

1 A quantum-like model for complementarity of
2 preferences and beliefs in dilemma games

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9 **Abstract**

We propose a formal model to explain the mutual influence between observed behavior and subjects' elicited beliefs in an experimental sequential prisoner's dilemma. Three channels of interaction can be identified in the data set and we argue that two of these effects have a non-classical nature as shown, for example, by a violation of the sure thing principle. Our model explains the three effects by assuming preferences and beliefs in the game to be complementary. We employ non-orthogonal subspaces of beliefs in line with the literature on positive-operator valued measure. Statistical fit of the model reveals successful predictions.

10 *Keywords:* positive-operator valued measure, preferences, beliefs, consensus
11 effect, sequential prisoner's dilemma

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12 **1. Introduction**

13 During the recent decade, there is an increasing interest in decision-making
14 and cognitive models that employ a quantum probabilistic (*QP*) framework. In
15 fact, the application of quantum-like concepts to portray human information
16 processing was considered since the early development of quantum mechanics.
17 For example, Bohr (1950) defended the idea that some aspects of quantum theory
18 could provide an understanding of cognitive processes but never provided a formal
19 cognitive model in light of a *QP* hypothesis. The so called quantum cognitive
20 theories have only begun to emerge as of late (Busemeyer and Bruza, 2012;
21 Deutsch, 1999; Haven and Khrennikov, 2013; Khrennikov, 2010; Pothos and
22 Busemeyer, 2013; Wang et al., 2014; Yearsley and Pothos, 2014).

23 *QP* is defined as the set of mathematical rules used to assign probabilities to
24 events from quantum mechanics (Hughes, 1989; Isham, 1989), but without any
25 of the physics. As it is derived from a different sets of axioms than classical
26 probability theory, it is subject to alternative constraints and has the potential to
27 be relevant in any area of science where a need to formalize uncertainty arises.
28 Since encoding uncertainty is a major aspect of cognitive functions in psychology,
29 *QP* shows potential for cognitive modeling. These studies are not about the use
30 of quantum physics in brain physiology, which is a disputable issue (Hameroff,
31 2007; Litt et al., 2006) about which we are skeptical. Rather, we are interested in
32 *QP* theory as a mathematical framework for cognitive modeling.

33 Applications of *QP* theory have been presented in decision-making (Bordley,
34 1998; Busemeyer et al., 2011, 2006; Lambert-Mogiliansky et al., 2009; Pothos
35 and Busemeyer, 2009; Trueblood and Busemeyer, 2011; White et al., 2014;
36 Yukalov and Sornette, 2011), conceptual combination (Aerts, 2009; Aerts and

37 Gabora, 2005; Blutner, 2008), memory (Bruza, 2010; Bruza et al., 2009), and
38 perception (Atmanspacher et al., 2004). For a detailed study on the potential
39 use of quantum modeling in cognition, see Busemeyer and Bruza (2012) and
40 Pothos and Busemeyer (2013). The majority of models presented in the quantum
41 cognition literature addresses standard aspects of decision-making processes:
42 similarity judgments (Barque-Duran et al., 2016; Pothos et al., 2015; Yearsley
43 et al., 2014), the constructive role of articulating impressions (White et al., 2015,
44 2014), and order effects in belief updating (Trueblood and Busemeyer, 2011)
45 among numerous other applications.

46 Little literature has focused on strategic decision-making or game theory.
47 Whenever two or more agents interact, one agent is not only reacting to the
48 information that he receives, but is likewise generating information towards
49 other players. These strategic environments are unique in relation to standard
50 decision-making scenarios under uncertainty, since every agent needs to reason
51 on two parts of the problem: his own actions and his expectations on the
52 opponent's actions. Few studies applying QP instruments to model the way
53 agents process the information in a game have been published with regards to this
54 particular matter: Pothos and Busemeyer (2009), Pothos et al. (2011), Busemeyer
55 and Pothos (2012), and Martínez-Martínez and Sánchez-Burillo (2016). Other
56 approaches in which the quantumness enters through an extension of the classical
57 space of strategies and/or signals have also been discussed, e.g., by La Mura
58 (2005), Brandenburger (2005), and Brunner and Linden (2013); as well as a
59 model to analyze games with agents exhibiting contextual preferences (Lambert-
60 Mogiliansky and Martínez-Martínez, 2015).

61 In this paper, we describe the application of QP theory to modeling the

62 mutual influence between preferences and beliefs in sequential social dilemmas.
63 This idea was first explored in Martínez-Martínez et al. (2015). We present
64 a quantum-like model for preferences and beliefs (QP&B) that replicates the
65 experimental results from Blanco et al. (2014) while providing a novel theoretical
66 approach on cognitive dynamics in strategic interactions. Our model asserts that
67 the relationship between a player's beliefs and his preferences is inherently non-
68 classical and continues the work done in Pothos and Busemeyer (2009) exploiting
69 the ideas of measurement utilized in quantum theory. We redefine these two
70 properties as complementary. In that capacity, they cannot be measured at the
71 same time, as the act of measuring one property alters the state of the other
72 property. The non-classical nature of such a relationship and its application in
73 cognition has already been discussed in, e.g., Denolf and Lambert-Mogiliansky
74 (2016).

75 **2. Experimental design**

76 The data set that our QP&B model deals with is provided by Blanco et al.
77 (2014). Their experiment was designed for explicitly testing different channels
78 through which preferences and beliefs of an agent immersed in a social dilemma
79 may influence each other. As the authors motivate, this experimental evidence
80 is novel and its main interest stems from the fact that previous analyses of
81 strategic interactions considered preferences and beliefs to be independent. This
82 fact implies that the choice of actions in environments with uncertainty can be
83 rationalized as just a best-response to some particular form of belief about the
84 possible states of the world or about the action that is expected to be played by an
85 opponent.

86 *2.1. Standard version of the prisoner's dilemma game*

87 The symmetric prisoner's dilemma game is a game involving two players,
 88 player *I* and player *II*, who can choose among two actions: cooperate (*C*) or defect
 89 (*D*). The normal form of this game is defined by the following 2×2 payoff matrix

		Player <i>II</i>		
		<i>C</i>	<i>D</i>	
Player <i>I</i>	<i>C</i>	(π_c, π_c)	(π_b, π_a)	(1)
	<i>D</i>	(π_a, π_b)	(π_d, π_d)	

90 where the payoff entries satisfy the inequalities $\pi_a > \pi_c > \pi_d > \pi_b$.

91 The scheme of possible results of payoffs is as follows. If player *I* decides to
 92 cooperate, *I* can receive the second best possible outcome π_c if the opponent *II*
 93 also cooperates, but *I*'s attempt to cooperate is exposed to being exploited by *II*
 94 if *II* decides to defect. In the latter scenario, *II* would collect the best outcome of
 95 value π_a while leaving *I* with the lowest payoff π_b . If player *I* decides to defect,
 96 then this player is guaranteed not to obtain the lowest payoff, but at least an amount
 97 π_d if player *II* defects as well. If player *II* decided to cooperate, then *I* is taking
 98 advantage of the situation and obtaining the maximum benefit π_a .

99 Technically, we say that mutual defection is the Nash equilibrium of this
 100 game because there is no unilateral deviation that could make the deviating player
 101 earn more, while mutual cooperation is the Pareto optimal situation. Therefore,
 102 this game represents a social dilemma for the players: the individual choice
 103 of defection dominates the attempt to cooperate for any given choice of the
 104 opponent, which is not socially optimal. Why is this a dilemma? Because this
 105 game formalizes a conflict between the individual (the Nash equilibrium) and the
 106 collective (Pareto optimal) level of reasoning: if both players actually choose to

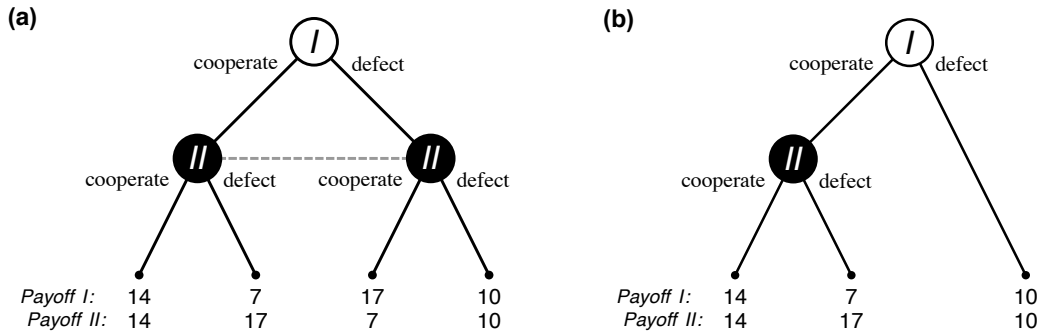


Figure 1: (a) Standard (simultaneous) Prisoner's Dilemma. (b) Sequential Prisoner's Dilemma.

107 defect, both of them generate a total payoff of $2 \times \pi_d$, which is by definition lower
 108 than the aggregate payoff if both of them coordinated in full cooperation, $2 \times \pi_c$.

109 The standard version of the prisoner's dilemma game is a one-shot strategic
 110 interaction with simultaneous moves by the opponents. This implies that both
 111 players make their own individual decision (whether to cooperate or not) without
 112 knowing what the opponent is choosing. Once both players have chosen their
 113 strategy, both actions become public and the payoffs are generated.

114 Each player reacts to his own belief or expectation on the opponent's intention,
 115 and as a consequence, the preferred action in the dilemma crucially depends on
 116 the way players form their beliefs about the opponent moves. Therefore, it is
 117 important to understand how beliefs and preferences do (or do not) influence each
 118 other in this decision-making process.⁵

119 2.2. Sequential prisoner's dilemma

120 The experiment conducted by Blanco et al. (2014) focuses on a variation of
 121 the Prisoner's Dilemma game discussed above: a sequential one. In Fig. 1 we

⁵See Blanco et al. (2014, Section 1) about possible correlations between preferences and beliefs in dilemmas with models of social preferences such as inequality aversion and reciprocal preferences.

122 show the game tree of the game played in this sequential experiment (b), and
123 compare it to its standard (simultaneous) counterpart with equivalent payoffs (a).
124 In the sequential version, the solution concept required is the Subgame Perfect
125 Nash Equilibrium (SPNE), a usual refinement of the Nash Equilibrium (NE) when
126 turning to sequential games. Solving by backwards induction, we see that it is in
127 the best interest of Player *II* to defect if given the chance to move, which would
128 leave Player *I* with a payoff of 7, and therefore *I* should choose defect at the
129 beginning of the tree, because 10 is a better outcome. Thus, the sequential game
130 maintains the content of the social dilemma because the SPNE implies that both
131 players' incentives drive them towards mutual defection, even though they could
132 obtain a higher social payoff if they coordinated on full cooperation.

133 On the one hand, one can see how in the sequential variation, only the player *I*
134 is bearing the risk of her cooperative choice being exploited by a selfish decision
135 of player *II*. In order to restore the symmetry between the players, all participants
136 in the experiment play the game twice. Once in role *I* and once in role *II*. After
137 all decisions have been made, the players are randomly matched into pairs, with
138 the assignment of roles being random as well. Subsequently, they earn the payoffs
139 determined by the relevant decisions, given their roles.

140 On the other hand, this procedural 'complication' is a small price to pay if
141 we compare it to the advantages it provides: because of the sequential structure
142 in the decision-making, each choice can be observed (measured) at a time. The
143 authors design three treatments that intersperse a belief-elicitation task with the
144 choices of actions.⁶ As we discuss now, the treatments differ in the order in which

⁶In the belief-elicitation task, the players were asked how many of the other participants (potential rivals for the play of the game) cooperate in the role of Player *II*. This task is incentivized

<i>Treatment</i>	<i>Baseline</i>	<i>Elicit Beliefs</i>	<i>True Distribution</i>
Task 1	2nd move (<i>II</i>)	2nd move (<i>II</i>)	2nd move (<i>II</i>)
Feedback on <i>II</i>	No	No	Yes
Task 2	1st move (<i>I</i>)	beliefs (about <i>II</i>)	1st move (<i>I</i>)
Task 3	beliefs (about <i>I</i>)	1st move (<i>I</i>)	beliefs (about <i>I</i>)
# Participants	40	60	60

Table 1: Experimental treatments in Blanco et al. (2014, Table 1).

145 each task is performed and this allows to measure different correlations between
146 actions (which are supposed to proxy the preferences of the players) and beliefs.
147 We now briefly explain the three different treatments, which are also summarized
148 in Table 1.

149 2.3. *Experimental treatments*

150 Ten subjects participate in each session. For each of the following treatments,
151 several sessions were conducted. The total numbers of participants are displayed
152 in Table 1.

153 *Baseline.* This treatment can be considered as a mere control group, such that
154 the subjects play the game in its natural structure, with no attention paid to
155 observing their beliefs. The players first choose what their action *II* will be
156 and no information is revealed to them so that the participants' beliefs are not
157 exogenously influenced. Subsequently, they choose what their action for the role
158 of *I* will be, and finally they are given a meaningless question about their beliefs on
159 the global rate of cooperation in the group of first movers. The informational gain
160 of this last task is void because its only use is to balance the different treatments
161 making their length comparable (both in time and the number of tasks).

with a quadratic scoring rule rewarding the accuracy of the stated beliefs: players earn more the closer their prediction is to the actual rivals' cooperation rate (Blanco et al., 2014, Equation 3).

<i>Treatment</i>	<i>Baseline</i>	<i>Elicit_Beliefs</i>	<i>True_Distribution</i>	<i>Total</i>
First mover (Player <i>I</i>)	27.5%	55.0%	56.7%	48.8%
Second mover (Player <i>II</i>)	55.0%	53.3%	55.0%	54.4%

Table 2: Average cooperation rates by treatment in the experiment by Blanco et al. (2014), also labeled as Table 2 in their original paper.

162 *Elicit_Beliefs*. In this treatment, the players first choose what their action *II* will
163 be, and then they have to reveal their belief about the rate of cooperation that they
164 will receive from the second movers. Finally, they have to choose their action *I*.
165 Thus, this treatment introduces a belief-measurement between the two choices of
166 actions. This allows us to explore the effect of a measurement of the beliefs about
167 the move by opponent *II* on the choice of action *I*.

168 *True_Distribution*. This treatment presents a somewhat ‘similar’ sequence of tasks
169 for the players compared to the previous treatment *Elicit_Beliefs*. The players
170 begin by choosing their action *II*. Then, they are told what the true cooperation
171 rate for action *II* was in their group. They finish by choosing the action *I*.
172 This treatment differs from the previous one in that this time, the forecast of the
173 opponents’ move is not a belief generated by the players themselves, but true
174 information being released to them exogenously.

175 **3. Aggregate behavior and basic modeling**

176 Table 2 presents the aggregate results of the three experimental treatments.
177 First off, we cannot observe any significant difference in the cooperation rates
178 as a second mover between treatments. This is to be expected as the question
179 (measurement) regarding the choice of action in the role of player *II* is identical

180 in all aspects over all treatments.⁷ The small variation in the proportion of
181 cooperation reported for the *Elicit_Beliefs* treatment (53.3% vs. 55% in the others)
182 can be attributed to sample variance.

183 The cooperation rates in the role of first mover (player *I*) show meaningful
184 differences. A chi square test across all three treatments yields a p-value of
185 0.007886 ($\chi^2 = 9.6853$, $df=2$). Starting with the first move cooperation rates
186 of the *Baseline* treatment (27.5%) and the *Elicit_Beliefs* treatment (55.0%), the
187 null hypothesis of no difference between these two proportions yields a p-value
188 of 0.007 ($\chi^2 = 7.3661$, $df=1$), clearly indicating a significant difference. There
189 is only one procedural variation between these two treatments: *Elicit_Beliefs*
190 includes the elicitation of beliefs about the cooperation rate expected from the
191 rivals *II* before the agents choose their action in the role of *I*. Thus, we can
192 attribute the difference in the player *I* cooperation rate to the effect that measuring
193 a subject's beliefs about the opponent *II* may have on his attitude toward the
194 actions as first mover.

195 A similar result can be found for the first move cooperation rates of the
196 *Baseline* treatment (27.5%) and the *True_Distribution* treatment (56.7%). The
197 null hypothesis claiming no difference between these two proportions can be
198 rejected, as it gives us a p-value of 0.004 ($\chi^2 = 8.2674$, $df=1$). For the first
199 move cooperation rates (role *I*) of the *Elicit_Beliefs* treatment (55.0%) and the
200 *True_Distribution* treatment (56.7%), the null hypothesis of no difference between
201 these proportions yields a p-value of 0.85 ($\chi^2 = 0.0351$, $df=1$), indicating no
202 significant difference between the result in the two treatments. In this sense, the

⁷Note especially that it is the first measurement performed in all treatments and therefore, it is not subject to the effects targeted by this experimental design.

203 incentivized elicitation of beliefs impacts the state of the subjects participating
204 in the experiment similarly to an update of beliefs via the acquisition of true
205 information revealed exogenously.

206 *3.1. Violation of the sure thing principle*

207 The differences in first move cooperation rates reveal the presence of a
208 violation of the sure thing principle in the data, as

$$27.5\% = p(C_I) \neq \sum_i p(C_I|B_i) = 55\%,$$

209 with C_I the event of the player cooperating on the first move and B_i the event of the
210 player answering that he thinks i opponents cooperate during the belief elicitation.
211 This in turn points out the interest in using a quantum-like model to describe the
212 behavior of the participants in this experiment, since classical statistics cannot
213 account for them in a simple manner, while quantum-like easily do.

214 *3.2. The simplest quantum-like model*

215 In the remaining of Section 3, we illustrate the basic mechanics of quantum-
216 like toy models designed to address the issue of measurement as well as construct
217 different building blocks that will be fully developed later. As the reader will see,
218 Section 4 integrates them in a unified model. Now, we only show which aspects
219 of quantum-like modeling can account for the empirical effects observed in the
220 data set, without taking into account how they correlate to form the proper model.

221 We introduce the most basic quantum-like model to represent concepts
222 such as actions, preferences and beliefs in quantum-like terms (observables,
223 measurements and orthonormal basis of their outcomes) and use projective
224 measurements (with their resulting probabilities) to explain the first results
225 observed in the data from Blanco et al. (2014). We consider the preferences

226 of an agent as the individual's attitude toward the different elements of a set of
 227 outcomes, to be reflected in the choices observed along the sequence of decisions
 228 (Lichtenstein and Slovic, 2006). In this case, and because of the strategic nature
 229 of this decision-making process, the outcomes (possible payoffs to be obtained)
 230 depend on the actions (cooperate or defect) a players chooses, but also on the
 231 choices made by a rival.

232 The actions of a player can be represented by two orthogonal vectors $|C\rangle$ (for
 233 cooperation) and $|D\rangle$ (for defection). The two vectors form an orthonormal basis
 234 and span a bi-dimensional Hilbert space \mathcal{H}_i with $i \in \{I, II\}$ denoting the role in the
 235 game as player I or II for which such action is chosen.⁸ The player is considered
 236 to be in a superposition over these actions, being represented by a normalized state
 237 vector $|S\rangle$. The projection of the state vector onto the elements of the orthonormal
 238 basis defines the probability that the player chooses each of the actions, as a proxy
 239 of her preferences.

240 We consider the beliefs as the distribution with which the agents judge
 241 the likelihood of realization of each possible relevant state of the world. The
 242 possible states in this setting concern the possible cooperation of opponents,
 243 as this, together with one's own actions, determines the outcome of the game.

⁸For the finite dimensional case, a Hilbert space \mathcal{H} is a linear space endowed with a scalar product $\langle\psi_1|\psi_2\rangle \in \mathbb{R}$. Its elements (or states) are denoted by $|\psi\rangle \in \mathcal{H}$. If the state of the system is $|\psi\rangle$ we say it is in a *pure state*. The projector $P_\psi = |\psi\rangle\langle\psi|$, an operator acting on \mathcal{H} as $P_\psi|\phi\rangle = \langle\psi|\phi\rangle|\psi\rangle$, has a bijective relation with $|\psi\rangle$, and we can describe the state $|\psi\rangle$ in terms of P_ψ . Any element or vector of the space of states is called a *ket*-vector and represented by $|\cdot\rangle$, and we have the dual space of the *bra*-vectors, symbolized by $\langle\cdot|$. Hilbert spaces are generally defined over the field of complex numbers, but in this paper it is enough to work only with reals. Note that given a state $|\psi\rangle$ associated to a vector $\psi \in \mathbb{R}^N$, we obtain $\langle\psi|$ associated to ψ^T , where T is the operation of vector transposition. The name of *bra-ket* (or Dirac's) notation comes from splitting the *bracket* $\langle\cdot|\cdot\rangle$ representing the scalar product, which is the crucial operation to compute probabilities in this framework.

244 These beliefs are also represented by a set of mutually orthogonal vectors $\{|B_j\rangle\}$,
 245 with the index j running from 0 to 9. This j represents how many of the
 246 opponents (maximum 9) are believed to cooperate. This orthonormal basis also
 247 spans a Hilbert space, \mathcal{H}_B , with the player's beliefs being represented by a
 248 normalized state vector: a superposition over the orthonormal basis of beliefs.
 249 Straightforwardly, $j/9$ is the expected share of cooperation among the opponents,
 250 and $1 - j/9$ is the expected rate of defection.

251 3.3. Projective measurement

252 Quantum-like models use projective measurements to represent measurements
 253 being performed on the system of interest.⁹ Here, we apply this to model the
 254 observed behavior in the choice of action as player II in the data from Blanco
 255 et al. (2014). The state of the player is represent by a normalized state vector $|S_{II}\rangle$
 256 in the two-dimensional Hilbert space \mathcal{H}_{II} :

$$|S_{II}\rangle = c_{II}|C_{II}\rangle + d_{II}|D_{II}\rangle. \quad (2)$$

257 The probability $p(C_{II})$ of the player choosing to cooperate is therefore:

$$p(C_{II}) = \|P_{C_{II}}|S_{II}\rangle\|^2 = \langle C_{II}|S_{II}\rangle^2 = c_{II}^2, \quad (3)$$

258 with $P_{C_{II}} = |C_{II}\rangle\langle C_{II}| = \text{diag}(1, 0)$ the projector on $|C_{II}\rangle$. This outcome would
 259 project the state vector onto its post-measurement state $|S'_{II}\rangle = |C_{II}\rangle$. The

⁹The probability of observing an outcome is calculated as the square of the norm of the projection of the state vector onto the subspace spanned by the vectors representing the outcome. When the outcome is represented by only one vector (simplest case), this calculation reduces to the square of the inner product of the state vector and the outcome vector. The act of measurement changes the state vector of the system from an initial state to a post-measurement state, by projecting (and normalizing) the state vector onto the subspace spanned by the outcome vectors. Projective measurements deal naturally with incompatible measurements, and note also that when they are performed on a density matrix diagonal in a particular basis, they are equivalent to Bayesian updates.

260 probability of the player defecting as second mover is:

$$p(D_{II}) = \|P_{D_{II}}|S_{II}\rangle\|^2 = \langle D_{II}|S_{II}\rangle^2 = d_{II}^2, \quad (4)$$

261 with $P_{D_{II}} = |D_{II}\rangle\langle D_{II}| = \text{diag}(0, 1)$ the projector on $|D_{II}\rangle$. This outcome would
 262 likewise project the state vector onto its post-measurement state $|S'_{II}\rangle = |D_{II}\rangle$. The
 263 normalization restriction on the state vector implies that total probabilities add up
 264 to one, $c_{II}^2 + d_{II}^2 = 1$. From the cooperation rates as player *II* reported in Table 2,
 265 we can estimate these through our sample as:

$$\hat{c}_{II}^2 = 0.544 \text{ and } \hat{d}_{II}^2 = 0.456. \quad (5)$$

266 Note that we estimate by taking the average cooperation rates across the
 267 treatments, because we have justified above that they are not significantly different
 268 from one another.

269 We can model the choice of the players for their action as player *I* in the
 270 *Baseline* condition in a Hilbert space $\mathcal{H}_I \equiv \mathbb{R}^2$, with the basis $\{|C_I\rangle, |D_I\rangle\}$. The
 271 state vector is now

$$|S_I\rangle = c_I|C_I\rangle + d_I|D_I\rangle, \quad (6)$$

272 and we can infer from the data (Table 2, column 1) that

$$\hat{c}_I^2 = 0.275, \text{ and } \hat{d}_I^2 = 0.725. \quad (7)$$

273 In this case, we only consider the cooperation and defection rates in the *Baseline*
 274 treatment. Because of the significant difference in the cooperation rate as player
 275 *I* across treatments, considering the average is not sensible (see discussion in
 276 Section 3).

#Cooperators (Belief)	0	1	2	3	4	5	6	7	8	9
Abs. frequency (out of 60 subjects)	5	2	5	5	12	9	9	6	4	3

Table 3: Number of players in treatment *Elicit_Beliefs* expecting each possible number of cooperators in their session.

277 Finally, we model the beliefs of the players in the Hilbert space \mathcal{H}_B , (spanned
278 by $\{|B_j\rangle\}$). The normalized state vector is

$$|S_B\rangle = \sum_{j=0}^9 b_j |B_j\rangle. \quad (8)$$

279 From the data regarding the *Elicit_Beliefs* treatment (see Table 3), we get that

$$\begin{aligned} \hat{b}_0^2 = 5/60, \hat{b}_1^2 = 2/60, \hat{b}_2^2 = 5/60, \hat{b}_3^2 = 5/60, \hat{b}_4^2 = 12/60, \\ \hat{b}_5^2 = 9/60, \hat{b}_6^2 = 9/60, \hat{b}_7^2 = 6/60, \hat{b}_8^2 = 4/60, \hat{b}_9^2 = 3/60. \end{aligned} \quad (9)$$

280 4. Building blocks

281 4.1. Three effects

282 **Effect 1 (Consensus effect).** Proof of and an extensive discussion on the presence
283 of this effect is presented in Blanco et al. (2014) where it is shown that players'
284 beliefs are biased towards their own actions. As such, a player who cooperates
285 as second mover will expect a higher second-mover cooperation rate amongst the
286 other players. A visualization of this effect can be found in Fig. 2. Viewing this in
287 light of the performed measurements, the consensus effect denotes the influence
288 of second mover action measurements on the beliefs of the same participant.

289

290 **Effect 2 (Reasoned player).** The second effect is the influence that belief
291 measurements have on action measurements. As these actions are driven by one's

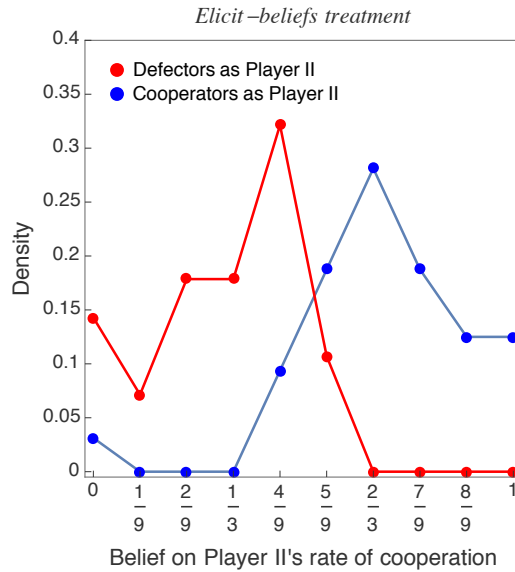


Figure 2: Second move defecting players (red line) believe that less opponents will cooperate. Second move cooperating players (blue line) believe more opponents will cooperate. The second move action was measured before the beliefs.

292 preferences, this effect encompasses the influence of the belief measurements
 293 on the preferences of the same player. We claim that the act of eliciting the
 294 beliefs of the player fundamentally changes this player even when disregarding
 295 the exact outcome of this belief measurement. When the player is asked to form
 296 an opinion about the cooperation rate of his opponents, this changes him into a
 297 more reasoned state about the opponent, in opposition to a more intuitive state
 298 when not explicitly asked to form this opinion. In the data, this can be viewed in
 299 the violation of the sure thing principle discussed in Section 3.1. The average first
 300 move cooperation rate of players, after forming explicitly their beliefs about the
 301 cooperation of the opponent (*Elicit_Beliefs*), is twice as large as the average first
 302 move cooperation rate of players, in which beliefs were not elicited (*Baseline*)
 303 (see Table 2). Nevertheless, this cooperation rate in the *Elicit_Beliefs* group is not

304 differing significantly from the cooperation rate in the *True_Distribution* group. In
305 this group, participants received full information about the cooperation rate of the
306 opponents and are therefore assumed to make a more deliberate decision. Since
307 these cooperation rates are similar, we can assume that players are in a similar
308 reasoned state in the *Elicit_Beliefs* group.

309

310 **Effect 3 (Classical correlation).** The third effect we discuss is the correlation
311 between a player's first and second move. This is observed in all three conditions,
312 as noted in Results 1, 2 and 3 from Blanco et al. (2014). That is, first move
313 cooperators are likely to also cooperate on the second move and vice versa. We
314 concur with Blanco et al. that this correlation is exhibited mostly through an
315 indirect belief-based channel. This way, we attempt to include the observed
316 correlation as a logical consequence of our previously described effects. The
317 second move action measurement influences the first move action measurement
318 through a player's beliefs. We can assume this correlation to be classical in nature,
319 as opposed to the two other effects.

320 4.2. *Compatible and incompatible measurements*

321 Roughly speaking, two measurements M_1 and M_2 are considered incompatible
322 if the order in which the measurements are done changes the outcome, as the act
323 of performing one measurement influences the other measurements regardless of
324 the outcome. Mathematically speaking, this means that one or more projector
325 matrices associated with outcomes of measurement M_1 do not commute with one
326 or more projector matrices associated with outcomes of measurement M_2 . If two
327 measurements are maximally incompatible, no projector matrix associated with an
328 outcome of measurement M_1 commutes with a projector matrix associated with an

329 outcome of measurement M_2 , and they are called complementary. As such, both
 330 measurements M_1 and M_2 cannot be performed together, as the *act* of performing
 331 one of the measurements (without specifying its outcome), influences the other
 332 measurement. These concepts elegantly deal with situations where violations of
 333 the sure thing principle emerge.

334 We will consider the belief elicitation to be complementary with the action
 335 measurements, as this explains both the consensus effect and the reasoned player
 336 effect. This approach should not come as a surprise. First, using complementarity
 337 as an explanation for the consensus effect is argued in Busemeyer and Pothos
 338 (2012) where the consensus effect is seen as a form of social projection. Second,
 339 the idea of the player being more reasoned can be seen as a violation of the
 340 sure thing principle. These violations are a prime indicator of measurements not
 341 commuting which is the definition of incompatible measurements. We will now
 342 show how the projective measurement formalism deals with our hypothetically
 343 compatible (first and second move actions) and incompatible (actions and beliefs)
 344 measurements.

345 When two measurements are considered compatible, the Hilbert spaces
 346 representing the outcomes of these measurements can be tensored to construct
 347 a larger Hilbert space spanned by vectors that now represent joint outcomes. As
 348 argued before, we assume the first move action and second move action to be
 349 compatible, as they are considered to be measurable at the same time. Therefore,
 350 the Hilbert space which models the relationship between both is $\mathcal{H}_I \otimes \mathcal{H}_{II}$, spanned
 351 by $\{|CC\rangle, |CD\rangle, |DC\rangle, |DD\rangle\}$, with $|CD\rangle = |C_I\rangle \otimes |D_{II}\rangle$ (other vectors defined
 352 similarly). The player is represented by a normalized state vector:

$$|S\rangle = s_{CC}|CC\rangle + s_{CD}|CD\rangle + s_{DC}|DC\rangle + s_{DD}|DD\rangle. \quad (10)$$

353 We now provide two examples of how probabilities are calculated within this
 354 Hilbert space. The other relevant probabilities are calculated in a similar way. The
 355 projector and probability associated with a player defecting on the role of I , but
 356 cooperating on the role of II is

$$P_{DC} = P_{D_I} \otimes P_{C_{II}} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (11)$$

357 so

$$p(DC) = \|P_{DC}|S\rangle\|^2 = s_{DC}^2. \quad (12)$$

358 The projector and probability associated with the player cooperating on the second
 359 move (without specifying a choice as player I), are:

$$P_{.C} = I^2 \otimes P_{C_{II}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (13)$$

360 and

$$p(.C) = \|P_{.C}|S\rangle\|^2 = s_{CC}^2 + s_{DC}^2. \quad (14)$$

361 Directly from the data (for the *Baseline* treatment), we derive

$$\hat{s}_{CC}^2 = 0.25, \quad \hat{s}_{CD}^2 = 0.025, \quad \hat{s}_{DC}^2 = 0.3, \quad \text{and} \quad \hat{s}_{DD}^2 = 0.425. \quad (15)$$

362 This models the (classical) correlation between first and second move, as noted
 363 above in Effect 3.

364 Incompatible measurements are represented by different bases in the same
365 Hilbert space (as opposed to one tensored basis for compatible measurements).
366 To model the relationship between the choice of action in the role of player I
367 and the beliefs that a player holds, we could use a Hilbert space $\mathcal{H}_{I,B}$ of large
368 enough dimensionality to present 10 orthogonal subspaces, each one representing
369 one belief. As such, we would need at least a 10-dimensional space, with 10
370 orthonormal vectors forming the belief basis. In such 10-dimensional Hilbert
371 space, the 2 possible outcomes of the first movement action are each represented
372 by orthogonal 5-dimensional subspaces.

373 The Hilbert space $\mathcal{H}_{II,B}$, which models the relationship between the belief
374 measurement and the second movement action would be similarly spanned by
375 10 orthonormal basisvectors, each one representing an outcome of the belief
376 measurement. The outcomes of the second movement action are also represented
377 by 5-dimensional subspaces. The rules for projection and calculating probabilities
378 remain the same. The probability of an outcome of a measurement is still the
379 square of the norm of the projection of the state vector on the relevant subspace.
380 The act of measuring still changes the superposition of the state vector, projecting
381 and normalizing it onto the relevant subspace.

382 In summary, the relationship between the belief and action measurement is
383 represented by the description of the action subspaces in terms of the belief basis.
384 In such setting, the consensus effect would be represented by the form of the 5-
385 dimensional action subspaces in $\mathcal{H}_{II,B}$, while the effect of the player becoming
386 more reasoned would be represented by the form of the 5-dimensional action
387 subspaces in $\mathcal{H}_{I,B}$.

388 *4.3. A very basic model*

389 We can attempt to construct a model which successfully incorporates all three
390 effects, by combining how we modeled the compatible action measurements, with
391 how we could model the incompatible belief and action measurements. The
392 standard procedure from quantum-like measurement theory tells us to construct
393 the Hilbert space $\mathcal{H}^{orth} = \mathcal{H}_{I,B} \otimes \mathcal{H}_{II,B}$. This is a 100-dimensional Hilbert space,
394 with 2 orthogonal 50-dimensional subspaces representing the actions in role *I*, 2
395 orthogonal 50-dimensional subspaces representing the actions in role *II*, and 10
396 orthogonal 10-dimensional subspaces representing the possible beliefs. As the
397 first and second move actions are considered compatible, they can be measured at
398 the same time. As such, the 4 possible joint outcomes of the action measurements
399 are represented by four 25-dimensional subspaces.

400 The player would be represented by a normalized state vector in this
401 100-dimensional Hilbert space, from which the relevant probabilities can be
402 calculated. From a statistical point of view this state vector already provides us
403 with 99 degrees of freedom (we lose 1 as the state vector is normalized), without
404 even delving into how many degrees of freedom pop up due to the different 10-,
405 25- and 50-dimensional subspaces used in this construction. As we have 160 data
406 points, this elementary model would be by no means elegant, and a statistical fit is
407 not feasible because of being greatly overparametrized. One solution is to impose
408 further restrictions on the state vector and/or on the different outcome subspaces,
409 for example, by allowing only state vectors within a certain subspace or assuming
410 a certain distribution over the resulting probabilities. The form of these restrictions
411 is, however, an open question at this point.

412 In the following section we show how a small deviation from the most

413 common quantum-like approach allows us to reduce the complexity of the total
414 Hilbert space to only four dimensions. We use a less structured set of planes to
415 represent beliefs which provides a truly intuitive connection between the different
416 elements of the model.

417 **5. The model for Quantum-like Preferences and Beliefs**

418 *5.1. A new belief basis*

419 To diminish the problematic dimensionality of \mathcal{H}_B we let the vectors $|B_i\rangle$
420 (the outcomes of the belief elicitation) be non-orthogonal because otherwise,
421 the 10 orthogonal vectors would span a 10-dimensional Hilbert space. Next to
422 making the dimension of \mathcal{H}_B sufficiently small, this modification will allow us to
423 model some implicit structure between the different outcomes and will link the
424 construction of these beliefs directly to the approach of Pothos and Busemeyer
425 (2009) to the standard prisoner dilemma. Roughly speaking, in Pothos and
426 Busemeyer (2009), the emergence and evolution of the player's beliefs about
427 his opponent's behavior is represented by a rotation of the state vector in the
428 Hilbert space. While in Pothos and Busemeyer (2009) this rotation is defined by a
429 Hamiltonian with a parameter γ , we now have the means to explicitly incorporate
430 the elicited beliefs into our model. To do so, we redefine the belief-vectors $|B_i\rangle$
431 in a 2-dimensional Hilbert space, with $|B_0\rangle$ and $|B_9\rangle$ orthogonal and the other
432 $|B_i\rangle$ in between them. For simplicity, we will assume the distribution of the $|B_i\rangle$
433 ($i \neq 0, 9$), to be uniform between $|B_0\rangle$ and $|B_9\rangle$, yielding an angle $\pi/(2 \times 9)$ between
434 all $|B_i\rangle$ and $|B_{i+1}\rangle$. This provides us with an elegant, parameter free (as the '9'
435 is endogenous to the game) form of the vectors representing the outcomes of
436 the belief elicitation. This is a simple first approach to the exact distribution of
437 the $\{|B_i\rangle\}_{i=0}^9$, which can be adjusted or made more complex if necessary. This

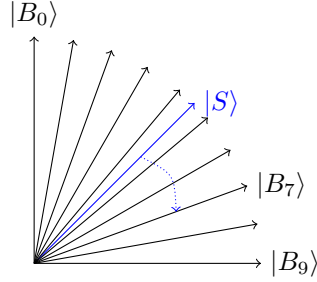


Figure 3: The redefined $|B_i\rangle$. A player thinking 7 out of 9 opponents cooperate projects the state vector onto $|B_7\rangle$.

438 effectively makes the players development of her explicit beliefs to be represented
 439 by a rotation of the state vector, as in Pothos and Busemeyer (2009). Our view
 440 differs from Pothos and Busemeyer (2009) in the sense that we still want to
 441 make predictions and derive probabilities from this rotation, using the standard
 442 rules of projective measurements: an outcome and its probability as defined by a
 443 projector on the relevant subspace. This approach also models an implicit order
 444 between the different outcome vectors (e.g. $|B_i\rangle$ being ‘in between’ $|B_{i-1}\rangle$ and
 445 $|B_{i+1}\rangle$), something lacking in the previous approach where all belief vectors were
 446 orthogonal. This idea is depicted in figure 3.

447 With the redefined 2-dimensional Hilbert Space \mathcal{H}_B , we rebuild the Hilbert
 448 space $\mathcal{H}_{QP\&B}$ which contains the representations of all measurements, as well
 449 as their correlations. As we redefined our vectors $|B_i\rangle$ representing the elicited
 450 beliefs, the projectors onto these vectors will also have a new form:

$$|B_i\rangle\langle B_i| = \begin{pmatrix} \cos^2\left(\frac{i\pi}{18}\right) & \cos\left(\frac{i\pi}{18}\right)\sin\left(\frac{i\pi}{18}\right) \\ \cos\left(\frac{i\pi}{18}\right)\sin\left(\frac{i\pi}{18}\right) & \sin^2\left(\frac{i\pi}{18}\right) \end{pmatrix}, \quad (16)$$

451 with $i = 0, 1, \dots, 9$.

452 To incorporate a measurement with non-orthogonal outcome vectors, we will
 453 go beyond the basic procedure of quantum measurement as done in Section 3.2.

454 To do so, we present two options, one favoring quantum theoretic consistency and
 455 one favoring a simpler experimental interpretation. Note that the resulting model
 456 and probabilities in these two options are identical. Readers not interested in the
 457 derivation and discussion of these options can skip to the last paragraph of this
 458 section.

459 In the first option we use positive-operator valued measures (POVMs), a well
 460 known measurement framework within quantum theory, in which non-orthogonal
 461 outcome vectors can be used. These POVMs allow us to easily build our smaller
 462 model with our newly defined belief space. For an introduction to these POVMs,
 463 as well as the mathematical details and recipe on how to construct them, we
 464 refer to Yearsley (2016, Section 4). The following derivations rely on the derived
 465 probabilities given in Yearsley (2016, Equation 56).

466 In short, when using the POVM framework, the measurement outcome is still
 467 represented by an outcome vector and its associated projector. If an outcome is
 468 observed, the state vector is still projected onto the relevant subspace; however,
 469 the probability of obtaining this outcome is calculated slightly differently. Assume
 470 that the player is represented by a state vector $|S\rangle$, the probability of the player
 471 thinking that i opponents have cooperated is now:

$$P'(B_i) = \frac{\langle B_i | S \rangle^2}{\sum_{j=0}^9 \langle B_j | S \rangle^2}. \quad (17)$$

472 This form deviates from the probabilities derived in section 3.2 only in the factor
 473 $\sum_{j=0}^9 \langle B_j | S \rangle^2$. This extra factor finds root in the fact that the projectors P_j forming
 474 a POVM need to adhere to *completeness*:

$$\sum_{j=0}^9 P_j = \mathbb{I},$$

475 with \mathbb{I} the identity matrix. Equation 16 shows that the projectors onto our belief
 476 vectors can never sum to the identity matrix, as the off-diagonal elements can
 477 never sum to zero. To make sure that the relevant projectors still form a POVM,
 478 a new projector (and outcome) is added to the formalism. This projector is
 479 associated with the outcome ‘measurement failed’. When this outcome is obtained
 480 the measurement is redone, ensuring *completeness*. For details, see again Yearsley
 481 (2016).

482 The second option dismisses the idea of an extra ‘measurement fails’ outcome
 483 and allows the set of projectors $|B_i\rangle\langle B_i|$ to violate the *completeness* criteria. This
 484 violation makes the probabilities of our possible belief outcomes not sum to one:

$$\sum_{j=0}^9 P(B_j) = \langle B_i|S\rangle^2 \neq 1. \quad (18)$$

485 From a modeling point of view, this requires the introduction of a scaling
 486 factor. This makes sure that the total sum of probabilities does sum to one, *after*
 487 the standard quantum measurement (calculating probabilities and projecting the
 488 state vector) is done. This scaling factor is defined as:

$$C = \sum_{j=0}^9 \langle B_j|S\rangle^2, \quad (19)$$

489 making the probability of eliciting belief i , given the state vector $|S\rangle$:

$$P'(B_i) = \langle B_i|S\rangle^2 / C. \quad (20)$$

490 It is vital to note that the end result of both approaches is identical. We have
 491 ten outcome vectors, representing the ten possible beliefs, in a two dimensional
 492 Hilbert Space \mathcal{H}_B . The probability of eliciting the belief that i opponents have

493 cooperated, given the state vector $|S\rangle$ is:

$$P'(B_i) = \frac{\langle B_i|S\rangle^2}{\sum_{j=0}^9 \langle B_j|S\rangle^2}. \quad (21)$$

494 If the result is i , the state vector gets projected onto $|B_i\rangle$. The difference between
495 the two options lies in the difference between an approach where we remain
496 firmly within the quantum theoretic setting at the cost of adding an ad hoc new
497 outcome (actually not present in the experimental setting) and an approach slightly
498 departing from the quantum sphere by redefining the probabilities with an ad hoc
499 scaling factor, but having a clear interpretation of all the elements of its machinery
500 regarding the experiment. The choice between the options has no effect on the rest
501 of the paper.

502 5.2. *The QP&B model*

503 With our belief measurement now adequately defined in the two dimensional
504 \mathcal{H}_B , we can define $\mathcal{H}_{QP\&B}$. We still assume the second move action and the
505 belief elicitation to be complementary, representing them by different bases in the
506 redefined 2 dimensional Hilbert Space $\mathcal{H}_{II,B}$. Additionally, we define the angle
507 between $|C_{II}\rangle$ and $|B_9\rangle$ as β_{SM} (see Figure 4) and derive estimated probabilities
508 for a player replying that he thinks i opponents cooperate, after the player has
509 cooperated or defected on his second move. As such, this models the consensus
510 effect. We expect β_{SM} to be close to 0, as the consensus effect tells us that people
511 who cooperate are more likely to assume that opponents cooperate as well.

512 Now we can derive the estimated probabilities for the beliefs of a player

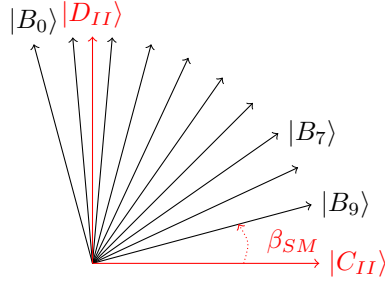


Figure 4: The redefined $\mathcal{H}_{II,B}$ with both an action-basis and the new belief-basis.

513 defecting on his second move (making the state vector $|S\rangle = |D_{II}\rangle$):

$$P(B_i|D) = \langle B_i|D_{II}\rangle^2 / \sum_{j=0}^9 \langle B_j|D\rangle^2 \quad (22)$$

$$= \cos^2\left(\beta_{SM} + \frac{i\pi}{9}\right) / \sum_{j=0}^9 \langle B_j|D\rangle^2 \quad (23)$$

514 and for the beliefs of a player cooperating on his second move (making the state
515 vector $|S\rangle = |C_{II}\rangle$):

$$P(B_i|C) = \langle B_i|C_{II}\rangle^2 / \sum_{j=0}^9 \langle B_j|C\rangle^2 \quad (24)$$

$$= \sin^2\left(\beta_{SM} + \frac{i\pi}{9}\right) / \sum_{j=0}^9 \langle B_j|C\rangle^2, \quad (25)$$

516 with $i \in \{0, \dots, 9\}$.

517 Similarly, we define $\mathcal{H}_{I,B}$ as 2-dimensional with both a first move action basis
518 and a belief basis, with β_{FM} the angle between $|C_{SM}\rangle$ and $|B_9\rangle$. Once again, we
519 assume β_{FM} close to zero, as players who explicitly think that their opponent will
520 defect are assumed to be more likely to defect as well. We can now derive the
521 estimated probabilities of a player cooperating or defecting on his first moves,
522 after replying that he thinks i opponents cooperated on their second move, which

523 made the state vector $|S\rangle = |B_i\rangle$. Note that this first move measurement uses the
 524 simple derived probabilities as defined in Section 3.2, as this measurement has
 525 both outcome vectors orthogonal.

$$P(D|B_i) = \langle D_I|B_i\rangle^2 \quad (26)$$

$$= \cos^2\left(\beta_{FM} + \frac{i\pi}{92}\right). \quad (27)$$

526 The first and second moves are still considered to be compatible, allowing
 527 for a tensoring of their respective Hilbert spaces to represent their correlation.
 528 The projectors and probabilities associated with these measurements are identical
 529 to the ones defined in the quantum-like model from section 3.2. This gives us
 530 a final model $\mathcal{H}_{QP\&B} = \mathcal{H}_{I,B} \otimes \mathcal{H}_{II,B}$. In $\mathcal{H}_{QP\&B}$, the belief that all opponents
 531 cooperate is represented by a plane \mathbf{B}_9 . The angle between \mathbf{B}_9 and the plane
 532 representing second move cooperation is β_{SM} . The angle between \mathbf{B}_9 and the plane
 533 representing first move cooperation is β_{FM} . This also defines the plane \mathbf{B}_0 , which
 534 is orthogonal to \mathbf{B}_9 , naturally representing the belief of all opponents defecting
 535 and the planes \mathbf{B}_i between \mathbf{B}_9 and \mathbf{B}_0 . This incorporates the representation of all
 536 three measurements and their relationships (compatible or complementary) into
 537 one 4-dimensional Hilbert space, with clear estimated probabilities resulting from
 538 this representation.

539 5.3. *Fitting the data*

540 We fit the experimental data of the three measurements to our model. Note that
 541 the proportions of the second move actions are already incorporated in the starting
 542 state vector (equation 15). Since we have derived concrete dependencies of the
 543 beliefs on the second moves, and of the first moves on the beliefs, we can formally

544 fit the experimental data of the *Elicit_Beliefs* group to our model. To do so, we
545 shall estimate an optimal value of β for the beliefs on the second moves, as well as
546 for the first moves on the beliefs. This can be achieved by minimizing the distance
547 between the counts observed in our data set and the expected frequencies based
548 on the equations derived above. The chi-squared test is typically used to check
549 whether or not an observed set of proportions sufficiently matches the expected
550 set, so we will focus on minimizing this statistic.

551 Let us first focus on the two contingency tables representing the dependencies
552 of the beliefs on the second moves (see Tables 4 and 5). When a specific value of
553 β_{SM} is provided, we can estimate the expected probabilities $P(B_i|D)$ and $P(B_i|C)$
554 based on equations (23) and (25), respectively, and subsequently evaluate a chi-
555 squared statistic for each of the two tables. In order to estimate an appropriate
556 β_{SM} , we optimize an algorithm in which the sum of the two chi-squared statistics
557 (one for the SM defectors and one for SM cooperators) is minimized over a range
558 of possible values for β_{SM} (ranging from $-\pi/2$ to $\pi/2$). The value of β_{SM} for which
559 this sum reaches its lowest point equals -0.2048 , corresponding to chi-squared
560 statistics of 14.13 and 14.24 for the two contingency tables (one concerning
561 the second move cooperators and one concerning the second move defectors),
562 respectively. As expected, our estimated β_{SM} is indeed near 0.

563 Under normal circumstances, these chi-squared statistics can be translated
564 into p-values, by relying on their asymptotic approximation of a chi-squared
565 distribution with $I - 1$ degrees of freedom (with $I = 10$ the number of possible
566 beliefs). For our data set, however, this asymptotic procedure can be problematic
567 because several of the expected frequencies fall below five. This induces concern
568 about the accuracy of any p-value obtained through asymptotic approximation;

i	0	1	2	3	4	5	6	7	8	9
Observed counts	1	0	0	0	3	6	9	6	4	3
Observed proportions	0.031	0.000	0.000	0.000	0.094	0.188	0.281	0.188	0.125	0.094
Expected proportions	0.011	0.000	0.005	0.025	0.058	0.099	0.144	0.187	0.223	0.248

Table 4: The observed counts, as well as the observed and expected proportions of the beliefs of second move cooperators

i	0	1	2	3	4	5	6	7	8	9
Observed counts	4	2	5	5	9	3	0	0	0	0
Observed proportions	0.143	0.071	0.179	0.179	0.321	0.107	0.000	0.000	0.000	0.000
Expected proportions	0.156	0.163	0.160	0.147	0.127	0.101	0.072	0.045	0.022	0.007

Table 5: The observed counts, as well as the observed and expected proportions of the beliefs of second move defectors

569 therefore, we resort to a more accurate estimation via Monte Carlo simulation.
570 This technique simulates the sampling distribution of the test statistic (in this
571 case, chi-squared) using Monte Carlo methods. We generate random contingency
572 tables with the same marginal distribution as our data (i.e. the same sample
573 size), and calculate their chi-squared statistic. Subsequently, it is determined
574 how many of these random samples display a test-statistic which is larger than
575 the one that was originally obtained. The resulting proportion of more extreme
576 chi-squared statistics represents our new and more accurate p-value. Note that
577 what can be calculated for one chi-squared statistic can also be achieved for
578 a sum of chi-squared statistics: we can simulate a p-value corresponding to
579 the proportion of summed test-statistics, which are larger than the original sum
580 (14.13 + 14.24 = 28.37). For our analyses, we chose to rely on 10^4 simulated
581 samples.

582 According to the reasoning in the previous paragraph, these two test-statistics
583 allow us to calculate a p-value through Monte Carlo simulation: we obtain one of
584 0.071 for both tables combined. Their observed counts, alongside the observed

i	Observed counts	Totals	Observed proportions	Expected proportions	χ^2
0	5	5	1.000	0.997	0.016
1	2	2	1.000	0.947	0.111
2	5	5	1.000	0.844	0.925
3	5	5	1.000	0.699	2.153
4	6	12	0.500	0.530	0.044
5	3	9	0.333	0.358	0.023
6	1	9	0.111	0.202	0.464
7	0	6	0.000	0.083	0.542
8	0	4	0.000	0.014	0.056
9	0	3	0.000	0.003	0.010

Table 6: The observed number of cooperators, total number of participants, and observed as well as expected proportions of and expected proportions of first move cooperators. Note that the observed number of defectors (as well the the respective observed and expected frequencies) are not mentioned in this table since this information is redundant (the observed counts of cooperators and defectors sum to the totals and the observed/expected proportions of both defectors and cooperators sum to one).

585 and expected frequencies, can be found in Tables 4 and 5. The p-value testing the
586 null hypothesis of no significant difference between our observed and expected
587 proportions on the $\alpha = 0.05$ level indicates an acceptable fit. As this p-value is
588 estimated using simulation, the degrees of freedom are not taken into account,
589 unlike a traditional (asymptotic) p-value where the chi-square distribution is used.
590 As such, this p-value does not take into account that 20 proportions are estimated
591 using only one free parameter, making our estimated p-value even more favorable
592 to accepting the null hypothesis than the value suggests at first sight. See Tables 4
593 and 5.

594 When we aim to establish an optimal value of β_{FM} for modeling the first move
595 actions, we see that we have to deal with ten different contingency tables: one for
596 each belief in the number of cooperators ($i = 0, \dots, 9$). Since the observed and
597 expected probabilities in each of these contingency tables sum to one, we only

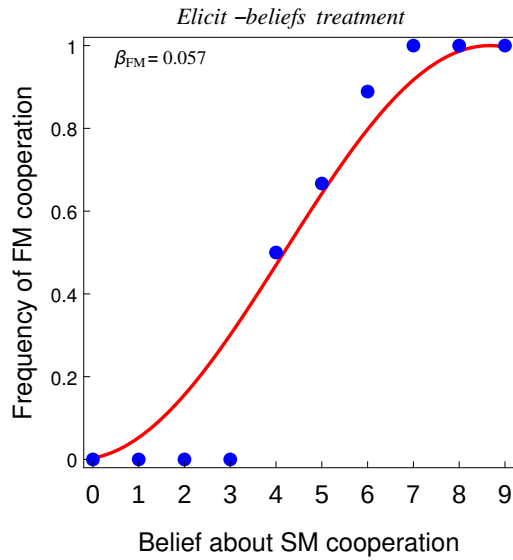


Figure 5: Observed frequency of FM cooperation versus elicited beliefs on SM cooperation (blue dots) and fitted model (equation 27, red line).

598 need to focus on the data counts and proportions for the cooperators $P(C|B_i)$.
 599 Similar to the beliefs of the second moves, we establish an optimal value of
 600 β_{FM} for the first move cooperators by minimizing the sum of the ten chi-squared
 601 statistics using equation (27). The optimal value of β_{FM} is 0.057 which is close
 602 to 0, as expected. Figure 5 plots the analytical prediction of the POVM model
 603 (equation 27) for the relationship between first move cooperation rates and stated
 604 beliefs about second mover cooperation with $\beta_{FM} = 0.057$, and compares it to the
 605 experimental observations.¹⁰ The chi-squared statistics and expected proportions
 606 are displayed in Table 6; and the corresponding simulated p-value equals 0.715
 607 indicating a very good fit.

¹⁰Blanco et al. (2014, Figure 3) explain the observed relationship between both experimental variables with a probit regression, obtaining a similar dependency. Nevertheless, our analytical curve has a deeper meaning because the functional form (equation 27) is a direct consequence of the geometrical structure of the POVM model.

608 **6. Discussion**

609 Our decision to abandon the restriction that outcome vectors coming from one
610 measurement are orthogonal to each other has consequences. The most important
611 one is the loss of the repeatability of outcomes.¹¹ Repeatability entails that when
612 a measurement is performed twice (without any manipulation or evolution of the
613 system between two measurements), the same outcome is observed twice. This
614 is assured in a standard quantum-like model, as the projection of a state vector
615 onto an orthogonal subspace gives the null vector. Repeatability seems a very
616 logical and sensible restriction but has been called into question, specifically when
617 applied in quantum cognition. See for example Khrennikov et al. (2014) for a
618 thorough discussion of this problem and Aliakbarzadeh and Kitto (2016) about
619 the use of POVMs, which lack repeatability, in Social Sciences.

620 In our context, the loss of repeatability in the measurement of beliefs means
621 that when a player replies that, e.g., ‘six’ opponents cooperate, he might reply
622 ‘seven’ when the question would be posed again. To justify this, we consider
623 the measurements to be *unsharp*. Unsharp measurements are measurements such
624 that the outcome represents a bigger subset of a (possible non-discrete) set of
625 outcomes. This is applicable in cases where a subject is asked to form a precise
626 opinion or belief but he is actually forming a broader opinion or belief. For
627 our example of interest, the subjects may think ‘most of them’ but have to give
628 a discrete number as an answer. We assume that when a player replies that,
629 e.g., six out of nine opponents cooperate, this indicates the player believing
630 ‘somewhere around six out of nine opponents cooperate’. This implies that he

¹¹Also called first kindness in Danilov and Lambert-Mogiliansky (2008).

631 would not necessarily disagree with the opinion that seven out of nine opponents
632 have cooperated. This structure can be viewed in the form of the belief vectors
633 $|B_i\rangle$. The state vector collapsing on $|B_6\rangle$ does not preclude the outcome associated
634 with $|B_7\rangle$, as they are close to each other, with the angle between them equaling
635 $\pi/18$. The closer two vectors are to being orthogonal, the more the outcomes
636 they represent do preclude each other. The vectors $|B_0\rangle$ and $|B_9\rangle$ are the limit
637 case: being orthogonal makes the events associated with them (the opponent
638 cooperating and defecting for sure) completely preclude each other.

639 The use of these non-orthogonal outcome vectors also opens up new research
640 possibilities within quantum cognition. Inflated dimensionality is a common
641 obstacle in elegant model building. Once multiple (compatible) measurements
642 with more than two possible outcomes are taken into account, any standard
643 quantum model would require high dimensionality. Next to the ease of reducing
644 dimensionality, some extra structure can be incorporated in the model. As
645 can be seen in our case, implicit relationships between different outcomes can
646 be represented. Our model can have the B_1 outcome be ‘closer’ to the B_2
647 outcome than it is to the B_8 outcome. In a standard quantum model, all outcome
648 vectors are orthogonal, so all outcomes play a similar role towards each other.
649 This works for all discrete examples in physics, but in decision-making there
650 are numerous examples of ordinal scales where a kind of structure is implied
651 between the different outcomes. While this might have seemed problematic at
652 first, there are other examples in which these techniques are successfully used,
653 again, see Aliakbarzadeh and Kitto (2016) and Yearsley (2016). As there is,
654 to our knowledge, no known way of simply incorporating ordinal scales into
655 the quantum framework (preserving some sense of the notion of an order),

656 constructing bases similar to the one in this paper seems to be an interesting
657 road for future research. One obvious candidate for this treatment would be the
658 quantum-like modeling of Likert scales. They allow for (mostly) 5 or 7 different
659 ordered outcomes and are ordinal scales widely used within cognition. Some first
660 steps for Likert scales of this form are presented in Yearsley (2016).

661 **7. Conclusion**

662 In this paper we constructed a quantum-like model for preferences and beliefs
663 in a social dilemma game. By taking a new look at the experimental data set
664 collected by Blanco et al. (2014) for a sequential prisoner's dilemma, we identified
665 and discussed three distinct effects. These effects are all explained as a specific
666 type of relationship between the measurements performed in the experiment.
667 First, there is a direct positive correlation between the player's first and second
668 move. As it is shown in Blanco et al. (2014), however, this does not provide a
669 complete picture of the subject's behavior because this correlation is also driven
670 by an indirect belief-based channel. This interaction is made up of the two other
671 effects: the influence of the second move on the beliefs of the player and the
672 influence of the beliefs of the player on his first move. The former effect is the so-
673 called consensus effect and we attributed the latter effect to the player becoming
674 more reasoned about his preferences.

675 The nature of these last two effects both point towards a quantum-like
676 model. The non-classical structure of the consensus effect is already discussed
677 in Busemeyer and Pothos (2012) where it is viewed as a form of social projection.
678 Busemeyer and Pothos (2012) represented the construction of a player's belief as
679 a rotation of the state vector in a Hilbert Space. In our case, the belief construction
680 and the second move action are non-commuting (and thus incompatible) in nature.

681 The effect of the player becoming more reasoned can be seen as a violation of
682 the sure thing principle. The act of belief elicitation significantly changes the
683 cooperation rate of the first move action, regardless of the beliefs elicitation
684 outcome. This also suggests viewing the belief elicitation and the first move
685 action as incompatible. We combined all these observations and constructed a
686 Hilbert Space in which the action measurements on the one hand and the belief
687 measurements on the other hand were viewed as incompatible measurements by
688 defining a different basis for each.

689 Following the more traditional recipe, we obtained a model within a 100-
690 dimensional Hilbert Space which was greatly overparametrized from a statistical
691 point of view. As a solution to this problem, we proposed to redefine the belief
692 basis as two-dimensional and considered two options. The first option constructed
693 a POVM, which framed our model neatly into conventional quantum theory, at
694 the cost of defining an ancillary outcome. The second option dismissed this
695 new outcome, staying closer to the actual experiment, at the cost of leaving the
696 standard quantum-like framework. Both options result in identical models and
697 diminish the problematic dimensionality. This model incorporates the three effects
698 observed in the experiment, and yields elegant dependencies between actions and
699 beliefs with successful statistical fit.

700 As not all vectors associated with outcomes of the belief measurement were
701 orthogonal, we lose repeatability of outcomes: obtaining an outcome does not
702 exclude obtaining a different outcome when the same measurement is performed
703 again immediately. We defined unsharp measurements of beliefs where forcing
704 the player to pick one outcome does not mean he disagrees with some other
705 possible outcome, thus relaxing the constraint of repeatability. For more on

706 the need (or lack thereof) of repeatability in psychological measurements, see
707 Khrennikov et al. (2014).

708 **8. Acknowledgements**

709 We are grateful to the authors of the original experiment (Blanco et al., 2014)
710 and especially to Hans-Theo Normann for providing access to their experimental
711 data set. We would also like to thank two anonymous referees, the Special Issue
712 Guest Editor, Sandro Sozzo, and the Associate Editor, Ehtibar Dzhafarov, for
713 helpful comments; as well as James Yearsley and Ariane Lambert-Mogiliansky
714 for fruitful discussions.

715 This work has been partly supported by COST Action IC1205 on
716 Computational Social Choice. ABD was supported by Air Force Office of
717 Scientific Research (AFOSR), Air Force Material Command, USAF, grant FA
718 8655-13-1-3044.

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