On the Construction of Associative, Commutative and Increasing Operations by Paving

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Abstract. Bodjanova, Kalina and Král' recently introduced a construction method, called paving, which enables to define a new associative, commutative and increasing operation from a given one and a discrete representable partial operation. As a matter of fact, not every discrete t-norm is representable, i.e. it can not always be generated by some additive generator, and this also holds for t-conorms and uninorms. Inspired by this fact and the method of paving, we construct some new associative, commutative and increasing operations on the unit interval from a t-norm on the unit interval and a discrete t-norm, t-superconorm, t-conorm or uninorm. Because of the duality between t-norms and t-conorms, we also define some operations from a t-conorm and a discrete t-norm, t-subnorm, t-conorm or uninorm.

Keywords: Associative operations · Uninorms · T-norms · T-conorms · Paving

1 Introduction

The associativity models the independence of the aggregation on the grouping of input values and it allows to investigate binary aggregation operators only (as far as their n-ary extensions are then determined uniquely). It is needless to emphasize the key role of associative operations (t-norms, t-conorms, uninorms, nullnorms, etc.) not only in fuzzy set theory, but also in many areas of application, especially in decision-making under uncertainty [5], image processing [1,6], fuzzy neural networks [7] and so on. The most important classes of associative, commutative, increasing operations in the framework of fuzzy sets is that of uninorms ([4,5,18]), which includes t-norms [10,17] and t-conorms [10] as two special classes. A large number of methods to construct uninorms (including t-norms and t-conorms) are introduced: Klement et al. [10], Schweizer and Sklar [17], Jenei [8], Ling [13], Maes and De Baets [11], Mas et al. [12], Mesiarová-Zemánková [14–16] and so on.

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Kalina et al. [2,9] introduced a construction method called paving. The main idea is as follows: the unit interval is split into countably many disjoint sub-intervals $(I_i)_{i \in J_n}$ with J_n an index-set and with the help of an appropriate operation $*'$ on J_n and a family of increasing transformations $\varphi_i : I_i \to [0,1]$, a new operation $\oplus$ is defined by

$$x \oplus y = \varphi_{i*'j}^{-1}(\varphi_i(x) * \varphi_j(y)), \quad x \in I_i,\ y \in I_j. \tag{1}$$

Unfortunately, Kalina et al. only consider discrete representable associative operations as operation $*'$, which is rather restrictive. For instance, not every discrete t-norm can be generated by some additive generator, and this applies to t-conorms and uninorms. Moreover, the operation $*'$ in [2] is not always internal on J_n. In this paper, we will consider a general discrete associative operation as operation $*'$ on J_n, to construct some new associative, commutative and increasing operations. The graphical schema of paving is depicted in Fig. 1

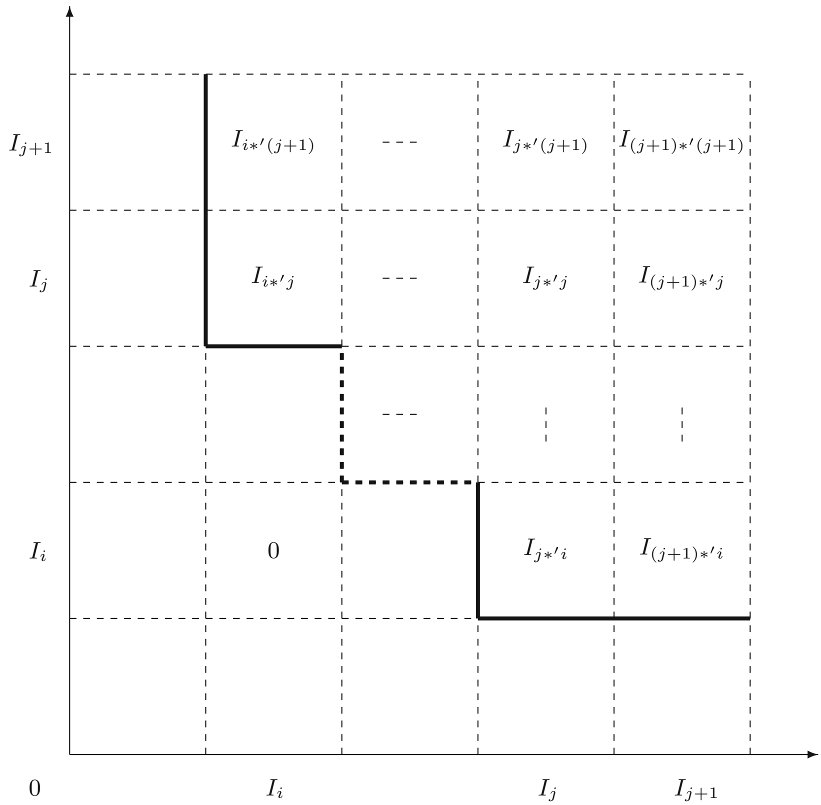

Fig. 1. The structure of $\oplus$, where the thick line is the boundary between $\{(i,j) \mid i *' j = 0\}$ and $\{(i,j) \mid i *' j > 0)\}$. Inside the blocks it is shown in which sub-interval the operation $\oplus$ takes its values.

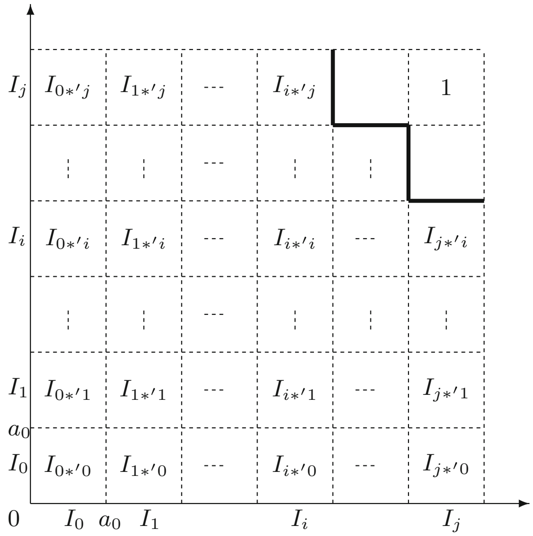

Fig. 2. The structure of $\oplus$, where the thick line is the boundary between $\{(i,j) \mid i *' j = n\}$ and $\{(i,j) \mid i *' j < n)\}$. Inside the blocks it is shown in which sub-interval the operation $\oplus$ takes its values.

(which depicts the construction of a conjunctive operation $\oplus$) and Fig. 2 (which depicts the construction of a disjunctive operation $\oplus$).

The paper is organized as follows. In Sect. 2, we present some preliminary notions and results that are necessary for the rest of the paper. Starting from Eq. (1), when $*$ is a t-norm and $*'$ is a discrete t-norm, t-superconorm, t-conorm or uninorm, we construct some new associative, commutative and increasing operations in Sect. 3. At the same time, all the dual constructions when $*$ is a t-conorm are also listed in Sect. 3.

2 Preliminaries

In this section we recall some basic notions and facts that are necessary for the understanding of what follows.

Definition 1 [10]**.** *A decreasing function $N : [0,1] \to [0,1]$ is called a fuzzy negation if $N(0) = 1$ and $N(1) = 0$. Moreover, a fuzzy negation N is called strong if it is involutive, i.e., if $N(N(x)) = x$ for all $x \in [0,1]$.*

Definition 2 [18]**.** *A binary operation $U : [0,1]^2 \to [0,1]$ is called a uninorm if it is associative, commutative, increasing and has a neutral element $e \in [0,1]$, i.e., $U(x,e) = x$ for all $x \in [0,1]$.*

A uninorm with neutral element $e = 1$ is a *t-norm* [10,17] and a uninorm with neutral element $e = 0$ is a *t-conorm* [10]. We say that a uninorm U is *proper* if $e \in]0,1[$. If $U(1,0) = 0$, then U is called *conjunctive*. If $U(1,0) = 1$, then U is called *disjunctive*. Conjunctive and disjunctive uninorms are dual to each other. For an arbitrary disjunctive uninorm U and a strong negation N, its N-dual conjunctive uninorm is given by

$$U_N^d(x,y) = N(U(N(x), N(y))). \tag{2}$$

For an overview of basic properties of uninorms, we refer to [3].

Remark 1. Note that, for a strong negation N, the N-dual operation to a t-norm T defined by $S(x,y) = N(T(N(x), N(y)))$ is a t-conorm. For more information, see, e.g., [10].

Definition 3 [8]**.** *(i) A binary operation $\widetilde{T} : [0,1]^2 \to [0,1]$ is called a triangular subnorm (t-subnorm, for short), if it is associative, commutative, increasing and fulfills the condition $\widetilde{T}(x,y) \le \min(x,y)$ for all $(x,y) \in [0,1]^2$.*

(ii) A binary operation $\widetilde{S} : [0,1]^2 \to [0,1]$ is called a triangular superconorm (t-superconorm, for short), if it is associative, commutative, increasing and fulfills the condition $\widetilde{S}(x,y) \ge \max(x,y)$ for all $(x,y) \in [0,1]^2$.

Definition 4. *Let $* : [0,1]^2 \to [0,1]$ be a commutative operation. Fix a value $a \in [0,1]$. We say that $x \in [0,1]$, $x \neq a$, is an a-divisor if there exists $y \in [0,1]$, $y \neq a$, such that*

$$x * y = a. \tag{3}$$

3 Construction of New Operations

The main idea of our construction method is described in Eq. (1) with the help of a discrete associative operation $*'$. For the rest of this paper, we adopt the following notations.

Let $\mathbb{N}$ be the set of all positive integers. We consider an index-set

$$J_n = \{0, 1, 2, \ldots, n\}$$

for some $n \in \mathbb{N}$.

We will split the interval $[0,1]$ into $n+1$ sub-intervals by choosing the end-points of the system of sub-intervals

$$0 = a_{-1} < a_0 < a_1 < a_2 < \ldots < a_{n-1} < a_n = 1.$$

Because of this partition, we will use half-open intervals, i.e., either left-open or right-open. We will use indexing of the chosen sub-intervals in accordance with the right end-point. For the case of left-open sub-intervals, $I_i =]a_{i-1}, a_i]$; for the case of right-open sub-intervals, $I_i = [a_{i-1}, a_i[$.

For a fixed system of right-open sub-intervals $(I_i)_{i=0}^n$, $\varphi_i : I_i \to [0,1[$ are increasing bijections. For a fixed system of left-open sub-intervals $(I_i)_{i=0}^n$, $\chi_i : I_i \to]0,1]$ are increasing bijections.

Remark 2 [2]. In order not to get out of the range of the transformations χ_i when using left-open sub-intervals, the starting operation $*$ (the basic paving stone) must be without zero-divisors. Similarly, when using right-open sub-intervals, $*$ must be without one-divisors.

Here, we consider to construct new associative, commutative and increasing operations from a given one $*$, and two certain cases of associative, commutative and increasing operations will be taken into account: the case that $*$ is a t-norm and the case that $*$ is a t-conorm.

3.1 The Case that $*$ Is a T-Norm

In this subsection, we construct some new associative, commutative and increasing operations on the unit interval from a t-norm on the unit interval and a discrete t-norm/t-superconorm/t-conorm/uninorm.

Firstly, we construct a new operation $\oplus$ from a t-norm $*$ and a discrete t-norm $*'$ in Eq. (1). Because of the partition of unit interval, we distinguish two cases: when right-open sub-intervals of $[0,1[$ and left-open sub-intervals of $]0,1]$.

Proposition 1. *Let $* : [0,1]^2 \to [0,1]$ be a t-norm, $(I_i)_{i=0}^n$ be a partition of $[0,1[$ consisting of right-open sub-intervals. Assume that $*'$ is a discrete t-norm on $J_n = \{0,\ldots,n\}$ such that $*'$ is strictly increasing on the domain $\{(i,j) \mid i,j \in J_n,\ i *' j > 0\}$. Then the operation $\oplus_1$ defined by*

$$x \oplus_1 y = \begin{cases} \varphi^{-1}_{i*'j}(\varphi_i(x) * \varphi_j(y)), & \text{if } x \in I_i,\ y \in I_j \text{ and } i *' j > 0, \\ \min(x,y), & \text{if } \max(x,y) = 1, \\ 0, & \text{otherwise,} \end{cases} \tag{4}$$

is a t-norm.

In fact, $\oplus_1$ is not always increasing without the condition that $*'$ is strictly increasing on the domain $\{(i,j) \mid i,j \in J,\ i *' j > 0\}$.

Example 1. Assume that $J_7 = \{0,1,2,\ldots,7\}$, $(I_i = [i/8,(i+1)/8[)_{i=0}^7$ is a partition of $[0,1[$. Let $*$ be the t-norm $T_M(x,y) = \min(x,y)$ on $[0,1]$, $*'$ be the discrete t-norm $T_M(i,j) = \min(i,j)$ on J_7, $\varphi_i(x) = \frac{x-a_{i-1}}{a_i-a_{i-1}}$. Define $x \oplus y$ as follows:

$$x \oplus y = \begin{cases} \varphi^{-1}_{\min(i,j)}(\min(\frac{x-a_{i-1}}{a_i-a_{i-1}}, \frac{y-a_{j-1}}{a_j-a_{j-1}})), & \text{if } x \in I_i,\ y \in I_j, \text{ and } \min(i,j) > 0, \\ \min(x,y), & \text{if } \max(x,y) = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Consider that $x = \frac{3}{16}$, $y = \frac{3}{16}$ and $z = \frac{1}{4}$, then we have that

$$x \oplus y = \varphi_1^{-1}(\frac{1}{2}) = \frac{3}{16} > \frac{1}{8} = \varphi_1^{-1}(0) = x \oplus z. \tag{5}$$

That is, $\oplus$ is not increasing.

By (4), we can see that for any t-norm $*$, its values on the upper right boundary of the unit square $[0,1]^2$ have no impact on the properties of $\oplus_1$. Moreover, It is obvious that associativity, commutativity and monotonicity of $\oplus_1$ are determined by the corresponding properties of $*$, respectively. Thus, we can easily obtain that Proposition 1 holds for t-subnorm instead of t-norm.

Example 2. Assume that $J_n = \{0, 1, 2, \ldots, n\}$, $(I_i)_{i=0}^n$ is a partition of $[0, 1[$ consisting of right-open sub-intervals. Let $*$ be the t-subnorm $\widetilde{T} = \max(\min(x, \frac{1}{2}) + \min(y, \frac{1}{2}) - \frac{3}{4}, 0)$ on $[0, 1]$, $*$ be the discrete t-norm $T_L(i,j) = \max(0, i+j-n)$ on J_n, $\varphi_i(x) = \frac{x-a_{i-1}}{a_i-a_{i-1}}$. Define $x \oplus y$ as follows:

$$x \oplus y = \begin{cases} \varphi^{-1}_{i+j-n}(\widetilde{T}(\frac{x-a_{i-1}}{a_i-a_{i-1}}, \frac{y-a_{j-1}}{a_j-a_{j-1}})), & \text{if } x \in I_i,\ y \in I_j \text{ and } i+j > n, \\ \min(x, y), & \text{if } \max(x, y) = 1, \\ 0, & \text{otherwise}, \end{cases} \tag{6}$$

is a t-norm.

As stated earlier, $*$ must be a t-norm without zero-divisors when left-open sub-intervals are taken into account. Similar to Proposition 1, the following proposition can be obtained:

Proposition 2. *Let $* : [0, 1]^2 \to [0, 1]$ be a t-norm without zero-divisors, $(I_i)_{i=0}^n$ be a partition of $]0, 1]$ consisting of left-open sub-intervals. Assume that $*'$ is a discrete t-norm on J_n such that $*'$ is strictly increasing on the domain $\{(i,j) \mid i, j \in J_n,\ i *' j > 0\}$. Then the operation $\oplus_2$ defined by*

$$x \oplus_2 y = \begin{cases} \min(x, y), & \text{if } \max(x, y) = 1, \\ \chi^{-1}_{i*'j}(\chi_i(x) * \chi_j(y)), & \text{if } x \in I_i \setminus \{1\},\ y \in I_j \setminus \{1\} \text{ and } i *' j > 0, \\ 0, & \text{otherwise}, \end{cases} \tag{7}$$

is a t-norm.

Next, we discuss the construction when $*$ is a t-norm and $*'$ is a discrete t-superconorm. Analogously, two cases of right-open sub-intervals of $[0, 1[$ and left-open sub-intervals of $]0, 1]$ are taken into account. We start with the case of the right-open sub-intervals.

Proposition 3. *Let $* : [0, 1]^2 \to [0, 1]$ be a t-norm, $(I_i)_{i=0}^n$ be a partition of $[0, 1[$ consisting of right-open sub-intervals. Assume that $*'$ is a discrete t-superconorm on J_n such that $*'$ is strictly increasing and $i *' j > \max(i, j)$ on the domain $\{(i,j) \mid i, j \in J_n,\ i *' j < n\}$. Then the operation $\oplus_3$ defined by*

$$x \oplus_3 y = \begin{cases} \varphi^{-1}_{i*'j}(\varphi_i(x) * \varphi_j(y)), & \text{if } x \in I_i,\ y \in I_j \text{ and } i *' j < n, \\ 1, & \text{otherwise}, \end{cases} \tag{8}$$

is a t-superconorm.

Without the condition that $*'$ is strictly increasing on the domain $\{(i,j) \mid i, j \in J_n,\ i *' j < n\}$, $\oplus_3$ is not always increasing. We have the following counterexample.

Example 3. Assume that $J_7 = \{0, 1, 2, \ldots, 7\}$, $(I_i = [i/8, (i+1)/8[)_{i=0}^{7}$ is a partition of $[0, 1[$. Let $*$ be the t-norm $T_M(x, y) = \min(x, y)$ on $[0, 1]$, $*'$ be the discrete t-superconorm $\widetilde{S} = \min(n, \max(i, j) + 4)$ on J_7, $\varphi_i(x) = \frac{x - a_{i-1}}{a_i - a_{i-1}}$. Define $x \oplus y$ as follows:

$$x \oplus y = \begin{cases} \varphi^{-1}_{\widetilde{S}(i,j)}(\min(\frac{x-a_{i-1}}{a_i-a_{i-1}}, \frac{y-a_{j-1}}{a_j-a_{j-1}})), & \text{if } x \in I_i,\ y \in I_j \text{ and } i *' j < n, \\ 1, & \text{otherwise.} \end{cases}$$

Consider that $x = \frac{1}{16}$, $y = \frac{1}{8}$ and $z = \frac{3}{16}$, then we have that

$$x \oplus z = \varphi_5^{-1}(\frac{1}{2}) = \frac{11}{16} > \frac{5}{8} = \varphi_5^{-1}(0) = y \oplus z. \tag{9}$$

Obviously, $\oplus$ is not increasing.

In Eq. (8), let $x \oplus_3 y = \max(x, y)$ on the domain $\{(x, y) \mid x, y \in [0, 1], \min(x, y) = 0\}$. We can easily prove that the operation $\oplus_3$ is a t-conorm by simple calculations.

Similarly, when left-open sub-intervals are taken into account, $*$ must be a t-norm without zero-divisors. Then, the following proposition can be obtained:

Proposition 4. *Let* $* : [0, 1]^2 \to [0, 1]$ *be a t-norm without zero-divisors,* $(I_i)_{i=0}^{n}$ *be a partition of* $]0, 1]$ *consisting of left-open sub-intervals. Assume that* $*'$ *is a discrete t-superconorm on* J_n *such that* $*'$ *is strictly increasing and* $i *' j > \max(i, j)$ *on the domain* $\{(i, j) \mid i, j \in J_n,\ i *' j < n\}$. *Then the operation* $\oplus_4$ *defined by*

$$x \oplus_4 y = \begin{cases} \chi^{-1}_{i*'j}(\chi_i(x) * \chi_j(y)), & \text{if } x \in I_i,\ y \in I_j \text{ and } i *' j < n, \\ \max(x, y), & \text{if } \min(x, y) = 0, \\ 1, & \text{otherwise,} \end{cases} \tag{10}$$

is a t-conorm.

In what follows, we construct a new operation from a t-norm $*$ and a discrete uninorm $*'$.

Proposition 5. *Let* $* : [0, 1]^2 \to [0, 1]$ *be a t-norm,* $(I_i)_{i=0}^{n}$ *be a partition of* $[0, 1[$ *consisting of right-open sub-intervals. Assume that* $*'$ *is a discrete uninorm on* J_n *with neutral element* h *such that* $*'$ *is strictly increasing on the domain* $\{(i, j) \mid i, j \in J_n,\ \max(i, j) \le h, i *' j > 0\}$ *and* $\{(i, j) \mid i, j \in J_n,\ \min(i, j) \ge h, i *' j < n\}$. *Then the operation* $\oplus_5$ *defined by*

$$x \oplus_5 y = \begin{cases} a_i, & \text{if } \min(x, y) < a_h \text{ and } a_h \le a_i \le \max(x, y) < a_{i+1}, \\ \varphi^{-1}_{i*'j}(\varphi_i(x) * \varphi_j(y)), & \text{if } x \in I_i, y \in I_j, \max(i, j) \le h \text{ and } i *' j > 0, \\ & \quad \text{or } h < \min(i, j) \text{ and } i *' j < n, \\ 1, & \text{if } x \in I_i, y \in I_j,\ h < \min(i, j) \text{ and } i *' j = n, \\ & \quad \text{or } \max(x, y) = 1, \\ 0, & \text{otherwise,} \end{cases} \tag{11}$$

is associative, commutative and increasing.

In fact, the similar proposition does not hold when $(I_i)_{i=0}^n$ is a partition of $]0,1]$ consisting of left-open sub-intervals. A counterexample is as follows:

Example 4. Assume that $J_4 = \{0,1,2,3,4\}$, $(I_i =]i/5,(i+1)/5])_{i=0}^4$ is a partition of $]0,1]$. Let $*$ be the t-norm $T_M(x,y) = \min(x,y)$ on $[0,1]$, $*'$ be the discrete uninorm U with neutral element 2:

$$U(i,j) = \begin{cases} T_L(i,j), & \text{if } 0 \le i,j \le 2, \\ S_L(i,j), & \text{if } 2 \le i,j \le 4, \\ \min(i,j), & \text{otherwise,} \end{cases}$$

where $T_L(i,j) = \max(0, i+j-2)$, $S_L(i,j) = \min(4, i+j-2)$.

Besides, $\varphi_i(x) = \frac{x-a_{i-1}}{a_i-a_{i-1}}$. Define $x \oplus y$ as follows:

$$x \oplus y = \begin{cases} a_{i+1}, & \text{if } \frac{3}{5} < \max(x,y) \text{ and } a_i < \min(x,y) \le a_{i+1} \le \frac{3}{5}, \\ \varphi^{-1}_{U(i,j)}(\varphi_i(x) * \varphi_j(y)), & \text{if } x \in I_i, y \in I_j, \max(i,j) \le 2 \text{ and } i *' j > 0, \\ & \text{ or } 2 < \min(i,j) \text{ and } i *' j < 4, \\ 0, & \text{if } x \in I_i, y \in I_j, \ \max(i,j) \le 2 \text{ and } i *' j = 0, \\ & \text{ or } \min(x,y) = 0, \\ 1, & \text{otherwise.} \end{cases}$$

Consider that $x = \frac{1}{2}$, $y = \frac{1}{2}$ and $z = \frac{4}{5}$, then we have that

$$(x \oplus y) \oplus z = a_{U(2,2)} = a_2 = \frac{3}{5} \ne \frac{1}{2} = x \oplus a_2 = x \oplus (y \oplus z). \tag{12}$$

Obviously, $\oplus$ is not associative.

When $*$ is a t-norm and $*'$ is a discrete t-conorm, we can construct some proper uninorms.

Proposition 6. *Let* $* : [0,1]^2 \to [0,1]$ *be a t-norm,* $(I_i)_{i=0}^n$ *be a partition of* $[0,1[$ *consisting of right-open sub-intervals. Assume that* $*'$ *is a discrete t-conorm on* J_n *such that* $*'$ *is strictly increasing on the domain* $\{(i,j) \mid i,j \in J_n,\ i *' j < n\}$. *Then the operation* $\oplus_6$ *defined by*

$$x \oplus_6 y = \begin{cases} \varphi^{-1}_{i*'j}(\varphi_i(x) * \varphi_j(y)), & \text{if } x \in I_i \setminus \{a_0\},\ y \in I_j \setminus \{a_0\} \text{ and } i *' j < n, \\ & \text{ or } \min(x,y) \in I_0,\ \max(x,y) \in I_n, \\ y, & \text{if } x = a_0, \\ x, & \text{if } y = a_0, \\ 1, & \text{otherwise,} \end{cases} \tag{13}$$

is a proper disjunctive uninorm with neutral element a_0 *if and only if* $*$ *has no zero-divisors.*

In what follows, we give an example to illustrate that $*'$ must be strictly increasing on the domain $\{(i,j) \mid i,j \in J_n,\ i *' j < n\}$.

Example 5. Assume that $J_4 = \{0,1,2,3,4\}$, $(I_i = [i/5,(i+1)/5[)_{i=0}^4$ is a partition of $[0,1[$. Let $*$ be the t-norm $T_M(x,y) = \min(x,y)$ on $[0,1]$, $*'$ be the discrete t-conorm $S_M = \max(i,j)$ on J_4, $\varphi_i(x) = \frac{x-a_{i-1}}{a_i-a_{i-1}}$. Define $x \oplus y$ as follows:

$$x \oplus y = \begin{cases} \varphi^{-1}_{\max(i,j)}(\min(\frac{x-a_{i-1}}{a_i-a_{i-1}}, \frac{y-a_{j-1}}{a_j-a_{j-1}})), & \text{if } x \in I_i \setminus \{\frac{1}{5}\},\ y \in I_j \setminus \{\frac{1}{5}\} \text{ and } \max(i,j) < 4, \\ & \text{or } \min(x,y) \in [0, \frac{1}{5}[,\ \max(x,y) \in [\frac{4}{5}, 1[, \\ y, & \text{if } x = \frac{1}{5}, \\ x, & \text{if } y = \frac{1}{5}, \\ 1, & \text{otherwise.} \end{cases}$$

Consider that $x = \frac{3}{10}$, $y = \frac{2}{5}$ and $z = \frac{1}{2}$, then we have that

$$x \oplus z = \varphi_2^{-1}\big(\frac{1}{2}\big) = \frac{1}{2} > \frac{2}{5} = \varphi_2^{-1}(0) = y \oplus z. \tag{14}$$

Obviously, $\oplus$ is not increasing.

Similar to Proposition 6, when the left-open sub-intervals are taken into account, we have the following result:

Proposition 7. *Let $* : [0,1]^2 \to [0,1]$ be a t-norm without zero-divisors, $(I_i)_{i=0}^n$ be a partition of $]0,1]$ consisting of left-open sub-intervals. Assume that $*'$ is a discrete t-conorm on J_n such that $*'$ is strictly increasing on the domain $\{(i,j) \mid i,j \in J_n,\ i *' j < n\}$. Then the operation $\oplus_7$ defined by*

$$x \oplus_7 y = \begin{cases} \chi^{-1}_{i*'j}(\chi_i(x) * \chi_j(y)), & \text{if } x \in I_i,\ y \in I_j \text{ and } i *' j < n, \\ & \text{or } \min(x,y) \in I_0,\ \max(x,y) \in I_n, \\ 0, & \text{if } \min(x,y) = 0, \\ 1, & \text{otherwise,} \end{cases} \tag{15}$$

is a proper conjunctive uninorm with neutral element a_0.

3.2 The Case that $*$ Is a T-Conorm

Taking into account the duality between t-norms and t-conorms, the results in the case that $*$ is a t-conorm are easily obtained.

Proposition 8. *Let $* : [0,1]^2 \to [0,1]$ be a t-conorm without one-divisors, $(I_i)_{i=0}^n$ be a partition of $[0,1[$ consisting of right-open sub-intervals. Assume that $*'$ is a discrete t-conorm on J_n such that $*'$ is strictly increasing on the domain $\{(i,j) \mid i,j \in J_n,\ i *' j < n\}$. Then the operation $\oplus^1$ defined by*

$$x \oplus^1 y = \begin{cases} \varphi^{-1}_{i*'j}(\varphi_i(x) * \varphi_j(y)), & \text{if } x \in I_i \setminus \{0\},\ y \in I_j \setminus \{0\} \text{ and } i *' j < n, \\ \max(x,y), & \text{if } \min(x,y) = 0, \\ 1, & \text{otherwise,} \end{cases}$$

is a t-conorm.

Similar to the case that $*$ is a t-norm, Proposition 8 holds for t-superconorm instead of t-conorm.

Proposition 9. *Let* $* : [0,1]^2 \to [0,1]$ *be a t-conorm,* $(I_i)_{i=0}^n$ *be a partition of* $]0,1]$ *consisting of left-open sub-intervals. Assume that* $*'$ *is a discrete t-conorm on* J_n *such that* $*'$ *is strictly increasing on the domain* $\{(i,j) \mid i,j \in J_n,\ i *' j < n\}$. *Then the operation* $\oplus^2$ *defined by*

$$x \oplus^2 y = \begin{cases} \max(x,y), & \text{if } \min(x,y) = 0, \\ \chi_{i*'j}^{-1}(\chi_i(x) * \chi_j(y)), & \text{if } x \in I_i,\ y \in I_j \text{ and } i *' j < n, \\ 1, & \text{otherwise,} \end{cases}$$

is a t-conorm.

Proposition 10. *Let* $* : [0,1]^2 \to [0,1]$ *be a t-conorm without one-divisors,* $(I_i)_{i=0}^n$ *be a partition of* $[0,1[$ *consisting of right-open sub-intervals. Assume that* $*'$ *is a discrete t-subnorm on* J_n *such that* $*'$ *is strict increasing and* $i *' j < \min(i,j)$ *on the domain* $\{(i,j) \mid i,j \in J_n,\ i *' j > 0\}$. *Then operation* $\oplus^3$ *defined by*

$$x \oplus^3 y = \begin{cases} \varphi_{i*'j}^{-1}(\varphi_i(x) * \varphi_j(y)), & \text{if } x \in I_i,\ y \in I_j \text{ and } i *' j > 0, \\ \min(x,y), & \text{if } \max(x,y) = 1, \\ 0, & \text{otherwise,} \end{cases}$$

is a t-norm.

Proposition 11. *Let* $* : [0,1]^2 \to [0,1]$ *be a t-conorm,* $(I_i)_{i=0}^n$ *be a partition of* $]0,1]$ *consisting of left-open sub-intervals. Assume that* $*'$ *is a discrete t-subnorm on* J_n *such that* $*'$ *is strictly increasing and* $i *' j < \min(i,j)$ *on the domain* $\{(i,j) \mid i,j \in J_n,\ i *' j > 0\}$. *Then the operation* $\oplus^4$ *defined by*

$$x \oplus^4 y = \begin{cases} \chi_{i*'j}^{-1}(\chi_i(x) * \chi_j(y)), & \text{if } x \in I_i,\ y \in I_j \text{ and } i *' j > 0, \\ 0, & \text{otherwise,} \end{cases}$$

is a t-subnorm.

Proposition 12. *Let* $* : [0,1]^2 \to [0,1]$ *be a t-conorm without one-divisors,* $(I_i)_{i=0}^n$ *be a partition of* $[0,1[$ *consisting of right-open sub-intervals. Assume that* $*'$ *is a discrete t-norm on* J_n *such that* $*'$ *is strictly increasing on the domain* $\{(i,j) \mid i,j \in J_n,\ i *' j > 0\}$. *Then the operation* $\oplus^5$ *defined by*

$$x \oplus^5 y = \begin{cases} \varphi_{i*'j}^{-1}(\varphi_i(x) * \varphi_j(y)), & \text{if } x \in I_i,\ y \in I_j \text{ and } i *' j > 0, \\ & \quad \text{or } \min(x,y) \in I_0,\ \max(x,y) \in I_n, \\ 1, & \text{if } \max(x,y) = 1, \\ 0, & \text{otherwise,} \end{cases}$$

is a proper disjunctive uninorm with neutral element a_{n-1}.

Proposition 13. *Let* $* : [0,1]^2 \to [0,1]$ *be a t-conorm,* $(I_i)_{i=0}^n$ *be a partition of* $]0,1]$ *consisting of left-open sub-intervals. Assume that* $*'$ *is a discrete uninorm on* J_n *with neutral element* h *such that* $*'$ *is strictly increasing on the domain* $\{(i,j) \mid i,j \in J_n,\ \max(i,j) \leq h, i *' j > 0\}$ *and* $\{(i,j) \mid i,j \in J_n,\ \min(i,j) \geq h, i *' j < n\}$*. Then the operation* $\oplus^6$ *defined by*

$$x \oplus^6 y = \begin{cases} a_{i+1}, & \text{if } \max(x,y) > a_{h-1} \text{ and } a_i < \min(x,y) \leq a_{i+1} \leq a_{h-1}, \\ \chi^{-1}_{i*'j}(\chi_i(x) * \chi_j(y)), & \text{if } x \in I_i, y \in I_j, \max(i,j) \leq h-1 \text{ and } i *' j > 0, \\ & \text{ or } h-1 < \min(i,j) \text{ and } i *' j < n, \\ 0, & \text{if } x \in I_i, y \in I_j, \max(i,j) \leq h-1 \text{ and } i *' j = 0, \\ & \text{ or } \min(x,y) = 0, \\ 1, & \text{otherwise,} \end{cases}$$

is associative, commutative and increasing.

Proposition 14. *Let* $* : [0,1]^2 \to [0,1]$ *be a t-conorm,* $(I_i)_{i=0}^n$ *be a partition of* $]0,1]$ *consisting of left-open sub-intervals. Assume that* $*'$ *is a discrete t-norm on* J_n *such that* $*'$ *is strictly increasing on the domain* $\{(i,j) \mid i,j \in J_n,\ i *' j > 0\}$*. Then the operation* $\oplus^7$ *defined by*

$$x \oplus^7 y = \begin{cases} \chi^{-1}_{i*'j}(\chi_i(x) * \chi_j(y)), & \text{if } x \in I_i \setminus \{a_{n-1}\},\ y \in I_j \setminus \{a_{n-1}\} \text{ and } i *' j > 0, \\ & \text{or } \min(x,y) \in I_0,\ \max(x,y) \in I_n, \\ y, & \text{if } x = a_{n-1}, \\ x, & \text{if } y = a_{n-1}, \\ 0, & \text{otherwise,} \end{cases}$$

is a proper conjunctive uninorm with neutral element a_{n-1} *if and only if* $*$ *has no one-divisors.*

Results

Inspired by the construction method of paving, we construct some new associative, commutative and increasing operations on the unit interval from a t-norm on the unit interval and a discrete t-norm/t-superconorm/t-conorm/uninorm. Similarly, we present the dual constructions from a t-conorm and a discrete t-norm/t-subnorm/t-conorm/uninorm.

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