Fischer decomposition for the symplectic group

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Abstract

We prove the Fischer decomposition for the space of spinor–valued polynomials, defined on Euclidean space of four–fold dimension, in terms of irreducible modules for the symplectic group, consisting of so–called $\mathfrak{osp}(4|2)$–monogenics.

1 Introduction

At the core of this paper are spaces of homogeneous quaternionic monogenic polynomials, i.e. polynomials defined in Euclidean space, the dimension of which is assumed to be a fourfold, taking their values in a Clifford algebra, or subspaces thereof, and which are null solutions of four first order differential operators: a quaternionic Dirac operator and three different conjugates of it. The associated function theory is called *quaternionic Clifford analysis*; it is the most recent branch in the still growing but already well established domain of Clifford analysis.

Standard Clifford analysis is, in its most basic form, a higher dimensional generalisation of holomorphic function theory in the complex plane, and a refinement of harmonic analysis. The fundamental notion in this function theory is that of a monogenic function, i.e. a Clifford algebra valued null solution of the Dirac operator $\underline{\partial} = \sum_{\alpha=1}^{m} e_\alpha \, \partial_{x_\alpha}$, where $(e_1, \ldots, e_m)$ is an orthonormal basis of $\mathbb{R}^m$, which underlies the construction of the real Clifford algebra $\mathbb{R}_{0,m}$. This elliptic version of the Dirac equation, which is the basic field equation for particles with spin $\frac{1}{2}$, is the model par excellence for the first order, elliptic, conformally invariant system of PDEs acting on functions defined in a Euclidean vector space and with values in the basic spinor representation of the corresponding spin group.

When taking the dimension to be even: $m = 2n$ and considering functions with values in the complex Clifford algebra $\mathbb{C}_{2n}$ or in complex spinor space, *hermitian Clifford analysis* arises as a first refinement of standard Clifford analysis by introducing an additional datum, a so–called complex structure $\mathbb{I}$, i.e. an SO-element squaring to minus the identity, which induces an associated, rotated, Dirac operator $\underline{\partial}_{\mathbb{I}}$. Hermitian monogenic functions then are simultaneous null solutions of the operators $\underline{\partial}$ and $\underline{\partial}_{\mathbb{I}}$; the fundamental group underlying this function theory is the unitary group $\mathrm{U}(n)$.

Quaternionic Clifford analysis is a further refinement of hermitian Clifford analysis, originating from the introduction of a second complex structure $\mathbb{J}$, anti–commuting with the first one $\mathbb{I}$, leading to the Dirac operators $\underline{\partial}$, $\underline{\partial}_{\mathbb{I}}$, $\underline{\partial}_{\mathbb{J}}$ and $\underline{\partial}_{\mathbb{IJ}}$. In a series of papers [3, 4, 5, 6] we have thoroughly studied the fundaments of this function theory, in particular aiming at decomposing spaces of spinor–valued homogeneous polynomials in terms of irreducible representations of the symplectic group $\mathrm{Sp}(p)$.

It turns out that in order to obtain Sp(p)–irreducibility in this Fischer decomposition, spaces of so–called $\mathfrak{osp}(4|2)$–monogenic polynomials, a subclass of the quaternionic monogenic polynomials, must be considered, the Lie superalgebra $\mathfrak{osp}(4|2)$ being the Howe dual partner to the symplectic group Sp(p). This new concept of $\mathfrak{osp}(4|2)$–monogenicity is defined by means of the four, already mentioned, quaternionic Dirac operators and two additional operators: a scalar Euler operator $\mathcal{E}$ underlying the notion of symplectic harmonicity (see [5]) and a multiplication operator P in the Clifford algebra, underlying the decomposition of spinor space $\mathbb{S}$ into symplectic cells $\mathbb{S}^r_s$, which are fundamental irreducible Sp(p)–representations (see [3]).

In [6] we have, a.o., conjectured the Fischer decomposition of the space $\mathcal{P}(\mathbb{R}^{4p};\mathbb{S})$ of spinor–valued polynomials in terms of spaces $\mathcal{S}^r_{a,b}$ of bi–homogeneous $\mathfrak{osp}(4|2)$–monogenic polynomials with values in the symplectic cell $\mathbb{S}^r_r$. However the conjectured form is not completely correct in some particular cases. The aim of the underlying paper is to formulate and prove a corrected version of this Fischer decomposition, which holds in *all* cases, while showing also the Sp(p)–irreducibility of the spaces $\mathcal{S}^r_{a,b}$. The latter is done in the spirit of Howe's invariant theory [11]. To make the paper self–contained we have included a section on hermitian, quaternionic and $\mathfrak{osp}(4|2)$–monogenicity, which is special in the sense that it presents an original point of view on the refinements of Clifford analysis alluded on at the beginning of this introduction, through the concept of symmetry reduction.

2 Hermitian, quaternionic and $\mathfrak{osp}(4|2)$–monogenicity

One way to introduce the refinements embodied in the hermitian and quaternionic monogenic function theories, is by answering the following fundamental question: *what is the interplay between systems of equations and their symmetries*? As mentioned above, classical Clifford analysis is centred around the Dirac equation $\underline{\partial} f(\underline{x}) = 0$ in $\mathbb{R}^m$, and the symmetry group for this equation is the conformal one. There are several approaches possible to explaining the meaning of this symmetry phenomenon. One can for instance use Vahlen matrices, which amounts to treating the conformal symmetry at the group level. Another approach consists in determining the so–called *generalised symmetries* for the Dirac operator and investigating the algebraic structure they generate. For the definition of generalised symmetries we refer to e.g. Miller's seminal work [14] in which the connection between these symmetries and the method of separation of variables was investigated (see also [8]). More recently, higher order (generalised) symmetries of e.g. the Laplace and the Dirac operator also appeared in the framework of higher spin symmetry algebras. For the Laplace operator in $\mathbb{R}^m$ we refer to [9] where also a nice explanation of the connection with these higher spin theories is given, and to e.g. [10, 13] for further generalisations.

Definition 1. *A linear differential operator φ is a generalised symmetry for the Dirac operator $\underline{\partial}$ if there exists another linear differential operator ψ such that $[\varphi, \underline{\partial}] = \psi\underline{\partial}$. In the case where $\psi(x) = 0$, or $[\varphi, \underline{\partial}] = 0$, one says that φ is a (proper) symmetry.*

The following result is proven by direct calculations.

Lemma 1. *The commutator $[\varphi_1, \varphi_2]$ of two generalised symmetries φ_1 and φ_2 is again a generalised symmetry.*

Note that the generalised symmetries φ under consideration belong to $\mathcal{W}(\mathbb{R}^m)\otimes\mathbb{C}_m$, where $\mathcal{W}(\mathbb{R}^m)$ stands for the Weyl algebra on $\mathbb{R}^m$ and where the Clifford algebra $\mathbb{C}_m$ enters the play as the algebra End($\mathbb{S}$) acting on the spinor–values.

Although higher-order symmetries have been studied in their own right (see the references mentioned above), it is worth focusing on first–order generalised symmetries, which are of the form

$$\sum_{j=1}^{m} a_j(x_1,\ldots,x_m)\partial_{x_j} + b(x_1,\ldots,x_m) \in \mathcal{W}(\mathbb{R}^m)\otimes\mathbb{C}_m$$

It is easily seen that, under composition as linear operators, they generate an algebra, and that its commutator algebra becomes a Lie algebra; the former also contains multiplication by a constant, which however vanishes when considering the commutator algebra. As is well–known, the first–order generalised symmetries of the Dirac equation are given by the $|1|$-graded real Lie algebra

$$\mathfrak{so}(1,m+1) = \mathfrak{g}_{-1}\oplus\mathfrak{g}_0\oplus\mathfrak{g}_{+1} \cong \mathbb{R}^m \oplus \Big(\mathfrak{so}(m)\oplus\mathbb{R}\Big)\oplus\mathbb{R}^m$$

The graded subspaces $\mathfrak{g}_{\pm1}$ are spanned by $\{\partial_{x_j} : 1\leq j\leq m\}$ and $\{\mathcal{I}\partial_{x_j}\mathcal{I} : 1\leq j\leq m\}$ respectively, where $\mathcal{I}$ denotes the so–called Kelvin inversion, defined as

$$f(\underline{x}) \mapsto \mathcal{I}[f(\underline{x})] = \frac{\underline{x}}{|\underline{x}|^m}\, f(\frac{\underline{x}}{|\underline{x}|^2})$$

More explicitly, one has that

$$\mathcal{I}\partial_{x_j}\mathcal{I} = -|\underline{x}|^2\partial_{x_j} + x_j(2\mathbb{E}_{\underline{x}} + m - 1) + \underline{x}\wedge e_j$$

where $\mathbb{E}_{\underline{x}}$ is the Euler operator in the variable $\underline{x}\in\mathbb{R}^m$ and the wedge product $''\wedge''$ of Clifford vectors $\underline{v}$ and $\underline{w}$ is given by $\underline{v}\wedge\underline{w} = \frac{1}{2}(\underline{v}\underline{w} - \underline{w}\underline{v})$. The action of $\mathcal{I}\partial_{x_j}\mathcal{I}$ on a k-homogeneous polynomial $M_k(\underline{x})\in\ker(\underline{\partial})$ can be seen as the projection (up to a constant) of $x_jM_k(\underline{x})$ onto $\ker(\underline{\partial})$, yielding a polynomial null solution of $\underline{\partial}$ of degree $k+1$. As to the graded subspace $\mathfrak{g}_0$, the copy of $\mathbb{R}$ is spanned by $2\mathbb{E}_{\underline{x}} + m - 1$ and this element plays a special role: it is the so-called grading element $E_{\mathfrak{g}}$ satisfying $[E_{\mathfrak{g}}, Y_a] = aY_a$ for all $Y_a\in\mathfrak{g}_a$. Finally, the copy of $\mathfrak{so}(m)$ is given by

$$\mathfrak{so}(m) \cong \mathrm{Alg}_{\mathbb{R}}\left(dL_{jk} := L_{jk} - \frac{1}{2}e_{jk} : 1\leq j<k\leq m\right)$$

where L_{ab} is the so–called momentum operator given by $L_{jk} = x_j\,\partial_{x_k} - x_k\,\partial_{x_j}$. Since $\mathfrak{so}(1,m+1)$ is a graded algebra, one has that $[\mathfrak{g}_a,\mathfrak{g}_b]\subset\mathfrak{g}_{a+b}$, where $a+b\equiv 0$ if the sum does not belong to $\{-1,0,+1\}$. In particular, one gets that $\mathfrak{g}_{\pm1}\cong\mathbb{R}^m$ defines a copy of the defining representation of $\mathfrak{so}(m)$. Moreover, one has that $[\partial_{x_j},\mathcal{I}\partial_{x_j}\mathcal{I}] = 2\mathbb{E}_{\underline{x}} + m - 1$, while for $j\neq k$ this relation becomes $[\partial_{x_j},\mathcal{I}\partial_{x_k}\mathcal{I}] = -2\,dL_{jk}$.

Remark 1. Note that the action of the graded subspaces on the full space $\mathcal{M}(\mathbb{R}^m,\mathbb{S})$ of polynomial solutions of the Dirac equation has a nice interpretation: the degree $a\in\{-1,0,+1\}$ indicates what happens with the degree of a polynomial under the action of $Y_a\in\mathfrak{g}_a$. As a matter of fact, this illustrates that $\mathcal{M}(\mathbb{R}^m,\mathbb{S})$ becomes an irreducible module for the conformal Lie algebra $\mathfrak{so}(1,m+1)$.

Remark 2. Also for the generalisations of both the Dirac operator and the Laplace operator, one finds that the first–order generalised symmetries give rise to a commutator algebra isomorphic to $\mathfrak{so}(1,m+1)$. This is why these operators are usually referred to as *conformally invariant operators*.

Remark 3. One also can realise the Lie algebra $\mathfrak{g} = \mathfrak{so}(1,m+1)$ in terms of bivectors: $\mathfrak{g}\cong\mathbb{R}^{(2)}_{1,m+1}$. Taking into account that $\mathbb{R}^{(2)}_{1,m+1}\subset\mathbb{R}_{1,m+1}\cong\mathbb{R}_{0,m}\otimes\mathbb{R}_{1,1}\cong\mathbb{R}_{0,m}\otimes\mathbb{R}^{2\times2}$, it becomes clear that the conformal Lie algebra can also be realised in terms of (2×2)-matrices with Clifford numbers in $\mathbb{R}_{0,m}$ as entries; these matrices will then have to satisfy extra conditions since we do not need the full algebra. This is precisely what happens when working with Vahlen matrices, for which we refer to e.g. [1, 15].

When refining the symmetry, one usually focuses on the subclass of (proper) symmetries of the Dirac equation given by $\mathfrak{so}(m)$. One indeed has that $[dL_{jk}, \underline{\partial}] = 0$.

Now the idea is to choose a specific subalgebra of $\mathfrak{so}(m)$, and to see how it affects the Dirac equation. Note that *reducing* the symmetry will allow us to *extend* the system of equations, which will then further *reduce* the class of solutions of this extended system. As a matter of fact, this last consequence hints towards one of the motivations for the programme we are about to carry out: as the space of k-homogeneous polynomial solutions for the Dirac operator defines an irreducible representation for the Lie algebra $\mathfrak{so}(m)$, these subclasses of solutions are natural candidates for irreducible representations for the specific Lie subalgebras of $\mathfrak{so}(m)$ considered.

A natural breeding ground for subalgebras is the framework of centralisers. Taking into account that the Lie algebra $\mathfrak{so}(m) \cong \mathbb{R}_m^{(2)}$ corresponds to $\mathrm{Spin}(m)$ under the exponential map and to $\mathrm{SO}(m)$ under the group morphism $\chi(\cdot) : \mathrm{Spin}(m) \to \mathrm{SO}(m)$, one can equivalently formulate it on the level of centralisers of subgroups:

$$\begin{aligned} S \subset G \quad &\Rightarrow \quad G^S := \{g \in G : [g, s] = 0, \forall s \in S\} \\ \Sigma \subset \mathfrak{g} \quad &\Rightarrow \quad \mathfrak{g}^\Sigma := \{\alpha \in \mathfrak{g} : [\alpha, \sigma] = 0, \forall \sigma \in \Sigma\} \end{aligned}$$

where both S and Σ are mere subsets of the group G or the algebra $\mathfrak{g}$ respectively; note that no further algebraic structure is required here. Labeling the cardinality of a set by a subscript, this actually provides a way to introduce "a nested series of function theory refinements" based on subgroups

$$S_1 \subset S_2 \subset \ldots \subset S_j \subset \ldots \subset \mathrm{SO}(m)$$

Obviously, one of the problems arising here is that for each j there are many ways to choose a subset S_j of $\mathrm{SO}(m)$ containing j elements. A natural question therefore is how to motivate this choice. Both the hermitian and the quaternionic refinement of the classical monogenic function theory arise when choosing specific subsets $S_1 \subset S_2 \subset \mathrm{SO}(m)$. In particular, as was elaborately explained in the first paper [3] of the series, S_1 contains the matrix in $\mathrm{SO}(2n)$ which translates the mapping

$$(z_1, z_2, \ldots, z_n) \ \mapsto \ (iz_1, iz_2, \ldots, iz_n)$$

into a matrix multiplication from the right, when identifying $z_j \mapsto (x_{2j-1}, x_{2j})$. This matrix, denoted by $\mathbb{I}$, is a so–called complex structure ($\mathbb{I}^2 = -\mathrm{Id}_m$) and it was then shown that for $S_1 = \{\mathbb{I}\}$ one has that

$$\mathrm{SO}(2n)^{S_1} := \mathrm{SO}_{\mathbb{I}} \cong \mathrm{U}(n)$$

Denoting $\Sigma_1 = \{\sigma_{\mathbb{I}}\}$, a similar conclusion can be drawn on the level of Lie algebras. Going back to symmetries, as we have reduced the symmetry group (and its corresponding Lie algebra), we can extend the system of equations, the new extended system becoming invariant under the reduced group. Indeed, as we only keep those group elements commuting with $\mathbb{I}$, we can consider the additional equation $\mathbb{I}[\underline{\partial}]f(\underline{x}) = 0$. As mentioned earlier, this reduces the class of null solutions to the so–called hermitian monogenic functions, a proper subset of the set of monogenic functions, which then can be used to defining irreducible modules for $\mathfrak{u}(n)$.

Similarly, going from S_1 to S_2 is done by adding an element to the set S_1. This extra element is again motivated by the idea that we want to mimic the multiplication

$$(q_1, q_2, \ldots, q_p) \mapsto \ (jq_1, jq_2, \ldots, jq_p)$$

by a matrix acting from the right, when identifying $q_a \mapsto (x_{4a-3}, x_{4a-2}, x_{4a-1}, x_{4a})$. One then arrives in a natural way at the set $S_2 = \{\mathbb{I}, \mathbb{J}\}$, where $\mathbb{J}$ is a second complex structure. We then have found that

$$\mathrm{SO}(4p)^{S_2} := \mathrm{SO}_Q(4p) \cong \mathrm{Sp}(p)$$

In sharp contrast to the previous case, where $\mathbb{I} \in \mathrm{SO}_{\mathbb{I}}(2n)$, we do not have that $S_2 \subset \mathrm{SO}(m)^{S_2}$ for the simple reason that the elements in S_2 do not commute amongst each other, but anti-commute instead, and this additional relation gives rise to the following observation:

$$S_2 \subset \{a_0 \,\mathrm{Id}_m + a_1 \,\mathbb{I} + a_2 \,\mathbb{J} + a_3 \,\mathbb{I}\mathbb{J} : \sum_j a_j^2 = 1\} \cong \mathrm{Sp}(1) \cong \mathrm{SU}(2)$$

where the last isomorphism was denoted by ψ_1 in [3]. In a sense, this group $\mathrm{Sp}(1)$ encodes all possible ways to choose a set S_2 satisfying the same algebraic relations.

Remark 4. Note that also in the hermitian case one could do something similar:

$$S_1 \subset \{a_0 \,\mathrm{Id}_m + a_1 \,\mathbb{I} : a_0^2 + a_1^2 = 1\} \cong \mathrm{U}(1) \cong \mathrm{SO}(2)$$

Returning to symmetries, as we have once more reduced the symmetry group (and the corresponding Lie algebra), we also can once more extend the system by adding extra equations. This time this results into the system of equations

$$\partial_x f(x) = \mathbb{I}[\partial_x]\, f(x) = \mathbb{J}[\partial_x]\, f(x) = \mathbb{K}[\partial_x]\, f(x) = 0$$

where we have put $\mathbb{K} = \mathbb{I}\,\mathbb{J}$, defining so–called quaternionic monogenic functions.

However in our quest for the Fischer decomposition of spaces of polynomials in terms of quaternionic monogenics, it became clear that spaces of quaternionic monogenic homogeneous polynomials with values in the symplectic cells of spinor space are reducible for the action of the symplectic group $\mathrm{Sp}(p)$, which lead us in [6] to the concept of $\mathfrak{osp}(4|2)$–monogenicity, the Lie superalgebra $\mathfrak{osp}(4|2)$ being the Howe dual partner of $\mathrm{Sp}(p)$. This notion is explained below, but first we introduce, in the unitary symmetry case, the main topic of this paper: the Fischer decomposition.

In the complexification $\mathbb{C}^{4p}$ of $\mathbb{R}^{4p}$ we consider the so–called Witt basis vectors, given by

$$\mathfrak{f}_k = -\frac{1}{2}(\mathbf{1} - i\,\mathbb{I})[e_{2k-1}] \quad \text{and} \quad \mathfrak{f}_k^\dagger = \frac{1}{2}(\mathbf{1} + i\,\mathbb{I})[e_{2k-1}] \quad (k = 1, \ldots, 2p)$$

The Witt basis vectors $(\mathfrak{f}_1, \ldots, \mathfrak{f}_{2p})$ on the one hand, and $(\mathfrak{f}_1^\dagger, \ldots, \mathfrak{f}_{2p}^\dagger)$ on the other, respectively, span isotropic subspaces W and $W^\dagger$ of $\mathbb{C}^{4p}$, such that $\mathbb{C}^{4p} = W \oplus W^\dagger$, those subspaces being eigenspaces of the complex structure $\mathbb{I}$ with respective eigenvalues $-i$ and i. They also generate the respective Grassmann algebras $\mathbb{C}\bigwedge_{2p}$ and $\mathbb{C}\bigwedge_{2p}^\dagger$. With the self-adjoint idempotents

$$I_j = \mathfrak{f}_j \mathfrak{f}_j^\dagger = \frac{1}{2}(1 - i e_{2j-1} e_{2j}), \qquad j = 1, \ldots, 2p$$

we compose the primitive self–adjoint idempotent $I = I_1 I_2 \cdots I_{2p}$, leading to the realization of spinor space as $\mathbb{S} = \mathbb{C}_{4p} I$. Since $\mathfrak{f}_j I = 0$, $j = 1, \ldots, 2p$, we also have $\mathbb{S} \simeq \mathbb{C}\bigwedge_{2p}^\dagger I$. When decomposing the Grassmann algebra as

$$\mathbb{C}\bigwedge\nolimits_{2p}^\dagger = \bigoplus_{r=0}^{2p} \left(\mathbb{C}\bigwedge\nolimits_{2p}^\dagger\right)^{(r)}$$

into its so–called homogeneous parts, where $\left(\mathbb{C}\bigwedge_{2p}^\dagger\right)^{(r)}$ is spanned by all products of r Witt basis vectors out of $(\mathfrak{f}_1^\dagger, \ldots, \mathfrak{f}_{2p}^\dagger)$, spinor space accordingly decomposes into

$$\mathbb{S} = \bigoplus_{r=0}^{2p} \mathbb{S}^r, \qquad \text{with} \quad \mathbb{S}^r \simeq \left(\mathbb{C}\bigwedge\nolimits_{2p}^\dagger\right)^{(r)} I$$

These homogeneous parts $\mathbb{S}^r$, $r = 0, \ldots, 2p$ provide models for fundamental $\mathrm{U}(2p)$–representations. Introducing the hermitian vector variables

$$\begin{aligned}
\underline{z} &= -\frac{1}{2}(\mathbf{1} - i\,\mathbb{I})[\underline{x}] = \sum_{k=1}^{2p} (x_{2k-1}\mathfrak{f}_k + x_{2k}(i\mathfrak{f}_k)) = \sum_{k=1}^{2p} z_k \mathfrak{f}_k \\
\underline{z}^\dagger &= \frac{1}{2}(\mathbf{1} + i\,\mathbb{I})[\underline{x}] = \sum_{k=1}^{2p} (x_{2k-1}\mathfrak{f}_k^\dagger + x_{2k}(-i\mathfrak{f}_k^\dagger)) = \sum_{k=1}^{2p} \overline{z}_k \mathfrak{f}_k^\dagger
\end{aligned}$$

and, correspondingly, the hermitian Dirac operators

$$\begin{aligned}
2\,\partial_{\underline{z}}^\dagger &= -\frac{1}{2}(\mathbf{1} - i\,\mathbb{I})[\underline{\partial}] = \sum_{k=1}^{2p} (\mathfrak{f}_k \partial_{x_{2k-1}} + i\mathfrak{f}_k \partial_{x_{2k}}) = \sum_{k=1}^{2p} \mathfrak{f}_k (\partial_{x_{2k-1}} + i\partial_{x_{2k}}) = 2\sum_{k=1}^{2p} \partial_{\overline{z}_k} \mathfrak{f}_k \\
2\,\partial_{\underline{z}} &= \frac{1}{2}(\mathbf{1} + i\,\mathbb{I})[\underline{\partial}] = \sum_{k=1}^{2p} (\mathfrak{f}_k^\dagger \partial_{x_{2k-1}} - i\mathfrak{f}_k^\dagger \partial_{x_{2k}}) = \sum_{k=1}^{2p} \mathfrak{f}_k^\dagger (\partial_{x_{2k-1}} - i\partial_{x_{2k}}) = 2\sum_{k=1}^{2p} \partial_{z_k} \mathfrak{f}_k^\dagger
\end{aligned}$$

we can redefine equivalently hermitian monogenic functions as simultaneous null solutions of these hermitian Dirac operators $\partial_{\underline{z}}$ and $\partial_{\underline{z}}^\dagger$, which are invariant under the action of the unitary group $\mathrm{U}(2p)$. This $\mathrm{U}(2p)$–symmetry also becomes apparent in the following result, obtained in [7], concerning the Fischer decomposition in terms of homogeneous hermitian monogenic polynomials. To that end we introduce the space $\mathcal{I}$ of all $\mathrm{Spin}_{\mathbb{I}}(4p)$–invariant polynomials, which is proven by invariance theory (see e.g. [11]) to be spanned by all words in the letters $\underline{z}$ and $\underline{z}^\dagger$:

$$\mathcal{I} = \mathrm{span}_{\mathbb{C}} \left(1, \underline{z}, \underline{z}^\dagger, \underline{z}\,\underline{z}^\dagger, \underline{z}^\dagger\underline{z}, \underline{z}\,\underline{z}^\dagger\underline{z}, \underline{z}^\dagger\underline{z}\,\underline{z}^\dagger, \underline{z}\,\underline{z}^\dagger\underline{z}\,\underline{z}^\dagger, \underline{z}^\dagger\underline{z}\,\underline{z}^\dagger\underline{z}, \cdots\right)$$

or alternatively $\mathcal{I} = \mathrm{span}_{\mathbb{C}} \left(w_l^{(i)}(\underline{z}, \underline{z}^\dagger) : l = 0, 1, 2, \ldots, i = 1, 2\right)$, where $w_0^{(1)} = w_0^{(2)} = 1$ and

$$\begin{aligned}
w_{2j}^{(1)}(\underline{z}, \underline{z}^\dagger) &= (\underline{z}\underline{z}^\dagger)^j = |\underline{z}|^{2j-2}\underline{z}\,\underline{z}^\dagger & \qquad w_{2j+1}^{(1)}(\underline{z}, \underline{z}^\dagger) &= |\underline{z}|^{2j}\underline{z} \\
w_{2j}^{(2)}(\underline{z}, \underline{z}^\dagger) &= (\underline{z}^\dagger\underline{z})^j = |\underline{z}|^{2j-2}\underline{z}^\dagger\underline{z} & \qquad w_{2j+1}^{(2)}(\underline{z}, \underline{z}^\dagger) &= |\underline{z}|^{2j}\underline{z}^\dagger\ .
\end{aligned}$$

Proposition 1. *The space $\mathcal{P}(\mathbb{R}^{4p}; \mathbb{S})$ of spinor–valued polynomials on $\mathbb{R}^{4p}$ can be decomposed according to the $\mathrm{U}(2p)$–action as*

$$\mathcal{P}(\mathbb{R}^{4p}; \mathbb{S}) = \bigoplus_{a,b=0}^{\infty} \bigoplus_{r=0}^{2p} \left(\mathcal{M}_{a,b}^r(\mathbb{R}^{4p}; \mathbb{S}) \oplus \bigoplus_{l=1}^{\infty} \bigoplus_{i=1,2} w_l^{(i)}(\underline{z}, \underline{z}^\dagger)\ \mathcal{M}_{a,b}^r(\mathbb{R}^{4p}; \mathbb{S}) \right) \tag{1}$$

with $\mathcal{M}_{a,b}^r(\mathbb{R}^{4p}; \mathbb{S})$ the space of (a, b)–homogeneous hermitian monogenic polynomials in the complex variables $(z_1, \cdots, z_{2p}, \overline{z}_1, \cdots, \overline{z}_{2p})$ with values in the homogeneous spinor subspace $\mathbb{S}^r$.

In the quaternionic refinement, when the decomposition of the space $\mathcal{P}(\mathbb{R}^{4p}; \mathbb{S})$ of spinor–valued polynomials into $\mathrm{Sp}(p)$–irreducibles is at stake, we should first take care of the irreducibility of the value space as an $\mathrm{Sp}(p)$–representation. To that end the $\mathrm{U}(2p)$–irreducible homogeneous parts $\mathbb{S}^r$ of spinor space $\mathbb{S}$, should further be decomposed into $\mathrm{Sp}(p)$–irreducibles, which we call *symplectic cells*. To this end we first introduce the $\mathrm{Sp}(p)$–invariant left multiplication operators

$$P = \mathfrak{f}_2\mathfrak{f}_1 + \mathfrak{f}_4\mathfrak{f}_3 + \ldots + \mathfrak{f}_{2p}\mathfrak{f}_{2p-1} \qquad \text{and} \qquad Q = \mathfrak{f}_1^\dagger\mathfrak{f}_2^\dagger + \mathfrak{f}_3^\dagger\mathfrak{f}_4^\dagger + \ldots + \mathfrak{f}_{2p-1}^\dagger\mathfrak{f}_{2p}^\dagger \ = \ P^\dagger$$

for which $P : \mathbb{S}^r \to \mathbb{S}^{r-2}$ and $Q : \mathbb{S}^r \to \mathbb{S}^{r+2}$. Note that together with the spin–Euler operator $\beta = \sum_{j=1}^{2p} \mathfrak{f}_j^\dagger \mathfrak{f}_j$, the operators P and Q generate a $\mathfrak{sl}_2(\mathbb{C})$–structure. Next we define, for $r = 0, \ldots, p$, the subspaces

$$\mathbb{S}^r_r = \ker\left(P|_{\mathbb{S}^r}\right), \qquad \mathbb{S}^{2p-r}_r = \ker\left(Q|_{\mathbb{S}^{2p-r}}\right)$$

and for $k = 0, \ldots, p-r$, the subspaces $\mathbb{S}^{r+2k}_r = Q^k\, \mathbb{S}^r_r$ and $\mathbb{S}^{2p-r-2k}_r = P^k\, \mathbb{S}^{2p-r}_r$, leading, for all $r = 0, \ldots, p$, to the following decompositions:

$$\mathbb{S}^r = \bigoplus_{j=0}^{\lfloor \frac{r}{2} \rfloor} \mathbb{S}^r_{r-2j}, \qquad \mathbb{S}^{2p-r} = \bigoplus_{j=0}^{\lfloor \frac{r}{2} \rfloor} \mathbb{S}^{2p-r}_{r-2j}$$

where now each of the symplectic cells $\mathbb{S}^r_s$ is an irreducible $\mathrm{Sp}(p)$–representation. Introducing the additional vector variables

$$\begin{aligned} \underline{z}^J &= \mathbb{J}[\underline{z}] = \sum_{j=1}^{p} (z_{2j}\, \mathfrak{f}^\dagger_{2j-1} - z_{2j-1}\, \mathfrak{f}^\dagger_{2j}) \\ \underline{z}^{\dagger J} &= \mathbb{J}[\underline{z}^\dagger] = \sum_{j=1}^{p} (\overline{z}_{2j}\, \mathfrak{f}_{2j-1} - \overline{z}_{2j-1}\, \mathfrak{f}_{2j}) \end{aligned}$$

and, correspondingly, the additional Dirac operators

$$\begin{aligned} \partial^J_{\underline{z}} &= \mathbb{J}[\partial_{\underline{z}}] = \sum_{j=1}^{p} (\partial_{z_{2j}} \mathfrak{f}_{2j-1} - \partial_{z_{2j-1}} \mathfrak{f}_{2j}) \\ \partial^{\dagger J}_{\underline{z}} &= \mathbb{J}[\partial^\dagger_{\underline{z}}] = \sum_{j=1}^{p} (\partial_{\overline{z}_{2j}} \mathfrak{f}^\dagger_{2j-1} - \partial_{\overline{z}_{2j-1}} \mathfrak{f}^\dagger_{2j}) \end{aligned}$$

we can equivalently redefine quaternionic monogenic functions as simultaneous null solutions of the differential operators $\partial_{\underline{z}}$, $\partial^\dagger_{\underline{z}}$, $\partial^J_{\underline{z}}$ and $\partial^{\dagger J}_{\underline{z}}$. Now turning our attention to the Howe dual partner of the invariance group $\mathrm{Sp}(p)$, we want these four operators and their algebraic counterparts $\underline{z}$, $\underline{z}^\dagger$, $\underline{z}^J$ and $\underline{z}^{\dagger J}$, which indeed all are $\mathrm{Sp}(p)$–invariant, to belong to (the odd part of) a Lie (super)algebra. Computing the anti–commutators of those differential and multiplication operators we find, next to the expected $\mathfrak{sl}(2, \mathbb{C})$ generators, viz $\mathbb{E}_{\underline{z}} + \mathbb{E}_{\underline{z}^\dagger} + 2p$, $|\underline{z}|^2 = r^2$, Δ_{4p}, and the operator $\mathbb{E}_{\underline{z}} - \mathbb{E}_{\underline{z}^\dagger}$, which already appear in the Howe dual partner $\mathfrak{sl}(1|2) = \mathfrak{osp}(2|2)$ of $\mathrm{U}(2p)$ in the framework of hermitian Clifford analysis, and next to the shifting operators P and Q used in the definition of the symplectic cells, also the new scalar differential operators

$$\mathcal{E} = \sum_{k=1}^{p} z_{2k-1}\, \partial_{\overline{z}_{2k}} - z_{2k}\, \partial_{\overline{z}_{2k-1}} \quad \text{and} \quad \mathcal{E}^\dagger = \sum_{k=1}^{p} \overline{z}_{2k-1}\, \partial_{z_{2k}} - \overline{z}_{2k}\, \partial_{z_{2k-1}}$$

This leads to the Howe dual partner of the Lie group $\mathrm{Sp}(p)$, viz the 17–dimensional orthosymplectic Lie superalgebra

$$\mathfrak{osp}(4|2) = \mathfrak{g}_0 \oplus \mathfrak{g}_1 = (\mathfrak{so}(4) \oplus \mathfrak{sp}(2)) \oplus \mathfrak{g}_1 = (\mathfrak{sl}(2) \oplus \mathfrak{sl}(2) \oplus \mathfrak{sl}(2)) \oplus \mathfrak{g}_1$$

with

$$\mathfrak{sl}(2) \cong \mathrm{Alg}_{\mathbb{C}}\left(\mathbb{E}_{\underline{z}} + \mathbb{E}^\dagger_{\underline{z}} + 2p, \frac{1}{2}|\underline{z}|^2, -\frac{1}{2}\Delta_{4p}\right) \cong \mathrm{Alg}_{\mathbb{C}}\left(\mathbb{E}_{\underline{z}} - \mathbb{E}^\dagger_{\underline{z}}, \mathcal{E}, \mathcal{E}^\dagger\right) \cong \mathrm{Alg}_{\mathbb{C}}\left(p - \beta, P, Q\right)$$

and

$$\mathfrak{g}_1 \cong \mathrm{span}_{\mathbb{C}}\left(\underline{z}, \underline{z}^\dagger, \partial_{\underline{z}}, \partial^\dagger_{\underline{z}}\right) \oplus \mathrm{span}_{\mathbb{C}}\left(\underline{z}^J, \underline{z}^{\dagger J}, \partial^J_{\underline{z}}, \partial^{\dagger J}_{\underline{z}}\right)$$

also inspiring the following definitions.

Definition 2.

(i) A differentiable function is called symplectic harmonic if it is simultaneously in the kernel of the operators Δ, $\mathcal{E}$, and P.

(ii) A differentiable function is called $\mathfrak{osp}(4|2)$–monogenic if it is simultaneously in the kernel of the operators $\partial_{\underline{z}}$, $\partial_{\underline{z}}^{\dagger}$, $\partial_{\underline{z}}^{J}$, $\partial_{\underline{z}}^{\dagger J}$, $\mathcal{E}$, and P.

In [6] we conjectured the Fischer decomposition of $\mathcal{P}(\mathbb{R}^{4p};\mathbb{S})$ in terms of spaces $\mathcal{S}^r_{a,b}$ of so–called $\mathfrak{osp}(4|2)$–monogenics, which are defined as follows.

Definition 3. *For arbitrary indices $(a,b;r)$ with $a \geq b \geq 0$ and $r \in \{0,1,\ldots,p\}$ one defines $\mathcal{S}^r_{a,b}$ to be the space of (a,b)–homogeneous $\mathfrak{osp}(4|2)$–monogenic polynomials defined in $\mathbb{R}^{4p}$ and with values in the symplectic cell $\mathbb{S}^r_r$.*

Remark 5. Note that in Definition 3 the condition $a \geq b$ is necessary because of the use of the operator $\mathcal{E}$. When $a \leq b$ one could work with $\mathcal{E}^{\dagger}$ instead (see also [5]).

In Section 4 we will now show the $\mathrm{Sp}(p)$–irreducibility of the spaces $\mathcal{S}^r_{a,b}$ using Howe's results in [11] on invariant theory. Furthermore, in Theorem 3, we will describe an irreducible decomposition of spaces of symplectic harmonics, and finally, in Theorem 4, we will prove the Fischer decomposition for the symplectic group $\mathrm{Sp}(p)$, i.e. an irreducible decomposition of the $\mathrm{Sp}(p)$–module $\mathcal{P}(\mathbb{R}^{4p},\mathbb{S})$ in terms of $\mathfrak{osp}(4|2)$–monogenics.

3 Howe harmonics

Throughout this section we will call $G = \mathrm{Sp}(p)$ and $\Gamma' = \mathfrak{osp}(4|2)$. This Lie superalgebra Γ', for which we have an operator realisation in terms of differential and multiplication operators (both in the co-ordinates and on the values, i.e. purely algebraic ones), can be decomposed as a super vector space in the following subspaces:

$$\Gamma' = \Gamma'(2,0) \oplus \Gamma'(1,1) \oplus \Gamma'(0,2)$$

where we have respectively introduced

$$\begin{aligned}
\Gamma'(2,0) &= \mathrm{span}(|\underline{z}|^2, Q) \oplus \mathrm{span}(\underline{z}^{\dagger}, \underline{z}^{J}) \\
\Gamma'(1,1) &= \mathrm{span}(\mathbb{E}+2p, \mathbb{E}_z - \mathbb{E}_z^{\dagger}, \mathcal{E}, \mathcal{E}^{\dagger}, p-\beta) \oplus \mathrm{span}(\underline{z}, \underline{z}^{\dagger J}, \partial_{\underline{z}}, \partial_{\underline{z}}^{\dagger J}) \\
\Gamma'(0,2) &= \mathrm{span}(\Delta, P) \oplus \mathrm{span}(\partial_{\underline{z}}^{\dagger}, \partial_{\underline{z}}^{J}) \ ,
\end{aligned}$$

and where the direct sum notation above refers to the decomposition $V = V_0 \oplus V_1$ of a super vector space into its even and odd subspaces.

Proposition 2. *The above decomposition of Γ' enjoys the following properties:*

(i) both $\Gamma'(2,0)$ and $\Gamma'(0,2)$ are abelian Lie superalgebras, meaning that the $\mathbb{Z}_2$-graded brackets of elements in these spaces are trivial (i.e. the elements all commute or anti-commute);

(ii) the spaces $\Gamma'(2,0)$ and $\Gamma'(0,2)$ are modules for the action of $\Gamma'(1,1)$:

$$\big[\Gamma'(1,1), \Gamma'(2,0)\big] \subset \Gamma'(2,0) \qquad \textit{and} \qquad \big[\Gamma'(1,1), \Gamma'(0,2)\big] \subset \Gamma'(0,2)$$

the notation $[\cdot,\cdot]$ referring to the $\mathbb{Z}_2$-graded bracket.

Also the space of polynomials we are working with can be related to the spaces Howe considers in his paper [11], which are of course very general. For a classical group G he considers the G–modules U and W which are formed by taking direct sums of the basic G–module V. Denoting by $\mathrm{Sym}(U)$ the symmetric algebra over U and by $\bigwedge(W)$ the exterior algebra over W, Howe then works with the space – actually an algebra –

$$\mathcal{A}(U,W) := \mathrm{Sym}(U) \otimes \bigwedge(W) = \mathrm{Sym}\Big(\bigoplus_{j=1}^{a} V\Big) \otimes \bigwedge\Big(\bigoplus_{j=1}^{b} V\Big)$$

In our case of interest we have that $V = \mathbb{C}^{2p}$ and $(a,b) = (2,1)$ since we are working with the space of spinor–valued polynomials defined in $\mathbb{R}^{4p}$:

$$\mathcal{P}(\mathbb{R}^{4p};\mathbb{S}) \ \subset\ \mathcal{P}(\mathbb{R}^{4p};\mathbb{C}) \otimes \mathbb{S} \ \cong\ \mathrm{Sym}(\mathbb{C}^{2p} \oplus \mathbb{C}^{2p}) \otimes \Lambda(\mathbb{C}^{2p})$$

As usual in Howe's approach to duality issues, he then defines 'harmonics' to be solutions for 'pure second order operators'. Following this idea we introduce the following space of *Howe harmonics*.

Definition 4. *We define*

$$\mathtt{H}(\mathbb{R}^{4p};\mathbb{S}) := \big\{P(\underline{z},\underline{z}^\dagger) \in \mathcal{P}(\mathbb{R}^{4p};\mathbb{S}) : DP(\underline{z},\underline{z}^\dagger) = 0, \forall D \in \Gamma'(0,2)\big\}$$

Note that we do not use here the typical notation $\mathcal{H}$ for harmonics, in order to avoid confusion.

Whereas $\Gamma'(0,2)$ contains all operators defining the space $\mathtt{H}(\mathbb{R}^{4p};\mathbb{S})$, the 'opposite space' $\Gamma'(2,0)$ will be referred to as the space $\mathtt{I}$ containing the (dual) invariants. We also index this space by a subscript $(a,b) \in \mathbb{N}_0 \times \mathbb{N}_0$ denoting the homogeneity bidegree of the polynomials and by a superscript $r \in \{0,1,\ldots,p\}$ denoting the homogeneity degree of the spinor space component values. Moreover, we will use the shorthand notation: $\mathtt{H}^r_{a,b} := \mathtt{H}_{a,b}(\mathbb{R}^{4p};\mathbb{S}^r)$ and similarly for the other spaces.

A natural question then is: how does the space $\mathtt{H}^r_{a,b}$ of Howe harmonics decompose in terms of the subspaces $\mathcal{S}^r_{a,b}$ of $\mathfrak{osp}(4|2)$-monogenics? A first result, on purely algebraic grounds, is the following.

Theorem 1. *For arbitrary indices $(a,b;r)$ with $a,b \geq 0$ and $r \in \{0,1,\ldots,p\}$ one has that*

$$\mathtt{H}^r_{a,b} = \mathcal{S}^r_{a,b} \oplus \left(\underline{z}\mathtt{H}^{r+1}_{a-1,b} + \underline{z}^{\dagger J}\mathtt{H}^{r+1}_{a,b-1} + \mathcal{E}^\dagger\mathtt{H}^r_{a+1,b-1}\right)$$

where it is tacitly assumed that if indices make no sense then the corresponding spaces do not appear in the decomposition (in particular, $\mathcal{S}^r_{a,b} = 0$ for $a < b$).

Proof
First note that the space between brackets at the right hand–side indeed is a subspace of the space $\mathtt{H}^r_{a,b}$ at the left hand–side, since the dual operator for each of the operators in $\{\partial_{\underline{z}}, \partial^{\dagger J}_{\underline{z}}, \mathcal{E}\}$ on the full space of polynomials is equal to the dual operator for the restriction of the Fischer inner product to the subspace $\mathtt{H}^r_{a,b}$. The desired result then follows from the fact that in $\mathtt{H}^r_{a,b}$

$$\mathcal{S}^r_{a,b} = \left(\underline{z}\mathtt{H}^{r+1}_{a-1,b} + \underline{z}^{\dagger J}\mathtt{H}^{r+1}_{a,b-1} + \mathcal{E}^\dagger\mathtt{H}^r_{a+1,b-1}\right)^\perp.$$

□

Now Theorem 1 can be refined to obtain a direct sum decomposition of Howe harmonics.

Theorem 2. *For arbitrary indices $(a,b;r)$ with $a,b \geq 0$ and $r \in \{0,1,\ldots,p\}$ one has that*

$$\mathtt{H}^r_{a,b} = \bigoplus_{k=0}^{b} \left((\mathcal{E}^\dagger)^k \, \mathcal{S}^r_{a+k,b-k} \oplus \underline{z}\, (\mathcal{E}^\dagger)^k \, \mathcal{S}^{r+1}_{a+k-1,b-k} \oplus \underline{z}^{\dagger J} \, (\mathcal{E}^\dagger)^k \, \mathcal{S}^{r+1}_{a+k,b-k-1} \oplus \underline{z}\, \underline{z}^{\dagger J} \, (\mathcal{E}^\dagger)^k \, \mathcal{S}^{r+2}_{a+k-1,b-k-1} \right) \tag{2}$$

with similar assumptions on the indices as in Theorem 1.

The proof of this theorem is developed in two steps. First we introduce the spaces

$$\mathtt{K}^r_{a,b} = \mathtt{H}^r_{a,b} \cap \ker(\partial_{\underline{z}}, \partial^{\dagger J}_{\underline{z}})$$

Lemma 2. *For arbitrary indices $(a,b;r)$ with $a,b \geq 0$ and $r \in \{0,1,\ldots,p\}$ one has that*

$$\mathtt{K}^r_{a,b} = \mathcal{S}^r_{a,b} \oplus \mathcal{E}^\dagger \mathcal{S}^r_{a+1,b-1} \oplus \cdots \oplus (\mathcal{E}^\dagger)^b \mathcal{S}^r_{a+b,0} \tag{3}$$

with similar assumptions on the indices as in Theorem 1.

Proof
First note that $\mathtt{K}^r_{c,d} \cap \ker \mathcal{E} = \mathtt{H}^r_{c,d} \cap \ker(\partial_{\underline{z}}, \partial^{\dagger J}_{\underline{z}}, \mathcal{E}) = \mathcal{S}^r_{c,d}$ for $c \geq d \geq 0$, and $\mathtt{K}^r_{c,d} \cap \ker \mathcal{E} = 0$ otherwise. This leads to the decomposition (3) (see [5]). Indeed, we have $\mathcal{S}^r_{a,b} = \left(\mathcal{E}^\dagger \mathtt{K}^r_{a+1,b-1}\right)^\perp$ in $\mathtt{K}^r_{a,b}$, whence

$$\mathtt{K}^r_{a,b} = \mathcal{S}^r_{a,b} \oplus \mathcal{E}^\dagger \mathtt{K}^r_{a+1,b-1},$$

which completes the proof. □

Lemma 3. *For arbitrary indices $(a,b;r)$ with $a,b \geq 0$ and $r \in \{0,1,\ldots,p\}$ one has that*

$$\mathtt{H}^r_{a,b} = \mathtt{K}^r_{a,b} \oplus \underline{z}\mathtt{K}^{r+1}_{a-1,b} \oplus \underline{z}^{\dagger J}\mathtt{K}^{r+1}_{a,b-1} \oplus \underline{z}\underline{z}^{\dagger J}\mathtt{K}^{r+2}_{a-1,b-1} \tag{4}$$

with similar assumptions on the indices as in Theorem 1.

Proof
It is easy to see that in $\mathtt{H}^r_{a,b}$ we have $\mathtt{K}^r_{a,b} = \left(\underline{z}\, \mathtt{H}^{r+1}_{a-1,b} + \underline{z}^{\dagger J}\, \mathtt{H}^{r+1}_{a,b-1}\right)^\perp$ and so

$$\mathtt{H}^r_{a,b} = \mathtt{K}^r_{a,b} \oplus \left(\underline{z}\, \mathtt{H}^{r+1}_{a-1,b} + \underline{z}^{\dagger J}\, \mathtt{H}^{r+1}_{a,b-1}\right).$$

Using the relations $\{\underline{z}, \underline{z}^{\dagger J}\} = 0$ and $\underline{z}^2 = 0 = (\underline{z}^{\dagger J})^2$, we get

$$\mathtt{H}^r_{a,b} = \mathtt{K}^r_{a,b} \oplus \left(\underline{z}\, \mathtt{K}^{r+1}_{a-1,b} + \underline{z}^{\dagger J}\, \mathtt{K}^{r+1}_{a,b-1} + \underline{z}\, \underline{z}^{\dagger J}\, \mathtt{K}^{r+2}_{a-1,b-1}\right)$$

which shows the decomposition (4), however without the directness of the sum between brackets. That last aspect now is tackled. Given four arbitrary polynomials $K_1 \in \mathtt{K}^r_{a,b}$, $K_2 \in \mathtt{K}^{r+1}_{a-1,b}$, $K_3 \in \mathtt{K}^{r+1}_{a,b-1}$, $K_4 \in \mathtt{K}^{r+2}_{a-1,b-1}$, we will first show that

$$0 = K_1 + \underline{z}K_2 + \underline{z}^{\dagger J} K_3 + \underline{z}\, \underline{z}^{\dagger J} K_4 \tag{5}$$

implies that $K_1 = K_2 = K_3 = K_4 = 0$. Acting on equation (5) with the operator $\partial^{\dagger J}_{\underline{z}} \partial_{\underline{z}}$, one arrives at

$$\begin{aligned} 0 = \partial^{\dagger J}_{\underline{z}} \partial_{\underline{z}} \underline{z}\, \underline{z}^{\dagger J} K_4 &= \partial^{\dagger J}_{\underline{z}} (-\underline{z}\, \partial_{\underline{z}} + \mathbb{E}_z + \beta) \underline{z}^{\dagger J} K_4 \\ &= (a+r)(\mathbb{E}^\dagger_z + \beta) K_4 + (\underline{z}\, \partial^{\dagger J}_{\underline{z}} - \mathcal{E}) \mathcal{E}^\dagger K_4 \\ &= (a+r)(b+r+1) K_4 - \mathcal{E}\mathcal{E}^\dagger K_4 \end{aligned}$$

from which it follows that $K_4 = 0$ on condition that $(a+r)(b+r+1)$ does not belong to the spectrum of the operator $\mathcal{E}\mathcal{E}^\dagger$ acting on the space $\mathsf{K}^{r+2}_{a-1,b-1}$. We now prove that this is indeed the case. Using the decomposition (3), we have

$$K_4(\underline{z}, \underline{z}^\dagger) = R_0(\underline{z}, \underline{z}^\dagger) + \mathcal{E}^\dagger R_1(\underline{z}, \underline{z}^\dagger) + \ldots + (\mathcal{E}^\dagger)^{b-1} R_{b-1}(\underline{z}, \underline{z}^\dagger)$$

where $R_j \in \mathcal{S}^{r+2}_{a-1+j,b-1-j}$ for $j = 0, \ldots, b-1$. Making use of the fact that the relation

$$[X, Y^k] = kY^{k-1}(H + 1 - k)$$

holds in the universal enveloping algebra $\mathcal{U}(\mathfrak{sl}(2))$ of $\mathfrak{sl}(2) = \mathrm{Alg}(X, Y, H)$, we obtain that

$$\mathcal{E}\mathcal{E}^\dagger(\mathcal{E}^\dagger)^j R_j = [\mathcal{E}, (\mathcal{E}^\dagger)^{j+1}] R_j = (j+1)(\mathcal{E}^\dagger)^j(\mathbb{E}_z - \mathbb{E}^\dagger_z - j)R_j = (j+1)(a-b+j)(\mathcal{E}^\dagger)^j R_j$$

It is clear that $(a+r)(b+r+1) > (a-1)b$, where $(a-1)b$ is the maximal value of the coefficient in the last equality above (obtained for $j = b-1$), unless $a = b = r = 0$, in which case the term involving K_4 does not appear in the equation (5). It thus follows that $K_4 = 0$. Now multiplying by $\underline{z}$ the remaining three terms in equation (5) leads to $\underline{z}K_1 + \underline{z}\underline{z}^{\dagger J} K_3 = 0$. Again acting with the operator $\partial^{\dagger J}_{\underline{z}} \partial_{\underline{z}}$ results into

$$0 = \big((a+r)(b+r) - \mathcal{E}\mathcal{E}^\dagger\big) K_3(\underline{z}, \underline{z}^\dagger)$$

from which it follows that $K_3 = 0$ by a similar argument as above, with $(a+r)(b+r) > (j+1)(1+a-b+j)$ with a maximal value ab of the right hand–side (obtained for $j = b-1$), on condition that $r > 0$. The exceptional case $r = 0$ will be treated below. Now we proceed as follows:

$$0 = K_1 + \underline{z}K_2 \;\Rightarrow\; 0 = \partial_{\underline{z}}\underline{z}(K_1 + \underline{z}K_2) = (\mathbb{E}_z + \beta)K_1 \;\Rightarrow\; (a+r)K_1 = 0$$

This implies that $K_1 = 0$, unless $a = r = 0$. And the resulting equation (5) $\underline{z}K_2 = 0$ allows us to deduce that $K_2(\underline{z}, \underline{z}^\dagger) = 0$ under the same restrictions on (a, r). But if $a = r = 0$, then the original equation (5) 'collapses', in the sense that K_2 and K_4 will a priori not appear, which means that the conclusion can be drawn without having to perform the steps involving this condition. We conclude that, indeed, equation (5) implies that $K_1 = K_2 = K_3 = K_4 = 0$, which proves the lemma in case $r > 0$. As mentioned above, we still have to investigate the special case of equation (5) for $r = 0$. Note that we can reduce the problem to

$$K^0_{a,b} + \underline{z}K^1_{a-1,b} + \underline{z}^{\dagger J} K^1_{a,b-1} = 0 \;\overset{?}{\Longrightarrow}\; K^0_{a,b} = K^1_{a-1,b} = K^1_{a,b-1} = 0$$

as the argument leading to $K_4(\underline{z}, \underline{z}^\dagger) = 0$ is equally valid. Recall that the polynomial $K^0_{a,b}(\underline{z}, \underline{z}^\dagger)$ belongs to the space $\mathsf{K}^0_{a,b} = \mathtt{H}_{a,b}(\mathbb{R}^{4p}, \mathbb{S}^0) \cap \ker(\partial_{\underline{z}}, \partial^{\dagger J}_{\underline{z}}) = \mathcal{P}(\mathbb{R}^{4p}; \mathbb{S}^0) \cap \ker(\Delta, \partial^\dagger_{\underline{z}}, \partial^J_{\underline{z}}, P, \partial_{\underline{z}}, \partial^{\dagger J}_{\underline{z}})$, which means that $K^0_{a,b}(\underline{z}, \underline{z}^\dagger)$ is anti-holomorphic in the variables $(\overline{z}_1, \ldots, \overline{z}_{2p})$ (see [2]). So the homogeneity degree a must be zero, the term $K^1_{a-1,b}$ does not appear and we have

$$K^0_{0,b} + \underline{z}^{\dagger J} K^1_{0,b-1} = 0.$$

By applying the operator $\partial^{\dagger J}_{\underline{z}}$ to this equality, we get $bK^1_{0,b-1} = 0$. This gives $K^1_{0,b-1} = 0 = K^0_{0,b}$, which finishes the proof. □

Proof of Theorem 2
It follows from Lemmata 2 and 3. □

4 Irreducibility of spaces of $\mathfrak{osp}(4|2)$–monogenics

In this section we prove the $\mathrm{Sp}(p)$–irreducibity of the modules $\mathcal{S}^r_{a,b}$ of $\mathfrak{osp}(4|2)$–monogenics. First, let us introduce some notations.

Notation 1. *Let Λ be the set of partitions $\lambda=(\lambda_1,\dots,\lambda_p)$ such that $\lambda_1\geq\lambda_2\geq\cdots\geq\lambda_p\geq 0$, $\lambda_1,\lambda_2\in\mathbb{Z}$ and $\lambda_j\in\{0,1\}$ for $j=3,\dots,p$. For $\lambda\in\Lambda$, denote*

$$\begin{aligned}\mathcal{S}_\lambda &= \mathcal{S}^0_{0,0} \quad \text{for } \lambda=(0,0);\\ &= \mathcal{S}^1_{a,0} \quad \text{for } \lambda=(a+1,0),\ a\geq 0;\\ &= \mathcal{S}^r_{a,b} \quad \text{for } \lambda=(a+1,b+1,1_{r-2}),\ a,b\geq 0, r\geq 2.\end{aligned}$$

We will use the same notation for other polynomial spaces indexed by the indices a,b,r as well.

Remark 6. Note that, in Corollary 1 below, we will show that $\mathcal{S}^0_{a,b}=0$ for $(a,b)\neq(0,0)$ and $\mathcal{S}^1_{a,b}=0$ for $b\neq 0$.

In [5], we showed that, for $a\geq b\geq 0$, the space $\mathcal{H}^S_{a,b}:=\mathcal{P}_{a,b}(\mathbb{R}^{4p};\mathbb{C})\cap\ker(\Delta,\mathcal{E})$ forms an irreducible $\mathrm{Sp}(p)$-module with highest weight (a,b). For $a\geq b\geq 0$ and $r=0,\dots,p$, denote by $\widetilde{\mathcal{S}}^r_{a,b}$ the Cartan product of $\mathcal{H}^S_{a,b}$ and $\mathbb{S}^r_r$, that is, the unique irreducible submodule of $\mathcal{H}^S_{a,b}\otimes\mathbb{S}^r_r$ with highest weight $(a,b)+(1_r)$. To prove that $\widetilde{\mathcal{S}}_\lambda=\mathcal{S}_\lambda$, for $\lambda\in\Lambda$, we need the following lemma.

Lemma 4. *For $\lambda\in\Lambda$ it holds that $0\neq\widetilde{\mathcal{S}}_\lambda\subset\mathcal{S}_\lambda$.*

Proof
It is easily verified that $h_{a,b}:=z_1^{a-b}(z_1\bar{z}_4-\bar{z}_2z_3)^b$ is a highest weight vector of $\mathcal{H}^S_{a,b}$ and that $v_r:=\mathfrak{f}_1^\dagger\mathfrak{f}_3^\dagger\cdots\mathfrak{f}_{2r-1}^\dagger I$ is a highest weight vector of $\mathbb{S}^r_r$. Then $s^r_{a,b}:=h_{a,b}\cdot v_r$ is a highest weight vector of $\widetilde{\mathcal{S}}^r_{a,b}$. Finally, it is easy to see that $s_\lambda\in\mathcal{S}_\lambda$, for all $\lambda\in\Lambda$, which completes the proof. □

We will now decompose the space of polynomials into isotypic components with respect to the symplectic group $\mathrm{Sp}(p)$.

Proposition 3. *One has*

$$\mathcal{P}(\mathbb{R}^m;\mathbb{S})=\bigoplus_{\lambda\in\Lambda}\mathtt{I}_\lambda \tag{6}$$

where $\mathtt{I}_\lambda$ is the $\mathrm{Sp}(p)$–isotypic component of $\mathcal{P}(\mathbb{R}^m,\mathbb{S})$ with highest weight λ. In addition, $\mathtt{I}_\lambda\neq 0$, for $\lambda\in\Lambda$.

Proof
Using the results from [5] on the Fischer decomposition for $\mathbb{C}$-valued polynomials, we obtained in [6] the following decomposition for the space of spinor–valued polynomials:

$$\mathcal{P}(\mathbb{R}^m;\mathbb{S})=\bigoplus_{j=0}^{\infty}\bigoplus_{a\geq b\geq 0}\bigoplus_{k=0}^{a-b}\bigoplus_{r=0}^{p}\bigoplus_{\ell=0}^{p-r}|\underline{z}|^{2j}(\mathcal{E}^\dagger)^kQ^\ell\big(\mathcal{H}^S_{a,b}\otimes\mathbb{S}^r_r\big). \tag{7}$$

Moreover, any irreducible submodule of $\mathcal{H}^S_{a,b}\otimes\mathbb{S}^r_r$ has the highest weight λ from Λ, that is, $\mathtt{I}_\lambda\neq 0$ implies $\lambda\in\Lambda$. Indeed, it is well known that any highest weight in $\mathcal{H}^S_{a,b}\otimes\mathbb{S}^r_r$ is a sum of the highest weight (a,b) of the first factor and a weight of the fundamental representation $\mathbb{S}^r_r$. But the components of all weights in a fundamental representation of $\mathrm{Sp}(p)$ are at most one. In addition, Lemma 4 shows that, for $\lambda\in\Lambda$, $\mathtt{I}_\lambda\neq 0$. □

In the previous section we obtained, in Theorem 2, the decomposition (2) of $\mathtt{H}^r_{a,b}$ in terms of spaces of $\mathfrak{osp}(4|2)$–monogenics, where the embedding factors contain 'words' involving an alphabet of (isotypic) letters. The final goal is of course to relate the decompositions (2) and (6). To that end we will first focus on the space $\mathtt{H}(\mathbb{R}^{4p};\mathbb{S}) = \mathcal{P}(\mathbb{R}^{4p};\mathbb{S}) \cap \ker(\Delta, P, \partial^\dagger_{\underline{z}}, \partial^J_{\underline{z}})$, which, quite naturally, can be decomposed as

$$\mathtt{H}(\mathbb{R}^{4p};\mathbb{S}) = \bigoplus_{a,b\geq 0} \bigoplus_{r=0}^{p} \mathtt{H}^r_{a,b}$$

Using (2) this decomposition takes the form

$$\bigoplus_{a,b\geq 0} \bigoplus_{r=0}^{p} \bigoplus_{j=0}^{b} ((\mathcal{E}^\dagger)^j \mathcal{S}^r_{a+j,b-j} \oplus \underline{z}\,(\mathcal{E}^\dagger)^j \mathcal{S}^{r+1}_{a+j-1,b-j} \oplus \underline{z}^{\dagger J}(\mathcal{E}^\dagger)^j \mathcal{S}^{r+1}_{a+j,b-j-1} \oplus \underline{z}\underline{z}^{\dagger J}(\mathcal{E}^\dagger)^j \mathcal{S}^{r+2}_{a+j-1,b-j-1})$$

where, as above, it again is tacitly assumed that if indices make no sense, then the corresponding spaces do not appear in the decomposition. Now slightly reordering the summands we arrive at

$$\mathtt{H}(\mathbb{R}^{4p},\mathbb{S}) = \bigoplus_{a\geq b\geq 0} \bigoplus_{r=0}^{p} \Theta^r_{a,b}$$

having defined the spaces

$$\Theta^r_{a,b} := \mathcal{S}^r_{a,b} \oplus \underline{z}\mathcal{S}^r_{a,b} \oplus \underline{z}^{\dagger J}\mathcal{S}^r_{a,b} \oplus \underline{z}\underline{z}^{\dagger J}\mathcal{S}^r_{a,b} \oplus \mathcal{E}^\dagger\mathcal{S}^r_{a,b} \oplus \underline{z}\mathcal{E}^\dagger\mathcal{S}^r_{a,b} \oplus \underline{z}^{\dagger J}\mathcal{E}^\dagger\mathcal{S}^r_{a,b} \oplus \underline{z}\underline{z}^{\dagger J}\mathcal{E}^\dagger\mathcal{S}^r_{a,b} \oplus \ldots$$

In other words, for defining $\Theta^r_{a,b}$ we have gathered all the 'shifted copies' of a fixed space $\mathcal{S}^r_{a,b}$ of $\mathfrak{osp}(4|2)$–monogenics into one space. For a given $\lambda \in \Lambda$, we use the notation Θ_λ according to Notation 1. At the same time we introduce the space $\widetilde{\Theta}_\lambda$ as a direct sum of shifted copies (in the same sense as above) of vector spaces $\widetilde{\mathcal{S}}_\lambda$, by

$$\widetilde{\Theta}_\lambda := \widetilde{\mathcal{S}}_\lambda \oplus \underline{z}\widetilde{\mathcal{S}}_\lambda \oplus \underline{z}^{\dagger J}\widetilde{\mathcal{S}}_\lambda \oplus \underline{z}\underline{z}^{\dagger J}\widetilde{\mathcal{S}}_\lambda \oplus \mathcal{E}^\dagger\widetilde{\mathcal{S}}_\lambda \oplus \underline{z}\mathcal{E}^\dagger\widetilde{\mathcal{S}}_\lambda \oplus \underline{z}^{\dagger J}\mathcal{E}^\dagger\widetilde{\mathcal{S}}_\lambda \oplus \underline{z}\underline{z}^{\dagger J}\mathcal{E}^\dagger\widetilde{\mathcal{S}}_\lambda \oplus \ldots$$

At this point we can only be certain about the following inclusion, as each space $\widetilde{S}_\lambda$ is a subspace of $\mathcal{S}_\lambda$ by Lemma 4:

$$\mathtt{H}(\mathbb{R}^{4p};\mathbb{S}) = \bigoplus_{a,b;r} \Theta^r_{a,b} \supset \bigoplus_{\lambda\in\Lambda} \Theta_\lambda \supset \bigoplus_{\lambda\in\Lambda} \widetilde{\Theta}_\lambda$$

The crux of the argument now comes from Howe's original paper [11], which contains the necessary material to prove that the utmost right–hand side is again equal to the space $\mathtt{H}(\mathbb{R}^{4p},\mathbb{S})$, from which the conclusion obviously follows. In particular, we need Theorem 9 (*ii*) of [11] stating that, with slightly different notations adapted to the present setting, we have the direct sum decomposition

$$\mathtt{H}(\mathbb{R}^{4p};\mathbb{S}) = \bigoplus_{\lambda\in\Lambda} \mathtt{H}(\mathbb{R}^{4p};\mathbb{S}) \cap \mathtt{I}_\lambda$$

where $\mathtt{I}_\lambda$ are the $\mathrm{Sp}(p)$–isotypic components introduced above. We now claim (see Lemma 5 below) that

$$\widetilde{\Theta}_\lambda = \mathtt{H}(\mathbb{R}^{4p};\mathbb{S}) \cap \mathtt{I}_\lambda$$

leading to

$$\mathtt{H}(\mathbb{R}^{4p};\mathbb{S}) = \bigoplus_{a,b;r} \Theta^r_{a,b} \supset \bigoplus_{\lambda\in\Lambda} \Theta_\lambda \supset \bigoplus_{\lambda\in\Lambda} \widetilde{\Theta}_\lambda = \mathtt{H}(\mathbb{R}^{4p};\mathbb{S})$$

from which we can indeed conclude that, for $\lambda \in \Lambda$, $\widetilde{\Theta}_\lambda = \Theta_\lambda$ and $\widetilde{\mathcal{S}}_\lambda = \mathcal{S}_\lambda$. In addition, it is clear that $\mathcal{S}^0_{a,b} = 0$ for $(a,b) \neq (0,0)$ and $\mathcal{S}^1_{a,b} = 0$ for $b \neq 0$.

Lemma 5. *With the notations from above, one has, for all $\lambda \in \Lambda$, that*

$$\widetilde{\Theta}_\lambda = \mathtt{H}(\mathbb{R}^{4p}, \mathbb{S}) \cap \mathtt{I}_\lambda$$

Proof

We refer to Theorem 9 (*iii*) of [11] where it is shown that a special role is played by the $\mathrm{Sp}(p) \times \Gamma'(1,1)$–irreducible submodules $\mathtt{H}(\mathbb{R}^{4p}, \mathbb{S}) \cap \mathtt{I}_\lambda$. To start with, we note that $\widetilde{\Theta}_\lambda$ is invariant under the action of $\Gamma'(1,1)$. Indeed, this is ensured by the commutation relation

$$\left[\Gamma'(1,1), \Gamma'(0,2)\right] \subset \Gamma'(0,2)$$

On the other hand, $\widetilde{\Theta}_\lambda$ is a non-trivial submodule of $\mathtt{H}(\mathbb{R}^{4p}; \mathbb{S}) \cap \mathtt{I}_\lambda$, which implies that they are equal since the latter is irreducible. $\square$

Corollary 1. *One has $\mathcal{S}^0_{a,b} = 0$ for $(a,b) \neq (0,0)$, and $\mathcal{S}^1_{a,b} = 0$ for $b \neq 0$. Otherwise, $\mathcal{S}^r_{a,b}$ is an* $\mathrm{Sp}(p)$*–irreducible module with highest weight $(a,b) + (1_r)$.*

Remark 7. Let $a \geq b \geq 0$. It is easily seen that

$$\begin{aligned} \tilde{\mathcal{S}}^0_{a,b} &= \underline{z}\, \mathcal{S}^1_{a-1,0} && \text{if } a > 0, b = 0; \\ &= \underline{z}\underline{z}^{\dagger J} \mathcal{S}^2_{a-1,b-1} && \text{if } b > 0; \\ \tilde{\mathcal{S}}^1_{a,b} &= (\underline{z}^{\dagger J} - \frac{1}{a-b+2} \mathcal{E}^\dagger \underline{z})\, \mathcal{S}^2_{a,b-1} && \text{if } b > 0. \end{aligned}$$

Indeed, the spaces at the right–hand side belong to $\ker(\Delta, \mathcal{E}, P)$, see [6] for details.

5 Decomposition of spaces of symplectic harmonics

In this section we prove the irreducible decomposition of spaces of symplectic harmonics under the action of $\mathrm{Sp}(p)$, see also [6, Proposition 9]. To that end we need to introduce the following projections. Denote by $\pi_{\ker(\Delta)}$, $\pi_{\ker(\mathcal{E})}$ and $\pi_{\ker(P)}$ the projection operators mapping $\mathcal{P}(\mathbb{R}^{4p}; \mathbb{S})$ onto $\ker(\Delta)$, $\ker(\mathcal{E})$ and $\ker(P)$, respectively. It is easy to see that these projections commute with each other and that they all are $\mathrm{Sp}(p)$-invariant, see [6] for details. Put $\pi = \pi_{\ker(\Delta)} \circ \pi_{\ker(P)} \circ \pi_{\ker(\mathcal{E})}$. Further on, for $A_1 := \{1, \underline{z}, \underline{z}^{\dagger J}, \underline{z}\underline{z}^{\dagger J}\}$ and $A_2 := \{1, \underline{z}^\dagger, \underline{z}^J, \underline{z}^\dagger \underline{z}^J\}$, put

$$A := \{w_1 w_2 |\ w_1 \in A_1, w_2 \in A_2\}.$$

Finally, for integers $a \geq b \geq 0$ and $r = 0, \dots, p$, we define the sets

$$\begin{aligned} A^r_{a,b} &:= A && \text{for } r \neq p \text{ and } a \neq b; \\ &:= A \setminus \{\underline{z}^{\dagger J} \underline{z}^J\} && \text{for } r = p \text{ or } a = b. \end{aligned}$$

Theorem 3. *Under the action of the group* Sp*(p), one has the irreducible decomposition*

$$\mathcal{H}^S_{a,b} \otimes \mathbb{S}^r_r = \bigoplus_{w \in A^r_{a,b}} \pi(\mathcal{S}^{r,w}_{a,b}) \tag{8}$$

where $\mathcal{S}^{r,w}_{a,b} := w\, \mathcal{S} \cap \mathcal{P}^r_{a,b}$. In addition, $\mathcal{S}^{r,w}_{a,b} = w\, \mathcal{S}^{r'}_{a',b'}$ for some (uniquely determined) a', b', r'. Moreover, for $r = p$ or $a = b$, one has $\pi(\underline{z}\underline{z}^\dagger \mathcal{S}^r_{a-1,b-1}) = \pi(\underline{z}^{\dagger J} \underline{z}^J \mathcal{S}^r_{a-1,b-1})$.

Remark 8. It is important to notice that in the direct sum (8) some summands might be trivial. For example, if $r = p$ and $a = b$, then $\pi(\underline{z}\underline{z}^\dagger \mathcal{S}^r_{a-1,b-1}) = \pi(\underline{z}^{\dagger J} \underline{z}^J \mathcal{S}^r_{a-1,b-1}) = 0$.

Remark 9. Obviously, the projection $\pi(w\,\mathcal{S}^{r'}_{a',b'})$ does not depend on the order of $\underline{z}, \underline{z}^\dagger, \underline{z}^J, \underline{z}^{\dagger J}$ in the word w. Indeed, $\underline{z}$ and $\underline{z}^\dagger$ anticommute with $\underline{z}^J$ and $\underline{z}^{\dagger J}$. Moreover, we have e.g.

$$\pi(\underline{z}\underline{z}^\dagger w'\,\mathcal{S}^{r'}_{a',b'}) = \pi(\underline{z}^\dagger \underline{z}\, w'\,\mathcal{S}^{r'}_{a',b'}) \quad \text{and} \quad \pi(w'\,\underline{z}^J \underline{z}^{\dagger J} \mathcal{S}^{r'}_{a',b'}) = \pi(w'\,\underline{z}^{\dagger J} \underline{z}^J \mathcal{S}^{r'}_{a',b'})$$

because $\underline{z}^\dagger \underline{z} = -\underline{z}\underline{z}^\dagger + r^2$ and $\underline{z}^{\dagger J}\underline{z}^J = -\underline{z}^J \underline{z}^{\dagger J} + r^2$.

Remark 10. In [6], explicit expressions for the projections $\pi(\mathcal{S}^{r,w}_{a,b})$ of (8) are given. However, the fact that, for $r = p$ or $a = b$, the projections $\pi(\underline{z}\underline{z}^\dagger \mathcal{S}^r_{a-1,b-1})$ and $\pi(\underline{z}^{\dagger J}\underline{z}^J \mathcal{S}^r_{a-1,b-1})$ coincide, was not observed.

We prove Theorem 3 in several steps.

Denoting by $\mathcal{Q} = \mathcal{Q}(\mathbb{R}^{4p}, \mathbb{S})$ the space of quaternionic monogenic polynomials on $\mathbb{R}^{4p}$, viz polynomials in $\mathcal{P}(\mathbb{R}^{4p}, \mathbb{S})$ which are in the kernel of the operators $\partial_{\underline{z}}$, $\partial^\dagger_{\underline{z}}$, $\partial^J_{\underline{z}}$, $\partial^{\dagger J}_{\underline{z}}$, we put, for $a, b \geq 0$ and $r \in \{0, 1, \ldots, 2p\}$, $\mathcal{Q}^r_{a,b} = \mathcal{Q} \cap \mathcal{P}_{a,b}(\mathbb{R}^{4p}, \mathbb{S}^r)$.

Lemma 6. *One has that*

$$\mathcal{P}^r_{a,b} = \mathcal{Q}^r_{a,b} \oplus \left(\underline{z}\mathcal{P}^{r+1}_{a-1,b} + \underline{z}^\dagger \mathcal{P}^{r-1}_{a,b-1} + \underline{z}^J \mathcal{P}^{r-1}_{a-1,b} + \underline{z}^{\dagger J}\mathcal{P}^{r+1}_{a,b-1}\right)$$

with similar assumptions on the indices as in Theorem 1.

Proof
The decomposition follows from the fact that

$$\mathcal{Q}^r_{a,b} = \left(\underline{z}\mathcal{P}^{r+1}_{a-1,b} + \underline{z}^\dagger \mathcal{P}^{r-1}_{a,b-1} + \underline{z}^J \mathcal{P}^{r-1}_{a-1,b} + \underline{z}^{\dagger J}\mathcal{P}^{r+1}_{a,b-1}\right)^\perp$$

with respect to the Fischer inner product on $\mathcal{P}^r_{a,b}$. □

Proposition 4. *One has that*

$$\mathcal{P}(\mathbb{R}^{4p}, \mathbb{S}) = \sum_{j,k,\ell \geq 0} \sum_{a \geq b \geq 0} \sum_{r=0}^{p} \sum_{w \in A} |\underline{z}|^{2j} (\mathcal{E}^\dagger)^k Q^\ell\, w\, \mathcal{S}^r_{a,b}$$

Proof
Using Lemma 6 inductively we obtain

$$\mathcal{P}(\mathbb{R}^{4p}, \mathbb{S}) = \sum_{a,b \geq 0} \sum_{r=0}^{2p} \sum_{w} w\, \mathcal{Q}^r_{a,b} \tag{9}$$

where the last sum is taken over all finite compositions w of the multiplicative operators $\underline{z}$, $\underline{z}^\dagger$, $\underline{z}^J$ and $\underline{z}^{\dagger J}$. We know that $\mathcal{Q}^r_{a,b} = \mathcal{Q}^{r,r}_{a,b} \oplus \cdots \oplus \mathcal{Q}^{2p-r,r}_{a,b}$ and, by [6, Corollary 7.1],

$$\mathcal{Q}^{r+2\ell,r}_{a,b} = \mathcal{E}^{\dagger(b-a)} Q^\ell \mathcal{S}^r_{b,a} \oplus \mathcal{E}^{\dagger(b-a+1)} Q^\ell \mathcal{S}^r_{b+1,a-1} \oplus \cdots \oplus \mathcal{E}^{\dagger b} Q^\ell \mathcal{S}^r_{b+a,0}.$$

Using the fact that Q and $\mathcal{E}^\dagger$ are commuting and invoking the commutation relations

$$\begin{array}{llll} [\mathcal{E}^\dagger, \underline{z}] = \underline{z}^{\dagger J} & [\mathcal{E}^\dagger, \underline{z}^\dagger] = 0 & [\mathcal{E}^\dagger, \underline{z}^J] = -\underline{z}^\dagger & [\mathcal{E}^\dagger, \underline{z}^{\dagger J}] = 0 \\ [Q, \underline{z}] = \underline{z}^J & [Q, \underline{z}^\dagger] = 0 & [Q, \underline{z}^J] = 0 & [Q, \underline{z}^{\dagger J}] = -\underline{z}^\dagger \end{array}$$

it is clear that we can refine decomposition (9) as

$$\mathcal{P}(\mathbb{R}^{4p},\mathbb{S}) = \sum_{k,\ell\geq 0}\sum_{a\geq b\geq 0}\sum_{r=0}^{p}\sum_{w}(\mathcal{E}^\dagger)^k\,Q^\ell\,w\,\mathcal{S}^r_{a,b}$$

where again the last sum is taken over all finite compositions w of the operators $\underline{z}$, $\underline{z}^\dagger$, $\underline{z}^J$ and $\underline{z}^{\dagger J}$. As the operators $\underline{z}$ and $\underline{z}^\dagger$ anticommute with $\underline{z}^J$ and $\underline{z}^{\dagger J}$ it suffices to take w of the form $\widetilde{w}_1\widetilde{w}_2$ where $\widetilde{w}_1$, respectively $\widetilde{w}_2$, is a finite composition of the operators $\underline{z}$ and $\underline{z}^\dagger$, respectively $\underline{z}^J$ and $\underline{z}^{\dagger J}$. It is easy to see that $\widetilde{w}_1=1$ or $\widetilde{w}_1=r^{2j_1}\,w_1$ for some $j_1\geq 0$ and $w_1\in\{\underline{z},\underline{z}^\dagger,\underline{z}\underline{z}^\dagger,\underline{z}^\dagger\underline{z}\}$, and $\widetilde{w}_2=1$ or $\widetilde{w}_2=r^{2j_2}\,w_2$ for some $j_2\geq 0$ and $w_2\in\{\underline{z}^J,\underline{z}^{\dagger J},\underline{z}^J\underline{z}^{\dagger J},\underline{z}^{\dagger J}\underline{z}^J\}$. By $r^2=\underline{z}\underline{z}^\dagger+\underline{z}^\dagger\underline{z}$ and $r^2=\underline{z}^J\underline{z}^{\dagger J}+\underline{z}^{\dagger J}\underline{z}^J$ the proof is concluded. □

Proof of Theorem 3
By Proposition 4, we have

$$\mathcal{P}_{a,b}(\mathbb{R}^{4p},\mathbb{S}^r) = \sum |\underline{z}|^{2j}\,(\mathcal{E}^\dagger)^k\,Q^\ell\,w\,\mathcal{S}^r_{a,b}$$

where the sum is taken over all j,k,ℓ,a,b,r,w such that $|\underline{z}|^{2j}\,(\mathcal{E}^\dagger)^k\,Q^\ell\,w\,\mathcal{S}^r_{a,b}\subset\mathcal{P}_{a,b}(\mathbb{R}^{4p},\mathbb{S}^r)$. Obviously, applying the projection π to this sum, we get

$$\mathcal{H}^S_{a,b}\otimes\mathbb{S}^r_r = \sum_{w\in A}\pi(\mathcal{S}^{r,w}_{a,b}) \tag{10}$$

where $\mathcal{S}^{r,w}_{a,b}:=w\,\mathcal{S}\cap\mathcal{P}^r_{a,b}$. The index set A has 16 elements and, for $w\in A$, we have $\mathcal{S}^{r,w}_{a,b}:=w\,\mathcal{S}^{r'}_{a',b'}$ where $(a',b',r')=(a,b,r)-\delta(w)$, with the shift $\delta(w)$ given in the following table:

(α)	$\delta(1)=(0,0,0)$,	$\delta(\underline{z}^{\dagger J})=(0,1,-1)$,	$\delta(\underline{z})=(1,0,-1)$,	$\delta(\underline{z}\underline{z}^{\dagger J})=(1,1,-2)$,
(β)	$\delta(\underline{z}^\dagger)=(0,1,1)$,	$\delta(\underline{z}^{\dagger J}\underline{z}^\dagger)=(0,2,0)$,	$\delta(\underline{z}\underline{z}^\dagger)=(1,1,0)$,	$\delta(\underline{z}\underline{z}^{\dagger J}\underline{z}^\dagger)=(1,2,-1)$,
(γ)	$\delta(\underline{z}^J)=(1,0,1)$,	$\delta(\underline{z}^{\dagger J}\underline{z}^J)=(1,1,0)$,	$\delta(\underline{z}\underline{z}^J)=(2,0,0)$,	$\delta(\underline{z}\underline{z}^{\dagger J}\underline{z}^J)=(2,1,-1)$,
(δ)	$\delta(\underline{z}^\dagger\underline{z}^J)=(1,1,2)$,	$\delta(\underline{z}^{\dagger J}\underline{z}^\dagger\underline{z}^J)=(1,2,1)$,	$\delta(\underline{z}\underline{z}^\dagger\underline{z}^J)=(2,1,1)$,	$\delta(\underline{z}\underline{z}^{\dagger J}\underline{z}^\dagger\underline{z}^J)=(2,2,0)$.

If $\pi(w\mathcal{S}^{r'}_{a',b'})\neq 0$ then $\pi(w\mathcal{S}^{r'}_{a',b'})$ is an Sp(p)–irreducible module with highest weight $(a',b')+(1_{r'})$. Moreover, it is obvious that if non-trivial submodules $\pi(w_1\mathcal{S}^{r'_1}_{a'_1,b'_1})$ and $\pi(w_2\mathcal{S}^{r'_2}_{a'_2,b'_2})$ in the decomposition are equivalent, then $(a'_1,b'_1,r'_1)=(a'_2,b'_2,r'_2)$ and $\delta(w_1)=\delta(w_2)$, i.e., either $w_1=w_2$ or $w_1,w_2\in\{\underline{z}\underline{z}^\dagger,\underline{z}^{\dagger J}\underline{z}^J\}$. Now, put $\widetilde{A}=A\setminus\{\underline{z}\underline{z}^\dagger,\underline{z}^{\dagger J}\underline{z}^J\}$; then we have that

$$\mathcal{H}^S_{a,b}\otimes\mathbb{S}^r_r = \bigoplus_{w\in\widetilde{A}}\pi(\mathcal{S}^{r,w}_{a,b})\oplus(\pi(\underline{z}\underline{z}^\dagger\mathcal{S}^r_{a-1,b-1})+\pi(\underline{z}^{\dagger J}\underline{z}^J\mathcal{S}^r_{a-1,b-1}))$$

First we show that, for $r=p$ or $a=b$, $\pi(\underline{z}\underline{z}^\dagger\mathcal{S}^r_{a-1,b-1})=\pi(\underline{z}^{\dagger J}\underline{z}^J\mathcal{S}^r_{a-1,b-1})$. In fact, we have $[Q,\underline{z}^{\dagger J}\underline{z}]=\underline{z}^{\dagger J}\underline{z}^J-\underline{z}^\dagger\underline{z}=\underline{z}^{\dagger J}\underline{z}^J-r^2+\underline{z}\underline{z}^\dagger$. Moreover, $[\mathcal{E}^\dagger,\underline{z}\underline{z}^J]=\underline{z}^{\dagger J}\underline{z}^J-\underline{z}\underline{z}^\dagger$. As $\mathcal{S}^p_{a-1,b-1}\subset\ker(Q)$ and $\mathcal{S}^r_{a-1,a-1}\subset\ker(\mathcal{E}^\dagger)$ we get

$$\underline{z}^{\dagger J}\underline{z}^J\mathcal{S}^p_{a-1,b-1}=(-\underline{z}\underline{z}^\dagger+r^2+Q\underline{z}^{\dagger J}\underline{z})\mathcal{S}^p_{a-1,b-1}\quad\text{and}\quad\underline{z}^{\dagger J}\underline{z}^J\mathcal{S}^r_{a-1,a-1}=(\underline{z}\underline{z}^\dagger+\mathcal{E}^\dagger\underline{z}\underline{z}^J)\mathcal{S}^r_{a-1,a-1}, \tag{11}$$

which implies the required equality. Moreover, if $r=p$ and $a=b$, then $\pi(\underline{z}\underline{z}^\dagger\mathcal{S}^r_{a-1,b-1})=\pi(\underline{z}^{\dagger J}\underline{z}^J\mathcal{S}^r_{a-1,b-1})=0$ since, by (11), we have

$$\begin{aligned}2\underline{z}^{\dagger J}\underline{z}^J\mathcal{S}^p_{a-1,a-1}&=(r^2+Q\underline{z}^{\dagger J}\underline{z}+\mathcal{E}^\dagger\underline{z}\underline{z}^J)\mathcal{S}^p_{a-1,a-1},\\ 2\underline{z}\underline{z}^\dagger\mathcal{S}^p_{a-1,a-1}&=(r^2+Q\underline{z}^{\dagger J}\underline{z}-\mathcal{E}^\dagger\underline{z}\underline{z}^J)\mathcal{S}^p_{a-1,a-1}.\end{aligned} \tag{12}$$

On the other hand, for $r = 0, \ldots, p-1$ and $a > b$,

$$\pi(\underline{z}\underline{z}^\dagger \mathcal{S}^r_{a-1,b-1}) \cap \pi(\underline{z}^{\dagger J}\underline{z}^J \mathcal{S}^r_{a-1,b-1}) = 0.$$

Indeed, assume that $2 \leq r \leq p-1$ and $a > b > 0$, otherwise we have $\mathcal{S}^r_{a-1,b-1} = 0$. Then $\pi(\underline{z}\underline{z}^\dagger \mathcal{S}^r_{a-1,b-1})$ and $\pi(\underline{z}^{\dagger J}\underline{z}^J \mathcal{S}^r_{a-1,b-1})$ are two disjoint irreducible submodules in the decomposition with the same highest weight $(a, b, 1_{r-2})$. Indeed, by [12, Proposition 5.5], the multiplicity of $(a, b, 1_{r-2})$ in the tensor product of $\mathrm{Sp}(p)$-modules $(a, b) \otimes (1_r)$ is 2. Namely, we get the Young diagram corresponding to $(a, b, 1_{r-2})$ from the Young diagram (a, b) by removing 1 box and then adding $r-1$ boxes just in two different ways if we can add at most one box in each row. This completes the proof. □

6 The Fischer decomposition for the symplectic group

In the previous sections we have obtained all necessary results in order to finally prove the symplectic Fischer decomposition.

Theorem 4. *With $A^r_{a,b}$ and $\pi(\mathcal{S}^{r,w}_{a,b})$ as in Theorem 3, the* $\mathrm{Sp}(p)$*–module* $\mathcal{P}(\mathbb{R}^{4p}, \mathbb{S})$ *shows the irreducible decomposition*

$$\mathcal{P}(\mathbb{R}^{4p}, \mathbb{S}) = \bigoplus_{j,k,\ell \geq 0} \bigoplus_{a \geq b \geq 0} \bigoplus_{r=0}^{p} \bigoplus_{w \in A^r_{a,b}} |\underline{z}|^{2j} \, (\mathcal{E}^\dagger)^k \, Q^\ell \, \pi(\mathcal{S}^{r,w}_{a,b})$$

Proof
We use the decomposition (7)

$$\mathcal{P}(\mathbb{R}^{4p}, \mathbb{S}) = \bigoplus_{j,k,\ell \geq 0} \bigoplus_{a \geq b \geq 0} \bigoplus_{r=0}^{p} |\underline{z}|^{2j} \, (\mathcal{E}^\dagger)^k \, Q^\ell \, (\mathcal{H}^S_{a,b} \otimes \mathbb{S}^r_r)$$

and decompose $\mathcal{H}^S_{a,b} \otimes \mathbb{S}^r_r$ according to Theorem 3. □

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