_1(x) \cap \Gamma_{n-1}(y_1) &= \{z_1, z_2, \ldots, z_{2^n-1}\}, \\ \Gamma_1(x) \cap \Gamma_{n-1}(y_2) &= \{z_1', z_2', \ldots, z_{2^n-1}'\} \end{aligned}$$

such that $z_i' = z_i$ for every $i \in \{1, 2, \ldots, 2^{n-1}-1\}$ and $\{x, z_i, z_i'\}$ is a line for every $i \in \{2^{n-1}, 2^{n-1}+1, \ldots, 2^n-1\}$. Let $\bar{v} \in V$ such that $\epsilon(x) = \langle \bar{v} \rangle$. For every $i \in \{1, 2, \ldots, 2^n-1\}$, let $\bar{v}_i, \bar{v}_i' \in V$ such that $\epsilon(z_i) = \langle \bar{v}_i \rangle$ and $\epsilon(z_i') = \langle \bar{v}_i' \rangle$. Then $\bar{v}_i = \bar{v}_i'$ for every $i \in \{1, 2, \ldots, 2^{n-1}-1\}$ and $\bar{v} + \bar{v}_i = \bar{v}_i'$ for every $i \in \{2^{n-1}, 2^{n-1}+1, \ldots, 2^n-1\}$. As 2^{n-1} is even, we have $\Omega_{x,y_1} = \sum_{i=1}^{2^n-1} \bar{v}_i = \sum_{i=1}^{2^n-1} \bar{v}_i' = \Omega_{x,y_2}$.

We next prove that $\Omega_{x_1,y} = \Omega_{x_2,y}$ if x_1, x_2, y are points of $DW(2n-1,2)$ such that $\mathrm{d}(x_1,y) = \mathrm{d}(x_2,y) = n$ and $\mathrm{d}(x_1,x_2) = 1$. Let x_3 denote the third point on the line x_1x_2. We have $\Big(\Gamma_1(x_1) \cap \Gamma_{n-1}(y)\Big) \cap \Big(\Gamma_1(x_2) \cap \Gamma_{n-1}(y)\Big) = \{x_3\}$. Every point of $\Big(\Gamma_1(x_1) \cap \Gamma_{n-1}(y)\Big) \setminus \{x_3\}$ is contained in a unique quad through x_1x_2 as well as every point of $\Big(\Gamma_1(x_2) \cap \Gamma_{n-1}(y)\Big) \setminus \{x_3\}$. So, it suffices to prove the following.

Suppose Q is a quad through x_1x_2. Let p_1, q_1 be the two points of $Q \cap \Gamma_1(x_1) \cap \Gamma_{n-1}(y)$ distinct from x_3, and let p_2, q_2 be the two points of $Q \cap \Gamma_1(x_2) \cap \Gamma_{n-1}(y)$ distinct from x_3. Then $\bar{u}_1 + \bar{v}_1 = \bar{u}_2 + \bar{v}_2$, where $\bar{u}_1, \bar{v}_1, \bar{u}_2, \bar{v}_2$ are the unique vectors of V such that $\langle \bar{u}_1 \rangle = \epsilon(p_1)$, $\langle \bar{u}_2 \rangle = \epsilon(p_2)$, $\langle \bar{v}_1 \rangle = \epsilon(q_1)$ and $\langle \bar{v}_2 \rangle = \epsilon(q_2)$.

Take such a quad Q. Let y' be the unique point of Q nearest to y, i.e. at distance $n-2$ from y. Then $\mathrm{d}(x_3, y') = 1$ and $\mathrm{d}(x_1, y') = \mathrm{d}(x_2, y') = 2$. Through the intersecting lines x_1x_2 and x_3y', there are two (3×3)-subgrids G and G' of Q. Without loss of generality, we may suppose that $p_1, p_2 \in G_1$ and $q_1, q_2 \in G_2$. Then $\{y', p_1, p_2\}$ and $\{y', q_1, q_2\}$ are lines. So, if $\bar{w}$ is the unique vector of V such that $\epsilon(y') = \langle \bar{w} \rangle$, then $\bar{w} + \bar{u}_1 = \bar{u}_2$ and $\bar{w} + \bar{v}_1 = \bar{v}_2$. It follows that $\bar{u}_1 + \bar{v}_1 = \bar{u}_2 + \bar{v}_2$, as we needed to prove.

Now, let Γ denote the graph whose vertices are the pairs (x, y) of opposite points of $DW(2n-1, 2)$, where two distinct pairs (x_1, y_1) and (x_2, y_2) are adjacent whenever either ($x_1 = x_2$ and $y_1 \sim y_2$) or ($x_1 \sim x_2$ and $y_1 = y_2$). By [10] (part 3 of the proof of Lemma 2.1), we know that Γ is connected. The validity of the lemma thus follows from its validity in the two special cases mentioned above. ■

The vector $\Omega_{x,y}$ in Proposition 4.1 will also be denoted by $\Omega_{\Delta,\epsilon}$, or shortly by Ω_ϵ if no confusion is possible.

The following are immediate consequences of Lemma 2.3 and Proposition 4.1.

Corollary 4.2 *Suppose $\epsilon : DW(2n-1, 2) \to \mathrm{PG}(V)$ is a homogeneous full projective embedding of $DW(2n-1, 2)$. Then the subspace $\langle \Omega_\epsilon \rangle$ is stabilized by the induced action of $Aut(DW(2n-1, 2))$ on $\mathrm{PG}(V)$.*

Corollary 4.3 *Suppose $\epsilon : DW(2n-1, 2) \to \mathrm{PG}(V)$ is a homogeneous full projective embedding of $DW(2n-1, 2)$ for which $\Omega_\epsilon \neq \bar{o}$. Then the point $\langle \Omega_\epsilon \rangle$ does not belong to $\overline{\mathrm{Im}(\epsilon)}$ and $\epsilon / \langle \Omega_\epsilon \rangle$ is a homogeneous full projective embedding whose vector dimension is equal to $\dim(V) - 1$.*

Proposition 4.4 *If $\epsilon : DW(2n-1, 2) \to \mathrm{PG}(V)$ is universal, then $\Omega_\epsilon \neq \bar{o}$.*

Proof. Let (x, y) be a pair of opposite points. Put $\Gamma_1(x) \cap \Gamma_{n-1}(y) = \{z_1, z_2, \ldots, z_{2^n-1}\}$ and let $\bar{v}_i$ with $i \in \{1, 2, \ldots, 2^n - 1\}$ be the unique vector of V such that $\epsilon(z_i) = \langle \bar{v}_i \rangle$. By McClurgh [16] and Li [15], $\langle \epsilon(x^\perp) \rangle$ has dimension $2^n - 1$. Hence, $\epsilon(x), \epsilon(z_1), \epsilon(z_2), \ldots, \epsilon(z_{2^n-1})$ are linearly independent points of $\mathrm{PG}(V)$ and $\Omega_\epsilon = \Omega_{x,y} = \bar{v}_1 + \bar{v}_2 + \cdots + \bar{v}_{2^n-1} \neq \bar{o}$. ■

In the case ϵ is the universal embedding of $DW(2n-1, 2)$, we denote the homogeneous embedding $\epsilon / \langle \Omega_\epsilon \rangle$ also by $\widetilde{\epsilon}_1$. By Corollary 4.3, we then know the following.

Corollary 4.5 *The homogeneous full projective embedding $\widetilde{\epsilon}_1$ has vector dimension $\frac{2^{2n-1} + 3 \cdot 2^{n-1} - 2}{3}$.*

If $\epsilon : DW(2n-1, 2) \to \mathrm{PG}(V)$ is a full projective embedding of $DW(2n-1, 2)$ and F is a nonempty convex subspace of $DW(2n-1, 2)$, then ϵ induces a full projective embedding ϵ_F of $\widetilde{F}$ in a subspace $\mathrm{PG}(V_F)$ of $\mathrm{PG}(V)$. If the diameter of F is at least 2, then by Proposition 4.1 we can define a vector $\Omega_{\widetilde{F}, \epsilon_F}$, which we will also denote by $\Omega_{F,\epsilon}$. The following is an immediate consequence of Lemma 2.6 and Proposition 4.4.

Corollary 4.6 *If ϵ is the universal embedding of $DW(2n-1, 2)$, then $\Omega_{F,\epsilon}$ is nonzero for every convex subspace F of diameter at least 2 of $DW(2n-1, 2)$.*

Lemma 4.7 (1) *Let $n \in \mathbb{N}^* \setminus \{0\}$ and $m \in \{-1, 0, \dots, n\}$. Then the number of m-dimensional subspaces of $\mathrm{PG}(n, 2)$ is odd.*

(2) *Let $n \in \mathbb{N}^* \setminus \{0\}$ and $m_1, m_2 \in \{-1, 0, \dots, n\}$ with $m_1 \leq m_2$. Then the number of m_2-dimensional subspaces of $\mathrm{PG}(n, 2)$ containing a given m_1-dimensional subspace of $\mathrm{PG}(n, 2)$ is odd.*

Proof. Part (1) follows from the fact that the Gaussian binomial coefficient $\begin{bmatrix} n+1 \\ m+1 \end{bmatrix}_2$ is odd. As for the second claim, this number is equal to the number of $(m_2 - m_1 - 1)$-dimensional subspaces of $\mathrm{PG}(n - m_1 - 1, 2)$ which is odd by (1). ■

The following can be derived from Lemma 4.7

Lemma 4.8 *Let F_1 and F_2 be two convex subspaces of respective diameters δ_1 and δ_2 such that $F_1 \subseteq F_2$. Let $\delta_3 \in \mathbb{N}$ such that $\delta_1 \leq \delta_3 \leq \delta_2$. Then the number of convex subspaces F_3 of diameter δ_3 satisfying $F_1 \subseteq F_3 \subseteq F_2$ is odd.*

Proof. Let α_1 and α_2 be the singular subspaces of $W(2n-1, 2)$ corresponding to F_1 and F_2, respectively. Then $\dim(\alpha_1) = n - 1 - \delta_1$, $\dim(\alpha_2) = n - 1 - \delta_2$ and $\alpha_2 \subseteq \alpha_1$. The convex subspaces F_3 of diameter δ_3 satisfying $F_1 \subseteq F_3 \subseteq F_2$ correspond to the singular subspaces α_3 of dimension $n - 1 - \delta_3$ of $W(2n-1, 2)$ satisfying $\alpha_2 \subseteq \alpha_3 \subseteq \alpha_1$, and by Lemma 4.7(2), we know that the number of such singular subspaces is odd. ■

Proposition 4.9 *Suppose $\epsilon : DW(2n-1, 2) \to \mathrm{PG}(V)$ is a homogeneous full projective embedding of $DW(2n-1, 2)$. If F is a convex subspace of diameter $\delta \geq 2$ of $DW(2n-1, 2)$ such that $\Omega_{F,\epsilon} = \bar{o}$, then $\Omega_{F',\epsilon} = \bar{o}$ for every convex subspace F' of diameter at least δ of $DW(2n-1, 2)$.*

Proof. Denote by δ' the diameter of F' and let x, y be two points of F' at distance δ' from each other. Let $\mathcal{F}$ denote the set of convex subspaces of diameter δ of $\widetilde{F'}$ that contain the point x. For every $G \in \mathcal{F}$, let z_G denote the unique point of G nearest to y. Then z_G is on a shortest path from y to x and lies at distance δ from x.

Let $\mathcal{L}$ denote the set of lines through x contained in F'. For every $L \in \mathcal{L}$, let x_L denote the unique point on L nearest to y, and let $\bar{v}_L$ denote the unique vector of V such that $\epsilon(x_L) = \langle \bar{v}_L \rangle$. As $\mathrm{d}(z_G, x) = \delta$ and z_G is on a shortest path from y to x, we also know that if $L \subseteq G \in \mathcal{F}$, then x_L is also the unique point on L nearest to z_G.

As ϵ is homogeneous, the fact that $\Omega_{F,\epsilon} = \bar{o}$ implies that $\Omega_{G,\epsilon} = \sum_{L \in \mathcal{L}, L \subseteq G} \bar{v}_L = \bar{o}$ for every $G \in \mathcal{F}$. By Lemma 4.8, we know that the number of elements of $\mathcal{F}$ through a given line of $\mathcal{L}$ is odd. This implies that $\sum_{G \in \mathcal{F}} \Omega_{G,\epsilon} = \sum_{L \in \mathcal{L}} \bar{v}_L$. By the above, the left hand side equals $\bar{o}$. The right hand side is precisely $\Omega_{F',\epsilon}$. ■

Suppose ϵ is a homogeneous full embedding of $DW(2n-1, 2)$ into a projective space $\mathrm{PG}(V)$. Let $\delta \in \{1, 2, \dots, n\}$ be the smallest number having the property that $\Omega_{F,\epsilon} = \bar{o}$ for every convex subspace F of diameter at least $\delta + 1$. Then ϵ is called a *homogeneous embedding of type δ*. This notion is equivalent with the one introduced in [13]. The following proposition provides some alternative proofs for some results in [13].

Proposition 4.10 (1) *The universal embedding of $DW(2n-1, 2)$ has type n.*

(2) *The spin-embedding of $DW(2n-1, 2)$ has type 1.*

(3) *The Grassmann embedding of $DW(2n-1,2)$ has type* 2.

Proof. (1) This is a consequence of Proposition 4.4.

(2) If Q is a quad of $DW(2n-1,2)$, then the full projective embedding of $\widetilde{Q}$ induced by the spin-embedding of $DW(2n-1,2)$ is isomorphic to the spin-embedding of $\widetilde{Q} \cong DW(3,2)$. So, it suffices to prove the claim in the case $n=2$.

The spin-embedding of $DW(3,2) \cong W(2)$ corresponds to the natural embedding ϵ of $W(2)$ in $\mathrm{PG}(3,2)$. This embedding has the property that if x and y are two noncollinear points of $W(2)$ and $x^\perp \cap y^\perp = \{z_1, z_2, z_3\}$, then $\{\epsilon(z_1), \epsilon(z_2), \epsilon(z_3)\}$ is a line of $\mathrm{PG}(3,2)$. This implies that the type of ϵ is equal to 1.

(3) If $n=2$, then the Grassmann embedding is isomorphic to the universal embedding and thus has type 2. In the sequel, we suppose that $n \geq 3$.

If F is a convex subspace of diameter 3 of $DW(2n-1,2)$, then the full projective embedding of $\widetilde{F}$ induced by the Grassmann embedding of $DW(2n-1,2)$ is isomorphic to the Grassmann embedding of $\widetilde{F}$. So, it suffices to prove the claim in the case $n=3$.

Let f be the nondegenerate alternating bilinear form on $V = V(6,2)$ defining $W(5,2)$ and $DW(5,2)$. Let $(\bar{e}_1, \bar{f}_1, \bar{e}_2, \bar{f}_2, \bar{e}_3, \bar{f}_3)$ be a hyperbolic basis of (V,f), i.e. an ordered basis satisfying $f(\bar{e}_i, \bar{f}_i) = 1$ and $f(\bar{e}_i, \bar{e}_j) = f(\bar{f}_i, \bar{f}_j) = f(\bar{e}_i, \bar{f}_j) = 0$ for all $i,j \in \{1,2,3\}$ with $i \neq j$. Then $x = \langle \bar{e}_1, \bar{e}_2, \bar{e}_3 \rangle$ and $y = \langle \bar{f}_1, \bar{f}_2, \bar{f}_3 \rangle$ are two opposite points of $DW(5,2)$. Put $\Gamma_1(x) \cap \Gamma_2(y) = \{z_1, z_2, \ldots, z_7\}$, where

$$z_1 = \langle \bar{e}_1, \bar{e}_2, \bar{f}_3 \rangle, \quad z_2 = \langle \bar{e}_1, \bar{f}_2, \bar{e}_3 \rangle, \quad z_3 = \langle \bar{f}_1, \bar{e}_2, \bar{e}_3 \rangle, \quad z_4 = \langle \bar{e}_1, \bar{e}_2 + \bar{e}_3, \bar{f}_2 + \bar{f}_3 \rangle,$$

$$z_5 = \langle \bar{e}_2, \bar{e}_1 + \bar{e}_3, \bar{f}_1 + \bar{f}_3 \rangle, \quad z_6 = \langle \bar{e}_3, \bar{e}_1 + \bar{e}_2, \bar{f}_1 + \bar{f}_2 \rangle, \quad z_7 = \langle \bar{e}_1 + \bar{e}_2, \bar{e}_2 + \bar{e}_3, \bar{f}_1 + \bar{f}_2 + \bar{f}_3 \rangle.$$

The Grassmann embedding ϵ of $DW(5,2)$ sends a point $\langle \bar{v}_1, \bar{v}_2, \bar{v}_3 \rangle$ of $DW(5,2)$ to the point $\langle \bar{v}_1 \wedge \bar{v}_2 \wedge \bar{v}_3 \rangle$ of $\mathrm{PG}(\bigwedge^3 V)$. The points z_1, z_2 and z_4 of $DW(5,2)$ are contained in the quad corresponding to the point $\langle \bar{e}_1 \rangle$ of $W(5,2)$. The fact that $\bar{e}_1 \wedge \bar{e}_2 \wedge \bar{f}_3 + \bar{e}_1 \wedge \bar{f}_2 \wedge \bar{e}_3 + \bar{e}_1 \wedge (\bar{e}_2 + \bar{e}_3) \wedge (\bar{f}_2 + \bar{f}_3) \neq \bar{o}$ implies that ϵ does not have type 1. The fact that $\bar{e}_1 \wedge \bar{e}_2 \wedge \bar{f}_3 + \bar{e}_1 \wedge \bar{f}_2 \wedge \bar{e}_3 + \bar{f}_1 \wedge \bar{e}_2 \wedge \bar{e}_3 + \bar{e}_1 \wedge (\bar{e}_2 + \bar{e}_3) \wedge (\bar{f}_2 + \bar{f}_3) + \bar{e}_2 \wedge (\bar{e}_1 + \bar{e}_3) \wedge (\bar{f}_1 + \bar{f}_3) + \bar{e}_3 \wedge (\bar{e}_1 + \bar{e}_2) \wedge (\bar{f}_1 + \bar{f}_2) + (\bar{e}_1 + \bar{e}_2) \wedge (\bar{e}_2 + \bar{e}_3) \wedge (\bar{f}_1 + \bar{f}_2 + \bar{f}_3) = \bar{o}$ then implies that ϵ has type 2. ∎

In Section 7 (Proposition 7.2), we show that the hyperbolic embedding ϵ_h has type n.

Proposition 4.11 *Suppose ϵ and ϵ' are two homogeneous full projective embeddings of $DW(2n-1,2)$ such that $\epsilon \leq \epsilon'$ and ϵ has type at most $n-1$. Then $\epsilon \leq \epsilon'/\langle \Omega_{\epsilon'} \rangle$.*

Proof. Suppose $\epsilon : DW(2n-1,2) \to \mathrm{PG}(V)$ and $\epsilon' : DW(2n-1,2) \to \mathrm{PG}(V')$. Let α be the unique subspace of $\mathrm{PG}(V')$ disjoint from $\overline{\mathrm{Im}(\epsilon')}$ such that ϵ is isomorphic to ϵ'/α. Since ϵ has type at most $n-1$, we have $\Omega_\epsilon = \bar{o}$. This implies that $\langle \Omega_{\epsilon'} \rangle \subseteq \alpha$. So, $\epsilon'/\alpha \cong \epsilon$ is isomorphic to a quotient of $\epsilon'/\langle \Omega_{\epsilon'} \rangle$. ∎

The following is an immediate consequence of Proposition 4.11.

Corollary 4.12 *Suppose ϵ is a homogeneous full projective embedding of $DW(2n-1,2)$ of type at most $n-1$. Then ϵ is isomorphic to a quotient of $\widetilde{\epsilon}_1$.*

Proposition 4.13 (1) *If $n = 2$, then $\widetilde{\epsilon}_1$ is isomorphic to the natural embedding of $DW(3,2) \cong W(2)$ in* PG(3, 2)*.*

(2) *If $n = 3$, then $\widetilde{\epsilon}_1$ is isomorphic to the Grassmann embedding of $DW(5,2)$.*

Proof. If $n = 2$, then by Corollary 4.5, $\widetilde{\epsilon}_1$ has vector dimension 4 and so is isomorphic to the natural embedding of $DW(3,2) \cong W(2)$ in PG(3, 2). If $n = 3$, then $\widetilde{\epsilon}_1$ has vector dimension 14 by Corollary 4.5. By Proposition 4.10(3) and Corollary 4.12, the Grassmann embedding is isomorphic to a quotient of $\widetilde{\epsilon}_1$. The Grassmann embedding, which also has vector dimension 14, must therefore be isomorphic to $\widetilde{\epsilon}_1$. ∎

5 The homogeneous embedding $\widetilde{\epsilon}_2$ of $DW(2n-1,2)$

Lemma 5.1 *Suppose $\{F_1, F_2, F_3\}$ is a hyperbolic set of maxes of $DW(2n-1,2)$, $n \geq 3$. Then for every full projective embedding $\epsilon : DW(2n-1,2) \to \mathrm{PG}(V)$, we have $\Omega_{F_1,\epsilon} + \Omega_{F_2,\epsilon} + \Omega_{F_3,\epsilon} = \bar{o}$.*

Proof. Let x_1 and y_1 be two points of F_1 at maximal distance $n-1$ from each other. For every $i \in \{2,3\}$, let x_i and y_i be the unique points of F_i collinear with respectively x_1 and y_1. Then x_i and y_i are two opposite points of F_i. Moreover, $\{x_1, x_2, x_3\}$ and $\{y_1, y_2, y_3\}$ are two lines of $DW(2n-1,2)$.

For every $i \in \{2,3\}$, let π_i be the projection map from $\widetilde{F_1}$ to $\widetilde{F_i}$. Then π_1 and π_2 are isomorphisms. So, the map π_i defines a bijection between the sets $\Gamma_1(x_1) \cap \Gamma_{n-2}(y_1)$ and $\Gamma_1(x_i) \cap \Gamma_{n-2}(y_i)$.

Now, take a $z_1 \in \Gamma_1(x_1) \cap \Gamma_{n-2}(y_1)$. Put $z_2 := \pi_2(z_1)$ and $z_3 := \pi_3(z_1)$, and let $\bar{v}_i$ with $i \in \{1,2,3\}$ be the unique vector of V such that $\epsilon(z_i) = \langle \bar{v}_i \rangle$. Since $\{z_1, z_2, z_3\}$ is a line of $DW(2n-1,2)$, we have that $A(z_1) := \bar{v}_1 + \bar{v}_2 + \bar{v}_3 = \bar{o}$. Taking the sum over all $z_1 \in \Gamma_1(x_1) \cap \Gamma_{n-2}(y_1)$, we find that $\bar{o} = \sum A(z_1) = \Omega_{F_1,\epsilon} + \Omega_{F_2,\epsilon} + \Omega_{F_3,\epsilon}$. ∎

Lemma 5.2 *Let F be a convex subspace of diameter $n-2$ of $DW(2n-1,2)$, $n \geq 3$, and let F_1, F_2, F_3 denote the three maxes through F. If ϵ is a full projective embedding of $DW(2n-1,2)$, then $\Omega_{F_1,\epsilon} + \Omega_{F_2,\epsilon} + \Omega_{F_2,\epsilon} = \Omega_\epsilon$.*

Proof. Let $x \in F$, let y be a point opposite to x, let y_i with $i \in \{1,2,3\}$ denote the unique point of F_i collinear with y, and let z denote the unique point of F at distance 2 from y. Then $\mathrm{d}(y,u_i) = \mathrm{d}(y,y_i) + \mathrm{d}(y_i,u_i) = 1 + \mathrm{d}(y_i,u_i)$ for every $u_i \in F_i$ and $\mathrm{d}(y,u) = \mathrm{d}(y,z) + \mathrm{d}(z,u) = 2 + \mathrm{d}(z,u)$ for every $u \in F$. This implies that $\mathrm{d}(x,y_1) = \mathrm{d}(x,y_2) = \mathrm{d}(x,y_3) = n-1$, $\mathrm{d}(x,z) = n-2$ and $z \in \Gamma_1(y_1) \cap \Gamma_1(y_2) \cap \Gamma_1(y_3)$. Moreover, each of the sets $\Gamma_{n-2}(y_1) \cap \Gamma_1(x) \subseteq F_1$, $\Gamma_{n-2}(y_2) \cap \Gamma_1(x) \subseteq F_2$, $\Gamma_{n-2}(y_3) \cap \Gamma_1(x) \subseteq F_3$, $\Gamma_{n-3}(z) \cap \Gamma_1(x) \subseteq F$ is contained in $\Gamma_{n-1}(y) \cap \Gamma_1(x)$. As every point of $\Gamma_1(x)$ is contained in $F_1 \cup F_2 \cup F_3$, we have $\Gamma_{n-1}(y) \cap \Gamma_1(x) = \Big(\Gamma_{n-1}(y) \cap F_1 \cap \Gamma_1(x)\Big) \cup \Big(\Gamma_{n-1}(y) \cap F_2 \cap \Gamma_1(x)\Big) \cup \Big(\Gamma_{n-1}(y) \cap F_3 \cap \Gamma_1(x)\Big) = \Big(\Gamma_{n-2}(y_1) \cap \Gamma_1(x)\Big) \cup \Big(\Gamma_{n-2}(y_2) \cap \Gamma_1(x)\Big) \cup \Big(\Gamma_{n-2}(y_3) \cap \Gamma_1(x)\Big)$. Also, as $\Gamma_{n-2}(y_1) \cap \Gamma_1(x) \subseteq F_1$, $\Gamma_{n-2}(y_2) \cap \Gamma_1(x) \subseteq F_2$, $\Gamma_{n-2}(y_3) \cap \Gamma_1(x) \subseteq F_3$, $F_1 \cap F_2 = F_1 \cap F_3 = F_2 \cap F_3 = F$ and $\Gamma_{n-2}(y_i) \cap \Gamma_1(x) \cap F = \Gamma_{n-3}(z) \cap \Gamma_1(x)$ for every $i \in \{1,2,3\}$, the sets $\Gamma_{n-2}(y_1) \cap \Gamma_1(x)$, $\Gamma_{n-2}(y_2) \cap \Gamma_1(x)$, $\Gamma_{n-2}(y_3) \cap \Gamma_1(x)$ mutually intersect in $\Gamma_{n-3}(z) \cap \Gamma_1(x)$. The latter fact in combination with $\Big(\Gamma_{n-2}(y_1) \cap \Gamma_1(x)\Big) \cup \Big(\Gamma_{n-2}(y_2) \cap \Gamma_1(x)\Big) \cup \Big(\Gamma_{n-2}(y_3) \cap \Gamma_1(x)\Big) = \Gamma_{n-1}(y) \cap \Gamma_1(x)$ implies that $\Omega_{F_1,\epsilon} + \Omega_{F_2,\epsilon} + \Omega_{F_3,\epsilon} = \Omega_\epsilon$. ∎

Lemma 5.3 *Suppose $\epsilon : DW(2n-1,2) \to \mathrm{PG}(V)$ is a homogeneous full projective embedding of $DW(2n-1,2)$, $n \geq 3$, of type at least $n-1$. Then $\Omega_{M_1,\epsilon} \neq \Omega_{M_2,\epsilon}$ for every two distinct maxes M_1 and M_2 of $DW(2n-1,2)$.*

Proof. We first prove this in the case when M_1 and M_2 are disjoint. Then M_1 and M_2 are contained in a hyperbolic set $\{M_1, M_2, M_3\}$ of maxes. From Lemma 5.1, we have $\Omega_{M_1,\epsilon} + \Omega_{M_2,\epsilon} + \Omega_{M_3,\epsilon} = \bar{o}$. Since $\Omega_{M_3,\epsilon} \neq \bar{o}$, we have $\Omega_{M_1,\epsilon} \neq \Omega_{M_2,\epsilon}$.

We next prove this in the case where M_1 and M_2 meet (necessarily in a convex subspace of diameter $n-2$). Suppose that $\Omega_{M_1,\epsilon} = \Omega_{M_2,\epsilon}$. Let M_3 denote a max disjoint from M_1 that intersects M_2 in a convex subspace of diameter $n-2$. Then there exists an automorphism of $DW(2n-1,2)$ fixing M_2 and mapping M_1 to M_3. So, there exists an automorphism of $\mathrm{PG}(V)$ fixing $\Omega_{M_2,\epsilon}$ and mapping $\Omega_{M_1,\epsilon}$ to $\Omega_{M_3,\epsilon}$. As $\Omega_{M_1,\epsilon} = \Omega_{M_2,\epsilon}$, we have $\Omega_{M_3,\epsilon} = \Omega_{M_2,\epsilon}$, i.e. $\Omega_{M_1,\epsilon} = \Omega_{M_3,\epsilon}$. This is in contradiction with the first part of the proof. So, also in this case, we have $\Omega_{M_1,\epsilon} \neq \Omega_{M_2,\epsilon}$. ■

Proposition 5.4 *Suppose $\epsilon : DW(2n-1,2) \to \mathrm{PG}(V)$ is a homogeneous full projective embedding of $DW(2n-1,2)$, $n \geq 3$, of type at least $n-1$. For every point x of $W(2n-1,2)$, put $\eta(x) := \langle \Omega_{M_x,\epsilon} \rangle$, where M_x is the max of $DW(2n-1,2)$ corresponding to x. Then η defines a full embedding of $\widetilde{W}(2n-1,2)$ into a subspace of $\mathrm{PG}(V)$. If the type of ϵ is equal to n, then η is isomorphic to the universal embedding of $\widetilde{W}(2n-1,2)$. If the type of ϵ is $n-1$, then η is isomorphic to the natural embedding of $\widetilde{W}(2n-1,2)$.*

Proof. The fact that η defines a full projective embedding is a consequence of Lemmas 5.1 and 5.3, taking into account that if $\{x_1, x_2, x_3\}$ is a hyperbolic line, then $\{M_{x_1}, M_{x_2}, M_{x_3}\}$ is a hyperbolic set of maxes. We have that ϵ has type $n-1$ if and only if $\Omega_\epsilon = \bar{o}$. Lemmas 5.1, 5.2 imply that η is the natural embedding of $\widetilde{W}(2n-1,2)$ if and only if $\Omega_\epsilon = \bar{o}$. All claims of the proposition are now clear. ■

Suppose $\epsilon : DW(2n-1,2) \to \mathrm{PG}(V)$ is a homogeneous full projective embedding of $DW(2n-1,2)$, $n \geq 3$. If the type of ϵ is at least $n-1$, then we denote by Σ_ϵ the subspace of $\mathrm{PG}(V)$ generated by all points $\langle \Omega_{M,\epsilon} \rangle$, where M is a max of $DW(2n-1,2)$. If the type of ϵ is at most $n-2$, then we define $\Sigma_\epsilon := \emptyset$. The following is an immediate consequence of Proposition 5.4.

Corollary 5.5 *Suppose ϵ is a homogeneous full projective embedding of $DW(2n-1,2)$, $n \geq 3$. Then the projective dimension of Σ_ϵ is equal to $2n$ if ϵ has type n, equal to $2n-1$ if ϵ has type $n-1$ and equal to -1 otherwise.*

Proposition 5.6 *Suppose $\epsilon : DW(2n-1,2) \to \mathrm{PG}(V)$ is a homogeneous full projective embedding of $DW(2n-1,2)$, $n \geq 3$. Then Σ_ϵ is disjoint from $\overline{\mathrm{Im}(\epsilon)}$ and the quotient embedding ϵ/Σ_ϵ is also homogeneous.*

Proof. As the number of points of Σ_ϵ is smaller than the total number $(2+1)(2^2+1)\cdots(2^n+1)$ of points of $DW(2n-1,2)$, Σ_ϵ must be a proper subspace of $\mathrm{PG}(V)$. By definition of Σ_ϵ, this subspace must be stabilized by the induced action of the automorphism group of $DW(2n-1,2)$ on $\mathrm{PG}(V)$. Lemma 2.3 then implies that Σ_ϵ is disjoint from $\overline{\mathrm{Im}(\epsilon)}$ and that the quotient embedding ϵ/Σ_ϵ is homogeneous. ■

In the case ϵ is the universal embedding of $DW(2n-1,2)$, $n \geq 3$, we denote the homogeneous embedding ϵ/Σ_ϵ also by $\widetilde{\epsilon}_2$. As the universal embedding of $DW(2n-1,2)$ has vector dimension $\frac{(2^n+1)(2^{n-1}+1)}{3}$, Proposition 4.10(1) and Corollary 5.5 imply the following.

Corollary 5.7 *The vector dimension of the homogeneous full projective embedding $\widetilde{\epsilon}_2$ is equal to $\frac{2^{2n-1}+3\cdot 2^{n-1}-2-6n}{3}$.*

Proposition 5.8 *Suppose ϵ and ϵ' are two homogeneous full projective embeddings of $DW(2n-1,2)$, $n \geq 3$, such that $\epsilon \leq \epsilon'$ and ϵ has type at most $n-2$. Then $\epsilon \leq \epsilon'/\Sigma_{\epsilon'}$.*

Proof. Suppose $\epsilon : DW(2n-1,2) \to \mathrm{PG}(V)$ and $\epsilon' : DW(2n-1,2) \to \mathrm{PG}(V')$. Let α be the unique subspace of $\mathrm{PG}(V')$ disjoint from $\overline{\mathrm{Im}(\epsilon')}$ such that ϵ is isomorphic to ϵ'/α. Since ϵ has type at most $n-2$, we have $\Omega_{M,\epsilon} = \bar{o}$ for every max M. This implies that $\langle \Omega_{M,\epsilon'} \rangle \subseteq \alpha$ for every max M, or equivalently that $\Sigma_{\epsilon'}$ is contained in α. So, $\epsilon'/\alpha \cong \epsilon$ is isomorphic to a quotient of $\epsilon'/\Sigma_{\epsilon'}$. ∎

The following is an immediate consequence of Proposition 5.8.

Corollary 5.9 *Suppose ϵ is a homogeneous full projective embedding of $DW(2n-1,2)$, $n \geq 3$, of type at most $n-2$. Then ϵ is isomorphic to a quotient of $\widetilde{\epsilon}_2$.*

Proposition 5.10 (1) *If $n = 3$, then $\widetilde{\epsilon}_2$ is isomorphic to the spin-embedding of $DW(5,2)$.*

(2) *If $n = 4$, then $\widetilde{\epsilon}_2$ is isomorphic to the Grassmann embedding of $DW(7,2)$.*

Proof. (1) By Proposition 4.10(2) and Corollary 5.9, we know that the spin-embedding of $DW(5,2)$ is isomorphic to a quotient of $\widetilde{\epsilon}_2$. But by Corollary 5.7, we know that both embeddings have the same vector dimension, namely 8. So, they must be isomorphic.

(2) A similar argument applies to the second claim of the proposition. By Proposition 4.10(3) and Corollary 5.9, we know that the Grassmann embedding of $DW(7,2)$ is isomorphic to a quotient of $\widetilde{\epsilon}_2$, and again both embeddings have the same dimension by Corollary 5.7. ∎

Proposition 5.11 *Suppose $\epsilon : DW(2n-1,2) \to \mathrm{PG}(V)$ is a homogeneous full embedding of type n of $DW(2n-1,2)$, $n \geq 3$, and put $\epsilon' := \epsilon/\langle \Omega_\epsilon \rangle$. Then the embeddings ϵ/Σ_ϵ and $\epsilon'/\Sigma_{\epsilon'}$ are isomorphic.*

Proof. The type of the homogeneous embedding ϵ/Σ_ϵ is smaller that n and so ϵ/Σ_ϵ is isomorphic to a quotient of ϵ' by Proposition 4.11. As ϵ/Σ_ϵ has type at most $n-2$, it is isomorphic to a quotient of $\epsilon'/\Sigma_{\epsilon'}$ by Proposition 5.8.

As $\epsilon'/\Sigma_{\epsilon'}$ is a quotient of ϵ', which itself is a quotient if ϵ, we see that $\epsilon'/\Sigma_{\epsilon'}$ is isomorphic to a quotient of ϵ. As $\epsilon'/\Sigma_{\epsilon'}$ has type at most $n-2$, we see that $\epsilon'/\Sigma_{\epsilon'}$ is isomorphic to a quotient of ϵ/Σ_ϵ by Proposition 5.8.

Combining the above two paragraphs, we thus see that the embeddings ϵ/Σ_ϵ and $\epsilon'/\Sigma_{\epsilon'}$ are isomorphic. ∎

Proposition 5.12 (1) *The embedding $\widetilde{\epsilon}_1$ has type $n-1$.*

(2) *If $n \geq 3$, then the embedding $\widetilde{\epsilon}_2$ has type $n-2$.*

Proof. (1) The embedding $\widetilde{\epsilon}_1$ has type at most $n-1$. If $\widetilde{\epsilon}_1$ would have type at most $n-2$, then $\widetilde{\epsilon}_1$ would be isomorphic to a quotient of $\widetilde{\epsilon}_2$ by Corollary 5.9. But that is impossible as the vector dimension of $\widetilde{\epsilon}_1$ is bigger than the one of $\widetilde{\epsilon}_2$.

(2) By Proposition 5.11 applied to $\epsilon = \widetilde{\epsilon}_0$, we know that $\widetilde{\epsilon}_2$ is isomorphic to $\widetilde{\epsilon}_1/\Sigma_{\widetilde{\epsilon}_1}$. As $\widetilde{\epsilon}_1$ has type $n-1$, we know from Proposition 5.4 that every point of $\Sigma_{\widetilde{\epsilon}_1}$ is of the form $\langle \Omega_{M,\widetilde{\epsilon}_1} \rangle$ for some max M of $DW(2n-1,2)$.

The embedding $\widetilde{\epsilon}_2$ has type at most $n-2$. Suppose $\widetilde{\epsilon}_2$ has type at most $n-3$. Then $n \geq 4$ and as $\widetilde{\epsilon}_2 \cong \widetilde{\epsilon}_1/\Sigma_{\widetilde{\epsilon}_1}$, there exists a convex subspace F of diameter $n-2$ and a max M of $DW(2n-1,2)$ such that $\langle \Omega_{F,\widetilde{\epsilon}_1} \rangle = \langle \Omega_{M,\widetilde{\epsilon}_1} \rangle$.

We prove that there exist maxes M' and M'' and a hyperbolic set $\{F, F', F''\}$ of maxes of $\widetilde{M'}$ such that $\mathcal{R}_{M''}(M) = M$ and $\mathcal{R}_{M''}(F) = F'$. We distinguish three cases.

(i) If $F \subseteq M$, then $M' := M$, $\{F, F', F''\}$ is an arbitrary hyperbolic set of maxes of $\widetilde{M}$ containing F and M'' is any max through F'' distinct from M.

(ii) If F is disjoint from M, then $F' := \mathcal{R}_M(F)$, $F'' := \pi_M(F)$, M' is the unique max containing $F \cup \pi_M(F)$ and $M'' := M$.

(iii) Suppose F meets M in a convex subspace of diameter $n-3$. Let M'' denote a max intersecting M in a convex subspace of diameter $n-2$ disjoint from $F \cap M$, put $F' := \mathcal{R}_{M''}(F)$, $F'' := \pi_{M''}(F)$ and let M' denote the unique max containing F, F' and F''.

So, let M', M'', F' and F'' as above. As $\widetilde{\epsilon}_1$ is homogeneous, $\langle \Omega_{F,\widetilde{\epsilon}_1} \rangle = \langle \Omega_{M,\widetilde{\epsilon}_1} \rangle$, $\mathcal{R}_{M''}(M) = M$ and $\mathcal{R}_{M''}(F) = F'$, we should have $\langle \Omega_{F',\widetilde{\epsilon}_1} \rangle = \langle \Omega_{M,\widetilde{\epsilon}_1} \rangle$, i.e. $\langle \Omega_{F',\widetilde{\epsilon}_1} \rangle = \langle \Omega_{F,\widetilde{\epsilon}_1} \rangle$. By Lemma 5.1 applied to the dual polar space $\widetilde{M'}$, we would then have $\Omega_{F'',\widetilde{\epsilon}_1} = \bar{o}$, in contradiction with the fact that $\widetilde{\epsilon}_1$ has type $n-1$.

So, $\widetilde{\epsilon}_2$ should have type $n-2$. ■

Remark. In Section 7.5 of [13], it was shown that for every $i \in \{1, 2, \ldots, n\}$, there exists a universal homogeneous projective embedding of type i of $DW(2n-1,2)$. By Corollaries 4.12, 5.9 and Proposition 5.12, we know that $\widetilde{\epsilon}_1$ (respectively, $\widetilde{\epsilon}_2$) is isomorphic to the universal homogeneous embedding of type $n-1$ (respectively, type $n-2$).

6 Another description of $\widetilde{\epsilon}_1$ in case universal hyperplanes exist

Lemma 6.1 *Let H be a hyperplane and M a max of $DW(2n-1,2)$, $n \geq 2$. If $M \subseteq H$, then $H * \mathcal{R}_M(H)$ is the whole point set $\mathcal{P}$. If M is not contained in H, then $H * \mathcal{R}_M(H)$ is the extension of the hyperplane $H \cap M$ of $\widetilde{M}$.*

Proof. Put $\mathcal{R} := \mathcal{R}_M$. Let $L = \{x, y, z\}$ be a line intersecting M in a point x. We distinguish three cases.

(1) $x \notin H$. Then without loss of generality we may suppose that y is the unique point of $L \cap H$. Then z is the unique point of $L \cap \mathcal{R}(H)$ and it follows that x is the unique point of L contained in $H * \mathcal{R}(H)$.

(2) $L \subseteq H$. Then $L \subseteq \mathcal{R}(H)$ and hence L is contained in $H * \mathcal{R}(H)$.

(3) $L \cap H = \{x\}$. Then $L \cap \mathcal{R}(H) = \{x\}$ and again it follows that $L \subseteq H * \mathcal{R}(H)$.

Note that every point u of $DW(2n-1,2)$ not contained in M is incident with a unique line meeting M (in the point $\pi_M(u)$). So, if $M \subseteq H$, then (2) and (3) imply that $H * \mathcal{R}_M(H)$ is the whole point set. If M is not contained in H, then (1), (2) and (3) imply that $H * \mathcal{R}_M(H)$ is the extension of the hyperplane $H \cap M$ of $\widetilde{M}$. ■

Lemma 6.2 *Suppose $\epsilon : DW(2n-1,2) \to \mathrm{PG}(V)$ is a homogeneous full projective embedding of $DW(2n-1,2)$, $n \geq 2$. Suppose $H \in \mathcal{H}_\epsilon$ and M is a max not contained in H. Then the extension of the hyperplane $H \cap M$ of $\widetilde{M}$ also belongs to $\mathcal{H}_\epsilon$.*

Proof. Let $\mathcal{R}$ denote the reflection about M. By Lemma 6.1, $H * \mathcal{R}(H)$ is the extension of the hyperplane $H \cap M$ of $\widetilde{M}$ (and so $H \neq \mathcal{R}(H)$). Since ϵ is homogeneous and $H \in \mathcal{H}_\epsilon$, we have $\mathcal{R}(H) \in \mathcal{H}_\epsilon$ by Lemma 2.2. By Lemma 2.5, we then know that also $H * \mathcal{R}(H) \in \mathcal{H}_\epsilon$. ■

Proposition 6.3 *Let H be a hyperplane that arises by extending a hyperplane of a max of $DW(2n-1,2)$, $n \geq 2$. Then $H \in \mathcal{H}_{\widetilde{\epsilon}_1}$.*

Proof. Suppose H is the extension of a hyperplane G of a max M of $DW(2n-1,2)$. Let M' be a max disjoint from M and put $H_1 := \mathcal{R}_{M'}(H)$. If $\{M, M', M''\}$ is the hyperbolic set of maxes containing M and M', then H_1 is the extension of the hyperplane $\pi_{M''}(G)$ of $\widetilde{M''}$ and $H_1 \cap M = G$. By Lemma 6.1, $H = H_1 * \mathcal{R}_M(H_1)$.

Now, as the embedding $\widetilde{\epsilon}_1$ has vector dimension one less that the embedding rank of $DW(2n-1,2)$, $\mathcal{H}_{\widetilde{\epsilon}_1} \cup \{\mathcal{P}\}$ is a hyperplane of $\widetilde{V}_0^*$. By Lemma 2.2, H_1 and $\mathcal{R}_M(H_1)$ belong both to $\mathcal{H}_{\widetilde{\epsilon}_1}$ or both to $\mathcal{H}_{\widetilde{\epsilon}_0} \setminus \mathcal{H}_{\widetilde{\epsilon}_1}$. In any case, we have that $H = H_1 * \mathcal{R}_M(H_1)$ belongs to $H_{\widetilde{\epsilon}_1}$. ■

The following is a consequence of Lemmas 2.4, 2.5 and Propositions 3.1, 3.3, 6.3.

Corollary 6.4 *Let $\mathcal{H}$ be the set*[1] *of hyperplanes of $DW(2n-1,2)$, $n \geq 2$, that arise by extending a hyperplane of a max of $DW(2n-1,2)$. Then $\epsilon_\mathcal{H}$ is isomorphic to a quotient of $\widetilde{\epsilon}_1$.*

Lemma 6.5 *Suppose H is a hyperplane of $DW(2n-1,2)$, $n \geq 2$, let $\mathcal{H}_1$ be the isomorphism class of hyperplanes containing H and let $\mathcal{H}_2$ be the smallest set of hyperplanes of $DW(2n-1,2)$ that satisfies the following:*

- *if M is a max not contained in H, then the extension of the hyperplane $H \cap M$ of $\widetilde{M}$ belongs to $\mathcal{H}_2$;*
- *if $H \in \mathcal{H}_2$, then also $H^\theta \in \mathcal{H}_2$ for every $\theta \in Aut(DW(2n-1,2))$.*

Then the subspace $\langle \mathcal{H}_2, H \rangle$ of $\widetilde{V}_0^*$ generated by $\mathcal{H}_2$ and H coincides with $\langle \mathcal{H}_1 \rangle = \overline{\mathcal{H}_1} \cup \{\mathcal{P}\}$.

Proof. By Lemma 6.1, $\mathcal{H}_2 \subseteq \overline{\mathcal{H}_1}$ and hence $\langle \mathcal{H}_2, H \rangle \subseteq \overline{\mathcal{H}_1} \cup \{\mathcal{P}\}$. In order to show that $\overline{\mathcal{H}_1} \cup \{\mathcal{P}\} \subseteq \langle \mathcal{H}_2, H \rangle$, it suffices to prove that $H * H^\theta \in \langle \mathcal{H}_2 \rangle$ for every automorphism θ of $DW(2n-1,2)$. We can put $\theta = \theta_1 \theta_2 \cdots \theta_k$, where each θ_i is a reflection about a max. But then

$$H * H^\theta = (H * H^{\theta_1}) * (H^{\theta_1} * H^{\theta_1 \theta_2}) * \cdots * (H^{\theta_1 \theta_2 \cdots \theta_{k-1}} * H^{\theta_1 \theta_2 \cdots \theta_k})$$

[1]This set of hyperplanes is easily seen to satisfy the property mentioned in Proposition 3.1.

By Lemma 6.1, each term $H^{\theta_1\cdots\theta_{i-1}} * H^{\theta_1\cdots\theta_i}$, $i \in \{1,2,\dots,k\}$, belongs to $\mathcal{H}_2 \cup \{\mathcal{P}\}$, implying that $H * H^\theta$ belongs to $\langle \mathcal{H}_2 \rangle$. ■

Proposition 6.6 *Let $\mathcal{H}$ be the set of hyperplanes of $DW(2n-1,2)$, $n \geq 2$, that arises by extending a hyperplane of a max of $DW(2n-1,2)$. If $DW(2n-1,2)$ has universal hyperplanes, then $\widetilde{\epsilon}_1$ is isomorphic to $\epsilon_{\mathcal{H}}$.*

Proof. Let H be a universal hyperplane of $DW(2n-1,2)$ and let $\mathcal{H}_2$ denote the set of hyperplanes of $DW(2n-1,2)$ as defined in Lemma 6.5. By Proposition 6.3, $\mathcal{H}_2 \subseteq \mathcal{H}_{\widetilde{\epsilon}_1}$. Let $\mathcal{H}_1$ be the isomorphism class of hyperplanes containing H. Then $\overline{\mathcal{H}_1} = \mathcal{H}_{\widetilde{\epsilon}_0}$ as H is universal. If $H \in \mathcal{H}_{\widetilde{\epsilon}_1}$, then Lemmas 2.2 and 2.5 imply that $\overline{\mathcal{H}_1} \subseteq \mathcal{H}_{\widetilde{\epsilon}_1}$, a contradiction. Hence, $H \notin \mathcal{H}_{\widetilde{\epsilon}_1}$. By Lemma 6.5, $\overline{\mathcal{H}_1} \cup \{\mathcal{P}\} = \mathcal{H}_{\widetilde{\epsilon}_0} \cup \{\mathcal{P}\} = \widetilde{V}_0^*$ is generated by H and $\mathcal{H}_2 \subseteq \mathcal{H}_{\widetilde{\epsilon}_1}$. As $\mathcal{H}_{\widetilde{\epsilon}_1} \cup \{\mathcal{P}\}$ is a hyperplane of $\mathcal{H}_{\widetilde{\epsilon}_0} \cup \{\mathcal{P}\}$ we thus have that $\overline{\mathcal{H}_2} = \mathcal{H}_{\widetilde{\epsilon}_1}$. We also know that $\overline{\mathcal{H}_2} \subseteq \overline{\mathcal{H}}$ and by Lemma 2.5 and Proposition 6.3, we know that $\overline{\mathcal{H}} \subseteq \mathcal{H}_{\widetilde{\epsilon}_1}$. It follows that $\overline{\mathcal{H}} = \mathcal{H}_{\widetilde{\epsilon}_1}$. Proposition 3.3 then implies that $\mathcal{H}_{\epsilon_{\mathcal{H}}} = \mathcal{H}_{\widetilde{\epsilon}_1}$. Hence, $\epsilon_{\mathcal{H}} \cong \widetilde{\epsilon}_1$ by Lemma 2.4. ■

7 Necessary and sufficient conditions for the hyperbolic embedding ϵ_h to be universal

Proposition 7.1 *A hyperbolic hyperplane of $DW(2n-1,2)$, $n \geq 2$, cannot belong to $\mathcal{H}_{\widetilde{\epsilon}_1}$.*

Proof. Suppose H is a hyperbolic hyperplane of $DQ(2n,2) \cong DW(2n-1,2)$. Then H arises from a hyperbolic quadric $Q^+(2n-1,2) \subseteq Q(2n,2)$. Let $x \in H$ and let α be the generator of $Q(2n,2)$ corresponding to x. As $x \in H$, the generator α is not contained in $Q^+(2n-1,2)$. The lines through x correspond to the hyperplanes of α, and we see that there is only one line through x not contained in H, namely the line corresponding to the hyperplane $\alpha \cap Q^+(2n-1,2)$ of α.

Suppose now that $H \in \mathcal{H}_{\widetilde{\epsilon}_1}$. Then there exists a unique hyperplane $\mathrm{PG}(W)$ of $\mathrm{PG}(\widetilde{V}_0)$ containing $\langle \Omega_{\widetilde{\epsilon}_0} \rangle$ such that $H = \widetilde{\epsilon}_0^{-1}(\widetilde{\epsilon}_0(\mathcal{P}) \cap \mathrm{PG}(W))$. Let x and y be opposite points of $DQ(2n,2)$ with $x \in H$, put $\Gamma_1(x) \cap \Gamma_{n-1}(y) = \{z_1, z_2, \dots, z_{2^n-1}\}$, and let $\bar{v}_i$ with $i \in \{1,2,\dots,2^n-1\}$ be the unique vector of $\widetilde{V}_0$ such that $\widetilde{\epsilon}_0(z_i) = \langle \bar{v}_i \rangle$. Without loss of generality, we may suppose that the lines xz_i, $i \in \{1,2,\dots,2^n-2\}$, are contained in H, while xz_{2^n-1} is not. Then $\bar{v}_1, \bar{v}_2, \dots, \bar{v}_{2^n-2} \in W$. As also $\Omega_{\widetilde{\epsilon}_0} = \bar{v}_1 + \bar{v}_2 + \cdots + \bar{v}_{2^n-1} \in W$, we must have $\bar{v}_{2^n-1} \in W$, i.e. $xz_{2^n-1} \subseteq H$, a contradiction. ■

Proposition 7.2 *The hyperbolic embedding ϵ_h of $DW(2n-1,2)$, $n \geq 2$, has type n.*

Proof. Suppose the hyperbolic embedding ϵ_h has type at most $n-1$. Then $\epsilon_h \leq \widetilde{\epsilon}_1$ by Corollary 4.12. Hence, $\mathcal{H}_{\epsilon_h} \subseteq \mathcal{H}_{\widetilde{\epsilon}_1}$ by Lemma 2.4. But this is impossible as no hyperbolic hyperplane belongs to $\mathcal{H}_{\widetilde{\epsilon}_1}$. ■

Proposition 7.3 *Let H be a hyperbolic hyperplane and M a max of $DW(2n-1,2)$, $n \geq 3$. Then either $M \subseteq H$ or $M \cap H$ is a hyperbolic hyperplane of $\widetilde{M}$.*

Proof. Suppose H is defined by a hyperbolic quadric $Q^+(2n-1,2) \subseteq Q(2n,2)$. Let x be the point of $Q(2n,2)$ corresponding to M. If $x \notin Q^+(2n-1,2)$, then no generator of $Q(2n,2)$ through x is contained in $Q^+(2n-1,2)$ and we have $M \subseteq H$. Suppose therefore that $x \in Q^+(2n-1,2)$.

The tangent hyperplane T_x at the point $x \in Q(2n,2)$ intersects $Q(2n,2)$ in a cone $xQ(2n-2,2)$ and $Q^+(2n-1,2)$ in a cone $xQ^+(2n-3,2)$. The points of M correspond to the generators through x and the points of $M \cap H$ correspond to the generators through x not contained in $Q^+(2n-1,2)$, i.e. with the maximal subspaces of $xQ(2n-2,2)$ through x not contained in $xQ^+(2n-3,2)$. We thus see that $M \cap H$ must be a hyperbolic hyperplane of $\widetilde{M}$. ■

Proposition 7.4 *Suppose ϵ is the hyperbolic embedding of $DW(2n-1,2)$, $n \geq 2$. Then for every convex subspace F of $DW(2n-1,2)$ of diameter at least 2, ϵ_F is isomorphic to the hyperbolic embedding of $\widetilde{F}$.*

Proof. It suffices to prove the proposition in the case where F is a max (then induction shows the validity of the proposition for all convex subspaces of diameter at least 2.)

Let $\mathcal{H}$ denote the set of all hyperbolic hyperplanes of $DW(2n-1,2)$ and let $\mathcal{H}'$ denote the set of all hyperplanes of $\widetilde{F}$ of the form $H \cap F$, where $H \in \mathcal{H}$. By Proposition 7.3, we know that $\mathcal{H}'$ coincides with the set of all hyperbolic hyperplanes of $\widetilde{F}$. The set $\mathcal{H}_{\epsilon_F}$ consists of all hyperplanes of $\widetilde{F}$ of the form $H \cap F$, where $H \in \mathcal{H}_\epsilon$. As $\mathcal{H}_\epsilon = \overline{\mathcal{H}}$, we have $\mathcal{H}_{\epsilon_F} = \overline{\mathcal{H}'}$ and so ϵ_F is isomorphic to the hyperbolic embedding of $\widetilde{F}$ by Lemma 2.4 and Proposition 3.3. ■

Lemma 7.5 *Let Π_1 and Π_2 be two distinct hyperplanes of $\mathrm{PG}(2n,2)$, $n \geq 2$, intersecting $Q(2n,2)$ in quadrics of type $Q^+(2n-1,2)$. Then $\Pi_1 \cap \Pi_2$ intersects $Q(2n,2)$ in a quadric of type $pQ^+(2n-3,2)$ and the hyperplane through $\Pi_1 \cap \Pi_2$ distinct from Π_1 and Π_2 intersects $Q(2n,2)$ in a quadric of type $pQ(2n-2,2)$.*

Proof. As Π_1 intersects $Q(2n,2)$ in a quadric of type $Q^+(2n-1,2)$, $\Pi_1 \cap \Pi_2$ should intersect $Q(2n,2)$ in a quadric of type $Q(2n-2,2)$ or a quadric of type $pQ^+(2n-3,2)$.

Suppose $\Pi_1 \cap \Pi_2$ intersects $Q(2n,2)$ in a quadric of type $Q(2n-2,2)$. As not every hyperplane through $\Pi_1 \cap \Pi_2$ is tangent to $Q(2n,2)$, the kernel k of $Q(2n,2)$ cannot belong to $\Pi_1 \cap \Pi_2$. But then $\Pi_1 \cap \Pi_2$ is contained in a unique tangent hyperplane, namely $\langle k, \Pi_1 \cap \Pi_2 \rangle$, and we would have $|Q(2n,2)| = |Q^+(2n-1,2)| + |Q^+(2n-1,2)| + |kQ(2n-2,2)| - 2 \cdot |Q(2n-2,2)|$, a contradiction, since $|Q(2n,2)| = 2^{2n}-1$, $|Q^+(2n-1,2)| = (2^n-1)(2^{n-1}+1)$, $|kQ(2n-2,2)| = 2^{2n-1}-1$ and $|Q(2n-2,2)| = 2^{2n-2}-1$.

Therefore $\Pi_1 \cap \Pi_2$ intersects $Q(2n,2)$ in a quadric of type $pQ^+(2n-3,2)$. As there are nontangent planes through $\Pi_1 \cap \Pi_2$, we have $k \notin \Pi_1 \cap \Pi_2$. So, the tangent hyperplane $\langle k, \Pi_1 \cap \Pi_2 \rangle$ is the third hyperplane through $\Pi_1 \cap \Pi_2$, and it intersects $Q(2n,2)$ in a quadric of type $pQ(2n-2,2)$. ■

Proposition 7.6 *Let Π_1 and Π_2 be two distinct hyperplanes of $\mathrm{PG}(2n,2)$, $n \geq 2$, intersecting $Q(2n,2)$ in quadrics of type $Q^+(2n-1,2)$. Then $\Pi_1 \cap \Pi_2$ intersects $Q(2n,2)$ in a quadric $pQ^+(2n-3,2)$. For every $i \in \{1,2\}$, let H_i be the hyperbolic hyperplane of $DQ(2n,2)$ associated with the hyperbolic quadric $\Pi_i \cap Q(2n,2)$. Let M denote the max consisting of all generators of $Q(2n,2)$ through p and let G denote the hyperbolic hyperplane of $\widetilde{M}$ consisting of all generators of $Q(2n,2)$ through p not contained in $pQ^+(2n-3,2)$. Then $H_1 * H_2$ equals the extension of the hyperplane G of $\widetilde{M}$.*

Proof. The tangent hyperplane T_p at the point p intersects $Q(2n,2)$ in $pQ(2n-2,2)$. Note that $T_p \cap \Pi_1 = \Pi_1 \cap \Pi_2 = T_p \cap \Pi_2$.

Let α be a point of M, i.e. a generator of $Q(2n,2)$ through p. If α is contained in $pQ^+(2n-3,2)$, then α does not belong to H_1 nor to H_2 and so belongs to $H_1 * H_2$. If $\alpha \subseteq T_p$ is not contained in $pQ^+(2n-3,2)$, then α belongs to H_1 and H_2 and hence also to $H_1 * H_2$.

Let α be a point of $DQ(2n,2)$ not contained in M, i.e. α is a generator of $Q(2n,2)$ not containing p. Let β denote the unique generator through p intersecting α in an $(n-2)$-dimensional subspace. We need to show that $\alpha \in H_1 * H_2$ if and only if β is not contained in $pQ^+(2n-3,2)$.

Suppose $\beta \subseteq T_p$ is not contained in $pQ^+(2n-3,2)$. Then $\alpha \cap \beta$ (and hence also α) contains points of $pQ(2n-2,2) \setminus pQ^+(2n-3,2)$. This implies that $\alpha \in H_1 \cap H_2$ and hence $\alpha \in H_1 * H_2$.

Suppose β is contained in $pQ^+(2n-3,2)$. The $(n-2)$-dimensional subspace $\alpha \cap \beta$ is contained in $Q_i^+(2n-1,2) := \Pi_i \cap Q(2n,2)$, $i \in \{1,2\}$, and hence is contained in two generators of $Q_i^+(2n-1,2)$. So, the $(n-2)$-dimensional subspace $\alpha \cap \beta$ is contained in three generators, namely $\langle p, \alpha\cap\beta\rangle = \beta$, a generator of $Q_1^+(2n-1,2)$ distinct from β and a generator of $Q_2^+(2n-1,2)$ distinct from β. Hence, α is contained in precisely one of $Q_1^+(2n-1,2)$, $Q_2^+(2n-1,2)$. It follows that $\alpha \in H_1 \Delta H_2$, i.e. $\alpha \notin H_1 * H_2$. ∎

Proposition 7.7 *Let $\mathcal{H}$ be the set*[2] *of hyperplanes of $DW(2n-1,2)$, $n \geq 3$, that arise by extending a hyperbolic hyperplane of a max. Then the following hold:*

(*a*) *$\epsilon_{\mathcal{H}}$ is isomorphic to a quotient of the hyperbolic embedding ϵ_h of $DW(2n-1,2)$;*

(*b*) *the vector dimension of $\epsilon_{\mathcal{H}}$ is one less than the vector dimension of ϵ_h;*

(*c*) *$\epsilon_{\mathcal{H}} \leq \widetilde{\epsilon}_1$;*

(*d*) *the hyperbolic embedding ϵ_h is universal if and only if $\epsilon_{\mathcal{H}} \cong \widetilde{\epsilon}_1$.*

Proof. (a) Let $\mathcal{H}'$ denote the set of all hyperbolic hyperplanes of $DW(2n-1,2)$. In order to show that $\epsilon_{\mathcal{H}}$ is isomorphic to a quotient of ϵ_h, it suffices by Lemma 2.4 to prove that $\mathcal{H}_{\epsilon_{\mathcal{H}}} \subseteq \mathcal{H}_{\epsilon_h}$. Taking into account Proposition 3.3, we thus need to prove that $\overline{\mathcal{H}} \subseteq \overline{\mathcal{H}'}$, or equivalently that $\mathcal{H} \subseteq \overline{\mathcal{H}'}$. But the inclusion $\mathcal{H} \subseteq \overline{\mathcal{H}'}$ is a consequence of Proposition 7.6.

(c) In order to show that $\epsilon_{\mathcal{H}} \leq \widetilde{\epsilon}_1$, it suffices by Lemma 2.4 to prove that $\mathcal{H}_{\epsilon_{\mathcal{H}}} \subseteq \mathcal{H}_{\widetilde{\epsilon}_1}$. Taking into account Proposition 3.3, we thus need to prove that $\overline{\mathcal{H}} \subseteq \mathcal{H}_{\widetilde{\epsilon}_1}$, or equivalently that $\mathcal{H} \subseteq \mathcal{H}_{\widetilde{\epsilon}_1}$. But the latter follows from Proposition 6.3.

(b) Every hyperbolic hyperplane belongs to $\mathcal{H}_{\epsilon_h}$ by Proposition 3.3. By Proposition 7.1, it cannot belong to $\mathcal{H}_{\widetilde{\epsilon}_1}$ and hence also not to $\mathcal{H}_{\epsilon_{\mathcal{H}}} = \overline{\mathcal{H}}$ by Lemma 2.4 and Part (c). So, $\epsilon_{\mathcal{H}}$ and ϵ_h cannot be isomorphic, and by (a) we then know that the vector dimension of $\epsilon_{\mathcal{H}}$ is at least one less than the one of ϵ_h. By Proposition 7.6, we know that the dimension of the subspace $\langle \mathcal{H}' \rangle = \overline{\mathcal{H}'} \cup \{\mathcal{P}\}$ of $\widetilde{V}_0^*$ is at most one more than the dimension of the subspace $\langle \mathcal{H} \rangle = \overline{\mathcal{H}} \cup \{\mathcal{P}\}$. As $\mathcal{H}_{\epsilon_{\mathcal{H}}} = \overline{\mathcal{H}}$ and $\mathcal{H}_{\epsilon_h} = \overline{\mathcal{H}'}$, we thus see that the vector dimensions of $\epsilon_{\mathcal{H}}$ and ϵ_h differ at most 1.

(d) Suppose the hyperbolic embedding ϵ_h is universal. By Part (b), we then know that the vector dimension of $\epsilon_{\mathcal{H}}$ is equal to $\frac{(2^n+1)(2^{n-1}+1)}{3} - 1$, i.e. equal to the vector dimension of $\widetilde{\epsilon}_1$ by Corollary 4.5. By Part (c), we then know that $\epsilon_{\mathcal{H}}$ and $\widetilde{\epsilon}_1$ are isomorphic.

Conversely, suppose that $\epsilon_{\mathcal{H}}$ and $\widetilde{\epsilon}_1$ are isomorphic. By Part (b) and Corollary 4.5, we then see that the vector dimension of ϵ_h is equal to $\frac{(2^n+1)(2^{n-1}+1)}{3}$, implying that ϵ_h is universal. ∎

[2]This set of hyperplanes is easily seen to satisfy the property mentioned in Proposition 3.1.

Lemma 7.8 *Let M be a max of $DW(2n-1,2)$, $n \geq 3$, and let G_1, G_2, G_3 be three hyperplanes of $\widetilde{M}$ such that $G_3 = M \setminus (G_1 \Delta G_2)$. For every $i \in \{1,2,3\}$, let H_i be the hyperplane of $DW(2n-1,2)$ that extends G_i. Then $H_3 = H_1 * H_2$.*

Proof. Every point x of M belongs to H_1, H_2 and H_3.

If x is a point of $DW(2n-1,2)$ not contained in M and $i \in \{1,2,3\}$, then $x \in H_i$ if and only if $\pi_M(x) \in G_i$. The equality $H_3 = H_1 * H_2$ then follows from the fact that G_3 is the complement of the symmetric difference of G_1 and G_2 (regarded as subsets of M). ■

Proposition 7.9 *Let $n \geq 3$. Then the following are equivalent:*

(1) *the hyperbolic embedding of $DW(2n-1,2)$ is universal;*

(2) *the hyperbolic embedding of $DW(2n-3,2)$ is universal and $DW(2n-1,2)$ has universal hyperplanes.*

Proof. Suppose the hyperbolic embedding of $DW(2n-1,2)$ is universal. By Lemma 2.6 and Proposition 7.4, we then know that the hyperbolic embedding of $DW(2n-3,2)$ is universal. By Proposition 3.5, we know that $DW(2n-1,2)$ has universal hyperplanes (e.g. the hyperbolic hyperplanes).

Conversely, suppose that the hyperbolic embedding of $DW(2n-3,2)$ is universal and that $DW(2n-1,2)$ has universal hyperplanes. Denote by $\mathcal{H}$ [resp. $\mathcal{H}'$] the set of hyperplanes of $DW(2n-1,2)$ that arise by extending a hyperplane [resp. hyperbolic hyperplane] of a max of $DW(2n-1,2)$. As the hyperbolic embedding of $DW(2n-3,2)$ is universal, all hyperbolic hyperplanes of $DW(2n-3,2)$ are universal by Proposition 3.5 and so $\overline{\mathcal{H}} = \overline{\mathcal{H}'}$ by Lemma 7.8. By Propositions 3.3 and 6.6, we also know that $\overline{\mathcal{H}} = \mathcal{H}_{\epsilon_{\mathcal{H}}} = \mathcal{H}_{\widetilde{\epsilon}_1}$. Hence, $\mathcal{H}_{\epsilon_{\mathcal{H}'}} = \overline{\mathcal{H}'} = \mathcal{H}_{\widetilde{\epsilon}_1}$, i.e. $\epsilon_{\mathcal{H}'} \cong \widetilde{\epsilon}_1$ by Lemma 2.4. By Proposition 7.7(d), we then know that the hyperbolic embedding of $DW(2n-1,2)$ is universal. ■

The following is a consequence of Propositions 3.5 and 7.9, taking into account that the hyperbolic embedding of $DW(3,2) \cong W(2)$ (which has type 2) is universal.

Corollary 7.10 *The following are equivalent:*

(1) *The hyperbolic embedding of $DW(2n-1,2)$ is universal for every $n \geq 2$.*

(2) *$DW(2n-1,2)$ has universal hyperplanes for every $n \geq 2$.*

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