Lexicographic choice functions without archimedeanicity

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Imprecise probabilities

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broaden probability theory in order to deal with imprecision and indecision.

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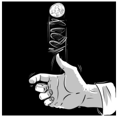
Basic idea of imprecise probabilities: decisions and choice.

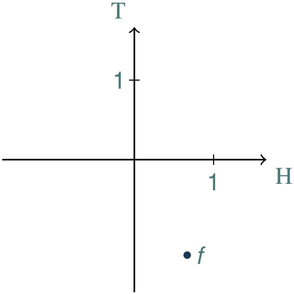
What do we choose between: gambles

The random variable X takes values in the possibility space $\mathscr{X}$.

A gamble $f\colon \mathscr{X} \to \mathbb{R}$ is an uncertain reward whose value is $f(X)$, and we collect all gambles in $\mathscr{L}$.



$\mathscr{X} = \{\mathrm{H}, \mathrm{T}\}$



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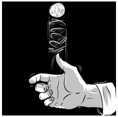

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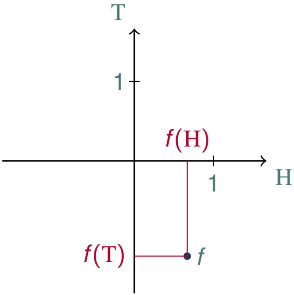

The simplest way out: binary choice

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We collect them in the agent's set of desirable gambles.

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$$f_1 \succ f_2 \Leftrightarrow f_1 - f_2 \succ 0 \Leftrightarrow f_1 - f_2 \in D$$

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To summarise:

$$D = \{f \in \mathscr{L} : f \succ 0\}.$$

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Sets of desirable gambles are more general than lower previsions or sets of probabilities.

Working with them is simple and elegant.

They form a strong belief structure: they generalise conservative logical inference (natural extension).

More general choice

A: option set of gambles (non-empty but finite)

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$\mathscr{Q}$: set of all the non-empty but finite subsets of $\mathscr{L}$.

A choice function C is a map

$$C\colon \mathscr{Q} \to \mathscr{Q} \cup \{\emptyset\} : A \mapsto C(A) \text{ such that } C(A) \subseteq A.$$

Axioms for “coherent”

We call a choice function C on $\mathscr{Q}$ coherent if for all A, A_1, A_2 in $\mathscr{Q}$, f, g in $\mathscr{L}$ and λ in $\mathbb{R}_{>0}$:

- C1 $C(A) \neq \emptyset$; [non-emptiness]
- C2 if $f < g$ then $\{g\} = C(\{f, g\})$; [non-triviality]
- C3a if $A \subseteq R(A_1)$ and $A_1 \subseteq A_2$ then $A \subseteq R(A_2)$; [Sen's condition α]
- C3b if $A_1 \subseteq R(A_2)$ and $A \subseteq A_1$ then $A_1 \setminus A \subseteq R(A_2 \setminus A)$; [Aizerman's condition]
- C4a if $A_1 \subseteq C(A_2)$ then $\lambda A_1 \subseteq C(\lambda A_2)$; [scaling]
- C4b if $A_1 \subseteq C(A_2)$ then $A_1 + \{f\} \subseteq C(A_2 + \{f\})$. [addition]

Coherent choice functions do not form a strong belief structure.

Convexity

We need something stronger: is a convexity axiom sufficient?
For all A and A_1 in $\mathscr{Q}$:

C5 if $A \subseteq A_1 \subseteq \mathrm{CH}(A)$ then $C(A) \subseteq C(A_1)$. [convexity]

Purely binary choice with choice functions

For all A in $\mathscr{Q}$

$C_D(A) := \{f \in A : (\forall g \in A) g - f \notin D\}$ "the undominated gambles in A"

But: not every choice function is binary.

Connection between desirability and choice functions

From D to C: $C_D(A) := \{f \in A : (\forall g \in A) g - f \notin D\}$ for all A in $\mathscr{Q}$

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Coherence is preserved.

Desirability and convexity

If D is coherent, then C_D is coherent.

What about C5?

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C_D is coherent + C5 $\Leftrightarrow$ D is coherent and $\mathrm{posi}(D^c) = D^c$.

Interestingly, $\mathrm{posi}(D^c) = D^c$ is a characterisation of lexicographic probability!

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We collect such D's in $\bar{\mathscr{D}}_{\mathrm{L}} := \{D \text{ coherent} : \mathrm{posi}(D^c) = D^c\}$, and call $\bar{\mathscr{C}}_{\mathrm{L}} := \{C_D : D \in \bar{\mathscr{D}}_{\mathrm{L}}\}$ the lexicographic choice functions.

Is every coherent and convex choice function an infimum of lexicographic choice functions?

Rejection sets: choice amongst three

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When $|\mathscr{X}| = 2$, choice functions are determined by choice amongst three options.

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The question becomes: "Is $0 \in R(\{0, f, g\})$?"

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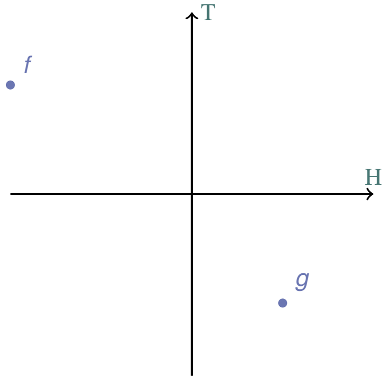

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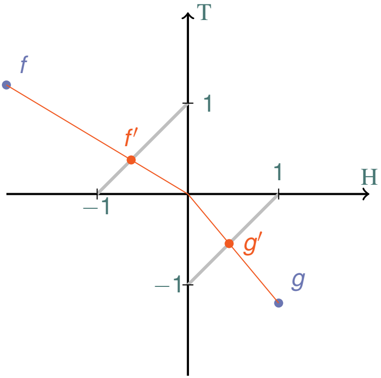

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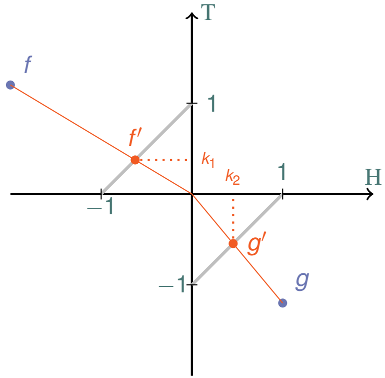

$$K = \{(k_1, k_2) \in [0,1)^2 : 0 \in R(\{0, (k_1 - 1, k_1), (k_2, k_2 - 1)\})\}$$

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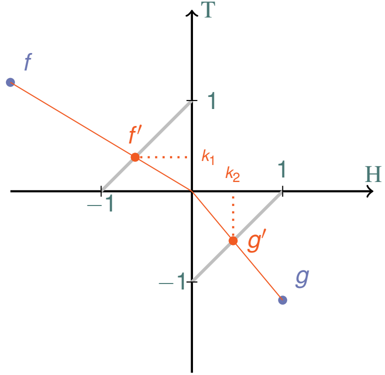

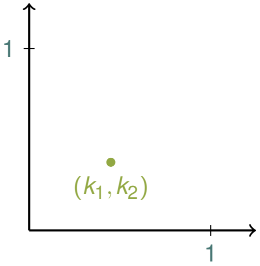

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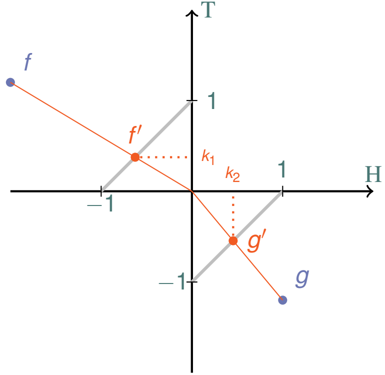

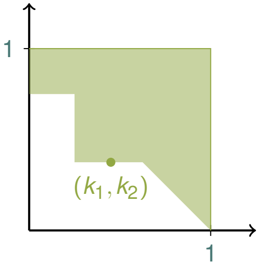

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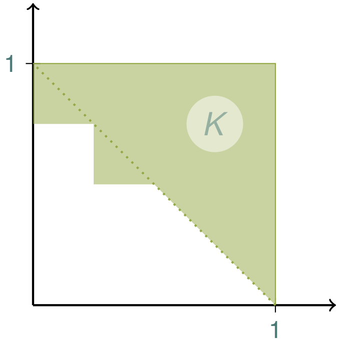
From coherence axioms to graphical properties



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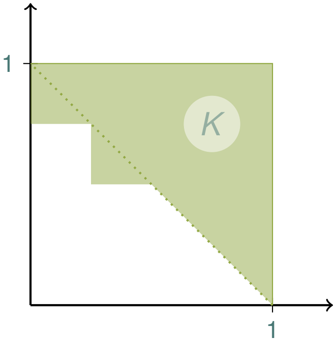
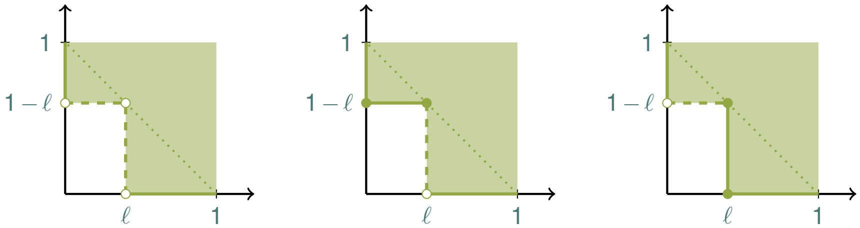


Properties:

- $(0,0) \notin K$;
- if $(k_1, k_2) \in K$ and $k_1 \le \ell_1$ and $k_2 \le \ell_2$, then $(\ell_1, \ell_2) \in K$ [K is increasing];
- C satisfies C5 $\Leftrightarrow \{(\ell_1, \ell_2) \in [0,1)^2 : \ell_1 + \ell_2 > 1\} \subseteq K$;
- below the diagonal, the border of K has only horizontal or vertical lines.

What do (infima of) lexicographic choice functions look like?



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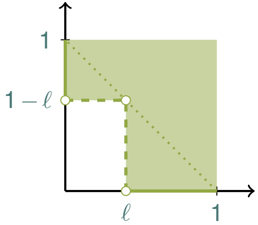

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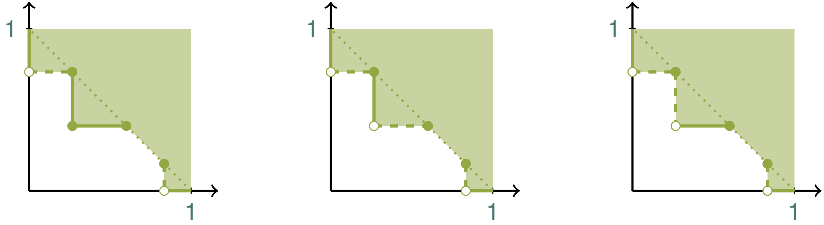

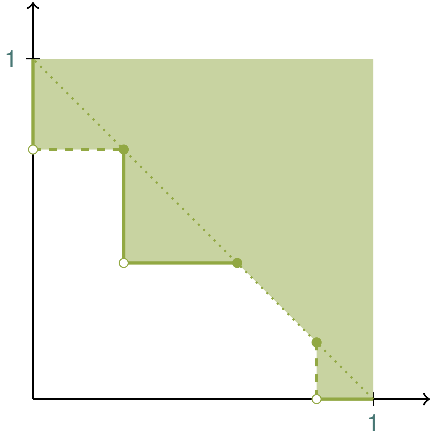
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Conclusion

Coherence and C5 is not sufficient to guarantee that C is an infimum of lexicographic choice functions.