DUALITIES AND DERIVED EQUIVALENCES
FOR CATEGORY $\mathcal{O}$

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ABSTRACT. We determine the Ringel duals for all blocks in the parabolic versions of the BGG category $\mathcal{O}$ associated to a reductive finite dimensional Lie algebra. In particular we find that, contrary to the original category $\mathcal{O}$ and the specific previously known cases in the parabolic setting, the blocks are not necessarily Ringel self-dual. However, the parabolic category $\mathcal{O}$ as a whole is still Ringel self-dual. Furthermore, we use generalisations of the Ringel duality functor to obtain large classes of derived equivalences between blocks in parabolic and original category $\mathcal{O}$. We subsequently classify all derived equivalence classes of blocks of category $\mathcal{O}$ in type A which preserve the Koszul grading.

1. INTRODUCTION AND MAIN RESULTS

For a reductive finite dimensional complex Lie algebra $\mathfrak{g}$ with a fixed triangular decomposition, consider the Bernstein-Gelfand-Gelfand category $\mathcal{O}$ from [BGG] (see also [Hu]) and its parabolic generalisation by Rocha-Caridi in [RC]. It is well-known that blocks in parabolic category $\mathcal{O}$ are described by quasi-hereditary algebras which possess a Koszul grading, see [Ba1, BGS, So1, Ma2]. Moreover, graded lifts of blocks of category $\mathcal{O}$ are Koszul dual to graded lifts of blocks in the parabolic versions of $\mathcal{O}$. It was proved in [So2] that parabolic category $\mathcal{O}$, as a whole, is Ringel self-dual. It is well-known that for full category $\mathcal{O}$, even every individual block is Ringel self-dual, as follows from [So1, Struktursatz 9]. The same holds for the *principal* block in parabolic category $\mathcal{O}$, see [MS3]. One of our observations which motivated the present paper is that arbitrary blocks in parabolic category $\mathcal{O}$ are, surprisingly, *not* Ringel self-dual in general. This is our first main result.

Theorem A. *Blocks in parabolic category $\mathcal{O}$ are, in general, not Ringel self-dual. However, every block is Ringel dual to another block in the same parabolic category. The Ringel dual of a block is also equivalent to a block with the same central character in a different parabolic version of $\mathcal{O}$.*

Ringel duality between two blocks always implies an equivalence, as triangulated categories, of the bounded derived categories of these blocks given by the derived *Ringel duality functor*, see [Rin, Ha, Ric1]. From our proof of Theorem A, it follows easily that, for parabolic category $\mathcal{O}$, this equivalence lifts to the graded setting for the Koszul grading. On the other hand, the *Koszul duality functor*, see [BGS, MOS], also induces an equivalence between the derived categories of a graded lift of a block

and its Koszul dual. However, this equivalence does not correspond to a derived equivalence in the ungraded sense. Moreover, we show explicitly that two Koszul dual algebras need not be derived equivalent in the sense of [Ric1], by considering an example for category $\mathcal{O}$.

We are interested in derived equivalences in the sense of [Ric1] which lift to the graded setting, see explicit Definition 3.2. We refer to this as a *gradable derived equivalence* or being *gradable derived equivalent*. Hence, for parabolic category $\mathcal{O}$, the Ringel duality functor induces a gradable derived equivalence, whereas the Koszul duality functor does not.

In the current paper we start the systematic study of derived equivalences for (parabolic) category $\mathcal{O}$. Some interesting derived equivalences for $\mathfrak{g} = \mathfrak{sl}(n)$ have been obtained by Chuang and Rouquier in [CR] and by Khovanov in [Kh]. By construction, the ones in [CR] are gradable. We take an approach independent from the previous results but recover the derived equivalences in [CR, Kh]. Our results are rather conclusive for category $\mathcal{O}$ for type A, whereas there remain open questions for the parabolic versions of $\mathcal{O}$ over other reductive Lie algebras.

Consider a reductive finite dimensional complex Lie algebra $\mathfrak{g}$ with a fixed Cartan subalgebra $\mathfrak{h}$ and a fixed Borel subalgebra $\mathfrak{b}$ containing $\mathfrak{h}$. Consider $\mathcal{O} = \mathcal{O}(\mathfrak{g}, \mathfrak{b})$ and let $W = W(\mathfrak{g} : \mathfrak{h})$ be the Weyl group and Λ_{int} the set of integral weights. For a coset $\Lambda \in \mathfrak{h}^*/\Lambda_{\mathrm{int}}$, we denote the corresponding integral Weyl group by W_Λ. For a dominant $\lambda \in \Lambda$, the stabiliser of λ in W_Λ under the dot action is denoted by $W_{\Lambda,\lambda}$ and the block in the category $\mathcal{O}(\mathfrak{g}, \mathfrak{b})$ containing the simple highest weight module with highest weight λ by $\mathcal{O}_\lambda(\mathfrak{g}, \mathfrak{b})$. Our second main result is an analogue of [So1, Theorem 11] for derived categories, restricted to type A.

Theorem B. *Consider two Lie algebras $\mathfrak{g}$ and $\mathfrak{g}'$ of type A, with respective Borel subalgebras $\mathfrak{b}$ and $\mathfrak{b}'$. Then there is a gradable derived equivalence*

$$\mathcal{D}^b(\mathcal{O}_\lambda(\mathfrak{g}, \mathfrak{b})) \;\cong\; \mathcal{D}^b(\mathcal{O}_{\lambda'}(\mathfrak{g}', \mathfrak{b}'))$$

for dominant $\lambda \in \Lambda$ and $\lambda' \in \Lambda'$, if and only if, for some decompositions

$$W_\Lambda \cong X_1 \times X_2 \times \cdots \times X_k \quad and \quad W_{\Lambda'} \cong X_1' \times X_2' \times \cdots \times X_m'$$

into products of irreducible Weyl groups, we have $k = m$ and there is a permutation φ on $\{1, 2, \ldots, k\}$ such that $W_{\Lambda,\lambda} \cap X_i \cong W_{\Lambda',\lambda'} \cap X_{\varphi(i)}'$ and $X_i \cong X_{\varphi(i)}'$, for all $i = 1, 2, \ldots, k$.

Note that, according to [So1, Theorem 11] (restricted to type A), there is an equivalence $\mathcal{O}_\lambda(\mathfrak{g}, \mathfrak{b}) \cong \mathcal{O}_{\lambda'}(\mathfrak{g}', \mathfrak{b}')$ if the following stronger condition is satisfied: There is a Coxeter group isomorphism $W_\Lambda \to W_{\Lambda'}$ which swaps $W_{\Lambda,\lambda}$ and $W_{\Lambda',\lambda'}$.

To present the remainder of the main results, we need more notation. We consider again an arbitrary complex reductive Lie algebras $\mathfrak{g}$ and henceforth only consider integral weights, which is justified by the results in [So1]. Hence $\Lambda = \Lambda_{\mathrm{int}}$ and we leave out the reference to Λ. For every integral dominant weight λ, the block $\mathcal{O}_\lambda$ now consists of all modules in $\mathcal{O}$ with the same generalised central character as the simple module with highest weight λ. To any integral dominant weight μ, we associate a parabolic subalgebra $\mathfrak{q}_\mu$ of $\mathfrak{g}$, uniquely defined by the fact that the Weyl group of its Levi factor is W_μ. The full subcategory of $\mathcal{O}$ consisting of all modules which are

locally $\mathfrak{q}_\mu$-finite is denoted by $\mathcal{O}^\mu$. The integral part of this category decomposes naturally into subcategories $\mathcal{O}^\mu_\lambda$. Graded lifts of the latter categories, with respect to the Koszul grading, are denoted by $^\mathbb{Z}\mathcal{O}^\mu_\lambda$.

Theorem C. *Consider the algebra* $\mathfrak{g} = \mathfrak{sl}(n)$ *and four integral dominant weights* $\lambda, \lambda', \mu, \mu'$. *If we have isomorphisms of groups*

$$W_\lambda \cong W_{\lambda'} \qquad and \qquad W_\mu \cong W_{\mu'},$$

then there is a gradable derived equivalence between $\mathcal{O}^\mu_\lambda$ *and* $\mathcal{O}^{\mu'}_{\lambda'}$. *In particular, we have equivalences of triangulated categories*

$$\mathcal{D}^b(\mathcal{O}^\mu_\lambda) \cong \mathcal{D}^b(\mathcal{O}^{\mu'}_{\lambda'}) \qquad and \qquad \mathcal{D}^b(^\mathbb{Z}\mathcal{O}^\mu_\lambda) \cong \mathcal{D}^b(^\mathbb{Z}\mathcal{O}^{\mu'}_{\lambda'}).$$

A more complicated formulation of this result for arbitrary $\mathfrak{g}$ can be found in Theorem 6.1. Theorem C generalises [Kh, Proposition 7], which corresponds precisely to the case $\lambda = \lambda' = 0$ in the ungraded setting. Furthermore, our approach gives an explicit form of the functor and the tilting complex, which describe the equivalence. It also provides a purely algebraic proof, whereas the proof in [Kh] depends on the geometric description of $\mathcal{O}$. Theorem C, for $\mu = \mu'$, gives the derived equivalences which can be constructed from [CR, Theorem 6.4 and Section 7.4]. It seems that in this case even the functors describing the derived equivalence are isomorphic, as can be checked by hand for small cases. In [CR] these functors have the elegant property that they are defined directly to act between the two relevant categories. On the other hand, they are defined in terms of total complexes for a complex of functors which becomes arbitrarily big. The functors in the current paper have the drawback that they are defined implicitly by using an auxiliary regular block in category $\mathcal{O}$, see Theorem D. The advantage is that the functor on the latter category inducing the equivalence is a well-understood functor with elegant properties.

Our results are formulated in terms of four types of functors on category $\mathcal{O}$, *viz.* projective, Zuckerman, twisting and shuffling functors, see [MS2] or the preliminaries for an overview. Twisting functors commute with projective functors, see [AS], and shuffling functors commute with Zuckerman functors, see [MS2]. The non-trivial commutation relations between twisting and Zuckerman functors, and between shuffling and projective functors, lie at the origin of the failure of blocks to be Ringel self-dual. It is also these commutation relation that we exploit to obtain the derived equivalences. As an extra result, we extend the Koszul duality between projective and Zuckerman functors of [Ry] to the full parabolic and singular setting.

For any $x \in W$, the derived twisting functor $\mathcal{L}T_x$ and shuffling functor $\mathcal{L}C_x$ are auto-equivalences of the category $\mathcal{D}^b(\mathcal{O}_0)$. For an integral dominant ν, we introduce the notation $W_\nu^\dagger$ for the subgroup of W generated by all simple reflections which are orthogonal to all reflections in W_ν. Let w_0^ν stand for the longest element of W_ν. Theorem C can be obtained by an iterative application of the following theorem.

Theorem D. *Consider* $\mathfrak{g}$ *a finite dimensional complex reductive Lie algebra and an integral dominant* $\nu, \lambda, \mu \in \mathfrak{h}^*$ *such that* W_ν *is of type A.*

(i) *Assume that* $W_\mu \subset W_\nu \times W_\nu^\dagger$. *Then there is a dominant integral* μ' *with* $W_{\mu'} = w_0^\nu W_\mu w_0^\nu$. *The auto-equivalence* $\mathcal{L}T_{w_0^\nu}$ *of* $\mathcal{D}^b(\mathcal{O}_0)$ *restricts to an equivalence*

of triangulated categories between two subcategories, equivalent to, respectively,
$\mathcal{D}^b(\mathcal{O}_\lambda^\mu)$ *and* $\mathcal{D}^b(\mathcal{O}_\lambda^{\mu'})$, *yielding a gradable derived equivalence* $\mathcal{D}^b(\mathcal{O}_\lambda^\mu) \xrightarrow{\sim} \mathcal{D}^b(\mathcal{O}_\lambda^{\mu'})$.

(ii) *Assume that* $W_\lambda \subset W_\nu \times W_\nu^\dagger$. *Then there is a dominant integral* λ' *with* $W_{\lambda'} = w_0^\nu W_\lambda w_0^\nu$. *The auto-equivalence* $\mathcal{LC}_{w_0^\nu}$ *of* $\mathcal{D}^b(\mathcal{O}_0)$ *restricts to an equivalence of triangulated categories between two subcategories, equivalent to, respectively,* $\mathcal{D}^b(\mathcal{O}_\lambda^\mu)$ *and* $\mathcal{D}^b(\mathcal{O}_{\lambda'}^\mu)$, *yielding a gradable derived equivalence* $\mathcal{D}^b(\mathcal{O}_\lambda^\mu) \xrightarrow{\sim} \mathcal{D}^b(\mathcal{O}_{\lambda'}^\mu)$.

Note that the two parts can be interpreted as Koszul duals of one another using [MOS, Section 6.5]. When $w_0^\nu = w_0$ and $\mu = 0$ in Theorem D(ii), the categories $\mathcal{O}_\lambda$ and $\mathcal{O}_{\lambda'}$ are equivalent by [So1, Theorem 11]. However, our equivalence of the derived categories is not induced by that equivalence, but is rather given by the derived Ringel duality functor.

The paper is organised as follows. In Section 2 we recall some results on category $\mathcal{O}$ and Koszul and Ringel duality. In Section 3 we give an explicit example of Koszul dual algebras which are not derived equivalent and introduce the notion of gradable derived equivalence. We use this to study shuffling in the parabolic setting. In Section 4 we obtain several results on the graded lifts of translation functors. In particular, we extend the Koszul duality of [Ry] between translation functors and parabolic Zuckerman functors to the generality we will need it. In Section 5 we study the commutation relations between shuffling and projective functors. In Section 6 we construct the derived equivalences between blocks in parabolic category $\mathcal{O}$, proving Theorems C and D. This is used in Section 7 to classify the blocks in category $\mathcal{O}$ for Lie algebras of type A up to gradable derived equivalence, proving Theorem B. In Section 8 we determine the Ringel duals of all blocks in parabolic category $\mathcal{O}$, proving Theorem A, and study the Koszul-Ringel duality functor.

2. Preliminaries

We work over $\mathbb{C}$. Unless explicitly stated otherwise, commuting diagrams of functors commute only up to a natural isomorphism. *Graded* always refers to $\mathbb{Z}$-*graded*.

2.1. **Category $\mathcal{O}$ and its parabolic generalisations.** We consider the BGG category $\mathcal{O}$, associated to a triangular decomposition of a finite dimensional complex reductive Lie algebra $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$, see [BGG, Hu]. For any weight $\nu \in \mathfrak{h}^*$, we denote the corresponding simple highest weight module by $L(\nu)$. We also introduce an involution on $\mathfrak{h}^*$ by setting $\widehat{\nu} = -w_0(\nu)$, with w_0 the longest element of the Weyl group $W = W(\mathfrak{g} : \mathfrak{h})$. We denote by $\langle \cdot, \cdot \rangle$ a W-invariant inner product on $\mathfrak{h}^*$ and the set of integral, not necessarily regular, dominant weights by Λ_{int}^+. For any $\lambda \in \Lambda_{\mathrm{int}}^+$, the indecomposable block in category $\mathcal{O}$ containing $L(\lambda)$ is denoted by $\mathcal{O}_\lambda$.

For B the set of simple positive roots and $\mu \in \Lambda_{\mathrm{int}}^+$, set $B_\mu = \{\alpha \in B \mid \langle \mu + \rho, \alpha \rangle = 0\}$. Let $\mathfrak{u}_\mu^-$ be the subalgebra of $\mathfrak{g}$ generated by the root spaces corresponding to the roots in $-B_\mu$. Then we have the parabolic subalgebra $\mathfrak{q}_\mu$ of $\mathfrak{g}$, given by

$$\mathfrak{q}_\mu := \mathfrak{u}_\mu^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+.$$

The full subcategory of $\mathcal{O}_\lambda$ with objects given by the modules in $\mathcal{O}_\lambda$ which are $U(\mathfrak{q}_\mu)$-locally finite is denoted by $\mathcal{O}_\lambda^\mu$. These are subcategories of parabolic category $\mathcal{O}$ as

introduced in [RC]. By construction, $\mathcal{O}_\lambda^\mu$ is a Serre subcategory of $\mathcal{O}_\lambda$. We denote the corresponding exact full embedding of categories by $\imath^\mu : \mathcal{O}^\mu \hookrightarrow \mathcal{O}$. The left adjoint of $\imath^\mu$ is the corresponding *Zuckerman functor*, denoted by Z^μ, it is given by taking the largest quotient inside $\mathcal{O}^\mu$. The categories $\mathcal{O}_0^\mu$ and $\mathcal{O}_\lambda$ are indecomposable. The category $\mathcal{O}_\lambda^\mu$ may decompose, e.g. in the case $\mathfrak{g} = B(2) = \mathfrak{so}(5)$, for $w_0^\lambda = s$ and $w_0^\mu = t$, where s and t are the two different simple reflections.

We define the set X_λ as the set of *longest* representatives in W of cosets in W/W_λ. The non-isomorphic simple objects in the category $\mathcal{O}_\lambda$ are indexed by X_λ:

$$\{L(w \cdot \lambda) \,|\, w \in X_\lambda\}.$$

Now, for $x \in X_\lambda$, the module $L(x \cdot \lambda)$ is an object of $\mathcal{O}_\lambda^\mu$ if and only if x is a *shortest* representative in W of a coset in $W_\mu \backslash W$. The set of such shortest representatives $x \in X_\lambda$ is denoted by X_λ^μ.

We denote by **d** the usual duality on $\mathcal{O}$, which restricts to $\mathcal{O}^\mu$ and to each block in these categories, see [Hu, Section 3.2]. For $x \in X_\lambda^\mu$, consider the following structural modules in $\mathcal{O}_\lambda^\mu$: the standard module (or generalised Verma module) $\Delta^\mu(x \cdot \lambda)$ with simple top $L(x \cdot \lambda)$, the costandard module $\nabla^\mu(x \cdot \lambda) := \mathbf{d}\Delta^\mu(x \cdot \lambda)$, the injective envelope $I^\mu(x \cdot \lambda)$ and projective cover $P^\mu(x \cdot \lambda)$ of $L(x \cdot \lambda)$, the indecomposable quasi-hereditary tilting module $T^\mu(x \cdot \lambda)$ with highest weight $x \cdot \lambda$.

Consider a minimal projective generator of $\mathcal{O}_\lambda^\mu$ given by

$$(1) \qquad P_\lambda^\mu \;:=\; \bigoplus_{x \in X_\lambda^\mu} P^\mu(x \cdot \lambda),$$

where $P^\mu(x \cdot \lambda)$ is the indecomposable projective cover of $L(x \cdot \lambda)$ in $\mathcal{O}_\lambda^\mu$, and the algebra $A_\lambda^\mu := \mathrm{End}_\mathfrak{g}(P_\lambda^\mu)$. Then we have the usual equivalence of categories

$$\mathcal{O}_\lambda^\mu \;\xrightarrow{\sim}\; \mathrm{mod}\text{-}A_\lambda^\mu; \qquad M \mapsto \mathrm{Hom}_\mathfrak{g}(P_\lambda^\mu, M).$$

We consider the Bruhat order $\leq$ on W, with the convention that e is the smallest element. It restricts to the Bruhat order on X_λ^μ. The order on the weights is defined by $x \cdot \lambda \leq y \cdot \lambda$ if and only if $y \leq x$. From the BGG Theorem on the structure of Verma modules, see e.g. [Hu, Section 5.1], it follows that the algebras A_λ^μ are quasi-hereditary with respect to the poset of weights $X_\lambda^\mu \cdot \lambda$. The standard modules coincide with the ones above.

Remark 2.1. We will use the term *(generalised) tilting module* for a module of a finite dimensional algebra satisfying properties (i)-(iii) in [Ha, Section III.3]. When we refer to the modules in (parabolic) category $\mathcal{O}$ that simultaneously admit a standard and a costandard filtration, see e.g. [Hu, Chapter 11], we will use the term *quasi-hereditary tilting module*, or q.h. tilting module. Then a q.h. tilting module is a (special case of) a *partial* generalised tilting module, whereas the *characteristic* q.h. tilting module is a generalised tilting module, see e.g. [Rin, Theorem 5].

When μ is regular, meaning that the corresponding parabolic category $\mathcal{O}^\mu$ is the usual category $\mathcal{O}$, we leave out the reference to μ. Similarly, we will leave out λ (as in $L(x) := L(x.\lambda)$), or replace it by 0, whenever it is regular. By application of [So1, Theorem 11], all categories $\mathcal{O}_\lambda^\mu$, with λ arbitrary integral regular dominant and μ fixed, are equivalent, justifying this convention.

Consider the translation functor $\theta_\lambda^{on} : \mathcal{O}_0 \to \mathcal{O}_\lambda$ to the λ-wall and also its adjoint $\theta_\lambda^{out} : \mathcal{O}_\lambda \to \mathcal{O}_0$, which is the translation out of the λ-wall, see [Hu, Chapter 7]. For $x \in W$, denote by θ_x the unique projective functor on $\mathcal{O}_0$ which maps $P(e)$ to $P(x)$, see [BGe]. Note that, in particular, $\theta_\lambda^{out} \circ \theta_\lambda^{on} = \theta_{w_0^\lambda}$. By [Hu, Theorem 7.9] or [Ja], for any $x \in W$, we have

$$(2) \qquad \theta_\lambda^{on} L(x) = \begin{cases} L(x \cdot \lambda), & x \in X_\lambda; \\ 0, & \text{otherwise.} \end{cases}$$

2.2. **Koszul duality for category $\mathcal{O}$.** Consider a finite dimensional algebra B. If B is a quadratic positively graded algebra, we denote its *quadratic dual* by $B^!$, as in [BGS, Definition 2.8.1]. If B is, moreover, Koszul, we denote its *Koszul dual* by $E(B) = \text{Ext}_B^\bullet(B_0, B_0)$. By Theorem 2.10.1 in [BGS], we have $E(B) = (B^!)^{\text{opp}}$ for any Koszul algebra B. For a positively graded algebra B, we denote by B-gmod its category of finitely generated graded modules.

For a complex $\mathcal{M}^\bullet$ of graded modules, that is $\mathcal{M}^j = \bigoplus_{i \in \mathbb{Z}} \mathcal{M}_i^j$, where $j \in \mathbb{Z}$, we use the following convention

$$(\mathcal{M}^\bullet[a]\langle b \rangle)_j^i = \mathcal{M}_{j-b}^{i+a},$$

for shift in position in the complex and degree in the modules. This corresponds to the conventions in [BGS], but differs slightly from [MOS]. A module M, regarded as an object in the derived category put in position zero, is denoted by $M^\bullet$.

For any Koszul algebra B, [BGS, Theorem 2.12.6] introduces the *Koszul duality functor* $\mathcal{K}_B$, a covariant equivalence of triangulated categories

$$(3) \qquad \mathcal{K}_B : \mathcal{D}^b(B\text{-gmod}) \stackrel{\sim}{\to} \mathcal{D}^b(B^!\text{-gmod}).$$

As in [MOS, Section 3] or [MSa, Section 2], we introduce the full subcategory of $\mathcal{D}^b(B\text{-gmod})$ of linear complexes of projective modules in B-gmod, which we denote by $\mathfrak{LP}_B$. Then [MSa, Theorem 2.4] (or, more generally, [MOS, Theorem 12]) establishes, for any quadratic algebra B, an equivalence

$$\epsilon_B : \mathfrak{LP}_B \stackrel{\sim}{\to} B^!\text{-gmod}.$$

From [MOS, Chapter 5] it follows that ϵ_B is isomorphic to the restriction of $\mathcal{K}_B$ in case B is Koszul.

As proved in [Ba1], A_λ^μ has a Koszul grading, where the Koszul dual algebra is $E(A_\lambda^\mu) \cong A_{\hat\mu}^\lambda$, see also [BGS, So1, Ma2]. We also have $(A_\lambda^\mu)^{\text{opp}} \cong A_\lambda^\mu$ (as graded algebras) as an immediate consequence of the duality functor $\mathbf{d}$. The algebras A_λ^μ are even standard Koszul in the sense of [ADL], see [Ma2].

The graded module categories are denoted by $\mathbb{Z}\mathcal{O}_\lambda^\mu := A_\lambda^\mu$-gmod. We will sometimes replace the notation $\text{Hom}_{\mathbb{Z}\mathcal{O}}$ by $\hom_\mathcal{O}$.

It is more convenient to work with the composition of the usual Koszul duality functor with the duality $\mathbf{d}$ to obtain a contravariant functor

$$(4) \qquad \mathcal{K}_\lambda^\mu := \mathbf{d}\mathcal{K}_{A_\lambda^\mu} : \mathcal{D}^b(\mathbb{Z}\mathcal{O}_\lambda^\mu) \stackrel{\sim}{\to} \mathcal{D}^b(\mathbb{Z}\mathcal{O}_{\hat\mu}^\lambda),$$

where we also identify the graded module categories corresponding to the isomorphism $E(A_\lambda^\mu) \cong A_{\hat\mu}^\lambda$. This functor satisfies

$$\mathcal{K}_\lambda^\mu(\mathcal{M}^\bullet[i]\langle j\rangle) = K_\lambda^\mu(\mathcal{M}^\bullet)[j-i]\langle j\rangle, \tag{5}$$

see [BGS, Theorem 3.11.1]. Similarly, we define $\epsilon_\lambda^\mu = \mathbf{d}\epsilon_{A_\lambda^\mu}$, as a contravariant equivalence of categories

$$\epsilon_\lambda^\mu : \mathfrak{LP}_\lambda^\mu \xrightarrow{\sim} {}^{\mathbb{Z}}\mathcal{O}_{\hat\mu}^\lambda. \tag{6}$$

We conclude this subsection with the introduction of the graded lifts to ${}^{\mathbb{Z}}\mathcal{O}$ of the translation functors on $\mathcal{O}$, as studied in [St1]. We denote them by the same symbols as on $\mathcal{O}$ and use the grading convention of [St1]. This means that the graded version of equation (2) is

$$\theta_\lambda^{on} L(x)\langle 0\rangle = L(x \cdot \lambda)\langle -l(w_0^\lambda)\rangle, \tag{7}$$

for all $x \in X_\lambda$, and that

$$\hom_{\mathcal{O}_0}(\theta_\lambda^{out} M, N) \cong \hom_{\mathcal{O}_\lambda}(M, \theta_\lambda^{on} N\langle l(w_0^\lambda)\rangle), \tag{8}$$

see also [MOS, Lemma 38]. This implies that

$$\theta_\lambda^{out} P(x \cdot \lambda)\langle 0\rangle = P(x)\langle 0\rangle. \tag{9}$$

2.3. **Twisting and shuffling functors.** We will use the twisting functor T_s, which is an endofunctor on each integral block $\mathcal{O}_\lambda$ corresponding to a simple reflection s, see [AS, MS2]. For any $w \in W$ with reduced expression $w = s_1 s_2 \cdots s_m$, we can define the functor

$$T_w = T_{s_1} T_{s_2} \cdots T_{s_m},$$

where the resulting functor does not depend on the choice of a reduced expression, see [KM, Corollary 11]. The functor T_w is right exact and its derived functor $\mathcal{L}T_w$ is an auto-equivalence of $\mathcal{D}^b(\mathcal{O}_0)$, see [AS, Corollary 4.2]. This property extends to a singular block of category $\mathcal{O}$, see e.g. [CM1, Proposition 5.11], so we have an auto-equivalence

$$\mathcal{L}T_w : \mathcal{D}^b(\mathcal{O}_\lambda) \xrightarrow{\sim} \mathcal{D}^b(\mathcal{O}_\lambda). \tag{10}$$

Twisting functors admit graded lifts, see [MO, Appendix] or [KM, Theorem 1.1]. By [AS, Theorem 3.2], the following diagram commutes:

$$
\begin{array}{ccc}
\mathcal{D}^b(\mathcal{O}_0) & \xrightarrow{\ \mathcal{L}T_w\ } & \mathcal{D}^b(\mathcal{O}_0) \\
{\scriptstyle \theta_\lambda^{out}}\big\uparrow & & {\scriptstyle \theta_\lambda^{out}}\big\uparrow \\
\mathcal{D}^b(\mathcal{O}_\lambda) & \xrightarrow{\ \mathcal{L}T_w\ } & \mathcal{D}^b(\mathcal{O}_\lambda).
\end{array}
\tag{11}
$$

The shuffling functor C_s corresponding to a simple reflection s, see [Ca, MS2], is the endofunctor of $\mathcal{O}_0$ defined as the cokernel of the adjunction morphism from the identity functor to the projective functor θ_s. For any $w \in W$ with reduced expression $w = s_1 s_2 \cdots s_m$, we can define the functor

$$C_w = C_{s_m} C_{s_{m-1}} \cdots C_{s_1},$$

where the resulting functor does not depend on the choice of a reduced expression, see [MS1, Lemma 5.10] or [KM, Theorem 2] and [MOS, Section 6.5]. The functor

C_w is right exact and $\mathcal{L}C_w$ is an auto-equivalence, with inverse $\mathbf{d}\mathcal{L}C_w\mathbf{d}$, of $\mathcal{D}^b(\mathcal{O}_0)$, see [MS1, Theorem 5.7], so

$$(12) \qquad \mathcal{L}C_w : \ \mathcal{D}^b(\mathcal{O}_0) \ \tilde{\to} \ \mathcal{D}^b(\mathcal{O}_0).$$

The two basic types of derived auto-equivalences in this sections will be exploited to obtain more complicated derived equivalences in the remainder of the paper. We note that they also appear in more abstract generality in [We, Theorem 6.15 and Proposition 9.18].

2.4. **Ringel duality for category $\mathcal{O}$.** For a quasi-hereditary algebra B, we denote its *Ringel dual* as in [Rin], by

$$R(B) := \operatorname{End}_B(T)^{\mathrm{opp}},$$

with T the characteristic q.h. tilting module in B-mod. The Ringel dual is again quasi-hereditary. If B is basic, we have $R(R(B)) \cong B$, see [Rin, Theorem 7].

We also consider the following covariant right exact functor

$$\mathcal{R}_B = \operatorname{Hom}_B(\cdot, T)^* : \quad B\text{-mod} \to R(B)\text{-mod},$$

with $*$ being the canonical duality functor from mod-$R(B)$ to $R(B)$-mod. We call $\mathcal{R}_B$ the Ringel duality functor. From [Rin, Theorem 6] and [MS3, Proposition 2.2], it follows that $\mathcal{R}_B$ restricts to an equivalence between the additive subcategories of projective modules of B and tilting modules of $R(B)$ and between the additive subcategories of tilting modules of B and injective modules of $R(B)$. It also restricts to an equivalence between the category of B-modules with standard flag and the category of $R(B)$-modules with costandard flag. Finally, its left derived functor induces an equivalence

$$\mathcal{L}\mathcal{R}_B : \ \mathcal{D}^b(B\text{-mod}) \ \tilde{\to} \ \mathcal{D}^b(R(B)\text{-mod}).$$

As $\mathcal{O}_\lambda$ and $\mathcal{O}_0^\mu$ are Ringel self-dual, it is natural to compose the Ringel duality functor with a fixed equivalence which realises the self-duality. In case $\lambda = 0$, we can choose the Ringel duality functor as

$$(13) \qquad \overline{\mathcal{R}}^\mu := \mathcal{L}_{l(w_0^\mu)}C_{w_0} : \ \mathcal{O}_0^\mu \to \mathcal{O}_0^\mu,$$

see [MS3, Proposition 4.4]. In particular, the restriction of $\mathcal{L}_{l(w_0^\mu)}C_{w_0}$ to the category $\mathcal{O}_0^\mu$ is a right exact functor.

In case $\mu = 0$, the Ringel duality functor can be interpreted as

$$(14) \qquad \underline{\mathcal{R}}_\lambda := T_{w_0} : \mathcal{O}_\lambda \to \mathcal{O}_\lambda,$$

see [MS3, Section 4.1].

2.5. **The centre and coinvariants.** For an integral dominant λ, consider B_λ and the corresponding semisimple Lie algebra $\mathfrak{g}_\lambda$, generated by the root spaces of $\mathfrak{g}$ corresponding to elements in $\pm B_\lambda$. The Weyl group of this algebra is isomorphic to W_λ. Then $\mathfrak{h}_\lambda := \mathfrak{g}_\lambda \cap \mathfrak{h}$ is a Cartan subalgebra in $\mathfrak{g}_\lambda$. Now we can consider the algebra of coinvariants for W_λ given by

$$\mathsf{C}(W_\lambda) := S(\mathfrak{h}_\lambda)/\langle S(\mathfrak{h}_\lambda)_+^{W_\lambda}\rangle.$$

The algebra $C(W_\lambda)$ inherits a positive grading from $S(\mathfrak{h}_\lambda)$ which is defined by giving constants degree 0 and elements of $\mathfrak{h}$ degree 2.

For the particular case $W_\lambda = W$, we set $C := C(W)$. For $w \in W$, let ${}^w C$ denote the C-C bimodule obtained from ${}_C C_C$ by twisting the left action of C by w (note that W acts on C by automorphisms). For X a subgroup of W, let C^X denote the algebra of X-invariants in C. Similarly, for a simple reflection s, we denote by C^s the algebra of s-invariants in C.

Consider again an integral dominant λ and set $C_\lambda := C^{W_\lambda}$. By [So1, Endomorphismensatz 7], we have $\mathrm{End}_{\mathcal{O}_\lambda}(P(w_0 \cdot \lambda)) \cong C_\lambda$. We also recall Soergel's combinatorial functor
$$\mathbb{V}_\lambda = \mathrm{Hom}_{\mathcal{O}}(P(w_0 \cdot \lambda), -) : \mathcal{O}_0 \to C_\lambda\text{-mod}$$
from [So1, Section 2.3]. By [So1, Theorem 10], we have

(15) $$\mathbb{V}\theta_\lambda^{out} \cong C \otimes_{C_\lambda} \mathbb{V}_\lambda \quad \text{and} \quad \mathbb{V}_\lambda \theta_\lambda^{on} \cong \mathrm{Res}_{C_\lambda}^{C} \mathbb{V},$$

for any integral dominant λ. All these statement admit canonical graded lifts, as the grading on $\mathcal{O}$ can be introduced via $\mathbb{V}_\lambda$, see e.g. [St1].

From $C_\lambda \cong \mathrm{End}_{\mathcal{O}_\lambda}(P(w_0 \cdot \lambda))$ and [Br, Lemma 6.2] or [MS3, Theorem 5.2(2)], it follows that C_λ is isomorphic to the centre of A_λ. More generally, we denote the centre of A_λ^μ by C_λ^μ.

In the remainder of this subsection we consider $\mathfrak{g} = \mathfrak{sl}(n)$. Then $W \cong S_n$ and C_λ^μ has been calculated in [Br, St3]. For an integral dominant λ, it follows from [So1, Theorem 11] that $\mathcal{O}_\lambda$ is uniquely determined, up to equivalence, by a composition $p(\lambda) = (p_1, \cdots, p_k)$ of n. This composition is defined by demanding that W_λ, as a subgroup of S_n, is naturally given by $S_{p_1} \times S_{p_2} \times \cdots \times S_{p_k}$. It is well-known, by a result of Borel, that C_λ is, as a graded algebra, isomorphic to the cohomology ring of a partial flag variety. The Hilbert-Poincaré polynomial for C_λ, with $p(\lambda)$ as above, is hence well-known, see e.g. [Ch], and is given by

(16) $$\sum_{i=0}^{\infty} \dim (C_\lambda)_{2i} \, z^i \;=\; \left(\prod_{i=1}^{n}(1 - z^i) \right) \Big/ \left(\prod_{j=1}^{k}\prod_{l=1}^{p_j}(1 - z^l) \right).$$

2.6. **Extension quivers in parabolic category $\mathcal{O}$.** We demonstrate the basic property that the Ext^1-quiver of a parabolic singular block can be read off immediately from the Ext^1-quiver of the principal block $\mathcal{O}_0$.

Proposition 2.2. *For $x, y \in X_\lambda^\mu$, we have an isomorphism*
$$\mathrm{Ext}^1_{\mathcal{O}_\lambda^\mu}(L(x \cdot \lambda), L(y \cdot \lambda)) \cong \mathrm{Ext}^1_{\mathcal{O}_0}(L(x), L(y)).$$

Proof. As $\mathcal{O}_\lambda^\mu$ is a Serre subcategory of $\mathcal{O}_\lambda$, we immediately have
$$\mathrm{Ext}^1_{\mathcal{O}_\lambda^\mu}(L(x \cdot \lambda), L(y \cdot \lambda)) \cong \mathrm{Ext}^1_{\mathcal{O}_\lambda}(L(x \cdot \lambda), L(y \cdot \lambda)).$$

Now, the Koszul duality of [BGS] implies an isomorphism
$$\mathrm{Ext}^1_{\mathcal{O}_\lambda}(L(x \cdot \lambda), L(y \cdot \lambda)) \cong \mathrm{Ext}^1_{\mathcal{O}_0^\lambda}(L(x^{-1}w_0), L(y^{-1}w_0)).$$

Applying the same procedure again, but now to the right-hand side above, concludes the proof. $\square$

The elements of $\mathrm{Ext}^1_{\mathcal{O}^\mu_\lambda}(L(x \cdot \lambda), L(y \cdot \lambda))$ correspond to morphisms from $P^\mu(y \cdot \lambda)$ to $P^\mu(x \cdot \lambda)$ such that the top of $P^\mu(y \cdot \lambda)$ maps to the top of the radical of $P^\mu(x \cdot \lambda)$. As the (Koszul) grading on A^μ_λ is such that the degree 0 part is semisimple and the algebra is generated by the degree 0 and 1 parts, this can also be expressed as

$$\mathrm{Ext}^1_{\mathcal{O}^\mu_\lambda}(L(x \cdot \lambda), L(y \cdot \lambda)) \cong \hom_{\mathcal{O}^\mu_\lambda}(P^\mu(y \cdot \lambda)\langle 1 \rangle, P^\mu(x \cdot \lambda)).$$

Proposition 2.2 and equation (9) then imply the following corollary.

Corollary 2.3. *For $x, y \in X^\mu_\lambda$ with $P := P^\mu(x \cdot \lambda)$ and $Q := P^\mu(y \cdot \lambda)$, the graded translation functor θ^{out}_λ induces an isomorphism*

$$\hom_{\mathcal{O}^\mu_\lambda}(P\langle 1 \rangle, Q) \cong \hom_{\mathcal{O}^\mu_0}(\theta^{out}_\lambda P\langle 1 \rangle, \theta^{out}_\lambda Q).$$

3. Graded versus non-graded derived equivalences

3.1. The example of the Koszul duality functor.
The Koszul duality functor in equation (3) gives an equivalence between bounded derived categories of graded modules. We demonstrate, however, that Koszul dual algebras are, in general, not derived equivalent as ungraded algebras.

Proposition 3.1. *Despite the existence of an equivalence of triangulated categories $\mathcal{D}^b(\mathbb{Z}\mathcal{O}^\mu_\lambda) \cong \mathcal{D}^b(\mathbb{Z}\mathcal{O}^\lambda_{\hat{\mu}})$ as given in (4), in general, the categories $\mathcal{D}^b(\mathcal{O}^\mu_\lambda)$ and $\mathcal{D}^b(\mathcal{O}^\lambda_{\hat{\mu}})$ are not equivalent as triangulated categories.*

Proof. Note that A^μ_λ and $A^\lambda_{\hat{\mu}} \cong E(A^\mu_\lambda)$ are not derived equivalent, if their centres are not isomorphic, see [Ric1, Proposition 9.2]. Consider the socle of the left regular module for the two centres. For A_λ this socle is simple, by Subsection 2.5. It follows easily from [Br, Lemma 6.2] that the number of simple modules in that socle for A^λ must be at least the number of non-isomorphic indecomposable projective-injective modules. This is the cardinality of the right Kazhdan-Lusztig cell of $w^\lambda_0 w_0$. □

3.2. Gradable derived equivalences.
In this subsection we define a special case of the concept of a derived equivalence (see [Ric1, Definition 6.5]), which we will investigate for category $\mathcal{O}$ further in the paper.

Definition 3.2. *Two finite dimensional graded algebras B and D are said to be gradable derived equivalent if one of the following equivalent properties is satisfied.*

(i) *There is a triangulated equivalence from $\mathcal{D}^b(B\text{-gmod})$ to $\mathcal{D}^b(D\text{-gmod})$ which commutes with the degree shift functor $\langle 1 \rangle$.*

(ii) *There is a triangulated equivalence from $\mathcal{D}^b(B\text{-mod})$ to $\mathcal{D}^b(D\text{-mod})$, with inverse G, such that F and G admit graded lifts.*

Remark 3.3. It is clear that the failure of the Koszul duality functor to satisfy the requirement in Definition 3.2(i), see equation (5), is precisely what prevents it from being the graded lift of an ungraded equivalence.

Now we state and prove two simple propositions which demonstrate equivalence of the two properties in Definition 3.2.

Proposition 3.4. *Consider two finite dimensional graded algebras B and D such that there is an equivalence*

$$\widetilde{F} : \mathcal{D}^b(B\text{-gmod}) \to \mathcal{D}^b(D\text{-gmod}).$$

If $\widetilde{F}$ commutes with $\langle 1 \rangle$, then $\tilde{F}$ is a graded lift of an equivalence

$$F : \mathcal{D}^b(B\text{-mod}) \to \mathcal{D}^b(D\text{-mod}).$$

Moreover, if $\widetilde{G}$ is inverse to $\widetilde{F}$, then $\widetilde{G}$ is the graded lift of an inverse G to F.

Proof. Take a minimal projective generator P_D for D-mod and the graded lift $\widetilde{P}_D$. Denote by $\mathcal{T}^\bullet$ the object in $\mathcal{D}^b(B\text{-mod})$ obtained by forgetting the grading on $\widetilde{F}^{-1}(P_D^\bullet)$. One deduces straightforwardly that $\mathcal{T}^\bullet$ is a tilting complex according to [Ric1, Definition 6.5] with $D^{\text{opp}} \cong \text{End}_{\mathcal{D}^b(B\text{-mod})}(\mathcal{T}^\bullet)$. The first assertion of the proposition then follows from [Ric1, Theorem 6.4].

Now we consider the inverse $\widetilde{G}$. From $\widetilde{F}\widetilde{G} \cong \text{Id} \cong \widetilde{G}\widetilde{F}$ and $\widetilde{F}\langle 1 \rangle \cong \langle 1 \rangle \widetilde{F}$ it follows that $\widetilde{G}\langle 1 \rangle \cong \langle 1 \rangle \widetilde{G}$. Hence we can use the first part to obtain that $\widetilde{G}$ is the graded lift of a triangulated functor G. The construction, moreover, implies that FG and GF acts as identity functors restricted to the subcategories of projective modules. As they are triangulated functors it follows that F and G are mutually inverse. $\square$

Proposition 3.5. *Let B and D be finite dimensional graded algebras and*

$$F : \mathcal{D}^b(B\text{-mod}) \to \mathcal{D}^b(D\text{-mod})$$

an equivalence with inverse G. Assume that F and G admit graded lifts $\widetilde{F}$ and $\widetilde{G}$, respectively. Then $\widetilde{F}$ gives an equivalence of triangulated categories

$$\widetilde{F} : \mathcal{D}^b(B\text{-gmod}) \to \mathcal{D}^b(D\text{-gmod}).$$

Proof. First we prove that $\widetilde{F}$ is essentially surjective (dense). Consider the forgetful functor $f_D : \mathcal{D}^b(D\text{-gmod}) \to \mathcal{D}^b(D\text{-mod})$ and an indecomposable object $\mathcal{X}^\bullet$ in $\mathcal{D}^b(D\text{-gmod})$ and set $\mathcal{Y}^\bullet := \widetilde{G}\mathcal{X}^\bullet$. As we have $f_D\widetilde{F}\widetilde{G} \cong f_D$, it follows that $f_D\widetilde{F}(\mathcal{Y}^\bullet) \cong f_D\mathcal{X}^\bullet$. By [BGS, Lemma 2.5.3], the proof of which extends easily to the derived category, we then find that there is $j \in \mathbb{N}$ such that $\mathcal{X}^\bullet \cong \widetilde{F}(\mathcal{Y}^\bullet)\langle j \rangle$, or

$$\widetilde{F}(\mathcal{Y}^\bullet\langle j \rangle) \cong \mathcal{X}^\bullet.$$

The density of $\widetilde{F}$ hence follows.

For $\mathcal{X}^\bullet, \mathcal{Y}^\bullet \in \mathcal{D}^b(B\text{-gmod})$, we have the commutative diagram

$$\begin{CD} \bigoplus_{j\in\mathbb{N}} \text{Hom}_{\mathcal{D}^b(B\text{-gmod})}(\mathcal{X}^\bullet, \mathcal{Y}^\bullet\langle j \rangle) @>{\widetilde{F}}>> \bigoplus_{j\in\mathbb{N}} \text{Hom}_{\mathcal{D}^b(D\text{-gmod})}(\widetilde{F}\mathcal{X}^\bullet, \widetilde{F}\mathcal{Y}^\bullet\langle j \rangle) \\ @V{f_B}VV @VV{f_D}V \\ \text{Hom}_{\mathcal{D}^b(B\text{-mod})}(f_B\mathcal{X}^\bullet, f_B\mathcal{Y}^\bullet) @>{F}>> \text{Hom}_{\mathcal{D}^b(D\text{-mod})}(f_D\widetilde{F}\mathcal{X}^\bullet, f_D\widetilde{F}\mathcal{Y}^\bullet) \end{CD}$$

Here F, f_B and f_D act by isomorphisms and hence so does $\widetilde{F}$. As $\widetilde{F}$ respects it then follows easily that $\widetilde{F}$ is full and faithful. Hence $\widetilde{F}$ is an equivalence of categories. By construction it is also an equivalence of triangulated categories. $\square$

Finally, we note the following immediate extension of [Ric1, Proposition 9.2].

Lemma 3.6. *Consider two finite dimensional graded algebras B, D which are gradable derived equivalent. Then there is an isomorphism of graded algebras $\mathcal{Z}(B) \cong \mathcal{Z}(D)$, with canonically inherited grading on the centres.*

3.3. **An application: derived shuffling in the parabolic setting.** We use the results of the previous subsection to extend (12) to the form of (10) and (11). First we point out two subtleties (a) and (b). Consider the exact inclusion $\imath^\mu : \mathcal{O}_0^\mu \to \mathcal{O}_0$ leading to a faithful (see e.g. [Ba1, Lemma 2.6]) triangulated functor

$$\imath^\mu : \ \mathcal{D}^b(\mathcal{O}_0^\mu) \ \to \ \mathcal{D}^b(\mathcal{O}_0).$$

Hence $\mathcal{D}^b(\mathcal{O}_0^\mu)$ is canonically equivalent to a subcategory of $\mathcal{D}^b(\mathcal{O}_0)$.

(a) By construction, $\mathcal{O}_0^\mu$ is a full Serre subcategory of $\mathcal{O}_0$ and C_s restricts to a right exact endofunctor of $\mathcal{O}_0^\mu$, which we denote by C_s^μ. However, its left derived functor $\mathcal{L}C_s^\mu$ need not be isomorphic to the restriction of $\mathcal{L}C_s$ to $\mathcal{D}^b(\mathcal{O}_0^\mu)$, viewed as a subcategory as above. A trivial example is $\mathfrak{g} = \mathfrak{sl}(2)$, as then $C_s^\mu \cong 0$ for μ singular. However, a restriction of $\mathcal{L}C_s$ is never zero as $\mathcal{L}C_s$ is an equivalence.

(b) The objects of the subcategory $\mathcal{D}^b(\mathcal{O}_0^\mu)$ are the complexes in $\mathcal{D}^b(\mathcal{O}_0)$ for which the module in each position is a module in the subcategory $\mathcal{O}_0^\mu$ of $\mathcal{O}_0$. However, this subcategory is neither *full*, nor *isomorphism closed*. It is not full as, for instance, there can be higher extensions in $\mathcal{O}_0$ between projective objects in $\mathcal{O}_0^\mu$. It is not isomorphism closed, see, for instance, the projective resolution in $\mathcal{D}^b(\mathcal{O}_0)$ of a module in $\mathcal{O}_0^\mu$. The functor $\mathcal{L}C_s$ maps, by the definition of a derived functor, objects in $\mathcal{D}^b(\mathcal{O}_0^\mu)$ to something only *isomorphic* to objects in $\mathcal{D}^b(\mathcal{O}_0^\mu)$. To properly define a restriction of $\mathcal{L}C_s$ to $\mathcal{D}^b(\mathcal{O}_0^\mu)$ is hence a non-trivial problem.

Proposition 3.7. *For any $w \in W$, there is an endofunctor $\mathcal{L}C_w$ of $\mathcal{D}^b(\mathcal{O}_0^\mu)$, which yields an auto-equivalence and admits a commuting diagram*

$$
\begin{array}{ccc}
\mathcal{D}^b(\mathcal{O}_0) & \xrightarrow{\ \mathcal{L}C_w\ } & \mathcal{D}^b(\mathcal{O}_0) \\
{\scriptstyle \imath^\mu}\uparrow & & {\scriptstyle \imath^\mu}\uparrow \\
\mathcal{D}^b(\mathcal{O}_0^\mu) & \xrightarrow{\ \mathcal{L}C_w\ } & \mathcal{D}^b(\mathcal{O}_0^\mu).
\end{array}
$$

The same holds in the graded setting.

Proof. An inverse of $\mathcal{L}T_w$ is given by by $\mathbf{d}\mathcal{L}T_w\mathbf{d}$, see [AS, Section 4]. Proposition 3.5 thus implies that the graded lift of $\mathcal{L}T_w$ induces an auto-equivalence of $\mathcal{D}^b(^\mathbb{Z}\mathcal{O}_\lambda)$. So the diagram (11) admits a graded lift with equivalences on the horizontal arrows. Then we apply [MOS, Sections 6.4 and 6.5], see also [Ry] or Proposition 4.7. This implies a commutative diagram, where the horizontal arrows are equivalences

$$
\begin{array}{ccc}
\mathcal{D}^b(^\mathbb{Z}\mathcal{O}_0) & \xrightarrow{\ \mathcal{L}C_w\ } & \mathcal{D}^b(^\mathbb{Z}\mathcal{O}_0) \\
{\scriptstyle \imath^\mu}\uparrow & & {\scriptstyle \imath^\mu}\uparrow \\
\mathcal{D}^b(^\mathbb{Z}\mathcal{O}_0^\mu) & \xrightarrow{\ \widetilde{F}\ } & \mathcal{D}^b(^\mathbb{Z}\mathcal{O}_0^\mu),
\end{array}
$$

for $\widetilde{F} := (\mathcal{K}^{\widehat{\mu}})^{-1} \circ \mathcal{L}T_w \circ \mathcal{K}^{\widehat{\mu}}$. As $\mathcal{L}T_w$ is a triangulated functor which commutes with $\langle 1 \rangle$, equation (5) implies that $\widetilde{F}$ commutes with $\langle 1 \rangle$, so Proposition 3.4 implies that $\widetilde{F}$ is the lift of an ungraded equivalence and hence a gradable derived equivalence. $\qquad\square$

Remark 3.8. Consider the complex $0 \to \mathrm{Id}_{\mathcal{O}_0} \to \theta_s \to 0$ of exact functors, where the non-zero map is given by the adjunction morphism, *cf.* [Ric2] and [MS1, Remark 5.8]. Applying this to a complex in $\mathcal{D}^b(\mathcal{O}_0)$ and taking the total complex defines a triangulated endofunctor of $\mathcal{D}^b(\mathcal{O}_0)$, isomorphic to $\mathcal{L}C_s$. This endofunctor, by construction, preserves the image of $\mathcal{D}^b(\mathcal{O}_0^\mu)$ under $\imath^\mu$, since both the identity and θ_s do. This gives an alternative construction of the equivalence in Proposition 3.7.

4. Graded translation functors

4.1. **Translation through the principal block.**

Proposition 4.1. *As graded functors, we have*

$$\theta_\lambda^{on}\theta_\lambda^{out}\langle 0\rangle \cong \bigoplus_{j\in\mathbb{N}} \mathrm{Id}_{\mathcal{O}_\lambda}^{\oplus c_j}\langle j - l(w_0^\lambda)\rangle, \quad with \quad c_j := \dim\left(\mathsf{C}(W_\lambda)\right)_j.$$

In particular, $\pm l(w_0^\lambda)$ *are exactly the extremal degrees in which* Id *appears.*

Proof. The ungraded statement follows easily from [BGe, Theorem 3.3] and [Ja, Formula 4.13(1)]. It thus suffices to prove that

$$\mathbb{V}_\lambda \theta_\lambda^{on}\theta_\lambda^{out} \cong \bigoplus_{j\in\mathbb{N}} \mathbb{V}_\lambda^{\oplus d_j}\langle j - l(w_0^\lambda)\rangle, \quad \text{for some} \quad d_j \geq \dim\left(\mathsf{C}(W_\lambda)\right)_j.$$

Using equation (15) and ignoring an overall grading shift, we find

$$\mathbb{V}_\lambda \theta_\lambda^{on}\theta_\lambda^{out} \cong \mathsf{C} \otimes_{\mathsf{C}_\lambda} \mathbb{V}_\lambda.$$

So it suffices to prove that

$$\dim \mathsf{C}_{2i} - \dim(\mathsf{C}_\lambda)_{2i} + \dim(\mathbb{C})_{2i} \geq \dim \mathsf{C}(W_\lambda)_{2i},$$

for all $i \in \mathbb{N}$, where, of course, $\dim(\mathbb{C})_{2i} = \delta_{i0}$. Consider a W_λ-equivariant morphism $\mathfrak{h} \twoheadrightarrow \mathfrak{h}_\lambda$. This extends to a graded morphism $S(\mathfrak{h}) \twoheadrightarrow S(\mathfrak{h}_\lambda)$ which we compose with the canonical surjection $S(\mathfrak{h}_\lambda) \twoheadrightarrow \mathsf{C}(W_\lambda)$ to find $\xi : S(\mathfrak{h}) \twoheadrightarrow \mathsf{C}(W_\lambda)$. By construction $S(\mathfrak{h})_+^W$, and hence the corresponding ideal, is in the kernel of ξ. This implies a morphism of graded algebras

$$\eta : \mathsf{C} \twoheadrightarrow \mathsf{C}(W_\lambda).$$

As $(\mathsf{C}_\lambda)_+ = \mathsf{C}_+^{W_\lambda}$ is in the kernel of η, this proves the desired inequalities. $\square$

As an application, we prove the following proposition, which we need later.

Proposition 4.2. *Consider two objects* $\mathcal{X}^\bullet, \mathcal{Y}^\bullet$ *of* $\mathcal{D}^b(\mathbb{Z}\mathcal{O}_\lambda^\mu)$ *such that*

$$\hom_{\mathcal{D}^b(\mathcal{O}_\lambda^\mu)}(\mathcal{X}^\bullet, \mathcal{Y}^\bullet\langle j\rangle) = 0 \qquad if \quad j > 0.$$

Then θ_λ^{out} *induces isomorphisms*

$$\hom_{\mathcal{D}^b(\mathcal{O}_0^\mu)}(\theta_\lambda^{out}\mathcal{X}^\bullet, \theta_\lambda^{out}\mathcal{Y}^\bullet) \cong \hom_{\mathcal{D}^b(\mathcal{O}_\lambda^\mu)}(\mathcal{X}^\bullet, \mathcal{Y}^\bullet),$$

$$\hom_{\mathcal{D}^b(\mathcal{O}_0^\mu)}(\theta_\lambda^{out}\mathcal{X}^\bullet\langle 1\rangle, \theta_\lambda^{out}\mathcal{Y}^\bullet) \cong \hom_{\mathcal{D}^b(\mathcal{O}_\lambda^\mu)}(\mathcal{X}^\bullet\langle 1\rangle, \mathcal{Y}^\bullet).$$

Proof. Consider $i \in \mathbb{Z}$, then equation (8) and Proposition 4.1 imply that

$$\hom_{\mathcal{D}^b(\mathcal{O}_0^\mu)}(\theta_\lambda^{out}\mathcal{X}^\bullet\langle i\rangle, \theta_\lambda^{out}\mathcal{Y}^\bullet) \cong \hom_{\mathcal{D}^b(\mathcal{O}_\lambda^\mu)}(\mathcal{X}^\bullet\langle i\rangle, \bigoplus_{j\in\mathbb{N}}(\mathcal{Y}^\bullet)^{\oplus c_j}\langle j\rangle).$$

Now, if $i = 0$, the result follows by the assumptions as $c_0 = 1$. If $i = 1$, the result follows from the assumptions and $c_1 = 0$ and $c_0 = 1$. $\qquad\square$

This proposition generalises Corollary 2.3 and hence provides an alternative proof.

4.2. **Translating standard modules.** In this subsection we completely describe the graded translation of standard modules to and from the wall. These results generalise [St1, Theorem 8.1(3) and Theorem 8.2(2)] to arbitrary walls and the parabolic setting. Consider the bijection

$$b_\lambda : W \to X_\lambda \times W_\lambda,$$

which is inverse to multiplication and denote by b_λ^1 and b_λ^2 the composition of b_λ with the projection on the X_λ-component and the W_λ-component, respectively.

Theorem 4.3. *For any $x \in X^\mu$, we have*

$$\theta_\lambda^{on}\Delta^\mu(x) \cong \begin{cases} \Delta^\mu(b_\lambda^1(x)\cdot\lambda)\langle l(b_\lambda^2(x)) - l(w_0^\lambda)\rangle, & \text{if } b_\lambda^1(x) \in X^\mu; \\ 0, & \text{otherwise.} \end{cases}$$

In particular, for any $y \in X_\lambda$ and $u \in W_\lambda$, we have

$$\theta_\lambda^{on}\Delta(yu) \cong \Delta(y\cdot\lambda)\langle l(u) - l(w_0^\lambda)\rangle.$$

Theorem 4.4. *For any $x \in X_\lambda^\mu$, the standard filtration of $\theta_\lambda^{out}\Delta^\mu(x\cdot\lambda)$ satisfies*

$$\left(\theta_\lambda^{out}\Delta^\mu(x\cdot\lambda) : \Delta^\mu(xu)\langle j\rangle\right) = \delta_{j,l(u)} \qquad \text{for all} \quad u \in W_\lambda,$$

moreover, there are no other standard modules appearing in the filtration.

In the remainder of this subsection we prove these two theorems.

Lemma 4.5. *For any $x \in X_\lambda$, $u \in W_\lambda$ and $j \in \mathbb{N}$, we have*

$$[\Delta(xu) : L(x)\langle j\rangle] = \delta_{j,l(u)}.$$

Proof. We prove the Koszul dual statement, which is

$$\dim \operatorname{Ext}_{\mathcal{O}}^j(\Delta(vy), L(y)) = \delta_{j,l(v)},$$

for any $y \in X^\lambda$, $v \in W_\lambda$ and $i \in \mathbb{N}$. That the left-hand side is zero, for $j > l(v)$, follows immediately from [Hu, Theorem 6.11]. For any simple reflection $s \in W_\lambda$ and $v \in W_\lambda$ such that $sv > v$, the procedure in the proof of [Ma1, Proposition 3] shows

$$\dim \operatorname{Ext}_{\mathcal{O}}^k(\Delta(svy), L(y)) = \dim \operatorname{Ext}_{\mathcal{O}}^{k-1}(\Delta(vy), L(y)),$$

which proves the claim inductively. $\qquad\square$

Proof of Theorem 4.3. It suffices to prove the non-parabolic case, as the remainder follows via an application of the Zuckerman functor, which commutes with translation functors. The non-parabolic result follows from equation (7), Lemma 4.5 and [Ja, Formula 4.12(2)]. $\qquad\square$

Proof of Theorem 4.4. For any $y \in X^\mu$, [Hu, Theorem 3.3(d)] and equation (8) yield
$$\left(\theta_\lambda^{out}\Delta^\mu(x \cdot \lambda) : \Delta^\mu(y)\langle j\rangle\right) = \hom_{\mathcal{O}_\lambda^\mu}(\Delta^\mu(x \cdot \lambda), \theta_\lambda^{on}\nabla^\mu(y)\langle j + l(w_0^\lambda)\rangle).$$

Theorem 4.3 then implies that the only y which can appear non-trivially are those for which we have $b_\lambda^1(y) = x$, which is precisely the set $xW_\lambda \subset X^\mu$. $\qquad\square$

Remark 4.6. By using Proposition 4.7, an alternative proof of Theorem 4.4 would be to determine a resolution of standard modules in $\mathcal{O}^\mu$ by Verma modules. This can be obtained by applying parabolic induction to $\mathfrak{g}$ of the BGG resolutions for the Levi subalgebra of $\mathfrak{q}_\mu$, see e.g. [Hu, Section 6].

4.3. **Koszul duality.** In this subsection we derive a slight generalisation of [Ry, Theorem 4.1] and [MOS, Theorem 35].

Proposition 4.7. *There are commutative diagrams of functors as follows:*

$$
\begin{array}{ccc}
\mathcal{D}^b(\mathbb{Z}\mathcal{O}_\lambda^\mu) & \xrightarrow{\imath^\mu} & \mathcal{D}^b(\mathbb{Z}\mathcal{O}_\lambda) \\
{\scriptstyle K_\mu^{\widehat{\lambda}}}\Big\uparrow & & \Big\uparrow{\scriptstyle K^{\widehat{\lambda}}} \\
\mathcal{D}^b(\mathbb{Z}\mathcal{O}_\mu^{\widehat{\lambda}}) & \xrightarrow{\theta_\mu^{out}} & \mathcal{D}^b(\mathbb{Z}\mathcal{O}_0^{\widehat{\lambda}})
\end{array}
\qquad
\begin{array}{ccc}
\mathcal{D}^b(\mathbb{Z}\mathcal{O}_\lambda) & \xrightarrow{\mathcal{L}Z^\mu} & \mathcal{D}^b(\mathbb{Z}\mathcal{O}_\lambda^\mu) \\
{\scriptstyle K^{\widehat{\lambda}}}\Big\uparrow & & \Big\uparrow{\scriptstyle K_\mu^{\widehat{\lambda}}} \\
\mathcal{D}^b(\mathbb{Z}\mathcal{O}_0^{\widehat{\lambda}}) & \xrightarrow{\theta_\mu^{on}\langle l(w_0^\mu)\rangle} & \mathcal{D}^b(\mathbb{Z}\mathcal{O}_\mu^{\widehat{\lambda}})
\end{array}
$$

Before proving this we list some consequences. Recall the c_j's from Proposition 4.1.

Corollary 4.8. *On $\mathcal{D}^b(\mathbb{Z}\mathcal{O}_\lambda^\mu)$, we have*
$$\mathcal{L}Z^\mu \circ \imath^\mu \cong \bigoplus_{j \in \mathbb{N}} \mathrm{Id}_{\mathbb{Z}\mathcal{O}_\lambda^\mu}^{\oplus c_j}[j]\langle j\rangle.$$

For any M in $\mathbb{Z}\mathcal{O}_\lambda$ which is locally $U(\mathfrak{q}^\mu)$-finite and for any $k \in \mathbb{N}$, we have
$$\mathcal{L}_k Z^\mu M \cong M^{\oplus c_k}\langle k\rangle.$$

Proof. This is a direct application of Proposition 4.7 to Proposition 4.1. In particular, the maps in the complex $\mathcal{L}Z^\mu \circ \imath^\mu(N^\bullet)$ are trivial for any $N \in {}^{\mathbb{Z}}\mathcal{O}_\lambda^\mu$, which yields the result about the cohomology functors. $\qquad\square$

Remark 4.9. Note that Corollary 4.8 also implies the isomorphism
$$\mathcal{L}Z^\mu \circ \imath^\mu \cong \bigoplus_{j \in \mathbb{N}} \mathrm{Id}_{\mathcal{O}_\lambda^\mu}^{\oplus c_j}[j] \qquad \text{on} \qquad \mathcal{D}^b(\mathcal{O}_\lambda^\mu)$$
as all projective modules and homomorphism spaces between them are gradable.

Corollary 4.10. *Consider two objects $\mathcal{X}^\bullet$ and $\mathcal{Y}^\bullet$ in $\mathcal{D}^b(\mathcal{O}_\lambda^\mu)$ such that*
$$\mathrm{Hom}_{\mathcal{D}^b(\mathcal{O}_\lambda^\mu)}(\mathcal{X}^\bullet, \mathcal{Y}^\bullet[k]) = 0, \qquad \text{for all} \quad k \neq 0.$$
Then $\imath^\mu$ induces an isomorphism
$$\imath^\mu : \mathrm{Hom}_{\mathcal{D}^b(\mathcal{O}_\lambda^\mu)}(\mathcal{X}^\bullet, \mathcal{Y}^\bullet) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{D}^b(\mathcal{O}_\lambda)}(\imath^\mu\mathcal{X}^\bullet, \imath^\mu\mathcal{Y}^\bullet).$$

Proof. As $\imath^\mu$ is faithful, the morphism is always injective. So it suffices to prove that the dimensions of the spaces of homomorphisms agree. This follows from
$$\mathrm{Hom}_{\mathcal{D}^b(\mathcal{O}_\lambda)}(\imath^\mu\mathcal{X}^\bullet, \imath^\mu\mathcal{Y}^\bullet) \cong \mathrm{Hom}_{\mathcal{D}^b(\mathcal{O}_\lambda)}(\mathcal{L}Z^\mu\imath^\mu\mathcal{X}^\bullet, \mathcal{Y}^\bullet),$$
given by adjunction, and Remark 4.9. $\qquad\square$

Corollary 4.11. *Consider two objects $\mathcal{X}^\bullet$ and $\mathcal{Y}^\bullet$ in $\mathcal{D}^b(\mathcal{O}_\lambda^\mu)$. If either*

$$\theta_\lambda^{out}\mathcal{X}^\bullet \cong \theta_\lambda^{out}\mathcal{Y}^\bullet \quad \text{inside } \mathcal{D}^b(\mathcal{O}_0^\mu) \qquad or \qquad \imath^\mu\mathcal{X}^\bullet \cong \imath^\mu\mathcal{Y}^\bullet \quad \text{inside } \mathcal{D}^b(\mathcal{O}_\lambda),$$

then $\mathcal{X}^\bullet$ and $\mathcal{Y}^\bullet$ are isomorphic in $\mathcal{D}^b(\mathcal{O}_\lambda^\mu)$. The same is true for $\mathcal{D}^b(\mathbb{Z}\mathcal{O}_\lambda^\mu)$.

Proof. The property in the graded setting follows immediately from Proposition 4.1 and Corollary 4.8. The ungraded version now follows from Remark 4.9. □

Now we turn to the proof of Proposition 4.7.

Lemma 4.12. *The translation functor $\theta_\lambda^{out} : \mathcal{D}^b(\mathbb{Z}\mathcal{O}_\lambda^\mu) \to \mathcal{D}^b(\mathbb{Z}\mathcal{O}_0^\mu)$ restricts to a full and faithful functor $\theta_\lambda^{out} : \mathfrak{LP}_\lambda^\mu \to \mathfrak{LP}^\mu$, the image of which is the full subcategory of $\mathfrak{LP}^\mu$ given by linear complexes of projective modules in $\mathrm{add}(\theta_\lambda^{out}P_\lambda^\mu)$.*

Proof. Recall that the categories of linear complexes of projective modules are full subcategories of the derived categories. By equation (9), θ_λ^{out} restricts to a functor from $\mathfrak{LP}_\lambda^\mu$ to $\mathfrak{LP}^\mu$. To prove that this is full and faithful, consider two linear complexes $\mathcal{P}^\bullet$ and $\mathcal{Q}^\bullet$ of projective modules in $\mathbb{Z}\mathcal{O}_\lambda^\mu$. Proposition 4.2 now implies

$$\theta_\lambda^{out} : \mathrm{hom}_{\mathfrak{LP}_\lambda^\mu}(\mathcal{P}^\bullet, \mathcal{Q}^\bullet) \xrightarrow{\sim} \mathrm{hom}_{\mathfrak{LP}^\mu}(\theta_\lambda^{out}\mathcal{P}^\bullet, \theta_\lambda^{out}\mathcal{Q}^\bullet).$$

The description of the image is a consequence of Corollary 2.3. □

Corollary 4.13. *The following is a commutative diagram of functors:*

$$
\begin{array}{ccc}
\mathbb{Z}\mathcal{O}_\lambda^\mu & \xrightarrow{\imath^\mu} & \mathbb{Z}\mathcal{O}_\lambda \\
{\scriptstyle (\epsilon_\mu^{\widehat{\lambda}})^{-1}}\downarrow & & \downarrow{\scriptstyle (\epsilon^{\widehat{\lambda}})^{-1}} \\
\mathfrak{LP}_\mu^{\widehat{\lambda}} & \xrightarrow{\theta_\mu^{out}} & \mathfrak{LP}^{\widehat{\lambda}}
\end{array}
$$

Proof. Using the standard properties of $\mathcal{K}_\mu^{\widehat{\lambda}}$, see [BGS, Theorem 3.11.1], and equation (9) implies that, for any $x \in X_\lambda^\mu$,

$$\theta_\mu^{out}(\epsilon_\mu^{\widehat{\lambda}})^{-1}L(x\cdot\lambda) \cong P^{\widehat{\lambda}}(w_0 x^{-1})^\bullet \cong (\epsilon^{\widehat{\lambda}})^{-1}L(x\cdot\lambda).$$

Corollary 2.3 can then be applied to show that the complexes corresponding to $\theta_\mu^{out}(\epsilon_\mu^{\widehat{\lambda}})^{-1}M$ and $(\epsilon^{\widehat{\lambda}})^{-1}\imath^\mu M$, for any M in $\mathbb{Z}\mathcal{O}_\lambda^\mu$, are isomorphic. As, by construction, this isomorphism is natural, this concludes the proof. □

Proof of Proposition 4.7. The first diagram follows from [MOS, Theorem 30 and Proposition 21], in combination with Corollary 4.13. The second diagram is the adjoint reformulation of the first. □

4.4. **A generalisation of Bott's theorem.** Take ρ to be the half of the sum of positive roots. Then $Z^{-\rho}$ is the Zuckerman functor to the category of finite dimensional modules. Bott's extension of the Borel-Weil theorem then reads

$$\mathcal{L}_k Z^{-\rho}(\Delta(x)) = \delta_{k,l(x)}L(e) \quad \text{and} \quad \mathcal{L}_k Z^{-\rho}(\Delta(y\cdot\lambda)) = 0, \quad \text{for all} \quad k \in \mathbb{N},$$

for all $x \in W$ and all $y \in X_\lambda$ with λ singular. For an overview of the approach with Zuckerman functors, see e.g. [Co, dS]. We can now generalise this. Let

$$b^\mu : W \to W_\mu \times X^\mu$$

denote the bijection defined as the inverse of multiplication. Define x_1 and x^1 by $b^\mu(x) = (x_1, x^1)$ as in [Le, Proposition 3.4].

Theorem 4.14. *For any $x \in X_\lambda$ and any integral dominant μ, we have*

$$\mathcal{L}_k Z^\mu(\Delta(x \cdot \lambda)) = \begin{cases} \delta_{k,l(x_1)}\Delta^\mu(x^1 \cdot \lambda), & \text{if } x^1 \in X_\lambda; \\ 0, & \text{otherwise.} \end{cases}$$

In particular, for any $y \in X^\mu$ and $u \in W_\mu$, we have

$$\mathcal{L}_k Z^\mu(\Delta(uy)) = \delta_{k,l(u)}\Delta^\mu(y).$$

Setting $\mu = -\rho$ (so $x = x_1$ and $x^1 = e$) yields the original result of Bott.

Proof. This follows from a direct application of the Koszul duality functor to the statement in Theorem 4.3 by using Proposition 4.7. $\square$

Remark 4.15. This could also be proved directly by using parabolic induction to reduce to the Bott's theorem for the Levi subalgebra. This would yield an alternative proof of Theorem 4.3 by applying Proposition 4.7.

4.5. **Translation through the wall.** In this subsection we gather some technical results on translation through the wall which will be needed later and which can be formulated most generally by ignoring grading.

Lemma 4.16. *Consider an indecomposable $\mathcal{X}^\bullet \in \mathrm{Ob}(\mathcal{D}^b(\mathcal{O}_0))$ such that we have $\theta_{w_0^\lambda}\mathcal{X}^\bullet \cong (\mathcal{X}^\bullet)^{\oplus|W_\lambda|}$. Then there is an indecomposable $\mathcal{Y}^\bullet \in \mathrm{Ob}(\mathcal{D}^b(\mathcal{O}_\lambda))$ and $k \in \mathbb{N}$ such that $\theta_\lambda^{on}\mathcal{X}^\bullet \cong (\mathcal{Y}^\bullet)^{\oplus k}$.*

Proof. We take $\mathcal{Y}^\bullet$ to be some indecomposable direct summand of $\theta_\lambda^{on}\mathcal{X}^\bullet$. Then there is some $\mathcal{A}^\bullet \in \mathrm{Ob}(\mathcal{D}^b(\mathcal{O}_\lambda))$ such that

$$\theta_\lambda^{on}\mathcal{X}^\bullet \cong \mathcal{Y}^\bullet \oplus \mathcal{A}^\bullet.$$

Applying θ_λ^{out} to the above isomorphism implies that there must be some $p \in \mathbb{N}$ for which $\theta_\lambda^{out}\mathcal{Y}^\bullet \cong (\mathcal{X}^\bullet)^{\oplus p}$. Applying θ_λ^{on}, while using the ungraded version of Proposition 4.1, to the latter isomorphism then yields

$$(\mathcal{Y}^\bullet)^{\oplus|W_\lambda|} \cong \theta_\lambda^{on}(\mathcal{X}^\bullet)^{\oplus p}.$$

This implies the claim with $k = |W_\lambda|/p$. $\square$

Corollary 4.17. *Consider indecomposable $\mathcal{X}^\bullet \in \mathrm{Ob}(\mathcal{D}^b(\mathcal{O}_0))$ such that we have $\theta_{w_0^\lambda}\mathcal{X}^\bullet \cong (\mathcal{X}^\bullet)^{\oplus|W_\lambda|}$. There is an indecomposable $\mathcal{Y}^\bullet \in \mathrm{Ob}(\mathcal{D}^b(\mathcal{O}_\lambda))$ such that $\theta_\lambda^{out}\mathcal{Y}^\bullet \cong \mathcal{X}^\bullet$ if and only if $\theta_\lambda^{on}\mathcal{X}^\bullet$ contains $|W_\lambda|$ indecomposable summands.*

Lemma 4.18. *Consider a module $M \in \mathrm{Ob}(\mathcal{O}_0)$ such that $\theta_{w_0^\lambda}M \cong M^{\oplus|W_\lambda|}$ and for which $\mathbb{V}$ yields an algebra isomorphism $\mathbb{V} : \mathrm{End}_{\mathcal{O}_0}(M) \xrightarrow{\sim} \mathrm{End}_{\mathsf{C}}(\mathbb{V}M)$. Then $\mathbb{V}_\lambda$ also induces an algebra isomorphism $\mathbb{V}_\lambda : \mathrm{End}_{\mathcal{O}_\lambda}(\theta_\lambda^{on}M) \xrightarrow{\sim} \mathrm{End}_{\mathsf{C}_\lambda}(\mathbb{V}_\lambda\theta_\lambda^{on}M)$.*

Proof. Equation (15) and the faithfulness of θ_λ^{out} and induction imply that we have a commuting diagram of algebra homomorphisms:

$$
\begin{array}{ccc}
\mathrm{End}_{\mathcal{O}_0}(M^{\oplus|W_\lambda|}) & \xrightarrow{\ \ \mathbb{V}\ \ } & \mathrm{End}_{\mathsf{C}}(\mathbb{V}M^{\oplus|W_\lambda|}) \\
{\scriptstyle \theta_\lambda^{out}}\Big\uparrow & & \Big\uparrow{\scriptstyle \mathrm{Ind}_{\mathsf{C}_\lambda}^{\mathsf{C}}} \\
\mathrm{End}_{\mathcal{O}_\lambda}(\theta_\lambda^{on}M) & \xrightarrow{\ \ \mathbb{V}_\lambda\ \ } & \mathrm{End}_{\mathsf{C}_\lambda}(\mathbb{V}_\lambda\theta_\lambda^{on}M)
\end{array}
$$

As the upper horizontal arrow is an isomorphism by assumption, so is $\mathbb{V}_\lambda$. $\qquad\square$

Lemma 4.19. *Consider objects $\mathcal{X}^\bullet$ of $\mathcal{D}^b(\mathcal{O}_\lambda)$ and $\mathcal{Y}^\bullet$ of $\mathcal{D}^b(\mathcal{O}_0^\mu)$ such that we have $\theta_\lambda^{out}\mathcal{X}^\bullet \cong \imath^\mu\mathcal{Y}^\bullet$ in $\mathcal{D}^b(\mathcal{O}_0)$. Then there exists $\mathcal{Z}^\bullet \in \mathcal{D}^b(\mathcal{O}_\lambda^\mu)$ such that*

$$
\mathcal{X}^\bullet \cong \imath^\mu\mathcal{Z}^\bullet \qquad and \qquad \mathcal{Y}^\bullet \cong \theta_\lambda^{out}\mathcal{Z}^\bullet.
$$

Proof. Applying θ_λ^{on} to $\theta_\lambda^{out}\mathcal{X}^\bullet \cong \imath^\mu\mathcal{Y}^\bullet$ and using the commutation of inclusion and translation, implies

$$
(\mathcal{X}^\bullet)^{\oplus|W_\lambda|} \cong \imath^\mu\theta_\lambda^{on}\mathcal{Y}^\bullet.
$$

Hence $\mathcal{X}^\bullet \cong \imath^\mu\mathcal{Z}^\bullet$, for some $\mathcal{Z}^\bullet$. Applying θ_λ^{out} to the latter yields an isomorphism $\imath^\mu\mathcal{Y}^\bullet \cong \imath^\mu\theta_\lambda^{out}\mathcal{Z}^\bullet$, so the conclusion follows from Corollary 4.11. $\qquad\square$

5. Shuffling and projective functors

Unlike twisting functors, shuffling functors do not commute with projective functors in general. Here we investigate their intertwining relations. We consider the principle block $\mathcal{O}_0$ for an arbitrary reductive Lie algebra $\mathfrak{g}$.

Theorem 5.1. *For two simple reflections $s, t \in W$, we have*

$$
\begin{cases}
\theta_s C_t \cong C_t \theta_s, & \text{if } st = ts; \\
\theta_s C_{st} \cong C_{st} \theta_t, & \text{if } s, t \text{ generate } S_3.
\end{cases}
$$

The above statement is formulated in [MS2, Section 11], however, the proof of [MS2, Section 11] is not complete (it proves only a special case of the statement). Below, we follow the alternative argument outlined in [MS2, Remarks 11.2 and 11.4].

Corollary 5.2. *Consider $\nu \in \Lambda_{\mathrm{int}}^+$ such that W_ν is of type A. For any simple reflection $s \in W_\nu \times W_\nu^\dagger$, we have $C_{w_0^\nu}\theta_s \cong \theta_{s'}C_{w_0^\nu}$, with s' the simple reflection defined as $s' = w_0^\nu s w_0^\nu$.*

Set $\mathcal{Q}_{(w)} := \mathrm{add}(C_w P)$, with $w \in W$, and P a projective generator for $\mathcal{O}_0$.

Proposition 5.3.

(i) *For $\nu \in \Lambda_{\mathrm{int}}^+$ and a simple reflection $s \in W_\nu \times W_\nu^\dagger$, we have $C_{w_0^\nu}\theta_s \cong \theta_{w_0^\nu s w_0^\nu}C_{w_0^\nu}$ if and only if the category $\mathcal{Q}_{(w_0^\nu)}$ is stable under action of $\theta_{w_0^\nu s w_0^\nu}$.*

(ii) *For any $y \in W$, we have $C_{w_0}\theta_y \cong \theta_{w_0 y w_0}C_{w_0}$.*

Remark 5.4. Note that the stability of $\mathcal{Q}_{(w)}$ is not an obvious property. Already for $\mathfrak{g} = \mathfrak{sl}(3)$, the category $\mathcal{Q}_{(s)}$ is not stable under projective functors.

Now we start the proofs. Define the graded lift of C_t via the exact sequence

$$(17) \qquad\qquad \mathrm{Id}\langle 1\rangle \xrightarrow{\mathrm{adj}} \theta_t \to C_t \to 0.$$

Lemma 5.5. *For a simple reflection t, define the functor F_t as the kernel of the adjunction morphism in* (17). *Then $F_t\theta_t \cong 0 \cong \theta_t F_t$.*

Proof. As Id is exact, F_t is left exact. Then $\theta_t T_t$ is also left exact and so is $T_t\theta_t$, as θ_t maps injective modules to injective modules. By construction, we have

$$F_t L = 0 \ \text{ if } \ \theta_t L \neq 0 \quad \text{and} \quad F_t L \cong L \ \text{ if } \ \theta_t L \cong 0.$$

This implies immediately that $\theta_t F_t \cong 0$.

To prove that $F_t\theta_t$ acts trivially on all simple modules, it suffices to consider a simple module L with $\theta_t L \neq 0$. But then $\theta_t L$ has simple socle L and the latter is not annihilated by θ_t. This means that the adjunction morphism is injective on both L and $\theta_t L$, which implies $F_t\theta_t L = 0$. The claim follows. $\qquad\square$

Proof of Theorem 5.1. Assume first that s and t commute but are distinct, in particular we have $\theta_s\theta_t \cong \theta_{ts} \cong \theta_t\theta_s$. We can compose (17) with θ_s on both sides. This yields exact sequences

$$\theta_s\langle 1\rangle \to \theta_{st} \to C_t\theta_s \to 0 \qquad \text{and} \qquad \theta_s\langle 1\rangle \to \theta_{st} \to \theta_s C_t \to 0.$$

From [BGe, Theorem 3.5] and Kazhdan-Lusztig combinatorics it follows that the morphism $\theta_s\langle 1\rangle \to \theta_{st}$ is unique, up to a non-zero scalar, implying $C_t\theta_s \cong \theta_s C_t$.

Now consider the case $s = t$. Using Lemma 5.5 and the relation $\theta_s^2 \cong \theta_s\langle 1\rangle \oplus \theta_s\langle -1\rangle$ we obtain two short exact sequences

$$0 \to \theta_s\langle 1\rangle \to \theta_s\langle 1\rangle \oplus \theta_s\langle -1\rangle \to \theta_s C_s \to 0;$$
$$0 \to \theta_s\langle 1\rangle \to \theta_s\langle 1\rangle \oplus \theta_s\langle -1\rangle \to C_s\theta_s \to 0.$$

The only possible way to have such an injection $\theta_s\langle 1\rangle \hookrightarrow \theta_s\langle 1\rangle \oplus \theta_s\langle -1\rangle$ corresponds to $\theta_s\langle 1\rangle \xrightarrow{\sim} \theta_s\langle 1\rangle$. This implies indeed $\theta_s C_s \cong \theta_s\langle -1\rangle \cong C_s\theta_s$.

Finally, assume that the simple reflections s and t together generate a group isomorphic to S_3. Using the definitions of C_s and C_t via (17), we construct a commutative diagram with exact rows and columns as follows:

$$
\begin{array}{ccccccc}
\mathrm{Id}\langle 2\rangle & \longrightarrow & \theta_s\langle 1\rangle & \longrightarrow & C_s\langle 1\rangle & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
\theta_t\langle 1\rangle & \longrightarrow & \theta_t\theta_s & \longrightarrow & \theta_t C_s & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
C_t\langle 1\rangle & \longrightarrow & C_t\theta_s & \longrightarrow & C_{st} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & & 0 & & 0 & &
\end{array}
$$

Now we compose every functor in the diagram with the exact functor θ_s and use the above result, for the case $s = t$, in the first row of the following diagram:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \theta_s\langle 2\rangle & \xrightarrow{\alpha_1} & \theta_s\langle 2\rangle \oplus \theta_s & \xrightarrow{\alpha_2} & \theta_s & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle g_1} & & \downarrow{\scriptstyle h_1} & & \\
 & & \theta_s\theta_t\langle 1\rangle & \xrightarrow{\beta_1} & \theta_s\theta_t\theta_s & \xrightarrow{\beta_2} & \theta_s\theta_t C_s & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle g_2} & & \downarrow{\scriptstyle h_2} & & \\
 & & \theta_s C_t\langle 1\rangle & \xrightarrow{\gamma_1} & \theta_s C_t\theta_s & \xrightarrow{\gamma_2} & \theta_s C_{st} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

This diagram contains a surjection $\theta_s\theta_t\theta_s \twoheadrightarrow \theta_s C_{st}$ given by $h_2 \circ \beta_2 = \gamma_2 \circ h_2$. The kernel of this map can be described as the direct sum of the image of β_1 with the preimage under β_2 of the image of h_1. Because the top row is a split exact sequence, the latter corresponds to the image of θ_s under g_1. This implies the exact sequence

$$\theta_s \oplus \theta_s\theta_t\langle 1\rangle \to \theta_s\theta_t\theta_s \to \theta_s C_{st} \to 0.$$

From Kazhdan-Lusztig combinatorics we know that $\theta_s\theta_t\theta_s \cong \theta_{sts} \oplus \theta_s$. As there is no graded morphism $\theta_s \to \theta_{sts}$ and the only graded morphism $\theta_s \to \theta_s$ is the identity, up to a scalar, we finally obtain an exact sequence

$$\theta_s\theta_t\langle 1\rangle \to \theta_{sts} \to \theta_s C_{st} \to 0.$$

Analogously we can compose θ_t with every functor in the first diagram, eventually yielding an exact sequence

$$\theta_s\theta_t\langle 1\rangle \to \theta_{sts} \to C_{st}\theta_t \to 0.$$

From Kazhdan-Lusztig combinatorics it, moreover, follows that there is only one graded morphism $\theta_s\theta_t\langle 1\rangle \to \theta_{sts}$, which concludes the proof. $\square$

Proof of Corollary 5.2. If $s \in W_\nu^\dagger$, the result follows immediately from Theorem 5.1. The case $s \in W_\nu$ can be reduced to $W_\nu \cong S_n$ generated by $s_1, \cdots, s_n$ and $s = s_i$, for some $1 \le i \le n$. In case $i \le n/2$, we take the reduced form

$$w_0^\nu = s_n s_{n-1} \cdots s_2 s_1 s_n s_{n-1} \cdots s_3 s_2 \cdots s_n s_{n-1} s_{n-2} s_n s_{n-1} s_n;$$

if $i \ge n/2$, we take

$$w_0^\nu = s_1 s_2 \cdots s_{n-1} s_n s_1 s_2 \cdots s_{n-2} s_{n-1} \cdots s_1 s_2 s_3 s_1 s_2 s_1.$$

In both cases, apply Theorem 5.1 repeatedly, giving the result with $s' = s_{n-i+1}$. $\square$

Lemma 5.6. *For any simple reflection $t \in W$, we have $\mathbb{V}C_t \cong {}^t\mathsf{C} \otimes_{\mathsf{C}} \mathbb{V}$.*

Proof. Since both sides are right exact, it is enough to check the isomorphism on the category of projective modules in $\mathcal{O}_0$. [Ba2, Theorem 4.9] implies that there is a unique injective bimodule homomorphism $\mathsf{C} \hookrightarrow \mathsf{C} \otimes_{\mathsf{C}^t} \mathsf{C}$. We claim that the cokernel of this maps is isomorphic to ${}^t\mathsf{C}$. Indeed, this cokernel is isomorphic to C both as a left and as a right C-module. As a C^t-C^t-bimodule, we obviously get a decomposition of the cokernel into a direct sum of two copies of C^t. To determine the action of the

remaining generator, it is enough to compute the rank two case. In that case the statement is easily checked by a direct computation. $\qquad\square$

Corollary 5.7. *For* $\nu \in \Lambda_{\mathrm{int}}^{+}$ *and a simple reflection* $s \in W_\nu \times W_\nu^{\dagger}$, *we have*

$$\mathbb{V}C_{w_0^\nu}\theta_s \;\cong\; \mathbb{V}\theta_{w_0^\nu s w_0^\nu}C_{w_0^\nu}.$$

Proof. By Subsection 2.5, we have

$$\mathbb{V}\theta_s \cong \mathsf{C} \otimes_{\mathsf{C}^s} \mathsf{C} \otimes_{\mathsf{C}} \mathbb{V} \quad \text{and} \quad \mathbb{V}\theta_{w_0^\nu s w_0^\nu} \cong \mathsf{C} \otimes_{\mathsf{C}^{w_0^\nu s w_0^\nu}} \mathsf{C} \otimes_{\mathsf{C}} \mathbb{V}.$$

By induction and the choice of a reduced expression for w_0^ν, Lemma 5.6 implies $\mathbb{V}C_{w_0^\nu} \cong {}^{w_0^\nu}\mathsf{C} \otimes_{\mathsf{C}} \mathbb{V}$. A straightforward computation gives an isomorphism

$$\mathsf{C} \otimes_{\mathsf{C}^{w_0^\nu s w_0^\nu}} {}^{w_0^\nu}\mathsf{C} \cong {}^{w_0^\nu}\mathsf{C} \otimes_{\mathsf{C}^s} \mathsf{C},$$

of C-C bimodules. Combining the three isomorphisms yields the claim. $\qquad\square$

Lemma 5.8. *Soergel's combinatorial functor* $\mathbb{V}$ *is full and faithful when restricted to the category* $\mathcal{Q}_{(w)}$, *for any* $w \in W$.

Proof. As $\mathcal{L}C_w$ is an auto-equivalence of $\mathcal{D}^b(\mathcal{O}_0)$, see equation (12), C_w provides an isomorphism from $\mathrm{Hom}_{\mathcal{O}_0}(P, P')$ to $\mathrm{Hom}_{\mathcal{O}_0}(C_w P, C_w P')$ for any two projective modules P, P' in $\mathcal{O}_0$. The claim hence reduces to the statement that the functor $\mathbb{V}C_w$ is full and faithful on the category of projective modules. By Lemma 5.6 this reduces to [So1, Struktursatz 9]. $\qquad\square$

Proof of Proposition 5.3. We prove claim (i) first. It is clear that if the proposed equivalence of functors holds, then the fact that the category $\mathcal{Q}_{(w_0^\nu)}$ is stable under $\theta_{w_0^\nu s w_0^\nu}$ follows from the fact that θ_s preserves the category of projective modules. Hence we focus on the other direction of the claim.

The two composed functors in the proposition are right exact, as shuffling functors are right exact and projective functors are exact functors mapping projective modules to projective modules. Hence it suffices to prove the isomorphism as functors restricted to $\mathcal{Q}_{(e)}$, the full subcategory of projective modules in $\mathcal{O}_0$.

By assumption both functors restrict to functors between $\mathcal{Q}_{(e)}$ and $\mathcal{Q}_{(w_0^\nu)}$. Hence, the combination of Lemma 5.8 and Corollary 5.7 concludes the proof of claim (i).

Now we consider claim (i) for $w_0^\nu = w_0$. As the category $\mathcal{Q}_{(w_0)}$ is the category of q.h. tilting modules, see e.g. [MS3, Section 4.2], it is stable under the action of all projective functors. Therefore claim (ii) holds for $y = s$ a simple reflection. The full statement then follows by induction on the length of y and [Ma1, Equation (1)]. $\qquad\square$

We have the following application of Proposition 5.3(2).

Corollary 5.9. *Consider* $\Delta^\mu(x \cdot \lambda)^\bullet$ *as an object of* $\mathcal{D}^b(\mathcal{O}_\lambda)$, *then*

$$\mathcal{L}C_{w_0}\theta_\lambda^{out}\Delta^\mu(x \cdot \lambda)^\bullet \;\cong\; \theta_{\widehat{\lambda}}^{out}\nabla^\mu(w_0^\mu x w_0^\lambda w_0 \cdot \widehat{\lambda})^\bullet[l(w_0^\mu)].$$

Moreover, the same equation holds considering $\Delta^\mu(x \cdot \lambda)^\bullet$ *as an object of* $\mathcal{D}^b(\mathcal{O}_\lambda^\mu)$ *and* $\mathcal{L}C_{w_0}$ *the endofunctor on* $\mathcal{D}^b(\mathcal{O}_0^\mu)$ *defined in Proposition 3.7.*

Proof. By Theorem 4.3 and Proposition 5.3(2), we have

$$\mathcal{L}C_{w_0}\theta_\lambda^{out}\Delta^\mu(x\cdot\lambda)^\bullet \cong \mathcal{L}C_{w_0}\theta_{w_0^\lambda}\Delta^\mu(x)^\bullet \cong \theta_{w_0^{\hat{\lambda}}}\mathcal{L}C_{w_0}\Delta^\mu(x)^\bullet.$$

Applying then [MS3, Proposition 4.4(1)], gives

$$\mathcal{L}C_{w_0}\theta_\lambda^{out}\Delta^\mu(x\cdot\lambda)^\bullet \cong \theta_{w_0^{\hat{\lambda}}}\nabla^\mu(w_0^\mu x w_0)^\bullet[l(w_0^\mu)].$$

The result now follows by applying Theorem 4.3. The reformulation inside $\mathcal{D}^b(\mathcal{O}_0^\mu)$ follows from the commuting diagram in Proposition 3.7 and Corollary 4.11. $\square$

Finally, we derive useful expression for regular parabolic q.h. tilting modules.

Proposition 5.10. *For any $x \in X^\mu$, we have $T^\mu(x) \cong \theta_{w_0 w_0^\mu x}L(w_0^\mu w_0)$.*

Proof. As $L(w_0^\mu w_0) \cong \Delta^\mu(w_0^\mu w_0) \cong \nabla^\mu(w_0^\mu w_0)$ is a tilting module in $\mathcal{O}_0^\mu$ and projective functors preserve tilting modules, it follows that $\theta_{w_0 w_0^\mu x}L(w_0^\mu w_0)$ is a tilting module in $\mathcal{O}_0^\mu$. From Kazhdan-Lusztig combinatorics, see e.g. [Ma2, Equations (2.1) and (2.2)], it follows by induction on the length $l(w_0 w_0^\mu x)$ that the highest weight of $\theta_{w_0 w_0^\mu x}L(w_0^\mu w_0)$ is $x \cdot 0$.

It remains to prove the indecomposability of $\theta_{w_0 w_0^\mu x}L(w_0^\mu w_0)$. By [Ma3, Theorem 16], the Koszul-Ringel self-duality of $\mathcal{O}_0$ maps $\theta_{w_0 w_0^\mu x}L(w_0^\mu w_0)$ to a module isomorphic to $\theta_{w_0^{\hat{\mu}}}L(w_0 x^{-1})$. This module is the translation through the wall of a simple module, which has simple top and hence is indecomposable. $\square$

6. CONSTRUCTION OF DERIVED EQUIVALENCES

Theorem 6.1. *Let $\mathfrak{g}$ be a reductive Lie algebra, $\lambda, \lambda', \mu, \mu' \in \Lambda_{int}^+$ and $\nu_1, \nu_2 \in \Lambda_{int}^+$, such that $W_{\nu_1} \cong S_{n_1}$ and $W_{\nu_2} \cong S_{n_2}$, for some $n_1, n_2 \in \mathbb{N}$. Assume that*

$$W_\lambda = G_1 \times G_2, \quad W_{\lambda'} = G_1' \times G_2', \quad W_\mu = H_1 \times H_2, \quad W_{\mu'} = H_1' \times H_2' \quad with$$

- *$G_2 = G_2' \subset W_{\nu_1}^\dagger$ and $H_2 = H_2' \subset W_{\nu_2}^\dagger$;*
- *$G_1 \cong G_1'$ and both are subgroups of W_{ν_1};*
- *$H_1 \cong H_1'$ and both are subgroups of W_{ν_2}.*

Then A_λ^μ and $A_{\lambda'}^{\mu'}$ are gradable derived equivalent, so, in particular, we have equivalences of triangulated categories

$$\mathcal{D}^b(\mathcal{O}_\lambda^\mu) \cong \mathcal{D}^b(\mathcal{O}_{\lambda'}^{\mu'}) \qquad and \qquad \mathcal{D}^b(\mathbb{Z}\mathcal{O}_\lambda^\mu) \cong \mathcal{D}^b(\mathbb{Z}\mathcal{O}_{\lambda'}^{\mu'}).$$

For $\mathfrak{sl}(n)$, the formulation of the theorem simplifies substantially. In this case, without loss of generality, we can take $\nu_1 = \nu_2 = -\rho$, so $G_2 = G_2' = H_2 = H_2' = \{e\}$. Then we obtain precisely Theorem C.

The remainder of this section is devoted to the proof of Theorem 6.1.

Lemma 6.2. *For $\lambda \in \Lambda_{int}^+$, assume there is $\nu \in \Lambda_{int}^+$ such that $w_0^\lambda \in W_\nu \times W_\nu^\dagger$ where W_ν is of type A. Then there is $\lambda' \in \Lambda_{int}^+$ such that $w_0^{\lambda'} = w_0^\nu w_0^\lambda w_0^\nu$ and a triangulated equivalence $F_{\lambda\lambda'} : \mathcal{D}^b(\mathcal{O}_\lambda) \to \mathcal{D}^b(\mathcal{O}_{\lambda'})$ leading to a commutative diagram of functors*

$$\begin{array}{ccc}
\mathcal{D}^b(\mathcal{O}_0) & \xrightarrow{\;\;\mathcal{L}C_{w_0^\nu}\;\;} & \mathcal{D}^b(\mathcal{O}_0) \\
\Big\uparrow{\scriptstyle \theta_\lambda^{out}} & & \Big\uparrow{\scriptstyle \theta_{\lambda'}^{out}} \\
\mathcal{D}^b(\mathcal{O}_\lambda) & \xrightarrow{\;\;F_{\lambda\lambda'}\;\;} & \mathcal{D}^b(\mathcal{O}_{\lambda'})
\end{array}$$

Proof. For $x \in X_\lambda$, set $S_x := C_{w_0^\nu}\theta_\lambda^{out}P(x \cdot \lambda)$. Corollary 5.2 yields

(18) $$\theta_{w_0^{\lambda'}}S_x \cong S_x^{\oplus|W_{\lambda'}|}.$$

By Lemma 5.8, $\mathrm{End}_{\mathcal{O}_0}(S_x) \cong \mathrm{End}_{\mathbb{C}}(\mathbb{V}S_x)$. Hence (18) and Lemma 4.18 imply

$$\mathrm{End}_{\mathcal{O}_{\lambda'}}(\theta_{\lambda'}^{on}S_x) \cong \mathrm{End}_{\mathbb{C}_{\lambda'}}(\mathbb{V}_{\lambda'}\theta_{\lambda'}^{on}S_x).$$

The algebra can then be computed by using equation (15) and Lemma 5.6,

$$\mathrm{End}_{\mathcal{O}_{\lambda'}}(\theta_{\lambda'}^{on}S_x) \cong \mathrm{End}_{\mathbb{C}_{\lambda'}}(\mathrm{Res}_{\mathbb{C}_{\lambda'}}^{\mathbb{C}}\,{}^{w_0^\nu}\mathbb{C} \otimes_{\mathbb{C}} \mathbb{C} \otimes_{\mathbb{C}_\lambda} \mathbb{V}_\lambda P(x \cdot \lambda)).$$

Under the isomorphism $\mathbb{C}_\lambda \cong \mathbb{C}_{\lambda'}$, we can identify the modules

$$\mathrm{Res}_{\mathbb{C}_{\lambda'}}^{\mathbb{C}}\,{}^{w_0^\nu}\mathbb{C} \otimes_{\mathbb{C}} \mathbb{C} \otimes_{\mathbb{C}_\lambda} \mathbb{V}_\lambda P(x \cdot \lambda) \quad \text{and} \quad \mathrm{Res}_{\mathbb{C}_\lambda}^{\mathbb{C}}\mathbb{C} \otimes_{\mathbb{C}_\lambda} \mathbb{V}_\lambda P(x \cdot \lambda).$$

Therefore we obtain

(19) $$\mathrm{End}_{\mathcal{O}_{\lambda'}}(\theta_{\lambda'}^{on}S_x) \cong \mathrm{End}_{\mathcal{O}_\lambda}(P(x \cdot \lambda)^{\oplus|W_\lambda|}).$$

This formula implies that $\theta_{\lambda'}^{on}S_x$ decomposes into $|W_{\lambda'}|$ indecomposable direct summands. Equation (18) and Corollary 4.17 thus imply that there is an indecomposable $T_x \in \mathcal{O}_{\lambda'}$ such that $S_x \cong \theta_{\lambda'}^{out}T_x$. Then we define

$$S = \bigoplus_{x \in X_\lambda} S_x \quad \text{and} \quad T = \bigoplus_{x \in X_\lambda} T_x.$$

Now equation (19) can be rewritten as

$$\mathrm{End}_{\mathcal{O}_{\lambda'}}(T_x^{\oplus|W_\lambda|}) \cong \mathrm{End}_{\mathcal{O}_\lambda}(P(x \cdot \lambda)^{\oplus|W_\lambda|}).$$

The same calculation for P_λ, rather than $P(x \cdot \lambda)$, implies

(20) $$\mathrm{End}_{\mathcal{O}_{\lambda'}}(T) \cong \mathrm{End}_{\mathcal{O}_\lambda}(P_\lambda) = A_\lambda.$$

As $\mathcal{L}C_{w_0^\nu}$ is an auto-equivalence of $\mathcal{D}^b(\mathcal{O}_0)$, the module S does not have self-extensions. As $\theta_{\lambda'}^{out} : \mathcal{D}^b(\mathcal{O}_{\lambda'}) \to \mathcal{D}^b(\mathcal{O}_0)$ is faithful, by construction, the module T then satisfies

(21) $$\mathrm{Ext}_{\mathcal{O}_{\lambda'}}^i(T, T) = 0, \quad \text{for } i > 0.$$

For any projective module P in $\mathcal{O}_0$, we claim that $\theta_{\lambda'}^{on}C_{w_0^\nu}P$ is a module in $\mathrm{add}(T)$. Indeed, by Corollary 5.2 we obtain

$$\left(\theta_{\lambda'}^{on}C_{w_0^\nu}P\right)^{\oplus|W_{\lambda'}|} \cong \theta_{\lambda'}^{on}C_{w_0^\nu}\theta_{w_0^\lambda}P,$$

where $C_{w_0^\nu}\theta_{w_0^\lambda}P$ is clearly a module in $\mathrm{add}(S)$ and $\theta_{\lambda'}^{on}S \cong T^{\oplus|W_\lambda|}$.

As $\mathcal{L}C_{w_0^\nu}$ is an auto-equivalence of $\mathcal{D}^b(\mathcal{O}_0)$, [Ric1, Theorem 6.4] implies that $C_{w_0^\nu}P_0$ is a generalised tilting module. Hence, P_0 has a coresolution by modules in $\mathrm{add}(C_{w_0^\nu}P_0)$. By the above paragraph, applying the exact functor $\theta_{\lambda'}^{on}$ leads to a coresolution of $\theta_{\lambda'}^{on}P_0$ by modules in $\mathrm{add}(T)$. As T has no self-extensions by equation (21), this

immediately implies that every indecomposable summand in $\theta_{\lambda'}^{on} P_0$ has such a coresolution. Thus, in particular, there is some $r \in \mathbb{N}$ for which there is an exact sequence

(22) $$0 \to P_{\lambda'} \to T_0 \to T_1 \to \cdots T_{r-1} \to T_r \to 0,$$

where $T_i \in \mathrm{add}(T)$.

Equations (21) and (22) imply that T is a generalised tilting module, or a tilting complex (contained in one position) in the sense of [Ric1]. The equivalence then follows from equation (20) and [Ha, Theorem III.2.10] or [Ric1, Theorem 6.4].

Finally, we prove the existence of the commutative diagram. It suffices to prove that $\mathcal{L}C_{w_0^\nu} \circ \theta_\lambda^{out} \cong \theta_{\lambda'}^{out} \circ F_{\lambda\lambda'}$, restricted as functors on the category of projective modules in $\mathcal{O}_\lambda$, as both are triangulated functors. By construction, the functor $F_{\lambda\lambda'}$ acts on the category of projective modules by mapping $P(x \cdot \lambda)$ to T_x and its action on morphisms corresponds to the algebra isomorphism determined in equation (20). The equivalence of the functors $\mathcal{L}C_{w_0^\nu} \circ \theta_\lambda^{out}$ and $\theta_{\lambda'}^{out} \circ F_{\lambda\lambda'}$ acting between $\mathrm{add}(P_\lambda)$ and $\mathrm{add}(S)$, thus follows from construction of the isomorphism (20). $\qquad\square$

Lemma 6.3. *In the setup of Lemma 6.2, any object $\mathcal{X}^\bullet$ in $\mathcal{D}^b(\mathcal{O}_\lambda)$ satisfies*

$$\mathcal{L}Z^\mu \imath^\mu F_{\lambda\lambda'} \mathcal{X}^\bullet \;\cong\; F_{\lambda\lambda'} \mathcal{L}Z^\mu \imath^\mu \mathcal{X}^\bullet.$$

Proof. According to Corollary 4.11, it is sufficient to prove that

$$\theta_{\lambda'}^{out} \mathcal{L}Z^\mu \imath^\mu F_{\lambda\lambda'} \mathcal{X}^\bullet \;\cong\; \theta_{\lambda'}^{out} F_{\lambda\lambda'} \mathcal{L}Z^\mu \imath^\mu \mathcal{X}^\bullet.$$

That this is true follows from the fact that Zuckerman and inclusion functors commute with translation functors (and hence also with shuffling functors), see e.g. [Ba1, Lemma 2.6(a)] and the diagram in Lemma 6.2. $\qquad\square$

Lemma 6.4. *Consider λ, ν as in Lemma 6.2 and an arbitrary $\mu \in \Lambda_{\mathrm{int}}^+$. There is a an equivalence $F_{\lambda\lambda'}^\mu : \mathcal{D}^b(\mathcal{O}_\lambda^\mu) \to \mathcal{D}^b(\mathcal{O}_{\lambda'}^\mu)$ of triangulated categories which admits a commutative diagram of functors*

$$
\begin{array}{ccc}
\mathcal{D}^b(\mathcal{O}_\lambda) & \xrightarrow{\;\; F_{\lambda\lambda'} \;\;} & \mathcal{D}^b(\mathcal{O}_{\lambda'}) \\
{\scriptstyle \imath^\mu} \big\uparrow & & {\scriptstyle \imath^\mu} \big\uparrow \\
\mathcal{D}^b(\mathcal{O}_\lambda^\mu) & \xrightarrow{\;\; F_{\lambda\lambda'}^\mu \;\;} & \mathcal{D}^b(\mathcal{O}_{\lambda'}^\mu).
\end{array}
$$

Proof. Consider the minimal projective generator P_λ^μ of $\mathcal{O}_\lambda^\mu$ in (1). By Lemma 6.2, there is a complex $\mathcal{S}^\bullet := F_{\lambda\lambda'} \, \imath^\mu P_\lambda^\mu$ in $\mathcal{D}^b(\mathcal{O}_{\lambda'})$ with

(23) $$\mathrm{End}_{\mathcal{D}^b(\mathcal{O}_{\lambda'})}(\mathcal{S}^\bullet) \cong \mathrm{End}_{\mathcal{D}^b(\mathcal{O}_\lambda)}(\imath^\mu P_\lambda^\mu) \cong \mathrm{End}_{\mathcal{D}^b(\mathcal{O}_\lambda^\mu)}(P_\lambda^\mu) \cong A_\lambda^\mu,$$

where the latter isomorphism follows from Corollary 4.10. Lemma 6.2 and Proposition 3.7 yield $\theta_{\lambda'}^{out} \mathcal{S}^\bullet \cong \imath^\mu \mathcal{L}C_{w_0^\nu} \theta_\lambda^{out} P_\lambda^\mu$. Lemma 4.19 thus implies the existence of $\mathcal{T}^\bullet \in \mathcal{D}^b(\mathcal{O}_\lambda^\mu)$ such that

(24) $$\imath^\mu \mathcal{T}^\bullet \;\cong\; \mathcal{S}^\bullet \qquad \text{and} \qquad \theta_{\lambda'}^{out} \mathcal{T}^\bullet \cong \mathcal{L}C_{w_0^\nu} \theta_\lambda^{out} P_\lambda^\mu.$$

The second property in (24) and the faithfulness of $\theta_{\lambda'}^{out}$ give an injection

$$\mathrm{Hom}_{\mathcal{D}^b(\mathcal{O}_{\lambda'}^\mu)}(\mathcal{T}^\bullet, \mathcal{T}^\bullet[k]) \hookrightarrow \mathrm{Hom}_{\mathcal{D}^b(\mathcal{O}_0^\mu)}(\mathcal{L}C_{w_0^\nu} \theta_\lambda^{out} P_\lambda^\mu, \mathcal{L}C_{w_0^\nu} \theta_\lambda^{out} P_\lambda^\mu[k]).$$

The right-hand side is zero, if $k \neq 0$, by Proposition 3.7. So we find

$$(25) \qquad \mathrm{Hom}_{\mathcal{D}^b(\mathcal{O}^\mu_{\lambda'})}(\mathcal{T}^\bullet, \mathcal{T}^\bullet[k]) = 0 \qquad \text{if} \quad k \neq 0.$$

The first property in equation (24) and equation (25) allow us to apply Corollary 4.10 and deduce, using equation (23), that

$$(26) \qquad \mathrm{End}_{\mathcal{D}^b(\mathcal{O}^\mu_{\lambda'})}(\mathcal{T}^\bullet) \cong \mathrm{End}_{\mathcal{D}^b(\mathcal{O}_{\lambda'})}(\mathcal{S}^\bullet) \cong A^\mu_\lambda.$$

By equations (25) and (26), the result would follow from [Ric1, Theorem 6.4], if we could show that $\mathrm{add}(\mathcal{T}^\bullet)$ generates $\mathcal{D}^b(\mathcal{O}^\mu_\lambda)$ as a triangulated category.

Now we focus on the latter statement. Remark 4.9 and Lemma 6.3 yield

$$\bigoplus_{j \in \mathbb{N}} \mathcal{L}Z^\mu F_{\lambda\lambda'}(P^\bullet_\lambda)^{\oplus c_j}[j] \cong \mathcal{L}Z^\mu F_{\lambda\lambda'} \imath^\mu \mathcal{L}Z^\mu P^\bullet_\lambda.$$

As, for every $x \in X_\lambda$,

$$\mathcal{L}Z^\mu P(x \cdot \lambda)^\bullet \cong \begin{cases} P^\mu(x \cdot \lambda)^\bullet, & \text{if } x \in X^\mu_\lambda; \\ 0, & \text{if } x \notin X^\mu_\lambda; \end{cases}$$

we find

$$\bigoplus_{j \in \mathbb{N}} \mathcal{L}Z^\mu F_{\lambda\lambda'}(P^\bullet_\lambda)^{\oplus c_j}[j] \cong \mathcal{L}Z^\mu \imath^\mu \mathcal{T}^\bullet \cong \bigoplus_{j \in \mathbb{N}} (\mathcal{T}^\bullet)^{\oplus c_j}[j].$$

This implies that $\mathcal{L}Z^\mu F_{\lambda\lambda'} P^\bullet_\lambda \in \mathrm{add}(\mathcal{T}^\bullet)$. The fact that $F_{\lambda\lambda'} P^\bullet_\lambda$ generates $\mathcal{D}^b(\mathcal{O}_{\lambda'})$ as a triangulated category (which is an immediate consequence of Lemma 6.2) hence implies that $\mathrm{add}(\mathcal{T}^\bullet)$ generates $\mathcal{D}^b(\mathcal{O}^\mu_{\lambda'})$ as a triangulated category. The existence of the commuting diagram then follows by the same arguments as in the proof of Lemma 6.2. $\qquad \square$

Lemma 6.5. *The functor $F^\mu_{\lambda\lambda'}$ from Lemma 6.4 admits a graded lift, yielding an equivalence of triangulated categories*

$$\widetilde{F}^\mu_{\lambda\lambda'} : \mathcal{D}^b(\mathbb{Z}\mathcal{O}^\mu_\lambda) \to \mathcal{D}^b(\mathbb{Z}\mathcal{O}^\mu_{\lambda'}).$$

Proof. As translation, shuffling and inclusion functors admit graded lifts, it follows easily that so does $F^\mu_{\lambda,\lambda'}$.

We can switch the roles of λ and λ' and construct a functor $F^\mu_{\lambda'\lambda}$. We define the (gradable) functor $G^\mu_{\lambda'\lambda} = \mathbf{d}F_{\lambda'\lambda}\mathbf{d}$. Then $G^\mu_{\lambda'\lambda}$ admits commutative diagrams with $\mathbf{d}\mathcal{L}C_{w^\nu_0}\mathbf{d}$, analogous to the ones in Lemmata 6.2 and 6.4 and is also gradable. By construction, $F^\mu_{\lambda\lambda'}G^\mu_{\lambda'\lambda}$ and, respectively, $G^\mu_{\lambda'\lambda}F^\mu_{\lambda\lambda'}$, are isomorphic to the respective identity functors when restricted to the categories of projective modules in $\mathcal{O}^\mu_{\lambda'}$ and $\mathcal{O}^\mu_\lambda$, respectively. Hence $G^\mu_{\lambda'\lambda}$ is an inverse to $F^\mu_{\lambda'\lambda}$. The result thus follows from Proposition 3.5. $\qquad \square$

Lemma 6.6. *Consider $\lambda, \lambda', \mu \in \Lambda^+_{\mathrm{int}}$ and $\nu_1 \in \Lambda^+_{\mathrm{int}}$ such that $W_{\nu_1} \cong S_n$, for some $n \in \mathbb{N}$. Assume that $W_\lambda = G_1 \times G_2$ and $W_{\lambda'} = G'_1 \times G'_2$, where*

- *$G_2 = G'_2$ is a subgroup of $W^\dagger_{\nu_1}$;*
- *$G_1 \cong G'_1$ and both are subgroups of W_{ν_1}.*

Then A^μ_λ and $A^\mu_{\lambda'}$ are gradable derived equivalent. In particular, we have equivalences of triangulated categories

$$\mathcal{D}^b(\mathcal{O}^\mu_\lambda) \cong \mathcal{D}^b(\mathcal{O}^\mu_{\lambda'}) \qquad and \qquad \mathcal{D}^b(\mathbb{Z}\mathcal{O}^\mu_\lambda) \cong \mathcal{D}^b(\mathbb{Z}\mathcal{O}^\mu_{\lambda'}).$$

Proof. There are $k \in \mathbb{N}$ and $p_i \in \mathbb{N}$, for $1 \le i \le k$, such that $p_1 + \cdots + p_k = n$ and

$$G_1 = S_{p_1} \times S_{p_2} \times \cdots \times S_{p_k}$$

as a subgroup of S_n. Then G_2, as a subgroup of S_n, is equal to

$$G_2 = S_{p_{\sigma(1)}} \times S_{p_{\sigma(2)}} \times \cdots \times S_{p_{\sigma(k)}},$$

for some $\sigma \in S_k$. It suffices to prove the lemma for σ such that it only interchanges two neighbors $(j, j+1)$ for some $1 \le j < k$. Then we choose an integral dominant ν such that W_ν is equal to the subgroup of $W_{\nu_1} \subset W$ defined as

$$\overbrace{S_1 \times S_1 \times \cdots \times S_1}^{p_1 + p_2 + \cdots + p_{j-1}} \times S_{p_j + p_{j+1}} \times \overbrace{S_1 \times \cdots S_1}^{p_{j+2} + \cdots + p_k}.$$

For the case we consider, this leads to $w_0^{\lambda'} = w_0^\nu w_0^\lambda w_0^\nu$ and the equivalences follow from Lemma 6.4, Lemma 6.5 and Definition 3.2. $\qquad\square$

Proof of Theorem 6.1. To prove the theorem it suffices to restrict to either $\lambda = \lambda'$ or $\mu = \mu'$ as the full statement follows by composition of the two. The case $\mu = \mu'$ is precisely Lemma 6.6.

The graded equivalence for $\lambda = \lambda'$ follows immediately from composing Lemma 6.6 with the equivalence in equation (4). The fact that the equivalence then descends to the non-graded categories follows from Proposition 3.4 and equation (5). $\qquad\square$

Finally, we note how the results in this section lead to a proof of Theorem D. Part (ii) follows immediately from Lemmata 6.2, 6.4 and 6.5. Part (i) then follows from Koszul duality as in the proof of Theorem 6.1. The link with derived twisting functors follows from [MOS, Theorem 39].

7. The classification for type A

In this section we prove the following theorem, which implies Theorem B.

Theorem 7.1. *Consider two Lie algebras $\mathfrak{g}, \mathfrak{g}'$ of type A with fixed Borel subalgebras $\mathfrak{b}, \mathfrak{b}'$, classes $\Lambda \in \mathfrak{h}^*/\Lambda_{\mathrm{int}}$ and $\Lambda' \in (\mathfrak{h}')^*/\Lambda'_{\mathrm{int}}$ with the corresponding integral Weyl groups $W_\Lambda, W'_{\Lambda'}$ and two dominant weights $\lambda \in \Lambda$, $\lambda' \in \Lambda'$. Then the following statements are equivalent:*

(I) *There is a gradable equivalence $\mathcal{D}^b(\mathcal{O}_\lambda(\mathfrak{g}, \mathfrak{b})) \cong \mathcal{D}^b(\mathcal{O}_{\lambda'}(\mathfrak{g}', \mathfrak{b}'))$.*

(II) *There is a graded algebra isomorphism $\mathcal{Z}(\mathcal{O}_\lambda(\mathfrak{g}, \mathfrak{b})) \cong \mathcal{Z}(\mathcal{O}_{\lambda'}(\mathfrak{g}', \mathfrak{b}'))$.*

(III) *For some decompositions*

$$W_\Lambda \cong X_1 \times X_2 \times \cdots \times X_k \qquad and \qquad W'_{\Lambda'} \cong X'_1 \times X'_2 \times \cdots \times X'_m$$

into products of irreducible Weyl groups, we have $k = m$ and there is a permutation φ on $\{1, 2, \ldots, k\}$ such that $W_{\Lambda,\lambda} \cap X_i \cong W_{\Lambda',\lambda'} \cap X'_{\varphi(i)}$ and $X_i \cong X'_{\varphi(i)}$ for all $i = 1, 2, \ldots, k$.

Note that, as we only consider type A in this section, we do not need to distinguish between isomorphisms of Coxeter groups and isomorphisms of Coxeter systems. Before proving Theorem 7.1 we need the following preparatory lemma.

Lemma 7.2. *Consider Coxeter groups W and W' of type A with respective Young subgroups X and X'. There is a graded algebra isomorphism $\mathtt{C}(W)^X \cong \mathtt{C}(W')^{X'}$ if and only if, for some decompositions*

$$W \cong X_1 \times X_2 \times \cdots \times X_k \quad and \quad W' \cong X'_1 \times X'_2 \times \cdots \times X'_m$$

into products of irreducible Weyl groups, we have $k = m$ and there is a permutation φ on $\{1, 2, \ldots, k\}$ such that $X \cap X_i \cong X' \cap X'_{\varphi(i)}$ and $X_i \cong X'_{\varphi(i)}$ for all $i = 1, 2, \ldots, k$.

Proof. First we assume that W and W' are simple. Without loss of generality, we assume that $W \cong S_n$ and $W' \cong S_m$ with $m \geq n$. Denote the compositions of n and m, corresponding to X and X', by $(p_1, \cdots, p_k)$ and $(q_1, \cdots, q_l)$, respectively. A graded algebra isomorphism implies, in particular, a graded vector space isomorphism, so equation (16) implies that

$$\prod_{j=1}^{l} \prod_{a=1}^{q_j} (1 - z^a) = \prod_{j=1}^{k} \prod_{b=1}^{p_j} (1 - z^b) \prod_{s=n+1}^{m} (1 - z^s)$$

must be equal in $\mathbb{C}[z]$. If $m > n$, then the irreducible factorisation of the right-hand side contains a term $(1 - \exp(\frac{2\pi i}{m})z)$, with $i^2 = -1$, which never occurs in the factorisation of the left-hand side. Hence $m = n$, or $W \cong W'$. The same argument can now be used if the maximum of $\{p_j, 1 \leq j \leq k\}$ were larger than the maximum of $\{q_j, 1 \leq j \leq l\}$. By similar reasoning and induction it then follows that the compositions are the same up to ordering and hence we find $X \cong X'$.

Now consider an arbitrary Coxeter group W of type A. It follows easily from the definitions that, if we consider the decomposition $W \cong W_1 \times W_2 \times \cdots \times W_k$, for some k, such that each W_i is irreducible and set $X_i = X \cap W_i$, then we have

$$\mathtt{C}(W)^X \cong \mathtt{C}(W_1)^{X_1} \oplus \mathtt{C}(W_2)^{X_2} \oplus \cdots \oplus \mathtt{C}(W_k)^{X_k}.$$

The general result now follows. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Proof of Theorem 7.1. By [So1, Theorem 11], we can take reductive Lie algebras $\tilde{\mathfrak{g}}$ and $\tilde{\mathfrak{g}}'$ for which there are Borel subalgebras $\tilde{\mathfrak{b}}$ and $\tilde{\mathfrak{b}}'$, Weyl groups W and $\tilde{W}$, and *integral* dominant weights $\tilde{\lambda}$ and $\tilde{\lambda}'$ such that

(a) $\mathcal{O}_\lambda(\mathfrak{g}, \mathfrak{b}) \cong \mathcal{O}_{\tilde{\lambda}}(\tilde{\mathfrak{g}}, \tilde{\mathfrak{b}})$ and $\mathcal{O}_{\lambda'}(\mathfrak{g}', \mathfrak{b}') \cong \mathcal{O}_{\tilde{\lambda}'}(\tilde{\mathfrak{g}}', \tilde{\mathfrak{b}}')$;

(b) $W_\Lambda \cong \tilde{W}$ and $W'_{\Lambda'} \cong \tilde{W}'$;

(c) $W_{\Lambda, \lambda} \to \tilde{W}_{\tilde{\lambda}}$ and $W'_{\Lambda', \lambda'} \to \tilde{W}'_{\tilde{\lambda}'}$ under the above isomorphisms.

Moreover, by [Mat, Proposition A.4], we can chose $\tilde{\mathfrak{g}}$ and $\tilde{\mathfrak{g}}'$ to be of type A. By (a), the properties in claim (I) and claim (II) are true if and only if they are true for $\mathcal{O}_{\tilde{\lambda}}(\tilde{\mathfrak{g}}, \tilde{\mathfrak{b}})$ and $\mathcal{O}_{\tilde{\lambda}'}(\tilde{\mathfrak{g}}', \tilde{\mathfrak{b}}')$. Furthermore (b) and (c) imply that also the property in claim (III) is true if and only if it holds for W and $\tilde{W}$. Therefore it suffices to prove the equivalences between the three statements restricted to the integral setting. So, henceforth we can assume $\Lambda = \Lambda_{\text{int}}$ and $\Lambda' = \Lambda'_{\text{int}}$.

Claim (I) implies claim (II) by Lemma 3.6. Now assume that claim (II) holds. The results repeated in Subsection 2.5 imply that there must be a graded isomorphism between $\mathtt{C}_\lambda(W)$ and $\mathtt{C}_{\lambda'}(W')$. Application of Lemma 7.2 then proves that claim (II) implies claim (III). The fact that claim (III) implies claim (I) is an immediate consequence of Lemma 6.6. □

8. Ringel duality

In this section we study Ringel duality for arbitrary blocks in parabolic category $\mathcal{O}$ for arbitrary reductive Lie algebras. We will obtain two version of the Ringel duality functor, which generalise the ones in equations (13) and (14).

8.1. **Ringel duality for parabolic category $\mathcal{O}$.** The main result here is:

Theorem 8.1. *Consider a reductive Lie algebra $\mathfrak{g}$ and $\lambda, \mu \in \Lambda^+_{\mathrm{int}}$.*

(i) *There are isomorphisms of (graded) algebras: $A^\mu_{\widehat{\lambda}} \cong R(A^\mu_\lambda) \cong A^{\widehat{\mu}}_\lambda$.*

(ii) *The restriction of the endofunctor $\mathcal{L}_{l(w^\mu_0)} T_{w_0}$ of $\mathcal{O}_\lambda$ to the full subcategory $\mathcal{O}^\mu_\lambda$ yields a Ringel duality functor $\underline{\mathcal{R}}^\mu_\lambda$ satisfying*

$$\underline{\mathcal{R}}^\mu_\lambda \colon \mathcal{O}^\mu_\lambda \to \mathcal{O}^{\widehat{\mu}}_\lambda \quad \text{and} \quad \underline{\mathcal{R}}^\mu_\lambda(\Delta^\mu(x \cdot \lambda)) \cong \nabla^{\widehat{\mu}}(w_0 w^\mu_0 x w^\lambda_0 \cdot \lambda).$$

Remark 8.2. This theorem easily leads to a proof of Theorem A. In particular, there are blocks $\mathcal{O}^\mu_\lambda$ which are not equivalent to $\mathcal{O}^{\widehat{\mu}}_\lambda$, as demonstrated explicitly in Subsection 8.4 for completeness.

Before proving Theorem 8.1, we establish some preparatory lemmata.

Lemma 8.3. *For $x \in X^\mu_\lambda$, consider $\Delta^\mu(x \cdot \lambda)^\bullet$ as an object in $\mathcal{D}^b(\mathcal{O}_\lambda)$. Then*

$$\mathcal{L} T_{w_0} \Delta^\mu(x \cdot \lambda)^\bullet \cong \nabla^{\widehat{\mu}}(w_0 w^\mu_0 x w^\lambda_0 \cdot \lambda)^\bullet [l(w^\mu_0)].$$

Proof. Using [MOS, Section 6.5] and Proposition 4.7, one can consider the Koszul dual of (the graded version of) Corollary 5.9. Corollary 4.11 and equation (11) then allow us to interpret this as the proposed isomorphism. We also provide the sketch of a proof without the use of Koszul duality, following the proof of [MS3, Proposition 4.4(1)]. Applying parabolic induction to the classical BGG resolution (see [Hu, Chapter 6] or [Le]) for the Levi subalgebra of $\mathfrak{q}_\mu$, yields an exact complex

$$(27) \qquad 0 \to C_{l(w^\mu_0)} \to \cdots \to C_j \to C_{j-1} \to \cdots C_0 \to \Delta^\mu(x \cdot \lambda) \to 0, \quad \text{with}$$

$$C_j = \bigoplus_{u \in W_\mu, l(u)=j} \Delta(ux \cdot \lambda).$$

As Verma modules are acyclic for T_{w_0}, see [AS, Theorem 2.2], we can compute $\mathcal{L} T_{w_0} \Delta^\mu(x \cdot \lambda)^\bullet$ by evaluating T_{w_0} at the complex (27) (where $\Delta^\mu(x \cdot \lambda)$ is deleted). Now [AS, Theorem 2.3] implies

$$T_{w_0} \Delta(ux \cdot \lambda) \cong \nabla(w_0 ux \cdot \lambda).$$

The complex $D_\bullet = \mathcal{L} T_{w_0} C_\bullet = T_{w_0} C_\bullet$ is hence of the form

$$D_j = \bigoplus_{u \in W_{\widehat{\mu}}, l(u)=j} \nabla(u w_0 x \cdot \lambda) = \bigoplus_{u \in W_{\widehat{\mu}}, l(u)=l(w^\mu_0)-j} \nabla(u w_0 w^\mu_0 x \cdot \lambda).$$

We have $w_0 w_0^\mu x \cdot \lambda = y \cdot \lambda$ with $y := w_0 w_0^\mu x w_0^\lambda \in X_\lambda^{\widehat{\mu}}$. As the maps in the complex satisfy a dual version of the ones in the complex $C_\bullet$, we find

$$\mathcal{L}T_{w_0} \Delta^\mu (x \cdot \lambda)^\bullet \cong \nabla^{\widehat{\mu}} (y \cdot \lambda)^\bullet [l(w_0^\mu)],$$

which concludes the proof. $\hfill\square$

Lemma 8.4. *The restriction of the endofunctor $\mathcal{L}_{l(w_0^\mu)} T_{w_0}$ of $\mathcal{O}_\lambda$ to the full subcategory $\mathcal{O}_\lambda^\mu$ is right exact.*

Proof. Lemma 8.3 implies that $\mathcal{L}_i T_{w_0} N = 0$, if $i \neq l(w_0^\mu)$, for any $N \in \mathrm{Ob}(\mathcal{O}_\lambda^\mu)$ with standard flag. So, in particular, for projective objects of $\mathcal{O}_\lambda^\mu$. The fact that $\mathcal{L}_i T_{w_0} M = 0$ for $i < l(w_0^\mu)$ and arbitrary $M \in \mathcal{O}_\lambda^\mu$ can then be derived by induction on the projective dimension of M in $\mathcal{O}_\lambda^\mu$, by considering short exact sequences. $\hfill\square$

Proof of Theorem 8.1. First we prove part (ii). By Lemmata 8.3 and 8.4, the endofunctor $\mathcal{L}_{l(w_0^\mu)} T_{w_0}$ of $\mathcal{O}_\lambda$ restricts to a right exact functor

$$\underline{\mathcal{R}}_\lambda^\mu : \mathcal{O}_\lambda^\mu \to \mathcal{O}_\lambda$$

which maps standard modules to modules contained in the subcategory $\mathcal{O}_\lambda^{\widehat{\mu}}$. Subsequently, all simple modules, being the tops of standard modules, are mapped to modules in this subcategory. As $\mathcal{O}_\lambda^{\widehat{\mu}}$ is a full Serre subcategory, this leads to the conclusion that we actually obtain a right exact functor

$$\underline{\mathcal{R}}_\lambda^\mu : \mathcal{O}_\lambda^\mu \to \mathcal{O}_\lambda^{\widehat{\mu}}.$$

By Lemma 8.3, this functor restricted to the full subcategory of modules in $\mathcal{O}_\lambda^\mu$ with a standard flag, yields an exact functor from this category to the category of modules in $\mathcal{O}_\lambda^{\widehat{\mu}}$ with costandard flag. This is an equivalence as it has as inverse, namely, the restriction of $\mathbf{d}\mathcal{L}_{l(w_0^\mu)} T_{w_0} \mathbf{d}$, by Lemma 8.3, equation (10) and the fact that $\mathcal{O}_\lambda^\mu$ is isomorphism closed in $\mathcal{O}_\lambda$. Part (ii) then follows from [MS3, Proposition 2.2].

In particular, this implies $R(A_\lambda^\mu) \cong A_\lambda^{\widehat{\mu}}$. By [So1, Theorem 11] there is an isomorphism $A_\lambda \cong A_{\widehat{\lambda}}$. Under this isomorphism, the set of idempotents corresponding to $X_\lambda^{\widehat{\mu}}$ are mapped to the ones corresponding to $X_{\widehat{\lambda}}^\mu$, implying that $A_\lambda^{\widehat{\mu}} \cong A_{\widehat{\lambda}}^\mu$. This proves the ungraded isomorphisms of algebras in (i).

By the previous conclusions in this proof, [MS3, Proposition 2.2(2)] implies that the functor $\mathcal{L}_{l(w_0^\mu)} T_{w_0}$ maps a minimal projective generator of $\mathcal{O}_\lambda^\mu$ to a characteristic q.h. tilting module of $\mathcal{O}_{\lambda'}^\mu$. As the derived functor is an equivalence between the derived categories and is gradable, it follows that $R(A_\lambda^\mu) \cong A_\lambda^{\widehat{\mu}}$ holds as graded algebras. $\hfill\square$

Remark 8.5. Moreover, [MS3, Proposition 2.2] also implies that

$$\mathcal{D}^b(\mathcal{O}_\lambda^\mu) \cong \mathcal{D}^b(\mathcal{O}_\lambda^{\widehat{\mu}}) \cong \mathcal{D}^b(\mathcal{O}_{\widehat{\lambda}}^\mu).$$

When $\mathfrak{g}$ is not of type A, this is outside the scope of Theorem 6.1. However it can be proved similarly as will be done in the next subsection.

Remark 8.6. An anonymous referee brought to our attention an insightful different method for proving Theorem 8.1(i). We set $\mathfrak{q}_\mu = \mathfrak{l} \oplus \mathfrak{w}^+$, with $\mathfrak{w}^+$ the radical and $\mathfrak{l}$ the Levi subalgebra of the parabolic subalgebra $\mathfrak{q}_\mu$, leading to parabolic decomposition $\mathfrak{g} = \mathfrak{w}^- \oplus \mathfrak{l} \oplus \mathfrak{w}^+$. Consider $A = U(\mathfrak{g})$, $B = U(\mathfrak{w}^+)$, $H = U(\mathfrak{l})$ and $\overline{B} = U(\mathfrak{w}^-)$. This

gives a triangular decomposition in the sense of [GGOR, Section 2.1], when taking into account footnote 1.

For a complex Lie algebra $\mathfrak{a}$ such that $\dim(\mathfrak{a}) = d < \infty$, the extension groups $\mathrm{Ext}^\bullet_{U(\mathfrak{a})}(\mathbb{C}, U(\mathfrak{a}))$ can be calculated from the Chevalley-Eilenberg complex

$$0 \to U(\mathfrak{a}) \to \mathfrak{a}^* \otimes U(\mathfrak{a}) \to \cdots \to \overset{j}{\bigwedge}(\mathfrak{a}^*) \otimes U(\mathfrak{a}) \to \cdots \to \overset{d}{\bigwedge}(\mathfrak{a}^*) \otimes U(\mathfrak{a}) \to 0.$$

This can be re-interpreted as the Koszul resolution of the trivial $\mathfrak{a}$-module $\mathbb{C}$, shifted over d positions;

$$0 \to \overset{d}{\bigwedge}(\mathfrak{a}) \otimes U(\mathfrak{a}) \to \overset{d-1}{\bigwedge}(\mathfrak{a}) \otimes U(\mathfrak{a}) \to \cdots \to \overset{d-j}{\bigwedge}(\mathfrak{a}) \otimes U(\mathfrak{a}) \to \cdots \to U(\mathfrak{a}) \to 0,$$

yielding $\dim \mathrm{Ext}^i_{U(\mathfrak{a})}(\mathbb{C}, U(\mathfrak{a})) = \delta_{i,d}$. Hence B is Gorenstein and one can apply [GGOR, Proposition 4.3], which implies that the Ringel dual of $\mathcal{O}^\mu_\lambda$ is the category of *right A-modules*, which are finitely generated, H-semisimple (and locally finite), B-locally nilpotent and which admit generalised central character χ_λ. To interpret these as left $A = U(\mathfrak{g})$-modules we can apply an anti-automorphism of $U(\mathfrak{g})$. The principal automorphism corresponding to $X \mapsto -X$ for $X \in \mathfrak{g}$ shows that the Ringel dual is equivalent to $\mathcal{O}^\mu_{\widehat{\lambda}}$. The Chevalley anti-automorphism of $U(\mathfrak{g})$ gives an equivalence with $\mathcal{O}^{\widehat{\mu}}_\lambda$.

8.2. **An alternative approach.** The following result is a reformulation of the results in the previous subsection, but we provide an alternative proof, stressing the link with our construction of derived equivalences.

Proposition 8.7. *We have $R(A^\mu_\lambda) \cong A^\mu_{\widehat{\lambda}}$. Moreover, the Ringel duality functor $\overline{\mathcal{R}}^\mu_\lambda : \mathcal{O}^\mu_\lambda \to \mathcal{O}^\mu_{\widehat{\lambda}}$ satisfies, for every $x \in X^\mu_\lambda$, the following:*

$$\overline{\mathcal{R}}^\mu_\lambda(\Delta^\mu(x \cdot \lambda)) \cong \nabla^\mu(w^\mu_0 x w^\lambda_0 w_0 \cdot \widehat{\lambda}).$$

Lemma 8.8. *For all integral dominant λ, μ, there is an equivalence of triangulated categories $F^\mu_\lambda : \mathcal{D}^b(\mathcal{O}^\mu_\lambda) \to \mathcal{D}^b(\mathcal{O}^\mu_{\widehat{\lambda}})$ such that the following diagram commutes:*

$$
\begin{array}{ccc}
\mathcal{D}^b(\mathcal{O}_0) & \xrightarrow{\mathcal{L}C_{w_0}} & \mathcal{D}^b(\mathcal{O}_0) \\
{\scriptstyle \theta^{out}_\lambda} \big\uparrow & & {\scriptstyle \theta^{out}_{\widehat{\lambda}}} \big\uparrow \\
\mathcal{D}^b(\mathcal{O}_\lambda) & \xrightarrow{F_\lambda} & \mathcal{D}^b(\mathcal{O}_{\widehat{\lambda}}) \\
{\scriptstyle \imath^\mu} \big\uparrow & & {\scriptstyle \imath^\mu} \big\uparrow \\
\mathcal{D}^b(\mathcal{O}^\mu_\lambda) & \xrightarrow{F^\mu_\lambda} & \mathcal{D}^b(\mathcal{O}^\mu_{\widehat{\lambda}}).
\end{array}
$$

Proof. This is proved identically to Lemmata 6.2 and 6.4, the only change being that Proposition 5.3(ii) is used rather than Proposition 5.3(i). $\qquad\square$

Corollary 8.9. *For all $x \in X^\mu_\lambda$, we have $F^\mu_\lambda(P^\mu(x \cdot \lambda)^\bullet) \cong T^\mu(w^\mu_0 x w^\lambda_0 w_0 \widehat{\lambda})^\bullet[l(w^\mu_0)]$, implying that $R(A^\mu_{\widehat{\lambda}}) := \mathrm{End}_{\mathcal{O}^\mu_{\widehat{\lambda}}}(T^\mu_{\widehat{\lambda}}) \cong A^\mu_\lambda$.*

Proof. Lemma 8.8, Proposition 5.3(ii) and Corollary 5.9 imply

$$\theta_\lambda^{out} \imath^\mu F_\lambda^\mu P^\mu(x \cdot \lambda)^\bullet \cong \mathcal{L}C_{w_0} \theta_x \Delta^\mu(e)^\bullet \cong \theta_{w_0 x w_0} L(w_0^\mu w_0)^\bullet [l(w_0^\mu)].$$

Proposition 5.10 then implies

$$\theta_\lambda^{out} \imath^\mu F_\lambda^\mu P^\mu(x \cdot \lambda)^\bullet \cong T^\mu(w_0^\mu x w_0)^\bullet [l(w_0^\mu)] \cong \theta_{\widehat{\lambda}}^{out} \imath^\mu T^\mu(w_0^\mu x w_0 w_0^{\widehat{\lambda}} \cdot \lambda)^\bullet [l(w_0^\mu)].$$

The equivalence thus follows from Corollary 4.11.

The isomorphism of the endomorphism algebras then follows from the fact that both modules correspond to complexes contained in one position exchanged by an equivalence of triangulated categories. $\qquad\square$

The proof of Proposition 8.7 then follows easily from this corollary.

8.3. **Parabolic and singular Koszul-Ringel duality.** In this subsection we compose the Koszul duality (4) with the Ringel duality. We also study the link with the category of linear complexes of q.h. tilting modules $\mathfrak{LT}_\lambda^\mu$, a full subcategory of $\mathcal{D}^b(^{\mathbb{Z}}\mathcal{O}_\lambda^\mu)$, see [MO]. By [MO, Corollary 6], the category $\mathfrak{LT}_\lambda^\mu$ is equivalent to gmod-$R(A_\lambda^\mu)^!$. Hence Theorem 8.1 and equation (6) implies that there are equivalences of categories

$$\mathfrak{LP}_\lambda^{\widehat{\mu}} \cong {}^{\mathbb{Z}}\mathcal{O}_\mu^\lambda \cong \mathfrak{LT}_\lambda^\mu \cong {}^{\mathbb{Z}}\mathcal{O}_{\widehat{\mu}}^{\widehat{\lambda}} \cong \mathfrak{LP}_{\widehat{\lambda}}^\mu.$$

We now give explicit descriptions of these equivalences, which will be useful for practical application in e.g. [CM2].

Lemma 8.10. *Consider integral dominant* λ, μ.

(i) *The functor* $\mathcal{L}\overline{\mathcal{R}}_\lambda^\mu$ *induces an equivalence of categories*

$$\mathcal{L}\overline{\mathcal{R}}_\lambda^\mu : \ \mathcal{D}^b(^{\mathbb{Z}}\mathcal{O}_\lambda^\mu) \ \xrightarrow{\sim} \ \mathcal{D}^b(^{\mathbb{Z}}\mathcal{O}_{\widehat{\lambda}}^\mu),$$

which restricts to an equivalence of categories

$$\mathfrak{LP}_\lambda^\mu \ \xrightarrow{\sim} \ \mathfrak{LT}_{\widehat{\lambda}}^\mu \quad with \quad P^\mu(x \cdot \lambda)^\bullet \ \mapsto \ T^\mu(w_0^\mu x w_0^\lambda w_0 \cdot \widehat{\lambda})^\bullet.$$

(ii) *The functor* $\mathcal{L}\underline{\mathcal{R}}_\lambda^\mu$ *induces an equivalence of categories*

$$\mathcal{L}\underline{\mathcal{R}}_\lambda^\mu : \ \mathcal{D}^b(^{\mathbb{Z}}\mathcal{O}_\lambda^\mu) \ \xrightarrow{\sim} \ \mathcal{D}^b(^{\mathbb{Z}}\mathcal{O}_\lambda^{\widehat{\mu}}),$$

which restricts to an equivalence of categories

$$\mathfrak{LP}_\lambda^\mu \ \xrightarrow{\sim} \ \mathfrak{LT}_\lambda^{\widehat{\mu}} \quad with \quad P^\mu(x \cdot \lambda)^\bullet \ \mapsto \ T^{\widehat{\mu}}(w_0 w_0^\mu x w_0^\lambda \cdot \lambda)^\bullet.$$

Proof. The equivalences of derived categories (in the ungraded sense) follow from [MS3, Proposition 2.2(2)] and Theorem 8.1. The graded version then follows from application of Proposition 3.5. Alternatively, the (graded) equivalence can be proved as in Section 6 by using Proposition 5.3(ii) rather than Corollary 5.2. It is a general feature that the Ringel duality functor maps indecomposable projective modules to indecomposable q.h. tilting modules, see e.g. [MS3, Proposition 2.1(2)]. The explicit description then follows from the action on standard modules in Theorem 8.1. $\qquad\square$

We compose functors to obtain the contravariant Koszul-Ringel duality functor

$$\Phi^\mu_\lambda = \mathcal{K}^{\widehat{\mu}}_\lambda \circ \left(\mathcal{L}\underline{\mathcal{R}}^{\widehat{\mu}}_\lambda \right)^{-1} \; : \; \mathcal{D}^b(^{\mathbb{Z}}\mathcal{O}^\mu_\lambda) \; \tilde{\to} \; \mathcal{D}^b(^{\mathbb{Z}}\mathcal{O}^\lambda_\mu),$$

yielding an equivalence of triangulated categories.

Corollary 8.11. *The contravariant functor Φ^μ_λ above restricts to an equivalence $\Phi^\mu_\lambda : \mathfrak{L}\mathfrak{T}^\mu_\lambda \to {}^{\mathbb{Z}}\mathcal{O}^\lambda_\mu$ which satisfies, for any $x \in X^\mu_\lambda$, the following:*

(1) $\Phi^\mu_\lambda(T^\mu(x \cdot \lambda)^\bullet) \cong L(w^\lambda_0 x^{-1} w^\mu_0 \cdot \mu)$.

(2) The complex $\mathcal{T}^\bullet_L$ in $\mathfrak{L}\mathfrak{T}^\mu_\lambda$, which is isomorphic to $L(x \cdot \lambda)^\bullet$ in $\mathcal{D}^b(^{\mathbb{Z}}\mathcal{O}^\mu_\lambda)$, satisfies $\Phi^\mu_\lambda(\mathcal{T}^\bullet_L) \cong T^\lambda(w^\lambda_0 x^{-1} w^\mu_0 \cdot \mu)$.

(3) The complex $\mathcal{T}^\bullet_\nabla$ in $\mathfrak{L}\mathfrak{T}^\mu_\lambda$, isomorphic to $\nabla^\mu(x \cdot \lambda)^\bullet$ in $\mathcal{D}^b(^{\mathbb{Z}}\mathcal{O}^\mu_\lambda)$, satisfies $\Phi^\mu_\lambda(\mathcal{T}^\bullet_\nabla) \cong \Delta^\lambda(w^\lambda_0 x^{-1} w^\mu_0 \cdot \mu)$.

(4) The complex $\mathcal{T}^\bullet_\Delta$ in $\mathfrak{L}\mathfrak{T}^\mu_\lambda$, isomorphic to $\Delta^\mu(x \cdot \lambda)^\bullet$ in $\mathcal{D}^b(^{\mathbb{Z}}\mathcal{O}^\mu_\lambda)$, satisfies $\Phi^\mu_\lambda(\mathcal{T}^\bullet_\Delta) \cong \nabla^\lambda(w^\lambda_0 x^{-1} w^\mu_0 \cdot \mu)$.

Proof. The first property follows from combination of Lemma 8.10(ii) and the well-known property $\mathcal{K}^{\widehat{\mu}}_\lambda(P(x \cdot \lambda)^\bullet) \cong L(x^{-1} w_0 \cdot \mu)^\bullet$, see e.g. [BGS, Theorem 3.11.1]. The other three properties then follow from the first one and [MO, Theorem 9]. Property (3) also follows from the combination of Theorem 8.1(ii) with the property $\mathcal{K}^\mu_\lambda(\Delta^\mu(x \cdot \lambda)^\bullet) \cong \Delta^\lambda(x^{-1} w_0 \cdot \widehat{\mu})^\bullet$, see [BGS, Theorem 3.11.1]. $\qquad\square$

The following is a standard property for categories of linear complexes.

Lemma 8.12. *Consider $\mathcal{M}^\bullet \in \mathrm{Ob}\mathfrak{L}\mathfrak{T}^\mu_\lambda$. Then $T^\mu(x \cdot \lambda)$ is a submodule of $\mathcal{M}^k$ if and only if $L(w^\lambda_0 x^{-1} w^\mu_0 \cdot \mu)\langle k \rangle$ is a subquotient of $\Phi^\lambda_\mu(\mathcal{M}^\bullet)$.*
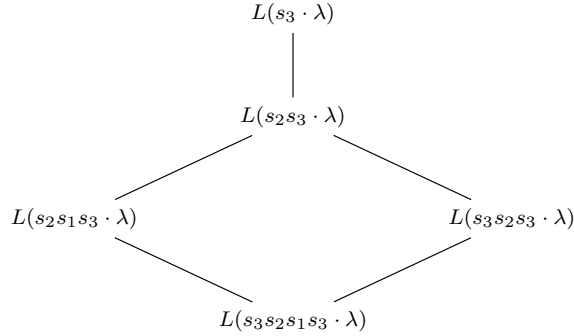
Proof. The property is clearly true for complexes contained in one position as, by construction,

$$\Phi^\mu_\lambda(T^\mu(x \cdot \lambda)^\bullet[k]\langle k \rangle) \cong L(w^\lambda_0 x^{-1} w^\mu_0 \cdot \mu)\langle k \rangle.$$
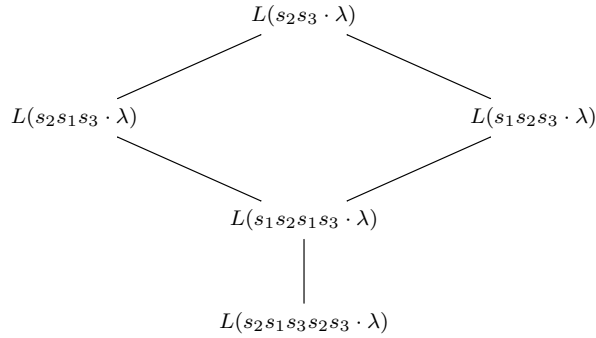
The full statement then follows by induction, using distinguished triangles in $\mathcal{D}^b(^{\mathbb{Z}}\mathcal{O}^\mu_\lambda)$ such that the cone belongs to $\mathfrak{L}\mathfrak{T}^\mu_\lambda$. $\qquad\square$

8.4. **A block which is not Ringel self-dual.** Consider $\mathfrak{g} = \mathfrak{sl}(4)$, with $W = S_4$ with simple reflections s_1, s_2, s_3 and choose $\lambda, \mu \in \Lambda^+_{\mathrm{int}}$ such that $w^\lambda_0 = s_3$ and $w^\mu_0 = s_1$. The Ext^1-quiver of $\mathcal{O}^\mu_\lambda$ and $\mathcal{O}^{\widehat{\mu}}_\lambda$ can be obtained from Proposition 2.2 and [St2, Appendix A]. As we have a duality preserving isomorphism classes of simple modules, we neglect the directions on the edges. We order the simple modules

according to the Bruhat order. The Ext^1-quiver of $\mathcal{O}_\lambda^\mu$ is given by

$$L(s_3 \cdot \lambda)$$
$$|$$
$$L(s_2 s_3 \cdot \lambda)$$
$$L(s_2 s_1 s_3 \cdot \lambda) \qquad\qquad L(s_3 s_2 s_3 \cdot \lambda)$$
$$L(s_3 s_2 s_1 s_3 \cdot \lambda)$$

We have $w_0^{\widehat{\mu}} = s_3$ and similarly we obtain the Ext^1-quiver of $\mathcal{O}_\lambda^{\widehat{\mu}}$,

$$L(s_2 s_3 \cdot \lambda)$$
$$L(s_2 s_1 s_3 \cdot \lambda) \qquad\qquad L(s_1 s_2 s_3 \cdot \lambda)$$
$$L(s_1 s_2 s_1 s_3 \cdot \lambda)$$
$$|$$
$$L(s_2 s_1 s_3 s_2 s_3 \cdot \lambda)$$

This implies that $\mathcal{O}_\lambda^\mu$ contains precisely one simple module which has a first extension with only one other simple module. Its projective cover is a non-simple standard module, so is not injective. Also $\mathcal{O}_\lambda^{\widehat{\mu}}$ has precisely one simple module which has a first extension with only one other simple module. This module is a simple standard module, which thus has an injective projective cover by [Ir]. This implies that $\mathcal{O}_\lambda^\mu$ is not equivalent to $\mathcal{O}_\lambda^{\widehat{\mu}}$.

REFERENCES

[ADL] I. Ágoston, V. Dlab, E. Lukács. Quasi-hereditary extension algebras. Algebr. Represent. Theory **6** (2003), no. 1, 97–117.

[AS] H. Andersen, C. Stroppel. Twisting functors on $\mathcal{O}$. Represent. Theory **7** (2003), 681–699.

[Ba1] E. Backelin. Koszul duality for parabolic and singular category O. Represent. Theory **3** (1999), 139–152.

[Ba2] E. Backelin. The Hom-spaces between projective functors. Represent. Theory **5** (2001), 267–283

[BGS] A. Beilinson, V. Ginzburg, W. Soergel. Koszul duality patterns in representation theory. J. Amer. Math. Soc. **9** (1996), no. 2, 473–527.

[BGe] I. Bernštein, S. Gel'fand. Tensor products of finite- and infinite-dimensional representations of semisimple Lie algebras. Compositio Math. **41** (1980), no. 2, 245–285.

[BGG] I. Bernštein, I. Gel'fand, S. Gel'fand. A certain category of $\mathfrak{g}$-modules. Funkcional. Anal. i Priložen. **10** (1976), no. 2, 1–8.

[Br]   J. Brundan. Symmetric functions, parabolic category O, and the Springer fiber. Duke Math.
         J. **143** (2008), no. 1, 41–79.
[Ca]   K. Carlin. Extensions of Verma modules. Trans. Amer. Math. Soc. **294** (1986), no. 1, 29–43.
[CR]   J. Chuang, R. Rouquier. Derived equivalences for symmetric groups and $\mathfrak{sl}_2$-categorification.
         Ann. of Math. (2) **167** (2008), no. 1, 245–298.
[Ch]   L. Chen. Poincaré polynomials of hyperquot schemes. Math. Ann. **321** (2001), no. 2, 235–
         251.
[Co]   K. Coulembier. Bott-Borel-Weil theory, BGG reciprocity and twisting functors for Lie su-
         peralgebras. arXiv:1404.1416.
[CM1]   K. Coulembier, V. Mazorchuk. Twisting functors, primitive ideals and star actions for
         classical Lie superalgebras. J. Reine Ang. Math., doi=10.1515/crelle-2014-0079.
[CM2]   K. Coulembier, V. Mazorchuk. Some homological properties of the category $\mathcal{O}$. IV.
         arXiv:1509.04391.
[dS]   J.C. de Sousa Oliveira Santos. Foncteurs de Zuckerman pour les superalgèbres de Lie., J.
         Lie Theory **9** (1999), 69–112.
[GGOR]   V. Ginzburg, N. Guay, E. Opdam, R. Rouquier. On the category $\mathcal{O}$ for rational Cherednik
         algebras. Invent. Math. **154** (2003), no. 3, 617–651.
[Ha]   D. Happel. Triangulated categories in the representation theory of finite-dimensional alge-
         bras. London Mathematical Society Lecture Note Series, **119**. Cambridge University Press,
         Cambridge, 1988.
[Hu]   J. Humphreys. Representations of semisimple Lie algebras in the BGG category O. Grad-
         uate Studies in Mathematics, **94**. American Mathematical Society, Providence, RI (2008).
[Ir]   R. Irving. Projective modules in the category $\mathcal{O}_S$: self-duality. Trans. Amer. Math. Soc.
         291 (1985), no. 2, 701–732.
[Ja]   J.C. Jantzen. Einhüllende Algebren halbeinfacher Lie-Algebren. Ergebnisse der Mathematik
         und ihrer Grenzgebiete, 3. Springer-Verlag, Berlin, 1983.
[KM]   O. Khomenko, V. Mazorchuk. On Arkhipov's and Enright's functors. Math. Z. **249** (2005),
         no. 2, 357–386.
[Kh]   M. Khovanov. Crossingless matchings and the cohomology of (n, n) Springer varieties. Com-
         mun. Contemp. Math. **6** (2004), no. 4, 561–577.
[Le]   J. Lepowsky. A generalization of the Bernstein-Gelfand-Gelfand resolution. J. Algebra **49**
         (1977), no. 2, 496–511.
[MSa]   R. Martinez Villa, M. Saorin. Koszul equivalences and dualities. Pacific J. Math. **214** (2004),
         no. 2, 359-378.
[Mat]   O. Mathieu. Classification of irreducible weight modules. Ann. Inst. Fourier (Grenoble) **50**
         (2000), no. 2, 537–592.
[Ma1]   V. Mazorchuk. Some homological properties of the category O. Pacific J. Math. **232** (2007),
         no. 2, 313–341.
[Ma2]   V. Mazorchuk. Applications of the category of linear complexes of tilting modules associated
         with the category $\mathcal{O}$. Algebr. Repres. Theory **12** (2009), no. 6, 489–512.
[Ma3]   V. Mazorchuk. Some homological properties of the category O. II. Represent. Theory **14**
         (2010), 249–263.
[MO]   V. Mazorchuk, S. Ovsienko. A pairing in homology and the category of linear complexes of
         tilting modules for a quasi-hereditary algebra. With an appendix by Catharina Stroppel.
         J. Math. Kyoto Univ. **45** (2005), no. 4, 711–741.
[MOS]   V. Mazorchuk, S. Ovsienko, C. Stroppel. Quadratic duals, Koszul dual functors, and appli-
         cations. Trans. Amer. Math. Soc. **361** (2009), no. 3, 1129–1172.
[MS1]   V. Mazorchuk, C. Stroppel. Translation and shuffling of projectively presentable modules
         and a categorification of a parabolic Hecke module. Trans. Amer. Math. Soc. **357** (2005),
         no. 7, 2939–2973.
[MS2]   V. Mazorchuk, C. Stroppel. On functors associated to a simple root. J. Algebra **314** (2007),
         no. 1, 97–128.
[MS3]   V. Mazorchuk, C. Stroppel. Projective-injective modules, Serre functors and symmetric
         algebras. J. Reine Angew. Math. **616** (2008), 131–165.
[Ric1]   J. Rickard. Morita theory for derived categories. J. London Math. Soc. (2) **39** (1989), no.
         3, 436–456.

[Ric2]   J. Rickard. Translation functors and equivalences of derived categories for blocks of algebraic groups. In: Finite-dimensional algebras and related topics (Ottawa, ON, 1992), 255–264, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., **424**, Kluwer Acad. Publ., Dordrecht, 1994.

[Rin]   C.M. Ringel. The category of modules with good filtrations over a quasi-hereditary algebra has almost split sequences. Math. Z. **208** (1991), no. 2, 209–223.

[RC]   A. Rocha-Caridi. Splitting criteria for g-modules induced from a parabolic and the Berňstein-Gel'fand-Gel'fand resolution of a finite-dimensional, irreducible $\mathfrak{g}$-module. Trans. Amer. Math. Soc. **262** (1980), no. 2, 335–366.

[Ry]   S. Ryom-Hansen. Koszul duality of translation- and Zuckerman functors. J. Lie Theory **14** (2004), no. 1, 151–163.

[So1]   W. Soergel. Kategorie O, perverse Garben und Moduln über den Koinvarianten zur Weylgruppe. J. Amer. Math. Soc. **3** (1990), no. 2, 421–445.

[So2]   W. Soergel. Charakterformeln für Kipp-Moduln über Kac-Moody-Algebren. Represent. Theory **1** (1997), 115–132.

[St1]   C. Stroppel. Category O: gradings and translation functors. J. Algebra **268** (2003), no. 1, 301–326.

[St2]   C. Stroppel. Category O: quivers and endomorphism rings of projectives. Represent. Theory **7** (2003), 322–345.

[St3]   C. Stroppel. Parabolic category O, perverse sheaves on Grassmannians, Springer fibres and Khovanov homology. Compos. Math. **145** (2009), no. 4, 954–992.

[We]   B. Webster. Knot invariants and higher representation theory. To appear in Memoirs of the AMS. arXiv:1309.3796v3.

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