Een minimale representatie voor de orthosymplectische Lie-superalgebra

A minimal representation of the orthosymplectic Lie superalgebra

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Woord vooraf

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[1] https://en.wikipedia.org/wiki/Abstract_nonsense

kunt eten in Gent. Me wijzen op het boek 'A Primer of Mathematical Writing' over hoe je wiskundige teksten behoort te schrijven had hij misschien wel beter gedaan voor ik aan deze thesis begon te schrijven in plaats van juist nadat ik het naar de jury had doorgestuurd.

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[2]Assisterend academisch personeel

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Contents

C'est véritablement utile puisque c'est joli.

Antoine de Saint-Exupéry,
Le petit prince

1 Introduction

Minimal representations

An old but still open problem in representation theory is a description of the equivalence classes of irreducible, unitary representations of (real reductive) Lie groups. One way to make progress on this problem is the orbit method, [Ki].

The orbit philosophy is a guiding principle in the representation theory of Lie groups and suggests a relation between irreducible unitary representations and coadjoint orbits. These are the orbits under the action of the Lie group on the dual of its Lie algebra. For nilpotent groups, or more generally solvable groups, it can be used to establish a bijective correspondence between coadjoint orbits and irreducible unitary representations, but already for the semisimple group $SL(2,\mathbb{R})$ this correspondence does not cover the whole unitary dual. One of the main problems is the quantisation of nilpotent coadjoint orbits of semisimple groups, which are expected to correspond to rather small unitary representations.

This is where minimal representations enter the picture. Minimal representations are the irreducible unitary representations of semisimple

Lie groups which are supposed to correspond to a minimal nilpotent coadjoint orbit. Prominent examples are the Segal–Shale–Weil representation of the metaplectic group $Mp(n, \mathbb{R})$, [Fo], which is a double cover of the symplectic group, or the more recently studied minimal representation of $O(p, q)$, [BZ, KM2, KØ].

More technically, a unitary representation of a simple real Lie group G is called minimal if the annihilator ideal of the derived representation of the universal enveloping algebra of the Lie algebra of G is the Joseph ideal. The Joseph ideal is the unique completely prime, two-sided ideal in the universal enveloping algebra such that the associated variety is the closure of the minimal nilpotent coadjoint orbit (see [GS]). Minimal representations have been constructed in various different ways, algebraically, analytically, or through Howe's theta correspondence.

L^2-models

For a minimal representation, the Gelfand–Kirillov dimension which measures the size of an infinite-dimensional representation attains its minimum among all infinite-dimensional unitary representations. Therefore, explicit geometric realisations of minimal representations are expected to have large symmetries and allow interactions with other mathematical areas such as conformal geometry, integral operators or special functions (see [KØ, KM2, HKMM]). In every known realisation, some aspects of the representations are rather clear to describe and some are more subtle. One realisation in which for instance the invariant inner product is particularly easy to see is the L^2-model (also called Schrödinger model) which is due to Vergne–Rossi [VR], Dvorsky–Sahi [DS] and Kobayashi–Ørsted [KØ]. Here the representation is realised on the Hilbert space $L^2(C)$ where C is a homogeneous space for a subgroup of G. The three constructions in [VR, DS, KØ] are different in nature, and only more recently a unified construction of L^2-models of minimal representations was developed in [HKM], using the framework of Jordan algebras. The approach consists of the following steps:

- Start from a simple real Jordan algebra.
- Consider the Tits–Kantor–Koecher (TKK) Lie algebra of the

Jordan algebra.

- Construct a representation of the TKK algebra on functions on the Jordan algebra.
- Identify a minimal orbit of the structure group of the Jordan algebra and restrict the representation to functions on this orbit.
- Find an admissible subrepresentation which integrates to the conformal group.
- Show that this representation is unitary with respect to an L^2-inner product on the minimal orbit.
- Show that the constructed representation is indeed a minimal representation.

We remark that the indefinite orthogonal groups $G = O(p, q)$ are special among the cases discussed above, since their corresponding minimal representations are in general neither spherical nor highest/lowest weight representations. This makes them harder to construct than in the remaining cases, but as a consequence their L^2-models allow a richer analysis.

Minimal representations of Lie supergroups

Supersymmetry is a framework introduced in the seventies to consider bosons and fermions at the same level [SS, WZ]. Lie supergroups and Lie superalgebras are the mathematical concepts underlying supersymmetry. Since the ingredients of the orbit method also exist in the super case, it is expected that the orbit method is a useful tool also in the study of irreducible representations of Lie supergroups [Ki, Chapter 6.3]. For example, the orbit method provides a classification of irreducible unitary representations of nilpotent Lie supergroups (see [Sal, NS]). With this perspective in mind, it is natural to ask for a super version of minimal representations.

The goal of this thesis is to start a systematic study of minimal representations of Lie supergroups. In particular, we will construct a minimal representation of the orthosymplectic Lie supergroup $OSp(p, q|2n)$

following the same approach as in [HKM]. Therefore we need the following concepts in the super case:

- Jordan superalgebras,
- TKK algebra of a Jordan superalgebra,
- a representation of the TKK algebra on functions on the Jordan superalgebra,
- a minimal orbit to which the representation restricts,
- an admissible subrepresentation which integrates to the group level,
- an invariant inner product,
- minimality of the constructed representation, i.e. show that the annihilator ideal is a Joseph-like ideal.

The notion of Jordan superalgebras is already well-developed (see e.g. [Ka2, CK, MZ, Sh]). For the TKK algebra different definitions exist in the literature [Ti, Kan1, Ko, Ka2, Kr]. For a simple Jordan algebra it is known that all definitions are equivalent. We will give an overview of the different constructions in the super case and show that for arbitrary Jordan superalgebras different definitions can become non-equivalent. We will also study when the constructions are still equivalent and give some links between them.

The next step is to construct a representation of TKK(J) on functions on the Jordan superalgebra J. Inspired by the original paper [Jo1] on the Joseph ideal, we study polynomial realisations of Lie superalgebras in Weyl superalgebras. This leads to a family of representations π_λ for general three-graded Lie superalgebras depending on a character λ of the zero-graded part. As a special case we recover the representation of the TKK algebra on functions on the Jordan algebra considered in [HKM].

Orbits under the action of a Lie supergroup on some supermanifold are delicate objects to handle. Supermanifolds, in contrast to ordinary manifolds, are not completely determined by their points, so in the super case we cannot define an orbit through a point x as all points given by $g \cdot x$ for g in G. Instead we will define an orbit through x as the quotient supermanifold of G and the stabiliser subgroup G_x. Using this definition, we can construct a minimal orbit for the $OSp(p, q|2n)$ case. This orbit can be characterised by $R^2 = 0$, where $R^2 = \sum_{ij} x_i \beta^{ij} x_j$, with β the defining orthosymplectic metric of our Jordan superalgebra. We then show that for precisely one value of the parameter λ the representation π_λ restricts to this minimal orbit.

To integrate the representation of the Lie superalgebra $\mathfrak{osp}(p, q|2n)$ to the group level, we make use of the theory of Harish-Chandra supermodules developed in [Al]. In particular, we explicitly construct an admissible submodule of our representation, which then, by the general theory in [Al], integrates to a representation of the Lie supergroup $OSp(p, q|2n)$. We remark that this construction only works for $p + q$ even, which is the same condition as for the existence of a minimal unitary representation of the Lie group $O(p, q)$.

A priori, we know that the representation of $OSp(p, q|2n)$ we construct cannot be unitary, since it was shown in [NS, Theorem 6.2.1] that there exist no unitary representations of $OSp(p, q|2n)$ if p, q and n are all different from zero. This shows that minimal representations of Lie supergroups cannot be expected to be unitary in the usual sense. It is our hope that the representation constructed in this thesis will be useful to find an appropriate replacement for the notion of unitarity for Lie supergroups. We remark that in [dGM], a new definition of Hilbert superspaces and unitary representations using the super version of Krein spaces is introduced, which allows for a more general notion of unitary representations of Lie superalgebras than the one considered in [NS, Theorem 6.2.1]. However, it seems that our representation is not unitary even with respect to this broadened notion of unitarity. Nevertheless, we are able to define a non-degenerate superhermitian, sesquilinear form for which the representation is skew-symmetric. This sesquilinear form is the analogue of the L^2-inner product on the minimal orbit in the classical case.

Finally, we compute the annihilator ideal of our representation and show that it agrees with one of the two Joseph-like ideals constructed in [CSS]. In this sense, our representation is the natural generalisation of a minimal representation to the context of Lie superalgebras.

Relation to other work

The representation we construct is a natural analogue of the minimal representation of the group $O(p,q)$ (see e.g. [KØ]). This highlights the first factor of the even part $O(p,q) \times Sp(2n,\mathbb{R})$ of the supergroup $OSp(p,q|2n)$. The second factor $Sp(2n,\mathbb{R})$ does not admit a minimal representation, but its double cover $Mp(2n,\mathbb{R})$ does, the Segal–Shale–Weil representation. An analogue of this representation in the super context was constructed in [dGM] (see also [Ni] for the corresponding Lie superalgebra representation of $\mathfrak{osp}(p,q|2n)$). Its annihilator ideal is equal to the second Joseph-like ideal constructed in [CSS].

We further remark that in [AS, Section 5.2] highest weight representations of the Lie algebra $\mathfrak{su}(p,p|2p)$ are considered. It seems likely that, for a specific parameter, their representation has a subrepresentation which is the analogue of the minimal representation of $\mathfrak{su}(p,p)$.

Structure of the thesis

This thesis is organised as follows. We start with a lengthy introduction to Lie superalgebras in Chapter 2. We introduce all finite-dimensional simple Lie superalgebras and give their (well-known) classification. We also give a basic introduction to the representation theory of Lie superalgebras.

The first new results are contained in Chapter 3. There we introduce Jordan superalgebras and compare a number of different definitions of structure algebras and TKK constructions for Jordan (super)algebras appearing in the literature. For unital Jordan superalgebras we find that all different definitions of the structure algebra reduce to two cases, see Proposition 3.3.7. Also for the TKK algebra we essentially have two distinct cases, see Propositions 3.4.3, 3.4.5, 3.5.1. We also give examples for the non-unital case which show that in that case the different definitions will lead to different outcomes.
We give an overview table containing the structure algebra and TKK algebra for the finite-dimensional Jordan superalgebras over $\mathbb{C}$ in Section 3.6.
For the spin factor Jordan superalgebra over $\mathbb{R}$, we also calculate the corresponding Lie superalgebras. In particular the Tits–Kantor–Koecher algebra of this Jordan superalgebra is given by $\mathfrak{osp}(p,q|2n)$.

In Chapter 4 we study realisations of Lie (super)algebras in Weyl (super)algebras. In particular we obtain in Section 4.5 a generalisation of the representations considered in [HKM] to the setting of *Jordan superpairs*. As another application we obtain small explicit realisations of the exceptional Lie superalgebras $D(2,1;\alpha)$, $F(4)$ and $G(3)$ in Section 4.6.

In Chapter 3 and Chapter 4 we prove our results for general Jordan and Lie superalgebras. In Chapter 5 and Chapter 6 we restrict our attention to the real Lie superalgebra $\mathfrak{osp}(p,q|2n)$.

Chapter 5 contains the construction of the minimal representation of $\mathfrak{osp}(p,q|2n)$. We start with the definition of Lie supergroups and collect some results on actions of Lie supergroups on supermanifolds. We use these results to introduce the orbit through a primitive idempotent under the action of the structure group on the real spin factor Jordan superalgebra (see Theorem 5.2.3). We also show that there exists a character for which the representation constructed in Chapter 4 can be restricted to this orbit (see Proposition 5.2.10).
We then introduce a submodule of this restricted representation and give a very explicit description of this submodule in Theorem 5.3.3. We also show that for $p+q$ even and $p-2n-3 \notin -2\mathbb{N}$, this module can be integrated to the group level (see Corollary 5.3.14).

We prove some properties of our representation in Chapter 6. We start by establishing the 'minimality' of our representation. We compute the annihilator ideal of our representation and show that it is equal to a Joseph-like ideal of $\mathfrak{osp}(p,q|2n)$ constructed in [CSS] (see Theorem 6.1.4) if $p+q-2n>2$. This links our representation to the definition of minimal representations in the classical case. The Gelfand–Kirillov dimension is computed and it equals $p+q-3$ (see Proposition 6.2.1), which is the same as the Gelfand–Kirillov dimension of the minimal representation of $\mathfrak{so}(p,q)$ and thus independent of the 'super part'.
In Section 6.3, we introduce a linear functional which defines a nondegenerate sesquilinear form resembling the L^2-inner product on the minimal orbit in the classical case. Our representation is shown to be skew-symmetric with respect to this form if $p+q-2n-6 \geq 0$ (see Theorem 6.3.9).

Finally in Chapter 7, we touch on some open questions and directions for future research.

In Appendix A, we briefly introduce supermanifolds. We also gather some results on Gegenbauer polynomials, Bessel functions and the generalised Laguerre functions introduced in [HKMM] in Appendix B. In particular, we also prove some new recursion relations for the generalised Laguerre functions, which are needed in Chapter 5.

Publications

Chapter 3 is based on the paper ‘On structure and TKK algebras for Jordan superalgebras’ which is joint work with Kevin Coulembier. Chapter 4 is based on the paper ‘Polynomial Realisations of Lie (Super)Algebras and Bessel Operators’ also joint with Kevin Coulembier. Chapter 5, Chapter 6 and the introduction are based on the paper ‘A minimal representation of the orthosymplectic Lie superalgebra’ which is joint work with Jan Frahm. In the context of this PhD, also the paper ‘The Joseph ideal for $\mathfrak{sl}(m|n)$’ was published. Since its subject did not fit in the topic of this thesis, it is not included in this work.

1. S. Barbier, K. Coulembier. Polynomial Realisations of Lie (Super)Algebras and Bessel Operators. Int. Math. Res. Not. IMRN 2017 (2017), no. 10, 3148–3179

2. S. Barbier, K. Coulembier. On structure and TKK algebras for Jordan superalgebras. Comm. Algebra **46** (2018), no 2, 684-704.

3. S. Barbier and J. Frahm. A minimal representation of the orthosymplectic Lie superalgebra, 45 pages, arXiv:1710.07271. Submitted for publication.

4. S. Barbier and K. Coulembier. The Joseph ideal for $\mathfrak{sl}(m|n)$. Lie theory and its applications in physics, 489499, Springer Proc. Math. Stat., 191, Springer, Singapore, 2016.

'Hell, I'm relieved to hear you say that,' said Ford. 'Why?' 'Because I thought I must be going mad.' 'Perhaps you are. Perhaps you only thought I said it.'

Douglas Adams,
The Hitchhiker's Guide to the Galaxy

2
Lie superalgebras

In this chapter we introduce (representation theory of) Lie superalgebras. We begin with an introduction to $\mathbb{Z}_2$-graded linear algebra in Section 2.1. Then we give two different, but equivalent, definitions of a Lie superalgebra and give some basic examples in Section 2.2. In Section 2.3 we give an overview of all simple finite-dimensional Lie superalgebras over $\mathbb{C}$. In Section 2.4 we introduce the basic notions of representations and modules. We explain the difference between irreducible and indecomposable modules and give some examples. Then we make a bit of a side tour in Section 2.5 where we give the super analogues of Cartan subalgebras, roots and Borel subalgebras. Using these concepts, we delve a bit deeper into representation theory in Section 2.6. There we introduce the universal enveloping algebra, weight modules and explain the procedure of inducing from a one-dimensional representation of a Borel subalgebra.

We refer to [CW, Mu] for a more advanced introduction to (Lie) superalgebras.
By $\langle A\rangle$ we will denote the linear span of a set A over some field $\mathbb{K}$. In our convention 0 is a natural number, so $\mathbb{N} = \{0, 1, 2, \dots\}$. We will use the notation $\mathbb{R}^+$ for $\{x \in \mathbb{R} \mid x > 0\}$ and $\mathbb{R}^m_{(0)}$ for $\mathbb{R}^m\backslash\{0\}$.

2.1 Introduction to $\mathbb{Z}_2$-graded linear algebra

2.1.1 Super-vector spaces

We start this chapter with an introduction to $\mathbb{Z}_2$-graded linear algebra. Throughout this thesis we use the notation $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z} = \{\bar{0}, \bar{1}\}$.

Let $\mathbb{K}$ be a field. A super-vector space is a $\mathbb{Z}_2$-graded vector space V over $\mathbb{K}$. Thus

$$V = \underbrace{V_{\bar{0}}}_{\text{even part}} \oplus \underbrace{V_{\bar{1}}}_{\text{odd part}} .$$

Elements in $V_{\bar{0}}$ are called even, elements in $V_{\bar{1}}$ odd and elements in $V_{\bar{0}} \cup V_{\bar{1}}$ homogeneous. We use the notation $|x|$ for the parity of a homogeneous element. So $|x| = 0$ for x even and $|x| = 1$ for x odd. We use the convention that the appearance of $|x|$ in a formula implies that we are considering homogeneous elements and the formula has to be extended linearly for arbitrary elements.

Write $\mathbb{K}^{m|n}$ for the super-vector space V with $V_{\bar{0}} = \mathbb{K}^m$ and $V_{\bar{1}} = \mathbb{K}^n$.

The dimension of V is given by the pair $(m|n)$ where $m = \dim V_{\bar{0}}$ and $n = \dim V_{\bar{1}}$. The superdimension is $M = m - n$.

Example 2.1.1. *Given $V = \mathbb{K}^{m|n}$, then*

$$\operatorname{Mat}\left(\mathbb{K}^{m|n}\right) = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \underbrace{\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}}_{\textit{even}} \oplus \underbrace{\begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}}_{\textit{odd}},$$

where $A \in \mathbb{K}^{m\times m}, B \in \mathbb{K}^{m\times n}, C \in \mathbb{K}^{n\times m}, D \in \mathbb{K}^{n\times n}$ is also a super-vector space.

A super-vector space is a vector space with extra structure, so morphisms and subspaces should preserve that structure. A super subspace $W \subseteq V$ is a super-vector space $W = W_{\bar{0}} \oplus W_{\bar{1}}$ such that $W \cap V_{\bar{0}} = W_{\bar{0}}$ and $W \cap V_{\bar{1}} = W_{\bar{1}}$.

Example 2.1.2. *Consider $\mathbb{K}^{1|1}$ with basis elements e, f spanning the even and odd part respectively. The one-dimensional space $\mathbb{K}(e + f)$ is an ordinary vector subspace but* not *a super subspace. In contrast, $\mathbb{K}e$ and $\mathbb{K}f$ are well-defined super subspaces.*

A morphism (homomorphism) is a linear map between two super-vector spaces that preserves the grading, i.e. it maps even to even and odd to odd elements. By $\mathrm{Hom}(V, W)$ we denote the set of all homomorphisms between V and W. We define $\mathrm{End}(V)$ as the set of *all* linear transformations. In particular $\mathrm{End}(V) \neq \mathrm{Hom}(V, V)$. The algebra $\mathrm{End}(V)$ is in itself a super-vector space with

$$\begin{aligned} \mathrm{End}(V)_{\bar{0}} &= \mathrm{Hom}(V, V) = \{f : V \to V \mid f \text{ even}\} \\ \mathrm{End}(V)_{\bar{1}} &= \{f : V \to V \mid f \text{ odd}\} \end{aligned}$$

where even maps preserve the parity of elements, i.e. even elements gets mapped to even elements and odd to odd, while odd maps change the parity, i.e. they map even to odd and odd to even.

Example 2.1.3. *Let $V = \mathbb{K}^{m|n}$ then*

$$\mathrm{Hom}(V, V) = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}$$

and

$$\mathrm{End}(V) = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

with A, B, C, D as above.

Remark 2.1.4. The set of all linear transformations between V and W is sometimes called the set of inner homomorphisms and denoted by $\underline{\mathrm{Hom}}(V, W)$. Some authors call all linear transformations morphisms. So one should always be careful which convention is used.

By convention, a *basis of a super-vector space* denotes a standard homogeneous basis, i.e. the basis elements are homogeneous and ordered in such a way that the even precede the odd.

2.1.2 Supertrace and supertranspose

For a matrix, we can consider its transpose and its trace. These operations can be used to define Lie subalgebras of the general Lie algebra. We now introduce the equivalent notions for $\mathbb{Z}_2$-graded matrices which can be used to define Lie subsuperalgebras of the general Lie superalgebra.

Given $A \in \mathrm{Mat}\left(\mathbb{K}^{m|n}\right)$ where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we define the supertrace of A as

$$\mathrm{str}(A) := \sum_i (-1)^{|i|} A_{ii} = \mathrm{tr}(a) - \mathrm{tr}(d)$$

and the supertranspose of A as

$$A^{ST} = \begin{pmatrix} a^t & c^t \\ -b^t & d^t \end{pmatrix}.$$

Observe that the supertranspose has order four.

Lemma 2.1.5. *The supertrace and supertranspose satisfy the following elementary properties:*

- $\mathrm{str}(AB) = (-1)^{|A||B|}\mathrm{str}(BA)$
- $(XY)^{ST} = (-1)^{|X||Y|} Y^{ST} X^{ST}$
- $\mathrm{str}(X^{ST}) = \mathrm{str}(X)$.

Proof. Straightforward verification. □

2.1.3 Superalgebras

A superalgebra A is a super-vector space with a product such that $A_i \cdot A_j \subseteq A_{i+j}$. We call a superalgebra associative if the underlying algebra is associative. It is commutative if $a \cdot b = (-1)^{|b||a|} b \cdot a$.

Example 2.1.6. $(\mathrm{Mat}(\mathbb{K}^{m|n}), \star)$ *is a commutative (but not associative) superalgebra for the product* $A \star B := \frac{AB + (-1)^{|A||B|} BA}{2}$.

Subalgebras and ideals of superalgebras should also be $\mathbb{Z}_2$-graded.

Definition 2.1.7. *A subalgebra B of a superalgebra A is a super subspace such that $BB \subseteq B$.*
A left ideal I of A is a subalgebra such that $AI \subseteq I$. A right and a two-sided ideal are defined similarly.

A superalgebra is *simple* if it does not contain proper non-trivial two-sided ideals.

A linear transformation that satisfies the super Leibniz rule is called a derivation.

Definition 2.1.8. *A derivation is an element D of* $\mathrm{End}(A)$ *for which* $D(ab) = D(a)b + (-1)^{|D||a|}aD(b)$ *for all a, b in A.*

We denote the set of all derivations of A by $\mathrm{Der}(A)$.

2.1.4 The tensor algebra and the symmetric algebra

We define direct sums and tensor products of super-vector spaces by the direct sums and tensor products of the underlying vector spaces. The $\mathbb{Z}_2$-grading is given by

$$\begin{aligned}(V \oplus W)_i &= V_i \oplus W_i \\ (V \otimes W)_i &= \sum_{j+k=i} V_j \otimes W_k, \qquad i, j, k \in \mathbb{Z}_2.\end{aligned}$$

We then define the tensor algebra $T(V)$ of a super-vector space V as

$$T(V) = \bigoplus_{n=0}^{\infty} V^{\otimes n}.$$

The product on $T(V)$ is defined in the usual way.

We have a natural action of the permutation group $\mathfrak{S}_n$ on $V^{\otimes n}$ which is given for each τ in $\mathfrak{S}_n$ by

$$\begin{aligned}\tau \colon V^{\otimes n} &\to V^{\otimes n} \\ v_1 \otimes v_2 \otimes \cdots \otimes v_n &\mapsto (-1)^{|\tau|} v_{\tau(1)} \otimes v_{\tau(2)} \otimes \cdots \otimes v_{\tau(n)}.\end{aligned}$$

Here $|\tau|$ counts the number of times we swapped two odd elements:

$$|\tau| = \sum_{i=1}^{n} \sum_{j=i+1}^{n} \left[\tau^{-1}(i) > \tau^{-1}(j) \text{ and } v_i, v_j \text{ are odd } \right], \tag{2.1}$$

where we used the Iverson bracket

$$[\text{expression}] = \begin{cases} 0 & \text{if the expression is false} \\ 1 & \text{if the expression is true.} \end{cases}$$

Note that $|\tau| = |\tau|(v_1, v_2, \ldots, v_n)$ depends on (the parity of) the v_i. Using this action of $\mathfrak{S}_n$ on $V^{\otimes n}$ we can define the supersymmetric n-power of V as

$$S^n(V) = V^{\otimes n}/W,$$

where W is the subspace in $V^{\otimes n}$ spanned by the elements

$$v_1 \otimes v_2 \otimes \cdots \otimes v_n - (-1)^{|\tau|} v_{\tau(1)} \otimes v_{\tau(2)} \otimes \cdots \otimes v_{\tau(n)}$$

for v_i in V and τ in $\mathfrak{S}_n$. We also define the supersymmetric algebra of V

$$S(V) = T(V)/I,$$

where I is the two-sided ideal generated by

$$v_1 \otimes v_2 - (-1)^{|v_1||v_2|} v_2 \otimes v_1, \text{ for } v_1, v_2 \in V.$$

We have $S(V) = \bigoplus_{n=0}^{\infty} S^n(V)$. Note that if V is purely odd, i.e. $V_{\bar{0}} = 0$ then $S(V)$ is equal to the Grassmann algebra $\Lambda(V)$ (where we see V as an ordinary vector space in $\Lambda(V)$).

2.2 Lie superalgebras

2.2.1 Definition

We start with a straightforward generalisation of the classical definition of a Lie algebra to a superalgebra. In Section 2.2.3 we will give another, alternative definition.

Definition 2.2.1. *A Lie superalgebra $\mathfrak{g}$ (sometimes also called super Lie algebra or $\mathbb{Z}_2$-graded Lie algebra) is a superalgebra endowed with a bilinear product $[.,.]$ for which*

1. $[x, y] = -(-1)^{|x||y|}[y, x]$ *(anti-commutativity)*
2. $(-1)^{|x||z|}[x, [y, z]] + (-1)^{|y||x|}[y, [z, x]] + (-1)^{|z||y|}[z, [x, y]] = 0$ *(super Jacobi identity).*

The first property means that homogeneous elements anti-commute, except if they are both odd, then they commute. The second property can be rewritten as

$$[x, [y, z]] = [[x, y], z] + (-1)^{|x||y|}[y, [x, z]],$$

i.e. the adjoint action ad x is a derivation, where

$$\begin{aligned} \text{ad}\colon \mathfrak{g} &\to \text{End}(\mathfrak{g}) \\ x &\to \text{ad}\, x \quad \text{ and ad}\, x(y) := [x, y]. \end{aligned}$$

We have the following basic examples of Lie superalgebras.

- Let A be an associative superalgebra. Define
$$[X,Y] := XY - (-1)^{|X||Y|}YX, \qquad \text{for } X,Y \in A.$$
Then $(A,[.,.])$ is a Lie superalgebra. In particular $(\mathrm{End}(V),[.,.])$ is a Lie superalgebra, which we denote by $\mathfrak{gl}(V)$. For $V=\mathbb{K}^{m|n}$ we get the general linear superalgebra $\mathfrak{gl}(m|n)$.
- The set of derivations $\mathrm{Der}(A)$ is a Lie subalgebra of $\mathfrak{gl}(A)$.
- The special linear superalgebra $\mathfrak{sl}(m|n)$ is defined as
$$\mathfrak{sl}(m|n) = \{X \in \mathfrak{gl}(m|n) \mid \mathrm{str}(X) = 0\}.$$
- Every Lie algebra is a Lie superalgebra with $\mathfrak{g}_{\bar{1}} = 0$.

2.2.2 Bilinear forms on super-vector spaces

We can introduce another class of Lie superalgebras using bilinear forms. A bilinear form is a map $\langle \cdot\,,\cdot\rangle : V \times V \to \mathbb{K}$ such that
$$\begin{aligned}\langle \lambda x + \mu x', y\rangle &= \lambda\langle x,y\rangle + \mu\langle x',y\rangle\\ \langle x, \lambda y + \mu y'\rangle &= \lambda\langle x,y\rangle + \mu\langle x,y'\rangle.\end{aligned}$$

Definition 2.2.2. *The bilinear form $\langle \cdot\,,\cdot\rangle$ is called*

- *(super)symmetric if $\langle x,y\rangle = (-1)^{|x||y|}\langle y,x\rangle$,*
- *skew symmetric if $\langle x,y\rangle = -(-1)^{|x||y|}\langle y,x\rangle$,*
- *even if $\langle x,y\rangle = 0$ for $|x|+|y| = 1$ and odd if $\langle x,y\rangle = 0$ for $|x|+|y| = 0$,*
- *non-degenerate if $\langle x,y\rangle = 0$ for all $y \in V$ implies $x = 0$ and $\langle x,y\rangle = 0$ for all $x \in V$ implies $y = 0$.*
- *If V is a Lie superalgebra, then we call $\langle \cdot\,,\cdot\rangle$ invariant if*
$$\langle [x,y],z\rangle = \langle x,[y,z]\rangle.$$

An even invariant bilinear form on a simple Lie superalgebra satisfies the following.

Proposition 2.2.3. *If $\langle \cdot\, , \cdot \rangle$ is an even bilinear invariant form on a simple Lie superalgebra $\mathfrak{g}$ then $\langle \cdot\, , \cdot \rangle$ is either zero or non-degenerate.*

Proof. If x is an element of the left radical of $\langle \cdot, \cdot \rangle$, i.e. $\langle x, y \rangle = 0$ for all $y \in \mathfrak{g}$, then $[x, z]$ is also an element of this left radical for all $z \in \mathfrak{g}$ since $\langle [x, z], y \rangle = \langle x, [z, y] \rangle = 0$ for all $y \in \mathfrak{g}$. If x is contained in the left radical, then also its even part $x_{\bar{0}}$ is contained in the left radical, since $\langle x_{\bar{0}}, y \rangle = \langle x, y \rangle = 0$ for y even and $\langle x_{\bar{0}}, y \rangle = 0$ for y odd since the form is even. In the same way also $x_{\bar{1}}$ is contained in the left radical. We conclude that the left radical is an ideal of $\mathfrak{g}$. Since $\mathfrak{g}$ is simple the left radical is therefore either zero or the whole Lie superalgebra. □

We then define the orthosymplectic and the periplectic Lie superalgebra using bilinear forms.

- The orthosymplectic Lie superalgebra.
  Let $\langle \cdot\, , \cdot \rangle$ be an even, non-degenerate, symmetric bilinear form on V. Then

  $$\mathfrak{osp}(V) := \{X \in \mathfrak{gl}(V) \mid \langle Xu, v \rangle = -(-1)^{|X||u|} \langle u, Xv \rangle\}.$$

  Note that $\dim V_{\bar{1}}$ is necessarily even, since the form is symplectic on $V_{\bar{1}}$.

- The periplectic Lie superalgebra.
  Let $\langle \cdot\, , \cdot \rangle$ be an odd, non-degenerate, symmetric bilinear form on V, where $\dim(V_{\bar{0}}) = \dim(V_{\bar{1}})$. Then

  $$\mathfrak{pe}(V) = \{X \in \mathfrak{gl}(V) \mid \langle Xu, v \rangle = -(-1)^{|X||u|} \langle u, Xv \rangle\}.$$

2.2.3 Alternative definition of a Lie superalgebra

There is also an equivalent definition of a Lie superalgebra, where we see $\mathfrak{g}_{\bar{1}}$ as a module for the Lie algebra $\mathfrak{g}_{\bar{0}}$.

Definition 2.2.4. *A super-vector space $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ is a Lie superalgebra if*

1. *$\mathfrak{g}_{\bar{0}}$ is a Lie algebra,*
2. *$\mathfrak{g}_{\bar{1}}$ is a $\mathfrak{g}_{\bar{0}}$-module.*

3. *There exists a* $\mathfrak{g}_{\bar{0}}$*-morphism* $p : S^2(\mathfrak{g}_{\bar{1}}) \to \mathfrak{g}_{\bar{0}}$.

4. *For all* $a, b, c \in \mathfrak{g}_{\bar{1}}$ *the morphism* p *satisfies*

$$[p(a,b),c] + [p(b,c),a] + [p(c,a),b] = 0,$$

where we denoted the $\mathfrak{g}_{\bar{0}}$*-action on* $\mathfrak{g}_{\bar{1}}$ *by* $[\cdot,\cdot]$.

Note that in $S^2(\mathfrak{g}_{\bar{1}})$, we see $\mathfrak{g}_{\bar{1}}$ as an ordinary vector space, so $S^2(\mathfrak{g}_{\bar{1}})$ is the symmetric second power and not the supersymmetric second power. Using this definition, the classical Lie superalgebras can be constructed as follows.

- Let $\mathfrak{g}_{\bar{0}} := \mathfrak{gl}(m) \oplus \mathfrak{gl}(n)$ and $\mathfrak{g}_{\bar{1}} := \mathbb{K}^m \otimes (\mathbb{K}^n)^* \oplus \mathbb{K}^n \otimes (\mathbb{K}^m)^*$. We can interpret an element in $\mathbb{K}^m \otimes (\mathbb{K}^n)^*$ as an $m \times n$ matrix B and an element in $\mathbb{K}^n \otimes (\mathbb{K}^m)^*$ as an $n \times m$ matrix C. Then

$$p(B + C, \tilde{B} + \tilde{C}) := \underbrace{B\tilde{C} + \tilde{B}C}_{\in \mathfrak{gl}(m)} + \underbrace{C\tilde{B} + \tilde{C}B}_{\in \mathfrak{gl}(n)}.$$

  This construction gives a Lie superalgebra, namely $\mathfrak{gl}(m|n)$.

- For $\mathfrak{g}_{\bar{0}} = \mathfrak{gl}(n)$ and $\mathfrak{g}_{\bar{1}} = S^2(\mathbb{K}^n) \oplus \Lambda^2(\mathbb{K}^n)$ where $\mathbb{K}^n$ is the natural representation of $\mathfrak{gl}(n)$, we obtain $\mathfrak{g} = \mathfrak{pe}(n)$. Here the morphism p is defined by

$$p(B + C, \tilde{B} + \tilde{C}) := B\tilde{C} + \tilde{B}C,$$

  for $B, \tilde{B} \in S^2(\mathbb{K}^n)$, $C, \tilde{C} \in \Lambda^2(\mathbb{K}^n)$, where we interpret $B, \tilde{B}, C, \tilde{C}$ as $n \times n$ matrices.

- Let $\mathfrak{g}_{\bar{0}} = \mathfrak{gl}(n)$ and $\mathfrak{g}_{\bar{1}} = \mathfrak{gl}(n)$ where $\mathfrak{gl}(n)$ acts with the adjoint representation on $\mathfrak{gl}(n)$. For $B, \tilde{B} \in \mathfrak{gl}(n)$ the morphism p is defined by

$$p(B, \tilde{B}) := B\tilde{B} + \tilde{B}B.$$

  This gives us the queer Lie superalgebra $\mathfrak{q}(n)$.

- For the orthosymplectic Lie superalgebra $\mathfrak{osp}(m|2n)$, we have $\mathfrak{g}_{\bar{0}} = \mathfrak{so}(m) \oplus \mathfrak{sp}(2n)$ and $\mathfrak{g}_{\bar{1}} = \mathbb{K}^m \otimes \mathbb{K}^{2n}$ where $\mathbb{K}^m$ is the natural representation of $\mathfrak{so}(m)$ and $\mathbb{K}^{2n}$ is the natural representation of $\mathfrak{sp}(2n)$.

2.3 The simple Lie superalgebras over $\mathbb{C}$

In this section we will give an overview of the simple finite-dimensional Lie superalgebras over $\mathbb{C}$. Different notations appear in literature; our nomenclature is based on [CW].

2.3.1 The special linear Lie superalgebra

We already defined the special linear superalgebra $\mathfrak{sl}(m|n)$ as

$$\mathfrak{sl}(m|n) = \{A \in \mathfrak{gl}(m|n) \mid \mathrm{str}A = 0\}.$$

We also have $[\mathfrak{gl}(m|n), \mathfrak{gl}(m|n)] = \mathfrak{sl}(m|n)$. If $m \neq n$ then $\mathfrak{sl}(m|n)$ is simple. If $m = n$ then $\langle I_{2n} \rangle$, with I_{2n} the identity matrix, is an ideal in $\mathfrak{sl}(n|n)$ and

$$\mathfrak{psl}(n|n) := \mathfrak{sl}(n|n)/\langle I_{2n} \rangle$$

is simple for $n > 1$. Similarly, we set

$$\mathfrak{pgl}(n|n) := \mathfrak{gl}(n|n)/\langle I_{2n} \rangle.$$

2.3.2 The orthosymplectic Lie superalgebra

The orthosymplectic superalgebra $\mathfrak{osp}(m|2n)$ can be defined as

$$\mathfrak{osp}(m|2n) = \{x \in \mathfrak{gl}(m|2n) \mid x^{ST}\Omega + \Omega x = 0\},$$

where $\Omega = \begin{pmatrix} I_m & 0 \\ 0 & J_{2n} \end{pmatrix}$ with

$$I_m = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \qquad \text{and} \qquad J_{2n} = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}.$$

If $n = 0$ then $\mathfrak{osp}(m|0) = \mathfrak{so}(m)$.
If $m = 0$ then $\mathfrak{osp}(0|2n) = \mathfrak{sp}(2n)$.

Example 2.3.1. $\mathfrak{osp}(1|2)$
A general matrix $X \in \mathbb{C}^{3\times 3}$ *is given by*

$$X = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

so its supertranspose reads

$$X^{ST} = \begin{pmatrix} a & d & g \\ -b & e & h \\ -c & f & i \end{pmatrix}.$$

Now $\Omega = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$ *and the condition* $X^{ST}\Omega + \Omega X = 0$ *reads*

$$\begin{pmatrix} a & g & -d \\ -b & h & -e \\ -c & i & -f \end{pmatrix} + \begin{pmatrix} a & b & c \\ -g & -h & -i \\ d & e & f \end{pmatrix} = 0,$$

thus leading to the conditions $\{a = 0, g = -b, c = d, i = -e\}$*. The orthosymplectic Lie superalgebra* $\mathfrak{osp}(1|2)$ *therefore consists of matrices*

$$X = \begin{pmatrix} 0 & b & c \\ c & e & f \\ -b & h & -e \end{pmatrix}$$

where $b, c, e, f, h \in \mathbb{C}$*.*

2.3.3 The periplectic Lie superalgebra

The periplectic Lie superalgebra is the subalgebra of $\mathfrak{gl}(n|n)$ defined as

$$\mathfrak{pe}(n) := \left\{ \begin{pmatrix} a & b \\ c & -a^t \end{pmatrix} \middle| \; a, b, c \in \mathbb{C}^{n\times n} \text{ with } b^t = b,\; c^t = -c \right\}.$$

The special periplectic Lie superalgebra is defined as

$$\mathfrak{spe}(n) := \{x \in \mathfrak{pe}(n) \mid \mathrm{tr}(a) = 0\},$$

and is simple for $n \geq 3$.

The relation between this matrix realisation and the definition of the periplectic Lie superalgebra using a bilinear form in Subsection 2.2.2 is obtained by using the bilinear form

$$\langle u, v\rangle = u^t \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix} v \quad \text{for } u, v \in \mathbb{C}^{n|n}.$$

2.3.4 The queer Lie superalgebra

The queer Lie superalgebra is the subalgebra of $\mathfrak{gl}(n|n)$ defined as

$$\mathfrak{q}(n) := \left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} \Bigm| a, b \in \mathbb{C}^{n\times n} \right\}.$$

Remark that $\mathrm{str}(X) = 0$ for all $X \in \mathfrak{q}(n)$. The special queer Lie superalgebra is defined as

$$\mathfrak{sq}(n) := \left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} \Bigm| a, b \in \mathbb{C}^{n\times n}, \mathrm{tr}(b) = 0 \right\}.$$

We have $[\mathfrak{q}(n), \mathfrak{q}(n)] \subseteq \mathfrak{sq}(n)$ because $\mathrm{tr}(ab' + ba' - a'b - b'a) = 0$. The projective special queer Lie superalgebra is defined as

$$\mathfrak{psq}(n) := \mathfrak{sq}(n)/\langle I_{2n}\rangle,$$

and is simple for $n \geq 3$. We also define the projective queer Lie superalgebra as

$$\mathfrak{pq}(n) := \mathfrak{q}(n)/\langle I_{2n}\rangle.$$

Also the queer Lie superalgebra can be more intrinsically defined. Let V be a super-vector space with $\dim(V_{\bar 0}) = \dim(V_{\bar 1}) = n$. Choose $P \in (\mathrm{End}(V))_{\bar 1}$ such that $P^2 = I_{n|n}$. Then

$$\mathfrak{q}(V) = \{X \in \mathfrak{gl}(V) \mid [X, P] = 0\}.$$

To get the previous matrix realisation, take $P = \imath \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$.

2.3.5 The exceptional Lie superalgebra $D(2,1;\alpha)$

There is a one-parameter family of 17-dimensional Lie superalgebras of rank 3 which are deformations of $D(2,1) = \mathfrak{osp}(4|2)$. These Lie superalgebras can be defined using a construction of Scheunert. We will use the notations of [Mu], where also more details can be found.

Let V be a two-dimensional vector space with basis u_+ and u_-. Let ψ be a non-degenerate skew-symmetric bilinear form with $\psi(u_+, u_-) = 1$. Consider $\mathfrak{sl}(V) = \mathfrak{sp}(\psi)$ the algebra of linear transformations preserving ψ. Denote by (V_i, ψ_i), $i = 1, 2$ or 3 three copies of (V, ψ).

We will define $D(2,1;\alpha)$ using Definition 2.2.4. Set

$$\mathfrak{g}_{\bar{0}} = \mathfrak{sp}(\psi_1) \oplus \mathfrak{sp}(\psi_2) \oplus \mathfrak{sp}(\psi_3)$$

and

$$\mathfrak{g}_{\bar{1}} = V_1 \otimes V_2 \otimes V_3.$$

The action of $\mathfrak{g}_{\bar{0}}$ on $\mathfrak{g}_{\bar{1}}$ is given by the outer tensor product:

$$(A, B, C) \cdot x \otimes y \otimes z = A(x) \otimes y \otimes z + x \otimes B(y) \otimes z + x \otimes y \otimes C(z).$$

Define $p_i \colon V_i \times V_i \to \mathfrak{sp}(\psi_i)$ by

$$p_i(x, y)z = \psi_i(y, z)x - \psi_i(z, x)y.$$

For $\sigma_i \in \mathbb{C}$ we define the $\mathfrak{g}_{\bar{0}}$-morphism p by

$$p(x_1 \otimes x_2 \otimes x_3, y_1 \otimes y_2 \otimes y_3) = \sigma_1 \psi_2(x_2, y_2)\psi_3(x_3, y_3)p_1(x_1, y_1)$$
$$+ \sigma_2 \psi_3(x_3, y_3)\psi_1(x_1, y_1)p_2(x_2, y_2) + \sigma_3 \psi_1(x_1, y_1)\psi_2(x_2, y_2)p_3(x_3, y_3).$$

The morphism p satisfies condition (4) in the definition of a Lie superalgebra if and only if $\sigma_1 + \sigma_2 + \sigma_3 = 0$ [Mu, Lemma 4.2.1].

So in that case the algebra $\Gamma(\sigma_1, \sigma_2, \sigma_3) = \mathfrak{g}_{\bar{0}} + \mathfrak{g}_{\bar{1}}$ is a Lie superalgebra. We have

$$\Gamma(\sigma_1, \sigma_2, \sigma_3) \cong \Gamma(\sigma_1', \sigma_2', \sigma_3')$$

if and only if there is a non-zero scalar c and a permutation π of $(1, 2, 3)$ such that $\sigma_i' = c\sigma_{\pi(i)}$ [Mu, Lemma 5.5.16]. If $\sigma_i = 0$ for $i = 1, 2$ or 3 then $\Gamma(\sigma_1, \sigma_2, \sigma_3)$ contains an ideal I such that

$$\Gamma(\sigma_1, \sigma_2, \sigma_3)/I \cong \mathfrak{sl}(V_i),$$

as one can easily deduce from the definition of p.

Define for $\alpha \in \mathbb{C}$

$$D(2,1;\alpha) = \Gamma\left(\frac{1+\alpha}{2}, \frac{-1}{2}, \frac{-\alpha}{2}\right).$$

Assume $\alpha \notin \{0,-1\}$, then $D(2,1;\alpha)$ is simple and $D(2,1;\alpha) \cong D(2,1;\beta)$ if and only if β is in the same orbit as α under the transformations $\alpha \mapsto \alpha^{-1}$ and $\alpha \mapsto -1-\alpha$.

Furthermore $D(2,1;1) \cong \mathfrak{osp}(4|2)$.

2.3.6 The exceptional Lie superalgebras $F(4)$ and $G(3)$

For a more detailed description of the exceptional Lie superalgebras $F(4)$ and $G(3)$ we refer to [FSS, Section 2.18 and 2.19]. Here we just give the even and odd parts.

For $F(4)$ we have $\mathfrak{g}_{\bar{0}} = \mathfrak{sl}(2) \oplus \mathfrak{so}(7)$ and $\mathfrak{g}_{\bar{1}}$ is the tensor product of the natural representation of $\mathfrak{sl}(2)$ and the simple $\mathfrak{so}(7)$-spin module. The Lie superalgebra $G(3)$ satisfies $\mathfrak{g}_{\bar{0}} = \mathfrak{sl}(2) \oplus G_2$. The odd part $\mathfrak{g}_{\bar{1}}$ is the tensor product of the natural representation of $\mathfrak{sl}(2)$ and the fundamental 7-dimensional G_2-module.

2.3.7 The Cartan types

Let $\Lambda(n)$ be the exterior algebra generated by $\xi_1, \ldots, \xi_n$. The indeterminates hence satisfy

$$\xi_i \xi_j = -\xi_j \xi_i.$$

This is an associative superalgebra where the generators are odd, $|\xi_i| = \bar{1}$. We also consider a compatible $\mathbb{Z}$-grading, by setting $\deg \xi_i = 1$. Denote by $W(n)$ the algebra of derivations of the associative superalgebra $\Lambda(n)$. The Lie superalgebra $W(n)$ is simple for $n \geq 2$.

Consider

$$D_f := \sum_{i=1}^{n} f_i \partial_{\xi_i}$$

with $f_i \in \Lambda(n)$ and ∂_{ξ_i} the unique derivation defined by $\partial_{\xi_i}\xi_j = \delta_{ij}$. Every element of $W(n)$ can be written is this way.

Define

$$S(n) := \left\{ D_f \in W(n) \mid \sum_{i=1}^{n} \partial_{\xi_i}(f_i) = 0 \right\}.$$

This defines a Lie subalgebra of $W(n)$, which is simple for $n \geq 3$. For n even we also set

$$\widetilde{S}(n) := \left\{ D_f \in W(n) \mid \sum_{i=1}^{n} \partial_{\xi_i}(\omega f_i) = 0 \right\},$$

where $\omega = 1 + \xi_1\xi_2 \dots \xi_n$. Also $\widetilde{S}(n)$ is a subalgebra of $W(n)$ which is simple for $n \geq 4$ (and even).

On $\Lambda(n)$, we define the following Poisson superbracket

$$\{f, g\} := (-1)^{|f|} \left((\partial_{\xi_{n-1}} f)(\partial_{\xi_n} g) + (\partial_{\xi_n} f)(\partial_{\xi_{n-1}} g) + \sum_{i=1}^{n-2} (\partial_{\xi_i} f)(\partial_{\xi_i} g) \right),$$

for f and g in $\Lambda(n)$. Then $(\Lambda(n), \{\cdot, \cdot\})$ becomes a Lie superalgebra with ideal $\langle 1 \rangle$. Consider the following Lie superalgebras

$$\widetilde{H}(n) := \Lambda(n)/\langle 1 \rangle \quad \text{and} \quad H(n) := [\widetilde{H}(n), \widetilde{H}(n)].$$

Note that $\widetilde{H}(n) = H(n) \oplus \mathbb{C}\xi_1 \cdots \xi_n$ as super-vector spaces. The Lie superalgebra $H(n)$ is simple for $n \geq 4$. We can embed $H(n)$ and $\widetilde{H}(n)$ into $W(n)$, using $f \mapsto \{f, \cdot\}$.

For later use in Section 3.6 we already introduce the following. Consider $C := \sum_{i=1}^{n} \xi_i \partial_{\xi_i} \in W(n)$, then the semi-direct product $\mathbb{C}C \ltimes \widetilde{H}(n)$ is naturally defined as a subalgebra of $W(n)$.

We also define the semidirect product $\widetilde{H}(n-2) \ltimes \Lambda(n-2)$, where the action of $\widetilde{H}(n-2)$ on $\Lambda(n-2)$ is given by the Poisson superbracket on $\Lambda(n-2)$, while the bracket of $\Lambda(n-2)$ is trivial. We further introduce, $\mathbb{C}C \ltimes \left(\widetilde{H}(n-2) \ltimes \Lambda(n-2) \right)$, where C acts by $[C, f] = (\deg f - 2) f$ for $f \in \widetilde{H}(n-2)$ and by $[C, g] = \deg g$ for $g \in \Lambda(n-2)$.

2.3.8 Classification

We will now give the classification of all simple finite-dimensional Lie superalgebras over $\mathbb{C}$.

Definition 2.3.2. *A simple Lie superalgebra* $\mathfrak{g}$ *is called classical if* $\mathfrak{g}_{\bar{1}}$ *is completely reducible as* $\mathfrak{g}_{\bar{0}}$*-module. Furthermore*

- $\mathfrak{g}$ *is called classical of type I if* $\mathfrak{g}_{\bar{1}}$ *is not irreducible as* $\mathfrak{g}_{\bar{0}}$*-module* $\mathfrak{g}$ *is called classical of type II if* $\mathfrak{g}_{\bar{1}}$ *is irreducible as* $\mathfrak{g}_{\bar{0}}$*-module*
- $\mathfrak{g}$ *is called basic if it is classical and admits an even, non-degenerate invariant bilinear form.*

By convention, the (non-simple) Lie superalgebra $\mathfrak{gl}(m|n)$ is sometimes also called basic.

Theorem 2.3.3 (Kac, 1977, [CW, Theorem 1.1])**.** *Every finite-dimensional simple Lie superalgebra over* $\mathbb{C}$ *is isomorphic to one of the following:*

- *A simple Lie algebra*
- *Classical type I*

$$\begin{aligned} A(m|n) &:= \mathfrak{sl}(m+1|n+1) && m > n \geq 0 \\ A(m|m) &:= \mathfrak{psl}(m+1|m+1) && m \geq 2 \\ C(n) &:= \mathfrak{osp}(2|2n-2) && n \geq 2 \\ P(n) &:= \mathfrak{spe}(n+1) && n \geq 2, \ \textit{(non-basic)} \end{aligned}$$

- *Classical type II*

$$\begin{aligned} &B(m|n) := \mathfrak{osp}(2m+1|2n) && m \geq 0, n \geq 1 \\ &D(m|n) := \mathfrak{osp}(2m|2n) && m \geq 2, n \geq 1 \\ &D(2,1;\alpha) && \\ &F(4) && \\ &G(3) && \\ &Q(n) = \mathfrak{psq}(n+1) && n \geq 2, \ \textit{(non-basic)} \end{aligned}$$

- *Cartan type*

$$W(n) \qquad n \geq 3,$$

$$\begin{array}{ll} S(n) & n \geq 3, \\ \widetilde{S}(2n) & n \geq 2, \\ H(n) & n \geq 4. \end{array}$$

We have the following isomorphisms between Lie superalgebras of low rank:

$$\mathfrak{osp}(2|2) \cong \mathfrak{sl}(2|1) \cong W(2), \quad \mathfrak{psl}(2|2) \cong H(4), \quad \mathfrak{spe}(3) \cong S(3).$$

2.4 Representations and modules

We will now give an introduction to the main subject of this thesis: representation theory of Lie superalgebras.

2.4.1 Definitions

A representation of a Lie superalgebra $\mathfrak{g}$ is defined as follows.

Definition 2.4.1. *Let $\mathfrak{g}$ be a Lie superalgebra and V a super-vector space. Let $\rho : \mathfrak{g} \to \mathrm{End}(V)$ be an even linear map such that*

$$\rho([x, y]) = \rho(x)\rho(y) - (-1)^{|x||y|}\rho(y)\rho(x)$$

i.e. ρ is a Lie superalgebra morphism between $\mathfrak{g}$ and $\mathfrak{gl}(V)$. Then (ρ, V) is a representation of $\mathfrak{g}$.

A $\mathfrak{g}$-module has the following definition.

Definition 2.4.2. *A super-vector space V is called a (left) $\mathfrak{g}$-module if we have a map $\mathfrak{g} \times V \to V : (x, v) \mapsto x \cdot v$ such that for all $x, y \in \mathfrak{g}$, $v, w \in V$, and $\lambda \in \mathbb{K}$,*

- $(x + y) \cdot v = x \cdot v + y \cdot v$
- $x \cdot (v + w) = x \cdot v + x \cdot w$
- $\lambda(x \cdot v) = (\lambda x) \cdot v = x \cdot (\lambda v)$
- $|x \cdot v| = |x| + |v|$
- $[x, y] \cdot v = x \cdot (y \cdot v) - (-1)^{|x||y|} y \cdot (x \cdot v)$.

Definition 2.4.3. *A $\mathfrak{g}$-module morphism is a morphism ϕ of super-vector spaces such that $\phi(x \cdot v) = x \cdot \phi(v)$.*

One can easily check that $\mathfrak{g}$-modules and representations of $\mathfrak{g}$ are equivalent: every $\mathfrak{g}$-module corresponds to a representation and vice versa. Therefore we shall often use $\mathfrak{g}$-module or $\mathfrak{g}$-representation interchangeably without stating it explicitly. Some easy examples of representations are as follows.

- If $\mathfrak{g}$ is a Lie subalgebra of $\mathfrak{gl}(m|n)$ we have the natural representation on $\mathbb{K}^{m|n}$ given by multiplication of a matrix and a column vector.
- Set $V := \mathfrak{g}$ and $x \cdot v = [x, v]$. This is the adjoint representation.

2.4.2 Irreducible and indecomposable modules

A submodule is defined as follows.

Definition 2.4.4. *Let V be a $\mathfrak{g}$-module. A space W is a submodule of V if W is a sub super-vector space of V and W is invariant under the restricted $\mathfrak{g}$-action, i.e. $\mathfrak{g} \cdot W \subseteq W$.*

Note that we in particular require a submodule to be also $\mathbb{Z}_2$-graded.

Definition 2.4.5. *We use the following terminology.*

- *A module V is decomposable if $V = W_1 \oplus W_2$ where W_1 and W_2 are submodules of V different from zero.*
- *A module V is indecomposable if it is not decomposable.*
- *A module V is irreducible (or simple) if V does not contain non-trivial submodules. Here the trivial submodules are the zero module and V.*

For classical finite-dimensional Lie algebra modules over a field of characteristic zero, we have Weyl's complete reducibility theorem.

Theorem 2.4.6. *Let $\mathfrak{g}$ be a semisimple Lie algebra. Every finite-dimensional $\mathfrak{g}$-module is isomorphic to a direct sum of simple modules, where the multiplicities of the summands are uniquely determined.*

This is no longer true for finite-dimensional Lie superalgebra modules. Therefore we introduce the following concept.

Definition 2.4.7. *A module V is called completely reducible if it is the direct sum of irreducible modules.*

We have the following example over $\mathbb{C}$ of an indecomposable but not irreducible module. Let

$$e := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \ f := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \ h := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \ I := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

then $\{e, f, h, I\}$ is a basis of $\mathfrak{gl}(2)$ and also of $\mathfrak{gl}(1|1)$. It is well known that $\mathfrak{gl}(2) = \mathfrak{sl}(2) \oplus \mathbb{C}I$, where both $\mathfrak{sl}(2) = \langle e, f, h\rangle$ and $\mathbb{C}I$ are simple. Hence $\mathfrak{gl}(2)$ is completely reducible as a $\mathfrak{gl}(2)$-module under the adjoint action. In contrast, consider $\mathfrak{gl}(1|1)$ as a $\mathfrak{gl}(1|1)$-module under the adjoint action. Then

$$[h, e] = 2e, \qquad [h, f] = -2f, \qquad [e, f] = I$$

are the only non-zero brackets. From this it follows easily that

- $\mathbb{C}I$ is a submodule,
- if a submodule contains e or f, it also contains I,
- if a submodule contains $\alpha e + \beta f$, with α, β non-zero, the submodule contains e and f,
- if a submodule contains h, it also contains e, f and thus also I,
- if a submodule contains $\alpha I + \beta e + \gamma f + \delta h$, with $\delta \neq 0$, then it also contains e and f and thus also I and h, so it is equal to the whole $\mathfrak{g}(1|1)$.

Summarising,

$$\begin{array}{ccccccccc} & & & & \langle I, e\rangle & & & & \\ & & & \nearrow & & \searrow & & & \\ \langle 0\rangle & \to & \langle I\rangle & & & & \langle I, e, f\rangle & \to & \mathfrak{gl}(1|1)\ . \\ & & & \searrow & & \nearrow & & & \\ & & & & \langle I, f\rangle & & & & \end{array}$$

So $\mathfrak{gl}(1|1)$ is clearly not irreducible. It is, however, indecomposable. If it were decomposable, one of its submodules would contain an element of the form $\alpha I + \beta e + \gamma f + \delta h$, with $\delta \neq 0$, but then this submodule would be $\mathfrak{gl}(1|1)$ and thus its complement would be zero.

Remark 2.4.8. Submodules of $\mathfrak{g}$ under the adjoint action correspond to ideals of $\mathfrak{g}$.

2.4.3 Tensor products and dual modules

Let V, W be $\mathfrak{g}$-modules, then we can make the tensor product $V \otimes W$ into a $\mathfrak{g}$-module by putting

$$x \cdot (v \otimes w) = (x \cdot v) \otimes w + (-1)^{|x||v|} v \otimes (x \cdot w).$$

Let V be a $\mathfrak{g}$-module, then we can define a $\mathfrak{g}$-module structure on the dual space V^* by $(x \cdot \alpha)v := -(-1)^{|\alpha||x|}(\alpha \circ L_x)v = -(-1)^{|\alpha||x|}\alpha(x \cdot v)$ for x in $\mathfrak{g}$, α in V^* and v in V. Here $L_x \colon V \to V$ is the operator given by the (left) action of x on V. This is indeed a $\mathfrak{g}$-module since

$$\begin{aligned}
([x, y] \cdot \alpha)v &= -(-1)^{|\alpha||x|+|\alpha||y|}(\alpha \circ L_{[x,y]})v \\
&= -(-1)^{|\alpha||x|+|\alpha||y|}(\alpha \circ L_x \circ L_y)v \\
&\quad + (-1)^{|\alpha||x|+|\alpha||y|+|x||y|}(\alpha \circ L_y \circ L_x)v \\
&= (-1)^{|\alpha||x|+|x||y|}(y \cdot (\alpha \circ L_x))v - (-1)^{|\alpha||y|}(x \cdot (\alpha \circ L_y))v \\
&= -(-1)^{|x||y|}(y \cdot (x \cdot \alpha))v + (x \cdot (y \cdot \alpha))v.
\end{aligned}$$

2.5 Cartan subalgebras and root systems

In this section, we assume that our ground field is $\mathbb{C}$.

2.5.1 The Cartan subalgebra

In classical Lie theory the Cartan subalgebra of a Lie algebra $\mathfrak{g}$ is an important tool to study $\mathfrak{g}$-representations. The definition of the Cartan subalgebra of a Lie superalgebra $\mathfrak{g}$ is as follows.

Definition 2.5.1. *A sub(super)algebra $\mathfrak{h} \subseteq \mathfrak{g}$ is a Cartan subalgebra if $\mathfrak{h}$ is a nilpotent, selfnormalising Lie sub(super)algebra of $\mathfrak{g}$.*

Here nilpotent and normaliser have the same meaning as in the non-super case. The algebra $\mathfrak{h}$ is nilpotent if, for certain ℓ, $\mathfrak{h}^\ell = 0$, where

$\mathfrak{h}^\ell = [\mathfrak{h}, \mathfrak{h}^{\ell-1}]$ and $\mathfrak{h}^0 = \mathfrak{h}$. The normaliser of a Lie sub(super)algebra $\mathfrak{h}$ in $\mathfrak{g}$ is defined by

$$N_{\mathfrak{g}}(\mathfrak{h}) := \{x \in \mathfrak{g} \mid [x, y] \in \mathfrak{h}, \forall y \in \mathfrak{h}\}.$$

It is the biggest subalgebra of $\mathfrak{g}$ such that $\mathfrak{h}$ is an ideal in this subalgebra.

Example 2.5.2. *Consider*

$$\begin{aligned} &\mathfrak{gl}(m|n) \\ &= \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \Big| \; A \in \mathbb{C}^{m\times m}, B \in \mathbb{C}^{m\times n}, C \in \mathbb{C}^{n\times m}, D \in \mathbb{C}^{n\times n} \right\}, \end{aligned}$$

then the subalgebra $\mathfrak{h}$ consisting of the diagonal matrices is a Cartan subalgebra. A basis of $\mathfrak{h}$ is given by $H_i := E_{ii}$ where $1 \leq i \leq m+n$. Remark that $\mathfrak{h} \subseteq \mathfrak{g}_{\bar{0}}$.

2.5.2 Roots

We can also generalise the concept of roots and root spaces to Lie superalgebras. In this section, let $\mathfrak{g}$ be a Lie superalgebra and $\mathfrak{h}$ its Cartan subalgebra.

Definition 2.5.3. *Let $\alpha \in \mathfrak{h}_{\bar{0}}^*$, then*

$$\mathfrak{g}_\alpha := \{X \in \mathfrak{g} \mid [H, X] = \alpha(H)X, \forall H \in \mathfrak{h}_{\bar{0}}\}.$$

If $\alpha \neq 0$ and $\mathfrak{g}_\alpha \neq 0$, then α is called a root and $\mathfrak{g}_\alpha$ a root space.

Note that α is contained in the dual space of the *even* part of $\mathfrak{h}$.

Example 2.5.4 ($\mathfrak{gl}(m|n)$)**.** *Consider the dual space $\mathfrak{h}^*$ of the Cartan subalgebra $\mathfrak{h}$ defined in Example 2.5.2. We define a basis of $\mathfrak{h}^*$ as follows:*

$$\epsilon_i(H_j) = \delta_{ij} \text{ for } 1 \leq i, j \leq m+n.$$

We will also use the notation

$$\delta_{i-m} := \epsilon_i \text{ for } i > m.$$

We have

$$[E_{ii}, E_{kl}] = \delta_{ik} E_{k\ell} - \delta_{i\ell} E_{k\ell},$$

which we can rewrite as

$$[H_i, E_{k\ell}] = (\epsilon_k - \epsilon_\ell)(H_i)E_{k\ell}.$$

Therefore we sometimes write $X_{\epsilon_k - \epsilon_\ell}$ *for* $E_{k\ell}$*. The roots are given by* $\epsilon_i - \epsilon_j$ *and the corresponding root spaces are one-dimensional*

$$\mathfrak{g}_{\epsilon_i - \epsilon_j} = \langle E_{ij} \rangle.$$

We have the decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha,$$

where Φ *is the set of all roots, called the root system.*

For Lie superalgebras we have the new concept of even and odd roots.

Definition 2.5.5. *A root* α *is called even if* $\mathfrak{g}_\alpha \cap \mathfrak{g}_{\bar{0}} \neq 0$ *and odd if* $\mathfrak{g}_\alpha \cap \mathfrak{g}_{\bar{1}} \neq 0$.

A root can be even and odd at the same time. This will occur for the queer Lie superalgebra $\mathfrak{q}(n)$.

For $\mathfrak{gl}(m|n)$ we have that the set of even roots is given by

$$\Phi_{\bar{0}} = \{\epsilon_i - \epsilon_j\} \cup \{\delta_k - \delta_\ell\}$$

for $1 \leq i, j \leq m, 1 \leq k, \ell \leq n$ with $i \neq j, k \neq \ell$ and the set of odd roots by

$$\Phi_{\bar{1}} = \{\pm(\epsilon_i - \delta_j)\}$$

for $1 \leq i \leq m, 1 \leq j \leq n$. Remark that for $\mathfrak{gl}(m|n)$

$$\Phi_{\bar{0}} \cap \Phi_{\bar{1}} = \emptyset, \quad -\Phi_{\bar{0}} = \Phi_{\bar{0}}, \quad -\Phi_{\bar{1}} = \Phi_{\bar{1}}, \quad \Phi = -\Phi.$$

The basic classical Lie superalgebras have the following structure, which is very similar to the structure of semisimple Lie algebras.

Proposition 2.5.6 ([CW, Theorem 1.15])**.** *Let* $\mathfrak{g}$ *be a* basic *Lie superalgebra with a Cartan subalgebra* $\mathfrak{h}$.

1. *We have a root space decomposition of* $\mathfrak{g}$ *with respect to* $\mathfrak{h}$*:*

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha.$$

The Cartan subalgebra $\mathfrak{h}$ *is even and*

$$\mathfrak{h} = \{X \in \mathfrak{g} \mid [H, X] = 0, \forall H \in \mathfrak{h}\}.$$

2. $\dim \mathfrak{g}_\alpha = 1$ *for* $\alpha \in \Phi$.
3. $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$ *for* $\alpha, \beta, \alpha+\beta \in \Phi$
4. *There exists a non-degenerate even invariant supersymmetric bilinear form* τ *on* $\mathfrak{g}$ *and the restriction of this bilinear form on* $\mathfrak{h}$ *is still non-degenerate.*
5. $\tau(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$ *unless* $\alpha = -\beta \in \Phi$.
6. *For every* α *there exists* $X_\alpha \in \mathfrak{g}_\alpha$ *such that*

$$[X_\alpha, X_{-\alpha}] = \tau(X_\alpha, X_{-\alpha}) H_\alpha,$$

where $\tau(H_\alpha, H) = \alpha(H)$ *for all* $H \in \mathfrak{h}^*$.

7. $\Phi = -\Phi, \Phi_{\bar{0}} = -\Phi_{\bar{0}}$ *and* $\Phi_{\bar{1}} = -\Phi_{\bar{1}}$.

This theorem is false for non-basic Lie superalgebras. For example for the queer Lie superalgebra the root spaces are no longer one-dimensional but two-dimensional. For the queer Lie superalgebra we also have that roots are even and odd at the same time and that the Cartan subalgebra $\mathfrak{h}$ is no longer contained in the even part. For the periplectic Lie superalgebra 7 does not longer hold, while 4 is wrong for both the periplectic and the queer Lie superalgebra.

Example 2.5.7. For the queer algebra $\mathfrak{q}(n)$, set

$$\begin{aligned} \mathfrak{h} &= \text{subset of } \mathfrak{q}(n) \text{ consisting of block diagonal matrices} \\ &= \left\{ \begin{pmatrix} \lambda & \mu \\ \mu & \lambda \end{pmatrix} \middle| \ \lambda, \mu \in \mathbb{C}^{n \times n}, \lambda, \mu \text{ diagonal matrices} \right\}. \end{aligned}$$

Remark that $\mathfrak{h}$ is not abelian. We define the following basis of $\mathfrak{q}(n)$:

$$\begin{aligned} \widetilde{E}_{ij} &= E_{ij} + E_{(n+i)(n+j)} & 1 \le i, j \le n \\ \overline{E}_{ij} &= E_{i(j+n)} + E_{(n+i)j} & 1 \le i, j \le n. \end{aligned}$$

Now $\mathfrak{h}$ consists of $H_i = \widetilde{H_i} = \widetilde{E}_{ii}$ and $\overline{H_i} = \overline{E}_{ii}$ for $1 \le i \le n$. Remark that $\mathfrak{h} \cap \mathfrak{g}_{\bar{1}} \neq \emptyset$ in contrast to the basic classical case.

Consider again the dual space of the even part of $\mathfrak{h}$, $\mathfrak{h}_{\bar{0}}^* = \{\epsilon_i \mid 1 \leq i \leq n\}$ where $\epsilon_i(H_j) := \delta_{ij}$. We have

$$\begin{aligned}
[H_i, \widetilde{E}_{k\ell}] &= \delta_{ik}\widetilde{E}_{k\ell} - \delta_{i\ell}\widetilde{E}_{k\ell} \\
&= \epsilon_k - \epsilon_\ell(H_i)\widetilde{E}_{k\ell} \\
[H_i, \overline{E}_{k\ell}] &= \delta_{ik}\overline{E}_{k\ell} - \delta_{i\ell}\overline{E}_{k\ell} \\
&= \epsilon_k - \epsilon_\ell(H_i)\overline{E}_{k\ell}.
\end{aligned}$$

Hence

$$\Phi = \{\epsilon_i - \epsilon_j \mid 1 \leq i, j \leq n, i \neq j\}, \quad \Phi_{\bar{0}} = \Phi, \quad \Phi_{\bar{1}} = \Phi.$$

We have $\mathfrak{g}_{\epsilon_i - \epsilon_j} = \langle \widetilde{E}_{ij} \rangle \oplus \langle \overline{E}_{ij} \rangle$, where $\langle \widetilde{E}_{ij} \rangle$ is even and $\langle \overline{E}_{ij} \rangle$ is odd. We still have

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha.$$

Therefore it follows immediately that $\mathfrak{h}$ is self-normalising. One can also easily check that $\mathfrak{h}$ is nilpotent. Hence $\mathfrak{h}$ is indeed a Cartan subalgebra.

2.5.3 Borel subalgebras

In this subsection, we will always assume $\mathfrak{g}$ to be a basic classical Lie superalgebra. Let Φ be a root system of $\mathfrak{g}$ and let E be the real vector space spanned by Φ. We will always assume that an ordering on E respects the vector space structure, i.e. $v \geq w$ and $v' \geq w'$ imply $v + v' \geq w + w'$ while $v \geq w$ implies $-v \leq -w$ and $\lambda v \geq \lambda w$ for $\lambda \in \mathbb{R}^+$.

Definition 2.5.8. *A subset Φ^+ of Φ is called a positive system when it contains precisely all $\alpha \in \Phi$ for which $\alpha > 0$ for some total ordering on E. A simple (or fundamental) system Π is the subset of Φ^+ such that every root in Π can not be written as the sum of two other roots in Φ^+. The elements of Π are called simple roots.*

The set Φ^- is defined similarly and we have $\Phi = \Phi^+ \sqcup \Phi^-$ where the elements of Φ^+ are called the positive roots and the elements of Φ^- the negative roots.

Example 2.5.9. *A simple system of* $\mathfrak{gl}(m|n)$ *is given by*

$$\begin{aligned}\Pi = \{\epsilon_i - \epsilon_{i+1} \mid 1 \leq i \leq m-1\} \cup \{\epsilon_m - \delta_1\} \\ \cup \{\delta_i - \delta_{i+1} \mid 1 \leq i \leq n-1\}.\end{aligned} \tag{2.2}$$

With each simple system we can give a decomposition of a basic simple Lie superalgebra $\mathfrak{g}$ as follows:

$$\mathfrak{n}^+ = \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha \qquad \mathfrak{n}^- = \bigoplus_{\alpha \in \Phi^-} \mathfrak{g}_\alpha \qquad \mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$$

and $\mathfrak{b} := \mathfrak{h} \oplus \mathfrak{n}^+$ is called a Borel subalgebra.

Example 2.5.10. *For* $\mathfrak{gl}(m|n)$ *the simple system in* (2.2) *corresponds to the triangular decompositions*

- $\mathfrak{n}^+ =$ *set of upper triangular matrices*
- $\mathfrak{h} =$ *set of diagonal matrices*
- $\mathfrak{n}^- =$ *set of lower diagonal matrices.*

Remark 2.5.11. The Weyl group of a Lie superalgebra $\mathfrak{g}$ is defined as the Weyl group of the underlying Lie algebra $\mathfrak{g}_{\bar{0}}$. One can show that the Weyl group acts on roots and on the set of simple systems. In contrast to the classical case, however, this action is not transitive. So there exist simple systems, and thus also Borel subalgebras, which are not conjugate to each other. We also mention that the Borel subalgebra is no longer maximal solvable since the result of adding the root space corresponding to a negative isotropic simple root to $\mathfrak{b}$ is still solvable, [CW, Remark 1.18].

2.5.4 Cartan matrix for a basic Lie superalgebra

Let Π be a simple system. For each $\alpha_i \in \Pi$, choose $E_{\alpha_i} \in \mathfrak{g}_{\alpha_i}, F_{\alpha_i} \in \mathfrak{g}_{-\alpha_i}$. Set $H_{\alpha_i} := [E_{\alpha_i}, F_{\alpha_i}]$, then H_{α_i} is defined up to a constant. If $\alpha_i(H_{\alpha_i}) \neq 0$, fix this constant by $\alpha_i(H_{\alpha_i}) = 2$. Then the Cartan matrix is defined by

$$A_{ij} = \alpha_j(H_i) \qquad (H_j = H_{\alpha_j}).$$

For $\mathfrak{gl}(m|n)$ we obtain the following Cartan matrix

$$\begin{pmatrix} 2 & -1 & & & & & & & & \\ -1 & 2 & -1 & & & & & & & \\ & \ddots & \ddots & -1 & & & & & & \\ & & -1 & 2 & -1 & & & & & \\ & & & -1 & 0 & -1 & & & & \\ & & & & -1 & 2 & -1 & & & \\ & & & & & -1 & \ddots & \ddots & & \\ & & & & & & -1 & 2 & -1 \\ & & & & & & & -1 & 2 \end{pmatrix}.$$

2.5.5 An example: $D(2,1;\alpha)$

Recall the construction of $D(2,1;\alpha)$ in Subsection 2.3.5. Consider the following matrix realisations for the basis elements $\{E_i, F_i, H_i\}$ of $\mathfrak{sl}(V_i)$.

$$E_i = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F_i = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad H_i = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then the realisation of the vector space V_i is given by

$$u_+ = (1,0)^t, \quad u_- = (0,1)^t.$$

In this realisation we have

$$p_i(u_+, u_+) = 2E_i, \quad p_i(u_+, u_-) = -H_i, \quad p_i(u_-, u_-) = -2F_i.$$

The Cartan subalgebra of $D(2,1;\alpha)$ is given by $\mathfrak{h} = \langle H_1, H_2, H_3\rangle$. If we define $\mathfrak{h}^* = \{\delta_1, \delta_2, \delta_3\}$ by

$$\delta_i(H_j) = \delta_{ij},$$

then the even and odd roots are given by

$$\Delta_0 = \{\pm 2\delta_1, \pm 2\delta_2, \pm 2\delta_3\} \qquad \Delta_1 = \{\pm\delta_1 \pm \delta_2 \pm \delta_3\}.$$

The corresponding root vectors are

$$X_{2\delta_i} = E_i, \quad X_{-2\delta_i} = F_i, \quad X_{\pm\delta_1\pm\delta_2\pm\delta_3} = u_\pm \otimes u_\pm \otimes u_\pm.$$

We will also use the notation H_{δ_i} for H_i.

Consider the simple root system

$$\Pi = \{2\delta_2, \delta_1 - \delta_2 - \delta_3, 2\delta_3.\}$$

We have

$$\begin{aligned}
[X_{2\delta_2}, X_{-2\delta_2}] &= H_2 \\
[X_{\delta_1-\delta_2-\delta_3}, X_{-\delta_1+\delta_2+\delta_3}] &= p(u_+ \otimes u_- \otimes u_-, u_- \otimes u_+ \otimes u_+) \\
&= \sigma_1 \psi(u_-, u_+)\psi(u_-, u_+)p_1(u_+, u_-) \\
&\quad + \sigma_2 \psi(u_-, u_+)\psi(u_+, u_-)p_3(u_-, u_+) \\
&\quad + \sigma_3 \psi(u_+, u_-)\psi(u_-, u_+)p_3(u_-, u_+) \\
&= -\sigma_1 H_1 + \sigma_2 H_2 + \sigma_3 H_3 \\
&= -\frac{1+\alpha}{2} H_1 - \frac{1}{2} H_2 - \frac{\alpha}{2} H_3 \\
[X_{2\delta_3}, X_{-2\delta_3}] &= H_3.
\end{aligned}$$

The Cartan matrix is given by

$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 0 & -\alpha \\ 0 & -1 & 2 \end{pmatrix}.$$

For $\alpha = -1$ this Cartan matrix corresponds to the Lie superalgebra $A(1,1) = \mathfrak{psl}(2,2)$. Remark that $D(2,1;-1)$ is non-simple since it contains an ideal I with $I \cong \mathfrak{psl}(2,2)$ and $D(2,1;-1)/I \cong \mathfrak{sl}(2)$, see [Se]. Denote by E_{ij} the matrix with the (i,j)th entry equal to one and all other entries zero. Then an explicit isomorphism between I and $\mathfrak{psl}(2,2)$ is given by

$$\begin{aligned}
H_{\delta_2} &= E_{11} - E_{22}, & X_{2\delta_2} &= E_{12}, & X_{-2\delta_2} &= E_{21} \\
H_{\delta_3} &= E_{33} - E_{44}, & X_{2\delta_3} &= E_{34}, & X_{-2\delta_3} &= E_{43} \\
X_{\delta_1+\delta_2-\delta_3} &= \frac{E_{13} + E_{42}}{\sqrt{2}} & X_{-\delta_1-\delta_2+\delta_3} &= \frac{-E_{31} + E_{24}}{\sqrt{2}} \\
X_{\delta_1+\delta_2+\delta_3} &= \frac{-E_{14} + E_{32}}{\sqrt{2}} & X_{-\delta_1+\delta_2+\delta_3} &= \frac{E_{32} + E_{14}}{\sqrt{2}} \\
X_{\delta_1-\delta_2-\delta_3} &= \frac{E_{23} - E_{41}}{\sqrt{2}} & X_{-\delta_1-\delta_2-\delta_3} &= \frac{-E_{41} - E_{23}}{\sqrt{2}} \\
X_{\delta_1-\delta_2+\delta_3} &= \frac{-E_{24} - E_{31}}{\sqrt{2}} & X_{-\delta_1+\delta_2-\delta_3} &= \frac{E_{42} - E_{13}}{\sqrt{2}}.
\end{aligned}$$

2.6 More on representation theory

In this section, we will again work over $\mathbb{C}$.

2.6.1 Weight modules

Let $\mathfrak{g}$ be a basic Lie superalgebra, $\mathfrak{h}$ a Cartan subalgebra, V a $\mathfrak{g}$-module. Weights, weight spaces and weight modules for Lie superalgebras are defined in the same way as for ordinary Lie algebras.

Definition 2.6.1. *The weight space V_μ for $\mu \in \mathfrak{h}^*$ is given by*

$$V_\mu := \{v \in V \mid H \cdot v = \mu(H)v, \forall H \in \mathfrak{h}\}.$$

An element μ in $\mathfrak{h}^*$ is called a weight of the representation V if $V_\mu \neq 0$. Set $\Pi(V) = \{\mu \in \mathfrak{h}^* \mid V_\mu \neq 0\}$ the set of all weights. Then V is called a weight module if $V = \bigoplus_{\mu \in \Pi(V)} V_\mu$.

Two examples of weight modules are the adjoint representation and the natural representation of $\mathfrak{gl}(m|n)$. Namely, consider $\mathfrak{g} = \mathfrak{gl}(m|n)$ as a $\mathfrak{g}$-module under the adjoint action. We find that the weight spaces are the root spaces and $\mathfrak{h}$, the weights are the roots and the zero weight, and $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$. Hence $\mathfrak{g}$ is a weight module because $\mathfrak{h} = \mathfrak{g}_0$, the zero weight space.

Let V be the natural representation of $\mathfrak{gl}(m|n)$ i.e. $V = \mathbb{C}^{m|n}$. Then the weights are ϵ_i, δ_j and

$$\begin{cases} V_{\epsilon_i} = \mathbb{C}e_i \\ V_{\delta_i} = \mathbb{C}e_{i+m} \end{cases}$$

with $(e_i)_{i=1,\ldots,m+n}$ the standard basis: $e_i = (0 \ \ldots \ 0 \ \underbrace{1}_{i^{\text{th}}\text{entry}} \ 0 \ \ldots \ 0)^T$.

2.6.2 The universal enveloping algebra

Definition 2.6.2. *The universal enveloping algebra of a Lie superalgebra $\mathfrak{g}$ is an associative superalgebra $U(\mathfrak{g})$, together with a homomorphism of Lie superalgebras $i : \mathfrak{g} \to U(\mathfrak{g})$ such that the following (universal) property holds: given an associative superalgebra A and* *a Lie superalgebra homomorphism* $\phi : \mathfrak{g} \to A$*, there exists a unique morphism* $\psi : U(\mathfrak{g}) \to A$ *of associative superalgebras for which*

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\;i\;} & U(\mathfrak{g}) \\ & \searrow^{\phi} & \downarrow{\scriptstyle \psi} \\ & & A \end{array}$$

commutes. The morphism i *is a monomorphism, i.e. we can embed* $\mathfrak{g}$ *in* $U(\mathfrak{g})$*.*

Alternatively we can define the universal enveloping algebra in the following way

$$U(\mathfrak{g}) = T(\mathfrak{g}) / \langle x \otimes y - (-1)^{|x||y|} y \otimes x - [x, y] \mid x, y \in \mathfrak{g} \rangle$$

where $T(\mathfrak{g}) = \sum_{k=0}^{+\infty} \bigotimes^k \mathfrak{g}$ is the tensor algebra of $\mathfrak{g}$.

Proposition 2.6.3. *Representations of* $\mathfrak{g}$ *are equivalent to representations of* $U(\mathfrak{g})$*.*

Proof. Given a representation of $\mathfrak{g}$, the definition of $U(\mathfrak{g})$ guarantees the existence of a representation $\psi : U(\mathfrak{g}) \to \mathrm{End}(V)$. Conversely, given such a $\psi : U(\mathfrak{g}) \to \mathrm{End}(V)$, we can define

$$\phi \colon \mathfrak{g} \to \mathrm{End}(V);\ x \mapsto \psi(i(x)).$$

For this map we have

$$\begin{aligned} \phi(x)\phi(y) - (-1)^{|x||y|}\phi(y)\phi(x) &= \psi(i(x)i(y) - (-1)^{|x||y|} i(y)i(x)) \\ &= \psi(i([x, y])) = \phi([x, y]). \end{aligned}$$

Hence it defines a Lie superalgebra morphism. □

We also have a Poincaré–Birkhoff–Witt Theorem for Lie superalgebras.

Theorem 2.6.4 ([CW, Theorem 1.32])**.** *Let* $\{x_1, \ldots, x_m\}$ *be a basis of* $\mathfrak{g}_{\bar{0}}$ *and* $\{y_1, \ldots, y_n\}$ *a basis of* $\mathfrak{g}_{\bar{1}}$*. Then a basis of* $U(\mathfrak{g})$ *is given by*

$$\{x_1^{s_1} \ldots x_m^{s_m} y_1^{t_1} \ldots y_n^{t_n} \mid s_1, \ldots, s_m \in \mathbb{N}, t_1, \ldots, t_n \in \{0, 1\}\}.$$

Also for $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ *we can decompose the universal enveloping algebra as* $U(\mathfrak{g}) = U(\mathfrak{n}^-)U(\mathfrak{h})U(\mathfrak{n}^+)$*.*

2.6.3 Induced modules

An important class of $\mathfrak{g}$-modules are the so-called induced modules. To construct these, we start from a $\mathfrak{k}$-module V for a subalgebra $\mathfrak{k}$ of $\mathfrak{g}$ and then set

$$\mathrm{Ind}_{\mathfrak{k}}^{\mathfrak{g}} V := U(\mathfrak{g}) \otimes_{U(\mathfrak{k})} V.$$

We will look at this construction in more detail for $\mathfrak{k}$ a Borel subalgebra and for V a one-dimensional module with trivial $\mathfrak{n}^+$-action.

Let $\mathfrak{g}$ be a basic Lie superalgebra, $\mathfrak{b}$ a Borel subalgebra: $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$. Let $\mathbb{C}_\lambda$ be the one-dimensional $\mathfrak{b}$-module defined by

$$\begin{aligned} \mathfrak{h} \cdot x &= \lambda(\mathfrak{h}) x \\ \mathfrak{n}^+ \cdot x &= 0 \end{aligned}$$

for a weight $\lambda \in \mathfrak{h}^*$. Then the Verma module $M(\lambda)$ is given by

$$M(\lambda) = \mathrm{Ind}_{\mathfrak{b}}^{\mathfrak{g}} \mathbb{C}_\lambda := U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda.$$

The interpretation of $U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda$ is as follows. We see $U(\mathfrak{g})$ as a right $U(\mathfrak{b})$-module and $\mathbb{C}_\lambda$ as a left $U(\mathfrak{b})$-module. Since we tensor over $U(\mathfrak{b})$, we get the equivalence relation

$$x \cdot y \otimes v_\lambda \sim x \otimes y \cdot v_\lambda$$

for x in $U(\mathfrak{g})$, y in $U(\mathfrak{b})$ and v_λ in $\mathbb{C}_\lambda$. Hence

$$\begin{aligned} M(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda &\cong U(\mathfrak{n}^-) U(\mathfrak{b}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda \\ &\cong U(\mathfrak{n}^-) \otimes_{U(\mathfrak{b})} U(\mathfrak{b}) \mathbb{C}_\lambda \cong U(\mathfrak{n}^-) \otimes \mathbb{C}_\lambda. \end{aligned}$$

We conclude that $M(\lambda)$ is a free $U(\mathfrak{n}^-)$-module of rank 1.

Let v_λ be a basis of $\mathbb{C}_\lambda$, then $1 \otimes v_\lambda$ satisfies

$$\begin{aligned} H \cdot (1 \otimes v_\lambda) &= H \otimes v_\lambda = \lambda(H) v_\lambda \qquad && \forall H \in \mathfrak{h} \\ x \cdot (1 \otimes v_\lambda) &= x \otimes v_\lambda = 0 && \forall x \in \mathfrak{n}^+. \end{aligned}$$

Therefore v_λ is a so-called highest weight vector.

Definition 2.6.5. *Consider $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ and let V be a $\mathfrak{g}$-module. Then $v \in V$ is called a maximal vector or a highest weight vector of weight $\lambda \in \mathfrak{h}^*$ if*

$$\mathfrak{h} \cdot v = \lambda(\mathfrak{h}) v \qquad \textit{and} \qquad \mathfrak{n}^+ \cdot v = 0.$$

If v generates V, i.e. $V = U(\mathfrak{g}) \cdot v = U(\mathfrak{n}^-) \cdot v$, then V is called a highest weight module.

Some examples of highest weight modules are:

- Verma modules are highest weight modules.
- Let $\mathfrak{g} = \mathfrak{gl}(m|n)$, V the natural representation. Then e_1 is a highest weight since $x \cdot e_1 = 0$ for all upper diagonal matrices and $h_i(e_1) = \epsilon_1(h_i)e_1$. Furthermore $e_i = E_{i1} \cdot e_1$, thus e_1 generates V. We conclude that V is a highest weight module.

Highest weight modules satisfy the following properties.

Theorem 2.6.6. *Let $\mathfrak{g}$ be a semisimple Lie algebra or a basic Lie superalgebra with decomposition $\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-$ and M a highest weight module of weight λ generated by v^+. Let $\alpha_1, \dots, \alpha_m$ be the set of positive roots and $y_{\alpha_i} = y_i$ the corresponding root vectors in $\mathfrak{n}^-$.*

1. *The module M is spanned by vectors $y_1^{z_1} \dots y_m^{z_m} \cdot v^+$ with $z_i \in \mathbb{N}$, having weights $\lambda - \sum_{i=1}^m z_i\alpha_i$.*
2. *All weights μ in M satisfy $\mu \leq \lambda$.*
3. *We have $\dim M_\mu < \infty$ and $\dim M_\lambda = 1$ and M is a weight module, locally $\mathfrak{n}^+$-finite. This also immediately implies that M is an element of category $\mathcal{O}$, see [Hu, Chapter 1] for a definition of category $\mathcal{O}$.*
4. *Each non-zero quotient of M is again a highest weight module.*
5. *Each submodule of M is a weight module, a submodule generated by a maximal vector of weight $\mu < \lambda$ is proper.*
6. *The module M has a unique maximal submodule and unique simple quotient. Therefore M is indecomposable.*
7. *All simple highest weight modules of weight λ are isomorphic.*

Proof. This is [Hu, Section 1.2] for Lie algebras. The proof in [Hu] can be carried over almost verbatim to the Lie superalgebra case. □

Proposition 2.6.7. *Let $\mathfrak{g}$ be a basic Lie superalgebra with a fixed Borel subalgebra $\mathfrak{b}$. Any finite-dimensional simple $\mathfrak{g}$-module is a highest weight module.*

To prove this proposition, we will use the following lemma. Remember that $\mathfrak{g}$ is solvable if for certain n we have $\mathfrak{g}^{(n)} = 0$ where $\mathfrak{g}^{(n)} := [\mathfrak{g}^{(n-1)}, \mathfrak{g}^{(n-1)}]$, $\mathfrak{g}^{(0)} = \mathfrak{g}$.

Lemma 2.6.8 ([CW, Lemma 1.33])**.** *Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a finite-dimensional solvable Lie superalgebra such that $[\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}] \subseteq [\mathfrak{g}_{\bar{0}}, \mathfrak{g}_{\bar{0}}]$. Then every finite-dimensional simple $\mathfrak{g}$-module is one-dimensional. A complete list of finite-dimensional simple $\mathfrak{g}$-modules is given by $\mathbb{C}_\lambda$ for $\lambda \in (\mathfrak{g}_{\bar{0}}/[\mathfrak{g}_{\bar{0}}, \mathfrak{g}_{\bar{0}}])^*$. Here $\mathbb{C}_\lambda$ is defined by*

$$\begin{aligned} x \cdot v_\lambda &= \lambda(x) v_\lambda && \text{for } x \in \mathfrak{g}_{\bar{0}} \\ y \cdot v_\lambda &= 0 && \text{for } y \in \mathfrak{g}_{\bar{1}}. \end{aligned}$$

The fact that the one-dimensional $\mathfrak{g}$-modules are given by $\mathbb{C}_\lambda$ for $\lambda \in (\mathfrak{g}_{\bar{0}}/[\mathfrak{g}_{\bar{0}}, \mathfrak{g}_{\bar{0}}])^*$ can be seen as follows. Let $V = \langle v \rangle$ be a one-dimensional $\mathfrak{g}$-module. We have $x \cdot v = 0$ if $|x| = 1$ since an odd element changes the parity of v and V is one-dimensional. For $x, y \in \mathfrak{g}_{\bar{0}}$ such that $x \cdot v = a_x \cdot v$ and $y \cdot v = a_y \cdot v$, for some constants $a_x, a_y \in \mathbb{C}$, we obtain

$$[x, y]v = x(yv) - y(xv) = (a_x a_y - a_y a_x)v = 0.$$

So $[\mathfrak{g}_{\bar{0}}, \mathfrak{g}_{\bar{0}}]v = 0$. Define $\lambda : \mathfrak{g}_{\bar{0}}/[\mathfrak{g}_{\bar{0}}, \mathfrak{g}_{\bar{0}}] \to \mathbb{C}$ by $\lambda(x) = a_x$.

Proof of 2.6.7, based on [CW, Proposition 1.35]. Let $\mathfrak{g}$ be a basic Lie superalgebra with Borel $\mathfrak{b}$. Then $\mathfrak{b}$ is solvable and

$$[\mathfrak{b}_{\bar{1}}, \mathfrak{b}_{\bar{1}}] = [\mathfrak{n}_{\bar{1}}, \mathfrak{n}_{\bar{1}}] \subseteq \mathfrak{n}_{\bar{0}} = [\mathfrak{h}, \mathfrak{n}_{\bar{0}}] \subseteq [\mathfrak{b}_{\bar{0}}, \mathfrak{b}_{\bar{0}}].$$

Let V be a finite-dimensional simple $\mathfrak{g}$-module. We can see V as a $\mathfrak{b}$-module and by Lemma 2.6.8 it contains a one-dimensional simple submodule $\mathbb{C}_\lambda$, where

$$\lambda : \underbrace{\mathfrak{h} \oplus \mathfrak{n}_{\bar{0}}/[\mathfrak{h} \oplus \mathfrak{n}_{\bar{0}}, \mathfrak{h} \oplus \mathfrak{n}_{\bar{0}}]}_{\cong \mathfrak{h}} \to \mathbb{C}.$$

We can thus interpret λ as an element of $\mathfrak{h}^*$. So there exists a $v_\lambda \in V$ for which

$$\begin{aligned} \mathfrak{h} \cdot v_\lambda &= \lambda(\mathfrak{h}) v_\lambda \\ x \cdot v_\lambda &= 0 && \text{for all } x \in \mathfrak{n}^+, \end{aligned}$$

and $U(\mathfrak{n}^-)v_\lambda$ is a submodule of V. Since V is simple, the module generated by v_λ has to be the whole of V and V is a highest weight module. $\square$

2.7 Polynomials and spherical harmonics

We can also realise the orthosymplectic Lie superalgebra as differential operators on polynomials. We will collect here also some results on spherical harmonics, which we will use later on in this thesis.

2.7.1 Another realisation of $\mathfrak{osp}$

Denote by $\mathcal{P}(\mathbb{R}^m)$ the space of complex-valued polynomials in m variables and by $\Lambda^{2n} = \Lambda(\mathbb{R}^{2n})$ the Grassmann algebra in $2n$ variables. Then we define the space of superpolynomials as

$$\mathcal{P}(\mathbb{R}^{m|2n}) := \mathcal{P}(\mathbb{R}^m) \otimes_{\mathbb{C}} \Lambda^{2n},$$

the space of complex-valued polynomials in m even and $2n$ odd variables. These variables satisfy the commutation relations

$$z_i z_j = (-1)^{|i||j|} z_j z_i.$$

We define the differential operator ∂^i as the unique derivation in $\mathrm{End}(\mathcal{P}(\mathbb{R}^{m|2n}))$ such that $\partial^i(z_j) = \delta_{ij}$.

Consider a supersymmetric, non-degenerate, even bilinear form $\langle \cdot, \cdot \rangle_\beta$ on $\mathbb{R}^{m|2n}$ with components β_{ij} and let β^{ij} be the components of the inverse matrix. So $\sum_j \beta_{ij}\beta^{jk} = \delta_{ik}$. Set $z^j = \sum_i z_i \beta^{ij}$. We also set $\partial_j = \sum_i \partial^i \beta_{ji}$. Then it satisfies $\partial_i(z^j) = \delta_{ij}$.

We can realise the orthosymplectic Lie superalgebra using differential operators acting on $\mathcal{P}(\mathbb{R}^{m|2n})$. A basis of the orthosymplectic Lie superalgebra in this realisation is given by

$$\begin{aligned} L_{i,j} &:= z_i\partial_j - (-1)^{|i||j|} z_j \partial_i, && \text{for } i < j \text{ and} \\ L_{i,i} &:= 2 z_i \partial_i && \text{for } |i| = 1. \end{aligned}$$

Define also operators by

$$R^2 := \sum_{i,j} \beta^{ij} z_i z_j, \quad \mathbb{E} := \sum_i z^i \partial_i \quad \text{and} \quad \Delta := \sum_{i,j} \beta^{ij} \partial_i \partial_j. \tag{2.3}$$

The operator R^2 acts through multiplication, $\mathbb{E}$ is called the Euler operator and Δ the Laplacian. We have the following.

Lemma 2.7.1. *The operators R^2, $\mathbb{E}$ and Δ commute with the orthosymplectic Lie superalgebra in* $\mathrm{End}(\mathcal{P}(\mathbb{R}^{m|2n}))$. *Furthermore, they satisfy*

$$[\Delta, R^2] = 4\mathbb{E} + 2M$$
$$[\Delta, \mathbb{E}] = 2\Delta$$
$$[R^2, \mathbb{E}] = -2R^2,$$

where $M = m - 2n$ is the superdimension.

Proof. A straightforward calculation or see, for example, [DeS]. □

In particular, Lemma 2.7.1 implies that $(R^2/2, \mathbb{E}+M/2, -\Delta/2)$ forms an $\mathfrak{sl}(2)$-triple.

Later on we will need these operators not only as operators acting on superpolynomials but as global differential operators acting on an affine superspace. (We refer to Appendix A for a definition of the affine superspace and for an explanation of the notations we use.) We can extend their definition as follows. Consider a finite-dimensional super-vector space V equipped with a supersymmetric, non-degenerate, even bilinear form $\langle \cdot, \cdot \rangle_\beta$. Denote by z_i the coordinate function on $\mathbb{A}(V^*)$ given by $z_i(v) = v_i$ for $v = \sum_i v_i e^i$, where $(e^i)_i$ is a homogeneous basis of V^*. Define ∂^i as the unique element of $\Gamma(\mathcal{D}_{\mathbb{A}(V^*)})$ which satisfies $\partial^i(z_j) = \delta_{ij}$. We define R^2, Δ, $\mathbb{E}$ and L_{ij} similarly as for the $\mathbb{R}^{m|2n}$ case. The operators L_{ij} will give a realisation of $\mathfrak{osp}(V)$ and Lemma 2.7.1 still holds.

2.7.2 Spherical harmonics

We write $\mathcal{P}_k(\mathbb{R}^{m|2n})$ for the space of homogeneous polynomials of degree k. These polynomials satisfy

$$\mathbb{E}f = kf \quad \text{for all } f \in \mathcal{P}_k(\mathbb{R}^{m|2n}).$$

The space of spherical harmonics $\mathcal{H}_k(\mathbb{R}^{m|2n})$ of degree k are the homogeneous polynomials of degree k which are also in the kernel of the Laplace operator:

$$\mathcal{H}_k(\mathbb{R}^{m|2n}) = \{f \in \mathcal{P}_k(\mathbb{R}^{m|2n}) \mid \Delta f = 0\}.$$

We have the following decomposition of $\mathcal{P}(\mathbb{R}^{m|2n})$, [DeS, Theorem 3]:

Proposition 2.7.2 (Fischer decomposition)**.** *If $m-2n \notin -2\mathbb{N}$, then $\mathcal{P}(\mathbb{R}^{m|2n})$ decomposes as*

$$\mathcal{P}(\mathbb{R}^{m|2n}) = \bigoplus_{k=0}^{\infty} \mathcal{P}_k(\mathbb{R}^{m|2n}) = \bigoplus_{k=0}^{\infty} \bigoplus_{j=0}^{\infty} R^{2j}\mathcal{H}_k(\mathbb{R}^{m|2n}).$$

Proposition 2.7.3. *If $m-2n \notin -2\mathbb{N}$ and $n \neq 0$, then $\mathcal{H}_k(\mathbb{R}^{m|2n})$ is an irreducible $\mathfrak{osp}(m|2n)$-module. If $n = 0$, then $\mathcal{H}_k(\mathbb{R}^m)$ is an irreducible $\mathfrak{so}(m)$-module if $m > 2$, while $\mathcal{H}_k(\mathbb{R}^2)$ decomposes as $\mathbb{C}z^k \oplus \mathbb{C}\bar{z}^k$, where $z = x + \imath y$, $\bar{z} = x - \imath y$, $(x, y) \in \mathbb{R}^2$.*

Proof. This is [Cou, Theorem 5.2] for the case $n \neq 0$ and [He, Introduction, Theorem 3.1] for the classical case. □

The dimension of the spherical harmonics of degree k is given in [DeS, Corollary 1].

Proposition 2.7.4. *The dimension of $\mathcal{H}_k(\mathbb{R}^{m|2n})$, for $m \neq 0$, is given by*

$$\dim \mathcal{H}_k(\mathbb{R}^{m|2n}) = \sum_{i=0}^{\min(k,2n)} \binom{2n}{i}\binom{k-i+m-1}{m-1} - \sum_{i=0}^{\min(k-2,2n)} \binom{2n}{i}\binom{k-i+m-3}{m-1}.$$

'There must be some kind of way out of here,' said the joker to the thief. 'There's too much confusion I can't get no relief.'

Bob Dylan,
All along the watchtower

3

Jordan superalgebras and their TKK constructions

3.1 Introduction

There is an acclaimed principle that associates a 3-graded Lie algebra to a Jordan algebra, as developed by Tits, Kantor and Koecher in three variations, see [Ti, Kan1, Ko]. These three constructions have natural analogues for Jordan superalgebras and some also extend to Jordan (super)pairs. The principle behind these three constructions, and further variations appearing in the literature, is loosely referred to as "the" TKK construction.

A common feature of TKK constructions is that, under the appropriate conditions, they associate *simple* Lie superalgebras to *simple* Jordan superalgebras or superpairs. They were as such used to classify simple Jordan superalgebras and superpairs, see [Ka2, CK, KMZ, Kan2, Kr], but also to study representations of Jordan superalgebras, see [MZ, Sh, KS]. When the constructions of Tits, Kantor and Koecher are applied to a simple finite-dimensional Jordan algebra over the field of complex numbers, they all yield the same Lie algebra, as follows *a posteriori* from the classification. However, if one applies the TKK constructions to more general algebras, they can yield different outcomes.

The aim of this chapter is to create more structure in this plethora of TKK constructions, by (dis)proving equivalences of some of the definitions under certain conditions and describing concrete links between the different constructions.

We will first consider the zero component of the 3-graded Lie (super)algebra associated to a Jordan (super)algebra, which is often referred to as the *structure algebra*. Next we construct the 3-graded Lie superalgebra out of the structure algebra and the Jordan superalgebra. We refer to this algebra as the *TKK algebra*.

We consider four definitions of the structure algebra and show that, for *unital* Jordan superalgebras, they lead to two non-equivalent versions of the structure algebra. For non-unital Jordan superalgebras, all four definitions are non-equivalent. For completeness, we also review two further definitions of structure algebras of unital Jordan superalgebras, with no direct link to TKK constructions, and prove that these are both equivalent to one of the above definitions. One of these definitions also applies to non-unital Jordan superalgebras, and we prove that it is non-equivalent to the previous definitions.

Then we consider the TKK algebras. First we introduce Kantor's construction. Koecher's construction appears in several forms in the literature, depending on the choice of structure algebra. Using two different structure algebras, we obtain two algebras $\mathrm{Ko}(V)$ and $\widetilde{\mathrm{Ko}}(V)$. Finally, the construction by Tits depends on the structure algebra and an auxiliary three-dimensional Lie algebra, which we assume to be $\mathfrak{sl}_2$ for now. The choice of $\mathrm{Inn}(V)$ or $\mathrm{Der}(V)$ as structure algebra leads to two algebras $\mathrm{Ti}(V, \mathrm{Inn}(V), \mathfrak{sl}_2)$ and $\mathrm{Ti}(V, \mathrm{Der}(V), \mathfrak{sl}_2)$. This yields 5 definitions of TKK superalgebras associated to a Jordan superalgebra V, corresponding to constructions of Tits, Koecher and Kantor:

$\mathrm{Ti}(V, \mathrm{Inn}(V), \mathfrak{sl}_2)$	$\mathrm{Ko}(V)$	$\mathrm{Kan}(V)$
$\mathrm{Ti}(V, \mathrm{Der}(V), \mathfrak{sl}_2)$	$\widetilde{\mathrm{Ko}}(V)$	

If V is a simple finite-dimensional Jordan algebra over the field of complex numbers, it is known that all five Lie algebras are isomorphic.

We prove that as long as V is unital, the three Lie superalgebras in the top row are isomorphic. Under the same assumption, the two algebras in the bottom row are then also isomorphic and given by the algebra of derivations of the Lie superalgebras in the top row. For arbitrary V, even when finite-dimensional, we show that all five algebras can be pairwise non-isomorphic and that the link between bottom and top row through derivations generally fails.

We derive these results for the super case, but they are already pertinent for ordinary Jordan algebras. However, the differences in definitions are more exposed for Jordan superalgebras, as they already appear for *finite-dimensional simple Jordan superalgebras over the field of complex numbers*. Contrary to simple Lie algebras, simple Lie superalgebras can admit outer derivations, and contrary to Jordan algebras, there is a simple finite-dimensional Jordan superalgebra which is non-unital.

Therefore we apply our results to obtain a table with all versions of the TKK construction for the simple finite-dimensional Jordan superalgebras over an algebraically closed field of characteristic zero. For this, we rely on the classification of simple Jordan superalgebras in [CK, Ka2] and the calculation of derivations in [Ka1, Sc].

We organise this chapter as follows. In Section 3.2 we introduce some concepts and terminology regarding Jordan superalgebras and superpairs. We also introduce the real spin factor Jordan superalgebra, which will be of importance in the rest of this thesis. In Section 3.3 we investigate the different definitions of the structure algebra. In Section 3.4 we compare the constructions of Tits, Kantor and Koecher. In Section 3.5 we study further variations of the Koecher construction, based on the choice of structure algebra. In Section 3.6 we use the above to list all the versions of the TKK algebras for the finite-dimensional simple Jordan superalgebras over an algebraically closed field of characteristic zero. Finally, in Section 3.7, we calculate explicitly the structure and TKK algebras for two more examples: the real spin factor Jordan superalgebra and the exceptional Jordan superalgebra D_t.

3.2 Jordan superalgebras and superpairs

In the following, we will consider a super-vector space $V = V_{\bar{0}} \oplus V_{\bar{1}}$ over a field $\mathbb{K}$. As is customary in the theory of Jordan superalgebras and superpairs, we will always assume that the characteristic of $\mathbb{K}$ is different from 2 and 3. At this stage we make no other assumptions on $\mathbb{K}$. We note furthermore that the main results of section 3.3, 3.4 and 3.5 still hold if we replace $\mathbb{K}$ by a ring containing $\frac{1}{2}$ and $\frac{1}{3}$.

3.2.1 Jordan superalgebras

Definition 3.2.1. *A Jordan superalgebra is a super-vector space V equipped with a bilinear product which satisfies*

- $V_iV_j \subset V_{i+j}, \qquad i,j \in \mathbb{Z}_2$
- $xy = (-1)^{|x||y|}yx \qquad$ *(commutativity)*
- $(-1)^{|x||z|}[L_x, L_{yz}] + (-1)^{|y||x|}[L_y, L_{zx}] + (-1)^{|z||y|}[L_z, L_{xy}] = 0$ *(Jordan identity),*

for $x, y, z \in V$. Here the operator $L_x \colon V \to V$ is defined by $L_x(y) = xy$ and $[\cdot\,,\cdot]$ is the supercommutator, i.e.

$$[L_x, L_y] := L_xL_y - (-1)^{|x||y|}L_yL_x.$$

A Jordan superalgebra V is unital if there exists an element $e \in V$ such that $ex = x = xe$ for all $x \in V$.

We stress that we do not restrict to finite-dimensional algebras.

A Jordan superalgebra satisfies the following relation, see [Ka2, Section 1.2],

$$[[L_x, L_y], L_z] = L_{x(yz)} - (-1)^{|x||y|}L_{y(xz)}. \tag{3.1}$$

Define the following operators on V:

$$D_{x,y} := 2L_{xy} + 2[L_x, L_y], \tag{3.2a}$$

$$P_{x,y} := L_xL_y + (-1)^{|x||y|}L_yL_x - L_{xy}. \tag{3.2b}$$

The Jordan triple product is given by

$$\{x, y, z\} := D_{x,y}z = 2\Big((xy)z + x(yz) - (-1)^{|x||y|}y(xz)\Big). \qquad (3.3)$$

This triple product satisfies the *symmetry property*

$$\{x, y, z\} = (-1)^{|x||y|+|y||z|+|x||z|}\{z, y, x\},$$

and the *5-linear Jordan identity*

$$\begin{aligned}&\{x, y, \{u, v, w\}\} - \{\{x, y, u\}, v, w\}\\ &= (-1)^{(|x|+|y|)(|u|+|v|)}(-\{u, \{v, x, y\}, w\} + \{u, v, \{x, y, w\}\}).\end{aligned}$$

The 5-linear identity can be rewritten as

$$\begin{aligned}[D_{x,y}, D_{u,v}] &= D_{\{x,y,u\},v} - (-1)^{(|x|+|y|)(|u|+|v|)}D_{u,\{v,x,y\}} \qquad (3.4)\\ &= D_{x,\{y,u,v\}} - (-1)^{(|x|+|y|)(|u|+|v|)}D_{\{u,v,x\},y}.\end{aligned}$$

We will now give some examples of Jordan superalgebras.

Full algebra. Let A be an associative superalgebra. Define A^+ as the same super-vector space with new product

$$x \cdot y = \frac{1}{2}(xy + (-1)^{|x||y|}yx).$$

Then A^+ is a Jordan superalgebra. The triple Jordan product then reduces to

$$\{x, y, z\} = xyz + (-1)^{|x||y|+|z||y|+|x||z|}zyx.$$

If $A = \mathrm{End}(\mathbb{K}^{m|n})$, we use the notation $gl(m|n)_+$ for the corresponding Jordan superalgebra.

Jordan algebras which can be embedded in a Jordan algebra A^+ for a certain associative algebra A are called special.

Hermitian Jordan algebras. Let A be an associative superalgebra with involution. Then the space of hermitian elements $\mathcal{H}(A, {}^*) = \{x \in A | x^* = x\}$ is a Jordan subalgebra of A^+.

Spin factors. Let V be a super-vector space with an even, supersymmetric bilinear form $\langle\cdot,\cdot\rangle$. Then $\mathbb{K} \oplus V$ with product

$$(\lambda, v) \cdot (\mu, w) = (\lambda\mu + \langle v, w\rangle, \lambda w + \mu v)$$

is a Jordan superalgebra. The spin factors are sometimes denoted by $\mathcal{JS}pin(V)$.

The exceptional Jordan superalgebra D_t. Let $t \in \mathbb{K}$. Then D_t is a unital Jordan superalgebra of dimension $(2|2)$. We use the realisation of D_t given in [CK]. We have

$$(D_t)_{\bar{0}} = \mathbb{K}e_1 + \mathbb{K}e_2 \text{ and } (D_t)_{\bar{1}} = \mathbb{K}\xi + \mathbb{K}\eta.$$

The multiplication table is given by

$$e_i e_i = e_i, \quad e_1 e_2 = 0, \quad e_i \xi = \tfrac{1}{2}\xi, \quad e_i \eta = \tfrac{1}{2}\eta, \quad \xi\eta = e_1 + te_2.$$

The Jordan superalgebra D_t is simple if $t \neq 0$ and $D_t \cong D_{t^{-1}}$. Remark that the unit is given by $1 = e_1 + e_2$.

If $t = -1$ then $D_t \cong gl(1|1)_+$, the full linear Jordan superalgebra of $(1|1) \times (1|1)$ matrices. The isomorphism is given by

$$e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad e_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$
$$\xi = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \eta = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}.$$

3.2.2 The spin factor over $\mathbb{R}$

We now introduce the real spin factor Jordan superalgebra associated with an orthosymplectic metric. This is the Jordan superalgebra associated to the orthosymplectic Lie superalgebra and thus it will be the Jordan algebra we will be mostly working with.

Let V be a real super-vector space with $\dim(V) = (p+q-3|2n)$ and a supersymmetric, non-degenerate, even, bilinear form $\langle \cdot, \cdot \rangle_{\tilde{\beta}}$ where the even part has signature $(p-1, q-2)$. We will always assume that $p \geq 2$ and $q \geq 2$. We choose a homogeneous basis $(e_i)_i$ of V. For $u = \sum_i u^i e_i$ and $v = \sum_i v^i e_i$ we then have

$$\langle u, v \rangle_{\tilde{\beta}} = \sum_{i,j} u^i \tilde{\beta}_{ij} v^j \quad \text{with} \quad \tilde{\beta}_{ij} := \langle e_i, e_j \rangle_{\tilde{\beta}}.$$

We have $\tilde{\beta}_{ij} = 0$ if $|i| \neq |j|$ since the form is even, while $\tilde{\beta}_{ij} = (-1)^{|i||j|}\tilde{\beta}_{ji}$ because it is supersymmetric and $\det((\tilde{\beta}_{ij})_{ij}) \neq 0$ since the form is non-degenerate.

We define the spin factor Jordan superalgebra $\mathcal{JS}pin_{p-1,q-2|2n}$ as

$$J := \mathbb{R}e \oplus V \text{ with } |e| = 0.$$

The Jordan product is given by

$$(\lambda e + u)(\mu e + v) = (\lambda\mu + \langle u, v\rangle_{\tilde{\beta}})e + \lambda v + \mu u \qquad \text{for } u, v \in V,\ \lambda, \mu \in \mathbb{R}.$$

Thus e is the unit of J.

We extend the homogeneous basis $(e_i)_{i=1}^{p+q-3+2n}$ of V to a homogeneous basis $(e_i)_{i=0}^{p+q-3+2n}$ of J by setting e_0 equal to the unit e.

Define $(\tilde{\beta}^{ij})_{ij}$ as the inverse of $(\tilde{\beta}_{ij})_{ij}$. Let $(e^i)_i$ be the right dual basis of $(e_i)_i$ with respect to the form $\langle\cdot,\cdot\rangle$, i.e.

$$\langle e_i, e^j\rangle = \delta_{ij} \qquad \text{with } \delta_{ij} \text{ the Kronecker delta.}$$

Then

$$e^j = \sum_i e_i \tilde{\beta}^{ij}.$$

Consider $J^* = \mathbb{R}e^* \oplus V^*$ the dual super-vector space of J with right dual basis $(e^i)_i$. Define a bilinear form on V^* by $\langle e^i, e^j\rangle := \langle e^i, e^j\rangle_{\tilde{\beta}} = \tilde{\beta}^{ji}$. Then we can make also J^* into a spin factor Jordan superalgebra with respect to this bilinear form.

3.2.3 Jordan superpairs

A Jordan superpair is a pair of super-vector spaces (V^+, V^-) equipped with two even trilinear products, known as the Jordan triple products,

$$\{\cdot,\cdot,\cdot\}^\sigma \colon V^\sigma \times V^{-\sigma} \times V^\sigma \to V^\sigma, \qquad \text{for} \quad \sigma \in \{+,-\}.$$

These triple products satisfy symmetry in the outer variables

$$\{x, y, z\}^\sigma = (-1)^{|x||y|+|y||z|+|z||x|}\{z, y, x\}^\sigma,$$

and the 5-linear identity

$$\begin{aligned}&\{x, y, \{u, v, w\}^\sigma\}^\sigma - \{\{x, y, u\}^\sigma, v, w\}^\sigma \\ &= (-1)^{(|x|+|y|)(|u|+|v|)}(-\{u, \{v, x, y\}^{-\sigma}, w\}^\sigma + \{u, v, \{x, y, w\}^\sigma\}^\sigma),\end{aligned}$$

for homogeneous $x, z, u, w \in V^\sigma$ and $y, v \in V^{-\sigma}$.

Example 3.2.2. *By the previous subsection, the doubling of a Jordan superalgebra V gives a Jordan superpair $(V^+, V^-) := (V, V)$ with products $\{x^\sigma, y^{-\sigma}, z^\sigma\}^\sigma := \{x, y, z\}$ for $\sigma \in \{+, -\}$.*

Here we use the notation x^+, resp. x^-, for an element $x \in V$ interpreted as in V^+, resp. V^-. When the context clarifies in which space we interpret $x \in V$, we will leave out the indices.

Example 3.2.3. *The Jordan superpair M_{pqst}. Consider the set of rectangular matrices of size $(p+s)\times(q+t)$ and the set of the transposed matrices. We can make these two sets into a Jordan superpair. Set $V^+ = \mathbb{K}^{p|s} \otimes (\mathbb{K}^{q|t})^*$ and $V^- = \mathbb{K}^{q|t} \otimes (\mathbb{K}^{p|s})^*$. Define the trilinear product using matrix multiplication*

$$\{x, y, z\} = xyz + (-1)^{|x||y|+|z||y|+|x||z|}zyx.$$

This makes $M_{pqst} = (V^+, V^-)$ into a Jordan superpair. Note that M_{pqst} can not be obtained from a Jordan superalgebra by the doubling procedure if $p \neq q$ or $s \neq t$.

Define the following operators

$$D^\sigma \colon V^\sigma \times V^{-\sigma} \to \mathrm{End}(V^\sigma); \qquad (x, y) \mapsto D^\sigma_{x,y},$$

where $D^\sigma_{x,y}(z) = \{x, y, z\}^\sigma$. We can rewrite the 5-linear identity with these operators as

$$\begin{aligned} [D^\sigma_{x,y}, D^\sigma_{u,v}] &= D^\sigma_{\{x,y,u\},v} - (-1)^{(|x|+|y|)(|u|+|v|)} D^\sigma_{u,\{v,x,y\}} \\ &= D^\sigma_{x,\{y,u,v\}} - (-1)^{(|x|+|y|)(|u|+|v|)} D^\sigma_{\{u,v,x\},y}. \end{aligned} \tag{3.5}$$

We also introduce the operators

$$\begin{aligned} &P^\sigma \colon V^\sigma \times V^\sigma \to \mathrm{Hom}(V^{-\sigma}, V^\sigma); \qquad (x, y) \mapsto P^\sigma_{x,y}, \\ &P^\sigma_{x,y}(z) := (-1)^{|y||z|}\frac{1}{2}D^\sigma_{x,z}(y) \qquad \text{for } x, y \in V^\sigma \text{ and } z \in V^{-\sigma}. \end{aligned}$$

In the following we will omit σ from the notation, as the upper index of $D_{x,y}$ and $P_{x,y}$ is determined by x and y, and similarly for $\{\cdot, \cdot, \cdot\}$.

3.3 Derivations and the structure algebra

In this section we show that the (inner) structure algebra of a unital Jordan superalgebra is isomorphic to the algebra of (inner) derivations of the corresponding superpair. We provide counterexamples to both claims when the Jordan superalgebra is non-unital.

3.3.1 The structure algebra

Definition 3.3.1. *Let V be a Jordan superalgebra. An element D in* $\mathrm{End}(V)$ *is called a* **derivation** *of V if*

$$D(xy) = D(x)y + (-1)^{|x||D|}xD(y).$$

We use the notation $\mathrm{Der}(V)$ *for the space of derivations, and* $\mathrm{Inn}(V)$ *for the subspace of* **inner derivations**, *which is spanned by the operators $[L_x, L_y]$ for $x, y \in V$.*

The condition on $D \in \mathrm{End}(V)$ to be a derivation is equivalent with

$$[D, L_x] = L_{D(x)} \qquad \text{for all } x \in V. \tag{3.6}$$

Hence equation (3.1) implies that $[L_x, L_y]$ is a derivation. One verifies easily that $\mathrm{Der}(V)$ is a subalgebra of $\mathfrak{gl}(V)$. The Jacobi identity on $\mathfrak{gl}(V)$ combined with equation (3.6), for any derivation D, implies that $\mathrm{Inn}(V)$ is an ideal in $\mathrm{Der}(V)$.

We will use the following definition for the structure algebra of Jordan superalgebras, since this is the one that will be required for the Kantor functor. There exist other definitions of the structure algebra in the literature which are not immediately connected to TKK constructions. We will review them in Section 3.3.4 and show that for unital Jordan superalgebras they are all equivalent to our definition.

Definition 3.3.2. *The structure algebra $\mathfrak{str}(V)$ is a subalgebra of $\mathfrak{gl}(V)$, defined as*

$$\mathfrak{str}(V) = \{L_x \mid x \in V\} + \mathrm{Der}(V).$$

Definition 3.3.3. *The inner structure algebra $\mathfrak{istr}(V)$ is a subalgebra of $\mathfrak{gl}(V)$, defined as*

$$\begin{aligned}\mathfrak{istr}(V) &= \{L_x \mid x \in V\} + \mathrm{Inn}(V)\\ &= \langle L_x, [L_x, L_y] \mid x, y \in V\rangle.\end{aligned}$$

By the above, $\mathfrak{istr}(V)$ is an ideal in $\mathfrak{str}(V)$.

Remark 3.3.4. For a unital Jordan superalgebra the sum in Definitions 3.3.2 and 3.3.3 is a direct sum of super-vector spaces:

$$\mathfrak{str}(V) = \{L_x \mid x \in V\} \oplus \mathrm{Der}(V),$$
$$\mathfrak{istr}(V) = \{L_x \mid x \in V\} \oplus \mathrm{Inn}(V),$$

since $D(e) = 0$ for all D in $\mathrm{Der}(V)$, while $L_x(e) = x$. For non-unital Jordan superalgebras the sums are not necessarily direct, as follows from Example 3.3.5 and Remark 3.3.18.

Example 3.3.5. *Consider the commutative three-dimensional algebra $V = \langle e_1, e_1, e_3\rangle$ with product*

$$e_1^2 = e_1, \quad e_1 e_2 = \frac{1}{2} e_2, \quad e_2^2 = e_3,$$

and all other products of basis elements zero. This is the Jordan algebra $\mathcal{J}_{19}$ in [KM, Section 3.3.3]. Because

$$[L_{e_1}, L_{e_2}] = -\frac{1}{2} L_{e_2},$$

we conclude that L_{e_2} is an element of $\mathrm{Inn}(V)$.

3.3.2 Derivations of Jordan superpairs

Definition 3.3.6. *Let (V^+, V^-) be a Jordan superpair. An element $\mathbb{D} = (D^+, D^-) \in \mathrm{End}(V^+) \oplus \mathrm{End}(V^-)$ is called a* **derivation** *of (V^+, V^-) if*

$$D^\sigma(\{x, y, z\}) = \{D^\sigma(x), y, z\} + (-1)^{|x||D^{-\sigma}|}\{x, D^{-\sigma}(y), z\}$$
$$+ (-1)^{(|x|+|y|)|D^\sigma|}\{x, y, D^\sigma(z)\}.$$

We use the notation $\mathrm{Der}(V^+, V^-)$ for the space of all derivations of (V^+, V^-) and the notation $\mathrm{Inn}(V^+, V^-)$ for the subspace of **inner derivations**, *which is spanned by the operators*

$$\mathbb{D}_{x,y} := (D_{x,y}, -(-1)^{|x||y|} D_{y,x}), \quad \textit{for} \quad x \in V^+, y \in V^-.$$

Observe that any derivation (D^+, D^-) can be written as the sum of derivations where D^+ and D^- have the same parity. The space $\mathrm{Der}(V^+, V^-)$ hence inherits a grading from the super-vector space $\mathrm{End}(V^+) \oplus \mathrm{End}(V^-)$.

By construction, the space $\mathrm{Der}(V^+, V^-)$ is a subalgebra of $\mathfrak{gl}(V^+) \oplus \mathfrak{gl}(V^-)$. The operator $\mathbb{D} = (D^+, D^-) \in \mathrm{End}(V^+) \oplus \mathrm{End}(V^-)$ is a derivation if and only if

$$[D^\sigma, D_{x,y}] = D_{D^\sigma(x),y} + (-1)^{|\mathbb{D}||x|} D_{x,D^{-\sigma}(y)}. \tag{3.7}$$

Using this, one verifies that $\mathrm{Inn}(V^+, V^-)$ is an ideal in $\mathrm{Der}(V^+, V^-)$.

3.3.3 Connections

The main result of this section is the following connection between the structure algebra of a unital Jordan superalgebra and the derivations of the associated Jordan superpair in Example 3.2.2.

Proposition 3.3.7. *For a unital Jordan superalgebra V we have*

1. $\mathfrak{str}(V) \cong \mathrm{Der}(V, V)$,
2. $\mathfrak{istr}(V) \cong \mathrm{Inn}(V, V)$.

Remark 3.3.8. For a unital Jordan algebra, we have that $\mathfrak{str}(V)$ is the Lie algebra of the structure group (Section 3.3.4) and that $\mathrm{Der}(V, V)$ is the Lie algebra of the automorphism group of the Jordan pair (V, V), [Lo, I.1.4]. Since the structure group is isomorphic to the automorphism group of the Jordan pair, [Lo, Proposition 1.8], we can immediately conclude that $\mathfrak{str}(V) \cong \mathrm{Der}(V, V)$.

Remark 3.3.9. Both parts of the proposition do not extend, as stated, to *non-unital* Jordan superalgebras. As a counterexample consider again Example 3.3.5. One can easily check that

$$\mathfrak{istr}(V) = \langle D_{x,y} \mid x, y \in V \rangle = \langle L_{e_1}, L_{e_2} \rangle \quad \text{and}$$
$$\mathrm{Inn}(V, V) = \langle (L_{e_1}, -L_{e_1}), (L_{e_2}, 0), (0, L_{e_2}) \rangle.$$

Define $A \in \mathrm{End}(V)$ by $A(e_1) = 0$, $A(e_2) = 2e_2$ and $A(e_3) = e_3$. Then we also obtain

$$\mathfrak{str}(V) = \mathfrak{istr}(V) + \langle A \rangle \quad \text{and}$$

$$\mathrm{Der}(V,V) = \mathrm{Inn}(V,V) + \langle (A,A),(A,-A)\rangle.$$

Subsection 3.6.2 also contains a counterexample where $\mathfrak{istr}(V) \cong \mathrm{Inn}(V,V)$ but $\mathfrak{str}(V) \ncong \mathrm{Der}(V,V)$.

Even without the existence of a multiplicative identity e, we still have a chain of inclusions if L_x is not a derivation for any x in V.

Proposition 3.3.10. *For a Jordan superalgebra V for which $L_x \notin \mathrm{Der}(V)$, for all x in V, we have*

$$\mathrm{Inn}(V,V) \;\subset\; \mathfrak{istr}(V) \;\subset\; \mathfrak{str}(V) \;\subset\; \mathrm{Der}(V,V).$$

Remark 3.3.11. Examples where the second inclusion is strict can be found in Subsection 3.6.1 while an example for the third inclusion to be strict can be found in Subsection 3.6.2.

Now we start the proofs of the propositions.

Lemma 3.3.12. *Let V be a Jordan superalgebra. For x in V and D in $\mathrm{Der}(V)$, we have that*

$$(L_x, -L_x) \qquad \textit{and} \qquad (D,D)$$

are elements of $\mathrm{Der}(V,V)$.

Proof. Using the Jordan identity and equation (3.1), we get for x,y,z in V

$$\begin{aligned}[L_x, D_{y,z}] &= 2[L_x, L_{yz}] + 2[L_x,[L_y,L_z]]\\ &= -2(-1)^{|x|(|y|+|z|)}[L_y, L_{zx}] - 2(-1)^{|z|(|x|+|y|)}[L_z, L_{xy}]\\ &\quad + 2L_{(xy)z} - 2(-1)^{|x||y|}L_{y(xz)}\\ &= D_{L_x(y),z} - (-1)^{|x||y|}D_{y,L_x(z)}.\end{aligned}$$

Thus $(L_x,-L_x)$ satisfies equation (3.7) and hence belongs to $\mathrm{Der}(V,V)$.

Let $D \in \mathrm{Der}(V)$. By equation (3.6), the Jacobi identity and the definition of $\mathrm{Der}(V)$, we find

$$\begin{aligned}[D, D_{x,y}] &= 2[D, L_{xy}] + 2[D,[L_x,L_y]]\\ &= 2L_{D(xy)} - 2(-1)^{|D|(|x|+|y|)}[L_x,[L_y,D]]\\ &\quad - 2(-1)^{|y|(|D|+|x|)}[L_y,[D,L_x]]\end{aligned}$$

$$
\begin{aligned}
&= 2L_{D(x)y} + 2[L_{D(x)}, L_y] + 2(-1)^{|D||x|}L_{xD(y)}\\
&\quad + 2(-1)^{|x||D|}[L_x, L_{D(y)}]\\
&= D_{D(x),y} + (-1)^{|x||D|}D_{x,D(y)}.
\end{aligned}
$$

Therefore, also (D, D) satisfies equation (3.7) and is thus an element of $\mathrm{Der}(V, V)$. □

Proof of Proposition 3.3.10. Since $D_{x,y} = 2L_{xy} + 2[L_x, L_y]$, the map

$$
\begin{aligned}
\psi\colon &\mathrm{Inn}(V, V) \to \mathfrak{istr}(V)\\
&\mathbb{D}_{x,y} = (D_{x,y}, -(-1)^{|x||y|}D_{y,x}) \mapsto D_{x,y}
\end{aligned} \tag{3.8}
$$

is well-defined and clearly a Lie superalgebra morphism. Assume $D_{x,y} = 0$. Then $L_{xy} = -[L_x, L_y]$ is a derivation. So by our assumption $L_{xy} = 0$, and thus also $D_{y,x} = 0$. Therefore ψ is injective.

From the definitions it follows immediately that $\mathfrak{istr}(V) \subset \mathfrak{str}(V)$. By assumption, $\mathfrak{str}(V)$ is a direct sum of $\{L_x \mid x \in V\}$ and $\mathrm{Der}(V)$. Together with Lemma 3.3.12 this implies that the map

$$
\begin{aligned}
\phi\colon &\mathfrak{str}(V) \to \mathrm{Der}(V, V)\\
&L_x + D \mapsto (L_x + D, -L_x + D),
\end{aligned} \tag{3.9}
$$

is well-defined. This map is clearly injective. A direct computation shows that it is also a Lie superalgebra morphism. This finishes the proof. □

The rest of this section is devoted to the proof of Proposition 3.3.7, so we consider a unital Jordan superalgebra V. From Remark 3.3.4 it follows then that the assumption of Proposition 3.3.10 is satisfied, so we can use that result. We will also use the following immediate consequences of equation (3.3),

$$
\frac{1}{2}\{x, e, y\} = xy = L_x(y) \quad \text{and} \quad \frac{1}{2}\{e, x, e\} = x. \tag{3.10}
$$

Consider the map σ

$$
\sigma\colon \mathrm{Der}(V, V) \to \mathrm{Der}(V, V); \quad (D^+, D^-) \mapsto (D^-, D^+).
$$

Then $\sigma^2 = \mathrm{id}$ and $\mathrm{Der}(V, V)$ decomposes in two subspaces

$$
\begin{aligned}
\mathfrak{h} &:= \{\mathbb{D} \in \mathrm{Der}(V, V) \mid \sigma(\mathbb{D}) = \mathbb{D}\} \quad \text{and}\\
\mathfrak{q} &:= \{\mathbb{D} \in \mathrm{Der}(V, V) \mid \sigma(\mathbb{D}) = -\mathbb{D}\}.
\end{aligned}
$$

Lemma 3.3.13. *We have*

$$\mathfrak{h} = \{\mathbb{D} \in \mathrm{Der}(V, V) \mid D^-(e) = 0\}.$$

Proof. Assume first that $D^-(e) = 0$. By equation (3.10), we have

$$D^-(e) = \frac{1}{2} D^-\{e, e, e\} = \frac{1}{2}\{e, D^+(e), e\} = D^+(e),$$

and hence also $D^+(e) = 0$. Then we also get for all x in V

$$2D^+(x) = D^+\{e, x, e\} = \{e, D^-(x), e\} = 2D^-(x).$$

Hence $D^+ = D^-$.

Now assume that $D^+ = D^-$. Equation (3.10) then implies

$$\begin{aligned} D^-(e) &= \frac{1}{2} D^-\{e, e, e\} \\ &= \frac{1}{2}\{D^-(e), e, e\} + \frac{1}{2}\{e, D^+(e), e\} + \frac{1}{2}\{e, e, D^-(e)\} \\ &= 3D^-(e). \end{aligned}$$

Hence $D^-(e) = 0$. This concludes the proof. □

Lemma 3.3.14. *We have a Lie superalgebra isomorphism*

$$\mathrm{Der}(V) \cong \mathfrak{h}, \qquad \textit{given by} \quad \phi \colon \mathrm{Der}(V) \to \mathfrak{h};\ D \mapsto (D, D).$$

Proof. The map ϕ is a restriction from $\mathfrak{str}(V)$ to $\mathrm{Der}(V)$ of the morphism defined in (3.9). Hence it is also injective. The image of ϕ is clearly contained in $\mathfrak{h}$. To show that ϕ is surjective, we let $\mathbb{D} = (D^+, D^-)$ be an element of $\mathrm{Der}(V^+, V^-)$ with $D^+ = D^-$, i.e. $\mathbb{D} \in \mathfrak{h}$. Then $D^-(e) = 0$ by Lemma 3.3.13. Hence, using equation (3.10),

$$\begin{aligned} D^+(xy) &= \frac{1}{2} D^+\{x, e, y\} \\ &= \frac{1}{2}\{D^+x, e, y\} + (-1)^{|D||x|}\frac{1}{2}\{x, D^-e, y\} \\ &\quad + (-1)^{|D||x|}\frac{1}{2}\{x, e, D^+y\} \\ &= D^+(x)y + (-1)^{|D||x|} x D^+(y). \end{aligned}$$

We conclude that $D^+ = D^-$ is an element of $\mathrm{Der}(V)$, so (D^+, D^-) is in the image of ϕ. □

Lemma 3.3.15. *We have an isomorphism of super-vector spaces*

$$\{L_x \mid x \in V\} \xrightarrow{\sim} \mathfrak{q}, \qquad \textit{given by} \qquad L_x \mapsto (L_x, -L_x).$$

Proof. The assignment $L_a \to (L_a, -L_a)$ is a restriction to $\{L_x \mid x \in V\}$ of the injective morphism $\phi\colon \mathfrak{str}(V) \to \mathrm{Der}(V,V)$ considered in (3.9). Its image is clearly contained in $\mathfrak{q}$. So the map is well-defined and injective. For an element $\mathbb{D} = (D, -D)$ in $\mathfrak{q}$ we claim that $(L_{D(e)}, -L_{D(e)}) = \mathbb{D}$. Indeed, using equation (3.10), we have

$$\begin{aligned} D(x) &= \frac{1}{2} D(\{e,x,e\}) \\ &= \frac{1}{2}\{D(e),x,e\} - \frac{1}{2}\{e,D(x),e\} + (-1)^{|D||x|}\frac{1}{2}\{e,x,D(e)\} \\ &= 2D(e)x - D(x), \end{aligned}$$

which implies that $D(x) = L_{D(e)}(x)$. This proves surjectivity. □

Proof of Proposition 3.3.7. Consider the injective morphism

$$\begin{aligned} \phi\colon \mathrm{Der}(V) \oplus \{L_x \mid x \in V\} &\to \mathrm{Der}(V,V) \\ L_x + D &\mapsto (L_x + D, -L_x + D) \end{aligned}$$

defined in (3.9). From Lemmata 3.3.14 and 3.3.15 it follows that ϕ is also surjective. This proves part (1) of the proposition. For part (2), consider the injective morphism

$$\psi\colon \mathrm{Inn}(V,V) \to \mathfrak{istr}(V); \quad (D_{x,y}, -(-1)^{|x||y|}D_{y,x}) \mapsto D_{x,y}.$$

defined in (3.8). Since

$$L_x = \frac{1}{2}D_{x,e} \quad \text{and} \quad [L_x, L_y] = \frac{1}{4}(D_{x,y}, -(-1)^{|x||y|}D_{y,x}),$$

the map ψ is surjective, which concludes the proof. □

3.3.4 Alternative definitions for the (inner) structure algebra

We review some further definitions appearing in the literature. We can rewrite the operator $P_{x,y}$ defined in equation (3.2b) as

$$P_{x,y}\colon V \to V;\ z \mapsto (-1)^{|y||z|}\{x,z,y\}.$$

Then for a unital Jordan superalgebra we define, see [GN, Section 3.1],

$$\begin{aligned}\widetilde{\mathfrak{str}}(V) := \{X \in \mathfrak{gl}(V) \mid P_{X(a),b} + (-1)^{|X||b|}P_{a,X(b)} \\ = XP_{a,b} + (-1)^{|X|(|a|+|b|)}P_{a,b}X^* \text{ for all } a, b \in V\},\end{aligned}$$

where $X^* = -X + 2L_{X(e)}$. In the non-super case, this algebra is the Lie algebra of the structure group, see [Ja, Section 9].

In the literature we did not find an explicit definition of the structure algebra for the non-unital case using this approach. However, we will define a natural generalisation which for a unital Jordan superalgebra will reduce to $\widetilde{\mathfrak{str}}(V)$. So, for V a Jordan superalgebra, define $\mathfrak{str}_w(V)$ as the Lie superalgebra consisting of the elements $(X, Y) \in \mathfrak{gl}(V) \oplus \mathfrak{gl}(V)^{op}$ for which

$$P_{X(a),b} + (-1)^{|X||a|}P_{a,X(b)} = XP_{a,b} + (-1)^{|Y|(|a|+|b|)}P_{a,b}Y \tag{3.11}$$

and

$$P_{Y(a),b} + (-1)^{|Y||a|}P_{a,Y(b)} = YP_{a,b} + (-1)^{|X|(|a|+|b|)}P_{a,b}X$$

hold for all a, b in V. If V is a Jordan algebra, one can check that $\mathfrak{str}_w(V)$ is the Lie algebra of the group consisting of pairs of 'weakly structural transformations', as defined in [McC, II.18.2].

Using the equality $P_{x,y}(z) = (-1)^{|y||z|}D_{x,z}(y)$, one finds that the defining conditions of $\mathfrak{str}_w(V)$ are equivalent with $(X, -Y) \in \mathrm{Der}(V, V)$. So we conclude that $\mathfrak{str}_w(V) \cong \mathrm{Der}(V, V)$ in full generality.

Lemma 3.3.16. *For a unital Jordan superalgebra V, we have*

$$\mathfrak{str}_w(V) \;\cong\; \widetilde{\mathfrak{str}}(V) \;\cong\; \mathfrak{str}(V).$$

Proof. Let X be an element of $\widetilde{\mathfrak{str}}(V)$. Note that by definition X satisfies equation (3.11) for $Y = -X + 2L_{X(e)}$. From Lemma 3.3.12 we know that $(L_{X(e)}, -L_{X(e)}) \in \mathrm{Der}(V, V)$. Combining this, one shows easily that for all $a, b \in V$

$$P_{Y(a),b} + (-1)^{|Y||b|}P_{a,X(b)} = YP_{a,b} + (-1)^{|X|(|a|+|b|)}P_{a,b}X$$

holds for $Y = -X + 2L_{X(e)}$. Thus the map

$$\varphi\colon \widetilde{\mathfrak{str}}(V) \to \mathfrak{str}_w(V);\; X \mapsto (X, -X + 2L_{X(e)})$$

is well-defined. Let $(X,Y) \in \mathfrak{str}_w(V)$. Then setting a and b equal to the unit e in equation (3.11) gives us $Y = -X + 2L_{X(e)}$, hence φ is an isomorphism. Since $\mathfrak{str}_w(V) \cong \mathrm{Der}(V,V)$, Proposition 3.3.7, immediately implies that $\mathfrak{str}_w(V)$ is also isomorphic to $\mathfrak{str}(V)$. □

The inner structure algebra is also often defined as the Lie superalgebra spanned by the operators $D_{x,y} \in \mathrm{End}(V)$, see for example [Ja, Section 9], [Sp, Chapter 4] and [GN, Section 3.1]. For this algebra we will use the notation

$$\widetilde{\mathfrak{istr}}(V) := \langle D_{x,y} \mid x,y \in V \rangle.$$

Lemma 3.3.17. *For a Jordan superalgebra V for which $L_x \notin \mathrm{Der}(V)$, for all x in V, we have*

$$\widetilde{\mathfrak{istr}}(V) \cong \mathrm{Inn}(V,V).$$

In particular, for a unital Jordan superalgebra, we have

$$\widetilde{\mathfrak{istr}}(V) \cong \mathfrak{istr}(V).$$

Proof. By assumption $L_{xy} \notin \mathrm{Der}(V)$, so we have that $D_{x,y} = 0$ implies $D_{y,x} = 0$. Hence the map

$$\mathrm{Inn}(V,V) \to \widetilde{\mathfrak{istr}}(V);\ (D_{x,y}, -(-1)^{|x||y|}D_{y,x}) \mapsto D_{x,y}$$

is bijective. It is also clearly an algebra morphism. This proves the first part of the lemma. Since unital Jordan superalgebras satisfy $L_x \notin \mathrm{Der}(v)$ for all x in V, Proposition 3.3.7 immediately implies the second part of the lemma. □

Remark 3.3.18. In the non-unital case we can both have

$$\widetilde{\mathfrak{istr}}(V) \ncong \mathfrak{istr}(V) \quad \text{and} \quad \widetilde{\mathfrak{istr}}(V) \ncong \mathrm{Inn}(V,V).$$

The example in Remark 3.3.9 is a counterexample for the second part, while a counterexample for the first part is as follows. Consider $V := t\mathbb{K}[t]/(t^k)$, the algebra of polynomials in the variable t without constant term, modulo the ideal $(t^k) = t^k\mathbb{K}[t]$ of polynomials without term in degree lower than k for some $k \in \mathbb{Z}_{>2}$. This is an (associative) Jordan algebra, for the standard multiplication of polynomials, which does not have multiplicative identity. In this example we have $D_{f,g} = 2L_{fg}$, for all $f, g \in V$. We hence find that

$$\mathrm{Inn}(V,V) \cong \widetilde{\mathfrak{istr}}(V) = \mathrm{Span}_{\mathbb{K}}\{L_{t^2}, L_{t^3}, \ldots, L_{t^{k-2}}\}.$$

On the other hand, by definition, we have

$$\mathfrak{istr}(V) = \mathrm{Span}_{\mathbb{K}}\{L_t, L_{t^2}, \ldots, L_{t^{k-2}}\}.$$

As the dimensions of both abelian Lie algebras do not agree, we find $\mathfrak{istr}(V) \ncong \widetilde{\mathfrak{istr}}(V)$. Observe further that $L_{t^{k-2}}$ is an element of $\mathrm{Der}(V)$. Therefore the structure algebra $\mathfrak{str}(V)$ also does not have a direct sum decomposition as in Remark 3.3.4.

3.4 The Tits–Kantor–Koecher construction

In this section, we will study the three different TKK constructions, dating back to Tits, Kantor and Koecher, and show that, for unital Jordan superalgebras, they are equivalent. Again this claim does not extend to non-unital cases.

3.4.1 TKK for Jordan superalgebras (Kantor's approach)

In [Ka2], Kac uses the "Kantor functor" Kan to classify simple finite-dimensional Jordan superalgebras over an algebraically closed field of characteristic zero. This functor is a generalisation to the super case of the one considered by Kantor in [Kan1]. In particular this functor provides a TKK construction, which we review for arbitrary Jordan superalgebras over arbitrary fields.

We associate to a Jordan superalgebra V, the 3-graded Lie superalgebra

$$\mathrm{Kan}(V) := \mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_+, \quad \text{with}$$

$$\mathfrak{g}_- := V \quad \text{and} \quad \mathfrak{g}_0 := \mathfrak{istr}(V) = \langle L_a, [L_a, L_b] \rangle \subset \mathrm{End}(\mathfrak{g}_-).$$

Finally, $\mathfrak{g}_+$ is defined as the subspace of $\mathrm{End}(\mathfrak{g}_- \otimes \mathfrak{g}_-, \mathfrak{g}_-)$, spanned by P and $[L_a, P]$ with

$$P(x, y) := xy,$$

$$[L_a, P](x, y) := a(xy) - (ax)y - (-1)^{|x||y|}(ay)x.$$

Note that $P = -[L_e, P]$ for a unital Jordan superalgebra.

As the notation suggests, $[L_a, P]$ corresponds to the superbracket of $L_a \in \mathfrak{g}_0$ and $P \in \mathfrak{g}_+$. The Lie superbracket is then completely defined by

- $[\mathfrak{g}_-, \mathfrak{g}_-] = 0 = [\mathfrak{g}_+, \mathfrak{g}_+]$.
- $[a, x] = a(x)$, for $a \in \mathfrak{g}_0, x \in \mathfrak{g}_-$.
- $[A, x](y) = A(x, y)$, for $A \in \mathfrak{g}_+, x, y \in \mathfrak{g}_-$.
- For $a \in \mathfrak{g}_0$, $B \in \mathfrak{g}_+$ and $x, y \in \mathfrak{g}_-$

$$\begin{aligned}[a, B](x, y) =& a(B(x, y)) - (-1)^{|a||B|}B(a(x), y) \\ &- (-1)^{|a||B|+|x||y|}B(a(y), x).\end{aligned}$$

To verify that $\mathrm{Kan}(V)$ is a Lie superalgebra, one can use the following relations (see Proposition 5.1 in [CK])

- $[P, x] = L_x$
- $[[L_a, P], x] = [L_a, L_x] - L_{ax}$
- $[L_a, [L_b, P]] = -[L_{ab}, P]$
- $[[L_a, L_b], P] = 0$
- $[[L_a, L_b], [L_c, P]] = (-1)^{|b||c|}[L_{a(cb)-(ac)b}, P]$.

3.4.2 TKK for Jordan superpairs (Koecher's approach)

In [Ko], Koecher defined a product on a triple consisting of two vector spaces and a Lie algebra acting on these vector spaces. This product makes the triple into a 3-graded anti-commutative algebra, which is a Lie algebra if and only if the vector spaces form a Jordan pair and the Lie algebra acts by derivations on the vector spaces. Hence the Koecher construction gives rise to a TKK construction, not only for Jordan algebras, but for Jordan pairs, which is the most natural formulation. Note that, as the concept of Jordan pairs was not yet studied at the time, Koecher did not use this terminology. This TKK construction can be generalised to the supercase, which was for example used by Krutelevich to classify simple finite-dimensional Jordan superpairs over an algebraically closed field in characteristic zero in [Kr].

We associate a 3-graded Lie superalgebra $\mathrm{TKK}(V^+, V^-)$ to the Jordan superpair (V^+, V^-) in the following way. As vector spaces we have

$$\mathrm{TKK}(V^+, V^-) = V^+ \oplus \mathrm{Inn}(V^+, V^-) \oplus V^-.$$

The Lie super bracket on $\mathrm{TKK}(V^+, V^-)$ is defined by

$$\begin{aligned}
[x, u] &= \mathbb{D}_{x,u}\\
[\mathbb{D}_{x,u}, y] &= \mathbb{D}_{x,u}(y) = \{x, u, y\}\\
[\mathbb{D}_{x,u}, v] &= \mathbb{D}_{x,u}(v) = -(-1)^{|x||u|}\{u, x, v\}\\
[\mathbb{D}_{x,u}, \mathbb{D}_{y,v}] &= \mathbb{D}_{\mathbb{D}_{x,u}(y),v} + (-1)^{(|x|+|u|)|y|}\mathbb{D}_{y,\mathbb{D}_{x,u}(v)}\\
[x, y] &= [u, v] = 0,
\end{aligned}$$

for $x, y \in V^+$, $u, v \in V^-$. Recall that

$$\mathbb{D}_{x,u} = (D_{x,u}, -(-1)^{|x||u|}D_{u,x}) \in \mathrm{Inn}(V^+, V^-).$$

In case V is a Jordan superalgebra, we simply write $\mathrm{TKK}(V)$ for $\mathrm{TKK}(V, V)$.

Conversely, with each 3-graded Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{+1}$ we can associate a Jordan superpair by $\mathcal{J}(\mathfrak{g}) = (\mathfrak{g}_{+1}, \mathfrak{g}_{-1})$ with the Jordan triple product given by

$$\{x^\sigma, y^{-\sigma}, z^\sigma\}^\sigma := [[x^\sigma, y^{-\sigma}], z^\sigma].$$

Definition 3.4.1. *A 3-graded Lie superalgebra $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_+$ is called **Jordan graded** if*

$$[\mathfrak{g}_+, \mathfrak{g}_-] = \mathfrak{g}_0 \quad \text{and} \quad \mathfrak{g}_0 \cap Z(\mathfrak{g}) = 0.$$

We have the following result by Lemmata 4 and 5 in [Kr].

Proposition 3.4.2. *For every Jordan superpair (V^+, V^-), we have*

$$\mathcal{J}(\mathrm{TKK}(V^+, V^-)) \cong (V^+, V^-).$$

Let $\mathfrak{g}$ be a Jordan graded Lie superalgebra, then $\mathrm{TKK}(\mathcal{J}(\mathfrak{g})) \cong \mathfrak{g}$.

Note that the main results in [Kr] are only concerned with finite-dimensional pairs, over algebraically closed fields with characteristic zero. However, the mentioned lemmata still hold for arbitrary Jordan superpairs over a field with characteristic different from 2 or 3.

3.4.3 Connection

The main result of this section is the following proposition, which shows that Kantor's and Koecher's constructions for unital Jordan superalgebras coincide.

Proposition 3.4.3. *For a unital Jordan superalgebra V, we have* $\mathrm{Kan}(V) \cong \mathrm{TKK}(V)$.

Proof. Let $\mathrm{Kan}(V) = \mathfrak{g}_+ \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_-$. The relations $[P, a] = L_a$ and $[[L_a, P], b] = [L_a, L_b] - L_{ab}$ in Subsection 3.4.1 imply $[\mathfrak{g}_-, \mathfrak{g}_+] = \mathfrak{g}_0$. For all $x \in \mathfrak{g}_0$, it follows from the definition of the bracket that, if $[x, \mathfrak{g}_-] = 0$, then $x = 0$. So $\mathfrak{g}_0 \cap Z(\mathfrak{g}) = 0$ and $\mathrm{Kan}(V)$ is Jordan graded. Hence Proposition 3.4.2 implies

$$\mathrm{TKK}(\mathcal{J}(\mathrm{Kan}(V))) \cong \mathrm{Kan}(V).$$

Set $(V^+, V^-) := \mathcal{J}(\mathrm{Kan}(V))$. Then $V^- = V$ and $V^+ = \langle P, [L_a, P] \mid a \in V\rangle$. One can check that the map $\phi\colon (V^+, V^-) \to (V, V)$ defined by

- $\phi(x) = x$ for $x \in V^-$,
- $\phi(P) = -\frac{e}{2}$, where e is the unit of V,
- $\phi([L_a, P]) = \frac{a}{2}$.

is an isomorphism of Jordan pairs. From this it follows that

$$\mathrm{TKK}(V) = \mathrm{TKK}(V, V) \cong \mathrm{TKK}(\mathcal{J}(\mathrm{Kan}(V))) \cong \mathrm{Kan}(V),$$

which proves the proposition. □

Remark 3.4.4. The proposition as stated does not extend to Jordan superalgebras without multiplicative identity e. If V is finite-dimensional but not unital, we will generally have

$$\dim \mathrm{Kan}(V)_+ \neq \dim V = \dim \mathrm{TKK}(V)_+,$$

and hence $\mathrm{Kan}(V) \ncong \mathrm{TKK}(V)$. This difference in dimension can for instance be caused by the occurrence of elements of V for which the left multiplication operator is trivial, since this lowers the dimension of $\langle P, [L_a, P]\rangle$, or by $P \notin \langle [L_a, P]\rangle$, which raises the dimension.

Another source of counterexamples comes from Jordan superalgebras V which satisfy $\mathrm{Inn}(V,V) \neq \mathfrak{istr}(V)$, see e.g. Remark 3.3.9.

3.4.4 Tits' approach.

There is a third version of the TKK construction, which appeared in [Ti] and historically was the first to appear. In this section, we will give the super version of this construction by Tits.

Consider an arbitrary Jordan superalgebra V. Let $\mathcal{D}$ be a Lie superalgebra, containing $\mathrm{Inn}(V)$, with a Lie superalgebra morphism

$$\psi : \mathcal{D} \to \mathrm{Der}(V); \qquad d \mapsto \psi_d,$$

such that ψ acts as the identity on the subalgebra $\mathrm{Inn}(V)$. Finally, let Y be an arbitrary three-dimensional simple Lie algebra Y. For example, for $\mathbb{K} = \mathbb{C}$, we only have $Y \cong \mathfrak{sl}_2(\mathbb{C})$ and for $\mathbb{K} = \mathbb{R}$ either $Y \cong \mathfrak{sl}_2(\mathbb{R}) \cong \mathfrak{su}(1,1)$ or $Y \cong \mathfrak{su}(2)$. Let $(y, y') := \frac{1}{2}\mathrm{tr}(\mathrm{ad}(y)\mathrm{ad}(y'))$ be the Killing form on Y.

Then we define a Lie superalgebra

$$\mathrm{Ti}(V, \mathcal{D}, Y) := \mathcal{D} \oplus (Y \otimes V),$$

where $\mathcal{D}$ is a subalgebra, and the rest of the multiplication is defined by

$$\begin{aligned} [d, y \otimes v] &= y \otimes \psi_d(v), \\ [y \otimes v, y' \otimes v'] &= (y, y')[L_v, L_{v'}] + [y, y'] \otimes vv', \end{aligned}$$

for arbitrary $d \in \mathcal{D}$, $y, y' \in Y$ and $v, v' \in V$. For $Y = \mathfrak{sl}_2(\mathbb{K})$ we can use the 3-grading on $\mathfrak{sl}_2(\mathbb{K})$ to define a 3-grading on $\mathrm{Ti}(V, \mathcal{D}, \mathfrak{sl}_2(\mathbb{K}))$:

$$\begin{aligned} \mathrm{Ti}(V, \mathcal{D}, \mathfrak{sl}_2(\mathbb{K}))_- &= \mathfrak{sl}_2(\mathbb{K})_- \otimes V, \\ \mathrm{Ti}(V, \mathcal{D}, Y)_0 &= \mathcal{D} \oplus (\mathfrak{sl}_2(\mathbb{K})_0 \otimes V), \\ \mathrm{Ti}(V, \mathcal{D}, \mathfrak{sl}_2(\mathbb{K}))_+ &= \mathfrak{sl}_2(\mathbb{K})_+ \otimes V. \end{aligned}$$

For unital Jordan superalgebras, this contains, as a special case, Koecher's and hence also Kantor's construction, as we prove in the following proposition.

Proposition 3.4.5. *For a unital Jordan superalgebra V we have*

$$\mathrm{Ti}\,(V, \mathrm{Inn}(V), \mathfrak{sl}_2(\mathbb{K})) \;\cong\; \mathrm{TKK}(V).$$

Proof. Consider a $\mathbb{K}$-basis e, f, h of $\mathfrak{sl}_2(\mathbb{K})$, such that $[e, f] = h$, $[h, e] = 2e$ and $[h, f] = -2f$. For a unital Jordan superalgebra V, we have $\mathrm{Inn}(V, V) \cong \mathfrak{istr}(V)$, by Proposition 3.3.7(2).

Then an isomorphism between $\mathrm{Ti}\,(V, \mathrm{Inn}(V), \mathfrak{sl}_2(\mathbb{K}))$ and $\mathrm{TKK}(V) = V^+ \oplus \mathfrak{istr}(V) \oplus V^-$ is given by

$$e \otimes a \mapsto a^+, \quad f \otimes a \mapsto a^-, \quad h \otimes a \mapsto 2L_a, \quad [L_a, L_b] \mapsto [L_a, L_b].$$

It follows from the definitions that this is a Lie superalgebra morphism. □

Remark 3.4.6. From the proof, it is clear that the proposition still holds for non-unital Jordan superalgebras as long as $\mathfrak{istr}(V) \cong \mathrm{Inn}(V, V)$.

Now we consider the opposite direction of the above construction. Let N be a Lie superalgebra and Y a simple Lie algebra of dimension 3. We say that Y acts on N if there is an (even) Lie superalgebra homomorphism from Y to $\mathrm{Der}(N)$. For example, we can define an action of Y on $\mathrm{Ti}\,(V, \mathcal{D}, Y)$ as follows

$$y \cdot (d + y' \otimes v) = [y, y'] \otimes v.$$

Under this action, $\mathrm{Ti}\,(V, \mathcal{D}, Y)$ viewed as an Y-module decomposes as a trivial part given by $\mathcal{D}$ and $\dim(V)$ copies of the adjoint representation.

Now consider an arbitrary Lie superalgebra N with Y-action which decomposes as above, *viz.* as a trivial representation $\mathcal{D}$ and some copies of the adjoint representation,

$$N = \mathcal{D} \oplus (Y \otimes A),$$

for some vector space A. As a direct generalisation of [Ti], we show that there is a Jordan algebra structure on A where $\mathcal{D}$ acts on A by derivations and $\mathrm{Ti}\,(V, \mathcal{D}, Y)$ is the inverse of this construction.

Proposition 3.4.7. *Let N be a Lie superalgebra and Y a three-dimensional simple Lie algebra which acts on N such that N decomposes as $N = \mathcal{D} \oplus (Y \otimes A)$ where $\mathcal{D}$ is a trivial representation and Y the adjoint representation. Then A is a Jordan superalgebra and $\mathcal{D}$ is a superalgebra containing the inner derivations on A equipped with a morphism $\psi\colon \mathcal{D} \to \mathrm{Der}(A)$, for which the restriction to the inner derivations is the identity. Furthermore*

$$N \cong \mathrm{Ti}(A, \mathcal{D}, Y).$$

Proof. Proposition 1 in [Ti] and its proof, which extend trivially to the super case, imply that under these conditions, $\mathcal{D}$ is a subalgebra of N, and there are bilinear maps $\alpha(\cdot,\cdot) : \mathcal{D} \times A \to A$, $\langle \cdot,\cdot \rangle : A \times A \to \mathcal{D}$ and $\mu : A \times A \to A$, such that

$$[y \otimes a] = y \otimes \alpha(d, a)$$
$$[y \otimes a, y' \otimes a'] = (y, y')\langle a, a' \rangle + [y, y']\mu(a, a').$$

Furthermore (A, μ) is a Jordan superalgebra and $d \mapsto \alpha(d, \cdot)$ is a Lie superalgebra morphism $\phi : \mathcal{D} \to \mathrm{Der}(A)$. Finally, by equation (2.6) in [Ti], we have

$$\phi(\langle a, b \rangle) = [L_a, L_b].$$

Comparison with the definition of $\mathrm{Ti}(A, \mathcal{D}, Y)$ concludes the proof. □

3.5 Further TKK constructions

In this section we consider variations of the TKK constructions for a Jordan superalgebra V, which also appear in the literature, by using $\mathfrak{str}(V)$ and $\mathrm{Der}(V, V)$, instead of $\mathfrak{istr}(V)$ and $\mathrm{Inn}(V, V)$.

3.5.1 Definition

The Lie superalgebra $\mathrm{Ti}(V, \mathcal{D}, \mathfrak{sl}_2)$ had more freedom compared to the constructions by Kantor and Koecher, due to the choice of $\mathcal{D}$. Also in the Koecher construction, we can replace $\mathrm{Inn}(V^+, V^-)$ by any Lie superalgebra containing $\mathrm{Inn}(V^+, V^-)$ with a morphism to $\mathrm{Der}(V^+, V^-)$ which restricts to the identity on the inner derivations. For example, we can set $\mathfrak{g}_0 = \mathrm{Der}(V^+, V^-)$ in the TKK construction of Section 3.4.2. This gives a 3-graded Lie superalgebra

$$\widetilde{\mathrm{TKK}}(V^+, V^-) = V^+ \oplus \mathrm{Der}(V^+, V^-) \oplus V^-,$$

see [GN] for more details. Remark that $\mathrm{TKK}(V^+, V^-)$ is an ideal in $\widetilde{\mathrm{TKK}}(V^+, V^-)$ by construction. We will again use the notation $\widetilde{\mathrm{TKK}}(V)$ for $\widetilde{\mathrm{TKK}}(V, V)$. In Subsection 3.5.3, we will prove that $\widetilde{\mathrm{TKK}}(V)$ is the superalgebra of derivations of $\mathrm{TKK}(V)$ for unital Jordan superalgebras. We can also relate $\widetilde{\mathrm{TKK}}(V)$ to Tits' construction in Subsection 3.4.4 as follows.

Proposition 3.5.1. *For a unital Jordan superalgebra V, we have*

$$\widetilde{\mathrm{TKK}}(V) \cong \mathrm{Ti}(V, \mathrm{Der}(V), \mathfrak{sl}_2(\mathbb{K})).$$

This will be proved in greater generality in Subsection 3.5.2.

3.5.2 Comparison of further TKK constructions

Let $\mathcal{D}$ be a Lie superalgebra containing $\mathrm{Inn}(V)$ with a morphism ψ to $\mathrm{Der}(V)$ such that $\psi_{|\mathrm{Inn}(V)} = \mathrm{id}$. Define the Lie superalgebra

$$\widetilde{\mathcal{D}} := \mathcal{D} \oplus \{L_x \mid x \in V\},$$

where $\mathcal{D}$ is a subalgebra of $\widetilde{\mathcal{D}}$, the product of L_x and L_y is given by $[L_x, L_y]$ interpreted via the embedding of $\mathrm{Inn}(V)$ in $\mathcal{D}$, and

$$[D, L_x] := L_{\psi(D)x} \qquad \text{for } D \in \mathcal{D},\ x \in V.$$

Note that if there exists an $x \in V$, such that L_x is in $\mathcal{D}$, then $\widetilde{\mathcal{D}}$ contains two copies of L_x, one in $\mathcal{D}$ and one in $\{L_x \mid x \in V\}$.

Set

$$\widetilde{\psi}\colon \widetilde{\mathcal{D}} \to \mathrm{Der}(V, V); \quad D + L_x \mapsto (\psi(D) + L_x, \psi(D) - L_x).$$

From Lemma 3.3.12, it follows that this map is well defined, while from the definition of the bracket on $\widetilde{\mathcal{D}}$ it follows that it is a Lie superalgebra morphism. The morphism $\widetilde{\psi}$ yields an action of $\widetilde{\mathcal{D}}$ on V^+ and V^-, which allows us to define a TKK construction similar to the Koecher construction in Subsection 3.4.2. Concretely, the bracket on

$$\mathrm{TKK}_{\mathcal{D}}(V) := V \oplus \widetilde{\mathcal{D}} \oplus V$$

is given by

$$\begin{aligned} [x,u] &= 2L_{xu} + 2[L_x, L_u], & [d,x] &= \widetilde{\psi}(d)x, & [d,u] &= \widetilde{\psi}(d)u, \\ [d_1,d_2] &= [d_1,d_2]_{\widetilde{\mathcal{D}}}, & [x,y] &= 0 = [u,v], \end{aligned}$$

for x, y in V^+, u, v in V^-, d, d_1, d_2 in $\widetilde{D}$ and $[\cdot,\cdot]_{\widetilde{\mathcal{D}}}$ the product in $\widetilde{\mathcal{D}}$.

Proposition 3.5.2. *Consider a (not necessarily unital) Jordan superalgebra V and a Lie superalgebra $\mathcal{D}$ as above. We have an isomorphism of Lie superalgebras*

$$\mathrm{Ti}(V, \mathcal{D}, \mathfrak{sl}_2(\mathbb{K})) \;\cong\; \mathrm{TKK}_{\mathcal{D}}(V).$$

Proof. The following generalisation of the map used in Proposition 3.4.5

$$e \otimes a \mapsto a^+, \quad f \otimes a \mapsto a^-, \quad h \otimes a \mapsto 2L_a, \quad D \mapsto D,$$

is an isomorphism between $\mathrm{Ti}(V, \mathcal{D}, \mathfrak{sl}_2(\mathbb{K}))$ and $\mathrm{TKK}_{\mathcal{D}}(V)$. □

The case $\mathcal{D} = \mathrm{Inn}(V)$ yields

$$\mathrm{Ti}(V, \mathrm{Inn}(V), \mathfrak{sl}_2(\mathbb{K})) \cong \mathrm{TKK}_{\mathrm{Inn}(V)}(V),$$

with $\widetilde{\mathrm{Inn}(V)} = \{L_x \mid x \in V\} \oplus \mathrm{Inn}(V)$. This is a generalisation of Proposition 3.4.5 to the non-unital case.

For unital Jordan superalgebras we have canonical isomorphisms

$$\widetilde{\mathrm{Inn}(V)} \cong \mathfrak{istr}(V) \text{ and } \widetilde{\mathrm{Der}(V)} \cong \mathfrak{str}(V),$$

by Remark 3.3.4, and thus

$$\mathrm{TKK}_{\mathrm{Inn}(V)}(V) = \mathrm{TKK}(V) \text{ and } \mathrm{TKK}_{\mathrm{Der}(V)}(V) = \widetilde{\mathrm{TKK}}(V).$$

Hence we find that Proposition 3.5.2 implies Proposition 3.5.1.

Remark 3.5.3. Let $\mathfrak{g}$ be an arbitrary 3-graded Lie superalgebra and set $(V^+, V^-) = \mathcal{J}(\mathfrak{g})$. Then we have a morphism of Lie superalgebras

$$\mathfrak{g}_0 \to \mathrm{Der}(V^+, V^-);\ x \mapsto (\mathrm{ad}_x |_{\mathfrak{g}_+}, \mathrm{ad}_x |_{\mathfrak{g}_-}),$$

and its kernel I is an ideal in $\mathfrak{g}_0$ and by construction even in $\mathfrak{g}$. By definition of $\widetilde{\mathrm{TKK}}(V^+, V^-)$, we have an embedding of $\mathfrak{g}/I$ into $\widetilde{\mathrm{TKK}}(V^+, V^-)$. If $\mathfrak{g} = \mathrm{Ti}\,(V, \mathcal{D}, \mathfrak{sl}_2)$ for a unital Jordan superalgebra V, then one can easily check that $I = 0$ (and thus $\mathcal{D} \subseteq \mathrm{Der}(V)$) is equivalent with the condition that the only ideal of $\mathfrak{g}$ contained in $\mathcal{D}$ is the zero ideal.

Another "universality property" of $\widetilde{\mathrm{TKK}}(V^+, V^-)$ will be discussed in Subsection 3.5.4.

3.5.3 Outer derivations

Definition 3.5.4 (See [AMR]). *For a Lie superalgebra $\mathfrak{g}$, denote the Lie superalgebra of derivations by* $\mathrm{Der}(\mathfrak{g})$. *The inner derivations* $\mathrm{Inn}(\mathfrak{g}) = \{\mathrm{ad}_X \mid X \in \mathfrak{g}\}$ *form an ideal isomorphic to the quotient of* $\mathfrak{g}$ *by its centre. The Lie superalgebra of outer derivations is* $\mathrm{Out}(\mathfrak{g}) = \mathrm{Der}(\mathfrak{g})/\mathrm{Inn}(\mathfrak{g})$.

An extension $\mathfrak{e}$ of a Lie superalgebra $\mathfrak{g}$ over a Lie superalgebra $\mathfrak{h}$ is a Lie superalgebra $\mathfrak{e}$ such that the following is a short exact sequence:

$$0 \to \mathfrak{h} \to \mathfrak{e} \to \mathfrak{g} \to 0.$$

In particular $\mathfrak{h}$ is an ideal in $\mathfrak{e}$.

Let $\mathfrak{h}$ be a Lie superalgebra with trivial centre. Then we will freely use the isomorphism between the space of extensions of $\mathfrak{g}$ over $\mathfrak{h}$, and the space of Lie superalgebra morphisms $\mathfrak{g} \to \mathrm{Out}(\mathfrak{h})$, see e.g. Corollary 8 in [AMR].

The main result of this section is the following proposition.

Proposition 3.5.5. *For a unital Jordan superalgebra V, we have*

$$\widetilde{\mathrm{TKK}}(V) \cong \mathrm{Der}(\mathrm{TKK}(V)),$$

and thus

$$\widetilde{\mathrm{TKK}}(V)/\mathrm{TKK}(V) \cong \mathfrak{str}(V)/\mathfrak{istr}(V) \cong \mathrm{Out}(\mathrm{TKK}(V)).$$

Remark 3.5.6. Again the assumption of a multiplicative identity is essential for this proposition. A counterexample of the statement for non-unital Jordan superalgebras is given in Subsection 3.6.2.

Remark 3.5.7. For any $\mathbb{Z}$-graded Lie superalgebra, the Lie superalgebra $\mathrm{Der}(\mathfrak{g}) \subset \mathrm{End}_{\mathbb{K}}(\mathfrak{g})$ is $\mathbb{Z}$-graded by construction. The endomorphisms in $\mathrm{Der}(\mathfrak{g})_i$ map elements in $\mathfrak{g}_j$ to elements in $\mathfrak{g}_{i+j}$. Clearly $\mathrm{Inn}(\mathfrak{g})$ is then a graded ideal in $\mathrm{Der}(\mathfrak{g})$, so that $\mathrm{Out}(\mathfrak{g})$ is also $\mathbb{Z}$-graded. In particular, when $\mathfrak{g}$ is 3-graded then $\mathrm{Der}(\mathfrak{g})$ and $\mathrm{Out}(\mathfrak{g})$ will be 5-graded.

The following reformulation of Proposition 3.5.5 holds for arbitrary Jordan superpairs and thus *a fortiori* also for non-unital Jordan superalgebras.

Proposition 3.5.8. *For a Jordan superpair (V^+, V^-), we have*

$$\begin{aligned}\widetilde{\mathrm{TKK}}(V^+, V^-)/\mathrm{TKK}(V^+, V^-) &\cong \mathrm{Der}(V^+, V^-)/\mathrm{Inn}(V^+, V^-)\\ &\cong \mathrm{Out}(\mathrm{TKK}(V^+, V^-))_0.\end{aligned}$$

In particular, for a (non-unital) Jordan superalgebra V we have that $\widetilde{\mathrm{TKK}}(V)$ is the extension over $\mathrm{TKK}(V)$ of $\mathrm{Out}(\mathrm{TKK}(V))_0$ corresponding to the embedding $\mathrm{Out}(\mathrm{TKK}(V))_0 \hookrightarrow \mathrm{Out}(\mathrm{TKK}(V))$.

The rest of the subsection is devoted to the proofs of Propositions 3.5.5 and 3.5.8.

Lemma 3.5.9. *We have a Lie superalgebra isomorphism*

$$\phi : \mathrm{Der}(V^+, V^-) \xrightarrow{\sim} \mathrm{Der}(\mathrm{TKK}(V^+, V^-))_0, \quad x \mapsto \mathrm{ad}_x|_{\mathrm{TKK}(V^+,V^-)}.$$

Proof. The map ϕ is well-defined since $\mathrm{TKK}(V^+, V^-)$ is an ideal in $\widetilde{\mathrm{TKK}}(V^+, V^-)$ and $\mathrm{Der}(V^+, V^-) \subset \widetilde{\mathrm{TKK}}(V^+, V^-)$ is the zero component of the $\mathbb{Z}$-grading. By construction it is an injective Lie superalgebra morphism.

Now let D be a $\mathbb{Z}$-grading preserving derivation of $\mathrm{TKK}(V^+, V^-)$, then $(D|_{V^+}, D|_{V^-})$ is an element of $\mathrm{Der}(V^+, V^-)$ since, using the definition of the bracket on $\mathrm{TKK}(V^+, V^-)$ in Subsection 3.4.2, we find

$$\begin{aligned}&D(\{x, y, z\}) = D([[x, y], z])\\ &= [[D(x), y], z] + (-1)^{|x||D|}[[x, D(y)], z] + (-1)^{(|x|+|y|)|D|}[[x, y], D(z)]\end{aligned}$$

$$= \{D(x), y, z\} + (-1)^{|x||D|}\{x, D(y), z\} + (-1)^{(|x|+|y|)|D|}\{x, y, D(z)\}.$$

One can check that

$$D = \mathrm{ad}_{(D|_{V^+}, D|_{V^-})} \text{ and } x = (\mathrm{ad}_x|_{V^+}, \mathrm{ad}_x|_{V^-}).$$

So we have indeed $\mathrm{Der}(V^+, V^-) \cong \mathrm{Der}(\mathrm{TKK}(V^+, V^-))_0$ as Lie superalgebras. □

Using this lemma, we can immediately prove Proposition 3.5.8.

Proof of Proposition 3.5.8. The first isomorphism follows immediately from Koecher's construction in Subsection 3.4.2. Furthermore, since the intersection of the centre of $\mathrm{TKK}(V^+, V^-)$ with $\mathrm{TKK}(V^+, V^-)_0$ is trivial, we have

$$\mathrm{Inn}(\mathrm{TKK}(V^+, V^-))_0 \cong \mathrm{TKK}(V^+, V^-)_0 = \mathrm{Inn}(V^+, V^-).$$

Hence we conclude that

$$\mathrm{Der}(V^+, V^-)/\mathrm{Inn}(V^+, V^-) \cong \mathrm{Out}(\mathrm{TKK}(V^+, V^-))_0,$$

from Lemma 3.5.9. □

To prove Proposition 3.5.5, we will show that, for a unital Jordan superalgebra V, all outer derivations of $\mathrm{TKK}(V)$ are grading preserving for the 3-grading we consider. This is not true for non-unital algebras, see Subsection 3.6.2.

Lemma 3.5.10. *For a unital Jordan superalgebra V, we have*

$$\mathrm{Der}(\mathrm{TKK}(V))_{-2} = 0 = \mathrm{Der}(\mathrm{TKK}(V))_2.$$

Proof. Let $D \in \mathrm{Der}(\mathrm{TKK}(V))_2$. First remark that D acts trivially on $\mathrm{TKK}(V)_0$ and $\mathrm{TKK}(V)_1$. We will show that it must also act trivially on $\mathrm{TKK}(V)_{-1}$. To use the definition of $\mathrm{TKK}(V)$ we use the Jordan superpair $(V^+, V^-) := (V, V)$. For $x \in V$, we use the notation x^+ and x^-, as in Example 3.2.2. We find, using the definition of the bracket on $\mathrm{TKK}(V)$ and the property $D(\mathrm{TKK}(V)_0) = 0$, that

$$D(e^-) = \frac{1}{2}D([e^-, \mathbb{D}_{e,e}]) = \frac{1}{2}[D(e^-), \mathbb{D}_{e,e}] + \frac{1}{2}[e^-, D(\mathbb{D}_{e,e})]$$

$$= \frac{1}{2}[D(e^-), \mathbb{D}_{e,e}] = -D(e^-).$$

Hence $D(e^-) = 0$, which then implies that

$$D(x^-) = \frac{1}{2}D([e^-, \mathbb{D}_{x,e}]) = \frac{1}{2}[D(e^-), \mathbb{D}_{x,e}] + \frac{1}{2}[e^-, D(\mathbb{D}_{x,e})] = 0.$$

We conclude that $D = 0$ for all $D \in \mathrm{Der}(\mathrm{TKK}(V))_2$. The proof that $\mathrm{Der}(\mathrm{TKK}(V))_{-2} = 0$ is completely similar. □

Lemma 3.5.11. *For a unital Jordan superalgebra V, we have isomorphisms*

$$V \tilde{\to} \mathrm{Der}(\mathrm{TKK}(V))_1; \quad x \mapsto \mathrm{ad}_{x^+} \quad \text{and}$$
$$V \tilde{\to} \mathrm{Der}(\mathrm{TKK}(V))_{-1}; \quad x \mapsto \mathrm{ad}_{x^-},$$

as super-vector spaces.

Proof. Let x^+ be an element of $\mathrm{TKK}(V)_1 = V^+$, then ad_{x^+} is an element in $\mathrm{Der}(\mathrm{TKK}(V))_1$. With an element D in $\mathrm{Der}(\mathrm{TKK}(V))_1$ we can associate the element $-\frac{1}{2}D(\mathbb{D}_{e,e}) \in V^+$. We will now show that $x^+ \mapsto \mathrm{ad}_{x^+}$ and $D \mapsto -\frac{1}{2}D(\mathbb{D}_{e,e})$ are each others inverse. This follows from

$$-\frac{1}{2}\mathrm{ad}_{x^+}(\mathbb{D}_{e,e}) = -\frac{1}{2}[x^+, \mathbb{D}_{e,e}] = x^+$$

and the following three calculations, for arbitrary $x, y \in V$,

$$\begin{aligned}
\tfrac{1}{2}\mathrm{ad}_{D(\mathbb{D}_{e,e})}(y^-) &= \tfrac{1}{2}[D(\mathbb{D}_{e,e}), y^-] \\
&= \tfrac{1}{2}D([\mathbb{D}_{e,e}, y^-]) - \tfrac{1}{2}[\mathbb{D}_{e,e}, D(y^-)] = -D(y^-) \\
\tfrac{1}{2}\mathrm{ad}_{D(\mathbb{D}_{e,e})}(L_x) &= \tfrac{1}{2}[D(\mathbb{D}_{e,e}), L_x] \\
&= \tfrac{1}{2}D([\mathbb{D}_{e,e}, L_x]) - \tfrac{1}{2}[\mathbb{D}_{e,e}, D(L_x)] = -D(L_x) \\
\tfrac{1}{2}\mathrm{ad}_{D(\mathbb{D}_{e,e})}([L_x, L_y]) &= \tfrac{1}{2}[D(\mathbb{D}_{e,e}), [L_x, L_y]] \\
&= \tfrac{1}{2}D([\mathbb{D}_{e,e}, [L_x, L_y]]) - \tfrac{1}{2}[\mathbb{D}_{e,e}, D([L_x, L_y])] \\
&= -D([L_x, L_y]).
\end{aligned}$$

We conclude that $V \cong \mathrm{Der}(\mathrm{TKK}(V))_1$. Similarly $\mathrm{TKK}(V)_{-1} \to \mathrm{Der}(\mathrm{TKK}(V))_{-1}$; $x^- \mapsto \mathrm{ad}_{x^-}$ is an isomorphism with inverse $D \mapsto \frac{1}{2}D(\mathbb{D}_{e,e})$. □

Proof of Proposition 3.5.5. Consider the following morphism of Lie superalgebras

$$\widetilde{\mathrm{TKK}}(V) \to \mathrm{Der}(\mathrm{TKK}(V));\ x \mapsto \mathrm{ad}_x|_{\mathrm{TKK}(V)}.$$

Combining Lemmata 3.5.9, 3.5.10 and 3.5.11, we see that this is an isomorphism. □

3.5.4 Alternative construction

The construction of $\widetilde{\mathrm{TKK}}(V^+, V^-)$ starting from $\mathrm{TKK}(V^+, V^-)$ in Proposition 3.5.8 fits into a more general construction. In [BDS, Section 4.1], the authors start from an arbitrary $(2n+1)$-graded Lie superalgebra $\mathcal{L} = \bigoplus_{i\in\mathbb{Z}} \mathcal{L}_i$ (strictly speaking only Lie algebras are considered, but the procedure carries over naturally to the super case). Then [BDS, Construction 4.1.2] constructs an extension $\overline{\mathcal{L}}$ over $\mathcal{L}$, which is again a $(2n+1)$-graded Lie superalgebra which satisfies $\overline{\mathcal{L}}_i = \mathcal{L}_i$ if $i \neq 0$.

It is not difficult to show that in the case of a 3-graded Lie superalgebra $\mathcal{L}$ we have $\overline{\mathcal{L}_0} = \mathrm{Der}(\mathcal{L}^+, \mathcal{L}^-)$ and hence

$$\overline{\mathcal{L}} = \widetilde{\mathrm{TKK}}(\mathcal{L}^+, \mathcal{L}^-) \qquad \text{with} \quad (\mathcal{L}^+, \mathcal{L}^-) := \mathcal{J}(\mathcal{L}),$$

the Jordan superpair associated with $\mathcal{L}$ in Subsection 3.4.2. In other words,

$$\widetilde{\mathrm{TKK}}(V^+, V^-) \;\cong\; \overline{\mathrm{TKK}(V^+, V^-)}.$$

This reveals a universality principle behind $\widetilde{\mathrm{TKK}}(V^+, V^-)$, as the construction of $\overline{\mathcal{L}}$ starting from $\mathcal{L}$ in [BDS] does not depend on $\mathcal{L}_0$.

An interesting consequence of [BDS, Lemma 4.1.3] is then

$$\mathrm{Out}(\widetilde{\mathrm{TKK}}(V^+, V^-)) \;=\; 0,$$

for arbitrary Jordan superpairs (V^+, V^-), so also for arbitrary (unital or non-unital) Jordan superalgebras.

3.6 Examples over $\mathbb{C}$

In this section, we use the results of the previous sections to calculate $\widetilde{\mathrm{TKK}}(V)$ for V any finite-dimensional simple Jordan superalgebra over an *algebraically closed field of characteristic zero.* We assume these conditions on the ground field for the entire section.

3.6.1 Unital finite-dimensional simple Jordan superalgebras.

A complete list of unital finite-dimensional simple Jordan superalgebras V and the corresponding $\mathrm{Kan}(V)$ is given in [Ka2, CK]. This gives us $\mathrm{TKK}(V)$ and $\mathrm{Ti}(V, \mathrm{Inn}(V), \mathfrak{sl}_2)$, by Propositions 3.4.3 and 3.4.5. For the Jordan superalgebras we use the notation of [CK], where also the definitions can be found. For the Lie superalgebras we use the notations introduced in Section 2.3. In [Ka1, Theorem 5.1.2] and [Sc, Chapter III, Proposition 3], $\mathrm{Der}(\mathfrak{g})$ is calculated for any simple finite-dimensional Lie superalgebra $\mathfrak{g}$. We can combine this with Proposition 3.5.5 and Proposition 3.5.1, to obtain $\widetilde{\mathrm{TKK}}(V) \cong \mathrm{Ti}(V, \mathrm{Der}(V), \mathfrak{sl}_2)$, leading to the following table.

V	$\mathrm{TKK}(V)$	$\widetilde{\mathrm{TKK}}(V)$	Remarks
$gl(m,n)_+$	$\mathfrak{sl}(2m\|2n)$		$m \neq n$
$gl(m,m)_+$	$\mathfrak{psl}(2m\|2m)$	$\mathfrak{pgl}(2m\|2m)$	$m > 1$
$osp(m,2n)_+$	$\mathfrak{osp}(4n\|2m)$		$(n,m) \neq (1,0)$
$(m-3,2n)_+$	$\mathfrak{osp}(m\|2n)$		$m \geq 3,\ (m,2n) \neq (4,0)$
$p(n)_+$	$\mathfrak{spe}(2n)$	$\mathfrak{pe}(2n)$	$n > 1$
$q(n)_+$	$\mathfrak{psq}(2n)$	$\mathfrak{pq}(2n)$	$n > 1$
D_t	$D(2,1;t)$		$t \notin \{0,-1\}$
E	E_7		
F	$F(4)$		
$JP(0,n-3)$	$H(0,n) = H(n)$	$\mathbb{K}C \ltimes \widetilde{H}(n)$	$n \geq 5$
$gl(1,1)_+$	$\mathfrak{psl}(2\|2)$	$D(2,1;-1)$	

When $\widetilde{\mathrm{TKK}}(V)$ is isomorphic to $\mathrm{TKK}(V)$, we only wrote it once.

Taking the zero component of the 3-graded algebras in the above table gives us $\mathfrak{istr}(V) \cong \mathrm{Inn}(V,V)$ and $\mathfrak{str}(V) \cong \mathrm{Der}(V,V)$. These are listed in the following table, where the same restrictions on the indices are assumed as in the previous table.

V	$\mathfrak{istr}(V)$	$\mathfrak{str}(V)$
$gl(m,n)_+$	$\mathfrak{sl}(m\|n)\oplus\mathfrak{sl}(m\|n)\oplus\mathbb{K}$	
$gl(m,m)_+$	$s(\mathfrak{gl}(m\|m)\oplus\mathfrak{gl}(m\|m))/\langle I_{4m}\rangle$	$(\mathfrak{gl}(m\|m)\oplus\mathfrak{gl}(m\|m))/\langle I_{4m}\rangle$
$osp(m,2n)_+$	$\mathfrak{gl}(2n\|m)$	
$(m-3,2n)_+$	$\mathfrak{osp}(m-2\|2n)\oplus\mathbb{K}$	
$p(n)_+$	$\mathfrak{sl}(n\|n)$	$\mathfrak{gl}(n\|n)$
$q(n)_+$	$s(\mathfrak{q}(n)\oplus\mathfrak{q}(n))/\langle I_{4n}\rangle$	$(\mathfrak{q}(n)\oplus\mathfrak{q}(n))/\langle I_{4n}\rangle$
D_t	$\mathfrak{sl}(2\|1)\oplus\mathbb{K}\cong\mathfrak{osp}(2\|2)\oplus\mathbb{K}$	
E	$E_6\oplus\mathbb{K}$	
F	$\mathfrak{osp}(2\|4)\oplus\mathbb{K}$	
$JP(0,n-3)$	$\widetilde{H}(n-2)\ltimes(\Lambda(n-2)/\langle\xi_1\cdots\xi_{n-2}\rangle)$	$\mathbb{K}C\ltimes\left(\widetilde{H}(n-2)\ltimes\Lambda(n-2)\right)$
$gl(1,1)_+$	$s(\mathfrak{gl}(1\|1)\oplus\mathfrak{gl}(1\|1))/\langle I_4\rangle$	$\mathfrak{sl}_2\ltimes\mathfrak{istr}(gl(1,1)_+)$

Again, if $\mathfrak{str}(V)$ is isomorphic to $\mathfrak{istr}(V)$, we only wrote it once. The action of $\mathfrak{sl}_2$ on $\mathfrak{istr}(gl(1,1)_+)$ is the adjoint action by using the embedding of $\mathfrak{sl}_2$ in $D(2,1;-1)$. The following isomorphisms exist in the list of Jordan superalgebras:

$$(1,2)_+\cong D_1,\quad D_t\cong D_{t^{-1}}.$$

Furthermore, also the simple Jordan superalgebras $JP(0,1)$ and D_{-1} appear in the literature, but they are isomorphic to $gl(1,1)_+$, so they are already included in the table.

3.6.2 The non-unital finite-dimensional simple Jordan superalgebra.

The full list of finite-dimensional simple Jordan superalgebras in [Ka2, CK] contains only one Jordan superalgebra which is non-unital. In [Ka2] it is denoted by K. The algebra K is defined as

$$K=\langle a\rangle\oplus\langle\xi_1,\xi_2\rangle,\quad |a|=\bar{0},\ |\xi_1|=|\xi_2|=\bar{1},$$

with multiplication satisfying $a^2=a$, $a\xi_i=\frac{1}{2}\xi_i$ and $\xi_1\xi_2=a$.

A straightforward calculation implies

$$\mathfrak{istr}(K)=\mathfrak{str}(K)=\mathrm{Inn}(K,K)\cong\mathfrak{sl}(1|2),\ \text{and}\ \mathrm{Der}(K,K)\cong\mathfrak{gl}(1|2).$$

This gives a counterexample to the statement in Proposition 3.3.7(1) for non-unital Jordan superalgebras. For K, the sums in Definitions 3.3.2 and 3.3.3 are direct.

One also finds

$$\mathrm{TKK}(K) \cong \mathfrak{psl}(2|2).$$

By construction, $\widetilde{\mathrm{TKK}}(K)$ is an extension over $\mathrm{TKK}(K)$. As $\mathfrak{istr}(K) \cong \mathrm{Inn}(K,K)$, it follows easily that the same is true for $\mathrm{Kan}(K)$. The algebras $\widetilde{\mathrm{TKK}}(K)$ and $\mathrm{Kan}(K)$ can hence be described in terms of $\mathrm{Out}(\mathrm{TKK}(K)) \cong \mathfrak{sl}_2$:

- $\widetilde{\mathrm{TKK}}(K) \cong \mathfrak{pgl}(2|2)$ is the extension of $\mathbb{K}$ over $\mathrm{TKK}(K)$ corresponding to the morphism $\mathbb{K} \to \mathfrak{sl}_2$, where $1 \in \mathbb{K}$ is mapped to a semisimple element of $\mathfrak{sl}_2$.
- $\mathrm{Kan}(K)$ is the extension of $\mathbb{K}$ over $\mathrm{TKK}(K)$ corresponding to the morphism $\mathbb{K} \to \mathfrak{sl}_2$, where $1 \in \mathbb{K}$ is mapped to a nilpotent element of $\mathfrak{sl}_2$.

In particular we find that

$$\widetilde{\mathrm{TKK}}(K) \ncong \mathrm{Der}(\mathrm{TKK}(K)) \quad \text{and} \quad \mathrm{Kan}(K) \ncong \mathrm{TKK}(K).$$

This gives counterexamples to the statements in Propositions 3.5.5 and 3.4.3, for non-unital Jordan superalgebras. By Remark 3.4.6 and the above, we do have

$$\mathrm{Ti}(K, \mathrm{Inn}(K), \mathfrak{sl}_2) \;\cong\; \mathrm{Ti}(K, \mathrm{Der}(K), \mathfrak{sl}_2) \;\cong\; \mathrm{TKK}(K) \;\cong\; \mathfrak{psl}(2|2).$$

For the 3-grading on $\mathfrak{psl}(2|2)$ corresponding to the interpretation as $\mathrm{TKK}(K)$, the algebra $\mathfrak{g} = \mathrm{Out}(\mathfrak{psl}(2|2) \cong \mathfrak{sl}_2$ is 3-graded where $\mathfrak{g}_i$ has dimension one for $i \in \{-1, 0, 1\}$. This is in sharp contrast with Lemma 3.5.11 for the unital case. By Proposition 3.5.8, $\widetilde{\mathrm{TKK}}(K)$ is the subalgebra of $\mathrm{Der}(\mathrm{TKK}(K))$ where only the degree 0 derivations are added to $\mathrm{TKK}(K)$. In the same way, $\mathrm{Kan}(K)$ is the subalgebra of $\mathrm{Der}(\mathrm{TKK}(K))$ where only the degree 1 derivations are added to $\mathrm{TKK}(K)$.

3.7 Other examples

3.7.1 The spin factor over $\mathbb{R}$

In this section J denotes the real spin factor defined in Section 3.2.2. Consider the orthosymplectic metric $\tilde{\beta}$ used in Section 3.2.2. We extend this form as follows. Set $\beta_{00} = -1$, $\beta_{i0} = 0 = \beta_{0i}$, $\beta_{ij} = \tilde{\beta}_{ij}$ for $i, j \in \{1, .., p+q-3+2n\}$. Then the corresponding form $\langle \cdot, \cdot \rangle_\beta$ is a supersymmetric, non-degenerate, even bilinear form on the super-vector space J where the even part has signature $(p-1, q-1)$. Consider the orthosymplectic Lie superalgebra $\mathfrak{osp}(J)$, i.e. the subalgebra of $\mathfrak{gl}(J)$ which leaves the form $\langle \cdot, \cdot \rangle_\beta$ invariant.

Proposition 3.7.1. *We have*

$$\mathfrak{istr}(J) = \mathfrak{osp}(J) \oplus \mathbb{R}L_e,$$

where the direct sum decomposition is as algebras. Furthermore

$$\mathrm{TKK}(J) = \mathfrak{osp}(p, q|2n).$$

Proof. From Section 3.6.1, it follows that for the complexified Jordan superalgebra $J_\mathbb{C}$ we have

$$\mathfrak{istr}(J_\mathbb{C}) = \mathfrak{osp}_\mathbb{C}(J) \oplus \mathbb{C} \text{ and } \mathrm{TKK}(J_\mathbb{C}) = \mathfrak{osp}_\mathbb{C}(p+q|2n).$$

For $n = 0$ we find

$$\mathrm{TKK}(J) = \mathfrak{so}(p, q),$$

see for example [KM2, Section 2.5]. One can check that the even part of $\mathrm{TKK}(J)$ still contains a component $\mathfrak{so}(p, q)$ if $n > 0$. For $p+q-2 > 0$, there is a unique real form of $\mathfrak{osp}_\mathbb{C}(p+q|2n)$ with contains the component $\mathfrak{so}(p, q)$, [Pa, Theorem 2.5]. So we can conclude

$$\mathrm{TKK}(J) = \mathfrak{osp}(p, q|2n).$$

The inner structure algebra is spanned by the operators L_{e_i}, $[L_{e_i}, L_{e_j}]$ for $i, j > 0$ and L_e. Observe that L_e is in the centre of $\mathfrak{istr}(J)$ since e is the unit. We defined $\langle \cdot, \cdot \rangle_\beta$ such that

$$\langle e_i e_j, e_k \rangle_\beta = 0, \quad \langle e_i e_0, e_j \rangle_\beta = \tilde{\beta}_{ij}, \quad \text{ and } \langle e_0, e_0 \rangle_\beta = -1,$$

for $i, j, k > 0$. Using this, one can show that the operators $X = L_{e_i}$ or $X = [L_{e_i}, L_{e_j}]$ for $i > 0$ satisfy

$$\langle X(u), v \rangle_\beta + (-1)^{|X||u|} \langle u, X(v) \rangle_\beta = 0.$$

Hence they form a subspace of $\mathfrak{osp}(J)$ and we obtain

$$\mathfrak{istr}(J) \subset \mathfrak{osp}(J) \oplus \mathbb{R}L_e.$$

Since $\mathfrak{istr}(J_\mathbb{C}) = \mathfrak{osp}_\mathbb{C}(J) \oplus \mathbb{C}$ we conclude that this inclusion is actually an equality. □

Corollary 3.7.2. *For the real spin factor J we have*

$$\begin{aligned}\mathfrak{str}(J) &\cong \mathfrak{istr}(J) \cong \mathrm{Der}(J,J) \cong \mathrm{Inn}(J,J),\\ \mathrm{TKK}(J) &\cong \mathrm{Kan}(J) \cong \mathrm{Ti}(J, \mathrm{Inn}(J), \mathfrak{sl}_2(\mathbb{R}))\\ &\cong \mathrm{Ti}(J, \mathrm{Der}(J), \mathfrak{sl}_2(\mathbb{R})) \cong \widetilde{\mathrm{TKK}}(J).\end{aligned}$$

Proof. From Section 3.6.1, we know that $\mathrm{TKK}(J_{\mathbb{C}}) \cong \widetilde{\mathrm{TKK}}(J_{\mathbb{C}})$. Then the corollary follows from the fact that J is unital and Proposition 3.3.7, Proposition 3.4.3, Proposition 3.4.5 and Proposition 3.5.1. □

Using the bilinear form

$$\overline{\beta} = \begin{pmatrix} 1 & & & \\ & \beta_{sym} & & \\ & & -1 & \\ & & & \beta_{asym} \end{pmatrix},$$

we saw in Section 2.7.1 that we have a realisation of $\mathfrak{osp}(p,q|2n)$ using differential operators

$$L_{i,j} = z_i\partial_j - (-1)^{|i||j|} z_j \partial_i.$$

An explicit isomorphism of $\mathrm{TKK}(J)$ with this realisation of $\mathfrak{osp}(p,q|2n)$ is given by

$$\begin{aligned} e_i^+ &\mapsto L_{\tilde{i},(p+q-1)} - L_{\tilde{i},0}\\ e_0^+ &\mapsto -L_{(p+q-2),(p+q-1)} - L_{(p+q-2),0}\\ L_{e_i} &\mapsto L_{\tilde{i},(p+q-2)}\\ L_{e_0} &\mapsto L_{0,(p+q-1)}\\ [L_{e_i}, L_{e_j}] &\mapsto L_{\tilde{i},\tilde{j}}\\ e_i^- &\mapsto L_{\tilde{i},(p+q-1)} + L_{\tilde{i},0}\\ e_0^+ &\mapsto L_{(p+q-2),(p+q-1)} + L_{(p+q-2),0}. \end{aligned}$$

Here $\tilde{i} = i$ if $|i| = 0$ and $\tilde{i} = i+1$ if $|i| = 1$. This yields another approach to show that $\mathrm{TKK}(J) \cong \mathfrak{osp}(p,q|2n)$.

3.7.2 The exceptional Jordan superalgebra D_t

The exceptional Jordan superalgebra D_t was introduced in Section 3.2.1. Since its dimension $(2|2)$ is small, it is possible to calculate the structure algebra and TKK algebra very explicitly. First note that D_t is unital with unit given by $e_1 + e_2$. In this section, we will always assume that $t \neq 0$ and that our field $\mathbb{K}$ contains $\sqrt{2}$ and $\sqrt{1+t}$. We have the following results.

Proposition 3.7.3. *For the simple, unital Jordan superalgebra D_t we have for $t \neq -1$*

$$\begin{aligned} \mathfrak{istr}(D_t) &= \mathfrak{str}(D_t) \cong \mathfrak{osp}(1,1|2) \oplus \mathbb{K} \\ \mathrm{TKK}(D_t) &= \widetilde{\mathrm{TKK}}(D_t) \cong D(2,1;t). \end{aligned}$$

For $t = -1$ we obtain

$$\begin{aligned} \mathfrak{istr}(D_{-1}) &\neq \mathfrak{str}(D_{-1}) \\ \widetilde{\mathrm{TKK}}(D_{-1}) &\cong D(2,1;-1) \\ \mathrm{TKK}(D_{-1}) &\cong \mathfrak{psl}(2|2). \end{aligned}$$

Here $\mathfrak{osp}(1,1|2n)$ is the orthosymplectic Lie superalgebra with bilinear form

$$\begin{pmatrix} 0 & 1 & & \\ 1 & 0 & & \\ & & 0 & -1 \\ & & 1 & 0 \end{pmatrix}.$$

In the rest of this section, we will prove this proposition. We will also give an explicit matrix realisation for the (inner) structure algebra. Since D_t is unital we also immediately have the following isomorphisms

$$\begin{aligned} \mathfrak{str}(D_t) &\cong \mathrm{Der}(D_t, D_t), \\ \mathfrak{istr}(D_t) &\cong \mathrm{Inn}(D_t, D_t), \\ \mathrm{TKK}(D_t) &\cong \mathrm{Kan}(D_t) \cong \mathrm{Ti}(D_t, \mathrm{Inn}(D_t), \mathfrak{sl}_2(\mathbb{R})) \\ \widetilde{\mathrm{TKK}}(D_t) &\cong \mathrm{Ti}(D_t, \mathrm{Der}(D_t), \mathfrak{sl}_2(\mathbb{R})). \end{aligned}$$

Set

$$e_1 = (1,\, 0,\, 0,\, 0)^t\,, \quad e_2 = (0,\, 1,\, 0,\, 0)^t\,,$$

$$\xi = (0,\,0,\,1,\,0)^t\,, \qquad \eta = (0,\,0,\,0,\,1)^t\,.$$

Then $\mathrm{End}(D_t)$ can be represented by four by four matrices. We immediately obtain for the left multiplication operators

$$L_{e_1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}, \quad L_{e_2} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix},$$
$$L_{\xi} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & t \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad L_{\eta} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & -t & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \end{pmatrix}.$$

We can characterise the derivations of D_t as follows.

Lemma 3.7.4. *The space* $\mathrm{Der}(D_t)$ *is given by*

$$\mathrm{Der}(D_t)_{\bar{0}} = \left\{ \begin{pmatrix} 0 & 0 & & \\ 0 & 0 & & \\ & & a & b \\ & & c & d \end{pmatrix} | a+d=0 \right\} \cong \mathfrak{sl}(2).$$

and

$$\mathrm{Der}(D_t)_{\bar{1}} = \left\{ \begin{pmatrix} 0 & 0 & a & b \\ 0 & 0 & -at & -bt \\ -\frac{b}{2} & \frac{b}{2} & & \\ \frac{a}{2} & -\frac{a}{2} & & \end{pmatrix} | a,b \in \mathbb{K} \right\}.$$

Proof. The condition on $D \in \mathrm{End}(V)$ to be a derivation is expressed by Equation (3.6),

$$[D, L_x] = L_{D(x)} \qquad \text{for all } x \in V.$$

Calculating the constraints this puts on D for x equal to e_1, e_2, ξ and η leads to the lemma. □

So $\mathrm{Der}(D_t)$ has dimension $(3|2)$. Since D_t had dimension $(2|2)$, we see that $\dim(\mathfrak{str}(D_t)) = (5|4)$.

We conclude

$$\mathfrak{str}(D_t) = \left\{ \begin{pmatrix} 2a & 0 & -2D & 2B \\ 0 & -2at & -2Ct & 2At \\ A & B & (1-t)a+e & f \\ C & D & g & (1-t)a-e \end{pmatrix} + \zeta \mathbb{I}_4 \right.$$
$$\left. \mid a,e,f,g,A,B,C,D,\zeta \in \mathbb{K} \right\}$$

Here $\mathbb{I}_4$ is the four by four identity matrix. If $t \neq -1$, then we have $\mathfrak{osp}(1,1|2) \oplus \mathbb{K} \cong \mathfrak{str}(D_t)$. An explicit isomorphism is given by

$$\begin{pmatrix} a & 0 & -D & B \\ 0 & -a & -C & A \\ A & B & e & f \\ C & D & g & -e \end{pmatrix} + \zeta \mapsto$$
$$\begin{pmatrix} \frac{2a}{1+t} & 0 & -B\frac{\sqrt{2}}{\sqrt{1+t}} & 2D\frac{\sqrt{2}}{\sqrt{1+t}} \\ 0 & \frac{-2at}{1+t} & At\frac{\sqrt{2}}{\sqrt{1+t}} & -2Ct\frac{\sqrt{2}}{\sqrt{1+t}} \\ -C\frac{\sqrt{2}}{\sqrt{1+t}} & D\frac{\sqrt{2}}{\sqrt{1+t}} & \frac{1-t}{1+t}a-e & 2g \\ -\frac{A}{2}\frac{\sqrt{2}}{\sqrt{1+t}} & \frac{B}{2}\frac{\sqrt{2}}{\sqrt{1+t}} & \frac{f}{2} & \frac{1-t}{1+t}a+e \end{pmatrix} + \zeta \mathbb{I}_4,$$

where we used the matrix realisation of $\mathfrak{osp}(1,1|2)$ one obtains using the bilinear form

$$\begin{pmatrix} 0 & 1 & & \\ 1 & 0 & & \\ & & 0 & -1 \\ & & 1 & 0 \end{pmatrix}.$$

To obtain the inner structure algebra we will calculate the matrix realisations of the operators

$$D_{x,y} = 2L_{xy} + 2[L_x, L_y]$$

explicitly. Since D_t is unital, these matrices span the inner structure algebra. We obtain

$$D_{e_1,e_1} = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad D_{e_1,e_2} = 0,$$

$$D_{e_1,\xi} = \begin{pmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad D_{e_1,\eta} = \begin{pmatrix} 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

$$D_{e_2,e_1} = 0, \qquad D_{e_2,e_2} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$D_{e_2,\xi} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2t \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad D_{e_2,\eta} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -2t & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

$$D_{\xi,e_1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2t \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad D_{\xi,e_2} = \begin{pmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$D_{\xi,\xi} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2t+2 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad D_{\xi,\eta} = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2t & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2t+2 \end{pmatrix},$$

$$D_{\eta,e_1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -2t & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \qquad D_{\eta,e_2} = \begin{pmatrix} 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

$$D_{\eta,\xi} = \begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & -2t & 0 & 0 \\ 0 & 0 & -2t-2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, D_{\eta,\eta} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -2t-2 & 0 \end{pmatrix}.$$

So we conclude that a basis of $\mathfrak{istr}(J)$ is given by

$$L_{e_1} = D_{e_1,e_1}, \quad L_{e_2} = D_{e_2,e_2}, \quad D_{\xi,\xi}, \quad D_{\eta,\eta}, \quad D_{\xi,\eta}$$
$$D_{e_1,\xi} = D_{\xi,e_2}, \quad D_{e_1,\eta} = D_{\eta,e_2}, \quad D_{e_2,\xi} = D_{\xi,e_1} \quad D_{e_2,\eta} = D_{\eta,e_1},$$

if $t \neq -1$ and then $\mathfrak{istr}(D_t)$ has dimension $(5|4)$. If $t = -1$, then furthermore $D_{\eta,\eta} = D_{\xi,\xi} = 0$ and $D_{\xi,\eta} = L_{e_1} - L_{e_2}$ and $\mathfrak{istr}(D_t)$ has dimension $(2|4)$.

So for $t \neq -1$,

$$\mathrm{Der}(D_t) = \langle [L_x, L_y] \mid x, y \in D_t \rangle \text{ and } \mathfrak{istr}(D_t) = \mathfrak{str}(D_t),$$

while for $t = -1$ the inclusions $\langle [L_x, L_y] \mid x, y \in D_t \rangle \subset \mathrm{Der}(D_t)$ and $\mathfrak{istr}(D_t) \subset \mathfrak{str}(D_t)$ are strict.

We use the explicit realisation of $D(2,1;\alpha)$ constructed in Section 2.5.5. We have $\widetilde{\mathrm{TKK}}(D_t) \cong D(2,1;t)$ where an explicit isomorphism is given as follows.

- For D_t^+

$$e_1 = X_{2\delta_2}, \quad e_2 = X_{2\delta_3}, \quad \xi = X_{-\delta_1+\delta_2+\delta_3} \quad \eta = 2X_{\delta_1+\delta_2+\delta_3}.$$

- For D_t^-

$$\begin{aligned} e_1 &= X_{-2\delta_2}, & e_2 &= X_{-2\delta_3}, \\ \xi &= -X_{-\delta_1-\delta_2-\delta_3} & \eta &= -2X_{\delta_1-\delta_2-\delta_3}. \end{aligned}$$

- For $\mathfrak{str}(D_t)$

$$\begin{aligned} &\begin{pmatrix} 2a & 0 & -2D & 2B \\ 0 & -2at & -2Ct & 2At \\ A & B & (1-t)a+e & f \\ C & D & g & (1-t)a-e \end{pmatrix} + \zeta \mathbb{I}_4 \mapsto \\ &a(H_{\delta_2} - tH_{\delta_3}) - \frac{e}{2}H_{\delta_3} + \frac{f}{2}X_{-2\delta_1} + 2gX_{2\delta_1} \\ &- AX_{-\delta_1-\delta_2+\delta_3} - BX_{-\delta_1+\delta_2-\delta_3} - 2CX_{\delta_1-\delta_2+\delta_3} \\ &- 2DX_{\delta_1+\delta_2-\delta_3} + \frac{\zeta}{2}(H_{\delta_2} - H_{\delta_3}). \end{aligned}$$

For $t \neq -1$ we then immediately also find that $\mathrm{TKK}(D_t) \cong D(2,1;t)$ since $\mathfrak{istr}(D_t) = \mathfrak{str}(D_t)$. For $t = -1$, we can use the same isomorphism restricted to $\mathfrak{istr}(D_{-1})$ for the zero graded part. Then we obtain $\mathrm{TKK}(D_{-1}) \cong \mathfrak{psl}(2,2)$.

If we increase the size of the penguin until it is the same height as the man and then compare the relative brain size, we now find that the penguin's brain is still smaller. But, and this is the point, it is larger than it was.

Monty Python

4

Polynomial realisations and Bessel operators

4.1 Introduction

Consider a complex simple Lie algebra $\mathfrak{g}$. In [Jo1], Joseph determined the minimal natural number $n = n_{\mathscr{D}}(\mathfrak{g})$ for which $\mathfrak{g}$ can be embedded in $\mathscr{D}_n$, where $\mathscr{D}_n$ is a canonically defined left quotient field of $\mathscr{A}_n$, the Weyl algebra of differential operators on $\mathbb{R}^n$ with complex polynomial coefficients. This embedding extends to a morphism from the universal enveloping algebra $U(\mathfrak{g})$ to $\mathscr{D}_n$ which has kernel J_0, known as the *Joseph ideal*.

In [Di], Dib introduced a second order differential operator on Jordan algebras, called the *Bessel operator*, yielding a system of differential equations generalising the Bessel differential equation. In [HKM], Hilgert, Kobayashi and Möllers obtained a unifying construction of "small" unitary representations, in particular minimal ones, of a large class of real simple Lie groups G, by using the Bessel operator on the Jordan algebra V, linked to G by the Tits–Kantor–Koecher (TKK) construction, as one of the main tools. Under this correspondence

Lie(G) is equal to the "conformal algebra" $\mathfrak{co}(V) = \mathrm{TKK}(V)$. The results in [HKM] fit into a large project, studying various properties of minimal representations using Bessel operators, on which remarkable progress has been made during the last decade. Bessel operators, also referred to as fundamental differential operators, appeared earlier in the study of specific examples of minimal representations in e.g. [Ko1, KØ, KM1, KM2, Sa]. Some recent further results on Bessel operators can be found in [HKMØ, Mö2, Ko3].

The purpose of the current chapter is threefold:

P1 Find a natural constructive way to introduce Bessel operators in the study of minimal representations of Lie groups.

P2 Start the systematic study of minimal representations for Lie *superalgebras*.

P3 Find compact explicit realisations of the exceptional Lie superalgebras $D(2,1;\alpha)$, $F(4)$ and $G(3)$.

To achieve part **P1** we will work out explicitly certain realisations of $\mathfrak{g}$, for any given three-term $\mathbb{Z}$-gradation, in a Weyl algebra. The existence of this realisation, guaranteed by general arguments in [Co], was used in [Jo1] to obtain an upper bound on $n_{\mathscr{D}}(\mathfrak{g})$. We will prove that this realisation is a generalisation of the representation of $\mathfrak{co}(V)$ in [HKM] to the setting of Jordan *pairs*, from the specific case of *simple unital* Jordan *algebras*. This construction makes a new direct link between [Jo1] and [HKM], which helps to further explain, from a different perspective, *why* the Bessel operators are so useful in the construction of minimal representations. At the same time, it now follows by construction that the Bessel operators lead to a representation of $\mathfrak{co}(V)$. Recently the construction in [HKM] has also been extended from simple unital Jordan algebras to the setting of Jordan pairs in [MS]. In that work the Bessel operators we obtain also appear.

To achieve the goal in part **P2**, we carry out the construction of **P1** immediately for the case where $\mathfrak{g}$ is a Lie superalgebra. For this, we need to generalise some technical results of [Be, Co], concerning universal enveloping algebras, to the case of ($\mathbb{Z}$-graded) superalgebras. In particular, we obtain a construction of small polynomial realisations for 3-graded Lie superalgebras, which will be the starting point of the study of an interesting class of representations of Lie supergroups following the spirit of [HKM]. In the next chapters we will work this out in particular for the orthosymplectic Lie supergroup.

The small polynomial realisations of Lie superalgebras have yet another application, as mentioned in aim **P3**. The exceptional simple basic classical Lie superalgebras $D(2,1;\alpha)$, $F(4)$ and $G(3)$, see e.g. [Mu], do not admit low dimensional representations, like the ones for the families $\mathfrak{osp}$ and $\mathfrak{sl}$. Therefore, there are no convenient matrix realisations available. Concretely, for the one parameter family $D(2,1;\alpha)$ of deformations of $\mathfrak{osp}(4|2)$, the smallest representation (the adjoint representation) of $D(2,1;\alpha)$ is 17-dimensional as soon as $\alpha \notin \mathbb{Q}$, see [VdJ]. This is in sharp contrast with the undeformed superalgebra $\mathfrak{osp}(4|2)$, which has a 6-dimensional representation, *viz.* the natural representation. So for generic $D(2,1;\alpha)$, contrary to $\mathfrak{osp}(4|2)$, the smallest matrix realisation is not convenient.

We apply our results to derive which convenient polynomial realisations exist for the exceptional Lie superalgebras and work them out very explicitly for the case $D(2,1;\alpha)$. For every fixed parameter α, this yields a one parameter family of realisations, as polynomial differential operators on $2|2$-dimensional superspace. We also determine when the corresponding representation on polynomials is irreducible. This reveals information on the expected structure of orbits on which representations can be constructed using the methods of [HKM] and yields a candidate for the minimal representation.

This chapter is organised as follows. In Section 4.2 we introduce some notations, define the symmetrisation map and explain what we mean by polynomial realisations. In Section 4.3 we study the universal enveloping algebra of Lie superalgebras and use this to construct useful embeddings of Lie superalgebras in (completions of) Weyl superalgebras. In Section 4.4 we carry out some technical calculations concerning enveloping algebras, which are essential for the construction in Section 4.3. In Section 4.5 we use a specific example of the aforementioned realisations, in the case of a 3-graded Lie superalgebra, to define the Bessel operators for the associated Jordan superpair. We show that this generalises the known Bessel operators for unital Jordan algebras. In Section 4.6 we use the results of Section 4.3 to construct a compact explicit realisation of the Lie superalgebras $D(2,1;\alpha)$.

4.2 Symmetrisation and polynomial realisations

4.2.1 Some conventions

In this chapter, we will always work over the field $\mathbb{K}$ which is either $\mathbb{R}$ or $\mathbb{C}$. Unless specified otherwise, Lie superalgebras and Jordan superalgebras are assumed to be finite-dimensional.

A $\mathbb{Z}$-grading of a Lie superalgebra $\mathfrak{g}$ is a decomposition as super-vector spaces

$$\mathfrak{g} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}_i, \tag{4.1}$$

where $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$. For such a grading we put

$$\mathfrak{g}_+ = \bigoplus_{i>0} \mathfrak{g}_i, \qquad \mathfrak{g}_- = \bigoplus_{i<0} \mathfrak{g}_i \qquad \text{and } \mathfrak{l} = \mathfrak{g}_0 \oplus \mathfrak{g}_+.$$

When $\mathfrak{g}_+ = \mathfrak{g}_1$ and $\mathfrak{g}_- = \mathfrak{g}_{-1}$, we say that $\mathfrak{g}$ is 3-graded. The grading in (4.1) is not to be confused with the $\mathbb{Z}_2$-grading $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$.

By a character of a $\mathbb{K}$-Lie superalgebra we mean an (even) Lie superalgebra morphism to $\mathbb{K}$. Note that we can extend any character λ of $\mathfrak{g}_0$, in the notation of (4.1), to a character $\lambda : \mathfrak{l} \to \mathbb{K}$ by setting $\lambda(\mathfrak{g}_+) = 0$. We will often silently make this identification. Moreover, for a character λ of $\mathfrak{g}_0$ we consider the one-dimensional $\mathfrak{l}$-module $\mathbb{K}_\lambda$, where $Xv = \lambda(X)v$ for all $X \in \mathfrak{l}$ and $v \in \mathbb{K}_\lambda$.

We will use (m, n)-multiple indices, which we define as

$$\mathbb{N}^{m|n} = \mathbb{N}^m \times \{0, 1\}^n, \quad K = (k_1, k_2, \ldots, k_m | k_{m+1}, \ldots, k_{m+n}) \in \mathbb{N}^{m|n}$$

and for which we introduce the notation

$$|K| = \sum_{i=1}^{m+n} k_i \qquad \text{and} \qquad K! = k_1! \cdots k_m!.$$

For $K, L \in \mathbb{N}^{m|n}$, we say that $L < K$ if $l_i \leq k_i$ for $i = 1, \ldots, m+n$ and $L \neq K$.

The Bernoulli numbers B_i are recursively defined by [IR, Chapter 15]

$$B_i = -\sum_{j=0}^{i-1} \binom{i}{j} \frac{B_j}{i-j+1} \text{ for } i > 1, \ B_0 = 1, \text{ and } \ B_1 = -1/2. \quad (4.2)$$

We will use the convention

$$\mathrm{ad}_Z(Y) := [Z, Y] \quad \text{and} \quad \widetilde{\mathrm{ad}}_Z(Y) := [Y, Z],$$

where $[\cdot, \cdot]$ denotes the Lie superbracket in case Y and Z are elements of a Lie superalgebra, or the supercommutator in case Y and Z are elements of an associative superalgebra.

4.2.2 Symmetrisation

Let A be an associative $\mathbb{K}$-superalgebra and let $S(A)$ be the supersymmetric algebra of the super-vector space A. To distinguish between multiplication in A and $S(A)$, we denote the product of two elements $\alpha, \beta \in S(A)$ by $\alpha \bullet \beta = (-1)^{|\alpha||\beta|}\beta \bullet \alpha$.

Consider the symmetrisation map σ from $S(A)$ to A. On elements of the form $a_1 \bullet \cdots \bullet a_{p+q}$ with homogeneous $a_i \in A$, ordered such that a_i is even for $i \leq p$ and odd for $i > p$, we have

$$\sigma(a_1 \bullet \cdots \bullet a_p \bullet a_{p+1} \bullet \cdots \bullet a_{p+q}) = \tfrac{1}{(p+q)!} \sum_{\tau \in \mathfrak{S}_{p+q}} (-1)^{|\tau|} a_{\tau(1)} \cdots a_{\tau(p+q)},$$

where $\mathfrak{S}_{p+q}$ is the permutation group of $p+q$ objects and $|\tau|$ is defined as in Equation 2.1.

The restriction of σ from $S(U(\mathfrak{g}))$ to $S(\mathfrak{g})$ yields a vector space isomorphism

$$\sigma : \ S(\mathfrak{g}) \tilde{\to} U(\mathfrak{g}). \quad (4.3)$$

In the multiplication on $S(\mathfrak{g})$ we will leave out $\bullet$ as there is no ambiguity.

A derivation D on a $\mathbb{K}$-superalgebra A is some $D \in \mathrm{End}_{\mathbb{K}}(A)$ satisfying the super Leibniz rule, see Definition 2.1.8. Let V be a finite-dimensional super-vector space and $X_1, \ldots, X_{m+n}$ a basis for V, where X_i is even for $i \leq m$ and odd for $i > m$. We define the partial derivatives ∂^i, as the unique derivation on the superalgebra $S(V)$ satisfying $\partial^i X_j = \delta_{ij}$. They satisfy the relations $\partial^i \partial^j = (-1)^{|X_i||X_j|} \partial^j \partial^i$ and hence canonically generate $S(V^*)$. We define a basis of $S(V^*)$ by

$$\partial^K = (\partial^1)^{k_1} \cdots (\partial^{m+n})^{k_{m+n}} \quad \text{with} \quad K \in \mathbb{N}^{m|n}.$$

In the next chapter we will work not only with polynomials but with smooth superfunctions on a (real) supermanifold. Therefore we extend our definition of derivatives if $\mathbb{K}$ is real. For a real super-vector space U, and for a homogeneous basis $X_j \in U^*$, the partial derivatives ∂^i defined in the above paragraph are elements of $\Gamma(\mathcal{D}_{\mathbb{A}(U)})$. Together with the elements of $\mathcal{O}_{\mathbb{A}(U)}$ they generate $\mathcal{D}_{\mathbb{A}(U)}$. Here $\mathbb{A}(U)$ is the affine superspace associated to U, see Appendix A.

4.2.3 Polynomial realisations

Consider a finite-dimensional $\mathbb{K}$-super-vector space V. We define the (super) Weyl algebra, also known as the Weyl–Clifford algebra, $\mathscr{A}(V)$ as the $\mathbb{K}$-subalgebra of $\mathrm{End}_{\mathbb{K}}(S(V))$ generated by multiplication with elements of V and the derivations on the algebra $S(V)$. In particular we have a natural identification of super-vector spaces

$$\mathscr{A}(V) \;\cong\; S(V) \otimes S(V^*) \;\subset \mathrm{End}(S(V)), \tag{4.4}$$

where V^* is interpreted as the space spanned by the partial derivatives. When we take $V = \mathbb{K}^{n|m}$ we denote this by $\mathscr{A}_{n|m}(\mathbb{K}) = \mathscr{A}(\mathbb{K}^{n|m})$. We will consider $\mathscr{A}(V)$ both as an associative algebra and as an infinite-dimensional Lie superalgebra with bracket given by the supercommutator. Note that we have a canonical embedding of $\mathscr{A}(V)$ into $\Gamma(\mathcal{D}_{\mathbb{A}(V^*)})$ for V real.

We define a polynomial realisation of a $\mathbb{K}$-Lie superalgebra $\mathfrak{g}$ to be an injective Lie superalgebra morphism $\phi : \mathfrak{g} \hookrightarrow \mathscr{A}_{n|m}(\mathbb{K})$ for some $n, m \in \mathbb{N}$. Note that if $\mathfrak{g}$ admits a faithful representation on a finite-dimensional vector space V then there is automatically a realisation in $\mathscr{A}(V)$, contained in $V \otimes V^*$ under (4.4). This is referred to as a matrix realisation.

Clearly the canonical representation of $\mathscr{A}(V)$ on $S(V)$ is faithful. This implies that for any associative algebra A with algebra morphism $\phi : A \to \mathscr{A}(V)$, the annihilator ideal in A of the induced representation on $S(V)$ is given by the kernel of ϕ.

We consider the Krull topology on $S(V^*)$, with respect to the maximal ideal of polynomials cancelling $0 \in V$. We define $\widehat{S}(V^*) \subset \mathrm{End}(S(V))$ as the completion of $S(V^*)$. Hence, $\widehat{S}(V^*)$ is the ring of formal power series corresponding to the polynomial ring $S(V^*)$. Finally we denote the subalgebra of $\mathrm{End}(S(V))$ generated by $\mathscr{A}(V)$ and $\widehat{S}(V^*)$ by $\widehat{\mathscr{A}(V)}$.

4.3 Construction of small polynomial realisations for Lie superalgebras

Our main result of this section is summarised in the following theorem.

Theorem 4.3.1. *Consider a finite-dimensional $\mathbb{K}$-Lie superalgebra $\mathfrak{g}$.*

1. *There is a Lie superalgebra embedding $\mathfrak{g} \hookrightarrow \widehat{\mathscr{A}(\mathfrak{g})}$.*
2. *For any $\mathbb{Z}$-grading of $\mathfrak{g}$, with $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{l}$ and any character λ of $\mathfrak{g}_0$, there is a Lie superalgebra morphism*

$$\phi_\lambda : \mathfrak{g} \to \mathscr{A}(\mathfrak{g}_-),$$

which is injective if and only if $U(\mathfrak{g}) \otimes_{U(\mathfrak{l})} \mathbb{K}_\lambda$ is a faithful $\mathfrak{g}$-module.

In case $\mathfrak{g}$ is a Lie algebra, Theorem 4.3.1(1) follows from Theorem 3 in [Be], while Theorem 4.3.1(2) can be obtained from (the proof of) Proposition 2.2 in [Co].

Furthermore, we will determine the explicit form of ϕ_λ in Theorem 4.3.1(2) in case of a 3-grading, which is new also for Lie algebras. In Section 4.5, this will lead to the natural appearance of Bessel operators, containing the ones in [Di, FK, HKM, KØ, Sa] as a special case.

4.3.1 Super version of a result of Berezin

In this section we generalise the approach of [Be] to superalgebras, where we work out most of the technical details in the next section, Section 4.4. We consider the left regular representation of a Lie superalgebra $\mathfrak{g}$ on $U(\mathfrak{g})$ by left multiplication. Using the isomorphism (4.3), this yields a $\mathfrak{g}$-representation π on $S(\mathfrak{g})$.

We prove that the image of $\pi : \mathfrak{g} \to \mathrm{End}(S(\mathfrak{g}))$ is actually contained in $\widehat{\mathscr{A}}(\mathfrak{g})$ and obtain an explicit expression in the following theorem. Therefore we choose a basis X_i, $1 \leq i \leq m+n$ of $\mathfrak{g}$, where X_i is even if $i \leq m$ and odd if $i > m$.

Theorem 4.3.2. *The action π defined by*

$$\pi(X)Y := \sigma^{-1}(X\sigma(Y)) \qquad \text{for all } X \in \mathfrak{g}, \quad Y \in S(\mathfrak{g})$$

is given by

$$\pi(X)Y = \sum_{K \in \mathbb{N}^{m|n}} \frac{(-1)^{|K|} B_{|K|}}{K!} s_{\mathfrak{g}}^K(X)\, \partial^K Y, \tag{4.5}$$

where the operator $s_{\mathfrak{g}}^K \in \mathrm{End}(\mathfrak{g})$ is defined as

$$s_{\mathfrak{g}}^K(X) := \sigma\left(\widetilde{\mathrm{ad}}_{X_1}^{\bullet k_1} \bullet \cdots \bullet \widetilde{\mathrm{ad}}_{X_{m+n}}^{\bullet k_{m+n}}\right) X \qquad \text{for all } X \in \mathfrak{g}, \tag{4.6}$$

and with $B_{|K|}$ the Bernoulli numbers (4.2).

First we observe that this theorem implies Theorem 4.3.1(i). Indeed, the expression in equation (4.5) confirms that $\pi(X) \in \widehat{\mathscr{A}}(\mathfrak{g})$ for any $X \in \mathfrak{g}$. Furthermore, the injectivity of $\pi : \mathfrak{g} \to \widehat{\mathscr{A}}(\mathfrak{g})$ follows immediately from the fact that the left regular representation (and hence π) is faithful.

Proof of Theorem 4.3.2. For Y of degree one, i.e. $Y \in \mathfrak{g}$, the claim reduces to

$$X\sigma(Y) = \sigma(XY) + \frac{1}{2}\sigma([X,Y]),$$

which is clearly true. Now assume (4.5) holds for all elements in $S(\mathfrak{g})$ which have lower degree than Y. We can rewrite Lemma 4.4.1 as

$$X\sigma(Y) = \sigma(XY) - \sum_{K>0} \frac{(-1)^{|K|}}{(|K|+1)K!} s_{\mathfrak{g}}^K(X)\sigma(\partial^K Y).$$

Since the degree of $\partial^K Y$ is now strictly lower than the degree of Y we can apply our induction hypothesis on

$$s_{\mathfrak{g}}^K(X)\sigma(\partial^K Y) = \sigma(\pi(s_{\mathfrak{g}}^K(X))\partial^K Y).$$

This leads to

$$\begin{aligned}
&X\sigma(Y)\\
&= \sigma(XY) - \sum_{K>0} \tfrac{(-1)^{|K|}}{(|K|+1)K!} \sum_{L\in\mathbb{N}^{m|n}} \tfrac{(-1)^{|L|}B_{|L|}}{L!}\sigma\left(s_{\mathfrak{g}}^L \circ s_{\mathfrak{g}}^K(X)\,\partial^L\partial^K Y\right)\\
&= \sigma(XY) - \sum_{K>0}\sum_{L<K} \tfrac{(-1)^{|K-L|+|L|}B_{|L|}}{(|K-L|+1)(K-L)!L!}\sigma\left(s_{\mathfrak{g}}^L \circ s_{\mathfrak{g}}^{K-L}(X)\,\partial^L\partial^{K-L} Y\right).
\end{aligned}$$

From Lemma 4.4.3, we have

$$\begin{aligned}
\sum_{L<K} \frac{(-1)^{|K-L|+|L|}B_{|L|}}{(|K-L|+1)(K-L)!L!} s_{\mathfrak{g}}^L \circ s_{\mathfrak{g}}^{K-L}(X)\,\partial^L\partial^{K-L}Y&\\
= -\frac{(-1)^{|K|}}{K!}B_{|K|}s_{\mathfrak{g}}^K(X)\partial^K(Y)&.
\end{aligned}$$

Using this, we obtain

$$X\sigma(Y) = \sigma(XY) + \sum_{K>0}\frac{(-1)^K B_{|K|}}{K!}\sigma(s_{\mathfrak{g}}^K(X)\partial^K Y), \tag{4.7}$$

which proves the theorem. □

Corollary 4.3.3. *We have*

$$\sigma(XY) = (-1)^{|X||Y|}\sigma(Y)X + \sum_{K>0}\frac{(-1)^{|K|}C_{|K|}}{K!}\sigma(s_{\mathfrak{g}}^K(X)\,\partial^K Y),$$

where $B_i = -C_i$ *for* $i \geq 2$ *and* $C_1 = B_1 = -1/2$.

Proof. Follows immediately from Lemma 4.4.2 and equation (4.7). □

4.3.2 A method to construct small polynomial realisations

Let $\mathfrak{g}$ be a finite-dimensional $\mathbb{Z}$-graded Lie superalgebra, where we maintain the notation of Subsection 4.2. We consider a character

$\lambda : \mathfrak{g}_0 \to \mathbb{K}$, interpreted as a character of $\mathfrak{l}$. For any such λ we will use Theorem 4.3.2 to construct a realisation of $\mathfrak{g}$ on $S(\mathfrak{g}_-)$, by reinterpreting the parabolic Verma module of scalar type $U(\mathfrak{g}) \otimes_{U(\mathfrak{l})} \mathbb{K}_\lambda$.

As vector spaces we have

$$U(\mathfrak{g}) \otimes_{U(\mathfrak{l})} \mathbb{K}_\lambda \cong U(\mathfrak{g}_-) \otimes \mathbb{K} \cong S(\mathfrak{g}_-).$$

The $\mathfrak{g}$-representation on $U(\mathfrak{g}) \otimes_{U(\mathfrak{l})} \mathbb{K}_\lambda$ hence yields a $\mathfrak{g}$-representation π on $S(\mathfrak{g}_-)$ using the symmetrisation map σ and

$$\mu \colon S(\mathfrak{g}_-) \otimes \mathbb{K} \tilde{\to} S(\mathfrak{g}_-)\ ;\ Y \otimes a \to aY.$$

We will prove that the image of π is contained in $\mathscr{A}(\mathfrak{g}_-)$.

Let $(r|s)$ be the dimension of $\mathfrak{g}_-$ and choose a basis X_i with $1 \leq i \leq r$ of the even part and a basis X_{j+r} with $1 \leq j \leq s$ of the odd part. To be able to give an explicit expression of this action we define the following elements of $\mathfrak{g}_-$ and $\mathfrak{l}$. For $1 \leq i \leq r+s$ and $X \in \mathfrak{l}$, we set

$$W_i(X) + H_i(X) := [X, X_i],$$

uniquely defined by the condition $W_i(X) \in \mathfrak{g}_-$ and $H_i(X) \in \mathfrak{l}$. For $K_1 \in \mathbb{N}^{r|s} \backslash \{0\}$, we put

$$W_{i,K_1}(X) + H_{i,K_1}(X) := \frac{(-1)^{|K_1|} C_{|K_1|}}{K_1!} s_{\mathfrak{g}_-}^{K_1}(H_i(X)),$$

with $W_{i,K_1}(X) \in \mathfrak{g}_-$ and $H_{i,K_1}(X) \in \mathfrak{l}$. The operator $s_{\mathfrak{g}_-}^{K_1}$ is defined as in (4.6), but now adjoining only the basis elements of $\mathfrak{g}_-$ instead of the whole $\mathfrak{g}$. Recursively we also set, for $K_1, \ldots, K_j \in \mathbb{N}^{r|s} \backslash \{0\}$,

$$\begin{aligned} &W_{i,K_1,\ldots,K_j}(X) + H_{i,K_1,\ldots,K_j}(X) \\ &\qquad := \frac{(-1)^{|K_j|} C_{|K_j|}}{K_j!} s_{\mathfrak{g}_-}^{K_j}(H_{i,K_1,\ldots,K_{j-1}}(X)), \end{aligned}$$

where again $W_{i,K_1,\ldots,K_j}(X) \in \mathfrak{g}_-$ and $H_{i,K_1,\ldots,K_j}(X) \in \mathfrak{l}$.

Theorem 4.3.4. *For any character $\lambda : \mathfrak{g}_0 \to \mathbb{K}$, the action π of $\mathfrak{g}$ on $S(\mathfrak{g}_-)$ defined by*

$$\pi(X)Y := \mu\left(\sigma^{-1} \otimes \mathrm{id}(X(\sigma(Y) \otimes 1))\right), \qquad Y \in S(\mathfrak{g}_-),$$

is given by

- $X \in \mathfrak{g}_-$ $\quad \pi(X) = X + \sum_{K>0} \frac{(-1)^{|K|} B_{|K|}}{K!} s^K_{\mathfrak{g}_-}(X)\, \partial^K$
- $X \in \mathfrak{g}_0$ $\quad \pi(X) = \lambda(X) + \sum_{i=1}^{r+s} [X, X_i] \partial^i$
- $X \in \mathfrak{g}_+$

$$\pi(X) = \sum_{i=1}^{r+s} \sum_{j=0}^{l} \sum_{K_1>0} \cdots \sum_{K_j>0} W_{i,K_1,\dots K_j}(X) \partial^{K_j} \cdots \partial^{K_1} \partial^i$$
$$+ \sum_{i=1}^{r+s} \sum_{j=0}^{l-1} \sum_{K_1>0} \cdots \sum_{K_j>0} \lambda(H_{i,K_1,\dots K_j}(X)) \partial^{K_j} \cdots \partial^{K_1} \partial^i$$

where the X_i form a homogeneous basis of $\mathfrak{g}_-$.

This theorem implies Theorem 4.3.1(ii). The expressions for $\pi(X)$ show that $\pi(X) \in \widehat{\mathscr{A}}(\mathfrak{g}_-)$. As $\mathfrak{g}$ is finite-dimensional, only a finite number of terms in its $\mathbb{Z}$-grading are non-zero. This implies that $s^K_{\mathfrak{g}_-}(X) = 0$ for any $X \in \mathfrak{g}_-$, for $|K|$ sufficiently large and likewise only a finite number of $H_{i,K_1,\dots K_j}(X)$ (and hence $W_{i,K_1,\dots K_j}(X)$) is non-zero for $X \in \mathfrak{g}_+$. Consequently $\pi(X) \in \mathscr{A}(\mathfrak{g}_-)$ for all $X \in \mathfrak{g}$.

Proof of Theorem 4.3.4. Let X be an element of $\mathfrak{g}_-$, Y an element of $S(\mathfrak{g}_-)$ and extend the basis (X_i) to a basis of $\mathfrak{g}$. Applying Theorem 4.3.2, we obtain

$$X\sigma(Y) = \sigma(XY) + \sum_{K \in \mathbb{N}^{m|n}, K>0} \frac{(-1)^{|K|} B_{|K|}}{K!} \sigma\left(s^K_{\mathfrak{g}}(X)\, \partial^K Y\right).$$

Since $Y \in S(\mathfrak{g}_-)$, the derivative $\partial^i Y = 0$ for all $i > r+s$. Therefore we can restrict to $K \in \mathbb{N}^{r|s}$ in the summation, and

$$\pi(X)Y = XY + \sum_{K>0} \frac{(-1)^{|K|} B_{|K|}}{K!} s^K_{\mathfrak{g}_-}(X)\, \partial^K Y.$$

Now, let X be a homogeneous element of $\mathfrak{g}_0$. Using Lemma 4.4.2, we find

$$X\sigma(Y) \otimes 1 = [X, \sigma(Y)] \otimes 1 + (-1)^{|X||Y|} \sigma(Y) X \otimes 1$$

$$= \left(\sum_{i=1}^{r+s} \sigma \left([X, X_i] \partial^i Y \right) + \lambda(X)\sigma(Y) \right) \otimes 1,$$

where we again restricted our summation to elements in $\mathfrak{g}_-$ since the partial derivatives of Y with respect to elements in $\mathfrak{l}$ are zero. We also used that $\lambda(X) \neq 0$ implies that $|X| = 0$ since λ is an even morphism. This proves the claim for X in $\mathfrak{g}_0$.

Finally, let X be a homogeneous element of $\mathfrak{g}_+$. From now on we will also drop the X in $W_{i,K_1,\dots K_j}(X)$ and $H_{i,K_1,\dots K_j}(X)$ for ease of notation. Using Lemma 4.4.2 and Corollary 4.3.3, we get

$$\begin{aligned}
X\sigma(Y) \otimes 1 &= \sigma(\sum_{i=1}^{r+s} [X, X_i] \partial^i Y) \otimes 1 + (-1)^{|X||Y|} \sigma(Y) X \otimes 1 \\
&= \sigma(\sum_{i=1}^{r+s} W_i \partial^i Y) \otimes 1 + \sum_{i=1}^{r+s} (-1)^{(|X|+|X_i|)|\partial^i Y|} \sigma(\partial^i Y) H_i \otimes 1 \\
&\quad + \sum_{i=1}^{r+s} \sum_{K_1 > 0} \frac{(-1)^{|K_1|} C_{|K_1|}}{K_1!} \sigma(s_{\mathfrak{g}_-}^{K_1}(H_i)\, \partial^{K_1} \partial^i Y) \otimes 1 \\
&= \sum_{i=1}^{r+s} \sigma(W_i \partial^i Y) \otimes 1 + \sum_{i=1}^{r+s} \lambda(H_i) \sigma(\partial^i Y) \otimes 1 \\
&\quad + \sum_{i=1}^{r+s} \sum_{K_1 > 0} \sigma\left((W_{i,K_1} + H_{i,K_1})\, \partial^{K_1} \partial^i Y \right) \otimes 1.
\end{aligned}$$

We will now repeatedly apply Corollary 4.3.3 to the part in $H_{i,K_1,\dots,K_j}$. This procedure finishes after l steps, since $H_{i,K_1,\dots K_l} = 0$. Thus

$$\begin{aligned}
X\sigma(Y) \otimes 1 &= \sum_{i=1}^{r+s} \sigma(W_i \partial^i Y) \otimes 1 + \sum_{i=1}^{r+s} \lambda(H_i) \sigma(\partial^i Y) \otimes 1 \\
&+ \sum_{i=1}^{r+s} \sum_{j=1}^{l} \sum_{K_1 > 0} \cdots \sum_{K_j > 0} \sigma(W_{i,K_1,\dots K_j}\, \partial^{K_j} \cdots \partial^{K_1} \partial^i Y) \otimes 1 \\
&+ \sum_{i=1}^{r+s} \sum_{j=1}^{l-1} \sum_{K_1 > 0} \cdots \sum_{K_j > 0} \lambda(H_{i,K_1,\dots K_j}) \sigma(\partial^{K_j} \cdots \partial^{K_1} \partial^i Y) \otimes 1.
\end{aligned}$$

This concludes the proof. $\square$

Remark 4.3.5. If $\mathfrak{g}$ is 3-graded, $\pi : \mathfrak{g} \to \mathscr{A}(\mathfrak{g}_{\text{-}})$ of Theorem 4.3.4 simplifies to

1. $X \in \mathfrak{g}_{-1}$ $\quad \pi(X) = X$
2. $X \in \mathfrak{g}_0$ $\quad \pi(X) = \lambda(X) + \sum_{i=1}^{r+s}[X, X_i]\partial^i$
3. $X \in \mathfrak{g}_{+1}$

$$\pi(X) = \sum_{i=1}^{r+s} \lambda([X, X_i])\partial^i + \frac{1}{2}\sum_{i,j=1}^{r+s}[[X, X_i], X_j]\partial^j\partial^i.$$

4.4 Three technical lemmata

In this section we obtain several technical results concerning the operator $s_{\mathfrak{g}}^K \in \mathrm{End}_{\mathbb{K}}(\mathfrak{g})$ of Section 4.3, for $\mathfrak{g}$ a $\mathbb{K}$-Lie superalgebra of dimension $m|n$ and $K \in \mathbb{N}^{m|n}$, defined as

$$s_{\mathfrak{g}}^K(X) := \sigma\left(\widetilde{\mathrm{ad}}_{X_1}^{\bullet k_1} \bullet \cdots \bullet \widetilde{\mathrm{ad}}_{X_{m+n}}^{\bullet k_{m+n}}\right) X \qquad \text{for all } X \in \mathfrak{g}, \tag{4.8}$$

where $\{X_i, 1 \leq i \leq m\}$ constitutes a basis of $\mathfrak{g}_{\bar{0}}$ and $\{X_{m+j,1\leq j\leq n}\}$ a basis of $\mathfrak{g}_{\bar{1}}$.

Lemma 4.4.1. *For $X \in \mathfrak{g}$ and Y in $S(\mathfrak{g})$, the following holds*

$$\sigma(XY) = \sum_{K\in\mathbb{N}^{m|n}} \frac{(-1)^{|K|}}{(|K|+1)K!} s_{\mathfrak{g}}^K(X)\sigma(\partial^K Y). \tag{4.9}$$

Proof. By linearity it suffices to consider $Y \in S(\mathfrak{g})$ of the form

$$Y = X_1^{\alpha_1} X_2^{\alpha_2} \cdots X_{m+n}^{\alpha_{m+n}}, \tag{4.10}$$

for some $\alpha \in \mathbb{N}^{m|n}$. Put $p = \sum_{i=1}^m \alpha_i$ and $q = \sum_{i=m+1}^{m+n} \alpha_i$. Then we write

$$Y = Z_1 Z_2 \cdots Z_{p+q},$$

where the $Z_i \in \mathfrak{g}$ are defined by

$$Z_1 = Z_2 = \cdots = Z_{\alpha_1} = X_1,$$
$$Z_{\alpha_1+1} = Z_{\alpha_1+2} = \cdots = Z_{\alpha_1+\alpha_2} = X_2,$$

$$\vdots$$
$$Z_{p+q-\alpha_{m+n}+1} = \cdots = Z_{p+q} = X_{m+n}.$$

Remark that Z_i is even for $i \leq p$ and odd for $i > p$.

Since (4.9) is also linear in X, we can assume X to be homogeneous. So, let X be a homogeneous element of $\mathfrak{g}$ and define $p+q$ indeterminates t_i, where t_i is even if $i \leq p$ and odd if $i > p$. Furthermore we define the indeterminate t to be even if X is even and odd if X is odd. Consider the supercommutative algebra T generated by $\{t_i, i = 1, \cdots, p+q\}$ and t. Then we define the following element of $U(\mathfrak{g}) \otimes T$

$$\kappa(t) = Xt + \sum_{i=1}^{p+q} Z_i t_i.$$

By construction we have $|\kappa(t)| = 0$, therefore $[W, \kappa(t)] = W\kappa(t) - \kappa(t)W$ for all $W \in U(\mathfrak{g}) \otimes T$.

We will calculate $\frac{\partial}{\partial t}\kappa(t)^{p+q+1}|_{t=0}$ in two different ways and then compare the term in $t_1 \cdots t_{p+q}$.

On the one hand, we have for $j \in \mathbb{N}$

$$\kappa(t)^j = \sum_{(\sum_{i=0}^{p+q} r_i)=j} \frac{j!}{r_0!r_1!r_2!\cdots r_p!}\, \sigma\left(X^{r_0} Z_1^{r_1} \cdots Z_{p+q}^{r_{p+q}}\right)\, t_{p+q}^{r_{p+q}} \cdots t_1^{r_1} t^{r_0}.$$

Setting $j = p+q+1$, we obtain

$$\begin{aligned} &\frac{\partial}{\partial t}\kappa(t)^{p+q+1}|_{t=0} \\ &= \sum_{(\sum_{i=1}^{p+q} r_i)=p+q} (-1)^{|X|} \frac{(p+q+1)!}{r_1!r_2!\cdots r_p!}\, \sigma\left(X Z_1^{r_1} \cdots Z_{p+q}^{r_{p+q}}\right)\, t_{p+q}^{r_{p+q}} \cdots t_1^{r_1}. \end{aligned}$$

Hence the term in $t_1 \dots t_{p+q}$ is given by

$$(-1)^{|X|}(p+q+1)!\sigma\left(X Z_1 \cdots Z_{p+q}\right)\, t_{p+q} \cdots t_1. \tag{4.11}$$

On the other hand, we can also calculate $\frac{\partial}{\partial t}\kappa(t)^{p+q+1}|_{t=0}$ using the expression

$$\kappa(t)^i W = \sum_{s=0}^{i} \binom{i}{s} \mathrm{ad}^s_{\kappa(t)}(W)\, \kappa(t)^{i-s},$$

which holds for all W in $U(\mathfrak{g}) \otimes T$. Using the super Leibniz rule, we find

$$\begin{aligned}\frac{\partial}{\partial t}\kappa(t)^{p+q+1} &= \sum_{i=0}^{p+q} \kappa(t)^i (-1)^{|X|} X \kappa(t)^{p+q-i}\\ &= (-1)^{|X|} \sum_{s=0}^{p+q} \sum_{i=s}^{p+q} \binom{i}{s} \mathrm{ad}^s_{\kappa(t)}(X) \kappa(t)^{p+q-s}\\ &= (-1)^{|X|} \sum_{s=0}^{p+q} \binom{p+q+1}{s+1} \mathrm{ad}^s_{\kappa(t)}(X) \cdot\\ &\sum_{(\sum_{i=0}^{p+q} r_i)=p+q-s} \frac{(p+q-s)!}{r_0! r_1! r_2! \cdots r_p!} \sigma\left(X^{r_0} Z_1^{r_1} \cdots Z_{p+q}^{r_{p+q}}\right) t_{p+q}^{r_{p+q}} \cdots t_1^{r_1} t^{r_0}.\end{aligned}$$

Setting $t = 0$, we obtain

$$\begin{aligned}\frac{\partial}{\partial t}\kappa(t)^{p+q+1} =& (-1)^{|X|} \sum_{s=0}^{p+q} \sum_{(\sum_{i=1}^{p+q} r_i)=p+q-s} \frac{(p+q+1)!}{(s+1)! r_1! r_2! \cdots r_p!} \cdot\\ & \mathrm{ad}^s_{\kappa(0)}(X) \sigma\left(Z_1^{r_1} \cdots Z_{p+q}^{r_{p+q}}\right) t_{p+q}^{r_{p+q}} \cdots t_1^{r_1}. \qquad (4.12)\end{aligned}$$

We will bring all the terms t_i which are still contained in $\mathrm{ad}^s_{\kappa(0)}$ to the right, so that we can compare it to (4.11). This will create many minus signs which we will calculate in several steps. Let $\{f(1), \ldots, f(s)\} \subset \{1, \ldots, p+q\}$ be a subset which is ordered, i.e. $f(i) < f(j)$ if $i < j$ and let τ be a permutation of $\mathfrak{S}_s$. As we will let τ act on products which are ordered as in Subsection 4.2, we can use the notation $|\tau|$ of equation (2.1). Furthermore we will want to manipulate expressions in a way that ignores the relations between the different Z_i. Therefore we consider the supersymmetric algebra $\mathcal{Z}$ generated by even variables z_i for $1 \le i \le p$ and odd variables z_{p+j} for $1 \le j \le q$. This comes with an algebra morphism $\xi_\alpha : \mathcal{Z} \to S(\mathfrak{g})$ defined by $\xi_\alpha(z_i) = Z_i$. Furthermore we introduce $\sigma_\alpha = \sigma \circ \xi_\alpha : \mathcal{Z} \to U(\mathfrak{g})$.

- Since $Z_j t_j$ is even, we have

$$\begin{aligned}&[Z_{\tau(f(1))} t_{\tau(f(1))}, [\cdots, [Z_{\tau(f(s))} t_{\tau(f(s))}, X] \cdots]]\\ &\quad= (-1)^s [[\cdots [X, Z_{\tau(f(s))} t_{\tau(f(s))}], \cdots], Z_{\tau(f(1))} t_{\tau(f(1))}]\\ &\quad= (-1)^s [[\cdots [X, Z_{\tau(f(s))}], \cdots], Z_{\tau(f(1))}]\ \ t_{\tau(f(1))} \cdots t_{\tau(f(s))}.\end{aligned}$$

- Denote by $\hat{Z}^f$ the product $Z_1 Z_2 \cdots Z_{p+q}$ after all terms in

$$\{Z_{f(i)} \,|\, i = 1, \cdots, s\}$$

are omitted and similarly by $\hat{t}^f$ the product $t_{p+q} \cdots t_1$ after removing $\{t_{f(i)} \,|\, i = 1, \cdots, s\}$

$$\begin{aligned} & t_{\tau(f(1))} \cdots t_{\tau(f(s))} \sigma(\hat{Z}^f)\, \hat{t}^f \\ & = \sigma \left(\xi_\alpha(\partial_{z_{\tau(f(1))}} \cdots \partial_{z_{\tau(f(s))}} z_{\tau(f(s))} \cdots z_{\tau(f(1))}) \hat{Z}^f \right) \hat{t}^f t_{\tau(f(1))} \cdots t_{\tau(f(s))} \\ & = \sigma_\alpha \left(\partial_{z_{\tau(f(1))}} \cdots \partial_{z_{\tau(f(s))}} z_1 \cdots \cdots z_{p+q} \right) t_{p+q} \cdots t_1. \end{aligned}$$

- Finally

$$\partial_{z_{\tau(f(1))}} \cdots \partial_{z_{\tau(f(s))}} = (-1)^{|\tau|} \partial_{z_{f(1)}} \cdots \partial_{z_{f(s)}}.$$

Combining these three calculations we conclude

$$\begin{aligned} & [Z_{\tau(f(1))} t_{\tau(f(1))}, [\cdots, [Z_{\tau(f(s))} t_{\tau(f(s))}, X] \cdots]]\, \sigma(\hat{Z}^f)\, \hat{t}^f \\ & = (-1)^{\delta(\tau,f)} [[\cdots [X, Z_{\tau(f(s))}], \cdots], Z_{\tau(f(1))}] \cdot \\ & \qquad \sigma_\alpha(\partial_{z_{f(1)}} \cdots \partial_{z_{f(s)}} z_1 \cdots z_{p+q})\, t_{p+q} \cdots t_1, \end{aligned}$$

where $\delta(\tau, f) = s + |\tau|$.

Therefore the term of $\frac{\partial}{\partial t} \kappa(t)^{p+q+1}|_{t=0}$ in $t_1 \cdots t_{p+q}$ is given by

$$\begin{aligned} & (-1)^{|X|} \sum_{s=0}^{p+q} \sum_{\substack{f \\ |f|=s}} \sum_{\tau} (-1)^{\delta(\tau,f)} \tfrac{(p+q+1)!}{(s+1)!} [[\cdots [X, Z_{\tau(f(s))}], \cdots], Z_{\tau(f(1))}] \\ & \sigma_\alpha(\partial_{z_{f(1)}} \cdots \partial_{z_{f(s)}} z_1 \cdots z_{p+q})\, t_{p+q} \cdots t_1, \end{aligned} \tag{4.13}$$

where we sum over all ordered subsets f and all possible permutations $\tau \in \mathfrak{S}_s$. By construction, (4.11) and (4.13) are identical, which implies

$$\begin{aligned} \sigma\left(X Z_1 \cdots Z_{p+q}\right) = & \sum_{s=0}^{p+q} \sum_{\substack{f \\ |f|=s}} \sum_{\tau} \tfrac{(-1)^{\delta(\tau,f)}}{(s+1)!} [[\cdots [X, Z_{\tau(f(s))}], \cdots], Z_{\tau(f(1))}] \\ & \sigma_\alpha(\partial_{z_{f(1)}} \cdots \partial_{z_{f(s)}} z_1 \cdots z_{p+q}). \end{aligned} \tag{4.14}$$

To write this in the proposed form, we associate with each f the unique $K_f = (k_1, \ldots, k_{m+n}) \in \mathbb{N}^{m|n}$ which satisfies

$$X_1^{k_1} \cdots X_{m+n}^{k_{m+n}} = Z_{f(1)} \cdots Z_{f(s)},$$

so in particular $|K_f| = s$. This definition implies

$$\xi_\alpha(\partial_{z_{f(1)}} \cdots \partial_{z_{f(s)}} z_1 \cdots z_{p+q}) = \frac{(\alpha - K_f)!}{\alpha!} \partial^{K_f} Y,$$

which allows to rewrite equation (4.14) as

$$\sigma(XY)$$
$$= \sum_{s=0}^{p+q} \sum_{\substack{f \\ |f|=s}} \sum_{\tau} \tfrac{(-1)^{\delta(\tau,f)}(\alpha - K_f)!}{(|K_f|+1)!\alpha!} [[\cdots [X, Z_{\tau(f(s))}], \cdots], Z_{\tau(f(1))}] \sigma(\partial^{K_f} Y).$$

Introducing the symmetrisation map σ then yields

$$\sigma(XY) = \sum_{s=0}^{p+q} \sum_{\substack{f \\ |f|=s}} \frac{(-1)^{|K_f|}(\alpha - K_f)! |K_f|!}{(|K_f|+1)!\alpha!} s_{\mathfrak{g}}^{K_f}(X)\, \sigma(\partial^{K_f} Y).$$

It hence remains to interpret the summation in the right-hand side and compare to the one in equation (4.9). Concretely we need to consider the map $q : f \mapsto K_f$. Firstly, this map implies that the summation in the above is not over all $K \in \mathbb{N}^{m|n}$, but only over K such that $K \leq \alpha$. However, when that condition on K is not satisfied we have $\partial^K Y = 0$. Secondly, the map is not injective. When $K \leq \alpha$, the cardinality of $q^{-1}(K)$ is clearly $\alpha!/(K!(\alpha - K)!)$. Hence we obtain precisely equation (4.9). □

Lemma 4.4.2. *Let $\mathfrak{g}$ be a Lie superalgebra with basis X_i, $1 \leq i \leq m+n$ and Y be an element of $S(\mathfrak{g})$. Then*

$$[X, \sigma(Y)] = \sigma\left(\sum_{i=1}^{m+n} [X, X_i] \partial^i Y\right) = \sigma\left(\sum_{|K|=1} s_{\mathfrak{g}}^K(X) \partial^K Y\right). \quad (4.15)$$

Proof. By linearity we can again assume X to be homogeneous and Y to be of the form $X_1^{\alpha_1} \cdots X_{m+n}^{\alpha_{m+n}}$. We will again write Y as $Z_1 \cdots Z_{p+q}$, where the Z_i are defined in the same way as in the proof of Lemma 4.4.1.

Assume X to be odd. The case X even can be shown in a similar way. Starting from the left-hand side of (4.15), we get

$$\begin{aligned}[X,\sigma(Y)] &= \tfrac{1}{(p+q)!}\sum_{\tau\in\mathfrak{S}_{p+q}}(-1)^{|\tau|}[X, Z_{\tau(1)}Z_{\tau(2)}\cdots Z_{\tau(p+q)}]\\ &= \tfrac{1}{(p+q)!}\sum_{\tau\in\mathfrak{S}_{p+q}}\sum_{i=1}^{p+q}(-1)^{\alpha(\tau,i)}Z_{\tau(1)}\cdots Z_{\tau(i-1)}[X,Z_{\tau(i)}]\\ &\quad Z_{\tau(i+1)}\cdots Z_{\tau(p+q)}, \qquad (4.16)\end{aligned}$$

where

$$\alpha(\tau,i)=\sum_{j=1}^{i-1}[\tau(j)>p]+\sum_{l=p+1}^{p+q-1}\sum_{j=l+1}^{p+q}[\tau^{-1}(l)>\tau^{-1}(j)].$$

We can rewrite the right-hand side of (4.15) using the notation

$$Z_1\cdots\hat{Z}_k\cdots Z_{p+q}$$

for the product of all Z_i without the term Z_k as

$$\begin{aligned}&\sigma\left(\sum_{i=1}^{m+n}[X,X_i]\partial^i Y\right)\\ &= \sigma\left(\sum_{k=1}^{p}[X,Z_k]Z_1\cdots\hat{Z}_k\cdots Z_{p+q}\right)\\ &\quad+\sigma\left(\sum_{k=p+1}^{p+q}(-1)^{k-1-p}[X,Z_k]Z_1\cdots\hat{Z}_k\cdots Z_{p+q}\right)\\ &= \frac{1}{(p+q)!}\sum_{k=1}^{p}\sum_{\tau\in\mathfrak{S}_{p+q}}(-1)^{\beta(\tau,k)}\bar{Z}^{(k)}_{\tau(1)}\cdots\bar{Z}^{(k)}_{\tau(p+q)}\\ &\quad+\frac{1}{(p+q)!}\sum_{k=p+1}^{p+q}\sum_{\tau\in\mathfrak{S}_{p+q}}(-1)^{\gamma(\tau,k)}\bar{Z}^{(k)}_{\tau(1)}\cdots\bar{Z}^{(k)}_{\tau(p+q)},\end{aligned}$$

where

- $\bar{Z}^{(k)}_i=\begin{cases}Z_i & \text{for } i\neq k\\ [X,Z_i] & \text{for } i=k.\end{cases}$

- $\beta(\tau,k) = \sum_{l=p+1}^{p+q}[\tau^{\text{-}1}(l) < \tau^{\text{-}1}(k)] + \sum_{l=p+1}^{p+q-1}\sum_{r=l+1}^{p+q}[\tau^{\text{-}1}(l) > \tau^{\text{-}1}(r)]$
- $\gamma(\tau,k) = k-1-p+\sum_{l=p+1,\, l\neq k}^{p+q}\sum_{r=l+1,\, r\neq k}^{p+q}[\tau^{\text{-}1}(l) > \tau^{\text{-}1}(r)]$.

One can calculate that $\beta(\tau,k) = \gamma(\tau,k) = \alpha(\tau,\tau^{\text{-}1}(k))$. Rewriting (4.16) as

$$[X,\sigma(Y)] = \tfrac{1}{(p+q)!}\sum_{k=1}^{p+q}\sum_{\tau\in\mathfrak{S}_{p+q}}(-1)^{\alpha(\tau,\tau^{\text{-}1}(k))}Z_{\tau(1)}\cdots[X,Z_k]\cdots Z_{\tau(p+q)},$$

concludes the proof. □

Lemma 4.4.3. *For any $K\in\mathbb{N}^{m|n}$, we have*

$$\sum_{L<K}\frac{B_{|L|}K!}{(|K-L|+1)(K-L)!L!}s_{\mathfrak{g}}^{L}s_{\mathfrak{g}}^{K-L}(X)\,\partial^L\partial^{K-L}Y = -B_{|K|}s_{\mathfrak{g}}^{K}(X)\partial^K(Y).$$

Proof. For any $K,L\in\mathbb{N}^{m|n}$ with $L<K$, we define $\gamma_{K,L}\in\mathbb{Z}_2$ by

$$\partial^L\partial^{K-L} = (-1)^{\gamma_{K,L}}\partial^K.$$

We claim that for every $i<|K|$,

$$|K|!s_{\mathfrak{g}}^{K}(X) = \sum_{L<K,|L|=i}|L|!|(K-L)|!\binom{K}{L}(-1)^{\gamma_{K,L}}s_{\mathfrak{g}}^{L}(s_{\mathfrak{g}}^{K-L}(X)). \tag{4.17}$$

Indeed, we start from equation (4.8) and consider one term in the expansion of the symmetrisation. This term corresponds to $|K|$ consecutive $\widetilde{\text{ad}}$-operators acting on X. We fix the first i operators from the left and now gather all other terms which start with this fixed sequence. This gives, up to an overall constant, the consecutive action of some $s_{\mathfrak{g}}^{L}$ with $|L|=|K|-i$, followed by the fixed i operators. Now we also consider all terms in the expansion of $s_{\mathfrak{g}}^{K}(X)$ where the first i of the $\widetilde{\text{ad}}$-operators correspond to a permutation of the ones we considered earlier. Adding all these together gives a term $s_{\mathfrak{g}}^{L}(s_{\mathfrak{g}}^{K-L}(X))$, again up to multiplicative constant. All the terms in $s_{\mathfrak{g}}^{K}(X)$ that have not yet been considered can also be gathered in such forms, for some different $L'\in\mathbb{N}^{m,n}$ with $|L'|=|K|-i$. Keeping track of all constants and signs then yields (4.17).

Now using the definition of the Bernoulli numbers (4.2) and equation (4.17), we obtain

$$\begin{aligned}
-B_{|K|}s_{\mathfrak{g}}^K(X)\partial^K(Y) &= \sum_{i=0}^{|K|-1} \frac{B_i|K|!}{(|K|-i+1)!i!} s_{\mathfrak{g}}^K(X)\partial^K(Y) \\
&= \sum_{L<K} \frac{B_{|L|}}{(|K-L|+1)!|L|!} |L|!|K-L|! \\
&\quad \binom{K}{L} s_{\mathfrak{g}}^L(x) s_{\mathfrak{g}}^{K-L}(X)\partial^L\partial^{K-L}(Y).
\end{aligned}$$

which proves the lemma. □

4.5 Bessel operators for Jordan superpairs

In this section we consider $\mathbb{K} = \mathbb{R}$.

4.5.1 The general case

Consider a real Jordan superpair $V = (V^+, V^-)$. Remark 4.3.5 then yields a representation π of $\mathfrak{g} := \mathrm{TKK}(V^+, V^-)$ on $S(V^-)$ for any character $\lambda\colon \mathrm{Inn}(V^+, V^-) \to \mathbb{R}$. This extends to a representation on $\mathcal{O}_{\mathbb{A}(V_-^*)}$, where we identify $S(V^-)$ with the polynomials on $\mathbb{A}(V_-^*)$ and we use the notation $V_-^* := (V^-)^*$.

Using the operators $D_{x,y}$ and $P_{x,y}$ introduced in Subsection 3.2.3, we can rewrite the representation

$$\pi : \mathfrak{g} = V^+ \oplus \mathrm{Inn}(V^+, V^-) \oplus V^- \;\to\; \mathscr{A}(V^-) \;\subset\; \Gamma(\mathcal{D}_{\mathbb{A}(V_-^*)})$$

into the following form:

1. $\pi(0, 0, u) = u$
2. $\pi(0, \mathbb{D}_{x,y}, 0) = \lambda(\mathbb{D}_{x,y}) - \sum_i (-1)^{|x||y|} D_{y,x}(e_i)\partial^i$
3. $\pi(v, 0, 0) = \sum_i \lambda(\mathbb{D}_{v,e_i})\partial^i - \sum_{i,j} (-1)^{|v|(|e_i|+|e_j|)} P_{e_i,e_j}(v)\partial^j\partial^i$,

for $x, v \in V^+$ and $u, y \in V^-$. Here $(e_i)_i$ is a homogeneous basis of V^- and $\partial^i \in \mathscr{A}(V^-) \subset \Gamma(\mathcal{D}_{\mathbb{A}(V_-^*)})$ the corresponding partial derivatives. For each $x \in V^+$ the expression for $\pi(x, 0, 0)$ in (3) gives a differential operator on the affine supermanifold $\mathbb{A}(V^*_-)$, which is of the same parity as x, in case x is homogeneous. We will use this expression to define the Bessel operator. This even operator is a global differential operator on $\mathbb{A}(V^*_-)$ taking values in the super-vector space $(V^+)^*$.

Definition 4.5.1. *Consider a Jordan superpair $V = (V^+, V^-)$ and a character $\lambda\colon \mathrm{Inn}(V^+, V^-) \to \mathbb{R}$. For any $u, v \in V^-$ we define $\lambda_u \in (V^+)^*$ and $\widetilde{P}_{u,v} \in V^- \otimes (V^+)^*$ by*

$$\lambda_u(x) = -\lambda(\mathbb{D}_{x,u}) \quad \textit{and} \quad \widetilde{P}_{u,v}(x) := (-1)^{|x|(|u|+|v|)} P_{u,v}(x)$$

*for all $x \in V^+$. Then we define the **Bessel operator***

$$\mathcal{B}_\lambda \in \Big(\Gamma(\mathcal{D}_{\mathbb{A}(V^*_-)}) \otimes (V^+)^*\Big)_{\bar{0}}, \qquad \textit{as} \quad \mathcal{B}_\lambda = \sum_i \lambda_{e_i} \partial^i + \sum_{i,j} \widetilde{P}_{e_i,e_j} \partial^j \partial^i.$$

In particular, by construction, we find

$$\mathcal{B}_\lambda(x) = -\pi(x, 0, 0) \qquad \forall x \in V^+.$$

These Bessel operators and the representation in (i)'-(iii)' can be viewed as a generalisation to the Jordan superpair setting of the construction in [HKM], as we will argue in the next subsection. Proposition 1.4 in [HKM] generalises to our setting.

Proposition 4.5.2. *Set $M = \mathbb{A}(V^*_-)$. The Bessel operator satisfies the following:*

1. *The family of operators $\mathcal{B}_\lambda(x) \in \Gamma(\mathcal{D}_M)$ for $x \in V^+$, supercommutes for fixed λ.*
2. *For any $\phi, \psi \in \mathcal{O}_M(U)$, with some open $U \subset |M|$, we have the product rule*

$$\mathcal{B}_\lambda(\phi\psi) = \mathcal{B}_\lambda(\phi)\psi + \phi\mathcal{B}_\lambda(\psi) + 2\sum_{i,j} (-1)^{|e_i||\phi|} \widetilde{P}_{e_i,e_j} \partial^j(\phi) \partial^i(\psi).$$

Proof. For the first statement, it suffices to note that for x_1 and x_2 in V^+ we have

$$[\pi(x_1), \pi(x_2)] = 0,$$

which follows by construction for a representation of a 3-graded Lie superalgebra.

For the second statement, we have

$$
\begin{aligned}
\mathcal{B}_\lambda(\phi\psi) &= \sum_i \lambda_{e_i}\partial^i(\phi\psi) + \sum_{i,j}\widetilde{P}_{e_i,e_j}\partial^j\partial^i(\phi\psi)\\
&= \sum_i \lambda_{e_i}\partial^i(\phi)\psi + \sum_{i,j}\widetilde{P}_{e_i,e_j}\partial^j(\partial^i(\phi))\psi\\
&\quad + \sum_i \lambda_{e_i}(-1)^{|e_i||\phi|}\phi\partial^i(\psi) + \sum_{i,j}(-1)^{(|e_i|+|e_j|)|\phi|}\widetilde{P}_{e_i,e_j}\phi\partial^j\partial^i(\psi)\\
&\quad + \sum_{i,j}(-1)^{|e_i||\phi|}\widetilde{P}_{e_i,e_j}\partial^j(\phi)\partial^i(\psi)\\
&\quad + \sum_{i,j}(-1)^{|e_j|(|e_i|+|\phi|)}\widetilde{P}_{e_i,e_j}\partial^i(\phi)\partial^j(\psi)\\
&= \mathcal{B}_\lambda(\phi)\psi + \phi\mathcal{B}_\lambda(\psi) + 2\sum_{i,j}(-1)^{|e_i||\phi|}\widetilde{P}_{e_i,e_j}\partial^j(\phi)\partial^i(\psi),
\end{aligned}
$$

where we used $P_{e_i,e_j} = (-1)^{|e_i||e_j|}P_{e_j,e_i}$. □

4.5.2 Example: the real spin factor

When the Jordan superpair is the doubling of the real spin factor Jordan superalgebra J defined in 3.2.2, we have, by Proposition 3.7.1,

$$\mathrm{TKK}(J) = \mathfrak{osp}(p,q|2n).$$

Recall $\mathrm{Inn}(J,J) = \mathfrak{osp}(J)\oplus\mathbb{R}L_e$ by Proposition 3.7.1 and Corollary 3.7.2. A character $\lambda\colon \mathrm{Inn}(J,J)\to\mathbb{R}$ is thus uniquely determined by its value on L_e, because $[\mathfrak{osp}(J),\mathfrak{osp}(J)] = \mathfrak{osp}(J)$. We will denote the value of $\lambda(2L_e)$ also by λ.

Up to an automorphism of $\Gamma(\mathcal{D}_{\mathbb{A}(J^*)})$ induced by $e_k \mapsto -\imath e_k$, the representation in 4.3.5 is given as follows

$$\pi_\lambda : \mathrm{TKK}(J) = J^+\oplus\mathrm{Inn}(J,J)\oplus J^- \quad\rightarrow\quad \Gamma(\mathcal{D}_{\mathbb{A}(J^*)})$$

1. $\pi_\lambda(0,0,e_k) = -\imath z_k$ for $e_k\in J^-$
2. $\pi_\lambda(0,L_{ij},0) = z_i\partial_{z^j} - (-1)^{|i||j|}z_j\partial_{z^i}$ for $L_{ij}\in\mathfrak{osp}(J)$
3. $\pi_\lambda(0,L_e,0) = \frac{\lambda}{2} - \mathbb{E}$
4. $\pi_\lambda(\bar{e}_k,0,0) = -\imath\mathcal{B}_\lambda(e_k)$ for $\bar{e}_k\in J^+$.

Here $(e_i)_{i=0}^{m+2n-1}$ is the homogeneous basis of $J^- = J$ introduced in Section 3.2.2. To simplify the expressions, we introduced a basis $(\bar{e}_i)_i$ of J^+ by $\bar{e}_i := e_i$ for $i > 0$ and $\bar{e}_0 := -e_0$. We get the following expressions for the Bessel operator

$$\mathcal{B}_\lambda(e_k) = (-\lambda + 2\mathbb{E})\partial_k - z_k\Delta, \tag{4.18}$$

where $\mathbb{E}$ and Δ are the Euler operator and Laplacian introduced in equation (2.3), page 41. We will also write $\mathcal{B}_\lambda(z_k)$ for $\mathcal{B}_\lambda(e_k)$.

4.5.3 A special case

Now assume that $V = (J, J)$, as in Example 3.2.2, with J a unital simple Jordan *algebra*. We will make freely use of Proposition 3.3.7 to identify $\mathrm{Inn}(J, J)$ with $\mathfrak{istr}(J)$. We will use concepts and nomenclature as in [FK]. We then make the extra assumption on J that the eigenspace for eigenvalue $+1$ of the *Cartan involution* α of J is a simple Jordan algebra. We also consider the symmetric bilinear form $\tau : J \times J \to \mathbb{R}$ known as the trace form, which is non-degenerate under the above assumptions. Furthermore we define the symmetric bilinear form $(\cdot|\cdot) = \tau(\cdot, \alpha\cdot)$, which is positive definite. We denote the dimension of J by n and its rank by r.

In order to compare the realisation (i)'-(iii)' with the one in [HKM], we need to adjust to the convention in [HKM], which considers a realisation of $\mathfrak{co}(J)$ on $\mathcal{O}_{\mathbb{A}(J)}$, rather than on $\mathcal{O}_{\mathbb{A}(J^*)}$. Therefore we define an isomorphism of vector spaces

$$D : J \to J^*; \qquad v \mapsto \tau(v, \cdot) \qquad \forall v \in J,$$

which extends to an isomorphism $D : S(J) \tilde{\to} S(J^*)$. The representation π on $S(J)$ in (i)'-(iii)' then leads to one on $S(J^*)$, defined as $D \circ \pi \circ D^{-1}$, which we also denote by π. This yields

1. $\pi(0, 0, u) = \tau(u, \cdot)$
2. $\pi(0, D_{x,y}, 0) = \lambda(D_{x,y}) - \sum_{i=1}^n \tau(D_{y,x}(e_i), \cdot)\partial^i$
3. $\pi(v, 0, 0) = \sum_{i=1}^n \lambda(D_{v,e_i})\partial^i - \sum_{i,j} \tau(P_{e_i,e_j}(v), \cdot)\partial^j\partial^i$

for $u, x, y, v \in J$, where the partial derivatives $\partial^i \in \Gamma(\mathcal{D}_{\mathbb{A}(J)})$ are taken with respect to the basis $\tau(e_i, \cdot)$ of J^*.

Now we make the further assumption that the character $\lambda : \mathfrak{istr}(J) \to \mathbb{R}$ is of the form

$$\lambda = -\frac{r}{2n}\lambda_0 \mathrm{Tr},$$

with Tr the trace of operators on J and $\lambda_0 \in \mathbb{R}$. The motivation to include the factor $r/2n$ comes from the observation that expression (3.2a) of the operator $D_{x,y}$ and Proposition III.4.2 in [FK] imply that for any $D_{x,y} \in \mathfrak{istr}(J)$

$$\mathrm{Tr}(D_{x,y}) = 2\mathrm{Tr}(L_{x*y}) = \frac{2n}{r}\tau(x,y).$$

Furthermore we also have $\tau(D_{x,y}u, v) = \tau(u, D_{y,x}v)$ and $\tau(P_{x,y}u, v) = \tau(P_{x,y}v, u)$, which follows easily if one uses the associativity of the trace form τ, see Proposition II.4.3 in [FK], and the expressions given in (3.2) for $D_{x,y}$ and $P_{x,y}$.

The Cartan involution θ, which is an involutive automorphism of $\mathfrak{co}(J)$, see e.g. Section 2.1.1 in [HKM] is given by

$$\theta(u, D_{x,y}, v) = (-\alpha v, -D_{\alpha y, \alpha x}, -\alpha u).$$

Then we rewrite the above representation in terms of $(\cdot|\cdot)$ and compose it with the Cartan involution, yielding another representation π' which takes the form we describe below.

Scholium 4.5.3. *The representation in Remark 4.3.5 contains as a special case the following situation. For any real Jordan algebra J, with assumptions as above, the Lie algebra $\mathfrak{co}(J)$ admits a representation π' on smooth functions on the manifold J, given by*

1. $\pi'(u,0,0) = -(u|\cdot)$
2. $\pi'(0, D_{x,y}, 0) = \frac{r\lambda_0}{2n}\mathrm{Tr}(D_{\alpha y,\alpha x}) + \sum_{i=1}^n \tau(e_i, D_{\alpha y,\alpha x}\cdot)\partial^i$
3. $\pi'(0,0,v) = \lambda_0 \sum_{i=1}^n (v|e_i)\partial^i + \sum_{i,j}(v|P_{e_i,e_j}\cdot)\partial^j\partial^i$,

for all $u,x,y,v \in J$. This is the representation occurring in Section 2.2 in [HKM]. Note that the action of $\mathfrak{co}(J)_{\pm 1}$ needs to be multiplied with an auxiliary constant $\mp\imath$ in order to get the exact same expressions. Furthermore we should point out that in [HKM] it is shown that this is not just a representation on functions on J, but that it also restricts to functions on certain orbits of the structure group.

4.6 A realisation of $D(2,1;\alpha)$

In this section we give polynomial realisations for the one parameter family of Lie superalgebras $D(2,1;\alpha)$ and discuss the reducibility of the corresponding representation on polynomials. We also comment on the other exceptional basic classical Lie superalgebras. In this section we set $\mathbb{K}=\mathbb{C}$.

4.6.1 Realisations of the exceptional Lie superalgebras

The Weyl superalgebra $\mathscr{A}(V)$, as a super-vector space, inherits a $\mathbb{Z}\times\mathbb{Z}$-grading from the natural gradings on $S(V)$ and $S(V^*)$ by equation (4.4).

Proposition 4.6.1.

1. *For any $\alpha\in\mathbb{C}\backslash\{0,-1\}$, the Lie superalgebra $D(2,1;\alpha)$ admits a one parameter family of realisations in $\mathscr{A}:=\mathscr{A}(\mathbb{C}^{2|2})$, which are contained in*

$$\mathscr{A}_{0,0}\oplus\mathscr{A}_{1,0}\oplus\mathscr{A}_{1,1}\oplus\mathscr{A}_{0,1}\oplus\mathscr{A}_{1,2}.$$

   One of those realisations is inside $\mathscr{A}_{1,0}\oplus\mathscr{A}_{1,1}\oplus\mathscr{A}_{1,2}$.

2. *The Lie superalgebra $G(3)$ admits a one parameter family of realisations in $\mathscr{A}:=\mathscr{A}(\mathbb{C}^{1|7})$, which are contained in*

$$\mathscr{A}_{0,0}\oplus\mathscr{A}_{1,0}\oplus\mathscr{A}_{1,1}\oplus\mathscr{A}_{0,1}\oplus\mathscr{A}_{1,2}\oplus\mathscr{A}_{0,2}\oplus\mathscr{A}_{1,3}.$$

   One of those is inside $\mathscr{A}_{1,0}\oplus\mathscr{A}_{1,1}\oplus\mathscr{A}_{1,2}\oplus\mathscr{A}_{1,3}$.

3. *The Lie superalgebra $F(4)$ admits a one parameter family of realisations in $\mathscr{A}:=\mathscr{A}(\mathbb{C}^{6|4})$, which are contained in*

$$\mathscr{A}_{0,0}\oplus\mathscr{A}_{1,0}\oplus\mathscr{A}_{1,1}\oplus\mathscr{A}_{0,1}\oplus\mathscr{A}_{1,2}.$$

   One of those realisations is inside $\mathscr{A}_{1,0}\oplus\mathscr{A}_{1,1}\oplus\mathscr{A}_{1,2}$.

Proof. Consider a simple $\mathbb{Z}$-graded Lie superalgebra $\mathfrak{g}=\oplus_{|j|\le d}\mathfrak{g}_j$, where $\mathfrak{g}_0$ is the direct sum of its (even) centre $\mathfrak{z}$ and some simple Lie superalgebras and set $k=\dim\mathfrak{z}$. Theorem 4.3.4 then implies that

$\mathfrak{g}$ has a k parameter family of realisations in $\mathscr{A} := \mathscr{A}(\mathfrak{g}_{-})$, which is contained in

$$(\oplus_{0\leq i\leq d-1}\mathscr{A}_{1,i}) \;+\; (\mathscr{A}_{0,0}\oplus\mathscr{A}_{1,1}) \;+\; (\oplus_{1\leq i\leq d+1}\mathscr{A}_{1,i} \;\oplus\; \oplus_{1\leq j\leq d}\mathscr{A}_{0,j}).$$

In the particular case that $\lambda = 0$, the realisation is actually contained in

$$(\oplus_{0\leq i\leq d-1}\mathscr{A}_{1,i}) \;+\; \mathscr{A}_{1,1} \;+\; (\oplus_{1\leq i\leq d+1}\mathscr{A}_{1,i}).$$

Now $D(2,1;\alpha)$ has a 3-term grading with $\mathfrak{g}_0 = \mathfrak{osp}(2|2)\oplus\mathbb{C}$ and $\mathfrak{z} = \mathbb{C}$, which will be considered explicitly in Subsection 4.6.2, the results follow then from using $d = 1$ in the above formulae.

The Lie superalgebra $G(3)$ has a 5-term grading with $\mathfrak{g}_0 \cong G(2)\oplus\mathbb{C}$ and $\mathfrak{z} = \mathbb{C}$, see Section 2.19 in [FSS], where $\mathfrak{g}_{-2}$ is one-dimensional and even and $\mathfrak{g}_{-1}$ is seven-dimensional and purely odd. Setting $d = 2$ in the above formulae, we get the result for $G(3)$.

For the last case, we remark that from Section 3.6.1 or from [Ka2] it follows that $F(4)$ is the Tits–Kantor–Koecher Lie superalgebra associated with the Jordan superalgebra F which has dimension $(6|4)$. The 3-term grading coming from this TKK-construction, can be derived from Proposition 1(I) in [Ka2]. The even and odd roots of $F(4)$ are given by

$$\Delta_0 = \{\pm\delta, \pm\epsilon_i \pm \epsilon_j, \pm\epsilon_i\} \quad i,j\in\{1,2,3\},$$
$$\Delta_1 = \{\frac{1}{2}(\pm\delta\pm\epsilon_1\pm\epsilon_2\pm\epsilon_3)\}.$$

Then the 3-term grading is given by

$$\begin{aligned}
\mathfrak{g}_{+} &= \langle X_{\epsilon_1}, X_{\delta}, X_{\epsilon_1\pm\epsilon_2}, X_{\epsilon_1\pm\epsilon_3}, X_{\frac{1}{2}(\delta+\epsilon_1\pm\epsilon_2\pm\epsilon_3)}\rangle\\
\mathfrak{g}_{-} &= \langle X_{-\epsilon_1}, X_{-\delta}, X_{-\epsilon_1\pm\epsilon_2}, X_{-\epsilon_1\pm\epsilon_3}, X_{-\frac{1}{2}(\delta+\epsilon_1\pm\epsilon_2\pm\epsilon_3)}\rangle\\
\mathfrak{g}_0 &= \langle H_\delta, H_{\epsilon_1}, H_{\epsilon_2}, H_{\epsilon_3},\\
&\qquad X_{\pm\epsilon_2}, X_{\pm\epsilon_3}, X_{\pm\epsilon_2\pm\epsilon_3}, X_{\frac{1}{2}(\delta-\epsilon_1\pm\epsilon_2\pm\epsilon_3)}, X_{\frac{1}{2}(-\delta+\epsilon_1\pm\epsilon_2\pm\epsilon_3)}\rangle.
\end{aligned}$$

We have that $\mathfrak{g}_0 = \mathfrak{osp}(2|4)\oplus\mathfrak{z}$. The center is given by $\mathfrak{z} = H_\delta + H_{\epsilon_1}$, hence one-dimensional. Again setting $d = 1$, we also obtain the last result. □

The specific form of the realisation of $D(2,1;\alpha)$ inside $\mathscr{A}_{1,0} \oplus \mathscr{A}_{1,1} \oplus \mathscr{A}_{1,2}$ is given in the following proposition.

Proposition 4.6.2. *Consider the differential operators*

$$\mathbb{E} = x\partial_x + y\partial_y + \eta\partial_\eta + \theta\partial_\theta, \qquad \Delta = \partial_x\partial_y + \partial_\eta\partial_\theta,$$
$$\mathbb{E}_\alpha = \alpha x\partial_x + y\partial_y + \alpha\eta\partial_\eta + \theta\partial_\theta, \qquad \Delta_\alpha = \partial_x\partial_y + \alpha\partial_\eta\partial_\theta$$

on $\mathbb{A}^{2|2}$. A realisation of $D(2,1;\alpha)$ is given by the operators

$$x, y, \theta, \eta, \theta\partial_\theta - \eta\partial_\eta, x\partial_x - y\partial_y,$$
$$\mathbb{E}, \theta\partial_\eta, \eta\partial_\theta, \eta\partial_y + \alpha x\partial_\theta, \theta\partial_x - y\partial_\eta, \theta\partial_y - \alpha x\partial_\eta, \eta\partial_x + y\partial_\theta$$

and

$$\mathbb{E}\partial_x - y\Delta,\ \mathbb{E}\partial_y - x\Delta_\alpha,\ \mathbb{E}_\alpha\partial_\theta + \eta\Delta,\ \mathbb{E}_\alpha\partial_\eta - \theta\Delta_\alpha.$$

This will be obtained as a special case of the realisations considered in the next subsection.

4.6.2 The realisations of $D(2,1;\alpha)$

We will again use the explicit realisation of $D(2,1;\alpha)$ for $\alpha \notin \{0,1\}$ constructed in Section 2.5.5. Recall that $D(2,1;\alpha) = \mathrm{TKK}(D_\alpha)$ by Proposition 3.7.3. We will use the 3-term grading induced from the TKK construction:

$$\begin{aligned}
\mathfrak{g}_+ &= \langle X_{2\delta_3}, X_{2\delta_2}, X_{-\delta_1+\delta_2+\delta_3}, X_{\delta_1+\delta_2+\delta_3}\rangle\\
\mathfrak{g}_- &= \langle X_{-2\delta_3}, X_{-2\delta_2}, X_{\delta_1-\delta_2-\delta_3}, X_{-\delta_1-\delta_2-\delta_3}\rangle\\
\mathfrak{g}_0 &= \langle H_{\delta_1}, H_{\delta_2}, H_{\delta_3},\\
&\qquad X_{2\delta_1}, X_{-\delta_1+\delta_2-\delta_3}, X_{\delta_1+\delta_2-\delta_3}, X_{-2\delta_1}, X_{\delta_1-\delta_2+\delta_3}, X_{-\delta_1-\delta_2+\delta_3}\rangle.
\end{aligned}$$

Here we have $\mathfrak{g}_0 = \mathfrak{osp}(2|2) \oplus \mathbb{C}$, where the ideal $\mathbb{C}$ is the centre of $\mathfrak{g}_0$ and $\mathfrak{osp}(2|2)$ is simple. We set $h := H_{\delta_2} + H_{\delta_3} \in \mathfrak{h}^* \subset \mathfrak{g}_0$ and the centre of $\mathfrak{g}_0$ is given by $\mathbb{C}h$. Hence, there is a bijection between characters $\lambda : \mathfrak{g}_0 \to \mathbb{C}$ and $\mathbb{C}$, which we normalise by $\lambda \mapsto \lambda(h)$.

We consider the realisation of $D(2,1;\alpha)$ in $\mathscr{A}(\mathrm{Span}_\mathbb{C}(x,y,\theta,\eta))$, where x, y are even and θ, η are odd, given by Remark 4.3.5. We add the character (complex number) λ in the notation. For $\mathfrak{g}_-$ we have

$$\pi_\lambda(X_{-2\delta_3}) = x, \qquad \pi_\lambda(X_{-2\delta_2}) = y,$$

$$\pi_\lambda(X_{\delta_1-\delta_2-\delta_3}) = \theta, \qquad \pi_\lambda(X_{-\delta_1-\delta_2-\delta_3}) = \eta.$$

For $\mathfrak{g}_0 \cong \mathbb{C} \oplus \mathfrak{osp}(2|2)$ we have

$$\begin{aligned}
\pi_\lambda(H_{\delta_1}) &= \theta\partial_\theta - \eta\partial_\eta,\\
\pi_\lambda(H_{\delta_2}) &= \lambda - 2y\partial_y - \theta\partial_\theta - \eta\partial_\eta\\
\pi_\lambda(H_{\delta_3}) &= \frac{\lambda}{\alpha} - 2x\partial_x - \theta\partial_\theta - \eta\partial_\eta,\\
\pi_\lambda(X_{2\delta_1}) &= (1+\alpha)\theta\partial_\eta,\\
\pi_\lambda(X_{-2\delta_1}) &= (1+\alpha)\eta\partial_\theta,\\
\pi_\lambda(X_{-\delta_1+\delta_2-\delta_3}) &= -\eta\partial_y - \alpha x\partial_\theta,\\
\pi_\lambda(X_{\delta_1-\delta_2+\delta_3}) &= \theta\partial_x - y\partial_\eta,\\
\pi_\lambda(X_{\delta_1+\delta_2-\delta_3}) &= \theta\partial_y - \alpha x\partial_\eta,\\
\pi_\lambda(X_{-\delta_1-\delta_2+\delta_3}) &= -\eta\partial_x - y\partial_\theta.
\end{aligned}$$

For $\mathfrak{g}_+$ we finally find

$$\begin{aligned}
\pi_\lambda(X_{2\delta_3}) &= \left(\frac{\lambda}{\alpha} - x\partial_x - \theta\partial_\theta - \eta\partial_\eta\right)\partial_x + y\partial_\eta\partial_\theta,\\
\pi_\lambda(X_{2\delta_2}) &= (\lambda - y\partial_y - \theta\partial_\theta - \eta\partial_\eta)\,\partial_y + \alpha x\partial_\eta\partial_\theta,\\
\pi_\lambda(X_{-\delta_1+\delta_2+\delta_3}) &= (-\lambda + \alpha x\partial_x + y\partial_y + (1+\alpha)\eta\partial_\eta)\partial_\theta + \eta\partial_x\partial_y,\\
\pi_\lambda(X_{\delta_1+\delta_2+\delta_3}) &= (\lambda - \alpha x\partial_x - y\partial_y - (1+\alpha)\theta\partial_\theta)\,\partial_\eta + \theta\partial_x\partial_y.
\end{aligned}$$

The restriction of the canonical representation of $\mathscr{A}_{2|2}$ on $S(\mathbb{C}^{2|2})$ to $U(\mathfrak{g})$, seen as a subalgebra through π_λ, leads to a representation of $\mathfrak{g} = D(2,1;\alpha)$ on $S(\mathbb{C}^{2|2})$, which we also denote by π_λ. By construction and Scholium 4.5.3, this is an analogue for superalgebras of the conformal representations considered in [HKM]. Another key step in the construction in *op. cit.* is the fact that for certain values of the parameter λ, the operators in the realisation are tangential to specific orbits of the structure group on the Jordan algebra. Consequently, the representation on functions on J is not irreducible and the representation of interest is a factor module of $\mathcal{C}^\infty(J)$. The set of parameters for which this occurs is directly linked to the Wallach set, see e.g. Theorem 1.12 of [HKM]. This motivates the question for which λ, the representation π_λ is irreducible in our example for $D(2,1;\alpha)$.

Proposition 4.6.3. *The representation π_λ of $D(2,1;\alpha)$ on $S(\mathbb{C}^{2|2})$ is irreducible if and only if*

$$\lambda \notin \mathbb{N} \quad \text{and} \quad \lambda/\alpha \notin \mathbb{N}.$$

If either $\lambda \in \mathbb{N}$ or $\lambda/\alpha \in \mathbb{N}$, the representation is indecomposable but not irreducible.

Proof. Set $\mathfrak{g} = D(2,1;\alpha)$. First we note that all modules are weight modules and that the weight corresponding to the constants in $\mathcal{P} := S(\mathbb{C}^{2|2})$ appears with multiplicity one. If $\mathcal{P}$ would be the direct sum of two $\mathfrak{g}$-modules, the space of constants would hence belong to precisely one of them. However, it is clear that $U(\mathfrak{g}_-)$-action on 1 generates $\mathcal{P}$, leading to a contradiction. Therefore the module is indecomposable. It is simple if and only if for any $P \in \mathcal{P}$ we have $1 \in U(\mathfrak{g}) \cdot P$.

Let P be a homogeneous polynomial of degree $l > 0$ in $\mathbb{C}[x, y, \theta, \eta]$. By a lengthy but straightforward calculation, one can show that $\mathfrak{g}_+$ acting trivially on P forces P to be zero unless $\lambda \neq i$ or $\lambda \neq i\alpha$ for some $i < l$, $i \in \mathbb{N}$.

First assume that both λ and λ/α are not in $\mathbb{N}$. For any homogeneous polynomial of degree l, there exists an element $v \in \mathfrak{g}_+$ such that $\pi_\lambda(v)P$ is a non-zero homogeneous polynomial of degree $l-1$. By induction, we can find $v_1, \ldots, v_l \in \mathfrak{g}_+$ such that $\pi_\lambda(v_1)\pi_\lambda(v_2)\cdots\pi_\lambda(v_l)P$ is a non-zero constant. For P an arbitrary polynomial we can consider the homogeneous polynomial $P_{\max}$ such that the polynomial $P - P_{\max}$ is of strictly lower degree than P. The above argument yields an element of $U(\mathfrak{g}_+)$ which annihilates $P - P_{\max}$ and maps $P_{\max}$ (and hence P) to a non-zero constant. We thus find that the representation is irreducible.

On the other hand, one can check directly that $\mathfrak{g}_+$ acts trivially on

$$P := ax + by + c\theta + d\eta,$$

if $\lambda = 0$, on

$$P := ay^{l+1} + by^l\theta + cy^l\eta + d(y^{l-1}\theta\eta - \frac{\alpha}{l}xy^l),$$

if $\lambda = l > 0$ and on

$$P := ax^{l+1} + bx^l\theta + cx^l\eta + d(x^{l-1}\theta\eta - \frac{1}{l}x^l y),$$

if $\lambda = l\alpha$ with $l > 0$. Here a, b, c, d are arbitrary complex constants. From the PBW-Theorem it follows that

$$U(\mathfrak{g}) \cdot P \cong U(\mathfrak{g}_-)U(\mathfrak{g}_0)U(\mathfrak{g}_+) \cdot P.$$

Since $\mathfrak{g}_+$ acts trivially, it follows that all polynomials in $U(\mathfrak{g}) \cdot P$ have degree higher or equal to P. Therefore $U(\mathfrak{g}) \cdot P$ is a proper submodule of $\mathcal{P}$. □

We conclude this section by focusing on the specific cases $\lambda = 1$ and $\lambda = \alpha$, as in the spirit of the above discussion the top of that module seems the first candidate for the 'minimal representation' of $D(2, 1; \alpha)$.

First assume that $\alpha = 1$, then the action of $\mathfrak{osp}(2|2)$, the semisimple part of $\mathfrak{g}_0$, on $\mathcal{P}$ reduces to the one studied in [Cou]. In particular it was derived that in this case the space $\mathcal{P}_2$ of homogeneous polynomials is indecomposable. This self-dual module has a simple socle given by the trivial representation generated by the polynomial $R^2 = xy + \eta\theta$. The calculations in the proof of Proposition 4.6.3 illustrate that the polynomials of degree 2 which generate the $D(2, 1; 1)$-submodule of $\mathcal{P}$ constitute a subspace of codimension 1. This is precisely the radical of the $\mathfrak{osp}(2|2)$-module $\mathcal{P}_2$, namely the solutions of the Laplace equation.

Now return to the case $\lambda = 1$ or $\lambda = \alpha$ with $\alpha \neq 1$. In this case the structure of the $\mathfrak{g}_0$-module clearly changes. There is no longer a one-dimensional submodule. But in both cases there is a five-dimensional submodule which generates the $D(2, 1; \alpha)$-submodule of $\mathcal{P}$. This five-dimensional submodule is generated either by $R^2 = xy + \eta\theta$, if $\lambda = \alpha$, or $R^2_\alpha = \alpha xy + \eta\theta$, if $\lambda = 1$.

Just because you can explain it doesn't mean it's not still a miracle.

Terry Pratchett, Small Gods

5

The minimal representation

We will now apply the results of the previous chapter to obtain a minimal representation for the orthosymplectic Lie superalgebra. In this chapter, we will construct the representation and in the next chapter we will show some properties of this representation.

Because we will focus on the orthosymplectic case from now on, J will always stand for the real spin factor Jordan superalgebra. For the Lie superalgebra $\mathfrak{osp}(p,q|2n)$ we always assume $p \geq 2$ and $q \geq 2$. Manifolds, affine spaces, Jordan and Lie algebras will be defined over the field of real numbers $\mathbb{R}$, while functions spaces will be over the complex field $\mathbb{C}$, unless otherwise stated.

We start this chapter with an introduction to Lie supergroups and actions of Lie supergroups on supermanifolds. We also define the structure and conformal group associated with J. In Section 5.2 we construct a minimal orbit on J under the action of the structure group. We show that the representation we defined in Section 4.5.2 can be restricted to functions defined on this minimal orbit. The next step is to integrate this restricted representation to group level. In order to do this we define in Section 5.3 a submodule W of the representation on the minimal orbit. We also introduce Harish-Chandra supermodules and then show that they can be used to integrate the module W to a representation of the conformal group.

5.1 Lie supergroups and their actions

5.1.1 Definitions

A Lie supergroup G is a group object in the category of smooth supermanifolds, i.e. there exist morphisms $\mu\colon G\times G \to G$, $i\colon G\to G$, $e\colon \mathbb{R}^{0|0}\to G$, called the multiplication, inverse and unit which satisfy the standard group properties. Alternatively, we can also characterise Lie supergroups in the following manner:

Definition 5.1.1. *A Lie supergroup G is a pair $(G_0, \mathfrak{g})$ together with a morphism $\sigma\colon G_0 \to \mathrm{End}(\mathfrak{g})$, where G_0 is a Lie group and $\mathfrak{g}$ is a Lie superalgebra for which*

- $\mathrm{Lie}(G_0)$ *is isomorphic to $\mathfrak{g}_{\bar{0}}$, the even part of the Lie superalgebra $\mathfrak{g}$.*
- *The morphism σ satisfies $\sigma(g)_{|\mathfrak{g}_{\bar{0}}} = \mathrm{Ad}(g)$ and $\mathrm{d}\sigma(X)Y = [X,Y]$ for all $g\in G_0$, $X\in\mathfrak{g}_{\bar{0}}$ and $Y\in\mathfrak{g}$. Here Ad is the adjoint representation of G_0 on $\mathrm{Lie}(G_0)\cong\mathfrak{g}_{\bar{0}}$.*

See [CCF, Chapter 7] for more details and the connection between those two approaches.

By a closed Lie subgroup H of a Lie supergroup G we mean a closed embedded submanifold of G that is also a subgroup. In the previous sentence we used submanifold instead of subsupermanifold and subgroup instead of supsupergroup. From now we will often omit the prefix super if it is clear from the context.

A (left) action of a Lie supergroup on a supermanifold is a morphism $a\colon G\times M\to M$ such that

- $a\circ(\mu\times \mathrm{id}_M) = a\circ(\mathrm{id}_G\times a)$
- $a\circ(e\times\mathrm{id}_M)\cong \mathrm{id}_M$, using $\mathbb{R}^{0|0}\times M\cong M$.

For every even point p of a supermanifold M we have a morphism $p_{\mathbb{R}^{0|0}}\colon \mathbb{R}^{0|0}\to M$ where $|p_{\mathbb{R}^{0|0}}|$ maps to p and $p^{\sharp}_{\mathbb{R}^{0|0}}$ is evaluation at p.

Then we define $a_p\colon G \to M$ for $p \in |M|$ and $a_g\colon M \to M$ for $g \in |G|$ by

$$a_p := a \circ (\mathrm{id}_G \times p_{\mathbb{R}^{0|0}}), \qquad a_g := a \circ (g_{\mathbb{R}^{0|0}} \times \mathrm{id}_M).$$

Also for actions, we can use the equivalent approach with pairs.

Definition 5.1.2. *An action a of a Lie supergroup $G = (G_0, \mathfrak{g})$ on a supermanifold M is a pair $(\underline{a}, \rho_a)$ where*

- *$\underline{a}\colon G_0 \times M \to M$ is an action of G_0 on M.*
- *$\rho_a\colon \mathfrak{g} \to \mathrm{Vec}_M$ is a Lie superalgebra anti morphism such that*

$$\begin{aligned} \rho_{a|\mathfrak{g}_{\bar 0}}(X) &= (X \otimes \mathrm{id}_{\mathcal{O}_M})\underline{a}^\sharp && \textit{for all } X \in \mathfrak{g}_{\bar 0}, \\ \rho_a(\sigma(g)Y) &= \underline{a}^\sharp_{g^{-1}}\rho_a(Y)\underline{a}^\sharp_g && \textit{for all } Y \in \mathfrak{g}, g \in G_0. \end{aligned}$$

Here Vec_M is the Lie superalgebra of vector fields on M, and we silently use the isomorphism $\mathfrak{g}_{\bar 0} \cong T_e G_0$.

See [CCF, Chapter 8] for more details.

By the reduced action $|a|$, we will mean the (ordinary) Lie group action $|\underline{a}|$ of $|G| = G_0$ on $|M|$. We have the two following propositions.

Proposition 5.1.3 ([CCF, Proposition 8.4.7])**.** *Let G be a supergroup with an action a on M and let $p \in |M|$. Set*

$$\widetilde{G_p} = \{g \in G_0 \mid |a|(g,p) = p\} \qquad \textit{and} \qquad \mathfrak{g}_p := \ker \mathrm{d}a_p.$$

Then $G_p = (\widetilde{G_p}, \mathfrak{g}_p)$ is a closed subgroup of $G = (G_0, \mathfrak{g})$.

Proposition 5.1.4 ([CCF, Proposition 9.3.7])**.** *Let G be a Lie supergroup and H a closed subgroup. There exists a supermanifold $X = (|G|/|H|, \mathcal{O}_X)$ and a morphism $\pi\colon G \to X$ such that*

- *The reduction $|\pi|\colon |G| \to |G|/|H|$ is the natural map.*
- *The morphism π is a submersion, i.e. for all $g \in |G|$ the map $\mathrm{d}\pi_g\colon T_g G \to T_{\pi(g)} X$ is surjective.*
- *There is an action $\beta\colon G \times X \to X$, which reduces to the action of $|G|$ on $|X|$ such that $\pi \circ \mu = \beta \circ (\mathrm{id}_G \times \pi)$, where μ is the multiplication on G.*

Moreover the pair (X, π) satisfying these properties is unique up to isomorphism.

These two propositions allow us to define the orbit through an even point p.

Definition 5.1.5. *Let G be a Lie supergroup with an action on a supermanifold M. Let $p \in |M|$. Let G_p the closed subgroup defined in Proposition 5.1.3. Then we define the orbit C_p through the point p as the manifold $X = (|G|/|G_p|, \mathcal{O}_X)$ defined in Proposition 5.1.4.*

5.1.2 The structure group

Define

$$\begin{aligned} O(p-1, q-1) &= \{X \in \mathbb{R}^{(p+q-2)\times(p+q-2)} \mid X^t \beta^s X = \beta^s\} \\ Sp(2n, \mathbb{R}) &= \{X \in \mathbb{R}^{(2n)\times(2n)} \mid X^t \beta^a X = \beta^a\}, \end{aligned}$$

where β^s, β^a are the matrices formed by the symmetric part and the anti-symmetric part of the bilinear form of J.

Set

$$Str(J)_0 := \mathbb{R}^+ \times O(p-1, q-1) \times Sp(2n, \mathbb{R})$$

and recall by Proposition 3.7.1

$$\mathfrak{istr}(J) = \mathfrak{osp}(J) \oplus \mathbb{R}L_e.$$

We embed $Str(J)_0$ in $\mathbb{R}^{(p+q-2+2n)\times(p+q-2+2n)}$ by associating to the triple $(\nu, k, h) \in Str(J)_0$ the matrix $\nu \begin{pmatrix} k & \\ & h \end{pmatrix}$ with $\nu \in \mathbb{R}^+$, $k \in \mathbb{R}^{(p+q-2)\times(p+q-2)}$ and $h \in \mathbb{R}^{(2n)\times(2n)}$. We will also interpret $X \in \mathfrak{osp}(J)$ as an $(p+q-2+2n) \times (p+q-2+2n)$ matrix.

For $\nu \in \mathbb{R}^+$, $k \in O(p-1, q-1)$ and $h \in Sp(2n, \mathbb{R})$, define $\sigma(\nu, k, h) \in \mathrm{End}(\mathfrak{istr}(J))$

$$\begin{aligned} &\sigma(\nu, k, h)L_e = L_e \quad \text{and} \\ &\sigma(\nu, k, h)X := \begin{pmatrix} k & \\ & h \end{pmatrix} X \begin{pmatrix} k^{-1} & \\ & h^{-1} \end{pmatrix} \text{ for } X \in \mathfrak{osp}(J). \end{aligned}$$

Then $Str(J) = (Str(J)_0, \mathfrak{istr}(J), \sigma)$ defines a Lie supergroup, in the sense of Definition 5.1.1. We call $Str(J)$ *the structure group.*

Next, we define an action of $Str(J)$ on $\mathbb{A}(J^*)$, the affine superspace associated to the dual super-vector space of J. Let z_i be the coordinate functions on J^*. For $x = \sum_i x_i e^i \in J^*$, we then have $z_j(x) = x_j$. By the global chart theorem, [CCF, Theorem 4.2.5], a morphism ϕ from a supermanifold M to an affine superspace is determined by the pullbacks of the coordinate functions. So we can define $\underline{a}\colon Str(J)_0 \times \mathbb{A}(J^*) \to \mathbb{A}(J^*)$ by

$$\underline{a}^\sharp(z_i) = g^{-1} z_i = \begin{pmatrix} (\nu k)^{-1} & \\ & (\nu h)^{-1} \end{pmatrix} (z_i) \in \mathcal{O}_{Str(J)_0} \hat{\otimes} \mathcal{O}_{\mathbb{A}(J^*)},$$

where $(\nu, k, h) = g = (g_{ij})_{1 \le i,j \le p+q-2+2n} \in Str(J)_0 \subset \mathbb{R}^{(p+q-2)\times(p+q-2)}$. We interpret the g_{ij} as coordinate functions on $\mathbb{R}^{(p+q-2)\times(p+q-2)}$ and then restrict them to functions in $\mathcal{O}_{Str(J)_0}$.

Set $\rho_a\colon \mathfrak{istr}(J) \to \mathrm{Vec}_{\mathbb{A}(J^*)}$

$$\rho_a(L_{ij}) = -(z_i \partial_j - (-1)^{|i||j|} z_j \partial_i) \text{ for } L_{ij} \in \mathfrak{osp}(J),$$
$$\rho_a(L_e) = -\mathbb{E}.$$

Proposition 5.1.6. *The pair $(\underline{a}, \rho_a)$ defines an action of the Lie supergroup $(Str(J)_0, \mathfrak{istr}(J))$ on $\mathbb{A}(J^*)$.*

Proof. The map ρ_a is clearly a Lie superalgebra anti-morphism from $\mathfrak{istr}(J)$ to $\mathrm{Vec}_{\mathbb{A}(J^*)}$. One can also check that $\underline{a}$ indeed defines an action of $Str(J)_0$ on M. So we only need to prove

$$\rho_{a|\mathfrak{istr}(J)_{\bar{0}}}(X) = (X \otimes \mathrm{id}_{\mathcal{O}_M})\underline{a}^\sharp, \qquad \rho_a(\sigma(g)Y) = \underline{a}^\sharp_{g^{-1}} \rho_a(Y) \underline{a}^\sharp_g$$

for $X \in \mathfrak{istr}(J)_{\bar{0}}$, $Y \in \mathfrak{istr}(J)$, $g \in Str(J)_0$. If we interpret $X \in \mathfrak{istr}(J)_{\bar{0}}$ as an element of $T_e Str(J)_0$, then it acts on the coordinate functions g_{ij} as $X(g_{ij}) = X_{ij}$. The map $g \mapsto g^{-1}$ corresponds on algebra level to $X \mapsto -X$, hence we also have $X((g^{-1})_{ij}) = -X_{ij}$. Thus

$$\begin{aligned}(X \otimes \mathrm{id}_{\mathcal{O}_M})\underline{a}^\sharp(z_i) &= X \otimes \mathrm{id}_{\mathcal{O}_M} \begin{pmatrix} (\nu k)^{-1} & \\ & (\nu h)^{-1} \end{pmatrix} (z_i) \\ &= -X(z_i) = \rho_a(X).\end{aligned}$$

Furthermore $\underline{a}^\sharp_g(z_i) = g^{-1} z_i$. We find

$$\underline{a}^\sharp_{g^{-1}} \rho_a(Y) \underline{a}^\sharp_g(z_i) = -g(Y(g^{-1}z_i)) = -(\sigma(g)Y)z_i = \rho_a(\sigma(g)Y)z_i.$$

We conclude that the pair $(\underline{a}, \rho_a)$ is an action. $\square$

5.1.3 The conformal group

We define the conformal group as follows. Define for $k \in O(p,q)$ and $h \in Sp(2n,\mathbb{R})$, $\sigma(k,h) \in \mathrm{End}(\mathfrak{osp}(p,q|2n))$ by

$$\sigma(k,h)X := \begin{pmatrix} k & \\ & h \end{pmatrix} X \begin{pmatrix} k^{-1} & \\ & h^{-1} \end{pmatrix} \text{ for } X \in \mathfrak{osp}(p,q|2n).$$

Then also $(O(p,q) \times Sp(2n,\mathbb{R}), \mathfrak{osp}(p,q|2n), \sigma)$ is a Lie supergroup, which we call the conformal group and denote by $OSp(p,q|2n)$.

5.2 The minimal orbit

We will use the action of the structure group $Str(J)$ on $\mathbb{A}(J^*)$ to construct a minimal orbit. In Section 5.2.3, we will then show that the representation constructed in 4.5.2 of $\mathfrak{osp}(p,q|2n)$ restricts to functions on this orbit.

For ordinary (i.e. not super) Jordan algebras the minimal orbit under the action of the structure group is the one through a primitive idempotent, see [Kane]. We will use this as a definition for the minimal orbit in our case.

Remark 5.2.1. If one looks at the action under the identity component of the structure group, as for example is done in [HKM], then this picture changes a bit. For non-Euclidean Jordan algebras there is still only one minimal orbit, but for Euclidean Jordan algebras we then have two minimal orbits, one through a primitive idempotent c and one through $-c$.

Let us first introduce the natural generalisations of primitive idempotents to Jordan superalgebras. An even element of a Jordan superalgebra is called an *idempotent* if it satisfies $x^2 = x$. An idempotent is *primitive* if it can not be written as the sum of two other (non-zero) idempotents. Two idempotents are called *orthogonal* if their product is zero. A *Jordan frame* is a collection of pairwise orthogonal primitive idempotents which sum to the unit [FK, Chapter IV].

Proposition 5.2.2. *For the spin factor Jordan superalgebra J^* it holds that an element $c = \lambda e + x \in J^*_{\bar{0}}$ is a non-zero idempotent iff*

$\lambda = \frac{1}{2}$ and x satisfies $\langle x, x\rangle = \frac{1}{4}$ or $c = e$ and $x = 0$. Here e denotes the unit of J^*.

All idempotents different from the unit are primitive and if $\frac{1}{2}e + x$ is an idempotent then $(\frac{1}{2}e + x, \frac{1}{2}e - x)$ is a Jordan frame.

Proof. Straightforward verification. □

Observe that the reduced action of the structure group $Str(J)$ on $|\mathbb{A}(J^*)|$ is equivalent with the action of $\mathbb{R}^+O(p-1, q-1)$ on $\mathbb{R}^{p+q-2}$ given by

$$(g, x) \to g^{-1}x,$$

since $Sp(2n, \mathbb{R})$ acts trivially on $\mathbb{R}^{p+q-2}$. Hence, for a primitive idempotent, the topological space underlying the orbit manifold is the same as in the classical case. This topological space is independent of the chosen idempotent and given by, see for example [HKM, Section 1.2],

$$\{x \in \mathbb{R}^{p+q-2}_{(0)} \mid R^2(x) = 0\},$$

where $R^2 = \sum_i z^i z_i$ is the superfunction defined in (2.3), page 41. We interpret R^2 not as an operator but as a function. Thus $R^2(x)$ denotes evaluating the function R^2 in the even point x. This also means that the odd component in R^2 does not play any role.

Let $\mathbb{A}(J^*)_{(0)}$ be the open submanifold of $\mathbb{A}(J^*)$ we get by excluding zero

$$\mathbb{A}(J^*)_{(0)} = (\mathbb{R}^{p+q-2}_{(0)}, \mathcal{C}^\infty_{\mathbb{R}^{p+q-2}_{(0)}} \otimes \Lambda^{2n}).$$

Denote by $\langle R^2\rangle$ the ideal in $\Gamma(\mathcal{O}_{\mathbb{A}(J^*)_{(0)}})$ generated by R^2. Set

$$|C| := \{x \in \mathbb{R}^{p+q-2}_{(0)} \mid R^2(x) = 0\}.$$

We will show that there is supermanifold C which has $|C|$ as its underlying topological space and $\Gamma(\mathcal{O}_{\mathbb{A}(J^*)_{(0)}})/\langle R^2\rangle$ as its global sections. By [CCF, Corollary 4.5.10], the global sections will determine the sheaf $\mathcal{O}_C$. The main theorem of this section establishes that $C = (|C|, \mathcal{O}_C)$ is the orbit through a primitive idempotent under the action of the structure group.

Theorem 5.2.3. *The space $C = (|C|, \mathcal{O}_C)$ is a well-defined supermanifold. Furthermore it is the orbit through a primitive idempotent* *of J^* under the action of the structure group $Str(J)$ on $\mathbb{A}(J^*)$ defined in Section 5.1.2. We will call C the minimal orbit.*

In the two following subsections we will prove this theorem.

5.2.1 The space C is a supermanifold

We first introduce the notion of a regular ideal, which we then use to show that C is a well-defined supermanifold.

Definition 5.2.4 ([CCF, Definition 5.3.6])**.** *Let M be a supermanifold with underlying topological space $|M|$. Let I be an ideal in $\Gamma(\mathcal{O}_M)$. For $m \in |M|$ denote by $\mathcal{J}_m$ the maximal ideal in $\Gamma(\mathcal{O}_M)$ given by the kernel of the morphism $\mathrm{ev}_m \colon \Gamma(\mathcal{O}_M) \to \mathbb{R}$ and by I_m the image of I in the stalk $\mathcal{O}_{M,m}$. Then I is called a* regular ideal *if*

- *For every $m \in |M|$ such that $I \subset \mathcal{J}_m$ there exist homogeneous $f_1, \ldots, f_n$ in I such that $[f_1]_m, \ldots, [f_n]_m$ generate I_m and $(df_1)_m, \ldots, (df_n)_m$ are linearly independent at m, where $[f_i]_m$ is the image of f_i in $\mathcal{O}_{M,m}$.*
- *If $\{f_i\}_{i\in\mathbb{N}}$ is a family in I such that any compact subset of M intersects only a finite number of supp f_i, then $\sum_i f_i$ is an element of I.*

Regular ideals can be used to define supermanifolds in the following manner.

Proposition 5.2.5 ([CCF, Proposition 5.3.8])**.** *Let M be a supermanifold and I a regular ideal in $\Gamma(\mathcal{O}_M)$. Then there exists a unique closed embedded supermanifold (N, j), where $j \colon N \to M$ is an embedding, such that*

$$\Gamma(\mathcal{O}_N) = \Gamma(\mathcal{O}_M)/I.$$

From the proof of the proposition it also follows that the underlying topological space $|N|$ of N is given by

$$|N| = \{m \in |M| \mid I \subset \mathcal{J}_m\} = \{m \in |M| \mid \mathrm{ev}_m(f) = 0 \text{ for all } f \in I\}.$$

Lemma 5.2.6. *The ideal I in $\Gamma(\mathcal{O}_{\mathbb{A}(J^*)_0})$ generated by R^2 is regular.*

Proof. For any m in $\mathbb{R}^{p+q-2}_{(0)}$ we have that I_m is generated by $[R^2]_m$. Furthermore $(dR^2)_m$ is different from zero if $m \neq 0$ and thus linearly independent. Since every f_i in I can be written as $R^2 g_i$, we have

$$\sum_i f_i = R^2 \sum_i g_i \in I.$$

So we conclude that I is a regular ideal. □

We have that

$$|C| = \{m \in \mathbb{R}^{p+q-2}_{(0)} \mid \mathrm{ev}_m(R^2) = 0\} = \{m \in \mathbb{R}^{p+q-2}_{(0)} \mid I \subset \mathcal{J}_m\}.$$

is the topological space corresponding to the regular ideal $I = \langle R^2 \rangle$.

Corollary 5.2.7. *The space $C = (|C|, \mathcal{O}_C)$ is the unique closed embedded submanifold of $\mathbb{A}(J^*)_0$ corresponding to the regular ideal $\langle R^2 \rangle$.*

Proof. This follows immediately from combining Proposition 5.2.5 and Lemma 5.2.6. □

We denote the embedding of C in $\mathbb{A}(J^*)_0$ by j_C.

5.2.2 The space C is an orbit

We will show that C is the orbit through a primitive idempotent in the sense of Definition 5.1.5. We introduce the following morphisms,

$$\begin{aligned} &a\colon Str(J) \times \mathbb{A}(J^*) \to \mathbb{A}(J^*) \\ &j\colon C \hookrightarrow \mathbb{A}(J^*). \end{aligned}$$

The morphism a is the action of $Str(J)$ on $\mathbb{A}(J^*)$ defined in Section 5.1.2. For the morphism j we combine the embedding j_C of C in $\mathbb{A}(J^*)_0$ with the embedding of $\mathbb{A}(J^*)_0$ in $\mathbb{A}(J^*)$. Define

$$b\colon Str(J) \times C \to \mathbb{A}(J^*); \qquad b = a \circ (\mathrm{id}_{Str(J)} \times j).$$

Then $b = (|b|, b^\sharp)$ with

$$|b| = |a| \circ (\mathrm{id}_{|Str(J)|} \times |j|) \text{ and } b^\sharp = (\mathrm{id}_{\mathcal{O}_{Str(J)}} \otimes j^\sharp) a^\sharp.$$

Lemma 5.2.8. *The morphism b takes values in C.*

Proof. We have to show that b factors as $j \circ \gamma$, with $\gamma\colon Str(J) \times C \to C$. This will be the case if $\mathrm{im}|b| \subset |C|$ and $b^\sharp(R^2) = 0$. On the topological level it is immediately clear that $|b|$ takes values in $|C|$. For the sheaf morphism, we will use the fact that for a Lie supergroup $G = (G_0, \mathfrak{g})$ we have [CCF, Remark 7.4.6]

$$\mathcal{O}_G(U) \cong \underline{\mathrm{Hom}}_{U(\mathfrak{g}_{\bar 0})}(U(\mathfrak{g}), \mathcal{C}^\infty_{G_0}(U)).$$

Note that by $\underline{\mathrm{Hom}}(V, W)$ we mean all linear maps from V to W including the odd ones. Using this isomorphism, an action $a = (\underline{a}, \rho_a)$ on M can be expressed in $\underline{a}$ and ρ_a as

$$\begin{aligned} a^\sharp\colon \Gamma(\mathcal{O}_M) &\to \underline{\mathrm{Hom}}_{U(\mathfrak{g}_{\bar 0})}(U(\mathfrak{g}), \mathcal{C}^\infty_{G_0}(G_0)\hat{\otimes}\Gamma(\mathcal{O}_M)) \\ f &\mapsto [X \mapsto (-1)^{|X|}(\mathrm{id}_{\mathcal{C}^\infty(G_0)} \otimes \rho_a(X))\underline{a}^\sharp f], \text{ with } X \in U(\mathfrak{g}) \end{aligned}$$

The Lie group $Str(J)_0$ preserves the orthosymplectic metric on J, so

$$\begin{pmatrix} (\nu k)^{-1} & \\ & (\nu h)^{-1} \end{pmatrix}^t \beta^{-1} \begin{pmatrix} (\nu k)^{-1} & \\ & (\nu h)^{-1} \end{pmatrix} = \beta^{-1}$$

with β the matrix corresponding to the metric. Hence

$$\underline{a}^\sharp(R^2) = \underline{a}^\sharp(z_i)\beta^{ij}\underline{a}^\sharp(z_j) = \mathrm{id}_{\mathcal{O}_{Str(J)_0}} \otimes z_i\beta^{ij}z_j = \mathrm{id}_{\mathcal{O}_{Str(J)_0}} \otimes R^2.$$

Therefore

$$a^\sharp(R^2) = [X \mapsto (-1)^{|X|}(1 \otimes \rho_a(X)R^2)].$$

So we get

$$b^\sharp(R^2) = (\mathrm{id}_{\mathcal{O}_{Str(J)}} \otimes j^\sharp)a^\sharp(R^2) = [X \mapsto (-1)^{|X|}(1 \otimes j^\sharp(\rho_a(X)R^2))] = 0,$$

since $\rho_a(X)R^2 = 0$ for $X \in \mathfrak{osp}(J)$ and for $X = L_e$ we use that R^2 evaluates to zero on $|C|$. □

For a primitive idempotent c of J^*, we define

$$\pi\colon Str(J) \to C \text{ by } \quad |\pi|g = |b|(g, c), \quad \pi^\sharp = (\mathrm{id}_{\mathcal{O}_{Str(J)}} \otimes \mathrm{ev}_c)b^\sharp.$$

Proposition 5.2.9. *The manifold C and the morphisms π and b satisfy the conditions of Proposition 5.1.4. In particular C is the manifold corresponding to the orbit through a primitive idempotent of J^* under the action of the structure group.*

Proof. Almost by definition, the map $|\pi|$ is the natural map from $|Str(J)|$ to $|C|$. To show that π is a submersion, we need

$$\mathrm{d}\pi_g \colon T_g Str(J) \to T_{|\pi|(g)} C$$

to be surjective for all $g \in |Str(J)|$. Consider $f \in \mathcal{O}_C(U)$ and let $\tilde{f} \in \mathcal{C}^\infty_{\mathbb{R}^{p+q-2}_{(0)}}(U) \otimes \Lambda^{2n}$ be a representative of f i.e. $f = \tilde{f} \mod R^2$. Let X be a vector field in $\mathfrak{istr}(J)$ and X_e the corresponding vector in $T_e Str(J)$. From [CCF, Proposition 7.2.3], we have $X_g := \mathrm{ev}_g X = \mathrm{ev}_g (1 \otimes X_e) \mu^\sharp$. Combining this with $\mathrm{ev}_c \circ j = \mathrm{ev}_c$, we compute

$$\begin{aligned} \mathrm{d}\pi_g(X_g) f &= X_g(\pi^\sharp f) \\ &= \mathrm{ev}_g(1 \otimes X_e)\mu^\sharp(\mathrm{id}_{\mathcal{O}_{Str(J)}} \otimes \mathrm{ev}_c) a^\sharp \tilde{f} \\ &= \mathrm{ev}_c(\mathrm{ev}_g \otimes \mathrm{id}_{\mathcal{O}_C})(\mathrm{id}_{\mathcal{O}_{Str(J)}} \otimes X_e \otimes \mathrm{id}_{\mathcal{O}_C})(\mu^\sharp \otimes \mathrm{id}_{\mathcal{O}_C}) a^\sharp \tilde{f}. \end{aligned}$$

For an action a of G on M it holds that $(\mu^\sharp \otimes \mathrm{id}_{\mathcal{O}_M}) a^\sharp = (\mathrm{id}_{\mathcal{O}_G} \otimes a^\sharp) a^\sharp$ and $\rho_a(X) = (X_e \otimes \mathrm{id}_{\mathcal{O}_M}) a^\sharp$. Thus we obtain

$$\begin{aligned} \mathrm{d}\pi_g(X_g) f &= \mathrm{ev}_c(\mathrm{ev}_g \otimes X_e \otimes \mathrm{id}_{\mathcal{O}_C})(\mathrm{id}_{\mathcal{O}_{Str(J)}} \otimes a^\sharp) a^\sharp \tilde{f} \\ &= \mathrm{ev}_c(X_e \otimes \mathrm{id}_{\mathcal{O}_C}) a^\sharp (\mathrm{ev}_g \otimes \mathrm{id}_{\mathcal{O}_C}) a^\sharp \tilde{f} \\ &= \mathrm{ev}_c(\rho_a(X) a^\sharp_g \tilde{f}). \end{aligned} \tag{5.1}$$

The map from $\mathfrak{osp}(p-1, q-1|2n)$ to $T_x \mathbb{R}^{p+q-2|2n}$ for $x \in \mathbb{R}^{p+q-2|2n}$ given by $L_{ij} \mapsto \mathrm{ev}_x \circ L_{ij}$ has codimension one. This follows for example from the fact that for i such that $\mathrm{ev}_x \circ z_i \neq 0$

$$\{\mathrm{ev}_x \circ \partial_i\} \cup \{\mathrm{ev}_x \circ L_{kl} \mid 0 < k, l \leq p+q-2+2n\}$$

span $T_x \mathbb{R}^{p+q-2|2n}$, so the codimension is less than or equal to one, while $L_{ij}(R^2) = 0$ implies that the codimension is not zero. Since $a^\sharp_g$ is surjective, we then conclude from equation (5.1) that

$$\dim\ \mathrm{im}\ (\mathrm{d}\pi_g \mid_{\mathfrak{osp}(J)}) = p + q - 3 + 2n.$$

Since the dimension of $T_{|\pi|(g)c} C$ is equal to $p+q-3+2n$ we conclude that also $\dim\ \mathrm{im}\ (\mathrm{d}\pi_g) = p+q-3+2n$ and $\mathrm{d}\pi_g$ is surjective.

Finally we have to show that b is an action that reduces to the natural action $|Str(J)| \times |C| \to |C|$ and $\pi \circ \mu = b \circ (\mathrm{id}_{Str(J)} \times \pi)$. We have

$|\pi| \circ |\mu|(g_1, g_2) = (g_1 g_2)c$ and $|b|(\mathrm{id}_{|Str(J)|} \times |\pi|)(g_1, g_2) = g_1(g_2 c)$. We also compute

$$\begin{aligned}\mu^\sharp \circ \pi^\sharp f &= \mu^\sharp(\mathrm{id}_{\mathcal{O}_{Str(J)}} \otimes \mathrm{ev}_c) b^\sharp f \\ &= (\mathrm{id}_{\mathcal{O}_{Str(J)}} \otimes \mathrm{id}_{\mathcal{O}_{Str(J)}} \otimes \mathrm{ev}_c)(\mu^\sharp \otimes \mathrm{id}_{\mathcal{O}_M}) a^\sharp f \\ &= (\mathrm{id}_{\mathcal{O}_{Str(J)}} \otimes \mathrm{id}_{\mathcal{O}_{Str(J)}} \otimes \mathrm{ev}_c)(\mathrm{id}_{\mathcal{O}_{Str(J)}} \otimes a^\sharp) a^\sharp f,\end{aligned}$$

and

$$\begin{aligned}(\mathrm{id}_{\mathcal{O}_{Str(J)}} \otimes \pi^\sharp) b^\sharp f &= (\mathrm{id}_{\mathcal{O}_{Str(J)}} \otimes \mathrm{id}_{\mathcal{O}_{Str(J)}} \otimes \mathrm{ev}_c)(\mathrm{id}_{\mathcal{O}_{Str(J)}} \otimes b^\sharp) b^\sharp f \\ &= (\mathrm{id}_{\mathcal{O}_{Str(J)}} \otimes \mathrm{id}_{\mathcal{O}_{Str(J)}} \otimes \mathrm{ev}_c)(\mathrm{id}_{\mathcal{O}_{Str(J)}} \otimes a^\sharp) a^\sharp f.\end{aligned}$$

Since b is almost by definition an action and reduces to the natural action on $|Str(J)| \times |C| \to |C|$, the proposition follows. □

5.2.3 Restriction to the minimal orbit

Recall that for a simple Jordan algebra the representation constructed in Section 4.5 corresponds to the representation of the conformal algebra considered in [HKM]. In the latter paper it is also shown that, for certain characters, this representation can be restricted to an orbit. We will show that for a specific character, also in our case the representation can be restricted to the minimal orbit defined in Section 5.2.

Consider the representation π_λ constructed in Section 4.5.2. For $\lambda = 2 - M$, with $M = p + q - 2 - 2n$ the superdimension of J, we can restrict the representation π_λ to the minimal orbit, as we will now show. We first prove that for this value of λ the Bessel operators are tangential to the minimal orbit.

Proposition 5.2.10. *The Bessel operators are tangential to the minimal orbit, i.e. they map $\langle R^2 \rangle$ into $\langle R^2 \rangle$, if and only if $\lambda = 2 - M$, with M the superdimension of J.*

Proof. Using the relations of Lemma 2.7.1 and equation (4.18) we obtain

$$[\mathcal{B}_\lambda(e_k), R^2] = z_k(-2\lambda + 4 - 2(p + q - 2 - 2n)) + 4R^2 \partial_k.$$

We conclude that $\langle R^2 \rangle$ gets mapped into $\langle R^2 \rangle$ if $\lambda = 2 - M$. □

The operators z_i, L_{ij} and $\mathbb{E}$ are also tangential to the orbit. Hence $\langle R^2\rangle$ gives a subrepresentation of π_{2-M}. Using the embedding j defined in Section 5.2.2, we set

$$\pi_C(X)f = j^\sharp(\pi_{2-M}(X)\tilde{f})$$

for f in $\Gamma(\mathcal{O}_C)$ and $\tilde{f}$ a representative from f in $\Gamma(\mathcal{O}_{\mathbb{A}(J^*)_0})$, i.e. $j^\sharp(\tilde{f}) = f$. Since all the operators occurring in π_{2-M} are tangential to C, this gives a well defined quotient representation

$$\pi_C\colon \mathrm{TKK}(J) \to \Gamma(\mathcal{D}_C)$$

on the orbit C. Here $\Gamma(\mathcal{D}_C)$ acts by differential operators on $\Gamma(\mathcal{O}_C)$, hence we found a representation of $\mathrm{TKK}(J)$ on functions on the minimal orbit.

5.3 Integration to the conformal group

We introduce the notations

$$\begin{aligned}
\mathfrak{g} &:= \mathrm{TKK}(J) = \mathfrak{osp}(p,q|2n),\\
\mathfrak{k} &:= \mathfrak{so}(p)\oplus\mathfrak{so}(q)\oplus\mathfrak{u}(n),\\
\mathfrak{k}' &:= \mathfrak{osp}(p|2n)\oplus\mathfrak{so}(q),\\
\mathfrak{k}_0' &:= \mathfrak{k}'\cap\mathfrak{istr}(J) = \mathfrak{osp}(p-1|2n)\oplus\mathfrak{so}(q-1).
\end{aligned}$$

Then $\mathfrak{k}$ is a maximal compact subalgebra of the even part of $\mathfrak{k}'$ and also a maximal compact subalgebra of the even part of $\mathfrak{g}$.

In this section we will integrate a subrepresentation π_C of $\mathfrak{g}$ on $\Gamma(\mathcal{O}_C)$ constructed in Section 5.2.3 to the conformal group $OSp(p,q|2n)$ using the concept of Harish-Chandra supermodules. To be able to do this we need a $(\mathfrak{g},\mathfrak{k})$-module W of $\mathfrak{k}$-finite vectors. As an intermediate step, we will first look for a $(\mathfrak{g},\mathfrak{k}')$-module of $\mathfrak{k}'$-finite vectors.

Remark 5.3.1. Our choice of $\mathfrak{k}'$ seems arbitrary, and one might as well work with $\mathfrak{osp}(q|2n)\oplus\mathfrak{so}(p)$. However, the same techniques can be used in this case which leads to similar results. Since $\mathfrak{osp}(p,q|2n)\cong\mathfrak{osp}(q,p|2n)$ it is enough to consider one of the two possible choices.

We start this section by introducing $\mathfrak{k}_0'$-invariant radial superfunctions.

5.3.1 Radial superfunctions

On $\mathbb{R}^{p-1}\oplus\mathbb{R}^{q-1}\oplus\mathbb{R}^{2n}$ we consider the supersymmetric, non-degenerate, even bilinear form β of signature $(p-1, q-1|2n)$ associated to $\mathfrak{osp}(p-1, q-1|2n)$. Choose a basis $(e_i)_i$, $(f_i)_i$, $(\theta_i)_i$ of $\mathbb{R}^{p-1}\oplus\mathbb{R}^{q-1}\oplus\mathbb{R}^{2n}$ such that

$$\langle e_i, e_j\rangle_\beta = \delta_{ij}, \qquad \langle f_i, f_j\rangle_\beta = -\delta_{ij}, \qquad \langle e_i, f_j\rangle_\beta = 0.$$

Let e^i, f^i and θ^i be the right duals of e_i, f_i and θ_i with respect to our form. Then $e^i = e_i$ and $f^i = -f_i$. We will use x_i, y_i and θ_i as the coordinates on $\mathbb{A}((\mathbb{R}^{p+q-2|2n})^*) \cong \mathbb{A}(\mathbb{R}^{p+q-2|2n})$ corresponding to this basis. Set

$$s^2 = \sum_{i=1}^{p-1} x_i^2, \quad t^2 = \sum_{j=1}^{q-1} y_j^2, \quad \theta^2 = \sum_{i=1}^{2n} \theta^i\theta_i.$$

For a function $h\colon \mathbb{R}\to\mathbb{R}$, $h\in\mathcal{C}^{2n}(\mathbb{R}_{(0)})$ and a superfunction $f = f_0 + \sum_{I\neq 0} f_I\theta^I$, with f_0 and f_I in $\mathcal{C}^\infty(\mathbb{R}^m_{(0)})$, a new superfunction $h(f)$ in $\mathcal{C}^\infty(\mathbb{R}^m_{(0)})\otimes\Lambda^{2n}$ is defined in [CDS2, Definition 3]

$$h(f) := \sum_{j=0}^{2n} \frac{(\sum_{I\neq 0} f_I\theta^I)^j}{j!} h^{(j)}(f_0).$$

We will use this definition to define radial functions depending on the superfunction $|X|$

$$|X| = \sqrt{\frac{s^2+t^2+\theta^2}{2}}.$$

Note that such a function $h(|X|)$ is $\mathfrak{k}_0'$ invariant since $|X|$ is $\mathfrak{k}_0'$ invariant.

Lemma 5.3.2. *Consider $h\colon \mathbb{R}\to\mathbb{R}$, $h\in\mathcal{C}^{2n+2}(\mathbb{R}_{(0)})$. The radial function $h(|X|)$ satisfies*

$$\partial_{x^i} h(|X|) = \frac{x_i}{2|X|}\partial_{|X|} h(|X|),$$
$$\partial_{y^i} h(|X|) = -\frac{y_i}{2|X|}\partial_{|X|} h(|X|),$$
$$\partial_{\theta^i} h(|X|) = \frac{\theta_i}{2|X|}\partial_{|X|} h(|X|),$$

$$\mathbb{E}h(|X|) = |X|\partial_{|X|}h(|X|),$$
$$\Delta h(|X|) = \tfrac{p-q-2n}{2|X|}\partial_{|X|}h(|X|) + \tfrac{R^2}{4|X|^2}\left(\partial^2_{|X|}h(|X|) - \frac{\partial_{|X|}h(|X|)}{|X|}\right),$$

and

$$(\mathcal{B}_\lambda(x_i) - x_i)h(|X|)$$
$$= x_i\left(\partial^2_{|X|}h(|X|) - (p-q-2n+\lambda)\frac{\partial_{|X|}h(|X|)}{2|X|} - h(|X|)\right),$$
$$(\mathcal{B}_\lambda(\theta_i) - \theta_i)h(|X|)$$
$$= \theta_i\left(\partial^2_{|X|}h(|X|) - (p-q-2n+\lambda)\frac{\partial_{|X|}h(|X|)}{2|X|} - h(|X|)\right),$$
$$(\mathcal{B}_\lambda(y_i) + y_i)h(|X|)$$
$$= -y_i\left(\partial^2_{|X|}h(|X|) + (p-q-2n-\lambda)\frac{\partial_{|X|}h(|X|)}{2|X|} - h(|X|)\right),$$

where the three last equalities are modulo R^2.

Proof. If f is an even superfunction and $h \in \mathcal{C}^{2n+1}(\mathbb{R}_{(0)})$, then we have the chain rule

$$\partial_{z^i}h(f) = \partial_{z^i}(f)h'(f).$$

Since $|X|$ is an even superfunction, the first three equations follow from this chain rule and

$$\partial_{x^i}|X| = \frac{x_i}{2|X|} \qquad \partial_{y^i}|X| = -\frac{y_i}{2|X|} \qquad \partial_{\theta^i}|X| = \frac{\theta_i}{2|X|}.$$

The other equations are then a straightforward corollary from these three equations. □

5.3.2 The $(\mathfrak{g}, \mathfrak{k}')$-module W

For our definition of W we start from a general $\mathfrak{k}'_0$-invariant function on $\Gamma(\mathcal{O}_C)$. Acting on this function with basis elements of $\mathfrak{k}'$ not in $\mathfrak{k}'_0$ leads to the differential equation (B.1) that the modified K-Bessel functions satisfy (see equation (5.7) in the proof of Lemma 5.3.6). So a natural ansatz for W is the $U(\mathfrak{g})$-module generated by $\widetilde{K}_\alpha$, the renormalised modified Bessel function of the third kind introduced in B.2.. Set

$$\mu = \max(p-2n-3, q-3), \qquad \nu = \min(p-2n-3, q-3).$$

We also set $\mathbb{R}^{\mu+2} = \mathbb{R}^{p-1|2n}$ and $\mathbb{R}^{\nu+2} = \mathbb{R}^{q-1}$ if $p-2n \geq q$ and $\mathbb{R}^{\mu+2} = \mathbb{R}^{q-1}$ and $\mathbb{R}^{\nu+2} = \mathbb{R}^{p-1|2n}$ if $p-2n < q$.

Let $\Lambda^{\mu,\nu}_{2,j}(|X|)$ be the radial superfunction defined using the generalised Laguerre function $\Lambda^{\mu,\nu}_{2,j}(z)$ introduced in B.3. Note that for $j=0$, we find $\Lambda^{\mu,\nu}_{2,0}(|X|) = \frac{1}{\Gamma(\frac{\mu+2}{2})}\widetilde{K}_{\frac{\nu}{2}}(|X|)$.

Define

$$W := U(\mathfrak{g})(\widetilde{K}_{\frac{\nu}{2}}(|X|) + \langle R^2\rangle) \subset \Gamma(\mathcal{O}_C), \tag{5.2}$$

where the $\mathfrak{g}$-module structure is given by the representation π_C. In the following, we will always work modulo $\langle R^2\rangle$ and drop $\langle R^2\rangle$ in our notation. So we write for example $\widetilde{K}_{\frac{\nu}{2}}(|X|)$ for $\widetilde{K}_{\frac{\nu}{2}}(|X|)+\langle R^2\rangle$.

Theorem 5.3.3. *Assume $\nu \notin -2\mathbb{N}$.*

1. *The decomposition of W as $\mathfrak{k}'$-module is given by*

$$W = \bigoplus_{j=0}^{\infty} W_j, \qquad \textit{where} \quad W_j = U(\mathfrak{k}')\Lambda^{\mu,\nu}_{2,j}(|X|).$$

2. *Assume $q \neq 3$ and $p \neq 3$. Then W is always indecomposable. It is furthermore a simple $\mathfrak{g}$-module if $p+q$ is odd or $\mu+\nu \geq 0$ or $q = p-2n = 2$ or $p = 2$.*

3. *If $p+q$ is even, then W_j and thus also W is $\mathfrak{k}'$-finite. An explicit decomposition of W_j into irreducible $\mathfrak{k}'_0$-modules is given by*

$$W_j = \bigoplus_{k=0}^{j} \bigoplus_{l=0}^{\frac{\mu-\nu}{2}+j} \Lambda^{\mu+2k,\nu+2l}_{2,j-k}(|X|)\mathcal{H}_k(\mathbb{R}^{\mu+2}) \otimes \mathcal{H}_l(\mathbb{R}^{\nu+2}). \tag{5.3}$$

   Here $\mathcal{H}_k(\mathbb{R}^{\mu+2})$ and $\mathcal{H}_l(\mathbb{R}^{\nu+2})$ are spaces of spherical harmonics introduced in Section 2.7.2. Furthermore, we also have the following $\mathfrak{k}'$-isomorphism

$$W_j \cong \mathcal{H}_j(\mathbb{R}^{\mu+3}) \otimes \mathcal{H}_{\frac{\mu-\nu}{2}+j}(\mathbb{R}^{\nu+3}).$$

4. *If $p+q$ is odd, the decomposition of W_j into irreducible $\mathfrak{k}_0'$-modules is given by*

$$W_j = \bigoplus_{k=0}^{j}\bigoplus_{l=0}^{\infty} \Lambda_{2,j-k}^{\mu+2k,\nu+2l}(|X|)\mathcal{H}_k(\mathbb{R}^{\mu+2}) \otimes \mathcal{H}_l(\mathbb{R}^{\nu+2}). \qquad (5.4)$$

If $p, q \neq 2$ then W_j is not $\mathfrak{k}'$-finite, while for $p = 2$ or $q = 2$, W_j is still $\mathfrak{k}'$-finite.

Proof. This is a combination of Proposition 5.3.7, Proposition 5.3.8, Corollary 5.3.9 and Proposition 5.3.10. □

Remark 5.3.4. This theorem gives new information even in the classical case (i.e. $n = 0$). Namely, for $\mathfrak{g} = \mathfrak{so}(\mathfrak{p}, \mathfrak{q})$ and $p+q$ even, it is well-known that the minimal representation W of $\mathfrak{g}$ decomposes as

$$W \simeq \bigoplus_{j=0}^{\infty} \mathcal{H}_j(\mathbb{R}^{\mu+3}) \otimes \mathcal{H}_{\frac{\mu-\nu}{2}+j}(\mathbb{R}^{\nu+3}),$$

and that for $p+q$ odd, W_0 is infinite-dimensional, but in the Schrödinger model only few $\mathfrak{k}$-finite vectors have been made explicit: The $\mathfrak{k}_0'$-invariant vectors $\Lambda_{2,j}^{\mu,\nu}$ were given in [HKMM, Corollary 8.2], which is the case $l = k = 0$. Further, the case $k = j$ and l arbitrary is contained in [KM2, Theorem 3.1.1]. However, the decomposition of W_j given in (5.3) and (5.4), which makes explicit all $\mathfrak{k}'$-finite vectors in the Schrödinger model if $p+q$ is even, is to the best of our knowledge new.

Remark 5.3.5. We have $\widetilde{K}_{\frac{\nu}{2}}(|X|) = \Gamma(\frac{\mu+2}{2})\Lambda_{2,0}^{\mu,\nu}(|X|)$. Hence

$$W_0 = U(\mathfrak{k}')\Lambda_{2,0}^{\mu,\nu}(|X|) = U(\mathfrak{k}')\widetilde{K}_{\frac{\nu}{2}}(|X|).$$

We start with a lemma which gives the action of some elements of $\mathfrak{k}'$ on a combination of Laguerre superfunctions with spherical harmonics. First we combine $\phi_k \in \mathcal{H}_k(\mathbb{R}^{p-1|2n})$ and $z_i \in \mathcal{P}_1(\mathbb{R}^{p-1|2n})$ to obtain spherical harmonics of degree $k+1$ and $k-1$. Namely we set

$$\begin{aligned} \phi_{k+1,i}^{+} &= z_i\phi_k - \frac{s^2+\theta^2}{p-3-2n+2k}\partial_{z^i}\phi_k, \\ \phi_{k-1,i}^{-} &= \frac{1}{p-3-2n+2k}\partial_{z^i}\phi_k, \end{aligned} \qquad (5.5)$$

for $p-3-2n+2k \neq 0$. One checks that $\phi^+_{k+1,i}$ and $\phi^-_{k-1,i}$ are contained in $\mathcal{H}_{k+1}(\mathbb{R}^{p-1|2n})$ and in $\mathcal{H}_{k-1}(\mathbb{R}^{p-1|2n})$ respectively. Similarly for $\phi_k \in \mathcal{H}_k(\mathbb{R}^{q-1})$, $y_i \in \mathcal{P}_1(\mathbb{R}^{q-1})$ and $q-3+2k \neq 0$, set

$$\phi^+_{k+1,i} = -y_i\phi_k - \frac{t^2}{q-3+2k}\partial_{y^i}\phi_k, \qquad \phi^-_{k-1,i} = \frac{1}{q-3+2k}\partial_{y^i}\phi_k.$$

In this case $\phi^+_{k+1,i}$ and $\phi^-_{k-1,i}$ are contained in $\mathcal{H}_{k+1}(\mathbb{R}^{q-1})$ and in $\mathcal{H}_{k-1}(\mathbb{R}^{q-1})$ respectively.

Lemma 5.3.6. *Let $\phi_k \in \mathcal{H}_k(\mathbb{R}^{\mu+2})$ and $\psi_l \in \mathcal{H}_l(\mathbb{R}^{\nu+2})$. Set*

$$\begin{aligned} &B^+_i := \mathcal{B}_\lambda(z_i) - z_i \quad \text{and} \quad B^-_i := \mathcal{B}_\lambda(y_i) + y_i \quad \text{if } p-2n \geq q,\\ &B^+_i := \mathcal{B}_\lambda(y_i) + y_i \quad \text{and} \quad B^-_i := \mathcal{B}_\lambda(z_i) - z_i \quad \text{if } p-2n < q, \end{aligned}$$

where $z_i = x_i$ and $z_{i+p-1} = \theta_i$. Then for $\nu + 2l \neq 0$ we have

$$\begin{aligned} B^+_i(\phi_k\psi_l\Lambda^{\mu+2k,\nu+2l}_{2,j-k}(|X|)) &= (j+\mu+k+1)\phi^+_{k+1,i}\psi_l\Lambda^{\mu+2(k+1),\nu+2l}_{2,j-(k+1)}(|X|)\\ &\quad + 4(j-k+1)\phi^-_{k-1,i}\psi_l\Lambda^{\mu+2(k-1),\nu+2l}_{2,j-(k-1)}(|X|)\\ B^-_i(\phi_k\psi_l\Lambda^{\mu+2k,\nu+2l}_{2,j-k}(|X|)) &= -(j+\tfrac{\mu-\nu}{2}-l)\phi_k\psi^+_{l+1,i}\Lambda^{\mu+2k,\nu+2(l+1)}_{2,j-k}(|X|)\\ &\quad - 4(j+\tfrac{\mu+\nu}{2}+l)\phi_k\psi^-_{l-1,i}\Lambda^{\mu+2k,\nu+2(l-1)}_{2,j-k}(|X|). \end{aligned}$$

Proof. We will only prove the case B^+_i for $p-2n \geq q$. The other cases are similar.

We first observe that the action of the Bessel operator on a product is given by

$$\begin{aligned} \mathcal{B}_\lambda(z_k)(fg) &= (-\lambda + 2\mathbb{E})\partial_k(fg) - z_k\Delta(fg) \qquad (5.6)\\ &= (\mathcal{B}_\lambda(z_k)f)g + (-1)^{|f||k|}f(\mathcal{B}_\lambda(z_k)g) + 2(-1)^{|f||k|}\mathbb{E}(f)\partial_k(g)\\ &\quad + 2\partial_k(f)\mathbb{E}(g) - 2z_k(-1)^{|f||j|}\beta^{ij}\partial_i(f)\partial_j(g). \end{aligned}$$

Note that this is a special case of Proposition 4.5.2.

From this product rule and Lemma 5.3.2 we obtain

$$\begin{aligned} &(\mathcal{B}_\lambda(z_i) - z_i)(\Lambda^{\mu+2k,\nu+2l}_{2,j-k}\phi_k\psi_l)\\ &= \left((\mathcal{B}_\lambda(z_i) - z_i)\Lambda^{\mu+2k,\nu+2l}_{2,j-k}\right)\phi_k\psi_l + \Lambda^{\mu+2k,\nu+2l}_{2,j-k}\mathcal{B}_\lambda(z_i)(\phi_k\psi_l)\\ &\quad + 2\mathbb{E}(\Lambda^{\mu+2k,\nu+2l}_{2,j-k})\partial_i(\phi_k\psi_l) + 2\partial_i(\Lambda^{\mu+2k,\nu+2l}_{2,j-k})\mathbb{E}(\phi_k\psi_l) \qquad (5.7) \end{aligned}$$

$$
\begin{aligned}
&- 2z_i\beta^{ab}\partial_a(\Lambda^{\mu+2k,\nu+2l}_{2,j-k})\partial_b(\phi_k\psi_l)\\
&= z_i\left((\Lambda^{\mu+2k,\nu+2l}_{2,j-k})'' + \frac{q-2}{|X|}(\Lambda^{\mu+2k,\nu+2l}_{2,j-k})' - \Lambda^{\mu+2k,\nu+2l}_{2,j-k}\right)\phi_k\psi_l \quad (5.8)\\
&\quad + \Lambda^{\mu+2k,\nu+2l}_{2,j-k}(\mu+\nu+2k+2l)\partial_i(\phi_k)\psi_l + 2\mathbb{E}(\Lambda^{\mu+2k,\nu+2l}_{2,j-k})\partial_i(\phi_k)\psi_l\\
&\quad + \frac{z_i}{|X|}(k+l)\Lambda^{\mu+2k,\nu+2l}_{2,j-k})'\phi_k\psi_l - \frac{z_i}{|X|}(k-l)(\Lambda^{\mu+2k,\nu+2l}_{2,j-k})'\psi_k\phi_l\\
&= (\phi^+_{k+1,i} + |X|^2\phi^-_{k-1,i})\psi_l(j-k+\mu+2k+1)\Lambda^{\mu+2k+2,\nu+2l}_{2,j-k-1}\\
&\quad + (\mu+2k)\phi^-_{k-1,i}\psi_l\,(\mu+2k+\nu+2l+2\mathbb{E})\,\Lambda^{\mu+2k,\nu+2l}_{2,j-k}\\
&= (j+\mu+k+1)\phi^+_{k+1,i}\psi_l\Lambda^{\mu+2(k+1),\nu+2l}_{2,j-(k+1)}\\
&\quad + 4(j-k+1)\phi^-_{k-1,i}\psi_l\Lambda^{\mu+2(k-1),\nu+2l}_{2,j-(k-1)}.
\end{aligned}
$$

In the last two steps we used Corollary B.3.2 and $s^2+\theta^2 = |X|^2 \mod R^2$. □

Proposition 5.3.7. *Assume $\nu \notin -2\mathbb{N}$. The decomposition of $W_j = U(\mathfrak{k}')\Lambda^{\mu,\nu}_{2,j}(|X|)$ as a $\mathfrak{k}'_0$-module is given by*

$$W_j = \bigoplus_{k=0}^{j}\bigoplus_{l=0}^{\frac{\mu-\nu}{2}+j}\Lambda^{\mu+2k,\nu+2l}_{2,j-k}(|X|)\mathcal{H}_k(\mathbb{R}^{\mu+2})\otimes\mathcal{H}_l(\mathbb{R}^{\nu+2}) \text{ if } p+q \text{ is even,}$$

$$W_j = \bigoplus_{k=0}^{j}\bigoplus_{l=0}^{\infty}\Lambda^{\mu+2k,\nu+2l}_{2,j-k}(|X|)\mathcal{H}_k(\mathbb{R}^{\mu+2})\otimes\mathcal{H}_l(\mathbb{R}^{\nu+2}) \text{ if } p+q \text{ is odd.}$$

If $p\neq 3$ and $q \neq 3$, then W_j is an indecomposable $\mathfrak{k}'$-module. If we also have $p+q$ odd or $j+\frac{\mu+\nu}{2}\geq 0$, then W_j is a simple $\mathfrak{k}'$-module. If $p=2$ or $q=2$, then W_j is always finite-dimensional.

Proof. Denote by (k,l) the space $\Lambda^{\mu+2k,\nu+2l}_{2,j-k}(|X|)\mathcal{H}_k(\mathbb{R}^{\mu+2})\otimes\mathcal{H}_l(\mathbb{R}^{\nu+2})$. We can extend a basis of $\mathfrak{k}'_0$ with the elements $\imath(\mathcal{B}_\lambda(x_i)-x_i)$, $i = 1,\ldots,p-1$, $\imath(\mathcal{B}_\lambda(y_i)+y_i)$, $i=1,\ldots,q-1$, $\imath(\mathcal{B}_\lambda(\theta_i)-\theta_i)$, $i=1,\ldots,2n$ to get a basis of $\mathfrak{k}'$.

First assume $q>3$ and $p>3$. In that case $\mathcal{H}_k(\mathbb{R}^{\mu+2})\otimes\mathcal{H}_l(\mathbb{R}^{\nu+2})$ is an irreducible $\mathfrak{k}'_0$-module. By Lemma 5.3.6, we can use B^+_i to go from (k,l) to $(k+1,l)$ and $(k-1,l)$. Note that $(k,l)=0$ if $k>j$, because then $\Lambda^{\mu+2k,\nu+2l}_{2,j-k}(|X|)=0$. Similar we can use B^-_i to go from (k,l) to $(k,l+1)$ and $(k,l-1)$ if $l\neq j+\frac{\mu-\nu}{2}$ and $l\neq -j-\frac{\mu+\nu}{2}$. If

$l = j + \frac{\mu-\nu}{2}$, then B_i^- maps (k,l) only to $(k,l-1)$ since the coefficient of the part in $(k,l+1)$ is zero. If $l = -j - \frac{\mu+\nu}{2}$ then B_i^- maps (k,l) to $(k,l+1)$ since now the coefficient of the part in $(k,l-1)$ is zero. Not that these two exceptional case can not occur if $p+q$ is odd, because then $\mu+\nu$ is also odd. The last case can also not occur if $j + \frac{\mu+\nu}{2} \geq 0$. Observe that W_j is the $\mathfrak{k}'$-module generated by $(0,0)$. Combining all this, we conclude that

$$W_j = \bigoplus_{k=0}^{j} \bigoplus_{l=0}^{\infty} (k,l),$$

if $p+q$ is odd. This module is $\mathfrak{k}'$-simple since the (k,l) are simple $\mathfrak{k}_0'$-modules and we can use B_i^+ and B_i^- to go from (k,l) to (k',l') for all $0 \leq k,k' \leq j$ and $0 \leq l,l' \leq \infty$. For $p+q$ even, we obtain

$$W_j = \bigoplus_{k=0}^{j} \bigoplus_{l=0}^{\frac{\mu-\nu}{2}+j} (k,l),$$

which is simple if $j + \frac{\mu+\nu}{2} \geq 0$. Otherwise W_j is still indecomposable but it has

$$\bigoplus_{k=0}^{j} \bigoplus_{l=-j-\frac{\mu+\nu}{2}}^{\frac{\mu-\nu}{2}+j} (k,l)$$

as a simple submodule.

If $q = 3$, then $\mathcal{H}_k(\mathbb{R}^{q-1})$ is no longer irreducible but decomposes in two submodules. However a real polynomial of degree k has components in both the subspaces of $\mathcal{H}_k(\mathbb{R}^{q-1})$ and if ϕ_{k-1} is real, also $\phi_{k,i}^+$ will be real. Therefore we conclude that the whole $\mathcal{H}_k(\mathbb{R}^{q-1})$ is contained in W_j and we still obtain

$$W_j = \bigoplus_{k=0}^{j} \bigoplus_{l=0}^{\infty} (k,l) \qquad \text{if } p+q \text{ is odd and}$$

$$W_j = \bigoplus_{k=0}^{j} \bigoplus_{l=0}^{\frac{\mu-\nu}{2}+j} (k,l) \qquad \text{if } p+q \text{ is even.}$$

However, these modules are no longer indecomposable, since $\phi_k \in \mathbb{C}(x \pm \imath y)^k$ implies $\phi_{k-1,i}^- \in \mathbb{C}(x \pm \imath y)^{k-1}$ and $\phi_{k+1,i}^+ \in \mathbb{C}(x \pm \imath y)^{k+1}$. So W_j decomposes in two submodules.

For $q = 2$ we have $\mathcal{H}_k(\mathbb{R}^{q-1}) = 0$ for $k \geq 2$. For $p - 2n \geq 2$ one still obtains,

$$W_j = \bigoplus_{k=0}^{j} \bigoplus_{l=0}^{\infty} (k,l) \qquad \text{if } p \text{ is odd and}$$

$$W_j = \bigoplus_{k=0}^{j} \bigoplus_{l=0}^{\frac{\mu-\nu}{2}+j} (k,l) \qquad \text{if } p \text{ is even}$$

but with $(k,l) = 0$ if $l \geq 2$. The module W_j is finite-dimensional and simple. For $p - 2n < 2$, the assumption $\nu \notin -2\mathbb{N}$ implies that p is even. We then get

$$W_j = \bigoplus_{k=0}^{\min(j,1)} \bigoplus_{l=0}^{\frac{\mu-\nu}{2}+j} (k,l),$$

which is simple if $j + \frac{\mu+\nu}{2} \geq 0$ or $j = 0$. For $j + \frac{\mu+\nu}{2} < 0$ and $j \neq 0$ it is indecomposable with $\bigoplus_{k=0}^{1} \bigoplus_{l=-j-\frac{\mu+\nu}{2}}^{\frac{\mu-\nu}{2}+j} (k,l)$ as simple submodule. For $p = 3$, the assumption $\nu \notin -2\mathbb{N}$ implies $2n = 0$ and $q = 2$. Then this case is considered in the case $q = 2$. However now W_j is not simple since $\mathcal{H}(\mathbb{R}^{p-1})$ is not simple. For $p = 2$ we have $\mathcal{H}_k(\mathbb{R}^{1|2n}) = 0$ if $k > 2n + 1$. Therefore we get

$$W_j = \begin{cases} \bigoplus_{k=0}^{j} \bigoplus_{l=0}^{2n+1} (k,l) & \text{if } q \text{ is odd} \\ \bigoplus_{k=0}^{j} \bigoplus_{l=0}^{\min(j+\frac{\mu-\nu}{2}, 2n+1)} (k,l) & \text{if } q \text{ is even.} \end{cases}$$

Then W_j is simple if $q \neq 3$. For $q = 3$, it decomposes in two simple submodules. □

Proposition 5.3.8. *If $\nu \notin -2\mathbb{N}$, then we have*

$$W = \bigoplus_{j=0}^{\infty} W_j.$$

Proof. Using $\pi_\lambda(L_e) = \frac{\lambda}{2} - \mathbb{E}$ and Proposition B.3.4, we obtain for $\lambda = -(\mu + \nu + 2)$

$$\pi_\lambda(-L_e) \left(\phi_k \psi_l \Lambda_{2,j-k}^{\mu+2k,\nu+2l}(|X|) \right)$$

$$
\begin{aligned}
&= \phi_k\phi_l \left(\left(\mathbb{E} + \frac{\mu+\nu+2}{2} \right) \Lambda^{\mu+2k,\nu+2l}_{2,j-k}(|X|) \right) \\
&\quad + (k+l)\phi_k\psi_l\Lambda^{\mu+2k,\nu+2l}_{2,j-k}(|X|) \\
&= \frac{(j-k+1)(j+k+\mu+1)}{2j+\mu+1}\phi_k\psi_l\Lambda^{\mu+2k,\nu+2l}_{2,j+1-k}(|X|) \\
&\quad - \frac{(j+l+\frac{\mu+\nu}{2})(j-l+\frac{\mu-\nu}{2})}{2j+\mu+1}\phi_k\psi_l\Lambda^{\mu+2k,\nu+2l}_{2,j-1-k}(|X|),
\end{aligned}
$$

if $j \neq 0$. If $j = 0$, then

$$\pi_\lambda(-L_e)\left(\psi_l\Lambda^{\mu,\nu+2l}_{2,0}(|X|)\right) = \psi_l\Lambda^{\mu,\nu+2l}_{2,1}.$$

So the result of the action of L_e on W_j has a non-zero component in W_{j+1}. By repeatedly acting by L_e on W_0 we obtain

$$\bigoplus_{j=0}^{\infty} W_j \subseteq W.$$

We will now show that the action of an element X in $\mathfrak{g}$ on W_j is contained in $W_{j-1} \oplus W_j \oplus W_{j+1}$. Here we set $W_{-1} = 0$. We have $\mathfrak{g} = \mathfrak{k}' + \mathfrak{p}$ with $\mathfrak{p} = [\mathfrak{k}', [\mathfrak{k}', L_e]]$. Hence, we can write every $X \in \mathfrak{g}$ as

$$X = Y_1 + [Y_2, [Y_3, L_e]],$$

where Y_1, Y_2, Y_3 are in $\mathfrak{k}'$. Because $\mathfrak{k}'$ leaves W_j invariant and L_e maps W_j into $W_{j-1} \oplus W_{j+1}$, also X maps W_j into $W_{j-1} \oplus W_j \oplus W_{j+1}$. Therefore

$$W \subseteq \bigoplus_{j=0}^{\infty} W_j,$$

which proves the proposition. □

Corollary 5.3.9. *Assume $\nu \notin -2\mathbb{N}$ and $q \neq 3$ and $p \neq 3$. Then W is a simple $\mathfrak{g}$-module if $p+q$ is odd or $\mu+\nu \geq 0$ or $q = p-2n = 2$ or $p = 2$. Otherwise it is still indecomposable and has*

$$\bigoplus_{j=0}^{\infty}\bigoplus_{k=0}^{j}\bigoplus_{l=\min(0,-j-\frac{\mu+\nu}{2})}^{\frac{\mu-\nu}{2}+j} \Lambda^{\mu+2k,\nu+2l}_{2,j-k}(|X|)\mathcal{H}_k(\mathbb{R}^{\mu+2}) \otimes \mathcal{H}_l(\mathbb{R}^{\nu+2}),$$

as simple submodule.

Proof. Remark that L_e maps the simple submodule of W_j into the simple submodules of W_{j-1} and W_{j+1} for $\frac{\mu+\nu}{2}+j<0$. Then the corollary follows simply from Proposition 5.3.7 and (the proof of) Proposition 5.3.8. □

Proposition 5.3.10. *For $p+q$ even and $\nu \notin -2\mathbb{N}$ we have*

$$W_j \cong \mathcal{H}_j(\mathbb{R}^{\mu+3}) \otimes \mathcal{H}_{\frac{\mu-\nu}{2}+j}(\mathbb{R}^{\nu+3}),$$

as $\mathfrak{k}'$-module.

Let s_0 and t_0 be the extra even coordinates from extending $\mathbb{R}^{\mu+2}$ and $\mathbb{R}^{\nu+2}$ to $\mathbb{R}^{\mu+3}$ and $\mathbb{R}^{\nu+2}$. Define also $S=\sqrt{s^2+\theta^2+s_0^2}$, $T=\sqrt{t^2+t_0^2}$ if $p-2n\geq q$ or $S=\sqrt{t^2+s_0^2}$, $T=\sqrt{s^2+\theta^2+t_0^2}$ if $p-2n<q$.

The $\mathfrak{k}'$-intertwining map is explicitly given by Φ:

$$\begin{aligned}
&\bigoplus_{k=0}^{j}\bigoplus_{l=0}^{\frac{\mu-\nu}{2}+j} \Lambda_{2,j-k}^{\mu+2k,\nu+2l}(|X|)\mathcal{H}_k(\mathbb{R}^{\mu+2})\otimes\mathcal{H}_l(\mathbb{R}^{\nu+2})\\
&\to \mathcal{H}_j(\mathbb{R}^{\mu+3})\otimes\mathcal{H}_{\frac{\mu-\nu}{2}+j}(\mathbb{R}^{\nu+3})\\
&\phi_k\psi_l\Lambda_{2,j-k}^{\mu+2k,\nu+2l}(|X|)\\
&\mapsto c_k d_l \phi_k\psi_l S^{j-k}T^{j-l+\frac{\mu-\nu}{2}}\widetilde{C}_{j-k}^{k+\frac{\mu+1}{2}}\left(\frac{s_0}{S}\right)\widetilde{C}_{j-l+\frac{\mu-\nu}{2}}^{l+\frac{\nu+1}{2}}\left(\frac{t_0}{T}\right),
\end{aligned} \tag{5.9}$$

where $\widetilde{C}_n^{\lambda}(z)$ are the normalised Gegenbauer polynomial introduced in B.1. The constants c_k and d_l are given by

$$c_k=\frac{(-4\imath)^k}{(\mu+j+1)_k}, \qquad d_l=\frac{(4\imath)^l}{(-j-\frac{\mu-\nu}{2})_l}$$

where we used the Pochhammer symbol $(a)_k=a(a+1)(a+2)\cdots(a+k-1)$.

Proof. Assume $p-2n\geq q$; the case $p-2n<q$ is again similar. A straightforward calculation shows that the right-hand side of (5.9) is indeed contained in $\mathcal{H}_j(\mathbb{R}^{p|2n})\otimes\mathcal{H}_{\frac{\mu-\nu}{2}+j}(\mathbb{R}^q)$. We need to prove

1. $L_{ij}\left(\Phi\left(\phi_k\psi_l\Lambda_{2,j-k}^{\mu+2k,\nu+2l}\right)\right)$
   $=\Phi\left(L_{ij}\left(\phi_k\psi_l\Lambda_{2,j-k}^{\mu+2k,\nu+2l}\right)\right)$ for $L_{ij}\in\mathfrak{k}_0'$,

2. $2L_{k0}\left(\Phi\left(\phi_k\psi_l\Lambda^{\mu+2k,\nu+2l}_{2,j-k}\right)\right)$
$= \Phi\left(\imath(\mathcal{B}_\lambda(z_k)-z_k)\left(\phi_k\psi_l\Lambda^{\mu+2k,\nu+2l}_{2,j-k}\right)\right)$
3. $2L_{kq}\left(\Phi\left(\phi_k\psi_l\Lambda^{\mu+2k,\nu+2l}_{2,j-k}\right)\right)$
$= \Phi\left(-\imath(\mathcal{B}_\lambda(y_k)+y_k)\left(\phi_k\psi_l\Lambda^{\mu+2k,\nu+2l}_{2,j-k}\right)\right).$

Because S and T are $\mathfrak{k}_0'$-invariant (1) follows immediately. For (2), we need the properties of the the Gegenbauer polynomial, see B.1,

$$\partial_z\widetilde{C}^\lambda_m(z) = 2\widetilde{C}^{\lambda+1}_{m-1}(z)$$

and

$$\begin{aligned}4(1-z^2)\widetilde{C}^{\lambda+1}_{m-1}(z) - 2z(2\lambda-1)\widetilde{C}^\lambda_m(z)&\\ = -(m+1)(2\lambda+m-1)\widetilde{C}^{\lambda-1}_{m+1}(z).&\end{aligned}$$

Then we compute, using $\phi^+_{k+1,i}$ and $\phi^-_{k-1,i}$ introduced in (5.5),

$$\begin{aligned}&L_{i0}\left(\phi_k\widetilde{C}^{k+\frac{\mu+1}{2}}_{j-k}\left(\frac{s_0}{S}\right)\right)\\ &= -s_0\partial_i(\phi_k)\widetilde{C}^{k+\frac{\mu+1}{2}}_{j-k}\left(\frac{s_0}{S}\right) + \frac{z_i}{S}\phi_k 2\widetilde{C}^{k+\frac{\mu+1}{2}+1}_{j-k-1}\left(\frac{s_0}{S}\right)\\ &= -s_0(\mu+2k)\phi^-_{k-1,i}\widetilde{C}^{k+\frac{\mu+1}{2}}_{j-k}\left(\frac{s_0}{S}\right) + \frac{2}{S}(\phi^+_{k+1,i}\\ &\quad + (S^2-s_0)^2\phi^-_{k-1,i})\widetilde{C}^{k+\frac{\mu+1}{2}+1}_{j-k-1}\left(\frac{s_0}{S}\right)\\ &= \frac{2}{S}\phi^+_{k+1,i}\widetilde{C}^{k+\frac{\mu+1}{2}+1}_{j-k-1}\left(\frac{s_0}{S}\right)\\ &\quad - (j-k+1)(\mu+k+j)\frac{S}{2}\phi^-_{k-1,i}\widetilde{C}^{k+\frac{\mu+1}{2}-1}_{j-k+1}\left(\frac{s_0}{S}\right).\end{aligned}$$

Thus we get

$$\begin{aligned}&2L_{i0}\left(\Phi\left(\phi_k\psi_l\Lambda^{\mu+2k,\nu+2l}_{2,j-k}\right)\right)\\ &= 4c_kd_l\phi^+_{k+1,i}\psi_lS^{j-k-1}T^{j-l+\frac{\mu-\nu}{2}}\widetilde{C}^{k+\frac{\mu+1}{2}+1}_{j-k-1}\left(\frac{s_0}{S}\right)\widetilde{C}^{l+\frac{\nu+1}{2}}_{j-l+\frac{\mu-\nu}{2}}\left(\frac{t_0}{T}\right)\\ &\quad - (j-k+1)(\mu+k+j)c_kd_l\phi^-_{k-1,i}\psi_lS^{j-k+1}T^{j-l+\frac{\mu-\nu}{2}}\cdot\\ &\quad \widetilde{C}^{k+\frac{\mu+1}{2}-1}_{j-k+1}\left(\frac{s_0}{S}\right)\widetilde{C}^{l+\frac{\nu+1}{2}}_{j-l+\frac{\mu-\nu}{2}}\left(\frac{t_0}{T}\right)\end{aligned}$$

$$
\begin{aligned}
&= \Phi\Big(\imath(j+\mu+k+1)\phi^{+}_{k+1,i}\psi_l\Lambda^{\mu+2k+2,\nu+2l}_{2,j-k-1}\\
&\quad +4\imath(j-k+1)\phi^{-}_{k-1,i}\psi_l\Lambda^{\mu+2k-2,\nu+2l}_{2,j-k+1}\Big)\\
&= \Phi\Big(\imath(\mathcal{B}_\lambda(z_i)-z_i)\phi_k\psi_l\Lambda^{\mu+2k,\nu+2l}_{2,j-k}\Big),
\end{aligned}
$$

where we used Lemma 5.3.6 in the last step. This proves (2). The case (3) is proven in a similar way. We conclude that Φ is a $\mathfrak{k}'$-intertwining map. One can check that there exists an element in the simple submodule of W_j with non-zero image. Hence Φ is injective. Since $\dim W_j = \dim \mathcal{H}_j(\mathbb{R}^{\mu+3}) \otimes \mathcal{H}_{\frac{\mu-\nu}{2}+j}(\mathbb{R}^{\nu+3})$, we conclude that Φ is an isomorphism. □

Remark 5.3.11. If $\nu \in -2\mathbb{N}-1$, then the $\mathfrak{osp}(p|2n)$-module $\mathcal{H}_k(\mathbb{R}^{\nu+3})$ is irreducible if $k > -1-\nu$ or $k < -\frac{1-\nu}{2}$. It is always indecomposable. This is [Cou, Theorem 5.2]. This is in correspondence with Proposition 5.3.7, since $\frac{\mu-\nu}{2} + j > -1-\nu$ is equivalent with $\frac{\mu+\nu}{2} + j \geq 0$, which was the condition for irreducibility of W_j.

5.3.3 Harish-Chandra supermodules

Let $G = (G_0, \mathfrak{g}, \sigma)$ be a Lie supergroup such that G_0 is almost connected and real reductive. Let K be a maximal compact subgroup of G_0.

Definition 5.3.12 ([Al, Definition 4.1])**.** *Let V be a complex super-vector space. Then V is a $(\mathfrak{g}, K)$-module if it is a locally finite K-representation which has also a compatible $\mathfrak{g}$-module structure, i.e. the derived action of K agrees with the $\mathfrak{k}$-module structure and*

$$k \cdot (X \cdot v) = (\sigma(k)X) \cdot (k \cdot v) \text{ for all } k \in K,\ X \in \mathfrak{g},\ v \in V.$$

A $(\mathfrak{g}, K)$-module is a Harish-Chandra supermodule if it is finitely generated over $U(\mathfrak{g})$ and is K-multiplicity finite.

A K-module W is K-multiplicity finite if every simple K-module occurs only a finite number of times in the decomposition of W.

Let $G = (SO(p,q)_e \times Sp(2n,\mathbb{R}), \mathrm{TKK}(J), \sigma)$ be the identity component of the conformal Lie supergroup defined in Section 5.1.3 and $K = SO(p) \times SO(q) \times U(n)$ be the maximal compact subgroup of $SO(p,q)_e \times Sp(2n,\mathbb{R})$. The Lie algebra of K is given by $\mathfrak{k} =$

$\mathfrak{so}(p) \oplus \mathfrak{so}(q) \oplus \mathfrak{u}(n)$. If $p+q$ is even and $\nu \notin -2\mathbb{N}$, we can define a K representation on the $\mathfrak{g}$-module W using the natural action of $SO(p) \times Sp(2n, \mathbb{R})$ on $\mathcal{H}_l(\mathbb{R}^{p|2n})$ and the action of $SO(q)$ on $\mathcal{H}_l(\mathbb{R}^q)$.

Proposition 5.3.13. *The module W is a Harish-Chandra supermodule if $p+q$ is even and $\nu \notin -2\mathbb{N}$.*

Proof. From the decomposition, Theorem 5.3.3,

$$W \cong \sum_{l=0}^{\infty} \mathcal{H}_{\frac{\mu-\nu}{2}+l}(\mathbb{R}^{p|2n}) \otimes \mathcal{H}_l(\mathbb{R}^q) \qquad \text{if } p-2n \leq q, \text{ or}$$
$$W \cong \sum_{l=0}^{\infty} \mathcal{H}_l(\mathbb{R}^{p|2n}) \otimes \mathcal{H}_{\frac{\mu-\nu}{2}+l}(\mathbb{R}^q) \qquad \text{if } p-2n \geq q,$$

it immediately follows that W is locally K-finite. This decomposition also implies that W is $SO(q)$-multiplicity finite since $\mathcal{H}_l(\mathbb{R}^q)$ is an irreducible $SO(q)$-module (or decomposes in just two irreducible subspaces if $q=2$) with $\mathcal{H}_l(\mathbb{R}^q) \ncong \mathcal{H}_k(\mathbb{R}^q)$ if $l \neq k$, while $\mathcal{H}_{\frac{\mu-\nu}{2}+l}(\mathbb{R}^{p|2n})$ is finite-dimensional with trivial $SO(q)$ action. If there would be a simple K-module which has infinite multiplicity in the decomposition of W, this would imply that all the simple $SO(q)$-modules contained in this K-module also have infinite multiplicity. Therefore we conclude that W is K-multiplicity finite.

By definition the derived action of K agrees with the action of $\mathfrak{k} \subset \mathfrak{g}$. To show $k \cdot (X \cdot v) = (\sigma(k)X) \cdot (k \cdot v)$ for all $k \in K$, $X \in \mathfrak{g}$, $v \in V$, we remark that we can write k as the product of elements of the form $\exp(Y)$ for $Y \in \mathfrak{k}$. Since Y satisfies

$$Y(X \cdot v) = \mathrm{ad}_Y(X) \cdot v + X(Y \cdot v),$$

it follows that

$$\exp Y(X \cdot v) = (\sigma(\exp Y)(X)) \cdot (\exp Y \cdot v).$$

This finishes the proof □

The importance of the previous proposition lies in the fact that it allows us to integrate our representation to group level. In [Al], it is proven that a Harish-Chandra supermodule has a (unique) smooth

Fréchet globalisation. This means that we have a Fréchet space and a representation of the Lie supergroup on this Fréchet space for which the space of K-finite vectors is the Harish-Chandra supermodule.

Corollary 5.3.14. *The* $(\mathfrak{g}, K)$*-module* W *integrates to a unique smooth Fréchet representation of moderate growth for the Lie supergroup* G*.*

Proof. This follows from combining Proposition 5.3.13 and Theorem A in [Al]. □

You look at where you're going and where you are and it never makes sense, but then you look back at where you've been and a pattern seems to emerge.

Robert Pirsig,
Zen and the art of motorcycle maintenance

6 Properties of the minimal representation

We will now investigate some properties that the minimal representation constructed in the previous chapter satisfies. In Section 6.1, we will show that the annihilator ideal is equal to a Joseph-like ideal if the superdimension satisfies $p + q - 2n > 2$. In this sense our representation is indeed a super version of a minimal representation since minimal representations for Lie groups are characterised by the property that their annihilator ideal is the Joseph ideal. The classical Joseph ideals and the Joseph ideal for $\mathfrak{osp}(m|2n)$ with $m - 2n > 2$ have the property that any ideal which contains the Joseph ideal and which has still infinite codimension is equal to the Joseph ideal, as follows from the characterisation by Garfinkle [Ga]. So the Joseph ideal is in this sense the biggest ideal with infinite codimension. If $p + q - 2n \leq 2$, we still have that the annihilator ideal contains the Joseph ideal, but due to the lack of the characterisation by Garfinkle in this case, we no longer know whether the annihilator ideal is equal to the Joseph ideal.

We will also calculate the Gelfand–Kirillov dimension of our representation in Section 6.2. We find that it is equal to $p - q - 3$ which is also the Gelfand–Kirillov dimension of the minimal representation of $O(p,q)$. As mentioned in Chapter 1, we know that there are no unitary representations of $\mathfrak{osp}(p,q|2n)$. However, we can still construct a non-degenerate, superhermitian, sesquilinear bilinear form. This is done in Section 6.3 where we also show that our representation is skew-symmetric with respect to this form if $p+q-2n \geq 6$.

6.1 The Joseph ideal for $\mathfrak{osp}(p,q|2n)$

In the classical case a minimal representation for a simple real Lie group G is a unitary representation such that the annihilator ideal of the derived representation in the universal enveloping algebra of $\mathrm{Lie}(G)_{\mathbb{C}}$ is the Joseph ideal. We will show that the representation π_C has as annihilator ideal the generalisation of the Joseph ideal for the $\mathfrak{osp}$-case. To do this, we will use the Fourier transformed representation. So we start this section by introducing the super Fourier transform.

6.1.1 The super Fourier transform

Consider $\mathcal{S}(\mathbb{R}^m)$ the Schwartz space of rapidly decreasing functions and the dual space $\mathcal{S}'(\mathbb{R}^m)$ of tempered distributions. The (even) Fourier transform $\mathbb{F}^{\pm}_{even}$ on $\mathcal{S}(\mathbb{R}^m)$ is given by

$$\mathbb{F}^{\pm}_{even} f(y) = \frac{1}{(2\pi)^{\frac{m}{2}}} \int_{\mathbb{R}^m} \exp\left(\pm\imath \sum_{i,j=1}^{m} z^i(x)\beta^s_{ij} z^j(y) \right) f(x)\mathrm{d}x,$$

where β^s is the symmetric part of the orthosymplectic metric. The Fourier transform can easily be extended to the dual space $\mathcal{S}'(\mathbb{R}^m)$, by duality. It satisfies $\mathbb{F}^{\pm}_{even}\mathbb{F}^{\mp}_{even} = \mathrm{id}$.

Let $\Lambda^{4n} := \Lambda(\mathbb{R}^{2n} \oplus \mathbb{R}^{2n})$ be generated by θ_i, η_i, $i = 1, \ldots, 2n$, with the relations $\theta_i\theta_j = -\theta_j\theta_i$, $\eta_i\eta_j = -\eta_j\eta_i$, $\theta_i\eta_j = -\eta_j\theta_i$. It contains two copies of $\Lambda^{2n} := \Lambda(\mathbb{R}^{2n})$, one generated by η_i and one generated by θ_i. Then we set $K^{\pm}(\theta,\eta) := \exp(\mp\imath \sum_{i,j} \theta^i \beta^a_{ij} \eta^j)$ where β^a is the antisymmetric part of the orthosymplectic metric. Define the odd

Fourier transform by

$$\mathbb{F}^{\pm}_{odd}\colon \Lambda^{2n} \to \Lambda^{2n}; \quad \mathbb{F}^{\pm}_{odd}(f) = \int_{B,\theta} K^{\pm}(\theta, \eta) f(\theta),$$

where $\int_{B,\theta} = \partial_{\theta_{2n}} \dots \partial_{\theta_1}$ is the Berezin integral, see e.g. [Le]. The odd Fourier transform satisfies $\mathbb{F}^{\pm}_{odd}\mathbb{F}^{\mp}_{odd} = \mathrm{id}$.

Then the super Fourier transform $\mathbb{F}^{\pm}\colon \mathcal{S}'(\mathbb{R}^m) \otimes \Lambda^{2n} \to \mathcal{S}'(\mathbb{R}^m) \otimes \Lambda^{2n}$ is given by

$$\mathbb{F}^{\pm}(f) := \sum_{I \in \mathbb{Z}_2^n} \mathbb{F}^{\pm}_{even}(f_I) \mathbb{F}^{\pm}_{odd}(\theta^I) \qquad \text{for } f = \sum_{I \in \mathbb{Z}_2^n} f_I \theta^I.$$

The super Fourier transform has the following properties.

Proposition 6.1.1. *Let $(e_k)_k$ be a basis of $\mathbb{R}^{m|2n}$ and $(e^k)_k$ its right dual basis with respect to the orthosymplectic metric and z_k, z^k the corresponding coordinate functions. Then we have*

$$\begin{aligned} \mathbb{F}^{\pm}(\partial_k f) &= \mp \imath z_k \mathbb{F}^{\pm}(f) \\ \mathbb{F}^{\pm}(z_k f) &= \mp \imath \partial_k \mathbb{F}^{\pm}(f) \\ \mathbb{F}^{\pm}\mathbb{F}^{\mp} &= \mathrm{id}. \end{aligned}$$

Proof. See [De, Theorem 7 and Lemma 3]. Remark that the metric used in op. cit. is not orthosymplectic. However, the same results and proofs still hold, mutatis mutandis. □

6.1.2 Fourier-transformed and adjoint representation

Using an isomorphism between $\mathbb{A}(J^*)$ and $\mathbb{A}^{p+q-2|2n}$, we can restrict the representation π_λ defined in Section 4.5.2 to $\mathcal{S}(\mathbb{R}^{p+q-2}) \otimes \Lambda^{2n}$. Then we can also immediately extend π_λ to $\mathcal{S}'(\mathbb{R}^{p+q-2}) \otimes \Lambda^{2n}$ since π_λ is given by differential operators. We will denote this extension also by π_λ. We will use the notations $\pi_{\lambda,\mathcal{S}}$ and $\pi_{\lambda,\mathcal{S}'}$ if we want to specify on which space the representations acts.

We define $\hat{\pi}_\lambda$ on $\mathcal{S}(\mathbb{R}^{p+q-2}) \otimes \Lambda^{2n}$ or $\mathcal{S}'(\mathbb{R}^{p+q-2}) \otimes \Lambda^{2n}$ using the super Fourier transform introduced in Section 6.1.1:

$$\hat{\pi}_\lambda(X) := \mathbb{F}^{-} \circ \pi_{-\lambda-2M}(X) \circ \mathbb{F}^{+}.$$

From Proposition 6.1.1 it follows that $\hat{\pi}_\lambda(X)$ is given by

1. $\hat{\pi}_\lambda(0,0,e_k) = \partial_k$ for $e_k \in J^-$
2. $\hat{\pi}_\lambda(0,L_{kl},0) = z_k\partial_l - (-1)^{|k||l|}z_l\partial_k$ for $L_{kl} \in \mathfrak{osp}_(J)$
3. $\hat{\pi}_\lambda(0,L_e,0) = -\frac{\lambda}{2} + \mathbb{E}$
4. $\hat{\pi}_\lambda(\bar{e}_k,0,0) = -z_k(2\mathbb{E}-\lambda) + R^2\partial_k$ for $\bar{e}_k \in J^+$.

Proposition 6.1.2. *The kernel of the Laplace operator Δ is a subrepresentation of $\hat{\pi}_\lambda$ if and only if $\lambda = 2-M$, with $M = p+q-2-2n$ the superdimension of J.*

Proof. Using Lemma 2.7.1, we find

$$\begin{aligned}[\Delta, \hat{\pi}_\lambda(\bar{e}_k,0,0)] &= [\Delta, -z_k(2\mathbb{E}-\lambda)] + [\Delta, R^2\partial_k]\\ &= 2(\lambda-2+M)\partial_k - 4z_k\Delta.\end{aligned}$$

Hence, $\hat{\pi}_\lambda(\bar{e}_k,0,0)$ preserves the kernel of Δ if and only if $\lambda = 2-M$. One verifies similarly that $\hat{\pi}_\lambda(0,0,e_k)$, $\hat{\pi}_\lambda(0,L_{kl},0)$ and $\hat{\pi}_\lambda(0,L_e,0)$ also preserve the kernel of Δ. □

Denote by $\langle\phi,f\rangle_{\mathcal{S}}$ the value of the action of $\phi \in \mathcal{S}'(\mathbb{R}^{p+q-2})\otimes\Lambda^{2n}$ on $f \in \mathcal{S}(\mathbb{R}^{p+q-2})\otimes\Lambda^{2n}$. If $\phi \in \mathcal{S}'(\mathbb{R}^{p+q-2})\otimes\Lambda^{2n}$ is an element of $\ker\Delta$, then for all $f \in \mathcal{S}(\mathbb{R}^{p+q-2})\otimes\Lambda^{2n}$,

$$0 = \langle\phi, \Delta\mathbb{F}^+ f\rangle_{\mathcal{S}} = -\langle\phi, \mathbb{F}^+R^2 f\rangle_{\mathcal{S}} = -\langle\mathbb{F}^+\phi, R^2 f\rangle_{\mathcal{S}}.$$

Hence the Fourier transform of $\ker\Delta$ consist of elements contained in $\mathcal{S}(\mathbb{R}^{p+q-2})'\otimes\Lambda^{2n}$ with support contained in the closure of $|C|$.

For $A \in \mathrm{End}(\mathcal{S}(\mathbb{R}^{p+q-2})\otimes\Lambda^{2n})$ define the adjoint operator $A^* \in \mathrm{End}(\mathcal{S}'(\mathbb{R}^{p+q-2})\otimes\Lambda^{2n})$ by

$$\langle\overline{A^*\phi}, f\rangle_{\mathcal{S}} = (-1)^{|A||\phi|}\langle\phi, Af\rangle_{\mathcal{S}},$$

for ϕ in $\mathcal{S}'(\mathbb{R}^{p+q-2})\otimes\Lambda^{2n}$ and f in $\mathcal{S}(\mathbb{R}^{p+q-2})\otimes\Lambda^{2n}$. Here $\overline{A^*\phi}$ is the complex conjugate of $A^*\phi$.

Proposition 6.1.3. *The adjoint operator $\pi_\lambda(X)^*$ is equal to the operator $-\pi_{-\lambda-2M}(X)$. Similar the adjoint operator $\hat{\pi}_\lambda(X)^*$ is equal to $-\hat{\pi}_{-\lambda-2M}(X)$.*

Proof. On $\mathcal{S}(\mathbb{R}^{p+q-2}) \otimes \Lambda^{2n}$ we have

$$\langle \partial_k \phi, f\rangle_{\mathcal{S}} := -(-1)^{|k||\phi|}\langle \phi, \partial_k f\rangle_{\mathcal{S}}$$
$$\langle z_k \phi, f\rangle_{\mathcal{S}} := (-1)^{|k||\phi|}\langle \phi, z_k f\rangle_{\mathcal{S}}.$$

Using this we obtain

$$\begin{aligned}\langle -iz_k\phi, f\rangle_{\mathcal{S}} &= -(-1)^{|k||\phi|}\langle \phi, iz_k f\rangle_{\mathcal{S}},\\ \langle (-\mathbb{E} - \frac{\lambda+2M}{2})\phi, f\rangle_{\mathcal{S}} &= -\langle \phi, (-\mathbb{E} + \frac{\lambda}{2}) f\rangle_{\mathcal{S}},\\ \langle L_{ij}\phi, f\rangle_{\mathcal{S}} &= -(-1)^{(|i|+|j|)|\phi|}\langle \phi, L_{ij} f\rangle_{\mathcal{S}},\\ \langle -i\mathcal{B}_{-\lambda-2M}(z_k)\phi, f\rangle_{\mathcal{S}} &= -(-1)^{|k||\phi|}\langle \phi, i\mathcal{B}_{\lambda}(z_k) f\rangle_{\mathcal{S}}.\end{aligned}$$

From this the proposition follows. □

6.1.3 Connection with a Joseph-like ideal

We will now quickly introduce the Joseph ideal. A more detailed account is given in [CSS]. Set $\mathfrak{g}_{\mathbb{C}} = \mathfrak{osp}_{\mathbb{C}}(p+q|2n)$. We choose a non-standard root system with the following simple positive roots

$$\epsilon_1 - \epsilon_2, \ldots, \epsilon_{\frac{p+q-3}{2}} - \epsilon_{\frac{p+q-1}{2}}, \epsilon_{\frac{p+q-1}{2}} - \delta_1, \delta_1 - \delta_2, \ldots, \delta_{n-1} - \delta_n, \delta_n,$$

for $p+q$ odd,

$$\epsilon_1 - \epsilon_2, \ldots, \epsilon_{\frac{p+q-2}{2}} - \epsilon_{\frac{p+q}{2}}, \epsilon_{\frac{p+q}{2}} - \delta_1, \delta_1 - \delta_2, \ldots, \delta_{n-1} - \delta_n, 2\delta_n,$$

for $p+q$ even. If $p+q-2n \notin \{1,2\}$, then the tensor product $\mathfrak{g}_{\mathbb{C}} \otimes \mathfrak{g}_{\mathbb{C}}$ contains a decomposition factor isomorphic to the simple $\mathfrak{g}_{\mathbb{C}}$-module of highest weight $2\epsilon_1 + 2\epsilon_2$, [CSS, Theorem 3.1]. We will denote this decomposition factor by $\mathfrak{g}_{\mathbb{C}} \circledcirc \mathfrak{g}_{\mathbb{C}}$.

Define a one-parameter family $\{\mathcal{J}_\mu \,|\, \mu \in \mathbb{C}\}$ of quadratic two-sided ideals in the tensor algebra $T(\mathfrak{g}_{\mathbb{C}}) = \oplus_{j\geq 0} \otimes^j \mathfrak{g}_{\mathbb{C}}$, where $\mathcal{J}_\mu$ is generated by

$$\{X \otimes Y - X \circledcirc Y - \frac{1}{2}[X,Y] - \mu\langle X, Y\rangle \mid X, Y \in \mathfrak{g}_{\mathbb{C}}\}$$
$$\subset \mathfrak{g}_{\mathbb{C}} \otimes \mathfrak{g}_{\mathbb{C}} \oplus \mathfrak{g}_{\mathbb{C}} \oplus \mathbb{C} \subset T(\mathfrak{g}_{\mathbb{C}}).$$

Here $X \circledcirc Y$ is the projection of $X \otimes Y$ on $\mathfrak{g}_\mathbb{C} \circledcirc \mathfrak{g}_\mathbb{C}$, and $\langle X, Y\rangle$ is the Killing form.

By construction there is a unique ideal J_μ in the universal enveloping algebra $U(\mathfrak{g}_\mathbb{C})$, which satisfies $T(\mathfrak{g}_\mathbb{C})/\mathcal{J}_\mu \cong U(\mathfrak{g}_\mathbb{C})/J_\mu$. Define

$$\mu^c := -\frac{p+q-4-2n}{4(p+q-1-2n)}.$$

Then J_μ has finite codimension for $\mu \neq \mu^c$ and infinite codimension for $\mu = \mu^c$, [CSS, Theorem 5.3]. We call J_{μ^c} the Joseph ideal of $\mathfrak{g}_\mathbb{C}$.

The annihilator ideal of a representation (π, V) of $\mathfrak{g}_\mathbb{C}$ is by definition the ideal in $U(\mathfrak{g}_\mathbb{C})$ given by

$$\mathrm{Ann}(\pi) := \{X \in U(\mathfrak{g}_\mathbb{C}) \mid \pi(X)v = 0 \text{ for all } v \in V\}.$$

Theorem 6.1.4. *If $p+q-2n > 2$, then*

$$\mathrm{Ann}(\pi_C) = J_{\mu^c}.$$

We also have $J_{\mu^c} \subseteq \mathrm{Ann}(\pi_C)$ if $p+q-2n \notin \{1,2\}$.

Proof. From [CSS, Corollary 5.8], it follows that

$$\hat{\pi}_{\lambda,\mathcal{S}}(X)\hat{\pi}_{\lambda,\mathcal{S}}(Y) = \hat{\pi}_{\lambda,\mathcal{S}}(X \circledcirc Y) + \frac{1}{2}\hat{\pi}_{\lambda,\mathcal{S}}([X,Y]) + \mu^c\langle X, Y\rangle$$

on $\mathcal{S} \cap \ker\Delta$ for $\lambda = 2-M$. Therefore

$$J_{\mu^c} \subseteq \mathrm{Ann}(\hat{\pi}_{\lambda,\mathcal{S}|\ker\Delta}) \qquad \text{for } \lambda = 2-M.$$

Proposition 6.1.3 implies

$$\mathrm{Ann}(\hat{\pi}_{\lambda,\mathcal{S}|\ker\Delta}) = \mathrm{Ann}(\hat{\pi}^*_{\lambda,\mathcal{S}'|\ker\Delta}) = \mathrm{Ann}(\hat{\pi}_{-\lambda-2M,\mathcal{S}'|\ker\Delta}).$$

We have

$$\pi_{\lambda,S'}(X)v = (\mathbb{F}^+ \circ \hat{\pi}_{-\lambda-2M,\mathcal{S}'}(X) \circ \mathbb{F}^-)v = 0$$

for all v in $\mathcal{S}' \cap \ker\Delta$ if and only if

$$\hat{\pi}_{-\lambda-2M,\mathcal{S}'}(X)v = 0$$

for all v in $\mathcal{S}'$ with support contained in the closure of $|C|$. Therefore

$$\mathrm{Ann}(\hat{\pi}_{-\lambda-2M,\mathcal{S}'|\ker\Delta}) = \mathrm{Ann}(\pi_{\lambda,\mathcal{S}'|\text{ supp contained in }|C|}).$$

Furthermore

$$\begin{aligned}\mathrm{Ann}(\pi_{\lambda,\mathcal{S}'|\text{ supp contained in }|C|}) &\subseteq \mathrm{Ann}(\pi_{\lambda,\mathcal{S}|\text{ supp contained in }|C|})\\ &= \mathrm{Ann}(\pi_C).\end{aligned}$$

We conclude

$$J_{\mu^c} \subseteq \mathrm{Ann}(\pi_C).$$

From [CSS, Theorem 5.4], it follows that every ideal with infinite codimension that contains the Joseph-like ideal J_{μ^c} is equal to J_{μ^c} if $p+q-2n>2$. Since $J_{\mu^c} \subseteq \mathrm{Ann}(\pi_C)$ and $\mathrm{Ann}(\pi_C)$ has infinite codimension, the theorem follows. □

6.2 The Gelfand–Kirillov dimension

The Gelfand–Kirillov dimension is a measure of the size of a representation. Suppose that R is a finitely generated algebra and M is a finitely generated R-module. Then the Gelfand–Kirillov dimension (GK-dimension) of M is defined by

$$GK(M) = \limsup_{k\to\infty}\Big(\log_k \dim(V^kF)\Big),$$

where V is a finite-dimensional subspace of R containing 1 and generators of R, and F is a finite-dimensional subspace of M which generates M as an R-module. This definition is independent of our choice of V and F, [Mu, Section 7.3].

Proposition 6.2.1. *The Gelfand–Kirillov dimension of the $U(\mathfrak{g})$-module W defined in* (5.2) *is given by*

$$GK(W) = p+q-3.$$

Proof. We choose W_0 for F and $\mathfrak{g}\oplus 1 \subset U(\mathfrak{g})$ for V. Then $V^k = U_k(\mathfrak{g})$, with $U_k(\mathfrak{g})$ the canonical filtration on the universal enveloping algebra. From the proof of Proposition 5.3.7, we obtain

$$U_k(\mathfrak{g})W_0 = \bigoplus_{j=0}^{k} W_j.$$

Using Proposition 5.3.10 and the dimension of the space of spherical harmonics given in Proposition 2.7.4, we compute the dimension of $U_k(\mathfrak{g})W_0$ for the case $p - 2n \geq q$

$$\begin{aligned}
&\dim U_k(\mathfrak{g})W_0 \\
&= \sum_{j=0}^{k} \left(\sum_{i=0}^{\min(j,2n)} \binom{2n}{i} \binom{j-i+p-1}{p-1} - \sum_{i=0}^{\min(j-2,2n)} \binom{2n}{i} \binom{j-i+p-3}{p-1} \right) \cdot \\
&\quad \left(\binom{\frac{\mu-\nu}{2}+j+q-1}{q-1} - \binom{\frac{\mu-\nu}{2}+j+q-3}{q-1} \right) \\
&= \sum_{i=0}^{2n} \binom{2n}{i} \left(\binom{k-i+p-1}{p-1} - \binom{k-i+p-3}{p-1} \right) \binom{\frac{\mu-\nu}{2}+k+q-1}{q-1} \\
&\quad + \sum_{i=0}^{2n} \binom{2n}{i} \left(\binom{k-i+p-2}{p-1} - \binom{k-i+p-4}{p-1} \right) \binom{\frac{\mu-\nu}{2}+k+q-2}{q-1} \\
&\quad - \binom{\frac{\mu-\nu}{2}+q-3}{q-1} - (p+2n) \binom{\frac{\mu-\nu}{2}+q-2}{q-1,}
\end{aligned}$$

where we assumed $k >> 2n$. By [Mu, Lemma 7.3.1], it is sufficient to know the highest exponent of k in the expression for $\dim U_k(\mathfrak{g})W_0$ to calculate $\limsup_{k\to\infty} \log_k \dim U_k(\mathfrak{g})W_0$. The highest exponent of k in $\left(\binom{k-i+p-1}{p-1} - \binom{k-i+p-3}{p-1} \right)$ is given by $p-2$, while in $\binom{\frac{\mu-\nu}{2}+k+q-1}{q-1}$ it is given by $q-1$. Therefore, we conclude $GK(W) = p + q - 3$. $\square$

6.3 Non-degenerate sesquilinear form

In this section we will define an 'integral' on the minimal orbit. More specifically, we will define a functional on a subspace of $\Gamma(\mathcal{O}_{\mathbb{A}_{(0)}^{p+q-2|2n}})$. This functional also leads to a functional on a subspace of $\Gamma(\mathcal{O}_C)$, which then can be used to define a sesquilinear form on W, where W is the submodule defined in Equation 5.2. Then we show that the representation π_C on W is skew-symmetric with respect to this sesquilinear form if $p + q - 2n - 6 \geq 0$.

We will use the same conventions as in Section 5.3.1 for s^2, t^2, and θ^2. Further, we also set

$$1 + \eta := \sqrt{1 - \frac{\theta^2}{2s^2}} = \sum_{j=0}^{n} \frac{1}{j!2^j} \left(\frac{-1}{2} \right)_j \frac{\theta^{2j}}{s^{2j}},$$

$$1 + \xi := \sqrt{1 + \frac{\theta^2}{2t^2}} = \sum_{j=0}^{n} \frac{(-1)^j}{j!2^j} \left(\frac{-1}{2}\right)_j \frac{\theta^{2j}}{t^{2j}},$$

where $\left(\frac{-1}{2}\right)_j$ is the Pochhammer symbol $\frac{-1}{2}\frac{1}{2}\frac{3}{2}\cdots\frac{-1+2(j-1)}{2}$. Note that η and ξ are nilpotent since $\eta^{n+1} = 0 = \xi^{n+1}$.

6.3.1 Bipolar coordinates

We use bipolar coordinates to define a morphism between certain algebras of superfunctions. More precisely, for $(x, y) \in \mathbb{R}^{p-1} \times \mathbb{R}^{q-1} = \mathbb{R}^{p+q-2}$ consider spherical coordinates by setting $x_i = s\omega_i^p$, and $y_j = t\omega_j^q$ with $\omega^p \in \mathbb{S}^{p-2}$ and $\omega^q \in \mathbb{S}^{q-2}$. We then define

$$\partial_u := \frac{1}{2}\partial_{t^2} - \frac{1}{2}\partial_{s^2} = \frac{1}{4t}\partial_t - \frac{1}{4s}\partial_s.$$

Lemma 6.3.1. *The morphism*

$$\begin{aligned} \phi^\sharp \colon \mathcal{C}^\infty(\mathbb{R}^{p+q-2}_{(0)}) \otimes \Lambda^{2n} &\to \mathcal{C}^\infty\left(\mathbb{R}^+ \times \mathbb{S}^{p-2} \times \mathbb{R}^+ \times \mathbb{S}^{q-2}\right) \otimes \Lambda^{2n} \\ f &\mapsto \phi^\sharp(f) = \exp \theta^2 \partial_u(f) = \sum_{j=0}^{n} \frac{\theta^{2j}}{j!} \left(\frac{1}{4t}\partial_t - \frac{1}{4s}\partial_s\right)^j (f), \end{aligned} \tag{6.1}$$

is a well-defined (algebra) morphism.

Proof. One can easily check that $\phi^\sharp$ is a linear map which satisfies $\phi^\sharp(fg) = \phi^\sharp(f)\phi^\sharp(g)$. Note that there are points of $\mathbb{R}^{p+q-2}_{(0)}$, for which $s = 0$ or $t = 0$ and in those points $1/s^k$ and $1/t^k$ are not well-defined. Therefore, we restricted the domain of the image to $s > 0$ and $t > 0$, where $1/s^k$ and $1/t^k$ are smooth. The product of a smooth function with a smooth function gives again a smooth function. For the partial derivatives we remark that $\partial_s = \sum_i \frac{x_i}{s}\partial_{x_i}$. Multiplication with x_i and derivations with respect to x_i are smooth operators, so ∂_s is a smooth operator. Similar we also have that ∂_t is smooth, which proves the lemma. □

The superalgebra morphism $\phi^\sharp$ satisfies the following properties:

Lemma 6.3.2. *We have*

$$\begin{aligned} \phi^\sharp x_i &= (1+\eta)x_i\phi^\sharp, & \phi^\sharp\partial_{x^i} &= \frac{1}{1+\eta}(\partial_{x^i} - x_i\frac{\theta^2}{s^2}\partial_{s^2})\phi^\sharp,\\ \phi^\sharp y_i &= (1+\xi)y_i\phi^\sharp, & \phi^\sharp\partial_{y^i} &= \frac{1}{1+\xi}(\partial_{y^i} - 2y_i\frac{\theta^2}{t^2}\partial_{t^2})\phi^\sharp,\\ \phi^\sharp\theta_k &= \theta_k\phi^\sharp, & \phi^\sharp\partial_{\theta^k} &= (\partial_{\theta^k} - 2\theta_k\partial_u)\phi^\sharp,\\ \phi^\sharp s &= (1+\eta)s\phi^\sharp, & \phi^\sharp\partial_s &= (1+\eta)\partial_s\phi^\sharp,\\ \phi^\sharp t &= (1+\xi)t\phi^\sharp, & \phi^\sharp\partial_t &= (1+\xi)\partial_t\phi^\sharp. \end{aligned}$$

Proof. We have $\partial_{t^2} = \frac{1}{2t}\partial_t = \frac{1}{2t^2}\sum_i y_i\partial_{y_i}$. Using this we obtain, for $l \in \mathbb{N}$.

$$\partial_{t^2}\left(\frac{y_k}{t^{2l}}\right) = (\tfrac{1}{2} - l)\frac{y_k}{t^{2l+2}} \text{ and } \partial_{t^2}^l(y_k) = (-1)^l\left(\frac{-1}{2}\right)_l\frac{y_k}{t^{2l}}.$$

Therefore

$$\phi^\sharp(y_k) = y_k\sum_{j=0}^{n}\frac{(-1)^j}{j!2^j}\left(\frac{-1}{2}\right)_j\frac{\theta^{2j}}{t^{2j}} = y_k(\xi+1).$$

Since $\phi^\sharp$ is an algebra morphism, we have

$$\phi^\sharp(y_i f) = \phi^\sharp(y_i)\phi^\sharp(f) = (1+\xi)y_i\phi^\sharp(f).$$

In the same way, we get $\phi^\sharp(x_i) = (1+\eta)x_i$, $\phi^\sharp(s) = (1+\eta)s$ and $\phi^\sharp(t) = (1+\xi)t$. Since $\partial_t = 2t\partial_{t^2}$, we obtain

$$\phi^\sharp(\partial_t) = 2t(\xi+1)\partial_{t^2}\phi^\sharp = (\xi+1)\partial_t\phi^\sharp.$$

Rewrite ∂_{y^i} as $\frac{-y_i}{t}\partial_t - \sum_k\frac{y^k}{t^2}L^q_{ki}$, with $L^q_{ki} = y_k\partial_{y^i} - y_i\partial_{y^k}$ and use the fact that $[\partial_{t^2}, L^q_{ki}] = 0$. Then we compute

$$\begin{aligned} \phi^\sharp\partial_{y^i} &= -(\xi+1)\frac{y_i}{t}\partial_t\phi^\sharp - \sum_k\frac{y^k}{(\xi+1)t^2}L^q_{ki}\phi^\sharp\\ &= \frac{1}{1+\xi}\partial_{y^i}\phi^\sharp - \frac{(\xi^2+2\xi)y_i}{(\xi+1)t}\partial_t\phi^\sharp. \end{aligned}$$

Because $\xi^2 + 2\xi = \frac{\theta^2}{2t^2}$, this proves it for ∂_{y^i}. Using $[\partial_{\theta^i}, \theta^{2k}] = 2k\theta_i\theta^{2k-2}$ we obtain

$$\phi^\sharp\partial_{\theta^i} = (\partial_{\theta^i} - 2\theta_i\partial_u)\phi^\sharp,$$

while the cases ∂_s and ∂_{x^i} are similar to ∂_t and ∂_{y^i}. □

6.3.2 The functional

In [KM2, Equation (2.2.3)] the following distribution on $\mathbb{R}^{p+q-2}_{(0)}$ is defined

$$\begin{aligned}\langle\delta(r^2), f\rangle &= \int_{|C|} f \\ &:= \frac{1}{2}\int_0^\infty \int_{\mathbb{S}^{p-2}} \int_{\mathbb{S}^{q-2}} f_{|s=t=\rho}\rho^{p+q-5}\mathrm{d}\rho\mathrm{d}\omega^p\mathrm{d}\omega^q,\end{aligned} \tag{6.2}$$

where f is a smooth function with compact support. The Berezin integral on Λ^{2n} is defined as

$$\int_B := \partial_{\theta_{2n}}\partial_{\theta_{2n-1}}\cdots\partial_{\theta_1}.$$

In the spirit of [CDS1], where integration over the supersphere was studied, we then define the following functional on $\mathcal{C}_c^\infty(\mathbb{R}^{p+q-2}_{(0)}) \otimes \Lambda^{2n}$, where $\mathcal{C}_c^\infty(\mathbb{R}^{p+q-2}_{(0)})$ stands for smooth functions on $\mathbb{R}^{p+q-2}_{(0)}$ with compact support,

$$\int_C f := \int_{|C|}\int_B (1+\eta)^{p-3}(1+\xi)^{q-3}\phi^\sharp(f) \tag{6.3}$$

$$\begin{aligned}&= \frac{1}{2}\int_0^\infty \int_{\mathbb{S}^{p-2}} \int_{\mathbb{S}^{q-2}} \int_B \rho^{p+q-5}\cdot \\ &\qquad (1+\eta)^{p-3}(1+\xi)^{q-3}\phi^\sharp(f)_{|s=t=\rho}\mathrm{d}\rho\mathrm{d}\omega^p\mathrm{d}\omega^q,\end{aligned} \tag{6.4}$$

with $\phi^\sharp(f)$ the morphism defined in equation (6.1).

Since the integral $\int_0^\infty \int_{\mathbb{S}^{p-2}} \int_{\mathbb{S}^{q-2}} \mathrm{d}\rho\mathrm{d}\omega^p\mathrm{d}\omega^q$ is convergent for smooth functions with compact support, Lemma 6.3.1 implies that the functional defined in (6.3) is well-defined. We found the definition of the integral $\int_C$ heuristically by setting

$$\int_C f = \int_{\mathbb{R}^{p+q-2|2n}} \delta(R^2) f.$$

Substituting

$$\delta(R^2) = \delta((s^2-t^2)+\theta^2) = \sum_{k=0}^n \frac{(-\theta^2)^k}{k!}\partial_u^k\delta(t^2-s^2),$$

in the expression for $\int_C f$, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^{p+q-2|2n}} \delta(R^2) f \\
&= \int_0^\infty \int_0^\infty \int_{\mathbb{S}^{p-2}} \int_{\mathbb{S}^{q-2}} \int_B \\
&\qquad \frac{1}{4}\delta(t^2 - s^2) \sum_{k=0}^{n} \frac{\theta^{2k}}{k!} (\partial_u)^k \, (t^{q-3}s^{p-3}f) \mathrm{d}s^2 \mathrm{d}t^2 \mathrm{d}\omega^p \mathrm{d}\omega^q \\
&= \int_0^\infty \int_0^\infty \int_{\mathbb{S}^{p-2}} \int_{\mathbb{S}^{q-2}} \int_B \frac{\delta(t-s)}{2s} st\phi^\sharp(t^{q-3}s^{p-3}f) \mathrm{d}s\mathrm{d}t\mathrm{d}\omega^p \mathrm{d}\omega^q \\
&= \frac{1}{2} \int_0^\infty \int_{\mathbb{S}^{p-2}} \int_{\mathbb{S}^{q-2}} \int_B \rho\phi^\sharp(s^{p-3}t^{q-3}f)_{|s=t=\rho} \mathrm{d}\rho \mathrm{d}\omega^p \mathrm{d}\omega^q,
\end{aligned}$$

which is equivalent to our definition of $\int_C f$.

We can extend the domain of our functional $\int_C$ from smooth functions with compact support to Bessel functions with polynomials of high enough degree.

Lemma 6.3.3. *Let P_k be a homogeneous polynomial of degree k in $\mathcal{P}(\mathbb{R}^{p+q-2|2n})$ and $\widetilde{K}_\alpha(|X|)$, $\widetilde{K}_\beta(|X|)$ Bessel functions with α, β in $\mathbb{R}$. If $p+q-2n-4+k > 2\max(\alpha,0) + 2\max(\beta,0)$, then we can extend the domain of our functional $\int_C$ such that*

$$\int_C P_k \widetilde{K}_\alpha(|X|) \widetilde{K}_\beta(|X|)$$

is also defined.

Proof. The morphism $\phi^\sharp$ leaves the degree of a polynomial unchanged. Hence, we can expand

$$(\phi^\sharp(P_k))_{s=t=\rho} = \sum_{j=0}^{k} \rho^{k-j} a_j(\theta) b_j(\omega^p, \omega^q),$$

where $a_j(\theta)$ is a polynomial in $\mathcal{P}(\mathbb{R}^{0|2n})$ of degree j and $b_j(\omega^p, \omega^q)$ is a function depending on the spherical coordinates ω^p and ω^q. Since $\partial_u(|X|) = 0$, we have

$$\phi^\sharp(\widetilde{K}_\alpha(|X|)) = \widetilde{K}_\alpha(|X|).$$

Set $\mathcal{L}_\alpha(2x^2) := \widetilde{K}_\alpha(x)$. Then $\mathcal{L}'_\alpha(x) = -\frac{1}{8}\mathcal{L}_{\alpha+1}(x)$. This follows directly from (B.2). Thus

$$\begin{aligned}\widetilde{K}_\alpha(|X|) = \mathcal{L}_\alpha(2|X|^2) &= \sum_{j=0}^{n} \frac{\theta^{2j}}{j!}\mathcal{L}_\alpha^{(j)}(t^2+s^2)\\ &= \sum_{j=0}^{n} \frac{(-1)^j\theta^{2j}}{j!8^j}\mathcal{L}_{\alpha+j}(t^2+s^2)\\ &= \sum_{j=0}^{n} \frac{(-1)^j\theta^{2j}}{j!8^j}\widetilde{K}_{\alpha+j}(\rho).\end{aligned}$$

We can expand

$$(1+\eta)^{p-3} = \left(1-\frac{\theta^2}{2s^2}\right)^{\frac{p-3}{2}} = \sum_{j=0}^{n} \frac{1}{j!}\left(\frac{3-p}{2}\right)_j \frac{\theta^{2j}}{2^j s^{2j}}, \tag{6.5}$$

$$(1+\xi)^{q-3} = \left(1+\frac{\theta^2}{2t^2}\right)^{\frac{q-3}{2}} = \sum_{j=0}^{n} \frac{(-1)^j}{j!}\left(\frac{3-q}{2}\right)_j \frac{\theta^{2j}}{2^j t^{2j}}. \tag{6.6}$$

Combining all this, we see that

$$\int_{|C|}\int_B (1+\eta)^{p-3}\,(1+\xi)^{q-3}\,\phi^\sharp(P_k\widetilde{K}_\alpha(|X|)\widetilde{K}_\beta(|X|))$$

converges if

$$\int_0^\infty\int_B \rho^{p+q-5+k-j_1-2j_2}a_{j_1}(\theta)\theta^{2j_2+2j_3+2j_4}\widetilde{K}_{\alpha+j_3}(\rho)\widetilde{K}_{\beta+j_4}(\rho)\mathrm{d}\rho$$

converges for all $0 \le j_1 \le k$, $0 \le j_2, j_3, j_4 \le n$. The Berezin integral is zero unless $j_1 + 2(j_2+j_3+j_4) = 2n$. The integral

$$\int_0^\infty \widetilde{K}_\alpha(\rho)\widetilde{K}_\beta(\rho)\rho^{\sigma-1}\mathrm{d}\rho$$

converges if $\sigma > 2\max(\alpha,0) + 2\max(\beta,0)$. This follows from the asymptotic behaviour of the Bessel functions, see Section B.2. Therefore we get the following condition

$$p+q-4+k-(j_1+2j_2) > 2\max(\alpha+j_3,0) + 2\max(\alpha+j_4,0),$$

with $j_1 + 2(j_2+j_3+j_4) = 2n$.

This is equivalent with

$$p+q-2n-4+k > 2\max(\alpha,-j_3)+2\max(\alpha,-j_4),$$

which proves the lemma. □

For future reference, we also need the following lemma

Lemma 6.3.4. *Let P_k be a homogeneous polynomial in $\mathcal{P}(\mathbb{R}^{p+q-2|2n})$ of degree k and $\widetilde{K}_\alpha(|X|)$, $\widetilde{K}_\beta(|X|)$ Bessel functions with α, β in $\mathbb{R}$. Then for $z_i = x_i$ or $z_i = y_i$*

$$\lim_{\rho\to 0}\int_B \rho^{p+q-5}(1+\xi)^{q-3}\frac{z_i}{\rho}\phi^\sharp(P_k\widetilde{K}_\alpha(|X|)\widetilde{K}_\beta(|X|)) = 0$$

if $p+q-2n-5+k > 2\max(\alpha,0)+2\max(\beta,0)$. The limit of ρ to infinity is always zero.

Proof. Similar as in the proof of Lemma 6.3.3, we have to calculate

$$\lim_{\rho\to 0}\rho^{p+q-5+k-2n+2j_1+2j_2}\widetilde{K}_{\alpha+j_1}(\rho)\widetilde{K}_{\beta+j_2}(\rho),$$

for $0 \leq j_1, j_2 \leq n$. Using the asymptotic behaviour at zero of the K-Bessel function, we obtain that this limit is zero if

$$p+q-5+k-2n+2j_1+2j_2 > 2\max(\alpha+j_1,0)+2\max(\beta+j_2,0).$$

This is equivalent with $M-3+k > 2\max(\alpha,0)+2\max(\beta,0)$. The Bessel function goes exponentially to zero at infinity. Hence the limit for ρ to infinity is also zero. □

As an example and to show that our functional is non-zero, we will now calculate the functional for the generating function of W.

Lemma 6.3.5. *We have*

$$\int_C K_{\frac{\nu}{2}}(|X|)K_{\frac{\nu}{2}}(|X|) = \frac{2^{\mu+\nu}}{n!}\left(\frac{3-p}{2}\right)_n \frac{\pi^{\frac{p+q-2}{2}}}{\Gamma(\frac{p-1}{2})\Gamma(\frac{q-1}{2})}\frac{\Gamma(\frac{\mu-\nu}{2}+1)\Gamma(\frac{\mu+\nu}{2}+1)\Gamma(\frac{\mu}{2}+1)\Gamma(\frac{\mu}{2}+1)}{\Gamma(\mu+2)}.$$

Note that $\left(\frac{3-p}{2}\right)_n = 0$ implies $\nu \in -2\mathbb{N}$. Thus for $\nu \notin -2\mathbb{N}$, $\int_C K_{\frac{\nu}{2}}(|X|)K_{\frac{\nu}{2}}(|X|)$ is non-zero.

Proof. Using

$$\phi^\sharp(\widetilde{K}_\alpha(|X|))_{|s=t=\rho} = \widetilde{K}_\alpha(|X|)_{|s=t=\rho} = \sum_{j=0}^{n} \frac{(-1)^j\theta^{2j}}{j!8^j}\widetilde{K}_{\alpha+j}(\rho)$$

and the expansion of $(1+\eta)^{p-3}$ and $(1+\xi)^{q-3}$ given in (6.5) and (6.6), we obtain

$$\begin{aligned}
&\int_C K_{\frac{\nu}{2}}(|X|)K_{\frac{\nu}{2}}(|X|)\\
&= \frac{1}{2}\int_0^\infty \int_{\mathbb{S}^{p-2}}\int_{\mathbb{S}^{q-2}}\int_B \rho^{p+q-5}\,\cdot\\
&\qquad (1+\eta)^{p-3}\,(1+\xi)^{q-3}\,\phi^\sharp(\widetilde{K}_{\frac{\nu}{2}}(|X|)\widetilde{K}_{\frac{\nu}{2}}(|X|))_{|s=t=\rho}\mathrm{d}\rho\mathrm{d}\omega^p\mathrm{d}\omega^q\\
&= \frac{1}{2}\frac{2\pi^{\frac{p-1}{2}}}{\Gamma(\frac{p-1}{2})}\frac{2\pi^{\frac{q-1}{2}}}{\Gamma(\frac{q-1}{2})}\int_0^\infty \mathrm{d}\rho\int_B\sum_{i,j,k,l=0}^{n}\Bigg(\frac{(-1)^{i+j+k}\theta^{2(i+j+k+l)}}{i!j!k!l!8^{i+j}2^{k+l}}\,\cdot\\
&\qquad \rho^{p+q-5-2k-2l}\left(\tfrac{3-q}{2}\right)_k\left(\tfrac{3-p}{2}\right)_l\widetilde{K}_{\frac{\nu}{2}+i}(\rho)\widetilde{K}_{\frac{\nu}{2}+j}(\rho)\Bigg)\\
&= \frac{1}{2}\frac{2\pi^{\frac{p-1}{2}}}{\Gamma(\frac{p-1}{2})}\frac{2\pi^{\frac{q-1}{2}}}{\Gamma(\frac{q-1}{2})}\int_0^\infty \mathrm{d}\rho\sum_{\substack{i,j,k,l=0,\\ i+j+k+l=n}}^{n}\Bigg(\frac{(-1)^{i+j+k}}{i!j!k!l!8^{i+j}2^{k+l}}\,\cdot\\
&\qquad \rho^{p+q-5-2k-2l}\left(\tfrac{3-q}{2}\right)_k\left(\tfrac{3-p}{2}\right)_l\widetilde{K}_{\frac{\nu}{2}+i}(\rho)\widetilde{K}_{\frac{\nu}{2}+j}(\rho)\Bigg)\\
&= \frac{1}{2}\frac{2\pi^{\frac{p-1}{2}}}{\Gamma(\frac{p-1}{2})}\frac{2\pi^{\frac{q-1}{2}}}{\Gamma(\frac{q-1}{2})}2^{p+q-7-3n}\sum_{\substack{i,j,k,l=0,\\ i+j+k+l=n}}^{n}\Bigg(\frac{(-1)^{i+j+k}}{i!j!k!l!}\left(\tfrac{3-q}{2}\right)_k\,\cdot\\
&\qquad \left(\tfrac{3-p}{2}\right)_l\frac{\Gamma(\frac{\mu+\nu}{2}+1+i+j)}{\Gamma(\mu+2+i+j)}\Gamma(\tfrac{\mu}{2}+1+i)\Gamma(\tfrac{\mu}{2}+1+j)\Gamma(\tfrac{\mu-\nu}{2}+1)\Bigg),
\end{aligned}$$

where we used, [EMOT, 10.3 (49)],

$$\begin{aligned}
&\int_0^\infty \rho^{\sigma-1}\widetilde{K}_\alpha(\rho)\widetilde{K}_\beta(\rho)\mathrm{d}\rho\\
&= 2^{\sigma-3}\frac{\Gamma(\frac{\sigma}{2})}{\Gamma(\sigma-\alpha-\beta)}\Gamma(\frac{\sigma-2\alpha}{2})\Gamma(\frac{\sigma-2\beta}{2})\Gamma(\frac{\sigma-2\alpha-2\beta}{2}).
\end{aligned}$$

Using $\frac{\Gamma(a+x)}{\Gamma(x)} = (x)_a$, we can rewrite this as

$$\int_C K_{\frac{\nu}{2}}(|X|)K_{\frac{\nu}{2}}(|X|)$$

$$= \frac{\pi^{\frac{p+q-2}{2}}}{\Gamma(\frac{p-1}{2})\Gamma(\frac{q-1}{2})} 2^{\mu+\nu-n} \cdot \frac{\Gamma(\frac{\mu-\nu}{2}+1)\Gamma(\frac{\mu+\nu}{2}+1)\Gamma(\frac{\mu}{2}+1)\Gamma(\frac{\mu}{2}+1)}{\Gamma(\mu+2)} \Sigma(p,q,n),$$

where

$$\Sigma(p,q,n) = \sum_{\substack{i,j,k,l=0,\\ i+j+k+l=n}}^{n} \frac{(-1)^{i+j+k}}{i!j!k!l!} \left(\frac{3-q}{2}\right)_k \left(\frac{3-p}{2}\right)_l \left(\frac{\mu}{2}+1\right)_i \left(\frac{\mu}{2}+1\right)_j \frac{\left(\frac{\mu+\nu}{2}+1\right)_{i+j}}{(\mu+2)_{i+j}}.$$

We have

$$\sum_{i=0}^{a} \binom{a}{i} (x)_i (y)_{a-i} = (x+y)_a.$$

Using this we can compute $\Sigma(p,q,n)$, with $a = i+j$,

$$\begin{aligned}
&\Sigma(p,q,n) \\
&= \sum_{l=0}^{n} \sum_{a=0}^{n-l} \sum_{i=0}^{a} \frac{(-1)^{n-l}}{(n-l-a)!l!} \left(\left(\frac{3-q}{2}\right)_{n-l-a} \left(\frac{3-p}{2}\right)_l \cdot \frac{\left(\frac{\mu+\nu}{2}+1\right)_a}{(\mu+2)_a} \frac{\left(\frac{\mu}{2}+1\right)_i \left(\frac{\mu}{2}+1\right)_{a-i}}{i!(a-i)!} \right) \\
&= \sum_{l=0}^{n} \frac{(-1)^{n-l}}{l!} \left(\frac{3-p}{2}\right)_l \sum_{a=0}^{n-l} \left(\frac{1}{(n-l-a)!} \cdot \left(\frac{3-q}{2}\right)_{n-l-a} \frac{\left(\frac{\mu+\nu}{2}+1\right)_a}{(\mu+2)_a} \frac{(\mu+2)_a}{a!} \right) \\
&= \sum_{l=0}^{n} \frac{(-1)^{n-l}}{l!} \left(\frac{3-p}{2}\right)_l \frac{1}{(n-l)!} \left(\frac{3-q}{2} + \frac{\mu+\nu}{2} + 1 \right)_{n-l} \\
&= \sum_{l=0}^{n} \frac{(-1)^{n-l}}{l!(n-l)!} \left(\frac{3-p}{2}\right)_l \left(\frac{p-3-2n}{2} + 1 \right)_{n-l} \\
&= \sum_{l=0}^{n} \frac{1}{l!(n-l)!} \left(\frac{3-p}{2}\right)_n = \frac{2^n}{n!} \left(\frac{3-p}{2}\right)_n .
\end{aligned}$$

This finishes the proof. □

The main proposition of this section is the following.

Proposition 6.3.6. *Let* $f = P_k \widetilde{K}_\alpha(|X|)\widetilde{K}_\beta(|X|)$, *with* P_k *a homogeneous polynomial of degree* k *with* $p+q-2n-5+k > 2\max(\alpha,0)+2\max(\beta,0)$ *or* f *be in* $\mathcal{C}_c^\infty(\mathbb{R}_{(0)}^{p+q-2}) \otimes \Lambda^{2n}$.

The integral $\int_C$ *has the following properties.*

1. *Only depends on the restriction of* f *to the minimal orbit* C*:*

$$\int_C R^2 f = 0.$$

2. *It is* $\mathfrak{osp}(p-1, q-1|2n)$ *invariant:*

$$\int_C X(f) = 0 \qquad \text{for all } X \text{ in } \mathfrak{osp}(p-1,q-1|2n).$$

3. *It satisfies*

$$\int_C (\mathbb{E} + M - 2)(f) = 0,$$

   where $M = m - 2n$ *is the superdimension of* $\mathbb{R}^{p+q-2|2n}$.

4. *The Bessel operators were given by*

$$\mathcal{B}_\lambda(e_k) = (-\lambda + 2\mathbb{E})\partial_k - e_k\Delta,$$

   where λ *is a complex parameter. Then the integral is symmetric with respect to the Bessel operators*

$$\int_C (\mathcal{B}_\lambda(e_k)f)g = (-1)^{|f||k|}\int_C f(\mathcal{B}_\lambda(e_k)g),$$

   for the critical value $\lambda = -M + 2$.

The integration of a derivative is as follows.

Lemma 6.3.7. *For* f *as in Proposition 6.3.6 it holds*

$$\int_C \partial_{z^i} f = \int_C (s\partial_s + p - 1)\frac{z_i}{2s^2} f - (t\partial_t + q - 1)\frac{z_i}{2t^2} f.$$

Proof. We will prove three different cases separately: z_i equal to x_i, y_i or θ_i.

We have $\partial_\rho(g_{|s=t=\rho}) = ((\partial_s + \partial_t)g)_{|\rho=s=t}$. Therefore

$$\int_{|C|}\int_B \Big((1+\eta)^{p-2}(1+\xi)^{q-3}\frac{x_i}{2s}\partial_s\phi^\sharp f\Big)_{|s=t=\rho}$$
$$= \int_{|C|}\int_B \Big((1+\eta)^{p-2}(1+\xi)^{q-3}\frac{x_i}{2s}\Big)_{|s=t=\rho} \cdot$$
$$\Big(\partial_\rho(\phi^\sharp f_{|s=t=\rho}) - (\partial_t\phi^\sharp f)_{|s=t=\rho}\Big).$$

Using integration by parts with respect to ∂_ρ and

$$\partial_\rho(1+\eta)^{p-2} = (p-2)(1+\eta)^{p-4}\frac{\theta^2}{2\rho^3},$$
$$\partial_\rho(1+\xi)^{q-3} = -(q-3)(1+\xi)^{q-5}\frac{\theta^2}{2\rho^3},$$

and $\partial_\rho(\frac{x_i}{\rho}) = 0$ we obtain

$$\int_{|C|}\int_B \Big((1+\eta)^{p-2}(1+\xi)^{q-3}\frac{x_i}{2s}\Big)_{|s=t=\rho}\partial_\rho(\phi^\sharp f_{|s=t=\rho})$$
$$= \Bigg[\frac{1}{2}\int_{\mathbb{S}^{p-2}}\int_{\mathbb{S}^{q-2}}\int_B$$
$$\Big(\rho^{p+q-5}(1+\eta)^{p-2}(1+\xi)^{q-3}\frac{x_i}{2\rho}\phi^\sharp(f)\Big)_{|s=t=\rho} \mathrm{d}\omega^p \mathrm{d}\omega^q\Bigg]_0^\infty,$$
$$-\frac{1}{2}\int_0^\infty\int_{\mathbb{S}^{p-2}}\int_{\mathbb{S}^{q-2}}\int_B \partial_\rho\left((\rho^{p+q-5}(1+\eta)^{p-2}\,(1+\xi)^{q-3}\,\frac{x_i}{2\rho}\right)\cdot$$
$$(\phi^\sharp f_{|s=t=\rho})\mathrm{d}\rho \mathrm{d}\omega^p \mathrm{d}\omega^q$$
$$= \Bigg[\frac{1}{2}\int_{\mathbb{S}^{p-2}}\int_{\mathbb{S}^{q-2}}\int_B$$
$$\Big(\rho^{p+q-5}(1+\eta)^{p-2}(1+\xi)^{q-3}\frac{x_i}{2\rho}\phi^\sharp(f)\Big)_{|s=t=\rho} \mathrm{d}\omega \mathrm{d}\eta\Bigg]_0^\infty,$$
$$+\int_{|C|}\int_B \frac{(1+\eta)^{p-2}}{(1+\xi)^{3-q}}\Bigg(-(p+q-5)-(p-2)\frac{\theta^2}{2(1+\eta)^2\rho^2}$$
$$+(q-3)\frac{\theta^2}{2(1+\xi)^2\rho^2}\Bigg)\frac{x_i}{2\rho^2}(\phi^\sharp f_{|s=t=\rho}).$$

The part between square brackets is zero. For functions with compact support this is immediate, while in the other case this follows from

Lemma 6.3.4. We conclude

$$\begin{aligned}\int_{|C|}\int_B \frac{(1+\eta)^{p-2}}{(1+\xi)^{3-q}}\frac{x_i}{2s}\partial_s\phi^\sharp f \\ &= -\int_{|C|}\int_B \frac{(1+\eta)^{p-2}}{(1+\xi)^{3-q}}\left(\frac{p-2}{(1+\eta)^2}+\frac{q-3}{(1+\xi)^2}\right)\frac{x_i}{2\rho^2}\phi^\sharp f \\ &\quad -\int_{|C|}\int_B \frac{(1+\eta)^{p-2}}{(1+\xi)^{3-q}}\frac{x_i}{2\rho}\partial_t\phi^\sharp f.\end{aligned}$$

Hence, using this and Lemma 6.3.2,

$$\begin{aligned}&\int_C \left((s\partial_s+p-1)\frac{x_i}{2s^2}-(t\partial_t+q-1)\frac{x_i}{2t^2}\right)f \\ &= \int_C \left(\frac{x_i}{2s}\partial_s+(p-2)\frac{x_i}{2s^2}-\frac{x_i}{2t}\partial_t-(q-3)\frac{x_i}{2t^2}\right)f \\ &= \int_{|C|}\int_B \frac{(1+\eta)^{p-3}}{(1+\xi)^{3-q}}(1+\eta)\left(\frac{x_i}{2s}\partial_s+(p-2)\frac{x_i}{2(1+\eta)^2s^2}\right. \\ &\qquad\qquad \left.-\frac{x_i}{2t}\partial_t-(q-3)\frac{x_i}{2(1+\xi)^2t^2}\right)\phi^\sharp f \\ &= \int_{|C|}\int_B \frac{(1+\eta)^{p-2}}{(1+\xi)^{3-q}}\left(\frac{x_i}{s}\partial_s+(p-2)\frac{x_i}{(1+\eta)^2s^2}\right)\phi^\sharp f.\end{aligned}$$

For x_i we have $\partial_{x^i}=\frac{x_i}{s}\partial_s+\sum_k\frac{x^k}{s^2}L^p_{ki}$, with $L^p_{ki}=x_k\partial_{x^i}-x_i\partial_{x^k}$. Then, using $\sum_k[x^k,L^p_{ki}]=(p-2)x_i$, the fact that $\int_{\mathbb{S}^{p-2}}L^p_{ki}f=0$ and Lemma 6.3.2, we obtain

$$\begin{aligned}\int_C \partial_{x^i}f &= \int_{|C|}\int_B \frac{(1+\eta)^{p-3}}{(1+\xi)^{3-q}}\frac{1}{1+\eta}\left(\partial_{x^i}-x_i\frac{\theta^2}{2s^2}\partial_{s^2}\right)\phi^\sharp f \\ &= \int_{|C|}\int_B \frac{(1+\eta)^{p-4}}{(1+\xi)^{3-q}}\left(\frac{x_i}{s}\partial_s+\sum_k\frac{1}{s^2}[x^k,L^p_{ki}]-x_i\frac{\theta^2}{2s^2}\partial_{s^2}\right)\phi^\sharp f \\ &= \int_{|C|}\int_B \frac{(1+\eta)^{p-4}}{(1+\xi)^{3-q}}\left(\frac{x_i}{s}(1+\eta^2)\partial_s+\frac{p-2}{s^2}x_i\right)\phi^\sharp f.\end{aligned}$$

We conclude

$$\int_C \partial_{x^i}f = \int_C \left((s\partial_s+p-1)\frac{x_i}{2s^2}-(t\partial_t+q-1)\frac{x_i}{2t^2}\right)f$$

For $z_i = y_i$, the proof is completely similar. Finally for θ_i we have, using Lemma 6.3.2 and integration by parts with respect to θ_i,

$$\begin{aligned}
&\int_C \partial_{\theta^i} f \\
&= \int_{|C|}\int_B \frac{(1+\eta)^{p-3}}{(1+\xi)^{3-q}} \left(\partial_{\theta^i} - 2\theta_i \partial_u\right) \phi^\sharp f \\
&= \int_{|C|}\int_B \frac{(1+\eta)^{p-3}}{(1+\xi)^{3-q}} \Bigg((p-3)\frac{\theta_i}{2(1+\eta)^2 s^2} \\
&\qquad\qquad -(q-3)\tfrac{\theta_i}{2(1+\xi)^2 t^2} + \frac{\theta_i}{2s}\partial_s - \frac{\theta_i}{2t}\partial_t \Bigg) \phi^\sharp f \\
&= \int_{|C|}\int_B \frac{(1+\eta)^{p-3}}{(1+\xi)^{3-q}} \phi^\sharp \left((p-3)\frac{\theta_i}{2s^2} - (q-3)\frac{\theta_i}{2t^2} + \frac{\theta_i}{2s}\partial_s - \frac{\theta_i}{2t}\partial_t \right) f \\
&= \int_C (s\partial_s + p - 1)\frac{\theta_i}{2s^2} f - (t\partial_t + q - 1)\frac{\theta_i}{2t^2} f.
\end{aligned}$$

This finishes the proof. □

Proof of Proposition 6.3.6, part (1)-(3) . From Lemma 6.3.2 we obtain that

$$\phi^\sharp R^2 = ((1+\eta)^2 s^2 - (1+\xi)^2 t^2 + \theta^2)\phi^\sharp = (s^2 - t^2)\phi^\sharp.$$

So $\phi^\sharp R^2{}_{|s=t} = 0$ and we conclude $\int_C R^2 f = 0$. This proves part (1) of the proposition.

The operators $L_{i,j} := z_i \partial_j - (-1)^{|i||j|} z_j \partial_i$ for $i \leq j$ span $\mathfrak{osp}(p-1, q-1|2n)$. We can rewrite the operator $L_{i,j}$ as follows

$$\begin{aligned}
L_{i,j} f &= (-1)^{|i||j|}\partial_j(z_i f) - (-1)^{|i||j|}\beta_{ji} f - \partial_i(z_j f) + \beta_{ij} f \\
&= (-1)^{|i||j|}\partial_j(z_i f) - \partial_i(z_j f).
\end{aligned}$$

Using Lemma 6.3.7, we thus get

$$\begin{aligned}
\int_C L_{i,j} f &= \int_C (-1)^{|i||j|}\partial_j(z_i f) - \partial_i(z_j f) \\
&= \int_C (-1)^{|i||j|}(s\partial_s + p - 1)\frac{z_j z_i}{2s^2} f - (-1)^{|i||j|}(t\partial_t + q - 1)\frac{z_j z_i}{2t^2} f \\
&\quad - \int_C (s\partial_s + p - 1)\frac{z_i z_j}{2s^2} f - (t\partial_t + q - 1)\frac{z_i z_j}{2t^2} f = 0.
\end{aligned}$$

This finishes the proof of part (2). For part (3), we use

$$\mathbb{E}f = \sum_k z^k \partial_k = \sum_k (-1)^{|k|} \partial_k (z^k f) - Mf.$$

Hence, again using Lemma 6.3.7, we find

$$\begin{aligned}
&\int_C \mathbb{E}f \\
&= \sum_k (-1)^{|k|} \int_C (s\partial_s + p - 1) \frac{z_k z^k}{2s^2} f - (t\partial_t + q - 1) \frac{z_k z^k}{2t^2} f - \int_C Mf \\
&= \int_C R^2 \left((s\partial_s + p - 1) \frac{1}{2s^2} f - (t\partial_t + q - 1) \frac{1}{2t^2} f \right) \\
&\quad + \int_C \frac{s^2}{s^2} f + \frac{t^2}{t^2} f - \int_C Mf \\
&= (2 - M) \int_C f,
\end{aligned}$$

where we used part (1) to eliminate the term with R^2. □

For the Laplacian and the Bessel operators we have the following.

Lemma 6.3.8. *We have*

$$\int_C \Delta f = -(M-4) \int_C (s\partial_s + p - 1) \frac{f}{2s^2} - (t\partial_t + q - 1) \frac{f}{2t^2}$$

and

$$\int_C \mathcal{B}_\lambda(z_k) f = 0,$$

for $\lambda = -M + 2$.

Proof. Using Lemma 6.3.7 and Proposition 6.3.6(3), we get for the Laplacian

$$\begin{aligned}
&\int_C \sum_{i,j} \beta^{ij} \partial_i \partial_j f \\
&\qquad = \sum_{i,j} \beta^{ij} \int_C (s\partial_s + p - 1) \frac{z_i}{2s^2} \partial_j f - (t\partial_t + q - 1) \frac{z_i}{2t^2} \partial_j f
\end{aligned}$$

$$
\begin{aligned}
&= \int_C (s\partial_s + p - 1)\frac{1}{2s^2}\mathbb{E}f - (t\partial_t + q - 1)\frac{1}{2t^2}\mathbb{E}f \\
&= \int_C (\mathbb{E}+2)\left((s\partial_s + p - 1)\frac{1}{2s^2} - (t\partial_t + q - 1)\frac{1}{2t^2}\right) f \\
&= -(M-4)\int_C (s\partial_s + p - 1)\frac{f}{2s^2} - (t\partial_t + q - 1)\frac{f}{2t^2}.
\end{aligned}
$$

For the Bessel operators $\mathcal{B}_\lambda(z_k) = (-\lambda + 2\mathbb{E})\partial_k - z_k\Delta$, we obtain

$$
\begin{aligned}
\int_C \mathcal{B}_\lambda(z_k)f &= (-M+2)\int_C \partial_k f - \int_C \Delta(z_k f) + 2\int_C \partial_k f \\
&= (-M+4)\int_C (s\partial_s + p - 1)\frac{z_k}{2s^2}f - (t\partial_t + q - 1)\frac{z_k}{2t^2}f \\
&\quad + (M-4)\int_C (s\partial_s + p - 1)\frac{z_k}{2s^2}f - (t\partial_t + q - 1)\frac{z_k}{2t^2}f \\
&= 0,
\end{aligned}
$$

where we used Proposition 6.3.6(3), Lemma 6.3.7 and $[\Delta, e_k] = 2\partial_k$.
□

Using this, we can prove the final part of Proposition 6.3.6.

Proof of Proposition 6.3.6, part (4). We will show

$$
\begin{aligned}
-2\int_C (\mathcal{B}_\lambda(z_k)f)g = \int_C \Big(& 2(-1)^{|f||k|}\mathbb{E}(f)\partial_k(g) + 2\partial_k(f)\mathbb{E}(g) \\
& - \sum_{i,j} 2z_k(-1)^{|f||j|}g^{ij}\partial_i(f)\partial_j(g)\Big).
\end{aligned} \tag{6.7}
$$

Combining this with Lemma 6.3.8 and the product rule given in (5.6), we can conclude

$$
\int_C (\mathcal{B}_\lambda(z_k)f)g - (-1)^{|f||k|}f(\mathcal{B}_\lambda(z_k)g) = 0,
$$

which proves part (4) of the proposition. To show (6.7), first remark that, using Lemma 6.3.7,

$$
\begin{aligned}
&\int_C \partial_k(\mathbb{E}(f)g) - \sum_{i,j}(-1)^{|j|(|i|+|k|)}\partial_j\left(z_k g^{ij}\partial_i(f)g\right) \\
&= \int_C (s\partial_s + p - 1)\frac{z_k}{2s^2}\mathbb{E}(f)g
\end{aligned}
$$

$$
\begin{aligned}
&- \sum_{i,j} \int_C (-1)^{|j|(|i|+|k|)} (s\partial_s + p - 1) \frac{z_j}{2s^2} z_k g^{ij} \partial_i(f) g \\
&- \int_C (t\partial_t + q - 1) \frac{z_k}{2t^2} \mathbb{E}(f) g \\
&+ \sum_{i,j} \int_C (-1)^{|j|(|i|+|k|)} (t\partial_t + q - 1) \frac{z_j}{2t^2} z_k g^{ij} \partial_i(f) g \\
= {} & 0.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\int_C (-1)^{|f||k|} \mathbb{E}(f) \partial_k(g) + \partial_k(f) \mathbb{E}(g) - \sum_{i,j} z_k (-1)^{|f||j|} g^{ij} \partial_i(f) \partial_j(g) \\
&= \int_C \partial_k(\mathbb{E}(f) g) - (\partial_k \mathbb{E}(f)) g + \mathbb{E}(\partial_k(f) g) - (\mathbb{E}\partial_k(f)) g - \sum_{i,j} \Bigg(\\
&\quad \int_C (-1)^{(|i|+|k|)|j|} \partial_j \left(g^{ij} z_k \partial_i(f) g \right) + (-1)^{(|i|+|k|)|j|} \partial_j \left(g^{ij} z_k \partial_i(f) \right) g \Bigg) \\
&= \int_C \left((\lambda - 2\mathbb{E}) \partial_k(f) \right) g - \partial_k(f) g + z_k \Delta(f) g + \partial_k(f) g \\
&= \int_C \left(-\mathcal{B}_\lambda(z_k) f \right) g,
\end{aligned}
$$

where we used the previous equation and part (3) of Proposition 6.3.6. This proves equation (6.7) and thus the proposition. □

6.3.3 The sesquilinear form

Define a sesquilinear form on the minimal orbit C using the functional $\int_C$

$$
\langle f, g \rangle := \int_C \overline{f} g.
$$

Theorem 6.3.9. *Suppose $\nu \notin -2\mathbb{N}$, $\mu + \nu$ even and $\mu + \nu = p + q - 2n - 6 \geq 0$. The representation π_C on W is skew-symmetric for the form $\langle \cdot, \cdot \rangle$, i.e. for $X \in \mathrm{TKK}(J)$, and f, g in W*

$$
\langle \pi_C(X) f, g \rangle + (-1)^{|X||f|} \langle f, \pi_C(X) g \rangle = 0.
$$

Proof. The theorem follows easily from Proposition 6.3.6. So we have to show that we can apply this proposition. We will prove

$$W \subset \sum_{a=0}^{\frac{\mu-\nu}{2}} \sum_{b=0}^{\infty} \widetilde{K}_{\frac{\nu}{2}+a+b}(|X|) \otimes \mathcal{P}_{\geq a+2b}(\mathbb{R}^{p+q-2|2n}). \tag{6.8}$$

If f and g are in the right-hand side, then $\overline{f}g$ is a linear combination of elements of the form $P_k \widetilde{K}_{\frac{\nu}{2}+a+b}(|X|)\widetilde{K}_{\frac{\nu}{2}+a'+b'}(|X|)$, where P_k is a homogeneous polynomial of degree k with $k \geq a + a' + 2b + 2b'$ and $a, a' \leq \frac{\mu-\nu}{2}$. We have

$$\begin{aligned} \mu + \nu + 1 > \max(\mu + \nu, 0) &\geq \max(\nu + a, 0) + \max(\nu + a', 0) \\ &\geq \max(\nu + a, -a - 2b) + \max(\nu + a, -a' - 2b'). \end{aligned}$$

Hence

$$\mu + \nu + 1 + k > \max(\nu + 2a + 2b, 0) + \max(\nu + 2a' + 2b', 0),$$

and $\overline{f}g$ satisfies Proposition 6.3.6.

To see (6.8), note that $W_0 = U(\mathfrak{k}')\widetilde{K}_{\frac{\nu}{2}}(|X|)$ is contained in the right-hand side. Using the differential relation of equation (B.2), we obtain $\mathbb{E}(\widetilde{K}_\alpha(|X|)) = -\frac{|X|^2}{2}\widetilde{K}_{\alpha+1}(|X|)$. Therefore the right-hand side is invariant for the action of $U(L_e)$, the associative algebra generated by powers of L_e. It is also clearly invariant for $U(J^-)$ which acts by multiplication with polynomials. By the Poincaré–Birkhoff–Witt theorem $W = U(\mathfrak{g})\widetilde{K}_{\frac{\nu}{2}} = U(J^-)U(L_e)U(\mathfrak{k}')\widetilde{K}_{\frac{\nu}{2}}$, hence equation (6.8) follows. □

We can use this skew-symmetry to show non-degeneracy of our form.

Lemma 6.3.10. *Assume $\nu \notin -2\mathbb{N}$, $p \neq 3$, $q \neq 3$, $p + q$ even and $p+q-2n-6 \geq 0$. The form $\langle , \rangle$ defines a sesquilinear, non-degenerate form on W, which is superhermitian, i.e.*

$$\langle f, g\rangle = (-1)^{|f||g|}\overline{\langle g, f\rangle}.$$

Proof. We see immediately that our form is sesquilinear and superhermitian. From Theorem 6.3.9, it follows that the radical of the form gives a subrepresentation. Namely if $\langle f, g\rangle = 0$ for all g in W, then also

$$\langle \pi_C(X)f, g\rangle = -(-1)^{|f||X|}\langle f, \pi_C(X)g\rangle = 0, \qquad \text{for all } g \in W.$$

So $\pi_C(X)f$ is also contained in the radical. By Corollary 5.3.9 W is simple for $\mu + \nu \geq 0$, hence the radical is zero or the whole W. Since $\int_C \widetilde{K}_{\frac{\nu}{2}}(|X|)\widetilde{K}_{\frac{\nu}{2}}(|X|) \neq 0$ by Lemma 6.3.5 we conclude that the radical is zero and the form is non-degenerate. $\square$

The only thing worse than being blind is having sight and no vision.

Helen Keller

7

Conclusions and open questions

In this thesis we constructed a minimal representation of the orthosymplectic Lie superalgebra. Of course, this is not the end of the story. As mentioned in the introduction, a natural goal is to construct minimal representations for all simple Lie superalgebras which can be obtained as the TKK algebra of a Jordan superalgebra. Or even more ambitiously, one can look at all simple Lie superalgebras obtained from Jordan superpairs.

The first steps in this direction were already taken in Chapter 3, where we investigated the different TKK constructions and in Chapter 4, where we constructed a representation of a simple Lie superalgebra on functions on the corresponding Jordan superalgebra (or Jordan superpair).
The next step would be to construct a minimal orbit and show that, for some character, we can restrict the representation to this minimal orbit. Abstractly, we can define this minimal orbit using Definition 5.1.5 and a primitive idempotent. To do this, we first need a definition of the structure group in the super case. One could, for example, define the even part of the structure group as the group of automorphisms of the associated Jordan superpair. Remark that it is also not

clear a priori that this definition of a minimal orbit will lead to the same minimal orbit for different primitive idempotents, a property which is known to hold in the non-super setting. But the difficult part would probably lie in showing that the differential operators occurring in our representation are tangential to the minimal orbit. This is already quite intricate in the classical case, for example in [HKM, Section 1.2.4] this is proved using equivariant measures. See also [MS] for another approach.

For the next step, integrating our representation to a representation of the conformal group, the approach using Harish-Chandra supermodules seems most promising. However, it is not clear how one finds such an admissible submodule. The construction in the orthosymplectic case depends on the intermediate algebra $\mathfrak{osp}(p|2n) \oplus \mathfrak{so}(q)$. A natural analogue of this intermediate algebra in the general case is not known. Therefore this construction can not be straightforwardly generalised to other Lie superalgebras.

We also remark that for Lie superalgebras the Joseph ideal has only been defined in the cases $\mathfrak{osp}(m|2n)$, [CSS], and $\mathfrak{sl}(m|n)$, [BC]. So one can no longer use the Joseph ideal to show 'minimality' of the constructed representations. However, one could reverse things, and use the annihilator ideal of the 'minimal' representations we construct to define the Joseph ideal. An interesting problem is then to investigate these annihilator ideals and see whether one can find an intrinsic characterisation.
We still do expect that the constructed representations have low Gelfand–Kirillov dimension. Minimal representations of real Lie groups have the lowest Gelfand–Kirillov dimension of all infinite-dimensional unitary representations. One could investigate whether a similar property holds for the minimal representations of Lie supergroups.
As mentioned in the introduction, these minimal representations will not be unitary. However, one could explore whether they share some unitary-like property, which then could be used to broaden the notion of unitarity in the super case.

We also mention two open question regarding the minimal representation of the orthosymplectic Lie superalgebra.

From [NS, Theorem 6.2.1], we know a priori that our representation is not unitary. However, we can still define a Hilbert superspace on the minimal orbit in the sense of the new definition introduced in [dGM]. Namely, we can 'pullback' the Hilbert superspace $L^2(\mathbb{R}^+, \rho^{p+q-5}\mathrm{d}\rho)\hat{\otimes}L^2(\mathbb{S}^{p-2})\hat{\otimes}L^2(\mathbb{S}^{q-2})$ using the isomorphism $\phi^\sharp$ defined in Section 6.3 to obtain a Hilbert superspace H on the minimal orbit. This defines a topology on H and W is contained in H. So a natural question to ask is whether W is dense in H with respect to this topology. We note that we were not able to show continuity of the operators in our representation with respect to this topology. This has to do with the fact that in the super case isometric operators are not necessarily continuous and that our operators do not respect the fundamental decomposition of H. So in particular we cannot use the standard techniques for integrating a $(\mathfrak{g}, \mathfrak{k})$-representation to a representation of the corresponding group on H. However, one could still investigate whether there is any connection between the Fréchet space on which we defined the representation of $OSp(p, q|2n)$ and the Hilbert superspace H.

At the moment our definition of W looks a bit arbitrary. In particular it depends on our choice of intermediate algebra $\mathfrak{k}' = \mathfrak{osp}(p|2n) \oplus \mathfrak{so}(q)$. Therefore an intrinsic characterisation of W, which could be generalised to other Lie superalgebras, would be interesting. In the classical case, W is simply the space of $\mathfrak{k}$-finite vectors in the representation π_C of $\mathfrak{g}$ on $\mathcal{C}^\infty(C)$, but a statement like this is not immediate in the super case.

A

The affine superspace and supermanifolds

A general introduction to supermanifolds can be found in [DM] and [CCF]. Here we quickly introduce definitions and notations.

Consider a topological space $|M|$. We associate a category $\mathcal{C}_{|M|}$ with it as follows. The objects of $\mathcal{C}_{|M|}$ are the open sets of $|M|$ and its morphisms are the inclusions. So if $U \subset V$ for U and V open sets in $|M|$, then there exists a unique morphism from U to V.

A *presheaf* (of superrings) on $|M|$ is a contravariant functor $\mathcal{O}$ from the category $\mathcal{C}_{|M|}$ to the category of superrings. This means that there corresponds a superring $\mathcal{O}(U)$ to each open set U in $|M|$ and that there exists a morphism $r_{U,V} \colon \mathcal{O}(V) \to \mathcal{O}(U)$ if $U \subset V$. These morphisms satisfy $r_{U,U} = \mathrm{id}$ and $r_{U,V} \circ r_{V,W} = r_{U,W}$ for $U \subset V \subset W$. We will often write $r_{U,V}(f)$ as $f_{|U}$ for a section f in $\mathcal{O}(V)$.

A presheaf $\mathcal{O}$ on $|M|$ is a *sheaf* if it has the following gluing property. Consider an open set U in $|M|$ and an open covering $\{U_i\}_{i \in I}$. Assume we have a family $\{f_i\}_{i \in I}$ of sections $f_i \in \mathcal{O}(U_i)$ for which $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ for all $i, j \in I$. Then the gluing property asserts that there exists a unique f in $\mathcal{O}(U)$ such that $f|_{U \cap U_i} = f_i|_{U \cap U_i}$ for all i.

For every x in $|M|$ we can define the stalk $\mathcal{O}_x$ as the direct limit

$$\varinjlim \mathcal{O}(U),$$

where we take the limit over all open neighbourhoods U of x. The idea is that the stalk captures the behaviour of the sheaf locally around the point x. It consists of sections defined on some neighbourhood of x and sections are considered equivalent if their restrictions on a smaller neighbourhood agree.

Definition A.0.1. *A superringed space $(|S|, \mathcal{O}_S)$ is a topological space $|S|$ and a sheaf $\mathcal{O}_S$ of superrings.*
A superspace $(|S|, \mathcal{O}_S)$ is a superringed space for which the stalk $\mathcal{O}_{S,x}$ is a local superring for all points $x \in |S|$.

A superring is *local* if it has a unique maximal ideal.

Let V be a real finite-dimensional super-vector space. Then the affine superspace is the superringed space

$$\mathbb{A}(V) = (V_{\bar{0}}, \mathcal{C}^\infty_{V_{\bar{0}}} \otimes_{\mathbb{R}} \Lambda V_{\bar{1}}^*),$$

where $\mathcal{C}^\infty_{V_{\bar{0}}}$ is the sheaf of smooth, complex-valued functions on $V_{\bar{0}}$ and $\Lambda V_{\bar{1}}^*$ is the Grassmann algebra of $V_{\bar{1}}^*$. In case $V = \mathbb{R}^{m|n}$ we also use the notation $\mathbb{A}^{m|n}$ for $\mathbb{A}(\mathbb{R}^{m|n})$.

A morphism $\phi = (|\phi|, \phi^\sharp)$ between two superspaces M and N is a continuous map $|\phi|\colon |M| \to |N|$ and a sheaf morphism $\phi^\sharp\colon \mathcal{O}_N \to |\phi|_* \mathcal{O}_M$. Here $|\phi|_* \mathcal{O}_M$ is the sheaf on $|N|$ given by $|\phi|_* \mathcal{O}_M(U) = \mathcal{O}_M(|\phi|^{-1}(U))$.

A (real smooth) supermanifold M is a superspace that is locally isomorphic to $\mathbb{A}^{m|n}$. We denote the underlying topological space by $|M|$ and the structure sheaf of commutative superrings by $\mathcal{O}_M$. The global sections are denoted by $\Gamma(\mathcal{O}_M)$. If M is an ordinary manifold, then we will also use the notation $\mathcal{C}^\infty(M)$ for $\Gamma(\mathcal{O}_M)$. Note that for supermanifolds the global sections $\Gamma(\mathcal{O}_M)$ determine the sheaf $\mathcal{O}_M$, [CCF, Corollary 4.5.10].

The elements in $\mathcal{O}_M(U)$ act by multiplication on $\mathcal{O}_M(U)$ and they form the differential operators of degree zero. The differential operators of degree k are defined inductively:

$$\mathcal{D}^k_M(U) := \{D \in \mathrm{End}(\mathcal{O}_M(U)) \mid [D, f] \in \mathcal{D}^{k-1}_M(U) \quad \forall f \in \mathcal{O}_M(U)\}.$$

Here $[D, f] = Df - (-1)^{|D||f|} fD$ is the supercommutator. The sheaf of differential operators $\mathcal{D}_M$ is then defined by

$$\mathcal{D}_M(U) = \bigcup_{i=0}^{\infty} \mathcal{D}_M^k(U).$$

We again use the notation $\Gamma(\mathcal{D}_M)$ for the global sections.

The product of supermanifolds M and N is given by

$$M \times N = (|M| \times |N|, \mathcal{O}_{M\times N}),$$

where $\mathcal{O}_{M\times N}(U \times V) := \mathcal{O}_M(U)\hat{\otimes}\mathcal{O}_N(V)$, for an open set $U \times V \in |M|\times|N|$. Here $\hat{\otimes}$ is the completion of the tensor product with respect to the projective tensor topology. This is the unique topology such that

$$C^\infty(U)\hat{\otimes}C^\infty(V) \cong C^\infty(U \times V)$$

for $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n$, [CCF, Section 4.5].

The following proposition tells us that for most practical purposes it is sufficient to only consider the tensor product.

Proposition A.0.2 ([CCF, Proposition 4.5.4])**.**

1. *The space of sections $\Gamma(\mathcal{O}_M) \otimes \Gamma(\mathcal{O}_N)$ is dense in $\Gamma(\mathcal{O}_{M\times N})$.*
2. *If $\phi_i \colon M_i \to N_i, i = 1, 2$ are supermanifold morphisms, then the sheaf morphism of the map $\phi_1 \times \phi_2 \colon M_1 \times M_2 \to N_1 \times N_2$ is given by $\phi_1^\sharp \hat{\otimes} \phi_2^\sharp$ which is in turn completely determined by $\phi_1^\sharp \otimes \phi_2^\sharp$.*

Let $\theta_1, \dots, \theta_n$ be a basis of $V_{\bar{1}}^*$. For a multi-index $I = (i_1, i_2, \dots, i_n) \in \mathbb{Z}_2^n$, we introduce the notation $\theta^I := \theta_1^{i_1}\theta_2^{i_2}\cdots\theta_n^{i_n}$. Then we can decompose every section $f \in \mathcal{O}_{\mathbb{A}(V)}(U)$ for U an open subspace of $V_{\bar{0}}$ as

$$f = f_0 + \sum_{I\in\mathbb{Z}_2^n\setminus\{0\}} f_I\theta^I,$$

where f_0, f_I are in $\mathcal{C}^\infty(U)$. The value of f at a point x in $V_{\bar{0}}$ is defined as

$$f(x) := \mathrm{ev}_x(f) := f_0(x).$$

Note that $\mathrm{ev}_x(fg) = \mathrm{ev}_x(f)\mathrm{ev}_x(g)$ for f, g in $\mathcal{O}_{\mathbb{A}(V)}(U)$.

B

Orthogonal polynomials and special functions

We will also need some orthogonal polynomials and special functions in this thesis which we introduce here.

B.1 Gegenbauer polynomials

For $n \in \mathbb{N}$ and $\lambda \in \mathbb{C}$, we define the Gegenbauer polynomial

$$C_n^\lambda(z) = \frac{1}{\Gamma(\lambda)} \sum_{k=0}^{n} \frac{(-1)^k \Gamma(\lambda+k)\Gamma(n+2\lambda+k)}{k!(n-k)!\Gamma(2\lambda+2k)} \left(\frac{1-z}{2}\right)^k .$$

We will use the normalised version

$$\widetilde{C}_n^\lambda(z) = \Gamma(\lambda) C_n^\lambda(z),$$

which, in contrast to $C_n^\lambda(z)$, is non-zero for $\lambda = 0$. We need the following two properties of the normalised Gegenbauer polynomial, [EMOT, 3.15(21) and 3.15(30)]:

$$\partial_z \widetilde{C}_m^\lambda(z) = 2\widetilde{C}_{m-1}^{\lambda+1}(z),$$

and

$$4(1-z^2)\widetilde{C}_{m-1}^{\lambda+1}(z) - 2z(2\lambda-1)\widetilde{C}_m^{\lambda}(z) \\ = -(m+1)(2\lambda+m-1)\widetilde{C}_{m+1}^{\lambda-1}(z).$$

B.2 Bessel functions

The modified Bessel function of the first kind or I-Bessel function is defined, for $z > 0$ and $\alpha \in \mathbb{C}$, by

$$I_\alpha(z) := \left(\frac{z}{2}\right)^\alpha \sum_{n=0}^\infty \frac{1}{n!\Gamma(n+\alpha+1)} \left(\frac{z}{2}\right)^{2n}$$

and the modified Bessel function of the third kind or K-Bessel function by

$$K_\alpha(z) := \frac{\pi}{2\sin(\pi\alpha)}(I_{-\alpha}(z) - I_\alpha(z)),$$

see [Wa, Section 3.7]. We will use the following renormalisations

$$\widetilde{I}_\alpha(z) := \left(\frac{z}{2}\right)^{-\alpha} I_\alpha(z), \qquad \widetilde{K}_\alpha(z) := \left(\frac{z}{2}\right)^{-\alpha} K_\alpha(z).$$

The functions $\widetilde{I}_\alpha(z)$ and $\widetilde{K}_\alpha(z)$ are linearly independent and solve the following second order differential equation

$$z^2\frac{\mathrm{d}^2u}{\mathrm{d}z^2} + (2\alpha+1)z\frac{\mathrm{d}u}{\mathrm{d}z} - z^2u = 0. \tag{B.1}$$

We also have the differential recurrence relations, [Wa, III.71 (6)]

$$\frac{\mathrm{d}}{\mathrm{d}z}\widetilde{I}_\alpha(z) = \frac{z}{2}\widetilde{I}_{\alpha+1}(z), \qquad \frac{\mathrm{d}}{\mathrm{d}z}\widetilde{K}_\alpha(z) = -\frac{z}{2}\widetilde{K}_{\alpha+1}(z). \tag{B.2}$$

Using these relations we can rewrite the second order differential equation as a recurrence relation, [Wa, III.71 (1)],

$$\begin{aligned} \frac{z^2}{4}\widetilde{I}_{\alpha+1}(z) + \alpha\widetilde{I}_\alpha(z) - \widetilde{I}_{\alpha-1}(z) &= 0, \\ \frac{z^2}{4}\widetilde{K}_{\alpha+1}(z) - \alpha\widetilde{K}_\alpha(z) - \widetilde{K}_{\alpha-1}(z) &= 0. \end{aligned} \tag{B.3}$$

The asymptotic behaviour of the K-Bessel function is given by, [Wa, Chapter III and VII],

$$\text{for } x \to 0: \quad \widetilde{K}_\alpha(x) = \begin{cases} \frac{\Gamma(\alpha)}{2}(\frac{x}{2})^{-2\alpha} + o(x^{-2\alpha}) & \text{if } \alpha > 0 \\ -\log(\frac{x}{2}) + o(\log(\frac{x}{2})) & \text{if } \alpha = 0 \\ \frac{\Gamma(-\alpha)}{2} + o(1) & \text{if } \alpha < 0. \end{cases}$$

$$\text{for } x \to \infty: \quad \widetilde{K}_\alpha(x) = \frac{\sqrt{\pi}}{2}\left(\frac{x}{2}\right)^{-\alpha-\frac{1}{2}} e^{-x}\left(1 + \mathcal{O}\left(\frac{1}{x}\right)\right).$$

B.3 Generalised Laguerre functions

Consider the generating function

$$G_2^{\mu,\nu}(t,x) := \frac{1}{(1-t)^{\frac{\mu+\nu+2}{2}}}\widetilde{I}_{\frac{\mu}{2}}\left(\frac{tx}{1-t}\right)\widetilde{K}_{\frac{\nu}{2}}\left(\frac{x}{1-t}\right),$$

for parameters $\mu, \nu \in \mathbb{C}$. This function $G_2^{\mu,\nu}$ is holomorphic near $t = 0$. We will define the generalised Laguerre functions $\Lambda_{2,j}^{\mu,\nu}(x)$ as the coefficients in the expansion

$$G_2^{\mu,\nu}(t,x) = \sum_{j=0}^{\infty} \Lambda_{2,j}^{\mu,\nu}(x)t^j. \tag{B.4}$$

Note that $\Lambda_{2,0}^{\mu,\nu}(x) = \frac{1}{\Gamma(\frac{\mu+2}{2})}\widetilde{K}_{\frac{\nu}{2}}(x)$. For notational convenience we set $\Lambda_{2,j}^{\mu,\nu} = 0$ for $j < 0$. We have some relations between the generating functions, which in turn lead to corresponding differential recurrence relations for the $\Lambda_{2,j}^{\mu,\nu}$.

Proposition B.3.1. *The generating functions satisfy*

$$\partial_x^2 G_2^{\mu,\nu}(x,t) + \frac{(\nu+1)}{x}\partial_x G_2^{\mu,\nu}(x,t) - G_2^{\mu,\nu}(x,t)$$
$$= t(\mathbb{E}_t + \mu + 2)G_2^{\mu+2,\nu}(x,t),$$
$$\partial_x^2 G_2^{\mu,\nu}(x,t) + \frac{(\mu+1)}{x}\partial_x G_2^{\mu,\nu}(x,t) - G_2^{\mu,\nu}(x,t)$$
$$= -(\mathbb{E}_t + \frac{\mu-\nu}{2})G_2^{\mu,\nu+2}(x,t),$$
$$t\left(\mu(\mu+\nu+2\mathbb{E}_x)G_2^{\mu,\nu}(x,t) + x^2(t\ (\mathbb{E}_t + \mu + 2))G_2^{\mu+2,\nu}(x,t)\right)$$

$$= 4\mathbb{E}_t G_2^{\mu-2,\nu}(x,t),$$

$$-\nu(\mu+\nu+2\mathbb{E}_x)G_2^{\mu,\nu}(x,t) + x^2(\mathbb{E}_t + \frac{\mu-\nu}{2})G_2^{\mu,\nu+2}(x,t)$$
$$= (4\mathbb{E}_t + 2(\mu+\nu))G_2^{\mu,\nu-2}(x,t),$$

where $\mathbb{E}_x = x\partial_x$ *and* $\mathbb{E}_t = t\partial_t$.

Proof. First one uses the differential recursion relations for the Bessel functions, equation (B.2), to calculate

$$\partial_x G_2^{\mu,\nu}(x,t) = \tfrac{x}{2(1-t)}(t^2 G_2^{\mu+2,\nu}(x,t) - G_2^{\mu,\nu+2}(x,t)),$$
$$\partial_x^2 G_2^{\mu,\nu}(x,t) = \tfrac{x^2t^4}{4(1-t)^2}G_2^{\mu+4,\nu}(x,t) + \tfrac{t^2}{2(1-t)}G_2^{\mu+2,\nu}(x,t)$$
$$- \tfrac{x^2t^2}{2(1-t)^2}G_2^{\mu+2,\nu+2}(x,t) - \tfrac{1}{2(1-t)}G_2^{\mu,\nu+2}(x,t) + \tfrac{x^2}{4(1-t)^2}G_2^{\mu,\nu+4}(x,t),$$
$$\partial_t G_2^{\mu,\nu}(x,t)$$
$$= \tfrac{\mu+\nu+2}{2(1-t)}G_2^{\mu,\nu}(x,t) + \tfrac{x^2t}{2(1-t)^2}G_2^{\mu+2,\nu}(x,t) - \tfrac{x^2}{2(1-t)^2}G_2^{\mu,\nu+2}(x,t).$$

From the recurrence relations (B.3) for the Bessel functions, we get the following recurrence relations for $G_2^{\mu,\nu}(x,t)$

$$\frac{x^2}{4(1-t)}G_2^{\mu,\nu+2}(x,t) - \frac{\nu}{2}G_2^{\mu,\nu}(x,t) - \frac{1}{1-t}G_2^{\mu,\nu-2}(x,t) = 0,$$
$$\frac{x^2t^2}{4(1-t)}G_2^{\mu+2,\nu}(x,t) + \frac{\mu}{2}G_2^{\mu,\nu}(x,t) - \frac{1}{1-t}G_2^{\mu-2,\nu}(x,t) = 0.$$

We can combine these relations with the expressions for the partial derivatives to obtain the proposition. □

Corollary B.3.2. *The generalised Laguerre functions satisfy*

$$\partial_x^2\Lambda_{2,j}^{\mu,\nu}(x) + \tfrac{(\nu+1)}{x}\partial_x\Lambda_{2,j}^{\mu,\nu}(x) - \Lambda_{2,j}^{\mu,\nu}(x) = (j+\mu+1)\Lambda_{2,j-1}^{\mu+2,\nu}(x) \tag{B.5}$$

$$\partial_x^2\Lambda_{2,j}^{\mu,\nu}(x) + \tfrac{(\mu+1)}{x}\partial_x\Lambda_{2,j}^{\mu,\nu}(x) - \Lambda_{2,j}^{\mu,\nu}(x) = -(j+\tfrac{\mu-\nu}{2})\Lambda_{2,j}^{\mu,\nu+2}(x)$$
$$\mu(\mu+\nu+2\mathbb{E}_x)\Lambda_{2,j}^{\mu,\nu} + (j+\mu+1)x^2\Lambda_{2,j-1}^{\mu+2,\nu} = 4(j+1)\Lambda_{2,j+1}^{\mu-2,\nu}$$
$$\nu(\mu+\nu+2\mathbb{E}_x)\Lambda_{2,j}^{\mu,\nu} + (-j-\tfrac{\mu-\nu}{2})x^2\Lambda_{2,j}^{\mu,\nu+2} = -4(j+\tfrac{\mu+\nu}{2})\Lambda_{2,j}^{\mu,\nu-2}.$$

Proof. The corollary follows from Proposition B.3.1 and the definition of $\Lambda_{2,j}^{\mu,\nu}$ in equation (B.3) as coefficients in the expansion of $G_2^{\mu,\nu}(x,t)$. □

In general we do not have of an explicit expression for the functions $\Lambda_{2,j}^{\mu,\nu}(x)$. However for our purposes it is sufficient to know when they are non-zero.

Corollary B.3.3. *Assume $\mu \notin -\mathbb{N}$ or $\mu + j \geq 0$. Then on every open interval the function $\Lambda_{2,j}^{\mu,\nu}$ is different from zero for $j \in \mathbb{N}$.*

Proof. Suppose $\Lambda_{2,j}^{\mu,\nu}(x) = 0$ for all $x \in I$, with $I \subset \mathbb{R}^+$ an open interval. Then from (B.5) it would follow that also $(j+\mu+1)\Lambda_{2,j-1}^{\mu+2,\nu}(x) = 0$. Since $\mu + 1 + j \neq 0$, we obtain $\Lambda_{2,j-1}^{\mu+2,\nu}(x) = 0$. This would again lead to $\Lambda_{2,j-2}^{\mu+4,\nu}(x) = 0$ and so on. Finally we get $\Lambda_{2,0}^{\mu+2j,\nu}(x) = 0$. This is a contradiction since $\Lambda_{2,0}^{\mu+2j,\nu}(x) = \frac{1}{\Gamma(\frac{\mu+2j+2}{2})}\widetilde{K}_{\frac{\nu}{2}}(x)$ and the Bessel function is different from zero on I. □

We also use the following recursion relation.

Proposition B.3.4. *For $\mu, \nu \in \mathbb{C}$, we have for $j \in \mathbb{Z}$*

$$\begin{aligned}&(2j + \mu + 1)\left(\mathbb{E}_x + \tfrac{\mu+\nu+2}{2}\right)\Lambda_{2,j}^{\mu,\nu}(x)\\ &= (j + 1)(j + \mu + 1)\Lambda_{2,j+1}^{\mu,\nu}(x) - \left(j + \tfrac{\mu+\nu}{2}\right)\left(j + \tfrac{\mu-\nu}{2}\right)\Lambda_{2,j-1}^{\mu,\nu}(x).\end{aligned}$$

For $j = 0$, we have

$$\left(\mathbb{E}_x + \tfrac{\mu+\nu+2}{2}\right)\Lambda_{2,0}^{\mu,\nu}(x) = \Lambda_{2,1}^{\mu,\nu}(x),$$

even for $\mu = -1$.

Proof. This is [Mö1, Proposition 3.6.1] and [Mö1, Example 3.3.1]. □

C
Nederlandstalige samenvatting

Het onderwerp van deze thesis betreft minimale representaties van Lie-superalgebra's.

We starten met een introductie tot Lie-superalgebra's in Hoofdstuk 2. Daarin geven we onder andere een overzicht van alle eindigdimensionale, enkelvoudige Lie-superalgebra's. Daarnaast wordt ook de basis gelegd van representatietheorie van Lie-superalgebra's.

Jordan-superalgebra's

In Hoofdstuk 3 vergelijken we de verschillende definities van structuuralgebra's en TKK-algebra's voor Jordan-superalgebra's die voorkomen in de literatuur. We tonen aan dat voor Jordan-superalgebra's die een eenheidselement bevatten de verschillende definities van de structuuralgebra's en TKK-algebra's kunnen gereduceerd worden tot twee onderscheiden gevallen. Bovendien kan de ene algebra dan bekomen worden als de Lie-superalgebra van superderivaties van de andere algebra. We tonen ook aan dat voor Jordan-superalgebras zonder eenheidselement meer definities niet equivalent worden. Als toepassing geven we een tabel van alle Lie-superalgebra's die overeenkomen met de enkelvoudige, eindigdimensionale Jordan-superalgebra's over een algebraïsch gesloten veld met karakteristiek nul.

Polynomiale realisaties en Bessel operatoren

Hoofdstuk 4 gaat over realisaties van Lie-(super)algebra's in Weyl-(super)algebra's en het verband met minimale representaties. Het belangrijkste resultaat is de constructie van kleine realisaties van Lie-superalgebras. We kunnen dit gebruiken voor twee doeleinden. Ten eerste introduceren en veralgemenen ze, op een heel natuurlijke wijze, de Bessel operatoren voor Jordanalgebra's die opduiken bij de studie van minimale representaties voor enkelvoudige Liegroepen. Ten twee, kunnen we deze theoretische realisatie uitwerken voor de exceptionele Lie-superalgebra $D(2,1;\alpha)$, hetgeen leidt tot een zeer concrete realisatie.

Een minimale representatie van de orthosymplectische Lie-superalgebra

In Hoofdstukken 5 en 6 construeren we een minimale representatie van de orthosymplectische Lie-supergroep $OSp(p,q|2n)$. Dit is een veralgemening van het Schrödingermodel van de minimale representatie van $O(p,q)$ naar het supergeval. De onderliggende representatie van de Lie-superalgebra wordt gerealiseerd op functies op de minimale baan bevat in de Jordan-superalgebra geassocieerd met $\mathfrak{osp}(p,q|2n)$. Op die manier past deze constructie in de 'orbit philosophy'. Het annihilatorideaal wordt gegeven door het Joseph-achtig ideaal van $\mathfrak{osp}(p,q|2n)$. Daardoor is deze representatie inderdaad een natuurlijke veralgemening van een minimale representatie in de context van Lie-superalgebras. We construeren ook een niet-ontaarde, sesquilineaire vorm. Met betrekking tot deze vorm is onze representatie scheefsymmetrisch en dit geeft een analogon voor het L^2-inproduct in het supergeval. We berekenen ook nog de Gelfand–Kirillov dimensie.

We eindigen deze thesis met enkele conclusies en open vragen in Hoofdstuk 7.

English summary

In this thesis we study minimal representations of Lie superalgebras.

An introduction to Lie superalgebras is given in Chapter 2. We give an overview of all simple finite-dimensional Lie superalgebras. We also explain some basic concepts of representation theory of Lie superalgebras used in this thesis.

Jordan superalgebras

In Chapter 3 we compare a number of different definitions of structure algebras and TKK constructions for Jordan (super)algebras appearing in the literature. We demonstrate that, for unital superalgebras, all the definitions of the structure algebra and the TKK constructions reduce to one of two cases. Moreover, one can be obtained as the Lie superalgebra of superderivations of the other. We also show that, for non-unital superalgebras, more definitions become non-equivalent. As an application, we obtain the corresponding Lie superalgebras for all simple finite-dimensional Jordan superalgebras over an algebraically closed field of characteristic zero.

Polynomial realisations and Bessel operators

Chapter 4 concerns realisations of Lie (super)algebras in Weyl (super)algebras and connections with minimal representations. The main result is the construction of small realisations of Lie superalgebras, which we apply for two distinct purposes. Firstly it naturally introduces, and generalises, the Bessel operators for Jordan algebras

in the study of minimal representations of simple Lie groups. Secondly, we work out the theoretical realisation concretely for the exceptional Lie superalgebra $D(2,1;\alpha)$, giving a useful hands-on realisation.

A minimal representation of the orthosymplectic Lie superalgebra

In Chapters 5 and 6 we construct a minimal representation of the orthosymplectic Lie supergroup $OSp(p,q|2n)$. This yields a generalisation of the Schrödinger model of the minimal representation of $O(p,q)$ to the super case. The underlying Lie algebra representation is realised on functions on the minimal orbit inside the Jordan superalgebra associated with $\mathfrak{osp}(p,q|2n)$, so that our construction is in line with the orbit philosophy. Its annihilator is given by a Joseph-like ideal for $\mathfrak{osp}(p,q|2n)$, and therefore the representation is a natural generalisation of a minimal representations to the context of Lie superalgebras. We also construct a non-degenerate sesquilinear form for which the representation is skew-symmetric and which is the analogue of an L^2-inner product in the super case, and calculate its Gelfand–Kirillov dimension.

We end this thesis by mentioning some open questions and directions for future research in Chapter 7.

Bibliography

[AMR] D. Alekseevsky, P. Michor, W. Ruppert. Extensions of super Lie algebras. J. Lie Theory **15** (2005), no. 1, 125–134.

[Al] A. Alldridge. Fréchet Globalisations of Harish-Chandra Supermodules. Int. Math. Res. Not. IMRN 2017, no. 17, 5182–5232.

[AS] A. Alldridge, Z. Shaikh. Superbosonisation, Riesz superdistributions, and highest weight modules. Advances in Lie superalgebras, Springer INdAM Ser. **7**, 1–18, Springer, Cham, 2014.

[AB] A. Astashkevich, R. Brylinski. Non-local equivariant star product on the minimal nilpotent orbit. Adv. Math. **171** (2002), no. 1, 86–102.

[BC] S. Barbier and K. Coulembier. The Joseph ideal for $\mathfrak{sl}(m|n)$. Lie theory and its applications in physics, 489–499, Springer Proc. Math. Stat., **191**, Springer, Singapore, 2016,

[Be] F. Berezin. Several remarks on the associative hull of a Lie algebra. Funktsional. Anal. i Prilozhen **1** (1967), no. 2, 1–14.

[BZ] B. Binegar, R. Zierau. Unitarization of a singular representation of $SO(p,q)$. Comm. Math. Phys. **138** (1991), no. 2, 245–258.

[BDS] L. Boelaert, T. De Medts, A. Stavrova. Moufang sets and structurable division algebras. To appear in Mem. Amer. Math. Soc.

[BJ] A. Braverman, A. Joseph. The minimal realization from deformation theory. J. Algebra **205** (1998), 113–36.

[CK] N. Cantarini, V. G. Kac. Classification of linearly compact simple Jordan and generalized Poisson superalgebras. J. Algebra **313** (2007), no. 1, 100–124.

[CCF] C. Carmeli, L. Caston, R. Fioresi. *Mathematical foundations of supersymmetry.* EMS Series of Lectures in Mathematics. European Mathematical Society (EMS), Zürich, 2011.

[CW] S.J. Cheng, W. Wang. *Dualities and representations of Lie superalgebras.* Graduate Studies in Mathematics, **144**. American Mathematical Society, Providence, RI, 2012.

[Co] N. Conze. Algèbres d'opérateurs différentiels et quotients des algèbres enveloppantes. Bull. Soc. Math. France **102** (1974), 379–415.

[Cou] K. Coulembier. The orthosymplectic superalgebra in harmonic analysis. J. Lie Theory **23** (2013), no. 1, 55–83.

[CDS1] K. Coulembier, H. De Bie, F. Sommen. Integration in superspace using distribution theory. J. Phys. A **42** (2009), no. 39, 395–206.

[CDS2] K. Coulembier, H. De Bie, F. Sommen. Orthosymplectically invariant functions in superspace. J. Math. Phys. **51** (2010), no. 8, 083504, 23 pp.

[CSS] K. Coulembier, P. Somberg, V. Souček. Joseph ideals and harmonic analysis for $\mathfrak{osp}(m|2n)$. Int. Math. Res. Not. IMRN (2014), no. 15, 4291–4340.

[De] H. De Bie. Fourier transform and related integral transforms in superspace. J. Math. Anal. Appl. **345** (2008), no. 1, 147–164.

[DeS] H. De Bie, F. Sommen. Spherical harmonics and integration in superspace. J. Phys. A **40** (2007), no. 26, 7193–7212.

[dGM] A. de Goursac, J. Michel. Superunitary representations of Heisenberg Supergroups. Preprint: arXiv:1601.07387

[DM] P. Deligne, J. W. Morgan. Notes On Supersymmetry (following Joseph Bernstein), in P. Deligne et. al., eds., Quantum Fields And Strings: A Course For Mathematicians, Vol. 1 (American Mathematical Society, 1999).

[Di] H. Dib. Fonctions de Bessel sur une algèbre de Jordan. J. Math. Pures Appl. (9) **69** (1990), no. 4, 403–448.

[DS] A. Dvorsky, S. Sahi. Explicit Hilbert spaces for certain unipotent representations II. Invent. Math. **138** (1999), no. 1, 203–224.

[ESS] M. Eastwood, P. Somberg, V. Souček. Special tensors in the deformation theory of quadratic algebras for the classical Lie algebras. J. Geom. Phys. **57** (2007) 2539–2546.

[EMOT] A. Erdélyi, W. Magnus, F. Oberhettinger, F. G. Tricomi. *Tables of integral transforms. Vol. II.* McGraw-Hill Book Company, Inc., New York-Toronto-London, 1954.

[FK] J. Faraut, A. Korányi. *Analysis on symmetric cones.* Oxford Mathematical Monographs. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, (1994)

[Fo] G. Folland. *Harmonic analysis in phase space.* Annals of Mathematics Studies, **122**. Princeton University Press, Princeton, NJ, 1989.

[FSS] L. Frappat, A. Sciarrino, P. Sorba. *Dictionary on Lie algebras and superalgebras.* Academic Press, Inc., San Diego, CA, 2000.

[GS] W. Gan, G. Savin. On minimal representations definitions and properties. Represent. Theory **9** (2005), 46–93.

[GN] E. García, E. Neher. Tits–Kantor–Koecher superalgebras of Jordan superpairs covered by grids. Comm. Algebra **31** (2003), no. 7, 3335–3375.

[Ga] D. Garfinkle. A new construction of the Joseph ideal. Massachusetts Institute of Technology. PhD-thesis. 1982.

[He] S. Helgason. *Groups and geometric analysis. Integral geometry, invariant differential operators, and spherical func-*

tions. Pure and Applied Mathematics, 113. Academic Press, Inc., Orlando, FL, 1984.

[HKMM] J. Hilgert, T. Kobayashi, G. Mano, J. Möllers. Special functions associated with a certain fourth-order differential equation. Ramanujan J. **26** (2011), no. 1, 1–34.

[HKM] J. Hilgert, T. Kobayashi, J. Möllers. Minimal representations via Bessel operators. J. Math. Soc. Japan **66** (2014), no. 2, 349–414.

[HKMØ] J. Hilgert, T. Kobayashi, J. Möllers, B. Ørsted. Fock model and Segal–Bargmann transform for minimal representations of Hermitian Lie groups. J. Funct. Anal. **263** (2012), no. 11, 3492–3563.

[Hu] J. Humphreys. *Representations of semisimple Lie algebras in the BGG category $\mathcal{O}$.* Graduate Studies in Mathematics, **94**. American Mathematical Society, Providence, RI, 2008.

[IR] K. Ireland, M. Rosen. *A classical introduction to modern number theory.* Second edition. Graduate Texts in Mathematics, **84**. Springer-Verlag, New York, 1990.

[Ja] N. Jacobson. Structure groups and Lie algebras of Jordan algebras of symmetric elements of associative algebras with involution. Advances in Math. **20** (1976), no. 2, 106–150.

[Jo1] A. Joseph. Minimal realizations and spectrum generating algebras. Comm. Math. Phys. **36** (1974), 325–338.

[Jo2] A. Joseph. The minimal orbit in a simple Lie algebra and its associated maximal ideal. Ann. Sci. École Norm. Sup. (4) **9** (1976), no. 1, 1–29.

[Ka1] V. G. Kac. Lie superalgebras. Advances in Math. **26** (1977), no. 1, 8–96.

[Ka2] V. G. Kac. Classification of simple Z-graded Lie superalgebras and simple Jordan superalgebras. Comm. Algebra **5** (1977), no. 13, 1375–1400.

[KMZ] V. G. Kac, C. Martinez, E. Zelmanov. Graded simple Jordan superalgebras of growth one. Mem. Amer. Math. Soc. **150** (2001), no. 711.

[Kane] S. Kaneyuki. The Sylvester's law of inertia in simple graded Lie algebras. J. Math. Soc. Japan **50** (1998), no. 3, 593–614.

[Kan1] I. L. Kantor. Transitive differential groups and invariant connections in homogeneous spaces. Trudy Sem. Vektor. Tenzor. Anal. **13** (1966) 310–398.

[Kan2] I. L. Kantor. Jordan and Lie superalgebras determined by a Poisson algebra. Amer. Math. Soc. Transl. Ser. 2, **151**, Amer. Math. Soc., Providence, RI, 1992.

[KM] I. Kashuba, M. E. Martin. The variety of three-dimensional real Jordan algebras. J. Algebra Appl. **15** (2016), no. 8, 1650158

[KS] I. Kashuba, V. Serganova. On the Tits–Kantor–Koecher construction of unital Jordan bimodules. J. Algebra **481** (2017), 420–463.

[Ki] A. A. Kirillov. *Lectures on the orbit method.* Graduate Studies in Mathematics, **64**. American Mathematical Society, Providence, RI, 2004.

[Ko1] T. Kobayashi. Algebraic analysis of minimal representations. Publ. Res. Inst. Math. Sci. **47** (2011), no. 2, 585–611.

[Ko2] T. Kobayashi. Varna lecture on L^2-analysis of minimal representations. Lie theory and its applications in physics, Springer Proc. Math. Stat. **36**, 77–93, Springer, Tokyo, 2013.

[Ko3] T. Kobayashi. Special functions in minimal representations. Perspectives in representation theory, 253–266, Contemp. Math., **610**, Amer. Math. Soc., Providence, RI, 2014.

[KM1] T. Kobayashi, G. Mano. The inversion formula and holomorphic extension of the minimal representation of the conformal group. *Harmonic Analysis, Group Representations, Automorphic Forms and Invariant Theory.* Singapore University Press and World Scientific Publishing, 2007, pp. 159–223.

[KM2] T. Kobayashi, G. Mano. The Schrödinger model for the minimal representation of the indefinite orthogonal group $O(p, q)$. Mem. Amer. Math. Soc. **212**, (2011), no. 1000.

[KØ] T. Kobayashi, B. Ørsted. Analysis on the minimal representation of $O(p, q)$. III. Ultrahyperbolic equations on $R^{p-1,q-1}$. Adv. Math. **180** (2003), no. 2, 551–595.

[Ko] M. Koecher. Imbedding of Jordan algebras into Lie algebras. I. Amer. J. Math. **89** (1967) 787–816.

[Kr] S. V. Krutelevich. Simple Jordan superpairs. Comm. Algebra **25** (1997), no. 8, 2635–2657.

[Le] D. A. Leĭtes. Introduction to the theory of supermanifolds. (Russian) Uspekhi Mat. Nauk **35** (1980), no. 1(211), 3–57, 255.

[Lo] O. Loos. *Jordan pairs.* Lecture Notes in Mathematics, Vol. 460. Springer-Verlag, Berlin-New York, 1975.

[MZ] C. Martinez, E. Zelmanov. Representation theory of Jordan superalgebras I. Trans. AMS, **362**, (2010), no.2, 815–846.

[McC] K. McCrimmon. *A taste of Jordan algebras.* Universitext, Springer-Verlag, New York, 2004.

[Mö1] J. Möllers. Minimal representations of conformal groups and generalized Laguerre functions. Universität Paderborn. PhD-thesis (2010), arXiv:1009.4549.

[Mö2] J. Möllers. Heat kernel analysis for Bessel operators on symmetric cones. J. Lie Theory **24** (2014), no. 2, 373–396.

[MS] J. Möllers, B. Schwarz. Bessel operators on Jordan pairs and small representations of semisimple Lie groups. J. Funct. Anal. **272** (2017), no. 5, 1892–1955.

[Mu] I. Musson. *Lie superalgebras and enveloping algebras.* Graduate Studies in Mathematics, **131**. American Mathematical Society, Providence, RI, 2012.

[NS] K.-H. Neeb, H. Salmasian. Lie supergroups, unitary representations, and invariant cones. Lecture Notes in Math., **2027**, 195–239, Springer, Heidelberg, 2011.

[Ni] K. Nishiyama. Oscillator representations for orthosymplectic algebras. J. Algebra **129** (1990), no. 1, 231–262.

[Pa] M. Parker. Classification of real simple Lie superalgebras of classical type. J. Math. Phys. **21** (1980), no. 4, 689–697.

[Sa] S. Sahi. Explicit Hilbert spaces for certain unipotent representations. Invent. Math. **110** (1992), no. 2, 409–418.

[SS] A. Salam, J. Strathdee. Super-gauge transformations. Nuclear Phys. **B76** (1974), 477–482.

[Sal] H. Salmasian. Unitary representations of nilpotent super Lie groups. Comm. Math. Phys. **297** (2010), no. 1, 189–227.

[Se] V. Serganova. Automorphism of Lie superalgebras. Izv. Akad. Nauk SSSR Ser. Mat. **48** (1984), no. 3, 585–598.

[Sc] M. Scheunert. *The theory of Lie superalgebras. An introduction.* Lecture Notes in Mathematics, 716. Springer, Berlin, 1979.

[Sh] A. S. Shtern. Representations of finite dimensional Jordan superalgebras of Poisson bracket. Comm. Algebra, **23**, (1995), no. 5, 1815–1823.

[Sp] T. Springer. *Jordan algebras and algebraic groups.* Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 75. Springer-Verlag, New York-Heidelberg, 1973

[Ti] J. Tits. Une classe d'algèbres de Lie en relation avec les algèbres de Jordan. Nederl. Akad. Wetensch. Proc. Ser. A 65 = Indag. Math. **24** (1962) 530–535.

[VdJ] J. Van der Jeugt. Irreducible representations of the exceptional Lie superalgebras $D(2,1;\alpha)$. J. Math. Phys. **26** (1985), no. 5, 913–924.

[VR] M. Vergne, H. Rossi. Analytic continuation of the holomorphic discrete series of a semi-simple Lie group. Acta Math. **136** (1976), no. 1–2, 1–59.

[Wa] G. N. Watson. *A Treatise on the Theory of Bessel Functions.* Cambridge University Press, Cambridge, England, 1944.

[WZ] J. Wess. B. Zumino. Supergauge transformations in four dimensions. Nuclear Phys. **B70** (1974), 39–50.