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# Eigenexpansions and ultradifferentiability

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*To my mother*

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Rastao sam sa samohranom majkom u verovatno najtežem periodu novije srpske istorije, u sredini neobično odbojnoj prema ženama koje su rekle “ne” posle izrečenog “da” i, uopšteno govoreći, prema onima koji misle ili delaju makar malo drugačije od onoga što se smatra standardom, ali sav taj teret tada gotovo da nisam osećao jer sam bio okružen ljubavlju. Moja majka je uvek bila tu, verovala je u mene i onda kada je sva moja vera presahla. Nikada nije preispitivala nijednu moju odluku, pa ni ovu da se bavim matematikom. Podržavala je svoje dete, osećala je i znala šta ono želi i nikada nije slušala hiljade glasova protiv, vrlo često nipodaštavajućih i omalovažavajućih. Moja majka je, najzad, podržala i moj odlazak iz Srbije, jer je verovala da ću ovde biti srećniji. I tri godine je uvek bila nasmejana na Skajpu. Za mene. I stoga najmanje što mogu da učinim za nju je da joj ovu tezu posvetim.

# Introduction

In this ground-level chapter, the reader will be informed about the structure of the thesis and the thesis itself. The goal (as well as purpose) of the pages the reader is about to explore is a thorough and clear presentation of the research author has conducted during his Ph.D. studies at Ghent University. Even at this starting point, considering the complexity of the task we are about to perform, namely, to write down and explain the result of studies that last more than 3 years, that includes results of three published articles and some results of still ongoing research, with a lot of different mathematical tools involved, some of them interesting in of themselves (especially for a curious reader), it becomes clear why the author himself had to contemplate a lot about the organization of this thesis.

Here is probably the best place to set the scene, and before any technical information regarding the organization of the structure of this thesis, to give a brief motivation and explain certain goals of the research that is to be presented.

The reader should be informed (or "warned" might be a better word) about mathematical background, or, more precisely, the machinery we (as well as (s)he) will use extensively in order to reach our desired goals. Of course, we were not able to present every single tool we used, so we decided to focus on just a few of them - namely, the ones that were omitted during bachelor and master studies so that the "average" graduate student might have not seen them during his studies. This is the content of Chapter 1. The reader is strongly encouraged to read it carefully. Some of the proofs are given so that the reader can feel the flavor of the read topics.

Very roughly and generally speaking, we will be interested in a various eigenexpansions, and, more accurately, eigenexpansions for ultradifferentiable functions and ultradistributions based on eigenfunctions of several differential operators, in order to achieve our goals. First we will use spherical harmonic expansions on the unit sphere $\mathbb{S}^{n-1}$ to characterize ultradifferentiable functions and ultradistributions on the sphere $\mathbb{S}^{n-1}$ which will lead us to the characterization of harmonic functions on the open ball $\mathbb{B}^n$ that admit ultradistributional boundary values on $\mathbb{S}^{n-1}$. These results will allow us to characterize rotational invariant ultradistributions. Other expansions used in this thesis will be eigenfunction expansions on $\mathbb{R}^n$ with respect to differential operators of Shubin type, in order to characterize some general classes of Gelfand-Shilov spaces. Finally, we will be interested in developing a pseudodifferential calculus for operators on $\mathbb{T}^n$ that will act continuously on the classes of our greatest interest - namely, classes of ultradifferentiable functions and ultradistributions.

In Chapter 2, we covered results from our article [65]. We are here interested in the boundary values of harmonic and analytic functions, which is a classical and important subject in distribution and ultradistribution theory. There is a vast literature dealing with boundary values on $\mathbb{R}^n$, see e.g. [1, 10, 11, 19, 22, 29, 44] and references therein. In the case of the unit sphere $\mathbb{S}^{n-1}$, the characterization of harmonic functions in the Euclidean unit ball of $\mathbb{R}^n$ having distributional boundary values on $\mathbb{S}^{n-1}$ was given by Estrada and Kanwal in [21]. In the article [25], González Vieli has used the Poisson transform to obtain a very useful description of the support of a Schwartz distribution on the sphere (cf. [62] for support characterizations on $\mathbb{R}^n$). Representations of analytic functionals on the sphere [38] as initial values of solutions to the heat equation were studied by Morimoto and Suwa [39].

Our first result in Chapter 2 is an explicit estimate for partial derivatives of spherical harmonics (which are of independent interest) that refine earlier estimates by Calderón and Zygmund. These estimates will be sharp enough for us to obtain a characterization of ultradifferentiable functions and ultradistributions on the sphere in terms of their spherical harmonic expansions. Once we have this characterization, the desired ultradistributional boundary value theory will follow naturally. Finally, we apply our results to characterize the support of ultradistributions on the sphere via Abel summability of their spherical harmonic expansions.

These results will also be essential for Chapter 3, where we discuss the problem of rotational invariant ultradistributions. Rotation invariant generalized functions have been studied by several authors, see e.g. [13, 59, 61]. The problem of the characterization of rotation invariant ultradistributions and hyperfunctions was considered by Chung and Na in [13]. There they showed that a non-quasianalytic ultradistribution or a hyperfunction is rotation invariant if and only if it is equal to its spherical mean. For continuous functions this result is clear, as a rotation invariant function must be radial and its spherical mean is given by $\varphi_S(x) = \frac{1}{|\mathbb{S}^{n-1}|}\int_{\mathbb{S}^{n-1}} \varphi(|x|\omega)d\omega$. Chung and Na's approach to the problem consists of reducing the case of rotation invariant generalized functions to that of ordinary functions. For ultradistributions, non-quasianalyticity was a crucial assumption for their method since they regularized by convolving with a net of compactly supported ultradifferentiable mollifiers. In the hyperfunction case they applied a similar idea, but this time based on Matsuzawa's heat kernel method. In Chapter 3, we show that the characterization of rotation invariant ultradistributions in terms of their spherical means remains valid for quasianalytic ultradistributions. Our approach differs from that of Chung and Na, and we also recover their results for non-quasianalytic ultradistributions and hyperfunctions. We will mention that the Chapter 3 contains results from our article [64].

Next, we will state a new problem, that will lead us to the topic of Chapter 4.

Back in 1969 Seeley characterized [57] real analytic functions on a compact analytic manifold via the decay of their Fourier coefficients with respect to eigenfunction expansions associated to a normal analytic elliptic differential operator. In recent times, this result by Seeley has attracted much attention and has been generalized in several directions. In the article [17], Dasgupta and Ruzhansky extended Seeley's work and achieved the eigenfunction expansion characterization of Denjoy-Carleman classes of ultradifferentiable functions, of both Roumieu and Beurling type, and the corresponding ultradistribution spaces on a compact analytic manifold. The reader can consult [16] for Gevrey classes on compact Lie groups.

Such results have also a global Euclidean counterpart. In this setting, it is natural to consider differential operators of Shubin type, that is, differential operators with polynomial coefficients

$$P = \sum_{|\alpha|+|\beta|\leq m} c_{\alpha\beta} x^{\beta} D^{\alpha}, \quad D^{\alpha} = (-i\partial_x)^{\alpha}. \tag{0.0.1}$$

In [26] Gramchev, Pilipović, and Rodino used this type of operators to give an analogue to Seeley's result for some classes of Gelfand-Shilov spaces.

The aim of the fourth chapter of this thesis is to extend the results from [26] by supplying a characterization of the general Gelfand-Shilov spaces $\mathcal{S}^{\{M_p\}}(\mathbb{R}^n) = \mathcal{S}^{\{M_p\}}_{\{M_p\}}(\mathbb{R}^n)$ and $\mathcal{S}^{(M_p)}(\mathbb{R}^n) = \mathcal{S}^{(M_p)}_{(M_p)}(\mathbb{R}^n)$ of ultradifferentiable functions of Roumieu and Beurling type [10, 12, 23, 24, 37].

After that, we will be moving to analysis on the torus, $\mathbb{T}^n$ and in the final Chapter 5 we present a theory of toroidal pseudodifferential operators that act continuously on the classes of ultradistributions on $\mathbb{T}^n$, both of Beurling and Roumieu type. The approach from [41, 58] for developing a similar theory on $\mathbb{R}^n$ is also followed here (consult also [47, 7, 8]). After defining symbols classes for our operators and a short survey through the topology of those symbols, we will present a symbolic calculus as a way of building an operator from a formal sum of symbols. The results of Chapter 5, unlike those from other chapters, have not been published elsewhere yet.

# Chapter 1

# Mathematical background

## 1.1 Spherical harmonics

In what follows, we will present the basics of the theory of spherical harmonics and the reader will shortly be convinced how powerful and elementary at the same time it is. Of course, we will present just a brief summary of a well developed theory (see e.g. [2, 3]).

The space of *solid spherical harmonics of degree* $j$ will be denoted by $\mathcal{H}_j(\mathbb{R}^n)$; its elements are nothing but the harmonic homogeneous polynomials of degree $j$ on $\mathbb{R}^n$.

A *spherical harmonic of degree* $j$ is the restriction to the unit sphere $\mathbb{S}^{n-1}$ of a solid harmonic of degree $j$ and we write $\mathcal{H}_j(\mathbb{S}^{n-1})$ for space of all spherical harmonics of degree $j$.

We will here follow the approach from [3] and present some interesting properties of these harmonic homogeneous polynomials. However, we first mention that the Poisson kernel of $\mathbb{S}^{n-1}$ is given by

$$P(x,\xi) = \frac{1}{|\mathbb{S}^{n-1}|}\frac{1-|x|^2}{|x-\xi|^n}, \quad \xi \in \mathbb{S}^{n-1},\ x \in \mathbb{B}^n, \tag{1.1.1}$$

where $\mathbb{B}^n$ stands for the unit ball in $\mathbb{R}^n$. This allows us to introduce the Poisson transform of a function $f \in L^2(\mathbb{S}^{n-1})$,

$$P[f](x) = \int_{\mathbb{S}^{n-1}} f(\xi)P(x,\xi)d\xi, \quad x \in \mathbb{B}^n. \tag{1.1.2}$$

This function in fact solves the Dirichlet problem for $\mathbb{B}^n$ with boundary data $f$, i.e.,

$$\Delta P[f] = 0, \quad P[f]|_{\mathbb{S}^{n-1}} = f.$$

We now see that the Poisson transform of a polynomial (restricted to $\mathbb{S}^{n-1}$) is, once again, a polynomial.

**Theorem 1.1.1.** *If $p$ is an arbitrary polynomial of degree $m$, then*

$$P[p|_{\mathbb{S}^{n-1}}] = (1 - |x|^2)q + p$$

*where $q$ is a polynomial of degree at most $m - 2$.*

*Proof.* The statement is trivially true for $m < 2$. Indeed, in those cases the polynomials $p$ are harmonic so the Poisson transform will not change them and we may just take $q = 0$.

In the case $m \geq 2$, note that, for an arbitrary polynomial $q$, $(1-|x|^2)q+p$ equals $p$ on $\mathbb{S}^{n-1}$. Therefore, finding a suitable $q$ for which $(1 - |x|^2)q + p$ is harmonic will actually solve the Dirichlet problem for $\mathbb{B}^n$ with the boundary data $p|_{\mathbb{S}^{n-1}}$. It remains to prove that there exists a polynomial $q$ with degree at most $m - 2$ such that $(1 - |x|^2)q + p$ is harmonic.

Consider the linear mapping

$$q \mapsto \Delta((1 - |x|^2)q)$$

from the space of polynomials of degree at most $m-2$ into itself. If $\Delta((1-|x|^2)q) = 0$ then $(1 - |x|^2)q$ is a harmonic function on $\mathbb{B}^n$ with boundary value 0; however, the maximum principle for harmonic functions forces it to be zero also on unit ball and therefore $q = 0$. The considered linear mapping is then injective from a finite-dimensional vector space into itself which automatically means it is also surjective, which allows us to solve equation

$$\Delta((1 - |x|^2)q) = -\Delta p$$

on the space of polynomials of degrees at most $m - 2$ and proves the theorem. $\square$

The previous theorem tells us that the Poisson transform of a polynomial is polynomial of a very special form.

One can ask himself if this statement can be refined for homogeneous polynomials.

First we will set some notation. For an arbitrary polynomial with degree $m$, there exist uniquely determined homogeneous polynomials $p_j$ of degree $j$, $j = 1, 2, \ldots, m$ such that $p = \sum_{j=0}^{m} p_j$. The homogeneous polynomial $p_j$ will be called the homogeneous part of $p$ of degree $j$. The vector space of all homogeneous polynomials of order $j$ will be denoted as $\mathcal{P}_j(\mathbb{R}^n)$.

**Theorem 1.1.2.** *If $m \geq 2$, then*

$$\mathcal{P}_m(\mathbb{R}^n) = \mathcal{H}_m(\mathbb{R}^n) \oplus |x|^2 \mathcal{H}_{m-2}(\mathbb{R}^n). \tag{1.1.3}$$

*Proof.* For an arbitrary $p \in \mathcal{P}_m(\mathbb{R}^n)$ note that

$$p = P[p|_{\mathbb{S}^{n-1}}] + |x|^2 q - q$$

for some $q$ of degree at most $m-2$ (Theorem 1.1.1). Taking homogeneous parts of order $m$ on both sides, we obtain

$$p = p_m + |x|^2 q_{m-2}$$

here $p_m$ is simply the homogeneous part of order $m$ of the harmonic polynomial $P[p|_{\mathcal{S}^{n-1}}]$ and $q_{m-2}$ is, naturally, the homogeneous part of $q$ of order $m-2$. This proves the existence of a decomposition, while the uniqueness can be easily concluded from the fact that no non-zero multiple of $|x|^2$ is harmonic, and that is a simple corollary of Theorem 1.1.1. $\square$

The previous theorem, with the aid of induction, proves the following result.

**Theorem 1.1.3.** *Every $p \in \mathcal{P}_m(\mathbb{R}^n)$ can be uniquely expressed in the form*

$$p = \sum_{j=0}^{\lfloor \frac{m}{2} \rfloor} p_{m-2j} |x|^{2j}, \tag{1.1.4}$$

*where* $p_j \in \mathcal{H}_j(\mathbb{R}^n)$ *for each* $j = 1, 2, \ldots, \lfloor \frac{m}{2} \rfloor$.

Keep the notation from the previous theorem, it is worth mentioning as a remark that, for $p \in \mathcal{P}_m(\mathbb{R}^n)$, we can easily obtain its Poisson transform:

$$P[p|_{\mathbb{S}^{n-1}}] = \sum_{j=0}^{\lfloor \frac{m}{2} \rfloor} p_j.$$

Standard combinatorial arguments will count $\dim(\mathcal{P}_j(\mathbb{R}^n))$ and equation (1.1.3) will help us calculate $\dim(\mathcal{H}_j(\mathbb{R}^n))$. Note that the injectivity of the mapping $p \mapsto p|_{\mathbb{S}^n}$ from $\mathcal{H}_j(\mathbb{R}^n)$ to $\mathcal{H}_j(\mathbb{S}^{n-1})$ implies that these two spaces have the same dimension. Here denote it as $d_j = \dim \mathcal{H}_j(\mathbb{S}^{n-1})$, then, in order to be explicit (cf. [3] or [56, Thm. 2, p. 117])

$$d_j = \frac{(2j + n - 2)(n + j - 3)!}{j!(n-2)!} \sim \frac{2j^{n-2}}{(n-2)!}.$$

From this exact formula, it is not hard to see that $d_j$ satisfies the bounds

$$\frac{2}{(n-2)!} j^{n-2} < d_j \leq n j^{n-2}, \quad \text{for all } j \geq 1. \tag{1.1.5}$$

Finally, we discuss the orthogonal decomposition of the space $L^2(\mathbb{S}^{n-1})$ using spherical harmonics. Let us just mention that $L^2(\mathbb{S}^{n-1})$ is the space of $L^2$-integrable, Borel measurable functions endowed with the Hermitian inner product

$$(f, g)_{L^2(\mathbb{S}^{n-1})} = \int_{\mathbb{S}^{n-1}} f(\xi)\overline{g(\xi)} d\xi.$$

First we need a little lemma.

**Lemma 1.1.4.** *If* $p$ *and* $q$ *are harmonic polynomials with different degrees, then* $(p|_{\mathbb{S}^{n-1}}, q|_{\mathbb{S}^{n-1}})_{L^2(\mathbb{S}^{n-1})} = 0$.

*Proof.* Using Green's identity, as well as the fact that we are dealing with the harmonic functions, we obtain

$$\int_{\mathbb{S}^{n-1}} \Big( p(\xi) \frac{\partial}{\partial n} q(\xi) - \frac{\partial}{\partial n} p(\xi) q(\xi) \Big) d\xi = 0,$$

where $\frac{\partial}{\partial n}$ stands for normal derivative. Note that homogeneity of the polynomial $p$ implies

$$\frac{\partial}{\partial n}p(\xi) = \frac{d(p(r\xi))}{dr}|_{r=1} = kp(\xi)$$

where $k$ stands for degree of $p$. If $l$ is the degree of $q$, then we directly conclude that $(k-l)(p,q)_{L^2(\mathbb{S}^{n-1})} = 0$, which proves the theorem. □

**Theorem 1.1.5.** *The Hilbert space $L^2(\mathbb{S}^{n-1})$ can be decomposed as follows*

$$L^2(\mathbb{S}^{n-1}) = \bigoplus_{j=0}^{\infty} \mathcal{H}_j(\mathbb{S}^{n-1}). \tag{1.1.6}$$

*Proof.* For every $j = 0, 1, \dots$, $\mathcal{H}_j(\mathbb{S}^{n-1})$ is finite-dimensional and therefore closed in $L^2(\mathbb{S}^{n-1})$. The previous theorem deals with the orthogonality of the spaces $\mathcal{H}_{j_1}(\mathbb{S}^{n-1})$ and $\mathcal{H}_{j_2}(\mathbb{S}^{n-1})$, $j_1 \neq j_2$. It remains to prove that the linear span of $\bigcup_{j=0}^{\infty} \mathcal{H}_j(\mathbb{S}^{n-1})$ is dense in $L^2(\mathbb{S}^{n-1})$. We have proved that every polynomial restricted to $\mathbb{S}^{n-1}$ is a finite sum of spherical harmonics. By the Stone-Weierstrass theorem [40, Thm. 4.15, pp. 24–25], polynomials are dense in $C(\mathbb{S}^{n-1})$; the $L^2$-norm is less or equal than the supremum norm on $\mathbb{S}^{n-1}$; finally, $C(\mathbb{S}^{n-1})$ is dense in $L^2(\mathbb{S}^{n-1})$ and, therefore, the theorem is proved. □

For an arbitrary $f \in L^2(\mathbb{S}^{n-1})$, the expansion inherited from (1.1.6) will be called the *harmonic expansion of a function* $f$; we now calculate the coefficients in the harmonic expansion of $f$ (i.e. orthogonal projection of $f$ onto $\mathcal{H}_j(\mathbb{S}^{n-1})$).

Let us consider the linear mapping $\Lambda : \mathcal{H}_j(\mathbb{S}^{n-1}) \to \mathbb{C}$ defined by $\Lambda(p) = p(\omega)$ for a given $\omega \in \mathbb{S}^{n-1}$; the finite dimensional vector (sub)space $\mathcal{H}_j(\mathbb{S}^{n-1})$ is endowed with the Hermitian inner product inherited from $L^2(\mathbb{S}^{n-1})$ and therefore there exists $Z_j(\cdot,\omega) \in \mathcal{H}_j(\mathbb{S}^{n-1})$,

$$\Lambda(p) = p(\omega) = \frac{1}{|\mathbb{S}^{n-1}|}\int_{\mathbb{S}^{n-1}} p(\xi)\overline{Z_j(\xi,\omega)}d\xi$$

The polynomial $Z_j(\cdot,\omega) \in \mathcal{H}_j(\mathbb{S}^{n-1})$ will be called *zonal harmonic* of degree $m$ with a pole $\omega$. Here we mention some properties of the zonal harmonics:

**Lemma 1.1.6** ([3, Proposition 5.27]). *Let $\xi, \omega \in \mathbb{S}^{n-1}$ and $j \geq 0$ be arbitrary.*

*a)* $Z_j(\xi,\omega) \in \mathbb{R}$;

*b)* $Z_j(\xi,\omega) = Z_j(\omega,\xi)$;

*c)* $Z_j(T(\xi),\omega) = Z_j(\omega, T^{-1}(\xi))$ *if $T$ is an orthogonal matrix of order $n$;*

*d)* $|Z_j(\xi,\omega)| \leq d_j$ *for every $\xi, \omega \in \mathbb{S}^{n-1}$.*

Thus, we then have that $|\mathbb{S}^{n-1}|^{-1} Z_j(\omega,\xi)$ is the reproducing kernel of $\mathcal{H}_j(\mathbb{S}^{n-1})$, namely,

$$Y_j(\omega) = \frac{1}{|\mathbb{S}^{n-1}|}\int_{\mathbb{S}^{n-1}} Y_j(\xi) Z_j(\omega,\xi)d\xi, \quad \text{for every } Y_j \in \mathcal{H}_j(\mathbb{S}^{n-1}). \tag{1.1.7}$$

Finally, let $f \in L^2(\mathbb{S}^{n-1})$ be arbitrary. Its orthogonal projection onto $\mathcal{H}_j(\mathbb{S}^{n-1})$ will always be denoted as $f_j$. Then, from the previous equation,

$$\begin{aligned} &f_j(\omega) = \frac{1}{|\mathbb{S}^{n-1}|}\int_{\mathbb{S}^{n-1}} f_j(\xi) Z_j(\omega,\xi)d\xi = \\ &\frac{1}{|\mathbb{S}^{n-1}|}\int_{\mathbb{S}^{n-1}} \Big(\sum_{k=0}^{\infty} f_k(\xi)\Big) Z_j(\omega,\xi)d\xi = \frac{1}{|\mathbb{S}^{n-1}|}\int_{\mathbb{S}^{n-1}} f(\xi) Z_j(\omega,\xi)d\xi, \end{aligned} \tag{1.1.8}$$

where we use Lemma 1.1.4.

An alternative way of a spherical harmonic expansion goes as follows. Fix an orthonormal basis $\{Y_{k,j}\}_{k=1}^{d_j}$ of each $\mathcal{H}_j(\mathbb{S}^{n-1})$, consisting of real-valued spherical harmonics. Hence, every function $f \in L^2(\mathbb{S}^{n-1})$ can be expanded as

$$f(\omega) = \sum_{j=0}^{\infty}\sum_{k=1}^{d_j} c_{k,j} Y_{k,j}(\omega)$$

with convergence in $L^2(\mathbb{S}^{n-1})$.

## 1.2 Tensor product of locally convex spaces

We follow approach from [60] in order to familiarize our reader with the tensor product of two locally convex spaces, along with the tensor product topologies. However, first we need to introduce the tensor product for two vector spaces, and we postpone further consideration regarding topological structures for later.

Therefore, consider two vector spaces $E, F$ (over $\mathbb{C}$). If $\phi$ is a bilinear map of $E \times F$ into a third vector space $M$, then $E$ and $F$ are $\phi$-linearly disjoint if the following holds.

$(LD)$ For a finite subset $\{x_1, \dots x_r\}$ of $E$ and $\{y_1, \dots, y_r\}$ of $F$ (note that we have the same number of elements) satisfying the relation $\sum_{j=1}^{r} \phi(x_j, y_j) = 0$, then linear independence of $x_1, \dots x_r$ implies that $y_1 = \cdots = y_r = 0$ and linear independence of $y_1, \dots y_r$ implies that $x_1 = \cdots = x_r = 0$.

Then the *tensor product* of $E$ and $F$ is a pair $(M, \phi)$ consisting of a vector space $M$ and a bilinear mapping $\phi : E \times F \to M$ such that the following holds:

$(TP\ 1)$ Linear span of the image $\phi(E \times F)$ is the whole $M$.

$(TP\ 2)$ $E$ and $F$ are $\phi$-linearly disjoint.

The tensor product of two vector spaces always exists and it is unique up to isomorphism.

**Theorem 1.2.1** ([60, Theorem 39.2]). *Let $E$ and $F$ be two vector spaces.*

$(a)$ *There exists a tensor product of $E$ and $F$.*

$(b)$ *Let $(M, \phi)$ be that tensor product, $G$ an arbitrary space and $b$ any bilinear mapping of $E \times F$ into $G$. There exists any unique linear* *map $\tilde{b} : M \to G$ such that the diagram*

$$\begin{tikzcd} E \times F \arrow[r, "b"] \arrow[d, "\phi"] & G \\ M \arrow[ur, "\tilde{b}"] & \end{tikzcd}$$

*is commutative.*

(c) *If $(M_1, \phi_1)$ and $(M_2, \phi_2)$ are two tensor products of $E$ and $F$, there is a one-to-one linear map $u : M_1 \to M_2$ such that the diagram*

$$\begin{tikzcd} E \times F \arrow[r, "\phi_2"] \arrow[d, "\phi_1"] & M_2 \\ M_1 \arrow[ur, "u"] & \end{tikzcd}$$

*is commutative.*

Property $(b)$ is called the universal property of the tensor product. The proof of these theorem from [60] provides an explicit construction of the vector space of $E$ and $F$, denoted by $E \otimes F$.

In order to endow the tensor product with a topological structure, we need some preparation.

Suppose $E$ are $F$ locally convex (see [60, 40]) and Hausdorff. Let $\mathcal{B}(E,F)$ be the (vector) space of separately continuous bilinear maps of $E \times F$ into $\mathbb{C}$, with the space $B(E,F)$ of continuous bilinear maps of $E \times F$ into $\mathbb{C}$ as a subspace. If we take $\mathcal{P}$ (resp. $\mathcal{Q}$), the family of bounded sets in $E$ (resp. $F$), then we may consider on $B(E,F)$ the topology of uniform convergence on subsets of the form $A \times B$, where $A \in \mathcal{P}$, $B \in \mathcal{Q}$. Then it is not hard to verify that the family

$$\mathcal{U}(A,B;W) = \{\Phi \in B(E,F); \Phi(A,B) \subset W\} \tag{1.2.1}$$

when $A$ $(B)$ varies over $\mathcal{P}$ (over $\mathcal{Q}$) and $W$ varies over a basis of neighborhood of 0 (in the space $\mathbb{C}$ with the standard topology), produces a basis of neighborhoods of zero for this topology.

The previous discussion does not hold for $\mathcal{B}(E,F)$ so the set of (just) separately continuous bilinear mappings instead of the space

$B(E,F)$. The above mentioned family (1.2.1) is not necessarily a basis of neighborhoods of zero. However, one can, carefully considering this "obstacle", conclude that boundedness of $\Phi(A,B)$ for every bounded $A$ and $B$ would be enough to assure that. This is true in case of hypocontinuity of $\Phi$ ([60, Chapter 41]), which will be obtained if $E$ and $F$ are barrelled.

On the other hand, [60, Proposition 42.1] shows that the definition of the topology on the space of separately bilinear mappings makes sense in one case particularly interesting for us, when the spaces do not need to be barrelled. Namely, if $E$ and $F$ are locally convex, $E'$ and $F'$ their strong duals (for the different locally convex topologies that can be imposed on the dual of locally convex spaces, see [60, Chapter 23] or [40, Chapter 19]), $\mathcal{P}$ (resp. $\mathcal{Q}$) is the family of equicontinuous subsets of $E'$ (resp. $F'$), then one can show that, for every $A' \in \mathcal{P}, B' \in \mathcal{Q}$ and every $\Phi$ separately continuous bilinear mapping of $E' \times F'$ into $\mathbb{C}$, $\Phi(A', B')$ is a bounded subset of $\mathbb{C}$, which will allow us to impose the topology of uniform convergence on the products $A' \times B'$ and turn $\mathcal{B}(E', F')$ into a locally convex $TVS$, denoted by $\mathcal{B}_\varepsilon(E', F')$. The subspace of this space $\mathcal{B}_\varepsilon(E'_\sigma, F'_\sigma)$, consisting of separately continuous bilinear mappings $E'_\sigma \times F'_\sigma \to \mathbb{C}$ (with the weak dual topologies), will be also interesting for us. It is easy to prove that these topologies are Hausdorff.

Considering two locally convex spaces, $E$ and $F$, there is a canonical bilinear mapping of $E \times F$ into $B(E'_\sigma, F'_\sigma)$ (let us remind our reader that this space consists of the continuous bilinear functions $E'_\sigma \times F'_\sigma \to \mathbb{C}$)

$$(x,y) \mapsto \phi_{x,y} \quad \phi_{x,y}(x',y') = \langle x', x\rangle \ \langle y', y\rangle.$$

Following the approach from ([60, pp. 431–432]), one can prove that $E$ and $F$ are $\phi$-linearly disjoint then the set $\{\phi_{x,y}, x, y \in E \times F\}$ will span the whole $B(E'_\sigma, F'_\sigma)$. Once the statement is proved, any arbitrary $\Phi \in B(E'_\sigma, F'_\sigma)$ can be written as a finite sum

$$\Phi(x',y') = \sum_{j=1}^{r} \sum_{k=1}^{s} \langle x', x_j\rangle \langle y', y_k\rangle.$$

This allows us (in light of what has been said so far, namely

Theorem 1.2.1) to treat $B(E'_\sigma, F'_\sigma)$ as a tensor product of $E$ and $F$ and to write $\phi_{x,y}$ simply as $x \otimes y$. Furthermore, this also allows us to endow the space $E \otimes F$ with a very important topology.

**Definition 1.2.2.** For two locally convex $TVS$, the $\varepsilon$-topology on $E\otimes F$ is the relative topology from $B(E'_\sigma, F'_\sigma)$, when the latter is regarded as a subspace of $\mathcal{B}_\varepsilon(E'_\sigma, F'_\sigma)$ with the inherited topology. The new TVS $E \otimes F$, equipped with the $\varepsilon$-topology, will be denoted by $E \otimes_\varepsilon F$.

It follows almost automatically that the canonical mapping $(x, y) \mapsto x \otimes y$ of $E \times F$ into $E \otimes_\varepsilon F$ is continuous. This motivates us to define the second (main) topology on tensor products.

**Definition 1.2.3.** On the space $E\otimes F$, the $\pi$-topology is the strongest locally convex topology such that the canonical bilinear mapping $(x, y) \mapsto x \otimes y$ of $E \times F$ into $E \otimes F$ is continuous. With this topology, $E \otimes F$ will be denoted by $E \otimes_\pi F$.

Given the fact that $E\times F \to E\otimes_\varepsilon F$ is continuous, the $\pi$-topology is finer than the $\varepsilon$-topology. Also, from the definition it follows how the neighborhoods of zero look like in the $\pi$-topology. Namely, a convex subset of $E\otimes F$ is a neighborhood of zero in this topology if and only if its inverse image under the mapping $(x, y) \mapsto x\otimes y$ contains a neighborhood of zero in $E \times F$, or, more operatively, it contains a set of the form

$$U \otimes V = \{x \otimes y \in E \otimes F; x \in U, y \in V\}$$

where $U$ (resp. $V$) is the neighborhood of zero in $E$ (resp. $F$).

Choosing a system of seminorms rather then the basis of convex zero neighborhoods, one can introduce seminorms for the $\pi$-topology. If $p$ (resp. $q$) is a seminorm on $E$, $U_p$ (resp. $V_p$) is its closed unit semiball, and $W$ is the balanced, convex hull of $U_p \otimes V_p$ that also absorbing (as one easily proves). We then define the tensor product of seminorms $p$ and $q$:

$$(p \otimes q)(\theta) = \inf_{\theta\in\rho W, \rho>0} \rho, \qquad \theta \in E \otimes F$$

and then it can be proved [60, Proposition 43.1] that

$$(p \otimes q)(\theta) = \inf \sum_j p(x_j)q(y_j),$$

where the infimum is taken over all finite sets of pairs $(x_j, y_j)$ for which $\theta = \sum_j x_j \otimes y_j$. In the special case, $(p \otimes q)(x \otimes y) = p(x)q(y)$.

The definition of the $\pi$-topology is almost enough to prove the so-called universal property:

**Theorem 1.2.4.** *For two locally convex spaces $E$ and $F$, $\pi$-topology is the only locally convex topology on $E \otimes F$ having the following property.*

*For every locally convex space $G$, the canonical isomorphism of the space of bilinear mappings $E \times F$ into $G$ onto the space of linear mappings from $E \otimes F$ into $G$ (seen in Thm. 1.2.1) will induce an isomorphism (in the algebraic sense) of the space of continuous bilinear mappings $E \times F$ into $G$ onto the space of continuous linear mappings of $E \otimes F$ into $G$.*

*In the special case $G = \mathbb{C}$ it follows as a corollary that the dual of $E \otimes_\pi F$ is canonically isomorphic to $B(E, F)$ (once again, isomorphism is considered in the algebraic sense).*

These topologies (like every locally convex topology) admit completion (see e.g. [40, Prop. 22.21]). The completion of $E \otimes_\varepsilon F$ (resp. $E \otimes_\pi F$) will be denoted by $E \hat{\otimes}_\varepsilon F$ (resp. $E \hat{\otimes}_\pi F$). In the case of two Fréchet spaces, $E$ and $F$, one is able to characterize the elements in the completion $E \hat{\otimes}_\pi F$ ([60, Theorem 45.1]). Namely, every $\theta \in E \hat{\otimes} F$ is the sum of an absolutely convergent series

$$\theta = \sum_{n=0}^{\infty} \lambda_n x_n \otimes y_n, \tag{1.2.2}$$

where $\{\lambda_n\}$ is a sequence of complex numbers for which $\sum_{n=0}^{\infty} |\lambda_n| < 1$ and $\{x_n\}$ (resp. $\{y_n\}$) is a zero-converging sequence in $E$ (resp. $F$). In a locally convex space, absolute convergence of a series $\sum_{n=0}^{\infty} x_n$ means that, for every continuous seminorm $p$ on $E$, $\sum_{n=0}^{\infty} p(x_n)$ converges.

In the next subsection we discuss the case of tensor products where the reader can, based on its convenience, use either $\pi$-or $\varepsilon$-topology.

First, we will introduce nuclear mappings, and, after them, nuclear spaces that will answer the question when $E\hat{\otimes}_{\varepsilon}F = E\hat{\otimes}_{\pi}F$.

### 1.2.1 Nuclear mappings

For the quick survey of nuclear mappings, the reader is advised to consult [60, Chapter 50] or [54, Chapter 7], or [45] for an extensive treatment. In what follows, we will introduce just the basic results. For the start, some basic facts will be recalled. In a Hausdorff locally convex space $E$, with a convex, balanced and bounded set $B$, define $E_B = \bigcup_{n\in\mathbb{N}} nB$, which is the subset of $E$ spanned by $B$. This space is normable, where the mapping $\|\cdot\|_B : E_B \to \mathbb{R}^+$, $\|x\|_B = \sup_{\rho>0, x\in\rho B}\rho$ (gauge-function) is a norm on $E_B$. Furthermore, completeness of $B$ in $E$ implies that the space $E_B$ is Banach. For $U$ closed, convex, balanced neighborhood of zero in $E$, $E_U = E$ in a set-theoretical sense, however, this set is not necessarily Hausdorff and complete. Therefore, we will turn it into a Hausforff space by taking quotient $E_U/\ker p$ (where $p$ is the seminorm corresponding to $U$) and then make its completion $\hat{E}_U$ ly Hausdorff and complete. Therefore, we will turn it into a Hausforff space by taking quotient $E_U/\ker p$ and then make its completion $\hat{E}_U$

$$E_U \xrightarrow{h_U} E_U/\ker p \xrightarrow{j_U} \hat{E}_U$$

where $h_U$ is the canonical mapping and $j_U$ is the injection of a space into its completion.

The tensor product $E' \otimes F$ of the dual of $E$ with $F$ can be considered as a subspace of the space of continuous linear maps $E \to F$, denoted by $L(E;F)$. To be more accurate, an arbitrary element $\sum_{j\in I} x'_j \otimes y_j$, where $I$ is some finite subset of $\mathbb{N}$, $x'_j \in E', y_j \in F$ defines the mapping

$$x \mapsto \sum_{j\in I}\langle x'_j, x\rangle y_j.$$

First we study the case of two Banach spaces $E$ and $F$. Here we have simply $L(E,F) = L_b(E,F)$ because every continuous mapping from $E$ into $F$ is automatically bounded. Moreover, the space $L(E,F)$

is also Banach, endowed with the operator norm. The bilinear map from $E' \times F$ into $L(E, F)$

$$(x', y) \mapsto (x \mapsto \langle x', x \rangle y)$$

is continuous with the norm $\leq 1$. Therefore, one can deduce easily, from the properties of the previously defined $\pi$-seminorm which is $\pi$-norm in this case ([60, Thm. 43.12(a), pp. 478]) that the norm induced by $L(E, F)$ on $E' \otimes F$ is $\leq \| \cdot \|_\pi$. Therefore, we can extend the injection of $E' \otimes F$ into $L(E; F)$ to be continuous linear (not necessarily injective) mapping $E' \hat{\otimes}_\pi F$ into $L(E, F)$. The image of $E' \hat{\otimes}_\pi F$ is denoted by $L^1(E, F)$ and its elements are *nuclear mappings* of $E$ into $F$.

Thus, $L^1(E, F)$ is isomorphic, in the sense of vector spaces, to $E' \hat{\otimes}_\pi F/N$ where $N$ is the kernel of the mapping $E' \hat{\otimes}_\pi F \to L(E, F)$. The norm from the factor space (that is apparently Banach) is called a *trace* norm, which is, restricted to $E' \otimes F$, nothing but the $\pi$-norm.

We now give the definition of a nuclear operator for the case of two locally convex Hausforff spaces $E$ and $F$. As we anticipated, take $U$, a convex, balanced, closed zero neighborhood in $E$, and $B$,a convex, balanced and bounded subset of $F$ such that $F_B$ is Banach. Let $u : \hat{E}_U \to F_B$ be a continuous linear map (between two Banach spaces). Then we may define a map $\tilde{u} : E \to F$ via the sequence.

$$E \xrightarrow{h_U} \hat{E}_U \xrightarrow{u} F_B \xrightarrow{i_B} F,$$

where $h_U$ is simply the canonical mapping $E \to \hat{E}_U$, while $i_B$ is the natural injection $F_B \to F$. Assuming $\tilde{u} = 0$ we conclude that $u$ vanishes on a dense subset of $\hat{E}_U$, therefore $u = 0$. This means that, via the mapping $u \mapsto \tilde{u}$, $L(\hat{E}_U, F_B)$ can be considered as the subspace of $L(E, F)$. For every such neighborhood $U \subseteq E$ and the set $B \subseteq F$, union of the (previously defined) subspaces $L^1(E, F) \subseteq L(\hat{E}_U, F_B) \subseteq L(E, F)$ is denoted by $L^1(E, F)$ and its elements are nuclear mappings of $E$ into $F$.

This definition is so complicated that we cannot even say whether $L^1(E, F)$ is a vector space or not without further discussion. However, this becomes evident after the next property.

**Proposition 1.2.5.** *If $E$ and $F$ are two locally convex Hausdorff spaces and $u : E \to F$ nuclear. If $G, H$ are two other locally convex Hausdorff spaces, and $g : G \to E, h : F \to H$ two mappings that are (just) continuous and linear. Then $h \circ u \circ g$ is nuclear.*

The proof can be found in [60, pp. 480].

The reader maybe recalls the same property for a compact mapping ([40, Chapters 15, 16 for Banach and Hilbert spaces], [60, Definition 47.4]), which is natural because every nuclear mapping is compact [60, Prop. 47.3].

The next theorem makes nuclearity easier to prove.

**Proposition 1.2.6** ([60, Prop. 47.2])**.** *If $E$ and $F$ are two locally convex Hausdorff spaces and $u : E \to F$ a continuous linear map. The following are equivalent.*

*a) $u$ is nuclear;*

*b) There exist an equicontinuous sequence $\{x'_k\}$ in $E'$ and a sequence $\{y_k\}$ in a convex balanced bounded set $B \subset F$ such that $F_B$ is Banach, as well as complex sequence $\{\lambda_k\}$, $\sum_k |\lambda_k| < \infty$ and the following holds:*

$$u(x) = \sum_k \lambda_k \langle x'_k, x\rangle y_k. \tag{1.2.3}$$

*Proof.* $(a) \Rightarrow (b)$ follows from the definition of nuclearity and the [60, Thm. 45.1]. To prove the converse, if $H'$ is the convex balanced weakly closed hull of the sequence $\{x'_k\}$, notice that $\sum_k \lambda_k x'_k \otimes y_k$ converges absolutely on $E'_{H'} \hat{\otimes} F_B$ (where the latter is, for now, a normed space).

The polar of $H'$, $U = H'^{\circ}$ is a convex balanced closed neighborhood of zero in $E$ [40, Chapter 23] and $E'_{H'}$ is the space of linear functions $E \to \mathbb{C}$ (linear functionals) continuous on $E_U$. This can be extended to the Banach space $\hat{E}_U$ from which it is clear that $E'_{H'}$ can be identified as a dual of a Banach space $\hat{E}_U$, therefore a Banach space itself. From the fact that the sum absolutely converges in $(\hat{E}_U)' \hat{\otimes} F_B$, the conclusion follows.

□

Let us just mention that, in the case of two Banach spaces, the previous theorem is valid if $\{x'_k\}$ (resp. $\{y_k\}$) belong to the closed unit ball in $E'$ (resp. $F$), which can be deduced easily.

### 1.2.2 Nuclear spaces

Bearing in mind that some of the spaces interesting for our research are nuclear and that we will exploit the properties arising from nuclearity throughout the thesis, here we define nuclear spaces.

We may allow ourselves a motivational example that will be the best explanation why nuclearity is so important for us.

Recall Schwartz kernels theorem. Basically, every continuous linear mapping of the space $\mathcal{D}_x(\mathbb{R})$ (for the purpose of our example, we do not need full generality) of test functions (in variable $x$) into space of distributions $\mathcal{D}'_y(\mathbb{R})$ in the variable $y$ can be also given by a (kernel) distribution $K_{x,y} \in \mathcal{D}'_{x,y}(\mathbb{R}^2)$:

$$(y \mapsto \phi(y)) = \phi \to \langle K_{x,y}, \phi(x) \rangle.$$

However, this representation fails to be true for the case of "classical" functions $L^2_x(\mathbb{R})$, namely, it is not true in general that every bounded operator $L^2_x(\mathbb{R})$ into $L^2_y(\mathbb{R})$ can be represented via kernel $K_{x,y} \in L^2_{x,y}(\mathbb{R}^2)$, therefore being a mapping $f \mapsto \int_{\mathbb{R}} K(x,y)f(x)dx$ because even the identity mapping cannot be represented in that way (the kernel for the identity mapping, when we drop "classical" nature of functions as a constraint is $K(x,y) = \delta(x-y)$).

Nuclearity is what lies beyond that strange difference between $\mathcal{D}'$ and $L^2$. The space $\mathcal{D}'$ is nuclear, while $L^2$ is not. A nuclear space is, basically, a locally convex Hausdorff space such that, for another locally convex space $F$, $E\hat{\otimes}_\pi F = E\hat{\otimes}_\varepsilon F$.

In our case of kernels, knowing that $\mathcal{D}'_{x,y}$ induces the $\pi$-topology (or the $\varepsilon$-topology) on a tensor space $\mathcal{D}'_x \otimes \mathcal{D}'_y$ we obtain $\mathcal{D}'_{x,y} \cong \mathcal{D}'_x \hat{\otimes} \mathcal{D}'_y$, with the luxury not to use indices $\pi$ or $\varepsilon$. This example also shows why nuclearity of the space of distributions is so convenient for the kernel theorem, namely, because it allows (in some sense) separation of the variables and factorizing the starting space into spaces with respect to every variable. In the discussion regarding tensor product of two mappings we may use either $\pi$-topology (that behaves well under homomorphisms "onto" [60, Prop. 43.7]) or $\varepsilon$-topology (that behaves well under isomorphisms "into" [60, Prop. 43.9]).

**Definition 1.2.7** ([60, Definition 50.1])**.** The locally convex Hausdorff space $E$ is *nuclear* if for every continuous seminorm $p$ on $E$ there exists another seminorm $q$, $q \geq p$, such that the canonical mapping $\hat{E}_q \to \hat{E}_p$ is nuclear.

The reader should recall the basics from the theory of a locally convex space [40, Chapter 22] and conclude that $E_p = E_{p^{-1}(\mathbb{D})}$ where $\mathbb{D} = \{z \in \mathbb{C} : |z| \leq 1\}$ is the unit disc in $\mathbb{C}$. From that one can conclude that $\hat{E}_p$ is just the completion of the space $E_p/\ker p$. What the canonical mapping $\hat{E}_q \to \hat{E}_p$ is should not be hard to guess. The assumption $q \geq p$ implies that $\ker q \subseteq \ker p$ which admits canonical mapping $E/\ker q$ onto $E/\ker p$; this mapping is actually continuous if the norm of the first space is $q/\ker q$ and the norm of the other one $p/\ker p$, which is then extended to the mentioned canonical mapping.

This theorem justifies the introduction of nuclear spaces

**Theorem 1.2.8** ([60, Thm. 40.1])**.** *The following properties of a locally convex space $E$ are equivalent:*

*a) $E$ is nuclear;*

*b) Every continuous linear map $E$ into a Banach space is nuclear.*

*c) For every Banach space $F$, the canonical map of $E\hat{\otimes}_\pi F$ into $E\hat{\otimes}_\varepsilon F$ is an isomorphism onto;*

*d) For every locally convex Hausdorff space $E$, the canonical map of $E\hat{\otimes}_\pi F$ into $E\hat{\otimes}_\varepsilon F$ is an isomorphism onto.*

As a consequence, for all locally convex Haudorff space $F$, $E \otimes_\pi F = E \otimes_\varepsilon F$.

Nuclearity is a property that remains under completion. Also, subspaces of a nuclear space are also nuclear, as well as quotients modulo a closed (linear) subspace. Moreover, product of nuclear spaces is nuclear as well, like their projective limit if it is Hausdorff. A countable inductive limit of nuclear spaces is nuclear and, finally, nuclearity of $E$ and $F$ implies nuclearity of $E\hat{\otimes}F$ [60, Prop. 50.1], [54, Thm. 7.4, 7.5].

As we mentioned, the space of distributions is nuclear. Some of the well-known nuclear spaces are the space $\mathcal{E}(\mathbb{R}^n)$ as well as the space of Schwartz functions $\mathcal{S}(\mathbb{R}^n)$ [54, pp. 106–107].

## 1.3 Ultradifferentiable functions and ultradistributions

We briefly review in the section the definition and some properties of the spaces of ultradifferentiable functions and ultradistributions [10, 33, 34].

Fix a positive sequence $(M_p)_{p\in\mathbb{N}}$ with $M_0 = 1$. We will make use of some of the following standard conditions on the weight sequence

$(M.0)$ $p! \subset M_p$.

$(M.1)$ $M_p^2 \leq M_{p-1}M_{p+1}$, $p \geq 1$. (logarithmic convexity)

$(M.1)^*$ $\left(\frac{M_p}{p!}\right)^2 \leq \frac{M_{p-1}}{(p-1)!} \cdot \frac{M_{p+1}}{(p+1)!}$, $p \geq 1$.

$(M.2)'$ $M_{p+1} \leq AH^pM_p$, $p \in \mathbb{N}$, for some $A, H > 0$. (stability under differential operators)

$(M.2)$ $M_p \leq AH^p \min_{1\leq q\leq p}\{M_qM_{p-q}\}$, $p \in \mathbb{N}$, for some $A, H > 0$. (stability under ultradifferential operators)

$(M.3)'$ $\sum_{p=1}^{\infty} M_{p-1}/M_p < \infty$. (non-quasianalyticity)

$(M.3)$ $\sum_{q=p+1}^{\infty} \frac{M_{p-1}}{M_p} \leq c_0 p \frac{M_p}{M_{p+1}}, p \geq 1,$ for some $c_0 > 0$. (strong non-quasianalyticity)

$(QA)$ $\sum_{p=1}^{\infty} M_{p-1}/M_p = \infty$. (quasianalyticity)

We will always impose $(M.1)$ while the other assumptions will be imposed when needed (which will be mentioned explicitly).

For an arbitrary $\mu > 0$, the reader can easily verify that the sequence $M_p = (p!)^\mu$, called Gevrey sequence, satisfies $(M.1)$ and $(M.2)$ while it satisfies also $(M.3)$ if $\mu > 1$. The relations $\subset$ and $\prec$ among sequences are defined as follows. One writes $N_p \subset M_p$ ($N_p \prec M_p$) if there are $C, \ell > 0$ (for each $\ell$ there is $C = C_\ell$) such that $N_p \leq C\ell^p M_p$, $p \in \mathbb{N}$. If $(M.3)'$ holds, we call $M_p$ non-quasianalytic; otherwise it is said to be quasianalytic.

We will also need the notion of associated function of the sequence, it is defined as

$$M(t) = \sup_{p\in\mathbb{N}} \log\left(\frac{t^p}{M_p}\right), \quad t > 0,$$

and $M(0) = 0$. This function is non-negative, continuous, increasing, vanishes for sufficiently small $t > 0$. In the particular case of Gevrey sequences, when $M_p = (p!)^s$, the associated function is $M(t) \asymp t^{1/s}$ [24].

See also [33] for the precise meaning of conditions of the weight sequences and how can they be translated into properties of $M$. In particular, properties $(M.1)$ and $(M.2)'$ imply

$$t^\eta e^{-M(H^\eta t)} \leq A^\eta e^{-M(t)}, \quad \text{for all } t > 0, \tag{1.3.1}$$

see [33, Eq. (3.13), p. 50]. We shall often make use of this inequality. We also point out that, under $(M.1)$, the condition $(M.0)$ becomes equivalent to the bound $M(t) = O(t)$ [33, Lemma 3.8].

Let $\Omega \subseteq \mathbb{R}^d$ be open. The space of all $C^\infty$-functions on $\Omega$ is denoted by $\mathcal{E}(\Omega)$. For $K \Subset \Omega$ (a compact subset with non-empty interior) and $h > 0$, one writes $\mathcal{E}^{\{M_p\},h}(K)$ for the space of all $\varphi \in \mathcal{E}(\Omega)$ such that

$$\|\varphi\|_{\mathcal{E}^{\{M_p\},h}(K)} := \sup_{\substack{x\in K\\ \alpha\in\mathbb{N}^n}} \frac{|\varphi^{(\alpha)}(x)|}{h^{|\alpha|}M_{|\alpha|}} < \infty,$$

and $\mathcal{D}_K^{\{M_p\},h}$ stands for the closed subspace of $\mathcal{E}^{\{M_p\},h}(K)$ consisting of functions with compact support in $K$ (if one also assumes $(M.1)$, its non-triviality is equivalent to $(M.3)'$, see the Denjoy-Carleman theorem [33, Thm. 4.2, pp. 56]). Set then

$$\mathcal{E}^{\{M_p\}}(K) = \varinjlim_{h\to\infty} \mathcal{E}^{\{M_p\},h}(K), \qquad \mathcal{E}^{(M_p)}(K) = \varprojlim_{h\to 0^+} \mathcal{E}^{\{M_p\},h}(K),$$

$$\mathcal{E}^{\{M_p\}}(\Omega) = \varprojlim_{K\Subset\Omega} \varinjlim_{h\to\infty} \mathcal{E}^{\{M_p\},h}(K), \quad \mathcal{E}^{(M_p)}(\Omega) = \varprojlim_{K\Subset\Omega} \varprojlim_{h\to 0^+} \mathcal{E}^{\{M_p\},h}(K),$$

and

$$\mathcal{D}^{\{M_p\}}(\Omega) = \varinjlim_{K\Subset\Omega} \varinjlim_{h\to\infty} \mathcal{D}_K^{\{M_p\},h} \quad \mathcal{D}^{(M_p)}(\Omega) = \varinjlim_{K\Subset\Omega} \varprojlim_{h\to 0^+} \mathcal{D}_K^{\{M_p\},h}.$$

In order to treat these spaces simultaneously we adopt standard notation and usually write $* = \{M_p\}, (M_p)$. The class $\mathcal{E}^*(\Omega)$ is called the class of ultradifferentiable functions on $\Omega$ of Roumieu type if $* = \{M_p\}$, or the class of Beurling type if $* = (M_p)$. In statements needing a separate treatment we will first state assertions for the Roumieu case, followed by the Beurling one in parenthesis.

A word about topologies on these spaces. The spaces $\mathcal{E}^{(M_p)}(K), \mathcal{E}^{(M_p)}(\Omega)$ and $\mathcal{D}_K^{(M_p)}$ are $FS$-spaces, Fréchet-Schwartz spaces. On the other hand, $\mathcal{E}^{\{M_p\}}(K)$, $\mathcal{D}_K^{\{M_p\}}(\Omega)$, are $DFS$-spaces (see [33, Thm. 2.5. pp. 44], definition and properties of these topologies can be found in [33, 40]). These spaces are separable complete bornological Montel and Schwartz spaces. If we additionally assume that $M_p$ satisfies $(M.2)'$, then all these spaces are nuclear (the reader is strongly advised to read how the nuclearity is being proved, [33, Proposition 2.4, pp. 43]).

The space $\mathcal{D}^{(M_p)}(\Omega)$, as the inductive limit of a strict injective sequence of $(FS)$-spaces, is an $(LFS)$-space.

It is not hard to prove that the space $\mathcal{D}^{\{M_p\}}(\Omega)$ is an inductive limit of a strict injective sequence of a $(DFS)$-spaces, therefore a $(DFS)$-space itself.

Condition $(M.3)'$ is very important because it, with $(M.1)$ assumed is equivalent to the existence of ultradifferentiable function of type $(M_p)$ with compact support. These functions are non-quasianalytic (see [52, 19.5, pp. 377] for quasi-analytic classes). Therefore $(M.3)'$ will be called non-quasianalyticity and its negation, $(QA)$, is quasianalyticity. The case of real analytic functions serves as a good example of quasianalytic functions.

If we assume $(M.1)$ and $(M.3)'$, when we are able to perform the partition of unity subordinate to the cover by ultradifferentiable functions ( [33, Lema 5.1, pp. 61]) and, like in distributional case, construct sheaves of ultradistributions, namely, (strong) duals of spaces of ultradifferentiable functions $\mathcal{D}^*(\Omega)$ which will lead to space of ultradistributions $\mathcal{D}'^*(\Omega)$, both of Beurling or Roumieu type [33].

In the special but very important case, when $* = \{p!\}$, we write $\mathcal{A}(\Omega) = \mathcal{E}^{\{p!\}}(\Omega)$, the space of real analytic functions on $\Omega$; its dual $\mathcal{A}'(\Omega)$ is then the space of analytic functionals on $\Omega$.

Note that $(M.0)$ implies that $\mathcal{A}(\Omega) \subseteq \mathcal{E}^*(\Omega)$, and, if in addition $(M.1)$ and $(M.2)'$ hold, $\mathcal{A}(\Omega)$ is densely injected into $\mathcal{E}^*(\Omega)$ because the polynomials are dense in both spaces; in particular, $\mathcal{E}^{*\prime}(\Omega) \subseteq \mathcal{A}'(\Omega)$ under these assumptions.

### 1.3.1 Ultradifferentiability on $\mathbb{S}^{n-1}$

In this subsection we focus on the unit sphere $\mathbb{S}^{n-1}$ . Let us first note that differentiability on $\mathbb{S}^{n-1}$ is defined in the usual way, as differentability on compact analytic manifolds via local analytic coordinates. We will use the typical notation, namely, $\mathcal{E}(\Omega) := C^\infty(\Omega)$ for the space of smooth functions on $\Omega$, where $\Omega$ is an open subset of $\mathbb{S}^{n-1}$. Given a function $\varphi$ on $\mathbb{S}^{n-1}$, its homogeneous extension (of order 0) is the function $\varphi^\upharpoonleft$ defined as $\varphi^\upharpoonleft(x) = \varphi(x/|x|)$ on $\mathbb{R}^n \setminus \{0\}$. It is easy to see that $\varphi \in \mathcal{E}(\mathbb{S}^{n-1})$ if and only if $\varphi^\upharpoonleft \in \mathcal{E}(\mathbb{R}^n \setminus \{0\})$. Furthermore, we define the differential operators $\partial^\alpha_{\mathbb{S}^{n-1}} : \mathcal{E}(\mathbb{S}^{n-1}) \to \mathcal{E}(\mathbb{S}^{n-1})$ via

$$(\partial^\alpha_{\mathbb{S}^{n-1}}\varphi)(\omega) = (\partial^\alpha \varphi^\upharpoonleft)(\omega), \quad \omega \in \mathbb{S}^{n-1}.$$

We can then consider $L(\partial_{\mathbb{S}^{n-1}})$ for any differential operator $L(\partial)$ defined on $\mathbb{R}^n \setminus \{0\}$. In particular, $\Delta_{\mathbb{S}^{n-1}}$ stands for the Laplace-Beltrami operator of the sphere.

Finally, if $F$ is a function on $\mathbb{R}^n$, we simply write $\|F\|_{L^q(\mathbb{S}^{n-1})}$ for the $L^q(\mathbb{S}^{n-1})$-norm of its restriction to $\mathbb{S}^{n-1}$.

Not just differentiability, but also ultradifferentiability can be defined on real analytic manifolds if one assumes $(M.0)$ (as we will always do when talking about ultradifferentiability on the sphere) for the sequence $M_p$. Indeed, the pullback of an invertible analytic change of variables $\Omega \to U$ becomes a TVS isomorphism between $\mathcal{E}^*(U)$ and $\mathcal{E}^*(\Omega)$ [29, Prop. 8.4.1]. Therefore, one can always define the spaces $\mathcal{E}^*(M)$ and $\mathcal{E}^{*\prime}(M)$ for $\sigma$-locally compact analytic manifolds $M$ via charts if $(M.0)$ holds. Note that $(M.0)$ is automatically
fulfilled if $(M.1)$ and $(M.3)'$ hold [33, Lemma 4.1].

However, in the sequel we will introduce ultradifferentiability on the sphere using homogeneous extensions. The reader will easily see, as we remark below, that these two concepts of defining ultradifferentiability are equivalent.

Define the space $\mathcal{E}^{\{M_p\}}(\mathbb{S}^{n-1})$ of ultradifferentiable functions of Roumieu type (or class $\{M_p\}$) as the space of all smooth functions $\varphi \in \mathcal{E}(\mathbb{S}^{n-1})$ such that

$$\sup_{\alpha\in\mathbb{N}} \frac{h^{|\alpha|}\|\partial^{\alpha}_{\mathbb{S}^{n-1}}\varphi\|_{L^\infty(\mathbb{S}^{n-1})}}{M_{|\alpha|}} < \infty, \tag{1.3.2}$$

for some $h > 0$. It is worth mentioning that the case $M_p = (p!)^s$ with $s \geq 1$ is important, when one recovers the spaces of Gevrey differentiable functions on the sphere. Among these, there is a special but very important case $M_p = p!$, we also write $\mathcal{A}(\mathbb{S}^{n-1}) = \mathcal{E}^{\{p!\}}(\mathbb{S}^{n-1})$; this is in fact the space of real analytic functions on $\mathbb{S}^{n-1}$ [38].

The space $\mathcal{E}^{(M_p)}(\mathbb{S}^{n-1})$ of ultradifferentiable functions of Beurling type (class $(M_p)$) is defined by requiring that (1.3.2) holds for every $h > 0$. Whenever we consider the Beurling case on the sphere, we suppose that $M_p$ satisfies the ensuing stronger assumption than $(M.0)$,

$(NA)$ For each $L > 0$ there is $A_L > 0$ such that $p! \leq A_L L^p M_p$, $p \in \mathbb{N}$ $(p! \prec M_p)$.

Notice $(M.1)$ implies that $(NA)$ is equivalent to $M(t) = o(t)$ as $t \to \infty$ [33, Lemma 3.10].

It should be noticed that the condition $(M.0)$ (the condition $(NA)$) implies that $\mathcal{A}(\mathbb{S}^{n-1})$ is the smallest among all spaces of ultradifferentiable functions that we consider here, that is, one always has the inclusion $\mathcal{A}(\mathbb{S}^{n-1}) \subseteq \mathcal{E}^*(\mathbb{S}^{n-1})$.

Let us remark that, since we have used the differential operators $\partial^\alpha_{\mathbb{S}^{n-1}}$ in (1.3.2), $*$-ultradifferentiability of $\varphi$ on $\mathbb{S}^{n-1}$ is the same as $*$-ultradifferentiability of its homogeneous extension (of order 0) on $\mathbb{R}^n \setminus \{0\}$, namely,

$$\varphi \in \mathcal{E}^*(\mathbb{S}^{n-1}) \quad \text{if and only if} \quad \varphi^{\upharpoonright} \in \mathcal{E}^*(\mathbb{R}^n \setminus \{0\}),$$

with the spaces of ultradifferentiable functions on an open subset of $\mathbb{R}^n$ defined as above. Moreover, in view of the analyticity of the mapping $x \to x/|x|$ and the fact that the pullbacks by analytic functions induce mappings between spaces of $*$-ultradifferentiable functions under the assumptions $(M.0)$ ($(NA)$ in the Beurling case), $(M.1)$ and $(M.2)'$ (cf. [29, Prop. 8.4.1], [34, p. 626], [50]), our definition of $\mathcal{E}^*(\mathbb{S}^{n-1})$ coincides with that of $*$- ultradifferentiable functions on compact analytic manifolds via local analytic coordinates.

### 1.3.2 Spaces of Gelfand-Shilov type and tempered ultradistributions

In the sequel, we will introduce spaces of type $\mathcal{S}$, introduced in [24] (see also [46] for the quasianalytic case). Dealing with the spaces that will be introduced in the sequel requires some assumptions on the weight sequence apart from the standard $(M.1)$ that we always assume. First we impose the essential assumption:

$$\sqrt{p!} \leq C_l l^p M_p, \quad \forall p \in \mathbb{N}_0 \quad \begin{array}{l}\text{(Roumieu case: for some } l, C_l > 0)\\ \text{(Beurling case: } \forall l > 0 \text{ there is } C_l > 0).\end{array} \tag{1.3.3}$$

With this assumption, we define the spaces of Gelfrand-Shilov type of ultradifferentiable functions of ultrapolynomial growth (that will serve as test spaces for the spaces of temperate ultradistributions), $\mathcal{S}^{\{M_p\}}(\mathbb{R}^n)$ and $\mathcal{S}^{(M_p)}(\mathbb{R}^n)$ as follows. First introduce the Banach space $\mathcal{S}_{L^2}^{\{M_p\},h}$, $h > 0$, consisting of all $f \in C^\infty(\mathbb{R}^n)$ such that

$$\|f\|_h := \sup_{\alpha,\beta\in\mathbb{N}_0^n} \frac{\|x^\beta \partial^\alpha f\|_{L^2(\mathbb{R}^n)}}{h^{|\alpha|+|\beta|} M_{|\alpha|+|\beta|}} < \infty \ ; \tag{1.3.4}$$

define then

$$\mathcal{S}^{\{M_p\}}(\mathbb{R}^n) = \varinjlim_{h\to\infty} \mathcal{S}_{L^2}^{\{M_p\},h} \quad \text{and} \quad \mathcal{S}^{(M_p)}(\mathbb{R}^n) = \varprojlim_{h\to 0} \mathcal{S}_{L^2}^{\{M_p\},h}, \tag{1.3.5}$$

If we additionally assume $(M.2)'$, these spaces are $(DFS)$- and $(FS)$-spaces, respectively. It is worth noticing that if $(M.2)'$ holds, using the norms $\| \ \|_{L^2(\mathbb{R}^n)}$ instead of $\| \ \|_{L^\infty(\mathbb{R}^n)}$ in (1.3.4) leads to an equivalent definition of $\mathcal{S}^*(\mathbb{R}^n)$.

As customary, one writes $\mathcal{S}^\mu_\mu(\mathbb{R}^n) = \mathcal{S}^{\{M_p\}}(\mathbb{R}^n)$ and $\Sigma^\mu_\mu(\mathbb{R}^n) = \mathcal{S}^{(M_p)}(\mathbb{R}^n)$ for the special case $M_p = (p!)^\mu$. Condition (1.3.3) yields $\mathcal{S}^{1/2}_{1/2}(\mathbb{R}^n) \subseteq \mathcal{S}^*(\mathbb{R}^n)$, which ensures the non-triviality of these spaces (as the function $\varphi(x) = e^{-|x|^2} \in \mathcal{S}^{\frac{1}{2}}_{\frac{1}{2}}(\mathbb{R}^n)$.)

The strong duals of the spaces $\mathcal{S}^*(\mathbb{R}^n)$ are the spaces of temperate ultradistributions $\mathcal{S}^*(\mathbb{R}^n)$ of Beurling or Roumieu type.

Although we have already mentioned (locally convex) topologies imposed on these spaces, namely the topology of projective and inductive limits of Banach spaces, in these particular cases we can be even more precise with the imposed topological structure.

In the sequel we define *tame continuity* of linear mappings for graded Fréchet spaces and inductive limits of Banach spaces. This notion is very important in the structure theory of Fréchet spaces (see e.g. [66]). A graded Fréchet space is a Fréchet space together with a choice of a non-decreasing sequence of seminorms defining its topology. A continuous linear mapping $T : (E, | \ |_j) \to (F, | \ |'_j)$ between two graded Fréchet spaces is called (linearly) tame if there are constants $L > 0$ and $j_0$ such that $|Tv|'_{Lj} \le C_j |v|_j$, for all $j \ge j_0$ and $v \in E$. Tame continuity for $(LB)$ spaces is defined similarly. Once one implicitly fixes the increasing sequences of Banach spaces, a mapping $T : E = \varinjlim_j E_j \to F = \varinjlim_j F_j$ is tamely continuous if there are $L$ and $j_0$ such that $\|Tv\|_{F_{Lj}} \leq C_j \|v\|_{E_j}$, for all $j \geq j_0$ and $v \in E_j$. Then the meaning of a tame isomorphism is clear.

In the next sections we always consider the grading of $\mathcal{S}^*(\mathbb{R}^n)$ given by (1.3.5), that is, the one provided by the Banach spaces $\mathcal{S}_{L^2}^{\{M_p\},h}$. Furthermore, with $(M.2)'$ assumed and using $\|\cdot\|_{L^r}$ instead of $\|\cdot\|_{L^2}$ norm in the definition of (1.3.4), $1 \leq r \leq \infty$, one can easily prove that modified system of norms is tamely equivalent to (1.3.4), as one easily verifies.

We also remark that our definition of the norms (1.3.4) does not separate between the behavior of derivatives and growth. On the other hand, if the sequence satisfies $(M.2)$, such behavior can be split and our system of norms becomes tamely equivalent to $\sup_{\alpha,\beta\in\mathbb{N}_0^n} \|x^\beta \partial^\alpha f\|_{L^2(\mathbb{R}^n)}/(h^{|\alpha|+|\beta|} M_{|\alpha|} M_{|\beta|})$. However, $(M.2)$ plays basically no role in our arguments when considering $\mathcal{S}^*(\mathbb{R}^n)$, we shall therefore not impose it and we choose to use the family of norms (1.3.4).

### 1.3.3 Ultradifferentiability on $\mathbb{T}^n$

In what follows, we present the basics of calculus on the torus $\mathbb{T}^n$ and some results from analysis on $\mathbb{T}^n$ that will be useful for us. We follow the approach from [53].

First we need to emphasize what will be the torus for us. Therefore, fix $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n = \left(\mathbb{R}/\mathbb{Z}\right)^n$. Often we will identify $\mathbb{T}^n$ with the hypercube $[0,1)^n \subset \mathbb{R}^n$ equipped with the restriction of the Lebesgue measure. With the equivalence relation defined as $x \sim y \Leftrightarrow x - y \in \mathbb{Z}^n$ on the Euclidean space $\mathbb{R}^n$, and the equivalence classes

$$[x] = \{y \in \mathbb{R}^n : x \sim y\} = \{x + k : k \in \mathbb{Z}^n\},$$

a point $x \in \mathbb{R}^n$ is naturally mapped to a point $[x] \in \mathbb{T}^n$, and we shall write $x \in \mathbb{T}^n$ instead of the actual $[x] \in \mathbb{T}^n$. Therefore, we may identify functions on $\mathbb{T}^n$ with $\mathbb{Z}^n$-periodic functions on $\mathbb{R}^n$ in a natural manner,

$\varphi : \mathbb{T}^n \to \mathbb{C}$ is identified with $\psi : \mathbb{R}^n \to \mathbb{C}$ satisfying $\psi(x) = \varphi([x])$ for all $x \in \mathbb{R}^n$. In such a case we should say that $\varphi \in C^\infty(\mathbb{T}^n)$ when actually $\psi \in C^\infty(\mathbb{R}^n)$. We rename $\mathcal{E}(\mathbb{T}^n) = C^\infty(\mathbb{T}^n)$.

As usual, we are also interested in ultradifferentiability. The reader should first consult Section 1.3; one may do this via $\varphi \in \mathcal{E}^*(\mathbb{T}^n)$ if and only if $\psi \in \mathcal{E}^*(\mathbb{R}^n)$ when regarded as a periodic function on $\mathbb{R}^n$. Let us be more explicit.

For a defining sequence $\{M_p\}$, first we introduce the Banach space

$$\mathcal{E}^{\{M_p\},h}(\mathbb{T}^n) = \{\varphi \in \mathcal{E}(\mathbb{T}^n) : \|\varphi\|_{\mathcal{E}^{\{M_p\},h}([0,1]^n)} < \infty\}.$$

And then we may define, as usual

$$\mathcal{E}^{\{M_p\}}(\mathbb{T}^n) = \varinjlim_{h\to\infty} \mathcal{E}^{\{M_p\},h}(\mathbb{T}^n), \qquad \mathcal{E}^{(M_p)}(\mathbb{T}^n) = \varprojlim_{h\to 0} \mathcal{E}^{\{M_p\},h}(\mathbb{T}^n).$$

The space of ultradifferentiable functions of Beurling type, $\mathcal{E}^{(M_p)}(\mathbb{T}^n)$ is an $(FS)$-space, while the space of Roumieu ultradifferentiable functions $\mathcal{E}^{\{M_p\}}(\mathbb{T}^n)$ is a $(DFS)$-space. Their duals are the spaces of ultradistributions of Roumieu and Beurling type [33]. In order to treat these spaces simultaneously we write $* = \{M_p\}, (M_p)$, as usual.

Now we need some properties of the integral lattice $\mathbb{Z}^n$ that will be recalled in the sequel. Denote the space of rapidly decaying (or Schwartz) functions $\varphi : \mathbb{Z}^n \to \mathbb{C}$ by $\mathcal{S}(\mathbb{Z}^n)$. More precisely, $\varphi \in \mathcal{S}(\mathbb{Z}^n)$ if and only if for every $M > 0$ there exists $c_{\varphi,M} > 0$ for which $|\varphi(\xi)| \le C_{\varphi,M}\langle\xi\rangle^{-M}$. The Schwartz space $\mathcal{S}(\mathbb{Z}^n)$ is an $FS$-space endowed by the seminorms $p_k(\varphi) = \sup_{\xi\in\mathbb{Z}^n}\langle\xi\rangle^k|\varphi(\xi)|$. As usual, the strong dual of $\mathcal{S}(\mathbb{Z}^n)$ will be denoted by $\mathcal{S}'(\mathbb{Z}^n)$.

In the sequel we should introduce the toroidal Fourier transform. For $u \in \mathcal{E}(\mathbb{T}^n)$ we shall use a notation $\mathcal{F}_{\mathbb{T}^n}u$ or simply $\hat{u}$ for $\hat{u}(\xi) = \int_{\mathbb{T}^n} e^{-2\pi i x\cdot\xi}u(x)dx$ where $x \cdot \xi$ stands for the standard inner product of $x, \xi \in \mathbb{R}^n$. It is easy to see that $\mathcal{F}_{\mathbb{T}^n}$ is a bijection and that the inverse mapping $\mathcal{F}_{\mathbb{T}^n}^{-1}$ is given by $\mathcal{F}_{\mathbb{T}^n}^{-1}\varphi(x) = \sum_{\xi\in\mathbb{Z}^n} e^{2\pi i x\cdot\xi}\hat{\varphi}(\xi)$ for $\varphi \in \mathcal{S}(\mathbb{Z}^n)$.

Since we are interested in the spaces of ultradifferentiable functions, it will be useful to see how the toroidal Fourier transform acts on them. Thus, we have the following lemma.

**Lemma 1.3.1.** *Suppose that the sequence* $\{M_p\}$ *satisfies*[1] $(M.1)$ *and* $(M.2)$ *and let* $u \in \mathcal{E}(\mathbb{T}^n)$. *Then* $u \in \mathcal{E}^{\{M_p\}}(\mathbb{T}^n)$ *(*$u \in \mathcal{E}^{(M_p)}(\mathbb{T}^n)$*) if and only if for some (for every)* $h > 0$ *there exists* $C_h > 0$ *such that* $\sum_{\xi\in\mathbb{Z}^n} |\hat{u}(\xi)|e^{M(h|\xi|)} \leq C_h$ *for every* $\xi \in \mathbb{R}^n$.

*Proof.* Let $u \in \mathcal{E}^{\{M_p\},h}(\mathbb{T}^n)$. Then, for $\xi \neq 0$ and the index $j$, $1 \leq j \leq n$ such that $|\xi_j| = \max_{k=1,\dots,n} |\xi_k|$, it is obvious that $|\xi_j| \geq \frac{|\xi|}{\sqrt{n}}$. It is now easy to see, with the help of $(M.2)$ and the integration by parts, that for arbitrary $p > 0$

$$\begin{aligned}
|\hat{u}(\xi)| &\leq \Big| \int_{\mathbb{T}^n} (-2\pi i \xi_j)^{-n-p-1} \left(\partial_{x_j}^{n+p+1} e^{-2\pi i x\cdot\xi}\right) u(x)dx \Big| \\
&\leq A\|u\|_{\mathcal{E}^{\{M_p\},h}(\mathbb{T}^n)} (\frac{\sqrt{n}Hh}{2\pi})^{n+p+1} M_{n+1}M_p|\xi|^{-p-n-1} \\
&\leq C_1 \frac{\|u\|_{\mathcal{E}^{\{M_p\},h}(\mathbb{T}^n)} (\frac{\sqrt{n}H}{2\pi})^p M_p}{|\xi|^{p+n+1}} \\
&\leq C_1 \|u\|_{\mathcal{E}^{\{M_p\},h}(\mathbb{T}^n)} \inf_{p\in\mathbb{N}} \left( \frac{(\frac{\sqrt{n}H}{2\pi})^p M_p}{|\xi|^p} \right) \cdot |\xi|^{-n-1} \\
&= C_1 \|u\|_{\mathcal{E}^{\{M_p\},h}(\mathbb{T}^n)} e^{-M(\frac{2\pi|\xi|}{H\sqrt{n}})} \cdot \frac{1}{|\xi|^{n+1}} \qquad (1.3.6)
\end{aligned}$$

where $A$ and $H$ are the constants from $(M.2)$, $C_1 = \frac{AM_{n+1}}{h^{n+1}}(\frac{\sqrt{n}H}{2\pi})^{n+1}$.

We now have

$$\sum_{\xi\in\mathbb{Z}^n} |\hat{u}(\xi)|e^{M(\frac{2\pi|\xi|}{\sqrt{n}Hh})} \leq C_2\|u\|_{\mathcal{E}^{\{M_p\},h}(\mathbb{T}^n)}$$

while $C_2 = C_1 \sum_{\xi\in\mathbb{Z}^n, \xi\neq 0} \frac{1}{|\xi|^{n+1}}$. This inequality practically proves the result.

The proof of the converse is similar. □

The proof of the previous lemma also shows that, if we define (for a fixed dimension $n$) Banach space with the weighted $l_1$-norm:

$$l_1^{\{M_p\},h} = \{(a_\xi)_{\xi\in\mathbb{Z}^n} : \|(a_\xi)_{\xi\in\mathbb{Z}^n}\|_{l_1^{\{M_p\},h}} := \sum_{\xi\in\mathbb{Z}^n} |a_\xi|e^{M(\frac{|\xi|}{h})} < \infty\}$$

[1] Condition $(M.2)'$, along with $(M.1)$, is enough; however, since our further calculations involve this result and a lot of other constants, it will be more convenient to perform the calculation here with the assumption $(M.2)$.

and then the graded spaces

$$l_1^{\{M_p\}} = \varinjlim_{h\to 0} l_1^{\{M_p\},h}; \qquad l_1^{(M_p)} = \varprojlim_{h\to\infty} l_1^{\{M_p\},h}$$

then the space $\mathcal{E}^*(\mathbb{T}^n)$ *is tamely isomorphic* to $l_1^*$ (the definition of the tame continuity was given in Subsection 1.3.2).

A simple property of the toroidal Fourier transform is that for $\varphi \in \mathcal{S}(\mathbb{Z}^n), u \in \mathcal{E}(\mathbb{T}^n)$:

$$\sum_{\xi\in\mathbb{Z}^n} \hat{u}(\xi)\varphi(\xi) = \int_{\mathbb{T}^n} u(x)\mathcal{F}_{\mathbb{T}^n}^{-1}\varphi(-x)dx \tag{1.3.7}$$

This allows us to define the Fourier transform for more general classes of functions, namely, (ultra)distributions on $\mathbb{T}^n$. Therefore, we extend the Fourier transform in the following way

$$\langle \hat{u}(\xi), \varphi(\xi)\rangle = \langle \mathcal{F}_{\mathbb{T}^n} u, \varphi\rangle = \langle u(x), (\mathcal{F}_{\mathbb{T}^n}^{-1}\varphi(\xi))(-x)\rangle$$

so that it is defined on the space $\mathcal{E}'(\mathbb{T}^n)$ and $\mathcal{E}'^*(\mathbb{T}^n)$. Note that $\hat{u}$ is a multisequence on $\mathbb{Z}^n$ and in fact $\hat{u}(\xi) = \langle u(x), e^{-2\pi i x\cdot\xi}\rangle$. Directly from the previous consideration we have the following corollary.

**Corollary 1.3.2.** *Suppose that the sequence $M_p$ satisfies $(M.1)$ and $(M.2)'$. Then $f \in \mathcal{E}'^{\{M_p\}}(\mathbb{T}^n)$ ($f \in \mathcal{E}'^{(M_p)}(\mathbb{T}^n)$) if and only if for every $L > 0$ (for some $L > 0$) there exists $C_L > 0$ such that*

$$|\hat{f}(\xi)| \le C_L e^{M(L|\xi|)}, \qquad \forall \xi \in \mathbb{Z}^n.$$

As in the case of Fourier transform on $\mathbb{R}^n$, we can extend the $\mathcal{F}_{\mathbb{T}^n}$ on $L^2(\mathbb{T}^n)$. Using standard notation

$$L^2(\mathbb{T}^n) = \{f \in C^\infty(\mathbb{T}^n) : \|f\|_{L^2(\mathbb{T}^n)} = \int_{\mathbb{T}^n} |f(x)|^2 dx < \infty\}$$

for the Hilbert space with the Hermitian inner product $(f,g)_{L^2(\mathbb{T}^n)} = \int_{\mathbb{T}^n} f(x)\overline{g(x)}dx$, then for $u \in L^2(\mathbb{T}^n)$ we can define (for every, $\xi \in \mathbb{Z}^n$), $\hat{u}(\xi) = \int_{\mathbb{T}^n} e^{-2\pi x\cdot\xi} u(x)dx$, and the partial sum of Fourier series $\sum_{\xi\in\mathbb{Z}^n} e^{2\pi i x\cdot\xi}\hat{u}(\xi)$ will converge to $u(x)$ in the norm $\|\cdot\|_{L^2(\mathbb{T}^n)}$. It follows than $\hat{u} \in l^2(\mathbb{Z}^n)$ and $\|u\|_{L^2(\mathbb{T}^n)} = \|\hat{u}\|_{l^2(\mathbb{Z}^n)}$, which is a discrete version of Plancherel's identity. Therefore, $\{e_\xi = e^{2\pi i x\cdot\xi}, \xi \in \mathbb{T}^n\}$ is an orthonormal basis for the space $L^2(\mathbb{T}^n)$.

## 1.4 Toroidal pseudodifferential operators

Let us inform our reader that the material presented here will only be used in Chapter 5.

We now introduce the calculus of finite differences, which will be needed to deal with pseudodifferential operators on $\mathbb{T}^n$. Let $\sigma : \mathbb{Z}^n \to \mathbb{C}$ and $1 \leq i, j \leq n$. If the multiindex $\delta_j \in \mathbb{N}^n$ is defined by

$$(\delta_j)_i = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases} .$$

Then we may define forward and backward partial difference operators $\Delta_{\xi_j}$ and $\overline{\Delta_{\xi_j}}$ by $\Delta_{\xi_j}\sigma(\xi) = \sigma(\xi+\delta_j)-\sigma(\xi)$ and $\overline{\Delta}_{\xi_j}\sigma(\xi) = \sigma(\xi)-\sigma(\xi-\delta_j)$ respectively. Just like the usual derivatives, for $\alpha \in \mathbb{N}_0^n$, we write $\Delta_\xi^\alpha = \Delta_{\xi_1}^{\alpha_1}\Delta_{\xi_2}^{\alpha_2}\cdots\Delta_{\xi_n}^{\alpha_n}$ and $\overline{\Delta}_\xi^\alpha = \overline{\Delta}_{\xi_1}^{\alpha_1}\overline{\Delta}_{\xi_2}^{\alpha_2}\cdots\overline{\Delta}_{\xi_n}^{\alpha_n}$. It is easy to see that

$$\Delta_\xi^\alpha \sigma(\xi) = \sum_{\beta \leq \alpha} (-1)^{|\alpha-\beta|} \binom{\alpha}{\beta} \sigma(\xi+\beta) \quad \text{and} \tag{1.4.1}$$

$$\overline{\Delta}_\xi^\alpha \sigma(\xi) = \sum_{\beta \leq \alpha} (-1)^{|\beta|} \binom{\alpha}{\beta} \sigma(\xi-\beta). \tag{1.4.2}$$

The reader can easily check discrete Leibniz formula, namely, the statement that holds for two functions $\phi$ and $\psi : \mathbb{Z}^n \to \mathbb{C}$

$$\Delta_\xi^\alpha(\phi\psi)(\xi) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \Delta_\xi^\beta \phi(\xi) \Delta_\xi^{\alpha-\beta}\psi(\xi+\beta). \tag{1.4.3}$$

We will sometimes use this useful discrete version of integration by parts. If $\phi, \psi : \mathbb{Z}^n \to \mathbb{C}$, then

$$\sum_{\xi \in \mathbb{Z}^n} \phi(\xi)\Delta^\alpha\psi(\xi) = (-1)^{|\alpha|} \sum_{\xi \in \mathbb{Z}^n} \left(\overline{\Delta^\alpha}\phi(\xi)\right)\psi(\xi) \tag{1.4.4}$$

provided that both series are absolutely convergent.

It is time for our reader to meet the discrete version of Taylor's formula.

**Theorem 1.4.1** ([53, Theorem 3.3.21]). *If $p : \mathbb{Z}^n \to \mathbb{C}$, then it holds*

$$p(\xi+\theta) = \sum_{|\alpha|<M} \frac{1}{\alpha!}\theta^{(\alpha)}\Delta_\xi^\alpha p(\xi) + r_M(\xi,\theta), \tag{1.4.5}$$

*where $\theta^{(\alpha)} = \binom{\theta}{\alpha}\alpha!$ and the remainder $r_M(\xi,\theta)$ satisfies the estimate*

$$|\Delta_\xi^\omega r_M(\xi,\theta)| \leq \sum_{|\alpha|=M} \frac{|\theta^{(\alpha)}|}{\alpha!} \max_{\nu\in Q(\theta)} |\Delta_\xi^{\alpha+\omega} p(\xi+\nu)| \tag{1.4.6}$$

*where*

$$Q(\theta) = \{\nu \in \mathbb{Z}^n : |\nu_j| \leq |\theta_j| : j = 1,2,\ldots,n\}.$$

Let us remind the reader Peetre's inequality [53, Proposition 3.3.31], dealing with the so called Japanese brackets. Namely, for $\xi \in \mathbb{R}^n$, define $\langle\xi\rangle = (1+|\xi|^2)^{\frac{1}{2}}$, and then it holds, for $\xi,\eta \in \mathbb{R}^n$, $s \in \mathbb{R}$

$$\langle\xi+\eta\rangle^s \leq 2^{|s|}\langle\xi\rangle^s\langle\eta\rangle^{|s|}. \tag{1.4.7}$$

We have everything we need to proceed furthermore. The class $\mathcal{E}(\mathbb{T}^n \times \mathbb{Z}^n)$ comprises functions $a(x,\xi), x \in \mathbb{T}^n, \xi \in \mathbb{Z}^n$ where $a(\cdot,\xi) \in \mathcal{E}(\mathbb{T}^n)$ for every $\xi \in \mathbb{Z}^n$. Fix $m \in \mathbb{R}$, $\rho,\sigma \in [0,1]$. Let $G^m_{\rho,\sigma}(\mathbb{T}^n \times \mathbb{Z}^n)$ be the class of functions $C^\infty(\mathbb{T}^n \times \mathbb{Z}^n) \ni a(x,\xi)$ such that for arbitrary multiindices $\alpha,\beta$ there exists $C_{\alpha,\beta} > 0$ for which the following inequality holds:

$$\sup_{(x,\xi)\in\mathbb{T}^n\times\mathbb{Z}^n} |\Delta_\xi^\alpha \partial_x^\beta a(x,\xi)| \leq C_{\alpha,\beta}\langle\xi\rangle^{m-\rho|\alpha|-\sigma|\beta|} \tag{1.4.8}$$

Then for a given $a(x,\xi) \in G^m_{\rho,\sigma}(\mathbb{T}^n \times \mathbb{Z}^n)$ we may define *the toroidal pseudodifferential operator* with the symbol $a$, $A = a(x,D) : \mathcal{E}(\mathbb{T}^n) \to \mathcal{E}(\mathbb{T}^n)$ :

$$a(x,D)u(x) = \sum_{\xi\in\mathbb{Z}^n} e^{2\pi i x\cdot\xi} a(x,\xi)\hat{u}(\xi). \tag{1.4.9}$$

This mapping is well defined, because the series is absolutely convergent, as one can easily prove. Moreover, given the mentioned canonical topology on $\mathcal{E}(\mathbb{T}^n)$, the mapping $a(x,D) : \mathcal{E}(\mathbb{T}^n) \to \mathcal{E}(\mathbb{T}^n)$ turns out to be continuous.

The Fourier transform may be written in the integral form and then we formally write

$$a(x,D)u(x) = \sum_{\xi\in\mathbb{Z}^n} \int_{\mathbb{T}^n} e^{2\pi i(x-y)\cdot\xi} a(x,\xi)u(y)dy \tag{1.4.10}$$

where the above form should be considered as a result of formal integration by parts, or, to be precise,

$$a(x,D)u(x) = \sum_{\xi\in\mathbb{Z}^n} \langle\xi\rangle^{-2q} \int_{\mathbb{T}^n} e^{2\pi i(x-y)\cdot\xi} a(x,\xi)\Big(I - \frac{\mathcal{L}_y}{4\pi^2}\Big)^q u(y)dy \tag{1.4.11}$$

for $q \in \mathbb{N}$ large enough, where $I$ is the identity operator, and $\mathcal{L}_y$ is the Laplacian with respect to the variable $y$ (let us remark that the abuse of notation occurs here because we use symbol $\Delta$ for differences).

Here we give another interpretation of pseudodifferential operators. If $\varphi, \psi \in \mathcal{E}(\mathbb{T}^n)$, $a(x,\xi) \in G^m_{\rho,\sigma}(\mathbb{T}^n \times \mathbb{Z}^n)$ and $A = a(x,D)$, then

$$\langle A\varphi, \psi\rangle = \langle K_A, \varphi \otimes \psi\rangle$$

where the distribution $K_A(x,y) \in \mathcal{E}'(\mathbb{T}^{2n})$ satisfies
$K_A(x,y) = (\mathcal{F}^{-1}_{\mathbb{T}^n} a(x,\xi))(x-y)$ or, explicitly,

$$K_A(x,y) = \sum_{\xi\in\mathbb{Z}^n} e^{2\pi i(x-y)\cdot\xi} a(x,\xi), \forall x,y \in \mathbb{T}^n \tag{1.4.12}$$

in the sense of distributions on $\mathbb{T}^n$.

This is a good place to say a word about the global counterpart of our discussion-namely, about the pseudodifferential operators on $\mathbb{R}^n$. We may consider the space $G^m_{\rho,\sigma}(\mathbb{R}^n \times \mathbb{R}^n)$ of functions $b(x,\xi) \in \mathcal{E}(\mathbb{R}^n \times \mathbb{R}^n)$, $m \in \mathbb{R}, \rho, \sigma \in [0,1]$ for which

$$\sup_{(x,\xi)\in\mathbb{R}^n\times\mathbb{R}^n} |\partial^\alpha_\xi \partial^\beta_x b(x,\xi)| \le C_{\alpha,\beta}\langle\xi\rangle^{m-\rho|\alpha|+\sigma|\beta|}$$

for some constant $C_{\alpha,\beta}$. If, additionally, $b(\cdot,\xi)$ is 1-periodic for every $\xi \in \mathbb{Z}^n$, we may write $b(x,\xi) \in G^m_{\rho,\delta}(\mathbb{T}^n \times \mathbb{R}^n)$ for this *Euclidean symbol on the torus* $\mathbb{T}^n$ defined in the following way

$$b(x,D)u(x) = \int_{\mathbb{T}^n}\int_{\mathbb{R}^n} e^{2\pi i(x-y)\cdot\xi} b(x,\xi)u(y)dyd\xi$$

in [53].

# Chapter 2

# Ultradistributional boundary values of harmonic functions on the sphere

## 2.1 Introduction

In this chapter, we will present a theory of ultradistributional boundary values for harmonic functions defined on the Euclidean unit ball, which we denote as $\mathbb{B}^n$.

The theory of spherical harmonic expansions of distributions was developed by Estrada and Kanwal in [21]. We generalize here their results to the framework of ultradistributions [33, 34] and supply a theory of ultradistributional boundary values of harmonic functions on $\mathbb{S}^{n-1}$. Our goal is to characterize all those harmonic functions $U$, defined in the unit ball, that admit boundary values $\lim_{r\to 1^-} U(r\omega)$ in an ultradistribution space $\mathcal{E}^{*\prime}(\mathbb{S}^{n-1})$. These considerations apply to both non-quasianalytic and quasianalytic ultradistributions, and, in particular, to analytic functionals. As an application, we also obtain a characterization of the support of a non-quasianalytic ultradistribution in terms of Abel summability of its spherical harmonic series expansion. Since Schwartz distributions are naturally embedded into the spaces of ultradistributions in a support preserving fashion, our support characterization contains as a particular instance that of González Vieli from the recent article [25].

The plan of the chapter goes as follows. In Section 2.2 we study spaces of ultradifferentiable functions and ultradistributions through spherical harmonics. Our main results there are descriptions of these spaces in terms of the decay or growth rate of the norms of the projections of a function or an ultradistribution onto the spaces of spherical harmonics. We also establish the convergence of the spherical harmonic series in the corresponding space. Note that eigenfunction expansions of ultradistributions on compact analytic manifolds have recently been investigated in [16, 17] with the aid of pseudodifferential calculus (cf. [63] for the Euclidean global setting). However, our approach here is quite different and is rather based on explicit estimates for partial derivatives of solid harmonics and spherical harmonics that are obtained in Section 2.2. Such estimates are of independent interest and refine earlier bounds by Calderón and Zygmund from [5].

Harmonic functions with ultradistributional boundary values are characterized in Section 2.4. The characterization is in terms of the growth order of the harmonic function near the boundary $\mathbb{S}^{n-1}$; we also show in Section 2.4 that a harmonic function satisfying such growth conditions must necessarily be the Poisson transform of an ultradistribution. In the special case of analytic functionals, our result yields as a corollary: *any* harmonic function on the unit ball arises as the Poisson transform of some analytic functional on the sphere. Finally, Section 2.5 deals with the characterization of the support of non-quasianalytic ultradistributions on $\mathbb{S}^{n-1}$.

## 2.2 Estimates for partial derivatives of spherical harmonics

Spherical harmonics were introduced in Section 2.1 of the first Chapter. This section deals with further new basic properties of spherical harmonics.

Calderón and Zygmund showed [5, Eq. (4), p. 904] the following estimates for the partial derivatives of a spherical harmonic $Y_j \in \mathcal{H}_j(\mathbb{S}^{n-1})$,

$$\|\partial^\alpha_{\mathbb{S}^{n-1}} Y_j\|_{L^\infty(\mathbb{S}^{n-1})} \leq C_{\alpha,n}\, j^{|\alpha|} \|Y_j\|_{L^\infty(\mathbb{S}^{n-1})}, \tag{2.2.1}$$

where the constants $C_{\alpha,n}$ depend on the order of differentiation and the dimension in an unspecified way. The same topic is treated in Seeley's article [56].

The goal of this section is to refine (2.2.1) by exhibiting explicit constants $C_{\alpha,n}$. We also give explicit bounds for the partial derivatives of spherical harmonics in spherical coordinates. Such estimates in spherical coordinates play an important role in the next section. We consider here $\mathfrak{p}(\theta) = (\mathfrak{p}_1(\theta), \dots, \mathfrak{p}_n(\theta))$,

$$\mathfrak{p}(\theta) = (\cos\theta_1, \sin\theta_1 \cos\theta_2, \dots, \prod_{k=1}^{n-2} \sin\theta_k \cdot \cos\theta_{n-1}, \prod_{k=1}^{n-1} \sin\theta_k),$$

where $\theta \in \mathbb{R}^{n-1}$. Naturally, the estimate (2.2.4) below also holds if we choose the north pole to be located at a point other than $(1, 0, \dots, 0)$.

We shall need the following lemma due to Seeley.

**Lemma 2.2.1.** *Let $Q_j \in \mathcal{H}_j(\mathbb{R}^n)$ and $\alpha \neq 0$. Then, for all multi-index $\beta$, with $|\beta| = |\alpha| - 1$ and $\beta \leq \alpha$, we have the inequality*

$$\int_{\mathbb{S}^{n-1}} |\partial^\alpha Q_j(\omega)|^2 d\omega \leq (j - |\alpha| + 1)(n + 2j - 2|\alpha|) \int_{\mathbb{S}^{n-1}} |\partial^\beta Q_j(\omega)|^2 d\omega, \tag{2.2.2}$$

*Proof.* Close inspection of the proof of part ($b$) of [56, Theorem 4] reveals that the proved inductive step also proves our inequality.

□

**Theorem 2.2.2.** *We have the bounds:*

*(a) For every solid harmonic $Q_j \in \mathcal{H}_j(\mathbb{R}^n)$ and all $\alpha \neq 0$,*

$$\|\partial^\alpha Q_j\|_{L^\infty(\mathbb{S}^{n-1})} \leq e^{\frac{n}{4} - \frac{1}{2}} \sqrt{n}\, 2^{\frac{|\alpha|}{2}} j^{|\alpha| + \frac{n}{2} - 1} \|Q_j\|_{L^\infty(\mathbb{S}^{n-1})}. \tag{2.2.3}$$

*(b) For all spherical harmonic* $Y_j \in \mathcal{H}_j(\mathbb{S}^{n-1})$ *and all* $\alpha \neq 0$,

$$\|\partial_\theta^\alpha (Y_j \circ \mathfrak{p})\|_{L^\infty(\mathbb{R}^{n-1})} \\ \leq e^{\frac{n}{4}-\frac{1}{2}}\sqrt{n}\left((n+1)^{|\alpha|}-1\right)2^{\frac{|\alpha|}{2}} j^{|\alpha|+\frac{n}{2}-1}\|Y_j\|_{L^\infty(\mathbb{S}^{n-1})}. \quad (2.2.4)$$

*(c) For all spherical harmonic* $Y_j \in \mathcal{H}_j(\mathbb{S}^{n-1})$, *all* $\alpha \neq 0$, *and any* $\varepsilon > 0$,

$$\|\partial_{\mathbb{S}^{n-1}}^\alpha Y_j\|_{L^\infty(\mathbb{S}^{n-1})} \\ \leq e^{n\left(\frac{1}{4}+\sqrt{2}+3\sqrt{2+4/\varepsilon}\right)-\frac{1}{2}} n^{\frac{|\alpha|+1}{2}}(2+\varepsilon)^{|\alpha|} j^{|\alpha|+\frac{n}{2}-1}|\alpha|!\|Y_j\|_{L^\infty(\mathbb{S}^{n-1})}. \\ (2.2.5)$$

*Proof.* (a) For (2.2.3), we assume that $|\alpha| \leq j$, otherwise the result trivially holds. Our starting point is Lemma 2.2.1.

Successive application of (5.3.10) leads to

$$\int_{\mathbb{S}^{n-1}} |\partial^\alpha Q_j(\omega)|^2 d\omega \leq \prod_{i=0}^{|\alpha|-1}(j-i)\cdot\prod_{i=1}^{|\alpha|}(n+2j-2i)\int_{\mathbb{S}^{n-1}}|Q_j(\omega)|^2 d\omega.$$

The coefficient in this bound can be estimated as follows,

$$\begin{aligned}\prod_{i=0}^{|\alpha|-1}(j-i)\cdot\prod_{i=1}^{|\alpha|}(n+2j-2i) &\leq j^{2|\alpha|}2^{|\alpha|}\prod_{i=1}^{|\alpha|}\left(1+\frac{n/2-i}{j}\right)\\ &\leq 2^{|\alpha|}j^{2|\alpha|}\left(1+\frac{n/2-1}{j}\right)^{|\alpha|}\\ &\leq 2^{|\alpha|}j^{2|\alpha|}e^{\frac{n}{2}-1}.\end{aligned}$$

Now, $\partial^\alpha Q_j \in \mathcal{H}_{j-|\alpha|}(\mathbb{R}^n)$ and $\|Z_{j-|\alpha|}(\omega,\,\cdot\,)\|^2_{L^2(\mathbb{S}^{n-1})} = d_{j-|\alpha|}|\mathbb{S}^{n-1}|$ for each $\omega \in \mathbb{S}^{n-1}$ (cf. [3, pp. 79–80]). Thus, we obtain (cf. (1.1.7)), for all $\omega \in \mathbb{S}^{n-1}$,

$$\begin{aligned}&|\partial^\alpha Q_j(\omega)| \leq \frac{1}{|\mathbb{S}^{n-1}|}\|\partial^\alpha Q_j\|_{L^2(\mathbb{S}^{n-1})}\|Z_{j-|\alpha|}(\omega,\,\cdot\,)\|_{L^2(\mathbb{S}^{n-1})}\\ &= \sqrt{\frac{d_{j-|\alpha|}}{|\mathbb{S}^{n-1}|}}\,\|\partial^\alpha Q_j\|_{L^2(\mathbb{S}^{n-1})} \leq e^{\frac{n}{4}-\frac{1}{2}}\sqrt{\frac{n}{|\mathbb{S}^{n-1}|}}\,2^{\frac{|\alpha|}{2}}j^{|\alpha|+\frac{n}{2}-1}\|Q_j\|_{L^2(\mathbb{S}^{n-1})},\end{aligned}$$

where we have used $d_{j-|\alpha|} \leq nj^{n-2}$ (see (1.1.5)). This shows (2.2.3).

(b) Our proof of (2.2.4) is based on the multivariate Faà di Bruno formula for the partial derivatives of the composition of functions. Let $m = |\alpha|$. Specializing [15, Eq. (2.4)] to $h = f \circ \mathfrak{p}$, where $f$ is a function on $\mathbb{R}^n$, we obtain

$$\partial_\theta^\alpha h = \sum_{1 \le |\lambda| \le m} (\partial_x^\lambda f) \circ \mathfrak{p} \sum_{(k,l) \in p(\alpha,\lambda)} \alpha! \prod_{j=1}^n \frac{[\partial_\theta^{l_j} \mathfrak{p}_j]^{k_j}}{(k_j!)[l_j!]^{|k_j|}},$$

where the set of multi-indices $p(\alpha, \lambda) \subset \mathbb{N}^{2n}$ is as described in [15, p. 506]. We also employ the identity [15, Cor. 2.9]

$$\alpha! \sum_{|\lambda|=k} \sum_{p(\alpha,\lambda)} \prod_{j=1}^n \frac{1}{(k_j!)[l_j!]^{|k_j|}} = n^k S(m,k),$$

where $S(m,k)$ are the Stirling numbers of the second kind. For such numbers [49, Thm. 3] we have the estimates[1]

$$S(m,k) \le \binom{m}{k} k^{m-k}, \quad 1 \le k \le m.$$

Since obviously $|\partial^{l_j} \mathfrak{p}_j(\theta)| \le 1$, we obtain

$$\|\partial_\theta^\alpha (f \circ \mathfrak{p})\|_{L^\infty(\Omega)} \le \sum_{k=1}^m \binom{m}{k} k^{m-k} n^k \max_{|\lambda|=k} \|\partial_x^\lambda f\|_{L^\infty(\mathfrak{p}(\Omega))}, \tag{2.2.6}$$

for any $\Omega \subseteq \mathbb{R}^{n-1}$ and the corresponding set $\mathfrak{p}(\Omega) \subseteq \mathbb{S}^{n-1}$. We now apply this inequality to estimate $\partial_\theta^\alpha (Y_j \circ \mathfrak{p})$. Let $Q_j \in \mathcal{H}_j(\mathbb{R}^n)$ be the solid harmonic corresponding to $Y_j$, clearly $Q_j \circ \mathfrak{p} = Y_j \circ \mathfrak{p}$ and $\|Q_j\|_{L^\infty(\mathbb{S}^{n-1})} = \|Y_j\|_{L^\infty(\mathbb{S}^{n-1})}$. Using (2.2.6) with $f = Q_j$, the bound (2.2.3), and the fact that $\partial_x^\lambda Q_j = 0$ if $|\lambda| > j$, we conclude that

$$\begin{aligned}\|\partial_\theta^\alpha (Y_j \circ \mathfrak{p})\|_{L^\infty(\mathbb{R}^{n-1})} &\le e^{\frac{n}{4}-\frac{1}{2}} \sqrt{n} 2^{\frac{m}{2}} j^{\frac{n}{2}-1} \|Y_j\|_{L^\infty(\mathbb{S}^{n-1})} \sum_{k=1}^m \binom{m}{k} n^k j^{m-k} j^k \\ &= e^{\frac{n}{4}-\frac{1}{2}} \sqrt{n} \left((n+1)^m - 1\right) 2^{\frac{m}{2}} j^{m+\frac{n-2}{2}} \|Y_j\|_{L^\infty(\mathbb{S}^{n-1})}.\end{aligned}$$

(c) We need to estimate the partial derivatives of $Y_j^\upharpoonright = Q_j \circ F$, where $Q_j \in \mathcal{H}_j(\mathbb{R}^n)$ and $F(x) = x/|x|$, $x \in \mathbb{R}^n \setminus \{0\}$. Instead of using

[1] Actually, $S(m,k) \le \frac{1}{2}\binom{m}{k} k^{m-k}$ holds for $1 \le k \le m-1$ if $m \ge 2$, and $S(m,m) = 1$.

the Faà di Bruno formula to handle directly the partial derivatives of this composition, we will adapt Hörmander's proof of [29, Prop. 8.4.1, p. 281] to our problem. Let $0 < r < 1/2$ and $\omega \in \mathbb{S}^{n-1}$. Note that if $|z-\omega| \leq r$ and we write $z = x + iy$, then

$$\Re e\,(z_1^2 + \cdots + z_n^2) = |x|^2 - |y|^2 \geq 1 - 2r > 0.$$

So, $F$ is holomorphic on this region of $\mathbb{C}^n$. For $m \geq 1$, we define the sequence of functions

$$G_m(z) = \sum_{|\beta| \leq m} \left(\partial^\beta Q_j\right)(F(\omega)) \frac{(F(z) - F(\omega))^\beta}{\beta!}.$$

Each $G_m$ is holomorphic when $|z - \omega| \leq r$ and the derivatives of $G_m$ of order $m$ at $z = \omega$ are the same as those of $Y_j^{\upharpoonright}(x)$ at $x = \omega$. We keep $|z - \omega| \leq r$. We have the bound

$$|F(z) - F(\omega)| \leq 1 + \frac{|z|}{\sqrt{\Re e\,(z_1^2 + \cdots + z_n^2)}} < 1 + \frac{3}{2\sqrt{1-2r}} = C_r,$$

and hence, by (2.2.3),

$$\begin{aligned} |G_m(z)| &\leq e^{\frac{n}{4} - \frac{1}{2}} \sqrt{n}\, \|Y_j\|_{L^\infty(\mathbb{S}^{n-1})} \sum_{|\beta| \leq \min\{m,j\}} j^{|\beta| + \frac{n}{2} - 1} \frac{(C_r\sqrt{2})^{|\beta|}}{\beta!} \\ &\leq e^{\frac{n}{4} - \frac{1}{2}} \sqrt{n}\, j^{m + \frac{n}{2} - 1} \|Y_j\|_{L^\infty(\mathbb{S}^{n-1})} \sum_{\beta \in \mathbb{N}^n} \frac{(C_r\sqrt{2})^{|\beta|}}{\beta!} \\ &= e^{n\left(\frac{1}{4} + \sqrt{2}C_r\right) - \frac{1}{2}} \sqrt{n}\, j^{m + \frac{n}{2} - 1} \|Y_j\|_{L^\infty(\mathbb{S}^{n-1})}. \end{aligned}$$

The Cauchy inequality applied in the polydisc $|z_j - \omega_j| \leq r/\sqrt{n}$ yields

$$\begin{aligned} &|\partial^\alpha_{\mathbb{S}^{n-1}} Y_j(\omega)| = |\partial^\alpha G_{|\alpha|}(\omega)| \\ &\leq e^{n\left(\frac{1}{4} + \sqrt{2}C_r\right) - \frac{1}{2}} n^{\frac{|\alpha|+1}{2}} r^{-|\alpha|} j^{|\alpha| + \frac{n}{2} - 1} \alpha! \|Y_j\|_{L^\infty(\mathbb{S}^{n-1})}. \end{aligned}$$

One obtains (2.2.5) upon setting $r = 1/(2 + \varepsilon)$. □

## 2.3 Spherical harmonic characterization of ultradifferentiable functions and ultradistributions

After the result from the last section, we are ready to characterize $\mathcal{E}^*(\mathbb{S}^{n-1})$ in terms of the norm decay of projections onto the spaces of spherical harmonics. Recall that our convention is to write $\varphi_j$ for the projection of $\varphi$ onto $\mathcal{H}_j(\mathbb{S}^{n-1})$.

**Theorem 2.3.1.** *Let $\varphi \in L^2(\mathbb{S}^{n-1})$ and let $1 \leq q \leq \infty$. The following statements are equivalent:*

*(i) $\varphi$ belongs to $\mathcal{E}^{\{M_p\}}(\mathbb{S}^{n-1})$ (to $\mathcal{E}^{(M_p)}(\mathbb{S}^{n-1})$) .*

*(ii) $\Delta^p_{\mathbb{S}^{n-1}}\varphi \in L^2(\mathbb{S}^{n-1})$ for all $p \in \mathbb{N}$ and there are $h, C > 0$ (for every $h > 0$ there is $C = C_h > 0$) such that*

$$\|\Delta^p_{\mathbb{S}^{n-1}}\varphi\|_{L^2(\mathbb{S}^{n-1})} \leq Ch^{-2p}M_{2p}. \tag{2.3.1}$$

*(iii) There are $C, h > 0$ (for every $h > 0$ there is $C = C_h > 0$) such that*

$$\|\varphi_j\|_{L^q(\mathbb{S}^{n-1})} \leq Ce^{-M(hj)}. \tag{2.3.2}$$

*Proof.* (i)$\Rightarrow$(ii). The proof of this implication is simple. Indeed, suppose that

$$|\partial^\alpha \varphi^{\upharpoonright}(x)| \leq Ch^{-|\alpha|}M_{|\alpha|}, \quad \text{for all } x \in \mathbb{R}^n \setminus \{0\}.$$

Since

$$\left(\frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}\right)^p = \sum_{\alpha_1+\cdots+\alpha_n=p} \frac{p!}{\alpha_1!\alpha_2!\ldots\alpha_n!}\frac{\partial^{2\alpha_1}}{\partial x_1^{2\alpha_1}}\cdots\frac{\partial^{2\alpha_n}}{\partial x_n^{2\alpha_n}}$$

and

$$\sum_{\alpha_1+\cdots+\alpha_n=p} \frac{p!}{\alpha_1!\alpha_2!\ldots\alpha_n!} = n^p,$$

the condition $(M.1)$ gives

$$\|\Delta^p_{\mathbb{S}^{n-1}}\varphi\|_{L^2(\mathbb{S}^{n-1})} \leq \frac{\|\Delta^p_{\mathbb{S}^{n-1}}\varphi\|_{L^\infty(\mathbb{S}^{n-1})}}{|\mathbb{S}^{n-1}|^{\frac{1}{2}}} \leq \frac{C}{|\mathbb{S}^{n-1}|^{\frac{1}{2}}}(h/\sqrt{n})^{-2p}M_p.$$

(ii)$\Rightarrow$(iii). Suppose (2.3.1) holds. The projection of $\varphi$ onto $\mathcal{H}_j(\mathbb{S}^{n-1})$ is

$$\varphi_j(\omega) = \frac{1}{|\mathbb{S}^{n-1}|}\int_{\mathbb{S}^{n-1}} \varphi(\xi)Z_j(\omega,\xi)d\xi. \tag{2.3.3}$$

We first assume that $j \geq 1$. The Laplace-Beltrami operator is self-adjoint [56, Lemma 1] and each spherical harmonic of degree $j$, such as $Z_j(\omega,\xi)$, is an eigenfunction of $\Delta_{\mathbb{S}^{n-1}}$ with eigenvalue $-j(j+n-2)$. Also, $\|Z_j(\omega,\,\cdot\,)\|_{L^2(\mathbb{S}^{n-1})} = \sqrt{d_j|\mathbb{S}^{n-1}|} \leq j^{\frac{n}{2}-1}\sqrt{n|\mathbb{S}^{n-1}|}$ (see (1.1.5) and [3, pp. 79–80]); therefore,

$$\begin{aligned}|\varphi_j(\omega)| &= \frac{1}{j^p(j+n-2)^p|\mathbb{S}^{n-1}|}\left|\int_{\mathbb{S}^{n-1}}(\Delta^p\varphi)(\xi)Z_j(\omega,\xi)d\xi\right| \\ &\leq \frac{C\sqrt{n}}{|\mathbb{S}^{n-1}|^{\frac{1}{2}}} j^{-2p+\frac{n}{2}-1}h^{-2p}M_{2p}.\end{aligned}$$

Taking supremum over $\omega$ and infimum over $p$, we conclude that

$$\|\varphi_j\|_{L^q(\mathbb{S}^{n-1})} \leq |\mathbb{S}^{n-1}|^{\frac{1}{q}}\|\varphi_j\|_{L^\infty(\mathbb{S}^{n-1})} \leq C|\mathbb{S}^{n-1}|^{\frac{1}{q}-\frac{1}{2}}\sqrt{n}\, j^{\frac{n}{2}-1}e^{-M(hj)}.$$

Taking $\eta = n/2-1$ in (1.3.1), we obtain

$$\|\varphi_j\|_{L^2(\mathbb{S}^{n-1})} \leq C|\mathbb{S}^{n-1}|^{\frac{1}{q}-\frac{1}{2}}\sqrt{n}(A/h)^{\frac{n}{2}-1}H^{(\frac{n}{2}-1)^2}e^{-M\left(jhH^{1-n/2}\right)}, \quad j\geq 1.$$

For $j=0$, using (2.3.3), we have $\|\varphi_j\|_{L^q(\mathbb{S}^{n-1})} \leq C|\mathbb{S}^{n-1}|^{\frac{1}{q}-\frac{1}{2}}$, thus

$$\|\varphi_j\|_{L^q(\mathbb{S}^{n-1})} \leq C_hCe^{-M\left(jhH^{1-n/2}\right)}, \quad j\geq 0.$$

with $C_h = |\mathbb{S}^{n-1}|^{\frac{1}{q}-\frac{1}{2}}\max\{1,\sqrt{n}(A/h)^{\frac{n}{2}-1}H^{(\frac{n}{2}-1)^2}\}$.

(iii)⇒(i). Assume now (2.3.2). In view of (2.3.3) and (1.3.1), we may also assume that $q=\infty$. We estimate the partial derivatives of $\varphi$ in spherical coordinates. Write $\tilde{\varphi} = \varphi\circ\mathfrak{p}$ and $\tilde{\varphi}_j = \varphi_j\circ\mathfrak{p}$. Let $\alpha\neq 0$. Let $r$ be an integer larger than $n/2+1$. If we combine the estimate (2.2.4) with (2.3.2), we obtain

$$\begin{aligned}\|\partial_\theta^\alpha\tilde{\varphi}_j\|_{L^\infty(\mathbb{R}^{n-1})} &\leq e^{\frac{n}{4}-\frac{1}{2}}\sqrt{n}\left(\sqrt{2}(n+1)\right)^{|\alpha|}j^{|\alpha|+\frac{n}{2}-1}\|\varphi_j\|_{L^\infty(\mathbb{S}^{n-1})} \\ &\leq C\frac{h^{-r}}{j^2}e^{\frac{n}{4}-\frac{1}{2}}\sqrt{n}\left(\sqrt{2}(n+1)/h\right)^{|\alpha|}M_{|\alpha|+r}, \quad j\geq 1.\end{aligned}$$

Calling $C_h = e^{\frac{n}{4}-\frac{1}{2}}h^{-r}\sqrt{n}\pi^2/6$, we conclude that

$$\|\partial_\theta^\alpha\tilde{\varphi}\|_{L^\infty(\mathbb{R}^{n-1})} \leq \sum_{j=1}^{\infty}\|\partial_\theta^\alpha\tilde{\varphi}_j\|_{L^\infty(\mathbb{R}^{n-1})} \leq C_h\left(\sqrt{2}(n+1)/h\right)^{|\alpha|}M_{|\alpha|+r}.$$

The assumption $(M.2)'$ implies $M_{p+r} \leq H^{rp} A^r H^{\frac{r(r-1)}{2}} M_p$, so

$$\|\partial_\theta^\alpha \tilde{\varphi}\|_{L^\infty(\mathbb{S}^{n-1})} \leq C C_h A^r H^{\frac{r(r-1)}{2}} \left(H^r \sqrt{2}(n+1)/h\right)^{|\alpha|} M_{|\alpha|}. \quad (2.3.4)$$

Setting the north pole at different points of the sphere induces an analytic atlas of $\mathbb{S}^{n-1}$ and $x \to x/|x|$ is analytic on $\mathbb{R}^n$. As previously mentioned, the conditions $(M.0)$ ($(NA)$ in the Beurling case), $(M.1)$, and $(M.2)'$ ensure that pullbacks by analytic functions preserve $*$-ultradifferentiability. So, $\varphi \in \mathcal{E}^*(\mathbb{S}^{n-1})$. The inequality (2.3.4) and the proof of [29, Prop. 8.1.4] give actually a more accurate result: There are constants $C_h'$ and $\ell$, depending also on the sequence $M_p$ and the dimension $n$ but not on $\varphi$, such that

$$\|\partial_{\mathbb{S}^{n-1}}^\alpha \varphi\|_{L^\infty(\mathbb{S}^{n-1})} \leq C C_h' (\ell h)^{-|\alpha|} M_{|\alpha|}.$$

□

The proof of Theorem 2.3.1 actually yields stronger information than what has been stated. The canonical topology of $\mathcal{E}^*(\mathbb{S}^{n-1})$ is defined as follows. For each $h > 0$, consider the Banach space $\mathcal{E}^{\{M_p\},h}(\mathbb{S}^{n-1})$ of all smooth functions $\varphi$ on $\mathbb{S}^{n-1}$ such that the norm

$$\|\varphi\|_h = \sup_{\alpha \in \mathbb{N}} \frac{h^{|\alpha|} \|\partial_{\mathbb{S}^{n-1}}^\alpha \varphi\|_{L^\infty(\mathbb{S}^{n-1})}}{M_{|\alpha|}} \quad (2.3.5)$$

is finite. As locally convex spaces, we obtain the $(DFS)$-space and $(FS)$-space

$$\mathcal{E}^{\{M_p\}}(\mathbb{S}^{n-1}) = \varinjlim_{h \to 0^+} \mathcal{E}^{\{M_p\},h}(\mathbb{S}^{n-1}), \quad \mathcal{E}^{(M_p)}(\mathbb{S}^{n-1}) = \varprojlim_{h \to \infty} \mathcal{E}^{\{M_p\},h}(\mathbb{S}^{n-1}).$$

What we have shown is that the family of norms (2.3.5) is tamely equivalent to the norms

$$\|\varphi\|_h' = \sup_{j \in \mathbb{N}} e^{M(hj)} \|\varphi_j\|_{L^q(\mathbb{S}^{n-1})}, \quad h > 0 \quad (1 \leq q \leq \infty), \quad (2.3.6)$$

in the sense that there are positive constants $\ell$ and $L$, only depending on the dimension $n$, the parameter $q$, and the weight sequence, such that one can find $C_h > 0$ and $c_h > 0$ with

$$c_h \|\cdot\|_{\ell h}' \leq \|\cdot\|_h \leq C_h \|\cdot\|_{Lh}', \quad \text{for all } h > 0.$$

Working with the family of norms (2.3.6) is more convenient than (2.3.5) when dealing with assertions about spherical harmonic expansions.

**Proposition 2.3.2.** *Let $\varphi \in \mathcal{E}^*(\mathbb{S}^{n-1})$. Then its spherical harmonic series expansion $\varphi = \sum_{j=0}^{\infty} \varphi_j$ converges in (the strong topology of) $\mathcal{E}^*(\mathbb{S}^{n-1})$.*

*Proof.* Let $h > 0$. Invoking (1.3.1) with $\eta = 1$,

$$\|\varphi - \sum_{j=0}^{k} \varphi_j\|'_h = \sup_{j>k} e^{M(hj)} \|\varphi_j\| \leq \frac{A}{kh} \|\varphi\|'_{Hh}, \quad \text{for each} \quad k \geq 1.$$

□

If we specialize our results to the space of real analytic functions and use the fact that the associated function of $p!$ is $M(t) \asymp t$, we obtain the following characterization of $\mathcal{A}(\mathbb{S}^{n-1}) = \mathcal{E}^{\{p!\}}(\mathbb{S}^{n-1})$.

**Corollary 2.3.3.** *A sequence of spherical harmonics with $\varphi_j \in \mathcal{H}_j(\mathbb{S}^{n-1})$ gives rise to a real analytic function $\varphi = \sum_{j=0}^{\infty} \varphi_j$ on $\mathbb{S}^{n-1}$ if and only if*

$$\limsup_{j \to \infty} \left( \|\varphi_j\|_{L^q(\mathbb{S}^{n-1})} \right)^{\frac{1}{j}} < 1.$$

Here is another application of the norms (2.3.6). The space of ultradistributions $\mathcal{E}^{*\prime}(\mathbb{S}^{n-1})$ (of class $*$) on $\mathbb{S}^{n-1}$ is the strong dual of $\mathcal{E}^*(\mathbb{S}^{n-1})$. When $* = \{p!\}$, one obtains the space of analytic functionals $\mathcal{A}'(\mathbb{S}^{n-1})$ [38]. Given $f \in \mathcal{E}^{*\prime}(\mathbb{S}^{n-1})$, we can also define its projection onto $\mathcal{H}_j(\mathbb{S}^{n-1})$ as

$$f_j(\omega) = \frac{1}{|\mathbb{S}^{n-1}|} \langle f(\xi), Z_j(\omega, \xi) \rangle,$$

where the ultradistributional evaluation in the dual pairing is naturally with respect to the variable $\xi$. Note that, clearly,

$$\langle f_j, \varphi \rangle = \int_{\mathbb{S}^{n-1}} f_j(\omega) \varphi(\omega) d\omega = \langle f, \varphi_j \rangle, \quad \text{for each } \varphi \in \mathcal{E}^*(\mathbb{S}^{n-1}). \tag{2.3.7}$$

**Theorem 2.3.4.** *Every ultradistribution* $f \in \mathcal{E}^{\{M_p\}'}(\mathbb{S}^{n-1})$ *(*$f \in \mathcal{E}^{(M_p)'}(\mathbb{S}^{n-1})$*) has spherical harmonic expansion*

$$f = \sum_{j=0}^{\infty} f_j, \tag{2.3.8}$$

*where its spherical harmonic projections* $f_j$ *satisfy*

$$\sup_{j \in \mathbb{R}} e^{-M(hj)} \|f_j\|_{L^q(\mathbb{S}^{n-1})} < \infty \qquad (1 \leq q \leq \infty), \tag{2.3.9}$$

*for all* $h > 0$ *(for some* $h > 0$*). Conversely, a series* (2.3.8) *converges in the strong topology of* $\mathcal{E}^{\{M_p\}'}(\mathbb{S}^{n-1})$ *(of* $\mathcal{E}^{(M_p)'}(\mathbb{S}^{n-1})$*) if the* $L^q(\mathbb{S}^{n-1})$*-norms of* $f_j$ *have the stated growth properties.*

*Proof.* Since $\mathcal{E}^*(\mathbb{S}^{n-1})$ are Montel spaces, the strong convergence of (2.3.8) follows from its weak convergence, and the latter is a consequence of Proposition 2.3.2 and (2.3.7). For the bound (2.3.9), the continuity of $f$ implies that for each $h > 0$ (for some $h > 0$) there is a constant $C_h$ such that

$$|\langle f, \varphi \rangle| \leq C_h \|\varphi\|'_h, \quad \text{for all } \varphi \in \mathcal{A}(\mathbb{S}^{n-1}).$$

We may assume that $j \geq 1$. Considering the case $q = 2$ of (2.3.6), taking $\varphi(\xi) = |\mathbb{S}^{n-1}|^{-1} Z_j(\omega, \xi)$, and using the inequalities (1.1.5) and (1.3.1), one has

$$\begin{aligned} \|f_j\|_{L^q(\mathbb{S}^{n-1})} &\leq |\mathbb{S}^{n-1}|^{\frac{1}{q}} \|f_j\|_{L^\infty(\mathbb{S}^{n-1})} \leq |\mathbb{S}^{n-1}|^{\frac{1}{q}-\frac{1}{2}} C_h \sqrt{n} j^{\frac{n}{2}-1} e^{M(hj)} \\ &\leq |\mathbb{S}^{n-1}|^{\frac{1}{q}-\frac{1}{2}} C_h (A/h)^{\frac{n}{2}-1} \sqrt{n} e^{M(jhH^{\frac{n}{2}-1})}. \end{aligned}$$

□

For analytic functionals we have,

**Corollary 2.3.5.** *A sequence* $f_j \in \mathcal{H}_j(\mathbb{S}^{n-1})$ *gives rise to an analytic functional* $f = \sum_{j=0}^{\infty} f_j$ *on* $\mathbb{S}^{n-1}$ *if and only if*

$$\limsup_{j \to \infty} \left( \|f_j\|_{L^q(\mathbb{S}^{n-1})} \right)^{\frac{1}{j}} \leq 1.$$

We mention that the strong topologies of the $(FS)$-space $\mathcal{E}^{\{M_p\}'}(\mathbb{S}^{n-1})$ and the $(DFS)$-space $\mathcal{E}^{(M_p)'}(\mathbb{S}^{n-1})$ can also be induced via the family of norms (2.3.9) as the projective and inductive limits of the Banach spaces of ultradistributions $f = \sum_{j=0}^{\infty} f_j$ satisfying (2.3.9).

For each $j \in \mathbb{N}$ select an orthonormal basis of real spherical harmonics $\{Y_{k,j}\}_{k=1}^{d_j}$ of $\mathcal{H}_j(\mathbb{S}^{n-1})$. It is then clear that every ultradistribution $f \in \mathcal{E}^{*\prime}(\mathbb{S}^{n-1})$ and every $\varphi \in \mathcal{E}^{*}(\mathbb{S}^{n-1})$ can be expanded as

$$f = \sum_{j=0}^{\infty}\sum_{k=1}^{d_j} c_{k,j} Y_{k,j} \tag{2.3.10}$$

and

$$\varphi(\omega) = \sum_{j=0}^{\infty}\sum_{k=1}^{d_j} a_{k,j} Y_{k,j}(\omega), \tag{2.3.11}$$

where the coefficients satisfy

$$\sup_{k,j} |c_{k,j}| e^{-M(hj)} < \infty$$

(for each $h > 0$ in the Roumieu case and for some $h > 0$ in the Beurling case), and

$$\sup_{k,j} |a_{k,j}| e^{M(hj)} < \infty$$

(for some $h > 0$ or for each $h > 0$, respectively). Conversely, any series (2.3.10) and (2.3.11) converge in $\mathcal{E}^{*\prime}(\mathbb{S}^{n-1})$ and $\mathcal{E}^{*}(\mathbb{S}^{n-1})$, respectively, if the coefficients have the stated growth properties. We have used here (2.3.2), (2.3.9), and (1.3.1).

From here one easily derives that $\mathcal{E}^{*}(\mathbb{S}^{n-1})$ (and hence $\mathcal{E}^{*\prime}(\mathbb{S}^{n-1})$) is a nuclear space. We also obtain that $\{Y_{k,j}\}$ is an absolute Schauder basis [54, p. 340] for both $\mathcal{E}^{*}(\mathbb{S}^{n-1})$ and $\mathcal{E}^{*\prime}(\mathbb{S}^{n-1})$. We end this section with a remark concerning Theorem 2.3.1.

*Remark* 2.3.6. It is very important to emphasize that Theorem 2.3.1 is no longer true without the assumption $(M.0)$.

To see that it is imperative to assume $(M.0)$, we give an example in which the implication (ii)$\Rightarrow$(i) fails without it. In fact, let $M_p$ be any weight sequence for which $(M.1)$ and $(M.2)'$ hold but $\lim_{p\to\infty}(M_p/p!)^{\frac{1}{p}} = 0$. (For example, the sequence $M_p = p!^s$ with $0 < s < 1$.) We consider $\varphi(\omega) = Y_1(\omega_1,\dots,\omega_n) = \omega_1$. This function is a spherical harmonic of degree 1, and thus it is an eigenfunction for the Laplace-Beltrami operator corresponding to the eigenvalue $-(n-2)$. Thus,

$$\|\Delta^p_{\mathbb{S}^{n-1}}\varphi\|_{L^2(\mathbb{S}^{n-1})} \leq \frac{n^p}{|\mathbb{S}^{n-1}|^{\frac{1}{2}}}$$

and in particular (2.3.1) is satisfied for $M_p$. If there would be an $h > 0$ such that (1.3.2) holds with $M_p = p!^s$, we would have for the function

$$f(t) = \frac{1}{\sqrt{t^2 + 1/2}}$$

that

$$\|f^{(p)}\|_{L^\infty(\mathbb{R})} = \sqrt{2}\sup_{t\in\mathbb{R}} |\partial^p_{x_2}\varphi^{\restriction}(\sqrt{2}/2, t, 0, \dots, 0)| \leq C' h^{-p} M_p, \quad \forall p \in \mathbb{N},$$

for some $C' > 0$. But then $f$ would be analytically continuable to the whole $\mathbb{C}$ as an entire function, which is impossible because $f$ has branch singularities at $t = \pm i\sqrt{2}/2$.

On the other hand, note that in establishing the implications (i)$\Rightarrow$(ii)$\Rightarrow$(iii) the condition $(M.0)$ plays no role because we have only made use there of $(M.1)$ and $(M.2)'$.

## 2.4 Boundary values of harmonic functions

We now generalize the results from [21] to ultradistributions. We shall characterize all those harmonic functions on the open unit ball $\mathbb{B}^n$ that admit ultradistributional boundary values on $\mathbb{S}^{n-1}$ in terms of their growth near the boundary. Our characterization applies for sequences satisfying the additional conditions discussed below.

Let us fix some notation and terminology. We write $\mathcal{H}(\mathbb{B}^n)$ for the space of all harmonic functions on $\mathbb{B}^n$. We say that $U \in \mathcal{H}(\mathbb{B}^n)$ has ultradistribution boundary values in the space $\mathcal{E}^{*\prime}(\mathbb{S}^{n-1})$ if there is

$f \in \mathcal{E}^{*\prime}(\mathbb{S}^{n-1})$ such that

$$\lim_{r\to 1^-} U(r\omega) = f(\omega) \quad \text{in } \mathcal{E}^{*\prime}(\mathbb{S}^{n-1}). \tag{2.4.1}$$

Since $\mathcal{E}^{*\prime}(\mathbb{S}^{n-1})$ is Montel, the converge of (2.4.1) in the strong topology is equivalent to weak convergence, i.e.,

$$\lim_{r\to 1^-} \langle U(r\omega), \varphi(\omega)\rangle = \lim_{r\to 1^-} \int_{\mathbb{S}^{n-1}} U(r\omega)\varphi(\omega)d\omega = \langle f, \varphi\rangle, \tag{2.4.2}$$

for each $\varphi \in \mathcal{E}^*(\mathbb{S}^{n-1})$.

We first show that (2.4.1) holds with $U$ being the Poisson transform of $f$. For this, our assumptions are the same as in the previous section, i.e., $(M.1)$, $(M.2)'$ and $(M.0)$ ($(NA)$ in the Beurling case). The Poisson kernel of $\mathbb{S}^{n-1}$ is [3]

$$P(x,\xi) = \frac{1}{|\mathbb{S}^{n-1}|}\frac{1-|x|^2}{|x-\xi|^n} = \frac{1}{|\mathbb{S}^{n-1}|}\sum_{j=0}^{\infty} |x|^j Z_j\left(\frac{x}{|x|}, \xi\right), \ \xi \in \mathbb{S}^{n-1},\ x \in \mathbb{B}^n. \tag{2.4.3}$$

Since $P$ is real analytic with respect to $\xi$, we can define the Poisson transform of $f \in \mathcal{E}^{*\prime}(\mathbb{S}^{n-1})$ as

$$P[f](x) = \langle f(\xi), P(x,\xi)\rangle, \quad x \in \mathbb{B}^n. \tag{2.4.4}$$

Clearly, $P[f] \in \mathcal{H}(\mathbb{B}^n)$ and, by (2.4.3), $P[f](r\omega) = \sum_{j=0}^{\infty} r^j f_j(\omega)$.

**Proposition 2.4.1.** *For each $f \in \mathcal{E}^{*\prime}(\mathbb{S}^{n-1})$ and $\varphi \in \mathcal{E}^*(\mathbb{S}^{n-1})$, we have*

$$\lim_{r\to 1^-} P[f](r\omega) = f(\omega) \quad \text{in } \mathcal{E}^{*\prime}(\mathbb{S}^{n-1}) \tag{2.4.5}$$

*and*

$$\lim_{r\to 1^-} P[\varphi](r\omega) = \varphi(\omega) \quad \text{in } \mathcal{E}^{*}(\mathbb{S}^{n-1}). \tag{2.4.6}$$

*Proof.* Due to the Montel property of these spaces (which also implies they are reflexive), it is enough to verify weak convergence of the Poisson transform in both cases in order to prove strong convergence of (2.4.5) and (2.4.6). By Theorem 2.3.4 (or Theorem 2.3.1), we have that $\langle f, \varphi\rangle =$

$\sum_{j=0}^{\infty}\langle f_j,\varphi_j\rangle$; Abel's limit theorem on power series then yields

$$\lim_{r\to 1^-}\int_{\mathbb{S}^{n-1}} P[f](r\omega)\varphi(\omega)d\omega$$
$$= \lim_{r\to 1^-}\langle f(\omega), P[\varphi](r\omega)\rangle = \lim_{r\to 1^-}\sum_{j=0}^{\infty} r^j\langle f_j,\varphi_j\rangle = \langle f,\varphi\rangle.$$

□

We now deal with the characterization of harmonic functions $U$ that satisfy (2.4.1). This characterization is in terms of the associated function of $M_p/p!$, which we denote by $M^*$ as in [33], i.e., the function

$$M^*(t) = \sup_{p\in\mathbb{N}}\log\left(\frac{p!t^p}{M_p}\right) \quad \text{for } t>0$$

and $M^*(0)=0$. We need two extra assumptions on the sequence, namely,

$(M.1)^*$ $M_p/p!$ satisfies $(M.1)$,

$(M.2)$ $M_{p+q}\le AH^{p+q}M_qM_q$, $p,q\in\mathbb{N}$, for some $A,H\ge 1$.

Naturally, $(M.1)^*$ implies $(M.0)$ and $(M.1)$ while $(M.2)$ is stronger than $(M.2)'$.

Note that $(M.1)^*$ delivers essentially two cases. Either $(NA)$ holds or there are constants such that $C_1L_1^p p!\le M_p\le C_2L_2^p p!$. In the latter case we may assume that $M_p=p!$ as for any such a sequence $\mathcal{E}^{\{M_p\}}(\mathbb{S}^{n-1})=\mathcal{A}(\mathbb{S}^{n-1})$. When $(NA)$ holds $M^*(t)$ is finite for all $t\in[0,\infty)$, whereas $M_p=p!$ gives $M^*(t)=0$ for $0\le t\le 1$ and $M^*(t)=\infty$ for $t>1$. In the $(NA)$ case we also have $M^*(t)=0$ for $t\in[0,M_1]$. The importance of the assumptions $(M.1)^*$ and $(M.2)$ lies in the ensuing lemma of Petzsche and Vogt:

**Lemma 2.4.2** ([44]). *Suppose that $M_p$ satisfies $(M.1)^*$ and $(M.2)$. Then, there are constants $L,\ell>0$ such that*

$$\inf_{y>0}(M^*(1/y)+ty)\le M(\ell t)+\log L, \quad \textit{for all } t>0.$$

We then have,

**Theorem 2.4.3.** *Assume $M_p$ satisfies $(M.1)^*$ and $(M.2)$. Then, a harmonic function $U \in \mathcal{H}(\mathbb{B}^n)$ admits boundary values in $\mathcal{E}^{\{M_p\}'}(\mathbb{S}^{n-1})$ (in $\mathcal{E}^{(M_p)'}(\mathbb{S}^{n-1})$) if and only if for each $h > 0$ there is $C = C_h > 0$ (there are $h > 0$ and $C > 0$) such that*

$$|U(x)| \leq Ce^{M^*\left(\frac{h}{1-|x|}\right)} \quad \textit{for all } x \in \mathbb{B}^n. \tag{2.4.7}$$

*In such a case $U = P[f]$, where $f$ is its boundary ultradistribution given by* (2.4.1).

*Proof.* Suppose $U(x) = P[f](x)$ with $f \in \mathcal{E}^{*\prime}(\mathbb{S}^{n-1})$. Then,

$$U(r\omega) = \sum_{j=0}^{\infty} r^j f_j(\omega).$$

If $\|f_j\|_{L^\infty(\mathbb{S}^{n-1})} \leq Ce^{M(hj)}$, for a fixed $h > 0$, the inequality (1.3.1) gives

$$\begin{aligned}|U(r\omega)| &\leq \sum_{j=0}^{\infty} |f_j(\omega)| r^j = C + \frac{A^2}{h^2}\sum_{j=1}^{\infty}\frac{1}{j^2}|f_j(\omega)|e^{-M(hj)}e^{M(hH^2j)}r^j \\ &\leq C\left(1 + \frac{A^2\pi^2}{6h^2}\right)\sup_{j\in\mathbb{N}} r^j e^{M(hH^2j)}.\end{aligned}$$

Now,

$$\sup_{j\in\mathbb{N}} r^j e^{M(hH^2j)} = \sup_{p\in\mathbb{N}} \frac{(H^2h)^p}{M_p} \sup_{j\in\mathbb{N}} r^j j^p$$

and

$$\begin{gathered}\sup_{j\in\mathbb{N}} r^j j^p \leq \sum_{j=0}^{\infty} r^j j^p \leq \sum_{j=0}^{\infty}\frac{(j+p)!}{j!}r^j = \\ \left(\frac{1}{1-r}\right)^{(p)} = \frac{p!}{(1-r)^{p+1}} < \frac{(p+1)!}{(1-r)^{p+1}}\,.\end{gathered}$$

Therefore, by $(M.2)'$,

$$|U(r\omega)| \leq C\frac{A}{H^3h}\left(1 + \frac{A^2\pi^2}{6h^2}\right)\sup_{p}\frac{(p+1)!(H^3h)^{p+1}}{M_{p+1}(1-r)^{p+1}} \leq CC_he^{M^*\left(\frac{H^3h}{1-r}\right)}.$$

Assume now that (2.4.7) holds for each $h > 0$ (for some $h > 0$). Every harmonic function on $\mathbb{B}^n$ can be written as

$$U(r\omega) = \sum_{j=0}^{\infty} r^j f_j(\omega),$$

with each $f_j$ a spherical harmonic of degree $j$. By Proposition 2.4.1, it is enough to check that $f = \sum_{j=0}^{\infty} f_j \in \mathcal{E}^{*\prime}(\mathbb{S}^{n-1})$, because in this case $U = P[f]$ and $f$ would be the boundary ultradistribution of $U$. By Theorem 2.3.4, it is then suffices to verify that the sequence $f_j$ satisfies the bounds (2.3.9) for each $h > 0$ (for some $h > 0$). Here we use $q = \infty$. Fix $h > 0$ and assume that (2.4.7) holds. One clearly has

$$f_j(\omega) = \frac{1}{r^j |\mathbb{S}^{n-1}|} \int_{\mathbb{S}^{n-1}} U(r\xi) Z_j(\omega, \xi) d\xi.$$

When $j = 0$, we obtain $f_0 = U(0)$ and so $\|f_0\|_{L^\infty(\mathbb{S}^{n-1})} \leq Ce^{M^*(h)}$. Keep now $j \geq 1$. Since the zonal harmonic satisfies $\|Z_j(\cdot, \xi)\|_{L^\infty(\mathbb{S}^{n-1})} = d_j \leq nj^{n-2}$ [3, p. 80], we obtain, for all $j \geq 1$,

$$\|f_j\|_{L^\infty(\mathbb{S}^{n-1})} \leq Cnj^{n-2} \inf_{0<r<1} r^{-j} e^{M^*\left(\frac{h}{1-r}\right)}.$$

Performing the substitution $r = e^{-y}$, and using Lemma 2.4.2 and $M^*(t) = 0$ for $t \leq M_1$,

$$\begin{aligned} \|f_j\|_{L^\infty(\mathbb{S}^{n-1})} &\leq CC_h nj^{n-2} \exp\left( \inf_{0<y<\infty} M^*(2h/y) + jy \right) \\ &\leq CC_h Lnj^{n-2} e^{M(2\ell hj)}. \end{aligned}$$

Finally, using the estimate (1.3.1), we conclude that there is $C'_h$ such that

$$\|f_j\|_{L^\infty(\mathbb{S}^{n-1})} \leq CC'_h e^{M(2\ell hj)}, \quad \text{for each } j \in \mathbb{N}.$$

□

When $M_p = p!$, the bound (2.4.7) holds for any arbitrary harmonic function since $M^*(t) = \infty$ for $t > 1$. Hence,

**Corollary 2.4.4.** *Any harmonic function $U \in \mathcal{H}(\mathbb{B}^n)$ can be written as the Poisson transform $U = P[f]$ of an analytic functional $f$ on $\mathbb{S}^{n-1}$.*

Suppose that $M_p$ satisfies $(NA)$. Consider the family of Banach spaces

$$\mathcal{H}^{M_p,h}(\mathbb{B}^n) = \{U \in \mathcal{H}(\mathbb{B}^n) : \|U\|_{\mathcal{H}^{M_p,h}(\mathbb{B}^n)} = \sup_{x\in\mathbb{B}^n} |U(x)|e^{-M^*(\frac{h}{1-|x|})} < \infty\}.$$

We define the Fréchet and $(LB)$-spaces of harmonic functions

$$\mathcal{H}^{\{M_p\}}(\mathbb{B}^n) = \varinjlim_{h\to 0^+} \mathcal{H}^{\{M_p\},h}(\mathbb{B}^n) \quad \text{and} \quad \mathcal{H}^{(M_p)}(\mathbb{B}^n) = \varprojlim_{h\to\infty} \mathcal{H}^{\{M_p\},h}(\mathbb{B}^n).$$

This definition still makes sense for $\{p!\}$ because for $h < 1$ we have

$$\sup_{x\in\mathbb{B}^n} |U(x)|e^{-M^*\left(\frac{h}{1-|x|}\right)} = \sup_{|x|\leq 1-h} |U(x)|.$$

In this case we obtain the space of all harmonic functions $\mathcal{H}(\mathbb{B}^n) = \mathcal{H}^{\{p!\}}(\mathbb{B}^n)$ with the canonical topology of uniform convergence on compact subsets of $\mathbb{B}^n$. By Theorem 2.4.3, the mapping $\mathrm{bv}(U) = f$, with $f$ given by (2.4.1), provides a linear isomorphism from $\mathcal{H}^*(\mathbb{B}^n)$ onto $\mathcal{E}^{*\prime}(\mathbb{S}^{n-1})$ if $M_p$ satisfies $(M.1)^*$ and $(M.2)$. Our proof given above actually yields a topological result:

**Theorem 2.4.5.** *Suppose $M_p$ satisfies $(M.1)^*$ and $(M.2)$. The boundary value mapping*

$$\mathrm{bv} : \mathcal{H}^*(\mathbb{B}^n) \to \mathcal{E}^{*\prime}(\mathbb{S}^{n-1})$$

*is a topological vector space isomorphism with the Poisson transform*

$$P : \mathcal{E}^{*\prime}(\mathbb{S}^{n-1}) \to \mathcal{H}^*(\mathbb{B}^n)$$

*as inverse.*

*Remark* 2.4.6. Suppose $M_p$ satisfies $(M.0)$ ($(NA)$ in the Beurling case). Theorem 2.4.5 is valid if one replaces $(M.1)^*$ by the condition

$(M.4)$ $M_p \leq L^{p+1} p! M_p^*$, $p \in \mathbb{N}$, for some $L \geq 1$.

Here $M_p^*$ is the convex regularization of $M_p/p!$, namely, the sequence

$$M_p^* = \sup_{t>0} \frac{t^p}{e^{M^*(t)}}.$$

In fact, $p!M_p^*$ satisfies $(M.1)^*$ and, under $(M.4)$, gives rise to the same ultradistribution spaces as $M_p$. We mention that strong non-quasianalyticity (i.e., Komatsu's condition $(M.3)$ [33]) automatically yields $(M.4)$, as was shown by Petzsche [43, Prop. 1.1]. Furthermore, Petzsche and Vogt [44, Sect. 5] proved under the assumption $(M.2)$ that $(M.4)$ is equivalent to the so-called Rudin condition:

$$(M.4)'' \quad \max_{q\leq p}\left(\frac{M_q}{q!}\right)^{\frac{1}{q}} \leq A\left(\frac{M_p}{p!}\right)^{\frac{1}{p}}, \quad p\in\mathbb{N}, \quad \text{for some} \quad A>0,$$

which is itself equivalent to the property that $\mathcal{E}^*(\mathbb{S}^{n-1})$ is inverse closed (cf. [48, 51]).

## 2.5 The support of ultradistributions on the sphere

This section is devoted to characterizing the support of non-quasianalytic ultradistributions in terms of (uniform) Abel-Poisson summability of their spherical harmonic expansions. Our assumptions on the weight sequence are $(M.1)$, $(M.2)'$ and $(M.3)'$. Note that $(NA)$ is automatically fulfilled because of $(M.3)'$ [33, Lemma 4.1, p. 56].

To emphasize we are assuming $(M.3)'$, we write $\mathcal{D}^{*\prime}(\mathbb{S}^{n-1}) = \mathcal{E}^{*\prime}(\mathbb{S}^{n-1})$. By the Denjoy-Carleman theorem [33], the support of an ultradistribution $f\in\mathcal{D}^{*\prime}(\mathbb{S}^{n-1})$ can be defined in the usual way. Since the natural inclusion $\mathcal{D}'(\mathbb{S}^{n-1})\subset\mathcal{D}^{*\prime}(\mathbb{S}^{n-1})$ is support preserving, Theorem 2.5.2 below contains González Vieli's characterization of the support of Schwartz distributions on the sphere [25]. The key to the proof of our generalization is the ensuing lemma about the Poisson kernel. Given a non-empty closed set $K\subset\mathbb{S}^{n-1}$ and a weight sequence $N_p$, we consider the family of seminorms

$$\|\varphi\|_{\mathcal{E}^{\{N_p\},h}(K)} = \sup_{\alpha\in\mathbb{N}^n}\frac{h^{|\alpha|}\|\partial^\alpha_{\mathbb{S}^{n-1}}\varphi\|_{L^\infty(K)}}{N_{|\alpha|}}.$$

**Lemma 2.5.1.** *Let $K_1$ and $K_2$ be two disjoint non-empty closed subsets of $\mathbb{S}^{n-1}$. Write $P_{r\omega}(\xi)=P(r\omega,\xi)$, regarded as a function in the variable*

$\xi \in \mathbb{S}^{n-1}$. *Then, there are two positive constants* $\ell$ *and* $C$, *only depending on* $K_1$ *and* $K_2$, *such that*

$$\|P_{r\omega}\|_{\mathcal{E}^{\{p!\},\ell}(K_2)} \leq C(1-r), \quad \textit{for all } \omega \in K_1 \textit{ and } \frac{1}{2} \leq r < 1.$$

*Proof.* For the sake of convenience, we introduce the spherical type distance

$$d(\omega, \xi) = 1 - \omega \cdot \xi.$$

Let $V \subset \mathbb{S}^{n-1}$ be open such that $K_1 \cap \overline{V} = \emptyset$ and $K_2 \subset V$. Set $\rho = d(K_1, V)$. Note that if $\omega \in K_1$ and $\xi \in V$, the term in the denominator of the Poisson kernel,

$$P(r\omega, \xi) = \frac{1}{|\mathbb{S}^{n-1}|} \frac{1-r^2}{(1 - 2r\omega \cdot \xi + r^2)^{\frac{n}{2}}} \,,$$

can be estimated by using the lower bound

$$1 - 2r\omega \cdot \xi + r^2 = (1-r)^2 + 2r(1 - \omega \cdot \xi) > 2r\rho.$$

We estimate the derivatives of the Poisson kernel in spherical coordinates $\mathfrak{p}(\theta)$ where the north pole is chosen to be located at an arbitrary point of the sphere. Keep $\omega \in K_1$ and $1/2 \leq r < 1$ arbitrary. Let $V' \subset \mathbb{R}^{n-1}$ be such that $V = \mathfrak{p}(V')$. Call $m = |\alpha|$. Using the estimate (2.2.6) and the obvious inequality $m^m \leq e^{m-1} m!$ , we obtain

$$\begin{aligned}
&\|\partial_\theta^\alpha (P_{r\omega} \circ \mathfrak{p})\|_{L^\infty(V')} \leq \frac{1-r^2}{|\mathbb{S}^{n-1}|} \sum_{k=1}^{m} \binom{m}{k} k^{m-k} n^k \frac{\Gamma\left(\frac{n}{2}+k\right)}{\Gamma\left(\frac{n}{2}\right)} \\
&\times \sup_{\xi \in V} \frac{(2r)^k |\omega|^k}{(1 - 2r\omega \cdot \xi) + r^2)^{k+\frac{n}{2}}} \\
&< \frac{1-r^2}{(2r\rho)^{\frac{n}{2}} |\mathbb{S}^{n-1}|} m^m \sum_{k=1}^{m} \binom{m}{k} \left(1 + \frac{\frac{n}{2}-1}{k}\right)^k \left(\frac{n}{\rho}\right)^k \\
&< \frac{3e^{\frac{n}{2}-2}}{2\rho^{\frac{n}{2}} |\mathbb{S}^{n-1}|} (1-r) m! \left(e\left(\frac{n}{\rho}+1\right)\right)^m = C_1 (1-r) \ell_1^{-|\alpha|} |\alpha|!.
\end{aligned}$$

Varying the north poles, we can cover $K_2$ by a finite number of open subsets of $V$, each of which parametrized by a system of invertible spherical coordinates. Inverting the polar coordinates on each of the open sets of this covering with the aid of [29, Prop. 8.1.4], we deduce that there are $\ell, C > 0$, depending only on $V$, such that

$$\frac{\ell^{\alpha} \| \partial^{\alpha}_{\mathbb{S}^{n-1}} P_{r\omega} \|_{L^{\infty}(K_2)}}{|\alpha|!} \leq C(1-r), \quad \text{for all } \alpha \in \mathbb{N}^n.$$

This completes the proof of the lemma. □

We are ready to state and prove our last result:

**Theorem 2.5.2.** *Let* $f = \sum_{j=0}^{\infty} f_j \in \mathcal{D}^{*\prime}(\mathbb{S}^{n-1})$ *and let* $\Omega$ *be an open subset of* $\mathbb{S}^{n-1}$*. If*

$$\lim_{r \to 1^-} \sum_{j=1}^{\infty} r^j f_j(\omega) = \lim_{r \to 1} P[f](r\omega) = 0 \tag{2.5.1}$$

*holds uniformly for* $\omega$ *on compact subsets of* $\Omega$*, then* $\Omega \subseteq \mathbb{S}^{n-1} \setminus \operatorname{supp} f$*.*

*Conversely,* (2.5.1) *holds uniformly on any compact subset of* $\mathbb{S}^{n-1} \setminus \operatorname{supp} f$*.*

*Proof.* The first part follows immediately from Proposition 2.4.1. Indeed, let $\varphi \in \mathcal{E}^*(\mathbb{S}^{n-1})$ be an arbitrary test function such that $\operatorname{supp} \varphi \subset \Omega$. Then,

$$\langle f, \varphi \rangle = \lim_{r \to 1^-} \int_{\mathbb{S}^{n-1}} P[f](r\omega)\varphi(\omega) d\omega = \lim_{r \to 1^-} \int_{\operatorname{supp} \varphi} P[f](r\omega)\varphi(\omega) d\omega = 0,$$

which gives that $f$ vanishes on $\Omega$.

Conversely, since we have the dense and continuous embeddings $\mathcal{E}^{(M_p)}(\mathbb{S}^{n-1}) \hookrightarrow \mathcal{E}^{\{M_p\}}(\mathbb{S}^{n-1})$ (by Proposition 2.3.2 the linear span of the spherical harmonics is dense in both spaces), we have the natural inclusion $\mathcal{E}^{\{M_p\}\prime}(\mathbb{S}^{n-1}) \to \mathcal{E}^{(M_p)\prime}(\mathbb{S}^{n-1})$ which is obviously support preserving. Thus, we may just deal with the case $f \in \mathcal{E}^{(M_p)\prime}(\mathbb{S}^{n-1})$. Let $K_1$ be closed such that $K_1 \cap \operatorname{supp} f = \emptyset$. Select a closed subset of the sphere $K_2$ such that $K_1 \cap K_2 = \emptyset$ and $\operatorname{supp} f \subset \operatorname{int} K_2$. There are then $C_1$ and $h > 0$ such that

$$|\langle f, \varphi \rangle| \leq C_1 \|\varphi\|_{\mathcal{E}^{\{M_p\},h}(K_2)}, \quad \text{for all } \varphi \in \mathcal{E}^{(M_p)}(\mathbb{S}^{n-1}).$$

The sequence $M_p$ satisfies $(NA)$, hence, given $\ell$, one can find $C_2 > 0$, depending only on $h$ and $\ell$, such that $\|\varphi\|_{\mathcal{E}^{\{M_p\},h}(K_2)} \leq C_2 \|\varphi\|_{\mathcal{E}^{\{p!\},\ell}(K_2)}$ for all $\varphi \in \mathcal{A}(\mathbb{S}^{n-1})$. Using this with $\varphi = P_{r\omega}$ and employing Lemma 2.5.1,

$$|P[f](r\omega)| = |\langle f, P_{r\omega}\rangle| \leq C_1 C_2 C(1-r), \quad \text{for all } \omega \in K_1 \text{ and } \frac{1}{2} \leq r < 1,$$

whence (2.5.1) holds uniformly for $\omega \in K_1$. $\square$

# Chapter 3

# Rotation invariant ultradistributions

## 3.1 Introduction

The aim of this chapter is to show that the characterization of rotation invariant ultradistributions due to Chung and Na in [13] in terms of their spherical means remains valid for quasianalytic ultradistributions. More precisely, we prove that an ultradistribution is rotation invariant if and only if it coincides with its spherical mean. For it, we study the problem of spherical representations of ultradistributions on $\mathbb{R}^n$. Our results apply to both the quasianalytic and the non-quasianalytic case, with the approach that differs from that of Chung and Na (see [13]).

Our method is based upon the study of spherical representations of ultradistributions, that is, the problem of representing an ultradistribution $f$ on $\mathbb{R}^n$ by an ultradistribution $g$ on $\mathbb{R} \times \mathbb{S}^{n-1}$ in such a way that $\langle f(x), \varphi(x) \rangle = \langle g(r, \omega), \varphi(r\omega) \rangle$. Spherical representations of distributions were studied by Drozhzhinov and Zav'yalov in [20]. We shall also exploit results on spherical harmonic expansions of ultradifferentiable functions and ultradistributions on the unit sphere $\mathbb{S}^{n-1}$ from the previous chapter.

The plan of the chapter is as follows. Section 3.2 discusses some background material on spherical harmonics and ultradistributions. Spherical representations of ultradistributions are studied in Section 3.3. We show in Section 3.4 that any ultradistribution is rotation invariant if and only if it coincides with its spherical mean. In the quasianalytic case we go beyond quasianalytic functionals by employing sheaves of quasianalytic ultradistributions.

## 3.2 Further remarks on ultradistributions and spherical harmonics

The spaces of ultradifferentiable functions and ultradistributions on $\mathbb{S}^{n-1}$ can be described in terms of their spherical harmonic expansions. We will apply results from the previous chapter in order to expand ultradifferentiable functions and ultradistributions on $\mathbb{R} \times \mathbb{S}^{n-1}$ in spherical harmonic series.

Recall that, under $(M.0)$, $(M.1)$, and $(M.2)'$ for the sequence $M_p$, if $\varphi \in L^2(\mathbb{S}^{n-1})$ has spherical harmonic expansion

$$\varphi(\omega) = \sum_{j=0}^{\infty} \sum_{k=1}^{d_j} a_{k,j} Y_{k,j}(\omega). \tag{3.2.1}$$

Then $\varphi \in \mathcal{E}^*(\mathbb{S}^{n-1})$ if and only if the estimate

$$\sup_{k,j} |a_{k,j}| e^{M\left(\frac{j}{h}\right)} < \infty \tag{3.2.2}$$

holds for some $h > 0$ (for all $h > 0$) (see Section 2.3).

We have also proved that every ultradistribution $f \in \mathcal{E}^{*\prime}(\mathbb{S}^{n-1})$ admits a spherical harmonic expansion

$$f(\omega) = \sum_{j=0}^{\infty} \sum_{k=1}^{d_j} c_{k,j} Y_{k,j}(\omega), \tag{3.2.3}$$

where the coefficients satisfy the estimate

$$\sup_{k,j} |c_{k,j}| e^{-M\left(\frac{j}{h}\right)} < \infty \tag{3.2.4}$$

for each $h > 0$ (for some $h > 0$). Conversely, any series (3.2.3) converges in $\mathcal{E}^{*\prime}(\mathbb{S}^{n-1})$ if the coefficients have the stated growth properties.

It is important to point out that we did not reveal all topological information encoded by the spherical harmonic coefficients. Denote as $\mathcal{E}_{sh}^{\{M_p\},h}(\mathbb{S}^{n-1})$ the Banach space of all (necessarily smooth) functions $\varphi$ on $\mathbb{S}^{n-1}$ having spherical harmonic expansion with coefficients $a_{k,j}$ satisfying (3.2.2) for a given $h$. One can then show, using results from the previous chapter

$$\mathcal{E}^{\{M_p\}}(\mathbb{S}^{n-1}) = \varinjlim_{h\to\infty} \mathcal{E}_{sh}^{\{M_p\},h}(\mathbb{S}^{n-1}), \quad \mathcal{E}^{(M_p)}(\mathbb{S}^{n-1}) = \varprojlim_{h\to 0^+} \mathcal{E}_{sh}^{\{M_p\},h}(\mathbb{S}^{n-1})$$

topologically. This for instance yields immediately the nuclearity of $\mathcal{E}^{*}(\mathbb{S}^{n-1})$ under the assumptions of Theorem 2.3.1. Observe also that the norm on the Banach space $\mathcal{E}_{sh}^{\{M_p\},h}(\mathbb{S}^{n-1})$ can be rewritten as

$$\|\varphi\|_{\mathcal{E}_{sh}^{\{M_p\},h}(\mathbb{S}^{n-1})} = \sup_{k,j} e^{M(\frac{j}{h})} \left| \int_{\mathbb{S}^{n-1}} \varphi(\omega) Y_{k,j}(\omega) d\omega \right|. \tag{3.2.5}$$

A similar topological description can be given for the ultradistribution space $\mathcal{E}^{*\prime}(\mathbb{S}^{n-1})$ by using the coefficient estimates (3.2.4).

### 3.2.1 Ultradistributions on $\mathbb{R} \times \mathbb{S}^{n-1}$

We also need some properties of the spaces $\mathcal{E}^{*}(\mathbb{R} \times \mathbb{S}^{n})$ and $\mathcal{E}^{*\prime}(\mathbb{R} \times \mathbb{S}^{n})$. Let us assume $(M.0)$, $(M.1)$, and $(M.2)$. We have

$$\mathcal{E}^{*}(\mathbb{R} \times \mathbb{S}^{n-1}) = \mathcal{E}^{*}(\mathbb{R}, \mathcal{E}^{*}(\mathbb{S}^{n-1})) = \mathcal{E}^{*}(\mathbb{S}^{n-1}, \mathcal{E}^{*}(\mathbb{R})) = \mathcal{E}^{*}(\mathbb{R}) \widehat{\otimes} \mathcal{E}^{*}(\mathbb{S}^{n-1}),$$

where the tensor product may be equally taken with respect to the $\pi$- or $\varepsilon$-topology in view of the nuclearity of these spaces. In fact, the first two equalities are completely trivial, while the third one follows because the linear span of terms of the form $p \otimes Y$, where $p$ is a polynomial on $\mathbb{R}$ and $Y$ a spherical harmonic, is dense in $\mathcal{E}^{*}(\mathbb{R} \times \mathbb{S}^{n-1})$. Moreover, this immediately gives (cf. (3.2.4)) that

$$\mathcal{E}^{\{M_p\}}(\mathbb{R} \times \mathbb{S}^{n-1}) = \varprojlim_{K \Subset \mathbb{R}} \varinjlim_{h\to\infty} \mathcal{E}_{sh}^{\{M_p\},h}(K \times \mathbb{S}^{n-1})$$

and

$$\mathcal{E}^{(M_p)}(\mathbb{R}\times\mathbb{S}^{n-1}) = \varprojlim_{K\Subset\mathbb{R}}\varprojlim_{h\to 0^+}\mathcal{E}_{sh}^{\{M_p\},h}(K\times\mathbb{S}^{n-1}),$$

where $\mathcal{E}_{sh}^{\{M_p\},h}(K\times\mathbb{S}^{n-1})$ is the space of functions $\Phi$ such that

$$\begin{aligned}&\|\Phi\|_{\mathcal{E}_{sh}^{\{M_p\},h}(K\times\mathbb{S}^{n-1})}\\ &= \sup_{k,j} e^{M\left(\frac{j}{h}\right)}\left\|\int_{\mathbb{S}^{n-1}}\Phi(\,\cdot\,,\omega)Y_{k,j}(\omega)d\omega\right\|_{\mathcal{E}^{\{M_p\},h}(K)} < \infty. \end{aligned}\tag{3.2.6}$$

These comments yield the following proposition.

**Proposition 3.2.1.** *Assume $M_p$ satisfies $(M.0)$, $(M.1)$, and $(M.2)$.*

*(i) Every $\Phi\in\mathcal{E}^*(\mathbb{R}\times\mathbb{S}^{n-1})$ has convergent expansion*

$$\Phi(r,\omega) = \sum_{j=0}^{\infty}\sum_{k=1}^{d_j} a_{k,j}(r)Y_{k,j}(\omega) \quad \textit{in } \mathcal{E}^*(\mathbb{R}\times\mathbb{S}^{n-1}),$$

*where $a_{k,j}\in\mathcal{E}^*(\mathbb{R})$ and for each $K\Subset\mathbb{R}$*

$$\sup_{k,j} e^{M\left(\frac{j}{h}\right)}\|a_{k,j}\|_{\mathcal{E}^{\{M_p\},h}(K)} < \infty \tag{3.2.7}$$

*for some $h>0$ (for all $h>0$). Conversely, any such series converges in the space $\mathcal{E}^*(\mathbb{R}\times\mathbb{S}^{n-1})$ if (3.2.7) holds.*

*(ii) Every ultradistribution $g\in\mathcal{E}^{*\prime}(\mathbb{R}\times\mathbb{S}^{n-1})$ has convergent expansion*

$$g(r,\omega) = \sum_{j=0}^{\infty}\sum_{k=1}^{d_j} c_{k,j}(r)\otimes Y_{k,j}(\omega) \quad \textit{in } \mathcal{E}^{*\prime}(\mathbb{R}\times\mathbb{S}^{n-1}),$$

*where $c_{k,j}\in\mathcal{E}^{*\prime}(\mathbb{R})$ and for any bounded subset $B\subset\mathcal{E}^*(\mathbb{R})$ one has*

$$\sup_{k,j}\; e^{-M\left(\frac{j}{h}\right)}\sup_{\varphi\in B}|\langle c_{k,j},\varphi\rangle| < \infty. \tag{3.2.8}$$

*for each $h$ (for some $h$). Conversely, any such series converges in the space $\mathcal{E}^{*\prime}(\mathbb{R}\times\mathbb{S}^{n-1})$ if (3.2.8) holds.*

*Proof.* For $(i)$, simply note that $a_{k,j}(r) = \int_{\mathbb{S}^{n-1}} \Phi(r,\omega)Y_{k,j}(\omega)d\omega$ and so (3.2.6) is the same as (3.2.7). The convergence of the series expansions of $\Phi$ is trivial to check via the seminorms (3.2.6). Part $(ii)$ follows from $(i)$ and the canonical identification
$\mathcal{E}^{*\prime}(\mathbb{R}\times\mathbb{S}^{n-1}) = \mathcal{E}^{*\prime}(\mathbb{S}^{n-1}, \mathcal{E}^{*\prime}(\mathbb{R}))(:= L_b(\mathcal{E}^{*}(\mathbb{S}^{n-1}), \mathcal{E}^{*\prime}(\mathbb{R})))$. □

Note that the same proposition holds for $\mathcal{D}^{*\prime}(\mathbb{R}\times\mathbb{S}^{n})$ if one additionally assumes $(M.3)'$.

## 3.3 Spherical Representations of Ultradistributions

It is easy to see that any $g \in \mathcal{E}^{*\prime}(\mathbb{R}\times\mathbb{S}^{n-1})$ gives rise to an ultradistribution $f$ on $\mathbb{R}^n$ via the formula

$$\langle f(x), \varphi(x)\rangle = \langle g(r,\omega), \varphi(r\omega)\rangle. \tag{3.3.1}$$

In fact, the assignment $g \mapsto f$ is simply the transpose of

$$\varphi \mapsto \Phi, \quad \Phi(r,\omega) := \varphi(r\omega), \tag{3.3.2}$$

which is obviously continuous $\mathcal{E}^{*}(\mathbb{R}^n) \to \mathcal{E}^{*}(\mathbb{R}\times\mathbb{S}^{n-1})$.

In this section we study the converse representation problem. That is, the problem of representing an $f \in \mathcal{E}^{*\prime}(\mathbb{R}^n)$ as in (3.3.1) for some ultradistribution $g$ on $\mathbb{R}\times\mathbb{S}^{n-1}$. We shall call any such $g$ a *spherical representation* of $f$. Naturally, the same considerations make sense for $f \in \mathcal{D}^{*\prime}(\mathbb{R}^n)$ in the non-quasianalytic case.

In order to fix ideas, let us first discuss the distribution case. The problem of finding a spherical representation of $f \in \mathcal{D}'(\mathbb{R}^n)$ can be reduced to the determination of the image of $\mathcal{E}(\mathbb{R}^n)$ under the mapping (3.3.2). Notice that the range of this mapping is obviously contained in the subspace of "even" test functions, namely,

$$\begin{aligned}&\mathcal{E}_e(\mathbb{R}\times\mathbb{S}^{n-1})\\ &= \{\Phi \in \mathcal{E}(\mathbb{R}\times\mathbb{S}^{n-1}) : \Phi(-r,-\omega) = \Phi(r,\omega), \forall (r,\omega) \in \mathbb{R}\times\mathbb{S}^{n-1}\}.\end{aligned}$$

In other words, one is interested here in characterizing all those $\Phi \in \mathcal{E}_e(\mathbb{R} \times \mathbb{S}^{n-1})$ such that

$$\varphi(x) = \Phi\left(|x|, \frac{x}{|x|}\right) \tag{3.3.3}$$

is a smooth function on $\mathbb{R}^n$. The solution to the latter problem is well-known:

**Proposition 3.3.1** ([20, 28])**.** *Let $\Phi \in \mathcal{E}_e(\mathbb{R} \times \mathbb{S}^{n-1})$. Then, $\varphi$ given by* (3.3.3) *is an element of $\mathcal{E}(\mathbb{R}^n)$ if and only if $\Phi$ has the property that for each $m \in \mathbb{N}$*

$$\frac{\partial^m \Phi}{\partial r^m}(0, \omega) \quad \textit{is a homogeneous polynomial of degree } m. \tag{3.3.4}$$

Write

$$\mathcal{V}(\mathbb{R} \times \mathbb{S}^{n-1}) := \{\Phi \in \mathcal{E}_e(\mathbb{R} \times \mathbb{S}^{n-1}) : \text{ (3.3.4) holds for each } m \in \mathbb{N}\}.$$

Hence $\mathcal{V}(\mathbb{R} \times \mathbb{S}^{n-1})$ is precisely the image of $\mathcal{E}(\mathbb{R}^n)$ under (3.3.2). Since it is obviously a closed subspace of $\mathcal{E}(\mathbb{R} \times \mathbb{S}^{n-1})$, one obtains from the open mapping theorem that $\mathcal{E}(\mathbb{R}^n)$ is isomorphic to $\mathcal{V}(\mathbb{R} \times \mathbb{S}^{n-1})$ via (3.3.2). Given $f \in \mathcal{D}'(\mathbb{R}^n)$, $\langle f(x), \Phi(|x|, x/|x|)\rangle$ defines a continuous linear functional on $\mathcal{D}(\mathbb{R} \times \mathbb{S}^{n-1}) \cap \mathcal{V}(\mathbb{R} \times \mathbb{S}^{n-1})$, and, by applying the Hahn-Banach theorem, one establishes the existence of a spherical representation $g \in \mathcal{D}'(\mathbb{R} \times \mathbb{S}^{n-1})$ for $f$.

We now treat the ultradistribution case. We consider

$$\mathcal{V}^*(\mathbb{R} \times \mathbb{S}^{n-1}) := \mathcal{V}(\mathbb{R} \times \mathbb{S}^{n-1}) \cap \mathcal{E}^*(\mathbb{R} \times \mathbb{S}^{n-1}),$$

a closed subspace of $\mathcal{E}^*(\mathbb{R} \times \mathbb{S}^{n-1})$. It is clear that (3.3.2) maps $\mathcal{E}^*(\mathbb{R}^n)$ continuously into $\mathcal{V}^*(\mathbb{R} \times \mathbb{S}^{n-1})$, but whether this mapping is surjective or not is not evident. The next theorem gives a partial answer to this question, which allows one to consider spherical representations of ultradistributions. We associate the weight sequence

$$N_p = \sqrt{p! M_p}$$

to $M_p$. Note that $N_p \subset M_p$ in the Roumieu case, while $N_p \prec M_p$ in the Beurling case. The symbol $\dagger$ stands for $\{N_p\}$ if $* = \{M_p\}$, while when $* = (M_p)$ we set $\dagger = (N_p)$.

**Theorem 3.3.2.** *Suppose that $M_p$ satisfies $(M.0)$, $(M.1)$, and $(M.2)$.*

*(i) The linear mapping $\Phi \to \varphi$, where $\varphi$ is given by (3.3.3), maps continuously $\mathcal{V}^\dagger(\mathbb{R} \times \mathbb{S}^{n-1})$ into $\mathcal{E}^*(\mathbb{R}^n)$.*

*(ii) Any ultradistribution $f \in \mathcal{E}^{*\prime}(\mathbb{R}^n)$ admits a spherical representation from $\mathcal{E}^{\dagger\prime}(\mathbb{R}\times\mathbb{S}^{n-1})$; more precisely, one can always find $g \in \mathcal{E}^{\dagger\prime}(\mathbb{R}\times \mathbb{S}^{n-1})$ such that (3.3.1) holds for all $\varphi \in \mathcal{E}^\dagger(\mathbb{R}^n)$.*

If $M_p$ additionally satisfies $(M.3)'$, one obviously obtains an analogous version of Theorem 3.3.2 for $\mathcal{D}^*(\mathbb{R}^n)$ and $\mathcal{D}^{*\prime}(\mathbb{R}^n)$. When $* = \{p!\}$, the sequence $N_p$ becomes equivalent to $p!$. We thus obtain the following corollary for real analytic functions and analytic functionals.

**Corollary 3.3.3.** *The linear mapping (3.3.2) is a (topological) isomorphism between the space the real analytic functions $\mathcal{A}(\mathbb{R}^n)$ and $\mathcal{V}^{\{p!\}}(\mathbb{R} \times \mathbb{S}^{n-1})$. Furthermore, any analytic functional $f \in \mathcal{A}'(\mathbb{R}^n)$ has a spherical representation $g \in \mathcal{A}'(\mathbb{R}\times\mathbb{S}^{n-1})$, so that (3.3.1) holds for all $\varphi \in \mathcal{A}(\mathbb{R}^n)$.*

The rest of this section is devoted to give a proof of Theorem 3.3.2. Note that $(ii)$ is a consequence of $(i)$ and the Hahn-Banach theorem (arguing as in the distribution case). In order to show $(i)$ we first need to establish a series of lemmas, some of them are interesting by themselves.

**Lemma 3.3.4.** *The space $\mathcal{V}^*(\mathbb{R}\times\mathbb{S}^{n-1})$ consists of all those $\Phi \in \mathcal{E}^*(\mathbb{R}\times \mathbb{S}^{n-1})$ whose coefficient functions $a_{k,j} \in \mathcal{E}^*(\mathbb{R})$ in the spherical harmonic expansion*

$$\Phi(r,\omega) = \sum_{j=0}^{\infty}\sum_{k=1}^{d_j} a_{k,j}(r)Y_{k,j}(\omega)$$

*satisfy that $a_{k,j}^{(m)}(0) = 0$ for each $m < j$, and $a_{k,j}$ is an even function if $j$ is even and $a_{k,j}$ is an odd function if $j$ is odd.*

*Proof.* Proposition 3.2.1 ensures that $\Phi$ has the spherical harmonic series expansion. Since $\Phi \in \mathcal{E}^*_e(\mathbb{R}\times\mathbb{S}^{n-1})$ we must necessarily have that $a_{k,j}$ is even when $j$ is even and $a_{k,j}$ is odd when $j$ is odd. Moreover, the other claim readily follows from the fact that for each $m \in \mathbb{N}$

$$\sum_{j=0}^{\infty}\sum_{k=1}^{d_j} a_{k,j}^{(m)}(0)Y_{k,j}(\omega)$$

needs to be the restriction to the sphere of a homogeneous polynomial of degree $m$, as for it $a_{k,j}^{(m)}(0)$ needs to be zero if $j > m$.

□

The latter suggests to study for each $j$ ultradifferentiable functions having the same properties as the coefficient functions $a_{k,j}$ from Lemma 3.3.4. Define the closed subspace

$$\mathcal{X}_j^* = \{\varphi \in \mathcal{E}^*(\mathbb{R}) :\ \varphi^{(m)}(0) = 0,\ \forall m < j\}.$$

**Lemma 3.3.5.** *Let $j \in \mathbb{N}$ and suppose $M_p$ satisfies $(M.0)$, $(M.1)$, and $(M.2)'$. The mapping*

$$\phi \mapsto \psi, \quad \psi(r) := \frac{\phi(r)}{r^j},$$

*is an isomorphism of TVS from $\mathcal{X}_j^*$ onto $\mathcal{E}^*(\mathbb{R})$. Moreover, giving a compact $K \subset \mathbb{R}$ and an arbitrary neighborhood $U$ of $K$ with compact closure, there is a constant $\ell$, only depending on $K$, $U$, and $M_p$ (but not on $j$), such that*

$$\|\psi\|_{\mathcal{E}^{\{M_p\},\ell h}(K)} \leq C_{h,U}\|\phi\|_{\mathcal{E}^{\{M_p\},h}(\overline{U})}, \quad \forall \phi \in \mathcal{X}_j^*. \tag{3.3.5}$$

*Proof.* The inverse mapping is obviously continuous, so it suffices to prove the last assertion. In order to treat the non-quasianalytic and quasianalytic cases simultaneously via a Paley-Wiener type argument, we use a Hörmander analytic cut-off sequence [29, 42]. So, find a sequence $\chi_p \in \mathcal{D}(\mathbb{R})$ such that $\chi_p \equiv 1$ on $K$, $\chi_p(x) = 0$ off $U$, and

$$\|\chi_p^{(m)}\|_{L^\infty(\mathbb{R})} \leq C(\ell_1 p)^m, \quad m \leq p.$$

By $(M.0)$ and $(M.1)$, we find with the aid of the Leibniz formula a constant $\ell_2$ such that the Fourier transform of $\phi_p = \chi_p\phi$ satisfies

$$|u^p\hat{\phi}_p(u)| \leq C' M_p(\ell_2 h)^p \|\phi\|_{\mathcal{E}^{\{M_p\},h}(\overline{U})}, \quad u \in \mathbb{R},\ p \in \mathbb{N}, \tag{3.3.6}$$

for all $\phi \in \mathcal{E}(\mathbb{R})$ with $C' = C'_{h,U}$. Consider now $\phi \in \mathcal{X}_j^*$ and the corresponding $\psi$. Setting $\psi_p = \chi_p \psi$, and Fourier transforming $r^j \psi_p(r) = \phi_p(r)$, we get $\hat{\psi}_p^{(j)}(u) = (i)^j \hat{\phi}_p(u)$. Thus, using the assumption $\varphi^{(m)}(0) = 0$ for $m < j$, we obtain

$$\begin{aligned}\hat{\psi}_p(u) &= i^j \int_{-\infty}^{u} \int_{-\infty}^{t_{j-1}} \dots \int_{-\infty}^{t_1} \hat{\varphi}_p(t_1) dt_1 \dots dt_j \\ &= (-i)^j \int_{u}^{\infty} \int_{t_{j-1}}^{\infty} \dots \int_{t_1}^{\infty} \hat{\varphi}_p(t_1) dt_1 \dots dt_j.\end{aligned}$$

Employing this expression for $\hat{\psi}_p$ and the fact that $\psi = \psi_p$ on $K$, one readily deduces (3.3.5) from (3.3.6) after applying the Fourier inversion formula and $(M.2)'$. □

Denote as $\mathcal{E}_e^*(\mathbb{R})$ the subspace of even $*$-ultradifferentiable functions.

**Lemma 3.3.6.** *Assume $M_p$ satisfies $(M.0)$, $(M.1)$, and $(M.2)$. The linear mapping*

$$\phi \mapsto \psi, \quad \psi(r) = \phi(\sqrt{|r|}),$$

*maps continuously $\mathcal{E}_e^\dagger(\mathbb{R})$ into $\mathcal{E}^*(\mathbb{R})$.*

*Proof.* We only give the proof in the non-quasianalytic case, the quasianalytic case can be treated analogously by using an analytic cut-off sequence exactly as in the proof of Lemma 3.3.5. Take an arbitrary even function $\phi \in \mathcal{D}^\dagger(K)$ with $\|\phi\|_{\mathcal{E}^{\{\sqrt{p!M_p}\},h}(K)} = 1$ and set $\psi(r^2) = \phi(r)$. We have

$$|u^{2p+1}\hat{\phi}(u)| \leq |K| h^{2p+1} \sqrt{(2p+1)!M_{2p+1}} \leq C'_h (\ell h^2)^p p! M_p. \tag{3.3.7}$$

with $C'_h = h|K|AH\sqrt{M_1}$ and $\ell = (2H)^{3/2}$, because of $(M.2)$. Consider

$$|u^p \hat{\psi}(u)| = \left| u^p \int_{-\infty}^{\infty} \phi(\sqrt{|r|}) e^{iru} dr \right| = 4 \left| u^p \int_0^\infty y\phi(y) \cos(y^2 u) dy \right|.$$

Integrating by parts the very last integral, we arrive at

$$|u^p \hat{\psi}(u)| = 2 \left| u^{p-1} \int_0^\infty \phi'(y) \sin(y^2 u) dy \right|.$$

Note that $\phi'$ is odd and so $\phi'(0) = 0$. Iterating this integration by parts procedure, we find that

$$|u^p \hat{\psi}(u)| = \frac{1}{2^{p-1}} \left| \int_0^\infty \mathcal{L}^{p-1}(\phi') G(y^2 u) dy \right| \leq |K| 2^{1-p} \|\mathcal{L}^{p-1}(\phi')\|_{L^\infty(K)} \tag{3.3.8}$$

where $G(t) = \sin t$ or $G(t) = \cos t$ and the differential operator $\mathcal{L}$ is given by

$$(\mathcal{L}\varphi)(y) = \frac{d}{dy}\left(\frac{\varphi(y)}{y}\right).$$

Note that $\mathcal{L}$ and their iterates are well-defined for smooth odd functions. Our problem then reduces to estimate $\mathcal{L}^{p-1}(\phi')$. Let $\eta_p$ be the Fourier transform of $\mathcal{L}^{p-1}(\phi')$, then

$$|\eta_p(u)| = |(T^{p-1}\widehat{(\phi')}(u)|,$$

where

$$(T\kappa)(u) = \begin{cases} \int_u^\infty t\kappa(t)dt & \text{for } u > 0 \\ \int_{-\infty}^u t\kappa(t)dt & \text{for } u < 0\ . \end{cases}$$

The inequality (3.3.7) then gives $(1 + |u|^2)|\eta_p|_{L^\infty(\mathbb{R})} \leq C_h''(\ell h^2)^p M_p$. Fourier inverse transforming and using (3.3.8), we see that $\|\psi^{(p)}\|_{L^\infty(\mathbb{R})} \leq C_h(\ell H h^2)^p M_p$, which shows the claimed continuity. $\square$

We need one more lemma. We denote as $B(0,r)$ the Euclidean ball with radius $r$ and center at the origin.

**Lemma 3.3.7.** *Given $r < 1$ there are constants $L = L_r$ and $C = C_r$ such that for any homogeneous harmonic polynomial $Q$ on $\mathbb{R}^n$ one has*

$$\|\partial^\alpha Q\|_{L^\infty(B(0,r))} \leq C L^{|\alpha|} \alpha! \|Q_{|\mathbb{S}^{n-1}}\|_{L^2(\mathbb{S}^{n-1})}.$$

*Proof.* By a result of Komatsu, one has that there is $L$, depending only on $r$, such that

$$\|\varphi\|_{\mathcal{E}^{\{p!\},Lh}(\overline{B(0,r)})} \leq C_h \sup_{p\in\mathbb{N}} \frac{\|\Delta^p \varphi\|_{L^2(B(0,1))}}{h^{2p} M_{2p}}.$$

(This actually holds for more general elliptic operators [32].) The estimate then follows by taking $h = 1$, $\varphi = Q$, using that $Q$ is harmonic, and writing out the integral in polar coordinates. $\square$

We are ready to prove Theorem 3.3.2:

*Proof of Theorem 3.3.2.* We have already seen that $(ii)$ follows from $(i)$. Let $\Phi \in V^{\dagger}(\mathbb{R} \times \mathbb{S}^{n-1})$ and set $\varphi$ as in (3.3.3). Since the change of variables $(r, \omega) \mapsto r\omega$ is analytic and invertible away from $r = 0$, it is enough to work with ultradifferentiable norms in a neighborhood of $x = 0$. Specifically, we estimate the ultradifferentiable norms of $\varphi$ on the ball $B(0, 1/2)$. Expand $\Phi$ as in Lemma 3.3.4 and assume that (cf. Proposition 3.2.1)

$$\|a_{k,j}\|_{\mathcal{E}^{\{\sqrt{p!M_p}\},h}([-1,1])} \leq e^{-M\left(\frac{j}{h}\right)}, \quad \forall j, k.$$

Combining Lemma 3.3.5 and Lemma 3.3.6, we can write

$$\frac{a_{k,j}(r)}{r^j} = b_{k,j}(r^2) \quad \text{with } b_{k,j} \in \mathcal{E}^*(\mathbb{R})$$

and

$$\|b_{k,j}\|_{\mathcal{E}^{\{M_p\},\ell_1 h^2}([-1/2,1/2])} \leq C'_h e^{-M\left(\frac{j}{h}\right)}, \quad \forall j, k.$$

where the constant $\ell_1$ does not depend on $h$. Therefore,

$$\varphi(x) = \varphi(r\omega) = \sum_{j=0}^{\infty} \sum_{k=1}^{d_j} B_{k,j}(x) P_{k,j}(x)$$

where $B_{k,j}(x) = b_{k,j}(|x|^2)$ and $P_{k,j}$ is the harmonic polynomial whose restriction to the unit sphere is $Y_{k,j}$. Since the mapping $x \mapsto |x|^2$ is analytic, the function $B_{k,j}$ is $*$-ultradifferentiable and furthermore we can find another constant $\ell_2$ such that

$$\|B_{k,j}\|_{\mathcal{E}^{\{M_p\},\ell_2 h^2}(B(0,1/2))} \leq C_h e^{-M\left(\frac{j}{h}\right)}, \quad \forall j, k.$$

Suppose $p! \leq C_{h_1} h_1^p M_p$. By $(M.1)$, Lemma 3.3.7, and the Leibniz formula,

$$\|\partial^{\alpha}\varphi\|_{L^{\infty}(B(0,1/2))} \leq C C_{h_1} C_h (L h_1 + \ell_2 h^2)^{|\alpha|} M_{|\alpha|} \sum_{j=0}^{\infty} d_j e^{-M\left(\frac{j}{h}\right)}$$

which completes the proof of Theorem 3.3.2 because $\log t = o(M(t))$ and $d_j = O(j^{n-2})$. $\square$

We end this section with two remarks. Remark 3.3.9 poses an open question.

*Remark* 3.3.8. The technique from this section leads to a new proof of Proposition 3.3.1 as well.

*Remark* 3.3.9. Whether Theorem 3.3.2 and Lemma 3.3.6 hold true or false with $\dagger = *$ is an open question. Notice that this holds when $* = \{p!\}$ (Corollary 3.3.3).

## 3.4 Rotation invariant ultradistributions

We now turn our attention to the characterization of rotation invariant ultradistributions via spherical means.

We begin with the case of ultradistributions from $\mathcal{E}^{*\prime}(\mathbb{R}^n)$. We say that $f \in \mathcal{E}^{*\prime}(\mathbb{R}^n)$ is rotation invariant if $f(x) = f(Tx)$ for all $T \in SO(n)$, the special orthogonal group, namely, if for every rotation $T$ and every $\varphi \in \mathcal{E}^*(\mathbb{R}^n)$

$$\langle f(x), \varphi(x) \rangle = \langle f(x), \varphi(T^{-1}x) \rangle.$$

Note that the mapping $\varphi \to \varphi_S$, where $\varphi_S$ is its spherical mean, is continuous from $\mathcal{E}^*(\mathbb{R}^n)$ into itself. This can easily be viewed from the alternative expression [28]

$$\varphi_S(x) = \int_{SO(n)} \varphi(Tx)dT,$$

where $dT$ stands for the normalized Haar measure of $SO(n)$. The spherical mean of $f \in \mathcal{E}^{*\prime}(\mathbb{R}^n)$ is the ultradistribution $f_S \in \mathcal{E}^{*\prime}(\mathbb{R}^n)$ defined by

$$\langle f_S, \varphi \rangle = \langle f, \varphi_S \rangle.$$

Clearly $f_S$ is rotation invariant. All these definitions also apply to $f \in \mathcal{D}^{*\prime}(\mathbb{R}^n)$ if $M_p$ is non-quasianalytic.

**Theorem 3.4.1.** *Suppose $M_p$ satisfies $(M.0)$, $(M.1)$, and $(M.2)'$. Then, $f \in \mathcal{E}^{*\prime}(\mathbb{R}^n)$ is rotation invariant if and only if $f = f_S$.*

*Proof.* We only need to show that if $f$ is rotation invariant then $f = f_S$. Furthermore, the general case actually follows from that of analytic functionals. In fact, suppose the theorem is true for $* = \{p!\}$. Since $\mathcal{A}(\mathbb{R}^n)$ is densely injected into $\mathcal{E}^*(\mathbb{R}^n)$, we have that $f \in \mathcal{E}^{*\prime}(\mathbb{R}^n)$ is rotation invariant if and only if it is rotation invariant when seen as an analytic functional. Furthermore, taking spherical mean commutes with the embedding $\mathcal{E}^{*\prime}(\mathbb{R}^n) \to \mathcal{A}'(\mathbb{R}^n)$, whence our claim follows.

Suppose that $f \in \mathcal{A}'(\mathbb{R})$ is rotation invariant. Applying Corollary 3.3.3 we can find a spherical representation $g \in \mathcal{A}'(\mathbb{R} \times \mathbb{S}^{n-1})$ for $f$. Using Proposition 3.2.1 we can expand $g$ as

$$g(r, \omega) = \sum_{j=0}^{\infty} \sum_{k=1}^{d_j} c_{k,j}(r) \otimes Y_{k,j}(\omega) \tag{3.4.1}$$

with convergence in $\mathcal{A}'(\mathbb{R}\times\mathbb{S}^{n-1})$ where $c_{k,j}$ are one-dimensional analytic functionals. Notice that if we also expand the polar coordinate expression of $\varphi \in \mathcal{E}^*(\mathbb{R}^n)$ as $\varphi(r\omega) = \sum_{j=0}^{\infty} \sum_{k=1}^{d_j} a_{k,j}(r) Y_{k,j}(\omega)$, we obtain that $\varphi_S(r\omega) = |\mathbb{S}^{n-1}|^{-1/2} a_{0,0}(r) = a_{0,0}(r) Y_{0,0}(\omega)$. The latter holds because $\int_{\mathbb{S}^{n-1}} Y_{k,j}(\omega) d\omega = 0$ for $j \geq 1$, which follows from the mean value theorem for harmonic functions. Thus, $c_{0,0} \otimes Y_{0,0}$ is a spherical representation for $f_S$. The result would then follow if we show that $c_{0,0} \otimes Y_{0,0}$ is also a spherical representation of $f$. By Lemmas 3.3.4-3.3.6 and the expansion (3.4.1), this would certainly be the case if we show that

$$\langle f(x), |x|^{2m} Q(x) \rangle = 0 \tag{3.4.2}$$

for every $m \in \mathbb{N}$ and every harmonic homogeneous polynomial $Q$ of degree $j \geq 1$. Since every such $Q$ can be written [3, Prop. 5.31] as

$$Q(x) = \int_{\mathbb{S}^{n-1}} Q(\omega) Z_j(x, \omega) d\omega,$$

where $Z_j(x, \omega)$ is the zonal spherical harmonic of degree $j$, we have that

$$\langle f(x), |x|^{2m} Q(x) \rangle = \int_{\mathbb{S}^{n-1}} Q(\omega) P_j(\omega) d\omega$$

with

$$P_j(\omega) := \langle f(x), |x|^{2m} Z_j(x, \omega) \rangle, \quad \omega \in \mathbb{S}^{n-1}.$$

So (3.4.2) would hold if we show that $P_j$ identically vanishes on $\mathbb{S}^{n-1}$ if $j \geq 1$. Observe that $P_j$ is a spherical harmonic of degree $j \geq 1$. On the other hand, $Z_j(T^{-1}x, \omega) = Z_j(x, T\omega)$ for every rotation $T$ [3, Prop. 5.27], and using the fact that $f$ is rotation invariant, we obtain $P_j(T\omega) = P_j(\omega)$ for all $\omega \in \mathbb{S}^{n-1}$ and $T \in SO(n)$. Due to the fact that the group $SO(n)$ acts transitively on $\mathbb{S}^{n-1}$, $P_j$ must be a constant function, and hence a spherical harmonic of degree 0. Since the spaces of spherical harmonics of different degrees are mutually orthogonal in $L^2(\mathbb{S}^{n-1})$, one concludes that $P_j \equiv 0$ if $j \neq 0$. This concludes the proof of the theorem.

□

In the non-quasianalytic case, we can use Theorem 3.4.1 to recover the result [13, Thm. 4.4] by Chung and Na quoted at the Introduction.

**Theorem 3.4.2.** *Suppose $M_p$ satisfies $(M.1)$, $(M.2)'$, and $(M.3)'$. An ultradistribution $f \in \mathcal{D}^{*\prime}(\mathbb{R}^n)$ is rotation invariant if and only if $f = f_S$.*

*Proof.* Using a partition of the unity, we can write any rotation invariant $f$ as a locally finite sum $\sum_{k=1}^{\infty} f_k$ with each $f_k \in \mathcal{E}^{*\prime}(\mathbb{R}^n)$ being also rotation invariant. By Theorem 3.4.1 we have $f_k = (f_k)_S$, and, consequently, $f_S = \sum_{k=1}^{\infty} (f_k)_S = \sum_{k=1}^{\infty} f_k = f$. □

We now discuss how one can extend Theorem 3.4.1 in the quasianalytic case of $\{M_p\}$ (including the hyperfunction case). From now on we assume that $M_p$ satisfies $(M.0)$, $(M.1)$, $(M.2)'$, and $(QA)$. Our next considerations are in terms of sheaves of quasianalytic ultradistributions[1], we briefly discuss their properties following the approach from [18, 30] (cf. [55] for hyperfunctions).

Let $f \in \mathcal{E}^{\{M_p\}'}(\mathbb{R}^n)$ (referred to as a $\{M_p\}$-quasianalytic functional hereafter). A compact $K \subseteq \mathbb{R}^n$ is called a $\{M_p\}$-*carrier* of $f$ if $f \in \mathcal{E}^{\{M_p\}'}(\Omega)$ for every open neighborhood $\Omega$ of $K$. If $f \in \mathcal{A}'(\mathbb{R}^n)$, it is well-known [29, Sect. 9.1] that there is a smallest compact $K \subseteq \mathbb{R}^n$ among all the $\{p!\}$-carriers of $f$, the $\{p!\}$-support of $f$ denoted by $\operatorname{supp}_{\mathcal{A}'} f$. It

[1] Also called sheaves of hyperfunctions

was noticed by Hörmander that a similar result basically holds for quasi-analytic functionals [30, Cor. 3.5], that is, for any $\{M_p\}$-quasianalytic functional there is a smallest $\{M_p\}$-carrier, say $\operatorname{supp}_{\mathcal{E}^{\{M_p\}'}} f$, and one has $\operatorname{supp}_{\mathcal{A}'} f = \operatorname{supp}_{\mathcal{E}^{\{M_p\}'}} f$. Hörmander only treats the Roumieu case in [30], but his proof can be modified to show the corresponding statement for the Beurling case [18, 27].

Denote as $\mathcal{E}^{\{M_p\}'}[K]$ the space of $\{M_p\}$-quasianalytic functionals with support in $K$. One can show that there is an (up to isomorphism) unique flabby sheaf $\mathfrak{B}^{\{M_p\}}$ whose space of global sections with support in $K$ is precise $\mathcal{E}^{\{M_p\}'}[K]$, for any compact $K$ of $\mathbb{R}^n$. We call $\mathfrak{B}^{\{M_p\}}$ the sheaf of $\{M_p\}$-quasianalytic ultradistributions. When $\{M_p\} = \{p!\}$, we simply write $\mathfrak{B} = \mathfrak{B}^{\{M_p\}}$, the sheaf of hyperfunctions. The existence of $\mathfrak{B}^{\{M_p\}}$ can be established exactly as for hyperfunctions with the aid of Hörmander support theorem by using the Martineau-Schapira method [55]; see e.g. [18]. Since it is important for us, we mention that on any bounded open set $\Omega$ the sections of $\mathfrak{B}^{\{M_p\}}$ are given by the quotient spaces

$$\mathfrak{B}^{\{M_p\}}(\Omega) = \mathcal{E}^{\{M_p\}'}[\overline{\Omega}]/\mathcal{E}^{\{M_p\}'}[\partial\Omega], \tag{3.4.3}$$

which reduces to the well-known Martineau theorem in the case of hyperfunctions. Finally, we call the space of global sections $\mathfrak{B}^{\{M_p\}}(\mathbb{R}^n)$ the space of $\{M_p\}$-quasianalytic ultradistributions on $\mathbb{R}^n$ (hyperfunctions if $\{M_p\} = \{p!\}$).

The operation of taking spherical mean preserves the space $\mathcal{E}^{\{M_p\}'}[K]$ if $K$ is a rotation invariant compact set. Because of (3.4.3), we can define the spherical mean $f_S \in \mathfrak{B}^{\{M_p\}}(\Omega)$ of $f \in \mathfrak{B}^{\{M_p\}}(\Omega)$ in a canonical manner if $\Omega$ is a bounded rotation invariant open subset of $\mathbb{R}^n$, namely, if $f = [g]$ with $g = \mathcal{E}^{\{M_p\}'}[\overline{\Omega}]$, we define $f_S = [g_S]$. Using the sheaf property, one extends the definition $f_S \in \mathfrak{B}^{\{M_p\}}(\mathbb{R}^n)$ for all $f \in \mathfrak{B}^{\{M_p\}}(\mathbb{R}^n)$. We say that $f \in \mathfrak{B}^{\{M_p\}}(\mathbb{R}^n)$ is rotation invariant if its restriction to $\Omega$ is rotation invariant for any rotation invariant bounded open set $\Omega$ (the latter makes sense because of (3.4.3)). Theorem 3.4.1 implies the following generalization:

**Theorem 3.4.3.** *Suppose $M_p$ satisfies $(M.0)$, $(M.1)$, $(M.2)'$, and $(QA)$. A quasianalytic ultradistribution $f \in \mathfrak{B}^{\{M_p\}}(\mathbb{R}^n)$ is rotation invariant if and only if $f = f_S$.*

We point out that Theorem 3.4.3 extends [13, Thm. 5.7], which was obtained for hyperfunctions.

# Chapter 4

# Eigenfunction expansions of ultradifferentiable functions and ultradistributions in $\mathbb{R}^n$

## 4.1 Introduction

In this chapter, we will obtain a characterization of $\mathcal{S}_{\{M_p\}}^{\{M_p\}}(\mathbb{R}^n)$ and $\mathcal{S}_{(M_p)}^{(M_p)}(\mathbb{R}^n)$, the general Gelfand-Shilov spaces of ultradifferentiable functions of Roumieu and Beurling type, in terms of decay estimates for the Fourier coefficients of their elements with respect to eigenfunction expansions associated to normal globally elliptic differential operators of Shubin type.

Moreover, we show that the eigenfunctions of such operators are absolute Schauder bases for these spaces of ultradifferentiable functions. This characterization extends earlier results by Gramchev et all [26] for Gevrey weight sequences. It also generalizes to $\mathbb{R}^n$ results by Dasgupta and Ruzhansky [17] which were obtained in the setting of compact manifolds.

Our characterization is as follows. Note that if $P$ is globally elliptic and normal ($PP^* = P^*P$), then there is an orthonormal basis of

$L^2(\mathbb{R}^n)$ consisting of eigenfunctions of $P$.

Let us mention that the properties of the Shubin type operators are very well explained in the textbooks [41, 58].

Our assumptions on the weight sequence are the standard $(M.1)$ and $(M.2)'$ Komatsu's conditions (logarithmic convexity and stability under differential operators [33]), together with the essential assumption (1.3.3)

**Theorem 4.1.1.** *Let $P$ be a normal globally elliptic differential operator of Shubin type* (0.0.1) *and let $\{u_j : j \in \mathbb{N}\}$ be an orthonormal basis of $L^2(\mathbb{R}^n)$ consisting of eigenfunctions of $P$. Let $f \in L^2(\mathbb{R}^n)$ have eigenfunction expansion*

$$f = \sum_{j=1}^{\infty} a_j u_j.$$

*Suppose that the weight sequence $M_p$ satisfies $(M.1)$, $(M.2)'$, and* (1.3.3). *Then,*

*(i) $f \in \mathcal{S}^{\{M_p\}}(\mathbb{R}^n)$ if and only if there are $\lambda > 0$ and $C_\lambda > 0$ such that*

$$|a_j| \leq C_\lambda e^{-M(\lambda j^{\frac{1}{2n}})}, \quad j \in \mathbb{N}. \tag{4.1.1}$$

*(ii) $f \in \mathcal{S}^{(M_p)}(\mathbb{R}^n)$ if and only if the estimate* (4.1.1) *holds for each $\lambda > 0$.*

Consequently, the global $M_p$ regularity and decay of a function $f$ are completely determined by the decay of its coefficients $a_j$. Since for Gevrey sequences $M_p = (p!)^\mu$ the associated function $M(t) \asymp |t|^{1/\mu}$ [24], our result includes as particular instances those from [26]. In the special case of the harmonic oscillator

$$-\Delta + |x|^2,$$

the eigenfunctions are given by the Hermite functions; Theorem 4.1.1 thus also recovers well-known results for Hermite expansions [10, 37, 67] (see also [31]).

It is important to point out that Theorem 4.1.1 does not reveal all topological information involved in the problem. In fact, in Section 4.3 we prove a much stronger result, namely, we shall show that the eigenfunctions $u_j$ are absolute Schauder bases for $\mathcal{S}^*(\mathbb{R}^n)$, where $* = \{M_p\}$ or $(M_p)$, and that these spaces become (tamely) isomorphic as topological vector spaces to sequence spaces canonically defined by the estimates (4.1.1). This will easily yield an eigenfunction expansion characterization of the ultradistribution spaces $\mathcal{S}^{*\prime}(\mathbb{R}^n)$. In Section 4.2 we characterize $\mathcal{S}^*(\mathbb{R}^n)$ via iterates of $P$; the characterization leads to the ensuing regularity result for solutions to the equation $Pu = f$.

**Theorem 4.1.2.** *Let $P$ be a globally elliptic operator of Shubin type* (0.0.1) *and let $M_p$ satisfy* $(M.1)$, $(M.2)'$, *and* (1.3.3). *If $u \in \mathcal{S}^{*\prime}(\mathbb{R}^n)$ is a solution to $Pu = f$ and $f \in \mathcal{S}^*(\mathbb{R}^n)$, then also $u \in \mathcal{S}^*(\mathbb{R}^n)$.*

We now derive a simple but very useful relation for sequences fulfilling $(M.1)$ and (1.3.3). This relation plays a crucial role in Section 4.2. Observe also that (4.1.2) obviously implies (1.3.3).

**Lemma 4.1.3.** *The conditions* $(M.1)$ *and* (1.3.3) *imply that*

$$\begin{aligned} &\sqrt{p+1}\,\frac{M_p}{M_{p+1}} \leq r, \quad p \in \mathbb{N}_0, \quad \textit{for some } r > 0 && (\textit{Roumieu case}),\\ &\lim_{p\to\infty} \sqrt{p+1}\,\frac{M_p}{M_{p+1}} = 0. && (\textit{Beurling case}) \end{aligned} \tag{4.1.2}$$

*Proof.* Stirling's formula yields $\sqrt{p+1} \leq C(\sqrt{p!})^{1/p}$ (for some constant $C$). Using $(M.1)$, we conclude that $(M_p/M_{p+1}) \leq M_p^{-1/p}$. Thus, (1.3.3) yields $\sqrt{p+1}M_p/M_{p+1} \leq CC_l^{1/p}l$. □

For the next section, the reader is advised to review the definition of spaces of Gelfand-Shilov type, $\mathcal{S}^{\{M_p\}}(\mathbb{R}^n)$ and $\mathcal{S}^{(M_p)}(\mathbb{R}^n)$ (Subsection 1.3.2) with the corresponding tame structure.

## 4.2 Iterates of the operator and regularity of solutions

In this section we exploit the iterative approach from [14, 26, 57] in order to obtain a structural characterization of $\mathcal{S}^*(\mathbb{R}^n)$ in terms of the growth of the $L^2$ norms of the iterates of the operator $P$. The regularity result Theorem 4.1.2 will readily follow from Theorem 4.2.5 below. We point out that these ideas go back to seminal works by Komatsu [32, 35] and Kotaké and Narasimhan [36].

We begin by introducing function spaces associated to the iterates of $P$. At this point, we do not need any ellipticity assumption on $P$. For $h > 0$, define the Banach space $\mathcal{S}_P^{\{M_p\},h}$ of all functions $f$ such $P^p f \in L^2(\mathbb{R}^n)$ for all $p \in \mathbb{N}_0$ and

$$\|f\|_{P,h} := \sup_{p\in\mathbb{N}_0} \frac{\|P^p f\|_{L^2(\mathbb{R}^n)}}{h^{mp} M_{mp}} < \infty; \tag{4.2.1}$$

set further,

$$\mathcal{S}_P^{\{M_p\}}(\mathbb{R}^n) = \varinjlim_{h\to\infty} \mathcal{S}_P^{\{M_p\},h} \quad \text{and} \quad \mathcal{S}_P^{(M_p)}(\mathbb{R}^n) = \varprojlim_{h\to 0^+} \mathcal{S}_P^{\{M_p\},h}.$$

We regard $\mathcal{S}_P^*(\mathbb{R}^n)$ as spaces graded by the norms (4.2.1). See Subsection 1.3.2 for the definition of tame continuity.

**Proposition 4.2.1.** *Suppose $M_p$ satisfies* (4.1.2). *Then, $\mathcal{S}^*(\mathbb{R}^n) \subseteq \mathcal{S}_P^*(\mathbb{R}^n)$ and the inclusion mapping $\mathcal{S}^*(\mathbb{R}^n) \to \mathcal{S}_P^*(\mathbb{R}^n)$ is tamely continuous.*

*Proof.* Fix $f \in \mathcal{S}_{L^2}^{\{M_p\},h}$ with $\|f\|_h = 1$. We are able to effectively calculate $P^p$. By employing the Leibniz formula, it is easy to see that

$$P^p u = \sum_{(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\tau})\in\mathcal{C}_p} q_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\tau}}(P) Q_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\tau}} L_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\tau}}(f), \tag{4.2.2}$$

where the summation extends over the set $\mathcal{C}_p$ of all $(3p-1)$-tuples of multi-indices $(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\tau}) = (\alpha_1,\dots,\alpha_p,\beta_1,\dots,\beta_p,\tau_1,\dots,\tau_{p-1})$ such that

$|\alpha_j|+|\beta_j| \leq m$ for each $j$, $\tau_{j-1} \leq \alpha_j$ for $j = 2, \ldots, p$, and $\tau_1 + \cdots + \tau_j \leq \beta_1 + \cdots + \beta_j$ for $j = 1, 2, \ldots, p-1$, and where

$$q_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\tau}}(P) := c_{\alpha_1,\beta_1} \prod_{j=2}^{p} c_{\alpha_j,\beta_j} \binom{\alpha_j}{\tau_{j-1}},$$

$$Q_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\tau}} := \prod_{j=1}^{p-1} \frac{(\beta_1 + \cdots + \beta_j - \tau_1 - \cdots - \tau_{j-1})!}{(\beta_1 + \cdots + \beta_j - \tau_1 - \cdots - \tau_{j-1} - \tau_j)!} \quad (\tau_0 := 0),$$

and the differential operator $L_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\tau}}$ is given by

$$L_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\tau}} := x^{\beta_1+\cdots+\beta_p-\tau_1-\cdots-\tau_{p-1}} D^{\alpha_1+\cdots+\alpha_p-\tau_1-\tau_2-\cdots-\tau_{p-1}}.$$

Set $C_P = \max_{|\alpha|+|\beta|\leq m}\{|c_{\alpha,\beta}|\}$. First note that $|q_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\tau}}(P)| \leq 2^{-m}(C_P 2^m)^p$, because of the well known estimate for binomial coefficients. We need an estimate on the number of elements of the set $\mathcal{C}_p$. The rough bound $|\mathcal{C}_p| \leq m^{-n}(2^{m+2n}m^n)^p$ suffices for our purposes. Indeed, for a fixed $j$, the number of multi-indices such that $|\alpha_j|+|\beta_j| \leq m$ is $\sum_{\nu=0}^{m} \binom{\nu+2n-1}{\nu} \leq 2^{m+2n}$ and number of $\tau_j$ is less than $m^n$. We conclude then

$$\|P^p f\|_{L^2(\mathbb{R}^n)} \leq m^{-n}2^{-m}(4^{1+n/m}m^{n/m}C_P^{1/m}h)^{mp}M_{mp} \max_{(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\tau})\in\mathcal{C}_p} Q'_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\tau}}(h),$$

where

$$Q'_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\tau}}(h) = \frac{h^{|\boldsymbol{\alpha}|+|\boldsymbol{\beta}|-2|\boldsymbol{\tau}|}M_{|\boldsymbol{\alpha}|+|\boldsymbol{\beta}|-2|\boldsymbol{\tau}|}}{h^{pm}M_{mp}} \times \prod_{j=1}^{p-1} \frac{(|\beta_1| + \cdots + |\beta_j| - |\tau_1| - \cdots - |\tau_{j-1}|)!}{(|\beta_1| + \cdots + |\beta_j| - |\tau_1| - \cdots - -|\tau_j|)!}.$$

We now estimate each of these terms. In order to treat both the Roumieu and Beurling case simultaneously, we rewrite the assumption (4.1.2) as $M_k/M_{k+1} \leq r_k/\sqrt{k+1}$, where in the Roumieu case $r_k = r$ and in the Beurling case $r_k$ is a non-increasing positive sequence tending to 0. We obtain

$$h^{|\boldsymbol{\alpha}|+|\boldsymbol{\beta}|-2|\boldsymbol{\tau}|-pm}\frac{M_{|\boldsymbol{\alpha}|+|\boldsymbol{\beta}|-2|\boldsymbol{\tau}|}}{M_{mp}} \leq \frac{M_{mp-2|\boldsymbol{\tau}|}}{h^{2|\boldsymbol{\tau}|}M_{mp}} \prod_{k=|\boldsymbol{\alpha}|+|\boldsymbol{\beta}|-2|\boldsymbol{\tau}|}^{mp-2|\boldsymbol{\tau}|-1} \frac{r_k}{h}$$
$$\leq \left(\prod_{k=|\boldsymbol{\alpha}|+|\boldsymbol{\beta}|-2|\boldsymbol{\tau}|}^{mp-1} \frac{r_k}{h}\right)\left(\prod_{\nu=mp-2|\boldsymbol{\tau}|}^{mp-1} \frac{1}{\sqrt{\nu+1}}\right)$$
$$\leq \left(\prod_{k=1}^{mp-|\boldsymbol{\alpha}|-|\boldsymbol{\beta}|+2|\boldsymbol{\tau}|} \frac{r_k}{h}\right)\left(\prod_{\nu=mp-2|\boldsymbol{\tau}|}^{mp-1} \frac{1}{\sqrt{\nu+1}}\right).$$

In the Beurling case we have that the sequence $\prod_{k=1}^{j}(r_k/h)$ is bounded by some $C'_h$ because $r_k \to 0$. In the Roumieu case this sequence is bounded by $C'_h = 1$ if we ask $h \geq r$ (we impose this condition in the Roumieu case in the rest of the proof). Further on, clearly

$$\frac{(|\beta_1|+\cdots+|\beta_j|-|\tau_1|-\cdots-|\tau_{j-1}|)!}{(|\beta_1|+\cdots+|\beta_j|-|\tau_1|-\cdots--|\tau_j|)!} \leq \frac{(mj)!}{(mj-|\tau_j|)!}.$$

Making use of $\tau_j \leq \alpha_{j+1}$ and $\sum_{k=1}^{j}\tau_i \leq \sum_{k=1}^{j}\beta_k$,

$$mp-2|\tau_{j+1}|-2|\tau_{j+2}|-\cdots-2|\tau_{p-1}| \geq jm,$$

and hence ($\tau_p := 0$)

$$\prod_{\nu=mp-2|\boldsymbol{\tau}|}^{mp-1}\frac{1}{\sqrt{\nu+1}} = \prod_{j=1}^{p-1}\prod_{\nu=mp-2|\tau_j|-\cdots-2|\tau_{p-1}|}^{mp-2|\tau_{j+1}|-\cdots-2|\tau_{p-1}|-1}\frac{1}{\sqrt{\nu+1}}$$
$$\leq \prod_{j=1}^{p-1}\sqrt{\frac{(mj-2|\tau_j|)!}{(mj)!}}.$$

Combining these two inequalities we will obtain, for $j > 2$

$$\frac{jm(jm-1)\cdots(jm-|\tau_j|+1)}{\sqrt{jm(jm-1)\dots(jm-2|\tau_j|+1)}} \leq \left(\frac{jm}{(j-2)m}\right)^m \leq 3^m,$$

if $j = 1$ this quantity does not exceed $m!$ while if $j = 2$ we have

$$\frac{2m(2m-1)\cdots(2m-|\tau_2|+1)}{\sqrt{2m(2m-1)\dots(2m-2|\tau_2|+1)}} \leq \sqrt{(2m)^m}$$

and therefore $Q'_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\tau}}(h) \leq 3^{mp}(2m^{3/2})^m C'_h/27$. Summarizing, in the Beurling case we have shown that $\|\cdot\|_{P,Lh} \leq C_h\|\cdot\|_h$ for all $h > 0$ where $L = 4^{1+n/m}3m^{n/m}C_P^{1/m}$, while in the Roumieu case such inequality is valid for all $h \geq r$. This establishes the claimed inclusion and its tame continuity. □

Our next goal is to show that actually $\mathcal{S}^*(\mathbb{R}^n) = \mathcal{S}^*_P(\mathbb{R}^n)$ whenever $P$ is globally elliptic. Recall [41, 58] that global ellipticity means that the principal symbol

$$\sum_{|\alpha|+|\beta|=m} c_{\alpha\beta}x^\beta\xi^\alpha \neq 0 \quad \text{for all} \quad (x,\xi) \neq (0,0). \tag{4.2.3}$$

Our starting point is the same as in [26], i.e., the interpolating inequality [26, Prop. 4.1]

$$|f|_{pm+j} \leq |f|_{pm} + C|f|_{(p+1)m} + C^{pm+j}((pm+j)!)^{1/2}\|f\|_{L^2(\mathbb{R}^d)}\,, \tag{4.2.4}$$

where $0 < j < m$ and $1 \leq C$, for the Sobolev type seminorms

$$|f|_s := \sum_{|\alpha|+|\beta|=s} \|x^\beta\partial^\alpha f\|_{L^2(\mathbb{R}^n)}.$$

We will prove a more general inequality. If $k = pm + r$ where $p \geq 1$ is an integer, $0 < r < m$, then for any $\varepsilon > 0$

$$|u|_k \leq \varepsilon|u|_{(p+1)m} + C^p\varepsilon^{-r/(m-r)}|u|_{pm} + C^k\sqrt{k!}\|u\|_{L^2(\mathbb{R}^d)}. \tag{4.2.5}$$

In order to prove it, we follow [6, Prop. 2.1]. Define

$$|u|^*_k = \||x|^k u\|_{L^2(\mathbb{R}^n)}, \quad |u|^{**}_k = \||D|^k u\|_{L^2(\mathbb{R}^n)}.$$

Where $|D|$ is the pseudodifferential operator (Fourier multiplier) with symbol $|\xi|$. We will utilize this simple inequality

$$\lambda^j \leq \varepsilon\lambda^p + \varepsilon^{-(j-R)/(p-j)}\lambda^R \tag{4.2.6}$$

for every $\varepsilon > 0, \lambda > 0$ and $0 \leq R < j < p$. The proof follows from the Jensen's inequality.

**Lemma 4.2.2** ([6, Lemma 2.2]). *There exists $C > 0$ such that for any integer $k \geq 0$*

$$|u|_k \leq C^k(|u|_k^* + |u|_k^{**} + \sqrt{k!}\|u\|_{L^2(\mathbb{R}^n)})$$

*Proof.* In the proof we utilize the so-called anti-Wick calculus from [41, Section 1.7]. Let $q_{\alpha\beta}(x, \xi)$ will be an anti-Wick symbol of the operator $D^\beta x^{2\alpha} D^\beta$, $|\alpha + \beta| = k$. Combining [41, Prop. 1.2.5] and [41, Prop. 1.8.2]), we conclude (after some calculation) that $q_{\alpha\beta}(x, \xi)$ is real-valued and

$$q_{\alpha\beta} = x^{2\alpha}\xi^{2\beta} + \sum_{\gamma<2\alpha,\delta<2\beta} A^{\alpha\beta}_{\gamma\delta} x^\gamma \xi^\delta$$

with the estimate $|A^{\alpha\beta}_{\gamma\delta}| \leq C_1^k H_1^{k-\frac{j}{2}} k^{k-\frac{j}{2}}$, $2k = |\alpha+\beta|, j = |\gamma+\delta|$, $C_1, H_1$ are positive constants.

Denote as $q_k(x)$ the anti-Wick symbol of the multiplication operator $|x|^{2k}$; then $q_k(\xi)$ will be the anti-Wick symbol of the operator $(-\Delta)^k$. The polynomial $q_k(x)$ is also real-valued and one can calculate that the same estimate for its coefficients holds, namely

$$q_k(x) = |x|^{2k} + \sum_{|\gamma|<2k} B^k_\gamma x^\gamma$$

where $B_\gamma \leq C_2^k H_2^{k-\frac{j}{2}} k^{k-\frac{k}{2}}, j = |\gamma|$. Here we are (trying to be as short as possible) performing necessary calculations anticipated before. For an operator $b(x, D) = (-\Delta)^k$ it is clear that it coincides with its Weyl symbol, $b^w = b(x, D)$ [41, Sect. 1.8]. Now we are able to perform [41, Thm. 1.8.2] and solve the equation $A_{q_k(x)} = b^w(x, D)$. Using the polynomial formula for expanding $|x|^{2k}$ as well as the powers of Laplacian, we obtain that $q_k(x) = |x|^{2k} + r_k(x)$, where

$$|r_k(x)| \leq \sum_{l=1}^{k} \sum_{l_1+l_2+\cdots+l_n=l} \frac{l!}{l_1! \cdots l_n!} \times \sum_{k_1+\cdots+k_n=k} \frac{k!}{k_1! \cdots k_n!} \frac{(\partial^{2l_1}_{x_1} x_1^{2k_1}) \cdots (\partial^{2l_n}_{x_n} x_n^{2k_n})}{4^l l!}.$$

We remark that the summation is finite. Also, we are summing over tuples $(l_1,\dots,l_n)$ for which $l_i \le k_i$, $i=1,\dots,n$. Using a standard combinatorial inequalities,

$$(2k_i)(2k_i-1)\cdots(2k_i-2l_i+1) = \binom{2k_i}{2l_i}\binom{2l_i}{l_i}(l_i!)^2 \le 4^{k_i+l_i}(l_i!)^2.$$

Therefore,

$$\begin{aligned}|r_k(x)| \le \sum_{l=1}^{k} \sum_{l_1+l_2+\cdots+l_n=l} \frac{l!}{l_1!\cdots l_n!}\times \\ \sum_{k_1+\cdots+k_n=k} \frac{k!}{k_1!\cdots k_n!}\frac{(\partial_{x_1}^{2l_1}x_1^{2k_1})\cdots(\partial_{x_n}^{2l_n}x_n^{2k_n})}{4^l l!}.\end{aligned}$$

The coefficients $B_\gamma$ in the expansion $q_k(x)=\sum_{0<|\gamma|<2k} B_\gamma x^\gamma$ are easy to estimate. First note that all $\gamma_i, i=1,2,\dots n$ are even. Then, for a fixed $\gamma$ (and, of course, $k$),the number of $(k_1,\dots,k_n)$ and $(l_1,\dots l_n)$ such that $2k-2l=|\gamma|$ is $\binom{2k-\gamma+n-1}{n-1} \le 4^{k-\frac{|\gamma|}{2}}2^{n-1}$. Also, $\prod_{i=1}^n l_i! \le (k-\frac{|\gamma|}{2})!$. Therefore,

$$|B_\gamma| \le 2^{n-1}(4n)^k 4^{k-\frac{|\gamma|}{2}} \le C_2 H_2^{k-\frac{|\gamma|}{2}} k^{k-\frac{|\gamma|}{2}}$$

as stated.

Consider now the operator with the anti-Wick symbol

$$a(x,\xi) = C^k(q_k(x)+q_k(\xi)+k!) - q_{\alpha\beta}(x,\xi).$$

Now we shall prove that $a(x,\xi)>0$ for sufficiently large $C$. Assuming it to be true for a moment, positivity of the operator $A$ with the anti-Wick symbol $a(x,\xi)$ follows from [41, Prop. 1.7.6] and then

$$\begin{aligned}&(Au,u) = C^k(|x|^{2k}u,u) + ((-\Delta)^k u,u) + k!(u,u) - (D^\beta x^{2\alpha} D^\beta u,u)\\ &= C^k((|u|_k^*)^2 + (|u|_k^{**})^2 + k!\|u\|^2_{L^2(\mathbb{R}^n)}) - \|x^\alpha D^\beta u\|^2_{L^2(\mathbb{R}^n)} \ge 0\end{aligned}$$

which leads to the conclusion.

It remains (just) to prove that $a(x,\xi)\ge 0$. First observe that $x^{2\alpha}\xi^{2\beta} < C^k(|x|^{2k}+|\xi|^{2k}), k=|\alpha+\beta|$ for a sufficiently large constant $C$. And we saw that the lower terms in the expression of polynomials

$q_k(x), q_k(\xi)$ and $q_{\alpha\beta}(x,\xi)$ are of the form $E_{\gamma\delta}x^\gamma\xi^\delta$ where $j = |\gamma+\delta| < 2k$ and $E_{\gamma\delta} \le C_3^k H_3^{k-\frac{j}{2}} k^{k-\frac{j}{2}}$ and therefore

$$|E_{\gamma\delta}x^\gamma\xi^\delta| \le C_3^k H_3^{k-\frac{j}{2}}(|x|^j + |\xi|^j).$$

However, from the inequality (4.2.6) we obtain $C_3^k H_3^{k-\frac{j}{2}} k^{k-\frac{j}{2}}|x|^j \le C_3^k(|x|^{2k} + H_3^k k^k)$ (take $\lambda = |x|, \varepsilon = H_3^{k-\frac{j}{2}} k^{\frac{j}{2}-k}, p = 2k, R = 0$) so that all lower terms can be estimated by $C_4^k(((|x|^{2k}+|\xi|^{2k})+ k!)$ for a suitable $C_4$. This yields the conclusion $a(x,\xi) \ge 0$ and we saw that is enough to prove our lemma. □

From the inequality (4.2.6), it is easily obtained that

$$|u|_k^* \le \varepsilon|u|^*_{(p+1)m} + \varepsilon^{\frac{r}{m-r}}|u|^*_{pm}, \qquad |u|_k^{**} \le \varepsilon|u|^{**}_{(p+1)m} + \varepsilon^{\frac{r}{m-r}}|u|^{**}_{pm}.$$

and then, applying the previous lemma, we obtain (4.2.5).

In the sequel, it will be convenient to consider the family of norms

$$\|f\|'_h = \sup_{p\in\mathbb{N}_0} \frac{|f|_{pm}}{h^{pm}M_{pm}}, \quad h > 0. \tag{4.2.7}$$

**Proposition 4.2.3.** *Under the assumptions* $(M.2)'$ *and* (1.3.3), *the family of norms* (1.3.4) *and* (4.2.7) *are tamely equivalent* (*both as* $h \to \infty$ *and* $h \to 0^+$).

*Proof.* Clearly, $\|\cdot\|'_h \le 2^{2n-1}\|\cdot\|_{h/2}$ without any assumption on $M_p$. In the Roumieu case, a routine computation with the aid of (4.2.4) shows that $\|\cdot\|_{H^m h} \le C'_h\|\cdot\|'_h$ for all $h \ge Cl$ with $C'_h = C(hAH^{(m-1)/2})^m + C_l + \max\{1,(r/h)^m\}$, where these are the constants occurring in $(M.2)'$, (1.3.3), (4.1.2), and (4.2.4). In the Beurling case we obtain $\|\cdot\|_{H^m h} \le C'_h\|\cdot\|'_h$ for all $h \le 1$ with $C'_h = C(AH^{(m-1)/2})^m + C_{h/C} + \max\{1,(r/h)^m\}$ where again $r$ is an upper bound for $\sqrt{p+1}M_p/M_{p+1}$. □

We need the ensuing adapted version of [26, Prop. 4.2]. Set

$$\sigma_p(f,h) = \frac{|f|_{mp}}{h^{mp}M_{mp}}, \quad p \in \mathbb{N}_0,$$

so that $\sigma_0(f,h) = \|f\|_{L^2(\mathbb{R}^2)}$. We also set $\sigma_{-1}(f,h) = 0$.

**Lemma 4.2.4.** *Let $P$ be globally elliptic and suppose that* (4.1.2) *holds. There is a constant $C'$ depending only on the operator and having the following properties:*

*(i) In the Roumieu case there is $h_0 > 0$ (depending only on $P$ and the weight sequence) such that for all $h \geq h_0$*

$$\sigma_{p+1}(f,h) \leq \frac{C' M_{pm}}{h^m M_{(p+1)m}} \sigma_p(Pf,h) + \frac{1}{3}(\sigma_p(f,h)+ \\ + \sigma_{p-1}(f,h) + \sigma_0(f,h)). \tag{4.2.8}$$

*(ii) In the Beurling case there is a positive non-increasing sequence $r_p$ tending to 0, which depends only on $P$ and the weight sequence, such that*

$$\sigma_{p+1}(f,h) \leq \frac{C' M_{pm}}{h^m M_{(p+1)m}} \sigma_p(Pf,h) + \frac{r_p}{3h^m}\sigma_p(f,h)+ \\ + \frac{r_p}{3h^{2m}}\sigma_{p-1}(f,h) + \sigma_0(f,h)\frac{r_1 \cdots r_p}{3h^{m(p+1)}}. \tag{4.2.9}$$

*Proof.* We closely follow the proof of [26, Prop. 4.2] with the required modifications. First notice that $P : Q^m(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is Fredholm, where $Q^m(\mathbb{R}^n)$ denotes the Sobolev type space consisting of functions with $\|u\|_{Q^m(\mathbb{R}^n)} = \sum_{j=0}^m |u|_j < \infty$, and actually $\operatorname{Ker} P$ is a finite dimensional subspace of the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ [41]. We may therefore assume for the sake of simplicity that $\operatorname{Ker} P = \{0\}$. Now, there is then a constant $C_1 > 0$ such that

$$\sum_{|\alpha|+|\beta| \leq m} \|x^\beta D^\alpha f\|_{L^2(\mathbb{R}^n)} = \sum_{s=0}^m |f|_s \leq C_1 \|Pf\|_{L^2(\mathbb{R}^d)}. \tag{4.2.10}$$

We will estimate exactly as in the proof of [26, Prop. 4.2] with the aid of commutators and (4.2.10) in order to obtain

$$|f|_{(1+p)m} \leq C'|Pf|_{pm} + C_2((pm)^{m/2}|f|_{pm}+ \\ (pm)^m |f|_{(p-1)m} + C_3^p((p+1)m)!^{1/2}|f|_0), \tag{4.2.11}$$

where the constants depend only on the operator and we may assume they are $\geq 1$.

First we consider the term $\|x^\beta D_x^\alpha f\|_{L^2(\mathbb{R}^n)}$ when $|\alpha+\beta| = (p+1)m$. Then we may write

$$x^\beta D_x^\alpha f = x^{\beta-\delta} x^\delta D_x^{\alpha-\gamma} D_x^\gamma f,$$

where we choose $\gamma \le \alpha, \delta \le \beta$ such that $|\gamma|+|\delta| = pm$ and $|\alpha-\gamma|+|\beta-\gamma| = m$. Then we use commutators and (4.2.10) in order to estimate

$$\begin{aligned}&\|x^\beta D_x^\alpha f\|_{L^2(\mathbb{R}^n)} \le \|x^{\beta-\delta} D_x^{\alpha-\gamma}(x^\delta D_x^\gamma f)\|_{L^2(\mathbb{R}^n)}\\ &+\|x^{\beta-\delta}[x^\delta, D_x^{\alpha-\gamma}] D_x^\gamma f\|_{L^2(\mathbb{R}^n)} \le C\|P(x^\delta D_x^\gamma f)\|_{L^2(\mathbb{R}^n)}+\\ &\|x^{\beta-\delta}[x^\delta, D_x^{\alpha-\gamma}] D_x^\gamma f\|_{L^2(\mathbb{R}^n)} \le I_1 + I_2 + I_3 \end{aligned} \tag{4.2.12}$$

where

$$I_1 = C\|x^\delta D_x^\gamma (Pf)\|_{L^2(\mathbb{R}^n)}, \qquad I_2 = C\|[P, x^\delta D_x^\gamma] f\|_{L^2(\mathbb{R}^n)}$$

$$I_3 = \|x^{\beta-\delta}[x^\delta, D_x^{\alpha-\gamma}] D_x^\gamma f\|_{L^2(\mathbb{R}^n)}.$$

Of course, if we sum $I_1, I_2, I_3$ over all tuples $(\alpha, \beta)$ such that $|\alpha+\beta| = (p+1)m$, we obtain the estimate for $|u|_{(p+1)m}$:

$$|u|_{(p+1)m} \le J_1 + J_2 + J_3. \tag{4.2.13}$$

We proceed further with the estimate. Let polynomial $P$ be
$P = \sum_{|\alpha|+|\beta|\le m} c_{\alpha\beta} x^\beta D_x^\alpha$. Then we have

$$[P, x^\delta D_x^\gamma] = \sum_{|\tilde\alpha|+|\tilde\beta|\le m} c_{\tilde\alpha\tilde\beta}[x^{\tilde\beta} D_x^{\tilde\alpha}, x^\delta D_x^\gamma].$$

And now it remains to calculate (or estimate) these "corner stone" commutators:

$$\begin{aligned}&[x^{\tilde\beta} D_x^{\tilde\alpha}, x^\delta D_x^\gamma] = \\ &\sum_{0\ne\tau\le\tilde\alpha, \tau\le\delta} C_{1\tilde\alpha\delta\tau} x^{\delta+\tilde\beta-\tau} D_x^{\gamma+\tilde\alpha-\tau} - \sum_{0\ne\tau\le\tilde\beta, \tau\le\gamma} C_{2\tilde\beta\gamma\tau} x^{\delta+\tilde\beta-\tau} D_x^{\gamma+\tilde\alpha-\tau}\end{aligned}$$

Leibniz rule and well-known estimates for binomials are enough to conclude that $C_{1\tilde\alpha\delta\tau}$ and $C_{2\tilde\beta\gamma\tau}$ are less than $C_3(pm)^{|\tau|}$. Then

$$\|[P, x^\delta D_x^\gamma] f\|_{L^2(\mathbb{R}^n)} \le C_4 \sum_{|\tilde\alpha|+|\tilde\beta|\le m} \sum_\tau (pm)^{|\tau|} \|x^{\delta+\tilde\beta-\tau} D_x^{\gamma+\tilde\alpha-\tau} f\|_{L^2(\mathbb{R}^n)}.$$

Here $0 \neq \tau \leq \tilde{\alpha}, \tau \leq \delta$ or $0 \neq \tau \leq \tilde{\beta}, \tau \leq \gamma$. If $s = |\delta+\tilde{\beta}-\tau|+|\gamma+\tilde{\alpha}-\tau| = pm + |\tilde{\alpha}| + |\tilde{\beta}| - 2|\tau|$. It follows easily that $(p-1)m \leq s < (p+1)m$ and $s \leq (p+1)m - 2|\tau|$, therefore $|\tau| \leq \frac{(p+1)m-s}{2}$. Then we may separate $J_2$ as follows

$$J_2 \leq C_5(J_2' + (pm)^{\frac{m}{2}}|u|_{pm} + J_2'')$$

where

$$J_2' = \sum_{pm<s<(p+1)m} (pm)^{\frac{(p+1)m-s}{2}}|f|_s, J_2'' \sum_{(p-1)m\leq s<pm} (pm)^{\frac{(p+1)m-s}{2}}|f|_s.$$

We now estimate $|f|_s$ using (4.2.5). For example, taking $\varepsilon = (pm)^{-\frac{(p+1)m-s}{2}}(4mC_5)^{-1}$ in $J_2'$ we obtain

$$J_2' \leq (4C_5)^{-1}|f|_{(p+1)m} + C_6(pm)^{\frac{m}{2}}|f|_{pm} + C_7^{p+1}((p+1)m)!^{\frac{1}{2}}|f|_0$$

and similar for $J_2''$, which proves desired inequality (4.2.11).

As in the proof of Proposition 4.2.1, the condition (4.1.2) ensures the existence of a non-increasing sequence of positive numbers $r_p'$ such that $\sqrt{p+1}M_p/M_{p+1} \leq r_p'$, $\forall p \in \mathbb{N}_0$, where in the Roumieu case we may take it to be constant $r_p' = r$ $(\geq 1)$, while in the Beurling case $r_p' \to 0^+$. Hence, (4.2.9) holds with any non-increasing sequence $r_p$ majorizing the three sequences $(3C_2)^{1/p}C_3b_p$, $3C_2b_pb_{p-1}$, and $3C_2b_p$, where $b_p = \prod_{\nu=pm}^{pm+m-1} r_\nu'$. In the Beurling case we can clearly choose $r_p \to 0^+$. In the Roumieu case (4.2.8) holds if we select $h_0 = (3C_2C_3)^{1/m}r^2$. □

We can now state and prove the main theorem of this section:

**Theorem 4.2.5.** *Let $P$ be globally elliptic and let $M_p$ satisfy* $(M.1)$, $(M.2)'$, *and* (1.3.3). *We have that* $\mathcal{S}_P^*(\mathbb{R}^n) = \mathcal{S}^*(\mathbb{R}^n)$ *and they are tamely isomorphic.*

*Proof.* We start with the Beurling case. Since the sequence $r_p \searrow 0$, we can find $p_h$ large enough such that (4.2.8) holds for all $p \geq p_h$. We may assume that $r_1 \geq 1$. We keep $h \leq r_1$. For $p \leq p_h$, one gets from (4.2.9)

$$\sigma_p(f,h) \leq \frac{C'M_{(p-1)m}}{h^m M_{pm}}\sigma_{p-1}(Pf,h) + \frac{r_1}{3h^m}\sigma_{p-1}(f,h) + \\ + \frac{r_1}{3h^{2m}}\sigma_{p-2}(f,h) + \sigma_0(f,h)\frac{r_1^{p-1}}{3h^{mp}}.$$

Iterating these two relations, one obtains

$$\sigma_{p+1}(f,h) \leq \frac{C_1^{p_h}}{h^{p_h m}}\sigma_0(f,h)+ \\ \frac{C_1}{h^m}\left(\sum_{q=p_h}^{p}\frac{M_{qm}}{M_{(q+1)m}}\sigma_q(Pf,h)+\sum_{q=0}^{p_h-1}\frac{C_1^{p_h-1-q}}{h^{(p_h-1-q)m}}\frac{M_{qm}}{M_{(q+1)m}}\sigma_q(Pf,h)\right) \tag{4.2.14}$$

where $C_1=\max\{r_1,C'\}$. Iterating once more, we have

$$\sigma_{p+1}(f,h)\leq\frac{C_1^{p_h}}{h^{p_h m}}\sum_{s=0}^{p}\binom{p}{s}C_1^s\frac{\sigma_0(P^s f,h)}{h^{sm}M_{sm}}, \tag{4.2.15}$$

for $h\leq C_1$. In fact, we check the latter inequality inductively. The assumption $(M.1)$ yields $M_{qm}/M_{(q+1)m}\leq M_{sm}/M_{(s+1)m}$ if $s\leq q$. By (4.2.14), (4.2.15) for $q\leq p$, and $h\leq C_1$

$$\begin{aligned}\sigma_{p+1}(f,h) &\leq \frac{C_1^{p_h}}{h^{p_h m}}\left(\sum_{q=0}^{p}\frac{C_1 M_{qm}}{h^m M_{(q+1)m}}\sigma_q(Pf,h)+\sigma_0(f,h))\right)\\ &=\frac{C_1^{p_h}}{h^{p_h m}}\left(\sigma_0(f,h))+\sum_{q=0}^{p}\frac{M_{qm}}{M_{(q+1)m}}\sum_{s=0}^{q}\binom{q}{s}\frac{C_1^{s+1}}{h^{m(s+1)}}\frac{\sigma_0(P^{s+1}f,h)}{M_{sm}}\right)\\ &\leq\frac{C_1^{p_h}}{h^{p_h m}}\left(\sigma_0(f,h))+\sum_{s=0}^{p}\binom{p+1}{s+1}\frac{C_1^{s+1}}{h^{m(s+1)}}\frac{\sigma_0(P^{s+1}f,h)}{M_{(s+1)m}}\right),\end{aligned}$$

which shows (4.2.15). It now follows immediately from (4.2.15) that $\|\cdot\|'_{hL}\leq C'_h\|\cdot\|_{P,h}$ for all $h\leq r_1$, where $C'_h=(h^{-m}C_1)^{p_h}$ and $L=(1+C_1)^{1/m}$. Combining this with Proposition 4.2.3, we obtain that $\mathcal{S}_P^{(M_p)}(\mathbb{R}^n)\subseteq\mathcal{S}^{(M_p)}(\mathbb{R}^n)$ and the inclusion mapping $\mathcal{S}_P^{(M_p)}(\mathbb{R}^n)\to\mathcal{S}^{(M_p)}(\mathbb{R}^n)$ is tamely continuous. The rest was already shown in Proposition 4.2.1, which completes the proof in the Beurling case.

The Roumieu case is simpler. We keep $h\geq h_0$, where $h_0$ is the constant occurring in part $(i)$ of Lemma 4.2.4. Iterating (4.2.8) in an analogous way as in the Beurling case, we obtain

$$\sigma_{p+1}(f,h)\leq\sum_{s=0}^{p}\binom{p}{s}(C')^s\frac{\|P^s f\|_{L^2(\mathbb{R}^n)}}{h^{sm}M_{sm}},$$

which implies that $\|\cdot\|'_{hL} \leq \|\cdot\|_{P,h}$ for all $h \geq h_0$, where $L = (1+C')^{1/m}$. The rest follows once again from Proposition 4.2.1 and Proposition 4.2.3. □

Theorem 4.1.2 is now an easy consequence of Theorem 4.2.5. In fact, if $Pu = f \in \mathcal{S}^*(\mathbb{R}^n)$, the standard result [41] yields membership to the Schwartz space, that is, $u \in \mathcal{S}(\mathbb{R}^n)$. Since $\|u\|_{P,h} = \max\{\|u\|_{L^2(\mathbb{R}^n)}, \|f\|_{P,h}\}$, we conclude $u \in \mathcal{S}^*_P(\mathbb{R}^n) = \mathcal{S}^*(\mathbb{R}^n)$. As a corollary, we recover a result first observed in [14]: *If $P$ is globally elliptic then all its eigenfunctions belong to* $\mathcal{S}^{\{(p!)^{1/2}\}}(\mathbb{R}^n) = \mathcal{S}^{1/2}_{1/2}(\mathbb{R}^n)$. Actually, we can strengthen this result by adding a bound on the partial derivatives of the eigenfunctions, the ensuing result is a direct corollary of the tame isomorphism established in this section (and inspection in the constants occurring in the proofs of the results for the Roumieu case).

**Corollary 4.2.6.** *Let $P$ be globally elliptic. There are constants $L_1$ and $L_2$ depending merely on $P$ such that if $u$ is a solution to $Pu = \lambda u$, $\lambda \in \mathbb{C}$, then*

(*i*) $\|x^\beta \partial^\alpha u\|_{L^2(\mathbb{R}^n)} \leq L_1^{|\alpha|+|\beta|}(\alpha!\beta!)^{1/2}\|u\|_{L^2(\mathbb{R}^n)}$ *if* $\lambda = 0$.

(*ii*) $\|x^\beta \partial^\alpha u\|_{L^2(\mathbb{R}^n)} \leq L_2|\lambda|(L_1|\lambda|^{\frac{1}{m}})^{|\alpha|+|\beta|}(\alpha!\beta!)^{1/2}\|u\|_{L^2(\mathbb{R}^n)}$ *if* $\lambda \neq 0$.

## 4.3 Eigenfunction expansions

We now study eigenfunction expansions of ultradifferentiable functions and ultradistributions.

Through the rest of the chapter we assume that $P$ is globally elliptic and normal. As pointed out in the Introduction, these two conditions on $P$ guarantee the existence of an orthonormal basis of $L^2(\mathbb{R}^n)$ consisting of eigenfunctions of $P$. We fix such an orthonormal basis of eigenfunctions $\{u_j : j \in \mathbb{N}\}$. For each $j$, let $\lambda_j$ be the eigenvalue corresponding to $u_j$. Since $PP^*$ is positive and self-adjoint, and has order $2m$ and eigenvalues $|\lambda_j|^2$, the Weyl asymptotic formula yields

$$|\lambda_j| \sim Bj^{\frac{m}{2n}}, \tag{4.3.1}$$

where the constant $B$ depends on the symbol of $PP^*$, see [4, 41, 58] for details. We introduce two (graded) sequence spaces suggested by the inequalities (4.1.1), that is, the (LB) space

$$\Lambda_n^{\{M_p\}} = \{(a_j)_{j\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}} : \sup_{j\in\mathbb{N}} |a_j| e^{M(j^{\frac{1}{2n}}/h)} < \infty \text{ for some } h > 0\},$$

and the Fréchet space

$$\Lambda_n^{(M_p)} = \{(a_j)_{j\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}} : \sup_{j\in\mathbb{N}} |a_j| e^{M(j^{\frac{1}{2n}}/h)} < \infty \text{ for every } h > 0\}.$$

The concept of absolute Schauder bases for locally convex spaces is defined in [40, p. 340].

**Theorem 4.3.1.** *Let $P$ be normal and globally elliptic and let $M_p$ satisfy* $(M.1)$*,* $(M.2)'$*, and* (1.3.3)*. The mapping*

$$f \mapsto ((f, u_j)_{L^2(\mathbb{R}^n)})_{j\in\mathbb{N}}$$

*is a tame isomorphism from $\mathcal{S}^*(\mathbb{R}^n)$ onto $\Lambda_n^*$. Moreover, the set of eigenfunctions $\{u_j : j \in \mathbb{N}\}$ is an absolute Schauder basis for $\mathcal{S}^*(\mathbb{R}^n)$.*

*Proof.* That $\{u_j : j \in \mathbb{N}_0\}$ is an absolute Schauder basis of $\mathcal{S}^*(\mathbb{R}^n)$ follows readily from the first assertion and the fact that it is an orthonormal basis of $L^2(\mathbb{R}^n)$, we leave details to the reader. Because of Theorem 4.2.5, we can work with the system of norms (4.2.1). Define the function

$$\widetilde{M}(t) := \sup_{p\in\mathbb{N}_0} \log \frac{t^{mp}}{M_{mp}}, \quad t > 0,$$

and notice that $\widetilde{M}(t) \leq M(t)$ and $M(t) \leq \widetilde{M}(H^m t) + \log(A^m H^{\frac{(m+2)(m-1)}{2}})$, as one readily verifies with the aid of $(M.2)'$. Thus, using $M$ for the definition of $\Lambda_n^*$ is tamely equivalent to using the function $\widetilde{M}$. Furthermore, the system of norms $\|(a_j)_j\|_{\infty,h} := \sup_{j\in\mathbb{N}} |a_j| e^{\widetilde{M}(j^{\frac{1}{2n}}/h)}$ for $\Lambda_n^*$ is tamely equivalent to $\|(a_j)_j\|_{2,h} := \|(a_j e^{\widetilde{M}(j^{\frac{1}{2n}}/h)})_j\|_{\ell^2(\mathbb{N})}$. In fact, we trivially have $\|(a_j)_j\|_{\infty,h} \leq \|(a_j)_j\|_{2,h}$ for all $h > 0$. On the other hand, the sequence $M_{mp}$ satisfies $M_{(p+1)m} \leq (AH^{\frac{m+1}{2}})^m H^{pm^2} M_{pm}$, and applying [33, Prop. 3.4, p. 50] to $M_{pm}$, we obtain

$$e^{\widetilde{M}(t)} \leq A^{2n} H^{n(m+1)} \frac{e^{\widetilde{M}(H^{2n}t)}}{t^{2n}}, \quad t > 0.$$

The latter inequality implies that

$$\|(a_j)_j\|_{2,h} \leq \|(a_j)_j\|_{\infty,H^{-2n}h}(AhH^{\frac{m+1}{2}})^{2n}\pi/\sqrt{6}$$

for all $h > 0$, showing the claimed tame equivalence. Write now $a_j = (f, u_j)_{L^2(\mathbb{R}^n)}$ and let $d = \dim(\operatorname{Ker} P)$. Employing the Weyl asymptotics (4.3.1), we have

$$B_1^2\|P^p f\|^2_{L^2(\mathbb{R}^n)} \leq \sum_{j=1}^{\infty} j^{\frac{mp}{n}}|a_j|^2 \leq d^{\frac{mp}{n}}\|f\|^2_{L^2(\mathbb{R}^2)} + B_2^2\|P^p f\|^2_{L^2(\mathbb{R}^n)},$$

whence $B_1\|f\|_{P,h} \leq \|(a_j)_j\|_{2,h}$ and $\|(a_j)_j\|_{\infty,h} \leq \|f\|_{P,h}\sqrt{B_2^2 + e^{2\widetilde{M}(d^{\frac{1}{2n}}/h)}}$ for all $h > 0$. This concludes the proof of the theorem. □

Observe that if $(M.2)'$ holds, the strong duals of $\Lambda_n^*$ are precisely

$$(\Lambda_n^{\{M_p\}})' = \{(a_j)_{j\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}} : \sup_{j\in\mathbb{N}} |a_j|e^{-M(j^{\frac{1}{2n}}/h)} < \infty \ \text{ for all } h > 0\},$$

and

$$(\Lambda_n^{(M_p)})' = \{(a_j)_{j\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}} : \sup_{j\in\mathbb{N}} |a_j|e^{-M(j^{\frac{1}{2n}}/h)} < \infty \ \text{ for some } h > 0\}.$$

Therefore, we obtain the following corollary from Theorem 4.3.1 for ultradistributions. Note that the ultradistributional evaluation $\langle f, \overline{u}_j\rangle = {}_{\mathcal{S}^{*\prime}}\langle f, \overline{u}_j\rangle_{\mathcal{S}^*}$ is well-defined in view of Corollary 4.2.6.

**Corollary 4.3.2.** *Under the assumptions of Theorem 4.3.1, every ultradistribution $f \in \mathcal{S}^{*\prime}(\mathbb{R}^n)$ has eigenfunction expansion*

$$f = \sum_{j=1}^{\infty} a_j u_j, \qquad a_j = \langle f, \overline{u}_j\rangle.$$

*Furthermore, $\{u_j : j \in \mathbb{N}\}$ is an absolute Schauder basis for $\mathcal{S}^{*\prime}(\mathbb{R}^n)$ and the mapping $f \mapsto (a_j)_{j\in\mathbb{N}}$ is a tame isomorphism from $\mathcal{S}^{*\prime}(\mathbb{R}^n)$ onto $\Lambda_n^{*\prime}$.*

We end this chapter with a specialized version of Corollary 4.2.6.

**Corollary 4.3.3.** *Let $P$ be normal and globally elliptic. Then, there is a constant $\ell = \ell_P$ such that*

$$\|x^\beta \partial^\alpha u\|_{L^2(\mathbb{R}^n)} \leq j^{\frac{m+|\alpha|+|\beta|}{2n}} \ell^{|\alpha|+|\beta|} (\alpha!\beta!)^{1/2} \|u\|_{L^2(\mathbb{R}^n)},$$

*for each eigenfunction $u$ with $Pu = \lambda_j u$.*

*Proof.* Apply Corollary 4.2.6 and the asymptotic estimate (4.3.1). $\square$

# Chapter 5

# Pseudodifferential operators in spaces of ultradistributions on $\mathbb{T}^n$

## 5.1 Introduction

In this chapter, a class of symbols and corresponding pseudodifferential operators of finite order on the torus $\mathbb{T}^n$ that act continuously on certain class of ultradifferentiable functions and ultradistributions on $\mathbb{T}^n$ will be studied and the corresponding symbolic calculus will be developed. We advise the reader to consult [41] in order to get in touch with the symbolic calculus for global symbols and, more accurately, their asymptotic expansions and its flavor. The reader should also review Section 1.4 where we briefly discussed symbol classes related to $C^\infty$-functions and distributions on $\mathbb{T}^n$.

Pseudodifferential operators that act continuously on Gevrey classes (see Section 1.3), both of finite or infinite order, have been studied over the years (see [8, 7] for the symbolic calculus). A similar approach was followed by Prangoski [47] for an extensive treatment of a class of pseudodiferential operators of infinite order in spaces of tempered ultradistributions, both of Beurling and Roumieu type, on $\mathbb{R}^n$. Finally, in [9], this treatment is finalized by constructing parametrices.

Our aim is to study the analogous problem for the torus $\mathbb{T}^n$. Our investigation reveals a subtle difference between analysis on $\mathbb{T}^n$ and its global counterpart in $\mathbb{R}^n$.

The plan of the chapter goes as follows. We first give a definition of symbol classes of operators on $\mathbb{T}^n$ and prove some mapping properties of the corresponding pseudodifferential operators. Then we will proceed to develop the symbolic calculus by defining formal sums and proving that we are able to build the operator from a given formal sum. This result is fundamental for any further consideration regarding these symbol classes. In the future the author plans to use the approach developed here to study composition and parametrix.

The notation for $|\xi|$ when $\xi \in \mathbb{Z}^n$ could be confusing here for the reader because it could mean either the length of a multiindex or its Euclidean norm. In order to avoid this, we will use $\|\cdot\|$ for the latter. More generally, when $\xi \in \mathbb{R}^n$, we denote (just in this chapter), $\|\xi\| = \left(\sum_{k=1}^n |\xi_k|^2\right)^{1/2}$.

## 5.2 Class of symbols

Our aim in this section is to construct periodic pseudodifferential operators that act continuously on ultradifferentiable classes on $\mathbb{T}^n$. In order to do that, we need a little preparation.

Let $A_p$ and $B_p$ be sequences that satisfy $(M.1)$, $(M.2)$, $(M.3)$, with the additional assumption that $A_0 = B_0 = 1$. For $0 < \rho \leq 1$, $m \in \mathbb{R}$, we shall define the toroidal class $\Gamma^m_{A_p,B_p,\rho}(\mathbb{T}^n \times \mathbb{Z}^n; h)$ as the set of functions $a(x,\xi) \in \mathcal{E}(\mathbb{T}^n)$ , $\forall \xi \in \mathbb{Z}^n$, for which

$$\|a\|_{\Gamma^m_{A_p,B_p,\rho}(\mathbb{T}^n\times\mathbb{Z}^n;h)}$$
$$= \sup_{(x,\xi)\in\mathbb{T}^n\times\mathbb{Z}^n} \sup_{\alpha,\beta} \frac{|\Delta_\xi^\alpha \partial_x^\beta a(x,\xi)|}{A_{|\alpha|}B_{|\beta|} \max_{0\leq\nu\leq\alpha}\langle \xi+\nu\rangle^{m-\rho|\alpha|} h^{|\alpha+\beta|}} < \infty$$

Recall that $\Delta_\xi^\alpha$ stands for the higher order forward difference operator defined in Section 1.4. It is easy to see that this is a Banach space. Define

$$\Gamma^m_{\{A_p,B_p\},\rho}(\mathbb{T}^n \times \mathbb{Z}^n) = \varinjlim_{h\to\infty} \Gamma^m_{A_p,B_p,\rho}(\mathbb{T}^n \times \mathbb{Z}^n; h).$$

$$\Gamma^m_{(A_p,B_p),\rho}(\mathbb{T}^n \times \mathbb{Z}^n) = \varprojlim_{h\to 0} \Gamma^m_{A_p,B_p,\rho}(\mathbb{T}^n \times \mathbb{Z}^n; h),$$

*Remark* 5.2.1. We can simultaneously define Euclidean counterparts of these operators. With the same assumptions, first define the Banach space $\Gamma^m_{A_p,B_p,\rho}(\mathbb{T}^n \times \mathbb{R}^n; h)$ as the set of functions $b(x,\xi) \in C^\infty(\mathbb{T}^n \times \mathbb{R}^n)$ for which

$$\|a\|_{\Gamma^m_{A_p,B_p,\rho}(\mathbb{T}^n\times\mathbb{R}^n;h)} = \sup_{\alpha,\beta} \sup_{(x,\xi)\in\mathbb{T}^n\times\mathbb{Z}^n} \frac{|\Delta_\xi^\alpha \partial_x^\beta a(x,\xi)|}{A_{|\alpha|}B_{|\beta|}\langle\xi\rangle^{m-\rho|\alpha|}h^{|\alpha+\beta|}}$$

is finite, and then define

$$\Gamma^m_{\{A_p,B_p\},\rho}(\mathbb{T}^n \times \mathbb{R}^n) = \varinjlim_{h\to\infty} \Gamma^m_{A_p,B_p,\rho}(\mathbb{T}^n \times \mathbb{R}^n; h),$$

$$\Gamma^m_{(A_p,B_p),\rho}(\mathbb{T}^n \times \mathbb{R}^n) = \varprojlim_{h\to 0} \Gamma^m_{A_p,B_p,\rho}(\mathbb{T}^n \times \mathbb{R}^n; h).$$

*Remark* 5.2.2. It might be useful to make a small remark regarding the choice of the symbol class. The reader could be puzzled with the factor $\max_{0\le\nu\le\alpha}\langle\xi+\nu\rangle$ that figures in the expression in the symbol class definition. Figuratively speaking, this factor is needed for a smooth transition between derivatives and differences as it reflects the connection between toroidal and Euclidean symbols and we definitely want to keep that connection. This becomes obvious from ([53, Proof of Theorem 4.5.3]), namely, from the fact (that is a consequence of the mean-value theorem) $\Delta_\xi^\alpha \partial_x^\beta a(x,\xi) = \partial_x^\alpha \partial_x^\beta a(x,\xi+\nu)$, for every $a(x,\xi) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ where $\nu$ lies on the line between $\xi$ and $\xi+\alpha$ so the restriction of a symbol $a \in \Gamma^m_{*,\rho}(\mathbb{T}^n \times \mathbb{Z}^n)$ to $\mathbb{Z}^n$ will give our toroidal symbol class.

In order to deal with the Roumieu and the Beurling classes simultaneously, we shall use the notation $\dagger = (A_p,B_p), \{A_p,B_p\}$.

Before any topological consideration, let us prove that $\Gamma^m_{\dagger,\rho}(\mathbb{T}^n \times \mathbb{Z}^n; h)$ are algebras under pointwise multiplication. Or, to be more precise, we will prove the following result.

**Lemma 5.2.3.** *If $a \in \Gamma^{m_1}_{A_p,B_p,\rho}(\mathbb{T}^n \times \mathbb{Z}^n; h_1)$ and $b \in \Gamma^{m_2}_{A_p,B_p,\rho}(\mathbb{T}^n \times \mathbb{Z}^n; h_2)$, then $ab \in \Gamma^{m_1+m_2}_{A_p,B_p,\rho}(\mathbb{T}^n \times \mathbb{Z}^n; 2(h_1+h_2))$.*

*Proof.* Combination of the Leibniz and the discrete Leibniz rule, with the help of $(M.1)$, gives:

$$
\begin{aligned}
&|\Delta_\xi^\alpha \partial_x^\beta (ab)(x,\xi)| \\
&\leq \sum_{\gamma\leq\alpha}\sum_{\delta\leq\beta}\binom{\alpha}{\gamma}\binom{\beta}{\delta}|\Delta_\xi^\gamma \partial_x^\delta a(x,\xi)||\Delta_\xi^{\alpha-\gamma}\partial_x^{\beta-\delta} b(x,\xi+\gamma)| \\
&\leq C_1 C_2 A_{|\alpha|} B_{|\beta|}(h_1+h_2)^{|\alpha+\beta|}\times \\
&\max_{\gamma\leq\alpha,\delta\leq\beta}\left(\max_{0\leq\nu_1\leq\gamma}\langle\xi+\nu_1\rangle^{m_1-\rho|\gamma|}\max_{0\leq\nu_2\leq\alpha-\gamma}\langle\xi+\nu_2+\gamma\rangle^{m_2-\rho|\alpha-\gamma|}\right) \\
&\leq C_1 C_2 A_{|\alpha|} B_{|\beta|} h^{|\alpha+\beta|} \\
&\max_{\gamma\leq\alpha,\delta\leq\beta}\left(\max_{0\leq\nu_1\leq\alpha}\langle\xi+\nu_1\rangle^{m_1-\rho|\gamma|}\max_{0\leq\nu_2\leq\alpha}\langle\xi+\nu_2\rangle^{m_2-\rho|\alpha-\gamma|}\right)
\end{aligned}
$$

with $h = h_1 + h_2$.

In the case when $(m_1 - \rho|\gamma|)(m_2 - \rho|\alpha-\gamma|) \geq 0$, then both $\langle\xi+\nu_1\rangle^{m_1-\rho|\gamma|}$ and $\langle\xi+\nu_2\rangle^{m_2-\rho|\alpha-\gamma|}$ attain their maximum at the same point $\nu_1 = \nu_2$ so we may simple multiply these two factors in order to obtain $\max_{0\leq\nu\leq\alpha}\langle\xi+\nu\rangle^{m_1+m_2-\rho|\alpha|}$. In the other case we will need to use Peetre's inequality (1.4.7). Assume that, e.g., $m_1 - \rho|\gamma| < 0$ and $m_2 - \rho|\alpha-\gamma| \geq 0$ then these factors attain maximum at the different points $\nu_1$ and $\nu_2$ and $\nu_1 \neq \nu_2$. However,

$$
\begin{aligned}
&\langle\xi+\nu_2\rangle^{m_2-\rho|\alpha-\gamma|} \leq 2^{m_2}\langle\xi+\nu_1\rangle^{m_2-\rho|\alpha-\gamma|}\cdot\langle\nu_1-\nu_2\rangle^{m_2} \\
&\leq 4^{m_2}\langle\alpha\rangle^{m_2}\max_{0\leq\nu\leq\alpha}\langle\xi+\nu\rangle^{m_2-\rho|\alpha-\gamma|}.
\end{aligned}
$$

Let $\langle\alpha\rangle^{m_2} \leq C_3 2^{|\alpha|}$ ($m_2$ is fixed). Therefore,

$$
\begin{aligned}
&|\Delta_\xi^\alpha \partial_x^\beta (ab)(x,\xi)| \leq 4^{m_2} C_1 C_2 C_3 A_{|\alpha|} B_{|\beta|}(2h)^{|\alpha+\beta|} \\
&\times \max_{0\leq\nu\leq\alpha}\langle\xi+\nu\rangle^{m_1+m_2-\rho|\alpha|}.
\end{aligned}
$$

$\square$

It follows that if $a_i \in \Gamma^{m_i}_{\dagger,\rho}(\mathbb{T}^n \times \mathbb{Z}^n)$, $i = 1,2$, then $a_1 \cdot a_2 \in \Gamma^{m_1+m_2}_{\dagger,\rho}(\mathbb{T}^n \times \mathbb{Z}^n)$.

The space $\Gamma^m_{(A_p,B_p),\rho}(\mathbb{T}^n \times \mathbb{Z}^n)$ is a Fréchet space. Both of these spaces are Hausdorff locally convex and bornological (see [54, Section 8.1]). Also, the space $\Gamma^m_{(A_p,B_p),\rho}(\mathbb{T}^n \times \mathbb{Z}^n)$ is barrelled, as being Fréchet, and $\Gamma^m_{\{A_p,B_p\},\rho}(\mathbb{T}^n \times \mathbb{Z}^n)$ is barrelled, as an inductive limit of barrelled spaces.

**Theorem 5.2.4.** *Let $a \in \Gamma^m_{\dagger,\rho}(\mathbb{T}^n \times \mathbb{Z}^n)$ and $M_p$ be a sequence such that $A_p \subseteq M_p$ and $B_p \subseteq M_p$.*

*Then $a(x,D) : \mathcal{E}^*(\mathbb{T}^n) \to \mathcal{E}^*(\mathbb{T}^n)$ is a continuous operator.*

*Proof.* Let $u \in \mathcal{E}^{\{M_p\},h}(\mathbb{T}^n)$. Let $l, c_l > 0$ be the constants such that $A_p \leq c_l l^p M_p$ and $B_p \leq c_l l^p M_p$.

Then, for $\xi \neq 0$, from the inequality (1.3.6) in the proof of Lemma 1.3.1

$$|\hat{u}(\xi)| \leq \frac{C_1 \|u\|_{\mathcal{E}^{\{M_p\},h}(\mathbb{T}^n)} (\frac{\sqrt{2n}Hh}{2\pi})^{p+n+1} M_p}{\langle \xi \rangle^{p+n+1}}$$

where the constants were taken from the mentioned inequality. Let us assume that $m \geq 0$, otherwise the calculation is even simpler. Using Leibniz rule (absolute convergence of these series allows us to apply it), we obtain

$$\begin{aligned}
&\left|\partial^\alpha \Big( a(x,D)u(x) - a(x,0)\hat{u}(0) \Big)\right| \\
&= \Big| \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \Big( \sum_{\xi \in \mathbb{Z}^n \setminus \{0\}} (2\pi i \xi)^{\alpha-\beta} \partial^\beta a(x,\xi) e^{2\pi i x \cdot \xi} \hat{u}(\xi) \Big) \Big| \\
&\leq C_1 \|a\|_{\Gamma^m_{A_p,B_p,\rho}(\mathbb{T}^n \times \mathbb{Z}^n; h_1)} \|u\|_{\mathcal{E}^{\{M_p\},h}(\mathbb{T}^n)} M_p \langle \xi \rangle^{-p-n-1} \\
&\times \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} (2\pi)^{|\alpha-\beta|} h_1^{|\beta|} B_{|\beta|} \sum_{\xi \in \mathbb{Z}^n \setminus \{0\}} \langle \xi \rangle^{m+|\alpha-\beta|} \times (\frac{\sqrt{2n}hH}{2\pi})^{p+n+1}
\end{aligned}$$

for some $h_1 > 0$.

Pick $p = |\alpha - \beta| + \lceil m \rceil$. Then, we have

$$\begin{aligned}
&\left|\partial^\alpha\Big(a(x,D)u(x) - a(x,0)\hat{u}(0)\Big)\right| \leq C_1 c_l\big(\frac{\sqrt{2n}hH}{2\pi}\big)^{\lceil m\rceil+n+1}\times \\
&\|a\|_{\Gamma^m_{A_p,B_p,\rho}(\mathbb{T}^n\times\mathbb{Z}^n;h_1)}\|u\|_{\mathcal{E}^{\{M_p\},h}(\mathbb{T}^n)}M_{|\beta|}M_{|\alpha-\beta|+\lceil m\rceil} \\
&\times\Big(\sum_{\beta\leq\alpha}\binom{\alpha}{\beta}(lh_1)^{|\beta|}(\sqrt{2n}hH)^{|\alpha-\beta|}\Big)\Big(\sum_{\xi\in\mathbb{Z}^n}\langle\xi\rangle^{-n-1}\Big) \qquad (5.2.1) \\
&\leq C_2\|a\|_{\Gamma^m_{A_p,B_p,\rho}(\mathbb{T}^n\times\mathbb{Z}^n;h_1)}\|u\|_{\mathcal{E}^{\{M_p\},h}(\mathbb{T}^n)}(2lh_1 + 2\sqrt{2n}hH)^{|\alpha|}\times \\
&M_{|\beta|}M_{|\alpha-\beta|+\lceil m\rceil}.
\end{aligned}$$

Finally $(M.1)$ and $(M.2)$ give

$$M_{|\beta|}M_{|\alpha-\beta|+\lceil m\rceil} \leq AH^{|\alpha|+\lceil m\rceil}M_{\lceil m\rceil}M_{|\alpha|}.$$

Therefore,

$$\begin{aligned}
&|\partial^\alpha\Big(a(x,D)u(x) - a(x,0)\hat{u}(0)\Big)| \leq C_3 M_{|\alpha|}\Big(2lh_1H + 2\sqrt{2n}hH^2\Big)^{|\alpha|}\times \\
&\|u\|_{\mathcal{E}^{\{M_p\},h}(\mathbb{T}^n)}\|a\|_{\Gamma^m_{A_p,B_p,\rho}(\mathbb{T}^n\times\mathbb{Z}^n;h_1)},
\end{aligned}$$

where $C_3$ does not depend on $h$. If $\xi = 0$ we have

$$\begin{aligned}
&|\partial^\alpha a(x,0)\hat{u}(0)| \leq |\hat{u}(0)|\ \|a\|_{\Gamma^m_{A_p,B_p,\rho}(\mathbb{T}^n\times\mathbb{Z}^n;h_1)}h_1^{|\alpha|}B_{|\alpha|} \\
&\leq c_l\|u\|_{\mathcal{E}^{\{M_p\},h}(\mathbb{T}^n)}\|a\|_{\Gamma^m_{A_p,B_p,\rho}(\mathbb{T}^n\times\mathbb{Z}^n;h_1)}(lh_1)^{|\alpha|}M_{|\alpha|} \\
&\leq c_l\|u\|_{\mathcal{E}^{\{M_p\},h}(\mathbb{T}^n)}\|a\|_{\Gamma^m_{A_p,B_p,\rho}(\mathbb{T}^n\times\mathbb{Z}^n;h_1)} \\
&\times\Big(lh_1H + 4\sqrt{2n}hH^2\Big)^{|\alpha|}M_{|\alpha|}, \qquad (5.2.2)
\end{aligned}$$

which already proves the continuity in the Roumieu case.

In the Beurling case, if $h > 0$ is arbitrary,

$$\frac{|\partial^\alpha a(x,D)u(x)|}{h^{|\alpha|}M_{|\alpha|}} \leq C_4\|u\|_{\mathcal{E}^{\{M_p\},\frac{h}{4\sqrt{2n}H^2}}(\mathbb{T}^n)}\|a\|_{\Gamma^m_{A_p,B_p,\rho}(\mathbb{T}^n\times\mathbb{Z}^n;\frac{h}{4lH})}. \qquad (5.2.3)$$

where $l > 0$ is fixed. Therefore,

$$\begin{aligned}
&\|a(x,D)u(x)\|_{\mathcal{E}^{\{M_p\},h}(\mathbb{T}^n)} \leq \\
&C_4\|u\|_{\mathcal{E}^{\{M_p\},\frac{h}{4\sqrt{2n}H^2}}(\mathbb{T}^n)}\|a\|_{\Gamma^m_{A_p,B_p,\rho}(\mathbb{T}^n\times\mathbb{Z}^n;\frac{h}{4lH})}, \qquad (5.2.4)
\end{aligned}$$

which ends the proof. □

By duality we obtain the following corollary,

**Corollary 5.2.5.** *Under the assumptions of Theorem 5.2.4, $a(x,D)$ : $\mathcal{E}'^{*}(\mathbb{T}^n) \to \mathcal{E}'^{*}(\mathbb{T}^n)$ is continuous as well.*

From the inequality (5.2.4) one can conclude that much more holds, namely, that the mapping $(a,u) \mapsto a(x,D)u$, $\Gamma^{m}_{\dagger,\rho}(\mathbb{T}^n \times \mathbb{Z}^n) \times \mathcal{E}^{*}(\mathbb{T}^n) \to \mathcal{E}^{*}(\mathbb{T}^n)$ is separately continuous which, along with the fact that these spaces are barrelled, proves that the mapping is also hypocontinuous (see [60, Theorem 41.2]).

## 5.3 Symbolic calculus

In this section we discuss the two basics of symbolic calculus for our symbol classes. For it, the concept of special formal series will be defined as follows. Here we restrict ourselves to the case $A_p = B_p$ in order to simplify the calculus. Recall that $m_p$ stands for $m_p = M_p/M_{p-1}$, with the convention $m_0 = 0$.

We consider here the case $A_p = B_p$ and assume that $A_p \subseteq M_p^{\rho}$, where $0 < \rho \leq 1$. Let us right now take care about the constants: $A_p \leq c_L L^p M_p^{\rho}$ for some $L > 0$ and we then take $c_0 = \max\{c_L, A, 1\}$ where the constant $A$ figures in $(M.2)$. We assume here that $M_p$ satisfies $(M.1)$, $(M.2)$ and $(M.3)$. We may actually assume that the sequence also satisfies $(M.1)^{*}$ because, as shown in [44], this set of assumptions implies that $M_p$ can be replaced by an equivalent sequence that satisfies $(M.1)^{*}$.

For $t > 0$ and a multi-index $\alpha \in \mathbb{N}^n$, we then set

$$(Q_t^{\alpha})^c = \mathbb{T}^n \times \{\xi \in \mathbb{Z}^n : \langle \xi + \eta \rangle > t \text{ for some } \eta \in \mathbb{N}^n,\ 0 \leq \eta \leq \alpha\}.$$

Let $FS^{m}_{A_p,M_p,\rho}(\mathbb{T}^n \times \mathbb{Z}^n; h)$ be the vector space of all formal series of the form $\sum_{j=0}^{\infty} a_j(x,\xi)$ such that, for some $B > 0$, each $a_j$ is smooth on each

$(Q^{\alpha}_{Bm_j})^c$ and

$$\sup_{j,\alpha,\beta}\ \sup_{(x,\xi)\in(Q^{\alpha}_{Bm_j})^c} \frac{|\Delta^{\alpha}_{\xi}\partial^{\beta}_{x}a_j(x,\xi)| \min\limits_{0\le\theta\le\alpha}\langle\xi+\theta\rangle^{\rho|\alpha|-m} \max\limits_{0\le\theta\le\alpha}\langle\xi+\theta\rangle^{\rho j}}{h^{|\alpha+\beta|+j}A_{|\alpha|}A_{|\beta|}A_j} < \infty,$$

where according to our convention $Q^{\alpha}_{Bm_0} = \mathbb{T}^n \times \mathbb{Z}^n$. We further set

$$FS^m_{A_p,\{M_p\},\rho}(\mathbb{T}^n \times \mathbb{Z}^n) = \varinjlim_{h\to\infty} FS^m_{A_p,M_p,\rho}(\mathbb{T}^n \times \mathbb{Z}^n; h),$$

$$FS^m_{A_p,(M_p),\rho}(\mathbb{T}^n \times \mathbb{Z}^n) = \varprojlim_{h\to 0} FS^m_{A_p,M_p,\rho}(\mathbb{T}^n \times \mathbb{Z}^n; h),$$

and use the common notation $FS^m_{A_p,*,\rho}(\mathbb{T}^n \times \mathbb{Z}^n)$ to include both cases. Of course, we may consider one symbol $a(x,\xi)$ as a formal sum $a(x,\xi) = a(x,\xi) + 0 + \cdots$.

Two formal series $\sum_{j=0}^{\infty} a_j(x,\xi)$ and $\sum_{j=0}^{\infty} b_j(x,\xi)$ in $FS^m_{A_p,*,\rho}(\mathbb{T}^n \times \mathbb{Z}^n)$ are said to be equivalent if there exist $C, h > 0$ (resp. for every $h > 0$ there exists $C > 0$) such that for every $(x,\xi) \in (Q^{\alpha}_{Bm_j})^c$

$$\Big|\Delta^{\alpha}_{\xi}\partial^{\beta}_{x}\Big(\sum_{j<N} a_j(x,\xi) - b_j(x,\xi)\Big)\Big| \le C h^{|\alpha|+|\beta|+N} A_{|\beta|}A_{|\alpha|}A_N \times \frac{\max_{0\le\theta\le\alpha}\langle\xi+\theta\rangle^{m-\rho|\alpha|}}{\max_{0\le\theta\le\alpha}\langle\xi+\theta\rangle^{\rho N}}.$$

We then write $\sum_{j=0}^{\infty} a_j(x,\xi) \sim \sum_{j=0}^{\infty} b_j(x,\xi)$. Below $\dagger$ stands for either $\{A_p, A_p\}$ or $(A_p, A_p)$, according to whether we consider the Roumieu or Beurling case of $*$.

**Theorem 5.3.1.** *Let the sequence $M_p$ satisfy the above mentioned assumptions. Then for every $\sum_{j=0}^{\infty} a_j$ in $FS^m_{A_p,*,\rho}(\mathbb{T}^n \times \mathbb{Z}^n)$ there exists a symbol $a \in \Gamma^m_{\dagger,\rho}(\mathbb{T}^n \times \mathbb{Z}^n)$ such that $a(x,\xi) \sim \sum_{j=0}^{\infty} a_j(x,\xi)$.*

*Proof.* Without loss of generality we suppose $m < 0$. Fix $\sum_{j=0}^{\infty} a_j(x,\xi) \in FS^m_{A_p,M_p,\rho}(\mathbb{T}^n \times \mathbb{Z}^n; h)$. Find $\varphi \in \mathcal{D}^{(A_p)}(\mathbb{R}^n)$ such that $\varphi(\xi) = 1$ for $\|\xi\| \ge \frac{1}{2}$, $\varphi(\xi) = 0$ if $\|\xi\| \le \frac{1}{4}$ (the Denjoy-Calerman theorem ensures the existence of such a function). We set $\varphi_0 = \varphi$ and for positive $j \in \mathbb{N}$ we consider $\varphi_j(\xi) = \varphi(\frac{\xi}{Rm_j})$, where $R > 0$ is a large (but fixed) number to be determined below. Then, we may define, for an arbitrary $(x, \xi) \in \mathbb{T}^n \times \mathbb{R}^n$,

$$a(x,\xi) = \sum_{j=0}^{\infty} \varphi_j(\xi) a_j(x,\xi)$$

where this series is actually a finite sum, for each fixed $\xi \in \mathbb{Z}^n$.

We will first prove that $a$ is a symbol in $\Gamma^m_{A_p,A_p,\rho}(\mathbb{T}^n \times \mathbb{Z}^n; h+h_1)$ for each arbitrary $h_1 > 0$. This will show the statement $a \in \Gamma^m_{\dagger,\rho}(\mathbb{T}^n \times \mathbb{Z}^n)$ simultaneously in the corresponding Roumieu and Beurling cases. We now fix an arbitrary $h_1 > 0$. By the mean-value theorem (see [53, Proof of Theorem 4.5.3]), with the aid of $(M.1)$ for the sequence $M_p$, it is easy to see that $(j > 0)$

$$|\Delta_\xi^\alpha \varphi_j(\xi)| \leq \frac{1}{(Rm_j)^{|\alpha|}} C_1 h_1^{|\alpha|} A_{|\alpha|} \tag{5.3.1}$$

for some $C_1 > 0$. Find $C_2$ such that, for each $j, \alpha, \beta$,

$$|\Delta_\xi^\alpha \partial_x^\beta a_j(x,\xi)| \leq C_2 h^{|\alpha+\beta|+j} A_{|\alpha|} A_{|\beta|} A_j \max_{0\leq\theta\leq\alpha} \langle \xi+\theta\rangle^{m-\rho|\alpha|} \max_{0\leq\theta\leq\alpha} \langle\xi+\theta\rangle^j, \tag{5.3.2}$$

for all $(x,\xi) \in (Q^\alpha_{Bm_j})^c$.

The claim $a \in \Gamma^m_{A_p,A_p,\rho}(\mathbb{T}^n \times \mathbb{Z}^n; h+h_1)$ would immediately follow if we establish the inequalities

$$|\Delta_\xi^\alpha \partial_x^\beta (\varphi_j a_j)(x,\xi)| \leq c_0 C_1 C_2 h^{|\beta|} (h+h_1)^{|\alpha|} A_{|\alpha|} A_{|\beta|} 2^{-j}, \tag{5.3.3}$$

for all $j > 0$, $\alpha, \beta \in \mathbb{N}^n$ and $(x,\xi) \in \mathbb{T}^n \times \mathbb{Z}^n$. Choose now $R$ such that $R \geq \max\{4B, 2, 8hL\}$.

We will show (5.3.3) by analyzing the terms that correspond to different multiindices $\gamma$ in the expression provided by the Leibniz rule for differences, that is,

$$\Delta_\xi^\alpha \partial_x^\beta (\varphi_j a_j)(x,\xi) = \sum_{\gamma\leq\alpha} \binom{\alpha}{\gamma} \Delta_\xi^\gamma \varphi_j(\xi) \cdot \Delta_\xi^{\alpha-\gamma} \partial_x^\beta a_j(x,\xi+\gamma). \tag{5.3.4}$$

We fix $\xi$, $\beta$, and $\alpha$ and consider case distinction accordingly to the size of $\max_{0\leq\theta\leq\alpha-\gamma}\langle\xi+\gamma+\theta\rangle$; more precisely, according to whether

$$\max_{0\le\theta\le\alpha-\gamma}\langle\xi+\gamma+\theta\rangle>\frac{Rm_j}{4} \tag{5.3.5}$$

holds or not.

Let then $A_{\xi,\alpha}$ be the set of multiindices $0\le\gamma\le\alpha$ such that (5.3.5) holds. It is clear that $(x,\xi+\gamma)\in(Q^{\alpha-\gamma}_{Bm_j})^c$ whenever $x\in\mathbb{T}^n$ and $\gamma\in A_{\xi,\alpha}$. Thus, we may estimate the term $\Delta_\xi^{\alpha-\gamma}\partial_x^\beta a_j(x,\xi+\gamma)$ in (5.3.4) via (5.3.2) for each $\gamma\in A_{\xi,\alpha}$. On the other hand, if $\gamma\notin A_{\xi,\alpha}$ we have that $\varphi_j(\xi+\gamma+\theta)\partial_x^\beta a_j(x,\xi+\gamma+\theta)=0$ for all $0\le\theta\le\alpha-\gamma$ due to the support of $\varphi_j$.

<u>Case I.</u> We now assume that

$$\min_{0\le\gamma\le\alpha}\max_{0\le\theta\le\alpha-\gamma}\langle\xi+\gamma+\theta\rangle>\frac{Rm_j}{4} \tag{5.3.6}$$

In this case every $\gamma, 0\le\gamma\le\alpha$ is in $A_{\xi,\alpha}$ and we do not have simplifications mentioned above.

Clearly, $\min_{0\le\theta\le\gamma}\|\xi+\theta\|\le Rm_j/2$ for every $\xi\in\operatorname{supp}\Delta_\xi^\gamma\varphi_j(\xi)$ or, in terms of Japanese brackets,

$$\min_{0\le\theta\le\alpha}\langle\xi+\theta\rangle\le 1+\min_{0\le\theta\le\gamma}\|\xi+\theta\|\le Rm_j.$$

In particular,

$$\min_{0\le\theta\le\alpha}\langle\xi+\theta\rangle^{\rho|\gamma|}\Big(\frac{1}{Rm_j}\Big)^{|\gamma|}\le 1, \tag{5.3.7}$$

which we use below. Taking into account (5.3.1), (5.3.2), (5.3.5), $(M.1)$ for $A_p$, and (5.3.7), we get

$$\begin{aligned}|\Delta_\xi^\alpha\partial_x^\beta a(x,\xi)| &\le C_1C_2h^{|\beta|}A_{|\beta|}A_{|\alpha|}\max_{0\le\theta\le\alpha}\langle\xi+\theta\rangle^{m-\rho|\alpha|}\sum_{\gamma\le\alpha}\binom{\alpha}{\gamma}h_1^{|\gamma|}h^{|\alpha-\gamma|}\\ &\quad\times\frac{1}{(Rm_j)^{|\gamma|}}h^j\Big(\min_{0\le\theta\le\alpha}\langle\xi+\theta\rangle\Big)^{\rho|\gamma|}\frac{A_j}{\max_{\gamma\le\theta\le\alpha}\langle\xi+\theta\rangle^{j\rho}}\\ &\le C_1C_2h^{|\beta|}A_{|\beta|}A_{|\alpha|}\max_{0\le\theta\le\alpha}\langle\xi+\theta\rangle^{m-\rho|\alpha|}(h_1+h)^{|\alpha|}\\ &\quad\times\Big(\frac{4h}{R}\Big)^j\frac{A_j}{M_j^\rho},\end{aligned}$$

whence (5.3.3) follows.

Case II. Assume that $\max_{0\le\theta\le\alpha-\gamma}\langle\xi+\gamma+\theta\rangle \le Rm_j/4$ for some $0\le\gamma\le\alpha$.

Let $\delta\in A_{\xi,\alpha}$ be the biggest multiindex (e.g. the multiindex of the biggest lenght) such that $\gamma\notin A_{\xi,\alpha}$ when $\gamma>\delta$. If $\delta=0$, then we are done, due to support of $\Delta_\xi^\gamma\varphi_j(\xi)$.

Therefore, suppose that $\delta\neq 0$. For every $\delta'\le\delta$, we may estimate $\Delta_\xi^{\delta'}\partial_x^\beta(\varphi_j a_j)(x,\xi)$ exactly as in the previous case, obtaining

$$|\Delta_\xi^{\delta'}\partial_x^\beta(\varphi_j(\xi)a_j(x,\xi))| \le c_0C_1C_2h^{|\beta|}(h+h_1)^{|\delta'|}A_{|\delta'|}A_{|\beta|}2^{-j}. \tag{5.3.8}$$

Moreover, similar calculation shows that

$$|\Delta_\xi^{\delta'-\gamma}\partial_x^\beta(\varphi_j(\xi)a_j(x,\xi+\gamma))| \le c_0C_1C_2h^{|\beta|}(h+h_1)^{|\delta'|}A_{|\delta'|}A_{|\beta|}2^{-j} \tag{5.3.9}$$

for every $0\le\gamma\le\delta'$, $|\delta'|\le|\delta|$. The reader recalls that $\partial_x^\beta a_j(x,\xi+\delta''+\theta)\varphi_j(\xi+\delta''+\theta)=0$ for every $\delta''$ with $|\delta|<|\delta''|$ and $0\le\theta\le\alpha-\delta''$. Our claim is that

$$|\Delta_\xi^{\delta''-\gamma}\partial_x^\beta(\varphi_j(\xi)a_j(x,\xi+\gamma))| \le c_0C_1C_2h^{|\beta|}(h+h_1)^{|\delta''|}A_{|\delta''|}A_{|\beta|}2^{-j} \tag{5.3.10}$$

for every $\delta''$, $|\delta|\le\ |\delta''|\le|\alpha|$. We perform the strong induction on $|\delta''|$. Since we already proved an inductive base, it remains to prove an inductive step. It will be enough if we prove the (5.3.10) when $\gamma=0$. Rename $\varphi_j(\xi)\partial_x^\beta a_j(x,\xi)=R_j(\xi)$. Suppose that (5.3.10) holds for some $\delta''<\alpha$. Then , for some $k, 1\le k\le n$, and $e_k$ being the $k$-th unit vector in $\mathbb{R}^n$, we have:

$$\begin{aligned}&\Delta_\xi^{\delta''+e_k}R_j(\xi)=\Delta_\xi^{\delta''-e_l+e_k}R_j(\xi+e_l)-\Delta_\xi^{\delta''-e_l+e_k}R_j(\xi)\\ &=\Delta_\xi^{\delta''-e_l-e_s+e_k}R_j(\xi+e_s+e_l)-\Delta_\xi^{\delta''-e_l-e_s+e_k}R_j(\xi+e_s)-\\ &\Delta_\xi^{\delta''-e_l+e_k}R_j(\xi),\ 1\le l,s\le n.\end{aligned}$$

In every step we are lowering the order of difference in the first member of the expression in order to obtain $\Delta_\xi^{\delta''-\delta}$. Iterating the previous argument we have:

$$\begin{aligned}&\Delta_\xi^{\delta''+e_k}R_j(\xi)=\Delta_\xi^{\delta''-\delta}R_j(\xi+\delta+e_k)-\Delta_\xi^{\delta''-\delta}R_j(\xi+\delta)\\ &-\cdots-\Delta_\xi^{\gamma'}R_j(\xi+\gamma_1)-\cdots-\Delta_\xi^{\delta''-e_l+e_k}R_j(\xi)\end{aligned}$$

where $|\gamma_1|+|\gamma|=|\delta''|$

Since $\Delta_\xi^{\delta''-\delta}R_j(\xi+\delta+e_k)=0$, and we have bounds from inductive step for the other members in expression, we estimate

$$|\Delta_\xi^{\delta''+e_k}R_j(\xi)|\leq c_0C_1C_2h^{|\beta|}(h+h_1)^{|\delta''|}(|\delta|+1)A_{|\delta''|}A_{|\beta|}2^{-j}.$$

From the fact that $(|\delta|+1)A_{|\delta''|}\leq|\delta''|A_{|\delta''|}\leq A_{|\delta''|+1}$, the claim follows.

This proves the first part of our theorem. A completely analogous argument yields $a(x,\xi)\sim\sum_{j=0}^{\infty}a_j(x,\xi)$, so we choose to omit details.

□

In the rest of this section we prove that $\sim$ is a relation up to $*$-regularizing operators. The reader with some experience in the theory of pseudodifferential operators will certainly recall what regularizing operator means. Namely, a toroidal pseudodifferential operator $a(x,D)$ is *$*$-regularizing* if it maps the space $\mathcal{E}'^*(\mathbb{T}^n)$ into the space of ultradifferentiable functions $\mathcal{E}^*(\mathbb{T}^n)$. To show this result we need a useful lemma.

**Lemma 5.3.2** ([47, Lemma 3.1]). *Let $M_p$ be a sequence that satisfies $(M.1)$, $(M.2)$, $(M.3)$ and $0<l\leq 1$ and $B>1$. There exists $C>0$ depending on $B,l,\{M_p\}$ and $\tilde m>0$ depending only on $B$ and $\{M_p\}$ and not on $l$ such that*

$$\inf\left\{\frac{M_p}{l^p\rho^p}|p\in\mathbb{Z}_+,\rho\geq Bm_p\right\}\leq Ce^{-M(l\tilde m\rho)},\ \textit{for all } \rho\geq BM_1.$$

**Theorem 5.3.3.** *If $a\in\Gamma^m_{\dagger,\rho}(\mathbb{T}^n\times\mathbb{Z}^n)$ is such that $a\sim 0$, then $a$ is a $*$-regularizing operator.*

*Proof.* We only prove the statement for the Roumieu case, the Beurling one is similar. Without loss of generality, suppose $m=0$. We will prove that the standard representation $Au=\sum_{\xi\in\mathbb{Z}^n}a(x,\xi)\hat u(\xi)$ will be well defined for $u\in\mathcal{E}'^{\{M_p\}}(\mathbb{T}^n)$ and, moreover, will give an element in $\mathcal{E}^{\{M_p\}}(\mathbb{T}^n)$. Our assumption gives

$$|\partial_x^\beta a(x,\xi)|\leq Ch^{-|\beta|}A_{|\beta|}\frac{A_j}{h^{\rho j}\langle\xi\rangle^{\rho j}},\qquad x\in\mathbb{T}^n,\ \langle\xi\rangle>Bm_j,$$

for some $h$ and $B$. Moreover,

$$\frac{A_j}{h^{\rho j}\langle\xi\rangle^{\rho j}} \leq C_0 \frac{L^j M_j^\rho}{h^{\rho j}\langle\xi\rangle^{\rho j}} \leq C_0\Big(\frac{L^{j/\rho}M_j}{h^j\langle\xi\rangle^j}\Big)^\rho.$$

Clearly, we may assume that $h > L^{1/\rho}$. Now Lemma 5.3.2 implies that for some $s > 0$:

$$|\partial_x^\beta a(x,\xi)| \leq C_0 C h^{-|\beta|} A_{|\beta|} e^{-\rho M(hs\langle\xi\rangle)}, \ p \in \mathbb{N} \text{ and } \langle\xi\rangle > BM_1.$$

On the other hand, by Lemma 1.3.1, an arbitrary $u \in \mathcal{E}'^{\{M_p\}}(\mathbb{T}^n)$ satisfies that for every $l > 0$ there exists $C_l > 0$ such that $|\hat{u}(\xi)| \leq C_l e^{M(l\langle\xi\rangle)}$. Therefore,

$$|\partial_x^\beta a(x,\xi)\hat{u}(\xi)| \leq C_0 C C_l h^{|\beta|} A_{|\beta|} e^{-\rho M(sh\langle\xi\rangle)} e^{M(l\langle\xi\rangle)}.$$

It remains to find a suitable $l > 0$ such that

$$\sum_{\xi\in\mathbb{Z}^n} e^{-\rho M(hs\langle\xi\rangle)} e^{M(l\langle\xi\rangle)}$$

is finite. In order to do that, we apply [33, Lemma 3.5], which holds provided that $(M.2)$ is satisfied. Namely, we have the inequality

$$2M(t) \leq M(Ht) + \log C_0, \qquad \forall t > 0, \tag{5.3.11}$$

where the constant $H > 0$ is the same as the one occurring in $(M.2)$. For a given $\lambda > 1$, iterating the previous inequality $n$ times, where $n \in \mathbb{N}$ with the property that $2^{n-1} \leq \lambda < 2^n$, we obtain

$$\lambda M(t) \leq M(\lambda^a t) + 2\lambda \log C_0$$

where $a$ is the constant $a = \log H/\log 2$. If $\lambda = 1/\rho$, renaming $\lambda^a t = u$ we obtain

$$-\rho M(u) \leq -M(u\rho^a) + 2\log C_0.$$

Now we have

$$e^{-\rho M(hs\langle\xi\rangle)} e^{M(l\langle\xi\rangle)} \leq C_0^2 e^{-M(\rho^a hs\langle\xi\rangle) + M(l\langle\xi\rangle)}.$$

If $l > 0$ satisfies $\rho^a hs = Hl$, we can use again (5.3.11) to conclude that the series with terms

$$e^{-M(\rho^a hs\langle\xi\rangle)+M(l\langle\xi\rangle)} \leq C_0 e^{-M(l\langle\xi\rangle)}$$

converges absolutely and the result now follows at once. □

Motivated by the last two results, then we have the right to say that a (formal) sum is an *asymptotic expansion* of a symbol $a$ if $a(x,\xi) \sim \sum_{j=0}^{\infty} a_j(x,\xi)$.

# Nederlandstalige samenvatting

In deze dissertatie zijn we geïnteresseerd in verscheidene eigenfunctie expansies van ultradifferentieerbare functies en ultradistributies. Daarenboven worden toroïdale pseudodifferentiaal operatoren van eindige orde die continu werken op zekere klassen van ultradifferentieerbare functies en ultradistributies op de torus $\mathbb{T}^n$ bestudeerd in het laatste hoofdstuk.

In Hoofdstuk 1 presenteren we de wiskundige tools die doorheen de dissertatie zullen gebruikt worden.

In Hoofdstuk 2 presenteren we een theorie van ultradistributionele randwaarden voor harmonische functies gedefinieerd op de Euclidische eenheidsbal $\mathbb{B}^n$. We veralgemenen de resultaten van Estrada en Kanwal [21] betreffende de expansie van distributies in sferische harmonieken naar de context van ultradistributies en bestuderen de ultradistributionele randwaarden van harmonische functies op de eenheidssfeer $\mathbb{S}^{n-1}$. Ons doel is de harmonische functies $U$, gedefinieerd op de eenheidsbal, te karakteriseren die een randwaarde $\lim_{r\to 1^-} U(r\omega)$ hebben in de ruimte van ultradistributies $\mathcal{E}^{*\prime}(\mathbb{S}^{n-1})$. Het eerste resultaat in dit hoofdstuk zijn expliciete begrenzingen voor de partiële afgeleiden van de sferische harmonieken; dit verfijnt eerder werk van Calderón and Zygmund. Deze begrenzingen laten ons toe om ultradifferentieerbare functies en ultradistributies op de sfeer te karakteriseren in termen van hun expansie in sferische harmonieken. Op basis van deze karakterisatie ontwikkelen we dan de gewenste theorie van ultradistributionele randwaarden voor harmonische functies. Ten slotte, gebruiken we onze resultaten om de drager van ultradistributies op de sfeer te karakteriseren in termen van de Abel sommeerbaarheid van hun expansie in sferische harmonieken.

In Hoofdstuk 3 bestuderen we rotationeel invariante ultradistributies. Karakterisaties van rotationeel invariante ultradistributies en hyperfuncties werden gegeven door Chung en Na in [13]. Meer precies toonden zij aan dat een niet-quasianalytische ultradistributie of hyperfunctie $f$ rotationeel invariant is als en slechts als $f$ gelijk is aan zijn sferisch gemiddelde. Wij tonen aan dat de karakterisatie van rotationeel invariante ultradistributies in termen van hun sferisch gemiddelde blijft gelden in het quasianalytische geval. Onze methode verschilt van die van Chung en Na en geeft een nieuw bewijs van hun voornoemde resultaten.

In Hoofdstuk 4 karakteriseren we de elementen van de algemene Gelfand-Shilov ruimten $\mathcal{S}^{\{M_p\}}_{\{M_p\}}(\mathbb{R}^n)$ en $\mathcal{S}^{(M_p)}_{(M_p)}(\mathbb{R}^n)$ in termen van begrenzingen van hun Fourier coëfficiënten met betrekking tot eigenfunctie expansies ten opzichte van normale globaal elliptische differentiaal operatoren van Shubin type. We tonen ook aan dat de eigenfuncties van zulke operatoren absolute Schauder basissen vormen voor deze ruimten van ultradifferentieerbare functies. Onze karakterisatie veralgemeent eerder werk van Gramchev et al. [26]. Daarenboven kan ze ook beschouwd worden als het analogon in $\mathbb{R}^n$ van recent werk van Dasgupta and Ruzhansky in de context van compacte variëteiten.

Ten slotte, in Hoofdstuk 5 presenteren we een theorie van toroïdale pseudodifferentiaal operatoren die continu werken op klassen van ultradifferentieerbare functies en ultradistributies op de torus $\mathbb{T}^n$; zowel het Beurling als het Roumieu geval worden behandeld. We volgen de methode van [41, 58], waar een gelijkaardige theorie op $\mathbb{R}^n$ wordt ontwikkeld. Nadat we de klassen van symbolen voor onze operatoren hebben gedefinieerd en kort hun topologie bespreken, ontwikkelen we een symbolische calculus en gebruiken deze om aan te tonen hoe we een operator kunnen verkrijgen uit een formele som van symbolen.

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