Choice functions and rejection sets

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Abstract We establish an equivalent representation of coherent choice functions in terms of a family of rejection sets, and investigate how each of the coherence axioms translates into this framework. In addition, we show that this family allows to simplify the verification of coherence in a number of particular cases.

1 Introduction

Coherent choice functions constitute an uncertainty model that is more general than sets of desirable gambles, while still preserving some of their nice properties, such as being able to deal effectively with sets of probability zero when conditioning. One of their drawbacks is the technical difficulty of verifying the coherence axioms. In this paper, we try to remedy this situation somewhat by providing an equivalent representation of choice functions in terms of those option sets that allow a subject to reject the zero gamble, which may be interpreted as those option sets that he should consider preferable to the status quo. As we shall see, this representation, in addition to capturing more intuitively the ideas underlying coherence, also helps to simplify the verification of coherence in a number of particular cases.

   This paper is organized as follows: in Section 2, we recall the basic aspects of coherent choice functions that we shall need in the rest of the paper. Our representation in terms of rejection sets is established in Section 3, where we also discuss two additional properties that seem of interest for choice functions. In Section 4, we look in more detail at a number of particular cases: choice functions on binary spaces (that is, when the experiment on which the outcomes of the options depend on can only take two values) and those defined by means of

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coherent sets of desirable gambles. The paper concludes with some additional remarks in Section 5.

2 Coherent choice functions

Let Ω be a possibility space. A *gamble* on Ω is a bounded map $f : \Omega \to \mathbb{R}$. We denote by $\mathscr{L}$, the set of all gambles on Ω. Gambles will also be called *options*. For any two gambles f and g, we denote $f \leq g$ if $f(\omega) \leq g(\omega)$ for every ω in Ω, and we collect all the gambles f for which $f \leq 0$ in $\mathscr{L}_{\leq 0}$. We let $f < g$ if $f \leq g$ and $f \neq g$, and collect all the gambles f for which $f < 0$ in $\mathscr{L}_{<0}$, and the gambles f for which $f > 0$ in $\mathscr{L}_{>0}$.

Choice functions are defined on finite collections of gambles. We collect all those collections in the set $\mathscr{Q}$.

Definition 1. A *choice function C* on a possibility space Ω is a map

$$C \colon \mathscr{Q} \to \mathscr{Q} \cup \{\emptyset\} \colon A \mapsto C(A) \text{ such that } C(A) \subseteq A.$$

We collect all the choice functions on Ω in $\mathscr{C}(\Omega)$, often denoted as $\mathscr{C}$ when the possibility space is clear from the context.

The idea underlying this simple definition is that a choice function C selects the set $C(A)$ of 'best' options in the *option set A*. Our definition resembles the one commonly used in the literature (Aizerman, 1985; Seidenfeld et al., 2010; Sen, 1977), except perhaps for an also not entirely unusual restriction to *finite* option sets (He, 2012; Schwartz, 1972; Sen, 1971).

Equivalently to a choice function C, we may consider its associated *rejection function R*, defined by $R(A) := A \setminus C(A)$ for all A in $\mathscr{Q}$. It returns the options $R(A)$ that are rejected—not selected—by C.

We focus here on a special class of choice functions, which we call *coherent*.

Definition 2. We call a choice function C on Ω *coherent* if for all A, A_1 and A_2 in $\mathscr{Q}$, all f and g in $\mathscr{L}$, and all λ in $\mathbb{R}_{>0}$:[1]
C_1. $C(A) \neq \emptyset$;
C_2. if $f < g$ then $\{g\} = C(\{f, g\})$;
C_3. a. if $C(A_2) \subseteq A_2 \setminus A_1$ and $A_1 \subseteq A_2 \subseteq A$ then $C(A) \subseteq A \setminus A_1$;
       b. if $C(A_2) \subseteq A_1$ and $A \subseteq A_2 \setminus A_1$ then $C(A_2 \setminus A) \subseteq A_1$;
C_4. a. if $A_1 \subseteq C(A_2)$ then $\lambda A_1 \subseteq C(\lambda A_2)$;
       b. if $A_1 \subseteq C(A_2)$ then $A_1 + \{f\} \subseteq C(A_2 + \{f\})$;

These axioms are a subset of the ones introduced in Seidenfeld et al. (2010), duly translated from horse lotteries to gambles. We have omitted two of the coherence axioms from Seidenfeld et al. (2010): one is the *Archimedean* axiom, be-

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[1] By $\mathbb{R}_{>0}$ we mean all the (strictly) positive real numbers.

cause it is not fully compatible with the idea of deriving choice functions from coherent sets of desirable gambles (Van Camp et al., 2015), which is one of the goals in our approach. The other one, which we shall consider later on, is the so-called *convexity* axiom. Although this axiom leads to a number of useful properties, and in particular to a connection with lexicographic probability systems (Van Camp et al., 2016), we have refrained from including it in the list of coherence axioms because it not satisfied by some interesting choice functions.

Equivalent formulations of these axioms, better suited for our subsequent proofs, are the following:

$$(C_3a) \Leftrightarrow (\forall A, A' \in \mathscr{Q}, \forall f \in A)\big((f \in R(A), A \subseteq A') \Rightarrow f \in R(A')\big),$$
$$(C_3b) \Leftrightarrow (\forall A \in \mathscr{Q}, \forall f \in A)\big(\{0, f\} \subseteq R(A) \Rightarrow 0 \in R(A \setminus \{f\})\big),$$
$$(C_4a) \Leftrightarrow (\forall A \in \mathscr{Q}, \forall \lambda > 0) R(\lambda A) = \lambda R(A),$$
$$(C_4b) \Leftrightarrow (\forall A \in \mathscr{Q}, \forall f \in \mathscr{L}) R(A + f) = R(A) + f.$$

3 A representation in terms of rejection sets

Next we give an equivalent representation of choice functions in terms of *rejection sets*. For any $f \in \mathscr{L}$ and any natural number i, we define

$$\mathbb{K}_f^i := \{A : f \in R(A), |A| = i\} \text{ and } \mathbb{K}_f := \cup_{i \in \mathbb{N}} \mathbb{K}_f^i. \tag{1}$$

We are going to characterize coherent choice functions in terms of these rejection sets. Our first result shows that we can restrict our attention to the case $f = 0$:

Proposition 1. *Let C be a choice function and consider the family of option sets* $\{\mathbb{K}_f : f \in \Omega\}$ *it induces by means of Eq.* (1). *Then*

$$C \text{ satisfies Axiom } C_4b \Leftrightarrow (\forall f \in \mathscr{L}) \, \mathbb{K}_0 + f = \mathbb{K}_f.$$

Proof. For necessity, consider an option set A that includes 0. Then the option set $A + f$ includes f, and since by C_4b it holds that $R(A + f) = R(A) + f$, we conclude that $A \in \mathbb{K}_0$ if and only if $A + f \in \mathbb{K}_f$.

Conversely, for sufficiency, consider an option set A and a gamble f. Take any $g \in R(A)$, then $A \in \mathbb{K}_g$, whence by assumption $A - g \in \mathbb{K}_0$ and as a consequence $A - g + (f + g) = A + f \in \mathbb{K}_{f+g}$. Then indeed $g + f \in R(A + f)$, whence Axiom C_4b holds. □

Taking this result into account, in what follows we shall restrict our attention to rejection sets $\mathbb{K}$ for which $\mathbb{K}_0 + f = \mathbb{K}_f$ for every f in $\mathscr{L}$. We can then simplify the notation above to

$$K^i := \mathbb{K}_0^i = \{A : 0 \in R(A), |A| = i\} \text{ and } K := \mathbb{K}_0 = \{A : 0 \in R(A)\}, \tag{2}$$

respectively, and denote $\mathcal{Q}_0$ the family of option sets that include the zero gamble. Our next result provides a characterisation of the different coherent axioms in terms of these sets:

Proposition 2. *Let C be a choice function satisfying Axiom C_4b, and consider the sets K^i, K defined in Eq. (2).*
1. C satisfies Axiom C_1 if and only if $(\forall A \in \mathcal{Q}_0)(\exists f \in A) A - f \notin K$.
2. C satisfies Axiom C_2 if and only if $(\forall f \in \mathcal{L}_{>0})\{f, 0\} \subseteq K^2$.
3. C satisfies Axiom C_3a if and only if $(\forall A \in K, \forall A' \in \mathcal{Q}_0)(A \subseteq A' \Rightarrow A' \in K)$.
4. C satisfies Axiom C_3b if and only if $(\forall A \in K, \forall f \in A)(A - f \in K \Rightarrow A \setminus \{f\} \in K)$.
5. C satisfies Axiom C_4a if and only if $(\forall A \in \mathcal{Q}_0, \forall \lambda > 0)(A \in K \Leftrightarrow \lambda A \in K)$.

Proof. 1. Taking Axiom C_4b into account, Axiom C_1 holds if and only if $C(A) \neq \emptyset$ for every $A \in \mathcal{Q}_0$. This in turn is equivalent to $(\exists f \in A) f \in C(A)$, which by C_4b is equivalent to $0 \in C(A - f)$ or, in other words, to $A - f \notin K$.
2. Under Axiom C_4b, Axiom C_2 is equivalent to $(\forall f \in \mathcal{L}_{>0})\{f\} = C(\{0, f\})$, or, in other words, to $(\forall f \in \mathcal{L}_{>0})\{f, 0\} \subseteq K^2$.
3. For necessity, consider any A in K and any A' in $\mathcal{Q}_0$ such that $A' \supseteq A$. Because $A \in K$, $0 \in R(A)$, whence, by Axiom C_3a, $0 \in R(A')$. Then indeed $A' \in K$.
   Conversely, for sufficiency, consider any A and A' in $\mathcal{Q}_0$ such that $A \subseteq A'$, and any f in $R(A)$. Then by Axiom C_4b, $0 \in R(A - f)$, so $A - f \in K$, whence also $A' - f \in K$, because $A' - f \supseteq A - f$. Then $0 \in R(A' - f)$, and applying again C_4b, indeed $f \in R(A')$.
4. For necessity, consider any A in K and f in A such that $A - f \in K$. Then $0 \in R(A - f)$, whence $f \in R(A)$, by Axiom C_4b. Applying Axiom C_3b, we deduce that $0 \in R(A \setminus \{f\})$, whence indeed $A \setminus \{f\} \in K$.
   Conversely, for sufficiency, consider any A in $\mathcal{Q}$ and f in A such that $\{0, f\} \subseteq R(A)$. Then $A \in K$ and by Axiom C_4b, $f \in R(A)$ implies that $0 \in R(A - f)$, so $A - f \in K$. Then $A \setminus \{f\} \in K$, or, in other words, indeed $0 \in R(A \setminus \{f\})$.
5. It suffices to note that under Axiom C_4b, Axiom C_4a is equivalent to $0 \in C(A) \Leftrightarrow 0 \in C(\lambda A)$ for every $\lambda > 0$ and every $A \in \mathcal{Q}_0$.   □

An immediate consequence is:

Corollary 1. *A choice function C is coherent if and only if it satisfies Axiom C_4b and the rejection set K it induces by Eq. (2) is increasing, scale invariant, includes $\{f, 0\}$ for every $f \in \mathcal{L}_{>0}$ and it satisfies the following two properties:*
• $(\forall A \in \mathcal{Q}_0)(\exists f \in A) A - f \notin K$.
• $(\forall A \in K, \forall f \in A)(A - f \in K \Rightarrow A \setminus \{f\} \in K)$.

Next we consider a couple of additional consistency axioms that were deemed interesting by Van Camp et al. (2016). The first one is the *convexity* axiom, which is given by:
C_5. if $A \subseteq A_1 \subseteq \text{CH}(A)$ then $C(A) \subseteq C(A_1)$, for all A and A_1 in $\mathcal{Q}$,
where $C(A) := \{\sum_{i=1}^{n} \alpha_i f_i : n \in \mathbb{N}, f_i \in A, \alpha_i \geq 0, \sum_{i=1}^{n} \alpha_i = 1\}$ is the *convex hull* of A.
In terms of rejection sets, it is characterized by the following proposition:

Proposition 3. *Let C be a choice function satisfying Axiom* C_4b. *Then C satisfies Axiom* C_5 *if and only if* $(\forall A_1 \in K, \forall A \in \mathcal{Q}_0)(A \subseteq A_1 \subseteq CH(A) \Rightarrow A \in K)$.

Proof. For necessity, application of Axiom C_5 tells us that, whenever $A \subseteq A_1 \subseteq CH(A)$ hold, $0 \in R(A_1)$ implies that $0 \in R(A)$, or, in other words, $A_1 \in K$ implies that $A \in K$.

Conversely, for sufficiency, consider two option sets A and A_1 such that $A \subseteq A_1 \subseteq CH(A)$, and let us show that $C(A) \subseteq C(A_1)$. Assume *ex absurdo* that there is some $f \in A$ such that $f \in R(A_1)$ and $f \in C(A)$. Then since $A - f \subseteq A_1 - f \subseteq CH(A - f)$, we can apply axiom C_4b and assume that, without loss of generality, $f = 0$. But then we obtain that $A_1 \in K$ while $A \notin K$, a contradiction. $\square$

A weaker property that is also useful is the so-called *separate homogeneity*, which means that for all n in $\mathbb{N}$, all $f_1, f_2, \ldots, f_n$ in $\mathscr{L}$ and all $\mu_1, \mu_2, \ldots \mu_n$ in $\mathbb{R}_{>0}$:

$$0 \in C(\{0, f_1, f_2, \ldots, f_n\}) \Leftrightarrow 0 \in C(\{0, \mu_1 f_1, \mu_2 f_2, \ldots, \mu_n f_n\}). \tag{3}$$

This property follows from axioms C_3a, C_4a, C_5 (Van Camp et al., 2016, Proposition 1). Moreover, and unlike C_5 that is linked to *lexicographic* choice functions, separate homogeneity is compatible with maximality as a decision rule, and therefore better suited for connecting choice functions with desirability. Furthermore, separate homogeneity is strictly weaker: there are classes of interesting coherent choice functions that satisfy Eq. (3) but not Axiom C_5. In terms of the rejection sets, it is trivial to prove that it can be expressed in the following manner:

Proposition 4. *Let C be a choice function satisfying Axiom* C_4b. *It satisfies separate homogeneity if and only if for all n in $\mathbb{N}$, all $f_1, f_2, \ldots, f_n$ in $\mathscr{L}$ and all $\mu_1, \mu_2, \ldots \mu_n$ in $\mathbb{R}_{>0}$, $\{0, f_1, f_2, \ldots, f_n\} \in K \Leftrightarrow \{0, \mu_1 f_1, \mu_2 f_2, \ldots, \mu_n f_n\} \in K$.*

4 Particular cases

In this section, we consider a number of particular cases of choice functions for which the representation in terms of rejection sets simplifies somewhat.

4.1 Coherent choice functions defined via maximality

We begin by considering choice functions defined via Walley's notion of maximality (Van Camp et al., 2015). A set of gambles $\mathscr{D}$ is called *coherent* when it is a convex cone that includes all non-negative gambles and does not include the zero gamble. We refer to (Walley, 1991; Miranda and Zaffalon, 2010; Quaeghebeur, 2014) for a study of the notion of desirability and its variants. In particular, any coherent set of desirable gambles can be used to define a coherent choice function, by means of the formula

$$C_{\mathscr{D}}(A) := \{f \in A : (\forall g \in A) g - f \notin \mathscr{D}\}. \tag{4}$$

Unlike general choice functions, the ones defined in the manner above are uniquely determined by binary comparisons. Thus, it is not surprising that for them the representation in terms of rejection sets takes a simpler form:

Proposition 5. *Let $\mathscr{D}$ be a coherent set of gambles and let $C_{\mathscr{D}}$ be the coherent choice function it induces by Eq. (4). Then $K = \{A \in \mathscr{Q}_0 : (\exists A_1 \in K^2) A_1 \subseteq A\}$ and $K^2 = \{\{0, f\} : f \in \mathscr{D}\}$.*

Proof. Consider an option set A in K. By Eq. (4), $0 \in R_{\mathscr{D}}(A)$ if and only if $A \cap \mathscr{D} \neq \emptyset$. If $|A| = 2$, then $A = \{0, f\}$ for some f in $\mathscr{D}$, and as a consequence $K^2 \supseteq \{\{0, f\} : f \in \mathscr{D}\}$. Conversely, consider any $A' \in K^2$. Then $A' = \{0, g\}$ for some g in $\mathscr{L}$. But since $0 \in R_{\mathscr{D}}(A')$, we have $g \in \mathscr{D}$, so $K^2 \subseteq \{\{0, f\} : f \in \mathscr{D}\}$, proving that indeed $K^2 = \{\{0, f\} : f \in \mathscr{D}\}$. If, on the other hand, $|A| \geq 3$, then $A \supseteq \{0, f\}$ for some f in $\mathscr{D}$. But then $0 \in R_{\mathscr{D}}(\{0, f\})$, so $A \supseteq A'$ for some $A' \in K^2$, and therefore indeed $K = \{A \in \mathscr{Q}_0 : (\exists A_1 \in K^2) A_1 \subseteq A\}$. $\square$

4.2 Coherent choice functions on binary spaces

Next, we consider coherent choice functions defined on binary spaces. It turns out that, under separate homogeneity, they are determined by rejection sets of cardinality two or three:

Proposition 6. *Let C be a coherent choice function on $\Omega = \{a, b\}$. If C satisfies Eq. (3), then*

$$K = \left\{A \in \mathscr{Q}_0 : (\exists A_1 \in K^2 \cup K^3) A_1 \subseteq A\right\}$$

Proof. Let us prove that for every A in K there exists a A_1 in $K^2 \cup K^3$ for which $A_1 \subseteq A$.

Consider thus A in K. By Axiom C_2, we find that $A \cap \mathscr{L}_{<0} \subseteq R(A \cap \mathscr{L}_{\leq 0})$, so Axiom C_3a implies that then $A \cap \mathscr{L}_{<0} \subseteq R(A)$. Since $A \in K$ and therefore also $0 \in R(A)$, by Axiom C_3b we find that then $0 \in (A \cap \mathscr{L}_{<0}^c)$, so we can assume without loss of generality that $A \cap \mathscr{L}_{<0} = \emptyset$. There are two possibilities.

If $A \cap \mathscr{L}_{>0} \neq \emptyset$, then for any f in $A \cap \mathscr{L}_{>0}$ it follows from Axiom C_2 that $0 \in R(\{0, f\})$, whence the set $\{0, f\} \subseteq A$ belongs to K^2. So we find indeed that $A_1 := \{0, f\}$ in K^2 for which $A_1 \subseteq A$.

If $A \cap \mathscr{L}_{>0} = \emptyset$, then we can denote $A = \{f_1, \ldots, f_n, g_1, \ldots, g_m\}$ for some $n \geq 0$ and $m \geq 0$ but $\max\{m, n\} \geq 1$, where f_i belongs to the second quadrant (i.e., $f_i(a) < 0 < f_i(b)$) for every i in $\{1, \ldots, n\}$ and g_j belongs to the fourth quadrant (i.e., $g_j(a) > 0 > g_j(b)$) for every j in $\{1, \ldots, m\}$. Let $\lambda_i := \frac{-1}{f_i(a)}$ and $\mu_j := \frac{1}{g_j(a)}$ for every i in $\{1, \ldots, n\}$ and j in $\{1, \ldots, m\}$. Then, applying Eq. (3),

$$0 \in R(\{0, \lambda_1 f_1, \ldots, \lambda_n f_n, \mu_1 g_1, \ldots, \mu_m g_m\}).$$

Infer that $\lambda_i f_i(a) = -1$ for every i in $\{1, \ldots, n\}$. Letting $i^* := \arg\max\{\lambda_i f_i(b) : i \in \{1, \ldots, n\}\}$, we infer that

$$\lambda_k f_k(b) < \lambda_{i^*} f_{i^*}(b) \Rightarrow \lambda_k f_k \in R(\{\lambda_k f_k, \lambda_{i^*} f_{i^*}\}) \Rightarrow \lambda_k f_k \in R(A),$$

where last implication follows from Axiom $C_3 a$. similarly, $\mu_j g_j(a) = 1$ for every j in $1, \ldots, m$, and letting $j^* := \arg\max\{\mu_j g_j(b) : j \in \{1, \ldots, m\}\}$, we infer that

$$\mu_j g_j(b) < \mu_{j^*} g_{j^*}(b) \Rightarrow \mu_j g_j \in R(\{\mu_j g_j, \mu_{j^*} g_{j^*}\}) \Rightarrow \mu_j g_j \in R(A),$$

where again last implication follows from Axiom $C_3 a$. If we now apply $C_3 b$, we deduce that $0 \in R(\{0, \lambda_{i^*} f_{i^*}, \mu_{j^*} g_{j^*}\})$, whence $0 \in R(\{0, f_{i^*}, g_{j^*}\})$, applying Eq. (3). Thus, there is a subset of A with cardinality three that also belongs to K. □

A key property in the proof of Proposition 6 is that separate homogeneity, together with Axiom C_2, allows to assume without loss of generality that an option set A that includes the zero gamble has at most one gamble f in the second quadrant (for which $f(a) < 0 < f(b)$) and one g in the fourth quadrant (for which $g(a) > 0 > g(b)$). Let us show that this need not happen without separate homogeneity:

Example 1. Consider $\Omega = \{a, b\}$ and let $\mathscr{D}$ be the coherent set of gambles $\mathscr{D} := \{f \in \mathscr{L} : f(a) < 0 < f(b) \text{ and } f(a) + f(b) > 0\} \cup \mathscr{L}_{>0}$. Let C be the choice function determined by the rejection function

$$0 \in R(A) \Leftrightarrow \left(A \cap \mathscr{D} \neq \emptyset \text{ or } (\exists \lambda_1 > \lambda_2 > 0)\{(-\lambda_1, \lambda_1), (-\lambda_2, \lambda_2)\} \subseteq A \right) \tag{5}$$

for all A in $\mathscr{Q}_0$. We extend the domain of R to $\mathscr{Q}$ by letting $f \in R(A) \Leftrightarrow 0 \in R(A - f)$ for all A in $\mathscr{Q}$ and f in A. Remark already that $(-\lambda, \lambda)$ lies on the border of $\mathscr{D}$ for every $\lambda > 0$: indeed, for every g in $\mathscr{D}$ we have that $(-\lambda, \lambda) + g \in \mathscr{D}$.

Let us show that C is a coherent choice function. Taking into account the last part of the definition, we see that $C_4 b$ holds, and we can restrict our attention to option sets in $\mathscr{Q}_0$. We show that C satisfies Axioms C_2, $C_3 a$, $C_3 b$, $C_4 a$, and C_1, in this order.

For Axiom C_2, consider any f in $\mathscr{L}_{>0}$. Then $f \in \mathscr{D}$, so indeed $0 \in R(\{0, f\})$.

For Axiom $C_3 a$, consider any A and A' in $\mathscr{Q}_0$ such that $A \subseteq A'$, and any f in $R(A)$. Using Axiom $C_4 b$, then $0 \in R(A - f)$, whence $(A - f) \cap \mathscr{D} \neq \emptyset$ or $\{(-\lambda_1, \lambda_1), (-\lambda_2, \lambda_2)\} \subseteq A - f$ for some $\lambda_1 > \lambda_2 > 0$. But $A' - f \supseteq A - f$, so also $(A' - f) \cap \mathscr{D} \neq \emptyset$ or $\{(-\lambda_1, \lambda_1), (-\lambda_2, \lambda_2)\} \subseteq A' - f$, and therefore $0 \in R(A' - f)$, whence, again by Axiom $C_4 b$, indeed $0 \in R(A')$.

For Axiom $C_3 b$, consider any A in $\mathscr{Q}_0$ and any f in A such that $\{0, f\} \subseteq R(A)$. We need to prove that then $0 \in R(A \setminus \{f\})$. Since $f \in R(A)$, then

(i) $\mathscr{D} \cap (A - f) \neq \emptyset$, or (ii) $\{(-\lambda_1, \lambda_1), (-\lambda_2, \lambda_2)\} \subseteq A - f$ for some $\lambda_1 > \lambda_2 > 0$.

Furthermore, since $0 \in R(A)$, then $\mathscr{D} \cap A \neq \emptyset$, or $\{(-\lambda_1, \lambda_1), (-\lambda_2, \lambda_2)\} \subseteq A$ for some $\lambda_1 > \lambda_2 > 0$. If $f \notin \mathscr{D} \cap A$ and $f \notin \{(-\lambda_1, \lambda_1), (-\lambda_2, \lambda_2)\}$, then also $\mathscr{D} \cap A \setminus \{f\} \neq \emptyset$ or

$\{(-\lambda_1, \lambda_1), (-\lambda_2, \lambda_2)\} \subseteq A \setminus \{f\}$, whence $0 \in R(A \setminus \{f\})$. So assume that (a) $f \in \mathscr{D}$ or (b) $f = (-\lambda, \lambda)$ for some $\lambda > 0$.

If (a) $f \in \mathscr{D}$, then (i) or (ii) must be the case. If (i) occurs, then there is some gamble g in $(\mathscr{D} + f) \cap A$, whence $g - f \in \mathscr{D}$ for some g in A. But since $f \in \mathscr{D}$, also $g = f + g - f \in \mathscr{D}$, and therefore $0 \in R(\{0, g\})$, whence by Axiom C$_3$a indeed $0 \in R(A \setminus \{f\})$. If (ii) occurs, then there are $\lambda_1 > \lambda_2 > 0$ such that $f + (-\lambda_1, \lambda_1), f + (-\lambda_2, \lambda_2) \in A$, whence, since $f \in \mathscr{D}$, by construction also $f + (-\lambda_1, \lambda_1) \in \mathscr{D}$. Therefore $0 \in R(\{0, f + (-\lambda_1, \lambda_1)\})$, whence by Axiom C$_3$a, also $0 \in R(A \setminus \{f\})$.

If (b) $f = (-\lambda, \lambda)$ for some $\lambda > 0$, then, similarly, (i) or (ii) must be the case. If (i) occurs, then there is some g in A such that $g - f \in \mathscr{D}$. Therefore by construction also $g = f + g - f \in \mathscr{D}$, whence $0 \in R(\{0, g\})$, and then by Axiom C$_3$a, also $0 \in R(A \setminus \{f\})$. If (ii) occurs, then there is some $\lambda_1 > 0$ and $\lambda_2 > 0$ for which $\{f + (-\lambda_1, \lambda_1), f + (-\lambda_2, \lambda_2)\} = \{(-\lambda - \lambda_1, \lambda + \lambda_1), f + (-\lambda - \lambda_2, \lambda + \lambda_2)\} \subseteq A$. Letting $\lambda_1' := \lambda + \lambda_1$ and $\lambda_2' := \lambda + \lambda_2$, we find that $\{(-\lambda_1', \lambda_1'), f + (-\lambda_2', \lambda_2')\} \subseteq A \setminus \{f\}$, whence $0 \in R(A \setminus \{f\})$.

Axiom C$_4$a follows from Equation (5), taking into account that $\mathscr{D}$ is a cone.

Finally, for Axiom C$_1$, assume *ex absurdo* that $C(A) = \emptyset$ for some A in $\mathscr{Q}_0$. Then $A = R(A)$ whence, by Axiom C$_3$b, $0 \in R(\{0\})$. But $0 \notin \mathscr{D}$ and $(-\lambda, \lambda) \notin \{0\}$ for every $\lambda > 0$, so $0 \notin R(\{0\})$, a contradiction.

On the other hand, it follows by Equation (5) that, given the option set $A = \{0, (-1, 1), (-2, 2)\}$, we obtain $C(A) = \{(-1, 1), (-2, 2)\}$. However, the same equation implies that $0 \in C(\{0, (-1, 1)\})$. This shows that C does not satisfy separate homogeneity, and also that we cannot reduce the intersection with the second quadrant to only one gamble. ♦

On the other hand, Proposition 6 also depends crucially on the assumption that $|\Omega| = 2$, as our next example shows:

Example 2. Consider a ternary space Ω, some n in $\mathbb{N}$, and let f_k be the gamble given by $f_k := (-1, \frac{k}{n}, -\frac{k^2}{n^2})$, for all k in $\{1, \ldots, n\}$. Let us show that for each k we can find a probability measure whose expectation operator P_k (called *linear prevision* in Walley's terminology) satisfies $P_k(f_k) > 0 > P_k(f_j)$ for every j in $\{1, \ldots, n\} \setminus \{k\}$.

To find such expectation operators, let P be the expectation operator associated with the mass function $(0, \frac{2k}{n+2k}, \frac{n}{n+2k})$. Then $P(f_k - f_j) = \frac{k-j}{n(n+2k)}(2k - (k+j))$, whence $P(f_k - f_j) > 0$ if $k \neq j$. Moreover, $P(f_k) = \frac{k^2}{n(n+2k)} > 0$.

If we now consider any $\lambda \in (0, 1)$ and define P_k as the expectation operator associated with the mass function $(\lambda, (1 - \lambda)\frac{2k}{n+2k}, (1 - \lambda)\frac{n}{n+2k})$, we obtain $P(f_k - f_j) = (1 - \lambda)P(f_k - f_j) > 0$ whenever $k \neq j$. Moreover,

$$P_k(f_k) = -\lambda + (1 - \lambda)P(f_k) > 0 \Leftrightarrow \lambda < \frac{P(f_k)}{1 + P(f_k)},$$

and similarly

$$P_k(f_j) = -\lambda + (1 - \lambda)P(f_j) < 0 \Leftrightarrow \lambda > \frac{P(f_j)}{1 + P(f_j)}.$$

Since, for every j in $\{1,\ldots,n\}\setminus\{k\}$, $\frac{P(f_j)}{1+P(f_j)} < \frac{P(f_k)}{1+P(f_k)}$ because $P(f_j) < P(f_k)$, we let

$$\lambda \in \left(\max_{j \in \{1,\ldots,n\}\setminus\{k\}} \frac{P(f_j)}{1 + P(f_j)}, \frac{P(f_k)}{1 + P(f_k)} \right),$$

and for this λ we obtain $P_k(f_k) > 0 > P_k(f_j)$ for every $j \neq k$.

Now, let $\mathscr{D}_k$ be the coherent set of gambles given by $\mathscr{D}_k := \{f \in \mathscr{L} : P_k(f) > 0\}$, and let $C_{\mathscr{D}_k}$ be the coherent choice function it induces by Eq. (4). Then the choice function C given by $C(A) := \bigcup_{k=1}^n C_{\mathscr{D}_k}(A)$ is also coherent (Van Camp et al., 2015, Proposition 3), and it can be checked to satisfy separate homogeneity because all $C_{\mathscr{D}_k}$ do. If we now consider the option set $A = \{0, f_1, \ldots, f_n\}$, we get that $C_{\mathscr{D}_k}(A) = \{f_k\}$ for every k, since $P_k(f_k) > 0 > P_k(f_j)$ implies that $f_k, f_k - f_j \in \mathscr{D}_k$ for every j. As a consequence, we obtain $C(A) = \{f_1, \ldots, f_n\}$, whence $A \in K$. However, for every k it holds that $C_{\mathscr{D}_k}(A \setminus \{f_k\}) = \{0\}$, using again that $P_k(f_j) < 0$ for every $j \neq k$, and therefore $C(A \setminus \{f_k\}) = A \setminus \{f_k\}$. Thus, A has no proper subset that also belongs to the rejection class K. ♦

5 Conclusions

It is a consequence of coherence that a choice function is uniquely determined by those option sets that allow us to reject the zero gamble, i.e., those that are considered preferable to the status quo. In this paper, we have investigated the structure of these sets and shown that the coherence axioms can be expressed more intuitively in terms of these sets. In addition, we have shown that all the necessary information is given by option sets of cardinality two when the choice function is defined via maximality, and with cardinality two or three in most (but not all) cases of interest when the possibility space is binary. Moreover, we have shown that this last result does not extend to larger possibility spaces; thus, determining an analogous representation for arbitrary spaces would be the main open problem for the future.

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On Pedro Gil

I was lucky to meet Pedro Gil in many different roles: first as a teacher, then as head of department, later as a supervisor, and finally, and all throughout, as a friend. He helped me in my first research work, on robust statistics for my final project during my BsC; and we taught together in a course on *Mathematics for everyday life* until a few weeks prior to his death.

He was brilliant as a professor, generous as a researcher, and charismatic as a leader. Few bring together unanimity in the manner he did; and still, or perhaps because of it, he was always unassuming: he would treat you the in the same manner irrespective of your position, and would make you feel that, whatever your problems, someone cared and would try to help. Because above all his many qualities, he had one that few possess: he was a good man.

He has not left a void because he is still very much present.

Enrique Miranda, April 2017