THE WAVELET TRANSFORMS IN GELFAND-SHILOV SPACES

STEVAN PILIPOVIĆ, DUŠAN RAKIĆ, NENAD TEOFANOV, AND JASSON VINDAS

Abstract. We describe local and global properties of wavelet transforms of ultradifferentiable functions. The results are given in the form of continuity properties of the wavelet transform on Gelfand-Shilov type spaces and their dual spaces. In particular, we introduce a new family of highly time-scale localized spaces on the upper half-space. We study the wavelet synthesis operator (the left-inverse of the wavelet transform) and obtain the resolution of identity (Calderón reproducing formula) in the context of ultradistributions.

1. Introduction

One of the most useful concepts in time-frequency analysis for signal analysts and engineers is the wavelet series expansion of a signal. The coefficients in such series, representing the discrete version of a signal, are then used in the signal analysis, processing and synthesis. The continuous versions of these discrete representations lead to the wavelet (analysis) transform $\mathcal{W}_\psi$ and the wavelet synthesis operator $\mathcal{M}_\phi$ [17]. The authors have studied both transforms in several papers, [29, 30, 31, 35, 40]. Although the continuous transforms are less popular in the literature than their discrete counterparts, studying the intrinsic properties of the continuous wavelet transform is also a very important subject. In particular, continuous transforms may potentially serve well in the study of microlocal and pointwise aspects of a signal, cf. [11, 18, 22, 33]. Microlocal aspects have also been recently studied by different authors via shearlet transforms, see e.g. [9, 15]. An interesting alternative approach to the wavelet transform in several variables with applications in microlocal analysis is performed in [11].

It is well known that smooth orthonormal wavelets cannot have exponential decay, cf. [8, 10, 16]. In this paper we study the wavelet transform defined by wavelets with almost exponential decay. In this context it is then natural to work with Gelfand-Shilov spaces as a functional-analytic groundwork. We shall prove continuity theorems for the wavelet transform and the wavelet synthesis operator on spaces of Gelfand-Shilov type, see Section 2 for definitions. In contrast to known results [17, 27, 29, 35], we introduce in the article a new family of (semi-)norms with additional parameters in the corresponding wavelet image space. These parameters measure fast decay or growth orders of the wavelet transform and the wavelet synthesis operator. Roughly speaking, our considerations are able to detect Gevrey ultra-differentiability properties (such as analyticity) via appropriate decay of the wavelet transform.

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Gelfand-Shilov spaces of ultra-differentiable functions were originally introduced in [12] as a tool to treat existence and uniqueness questions for parabolic initial-value problems. Such spaces, consisting of Gevrey ultra-differentiable functions, are also very useful in hyperbolic and weak hyperbolic problems, see [1, 13, 36] and the references therein. Exponential decay and holomorphic extension of solutions to globally elliptic equations in terms of Gelfand-Shilov spaces have been recently studied in [3, 4], see also [1]. We refer to [26] for an overview of results in this direction and for applications in quantum mechanics and traveling waves. On the other hand, in the context of time-frequency analysis, the Gelfand-Shilov spaces have recently captured much attention in connection with modulation spaces [14], localization operators [7], and the corresponding pseudodifferential calculus [32, 38, 39]. We follow here Komatsu's approach [20] to spaces of ultra-differentiable functions. Another widely used approach is that of Braun, Meise, Taylor, Vogt and their collaborators, see e.g. [2] and the recent contribution [34]. These two approaches are equivalent in many interesting situations, cf. [23] for more details.

We remark that the wavelet transform in the context of Gelfand-Shilov spaces was already studied in [27, 28] in dimension $n = 1$. In the present article we propose and develop an intrinsically different approach, which also covers the multidimensional case. We employ here wavelets with all vanishing moments. The advantage of this condition is that one is able to translate ultra-differentiability and subexponential decay of functions into sharper localization properties in the scale variable of the wavelet transform. Our approach also provides the resolution of the identity (Calderón reproducing formula) for ultradistributions. As a matter of fact, this inversion formula for the wavelet transform of ultradistributions seems to be out of reach of the results from [27, 28].

We point out that the number of vanishing moments (called *cancellations* in [25]) of a wavelet ψ is intimately related to *the order of approximation* of the corresponding wavelet series via the so-called Strang-Fix condition. In particular, wavelets with many vanishing moments are appropriate when dealing with objects which are very regular except for a few isolated singularities, cf. [8, 24]. It is also well known [16] that as soon as an orthogonal wavelet belongs to the Schwartz class $\mathcal{S}(\mathbb{R}^n)$ then all its moments must vanish. In [17] wavelets with all vanishing moments were used to develop a distributional framework for the wavelet transform in the context of Lizorkin spaces. Here we shall develop an new ultradistributional framework.

The paper is organized as follows. In Section 2 we explain some facts about Gelfand-Shilov type spaces. In particular, we introduce a new four-parameter family $\mathcal{S}^s_{t,\tau_1,\tau_2}(\mathbb{H}^{n+1})$ of function spaces on the upper-half space and study its properties (see Subsection 2.1). Section 3 contains our main continuity results, Theorems 1 and 2, which imply Calderón reproducing formulas for ultradistributions (Theorem 3 and Corollary 1). Finally, in Section 4 we collect the proofs of the main results.

1.1. **Notation and notions.** We denote by $\mathbb{H}^{n+1} = \mathbb{R}^n \times \mathbb{R}_+$ the upper half-space and $\mathbb{N} = \{0, 1, 2, \dots\}$. The unit sphere in $\mathbb{R}^n$ is denoted by $\mathbb{S}^{n-1}$. When $x, y \in \mathbb{R}^n$ and $m \in \mathbb{N}^n$, $|x|$ denotes the Euclidean norm, $\langle x \rangle = (1 + |x|^2)^{1/2}$, $xy = x_1y_1 + x_2y_2 + \cdots + x_ny_n$, $x^m = x_1^{m_1} \dots x_n^{m_n}$, $m! = m_1!m_2! \dots m_n!$, $\partial^m = \partial^m_x = \partial^{m_1}_{x_1} \dots \partial^{m_n}_{x_n}$, and $\triangle$ denotes the

Laplacian: $\triangle = \triangle_x = \partial^2_{x_1^2} + \cdots + \partial^2_{x_n^2}$. By a slight abuse of notation, the length of a multi-index $m \in \mathbb{N}^n$ is denoted by $|m| = m_1 + \cdots + m_n$, and the meaning of $|\cdot|$ shall be clear from the context. We denote by $C, h, \ldots$ positive constants which may be different in various occurrences; $A \lesssim B$ means that $A \leq C \cdot B$ for some positive constant C. If $A \lesssim B$ and $B \lesssim A$ we write $A \asymp B$. The dual pairing between a test function space $\mathcal{A}$ and its dual space of (ultra)distributions $\mathcal{A}'$ is denoted by $\langle \cdot , \cdot \rangle =_{\mathcal{A}'} \langle \cdot , \cdot \rangle_{\mathcal{A}}$.

When α and β are multi-indices and n is the space dimension, we have

$$|\alpha|! \leq n^{|\alpha|}\alpha!, \quad \alpha!\beta! \leq (\alpha+\beta)! \leq 2^{|\alpha|+|\beta|}\alpha!\beta!.$$

2. Gelfand-Shilov type spaces

In this section we discuss definitions and properties of the test function spaces that will be employed in our study of the wavelet transform.

For the reader's convenience, and in order to be self-contained, we first recall various spaces of rapidly decreasing functions that were considered in the context of wavelet transform in e.g. [17, 29].

The moments of $\varphi \in \mathcal{S}(\mathbb{R}^n)$, the Schwartz space of rapidly decreasing smooth test functions, are denoted by $\mu_m(\varphi) = \int_{\mathbb{R}^n} x^m \varphi(x)dx$, $m \in \mathbb{N}^n$. We fix constants in the Fourier transform as follows: $\hat{\varphi}(\xi) = \int_{\mathbb{R}^n} \varphi(x)e^{-ix\cdot\xi}\mathrm{d}x$, $\xi \in \mathbb{R}^n$.

The Lizorkin space $\mathcal{S}_0(\mathbb{R}^n) = \{\varphi \in \mathcal{S}(\mathbb{R}^n) : \mu_m(\varphi) = 0, \ \forall m \in \mathbb{N}^n\}$ is a closed subspace of $\mathcal{S}(\mathbb{R}^n)$ equipped with the relative topology inhered from $\mathcal{S}(\mathbb{R}^n)$, [17, 37].

The space $\mathcal{S}(\mathbb{H}^{n+1})$ of "highly localized functions over the half-space" [17] consists of $\Phi \in C^\infty(\mathbb{H}^{n+1})$ such that the seminorms

$$p^{l,k}_{\alpha,\beta}(\Phi) = \sup_{(b,a)\in\mathbb{H}^{n+1}} \left(a^l + \frac{1}{a^l}\right) \langle b\rangle^k \left|\partial^\alpha_a \partial^\beta_b \Phi(b,a)\right| \tag{1}$$

are finite for all $l, k, \alpha \in \mathbb{N}$ and for all $\beta \in \mathbb{N}^n$.

When $(b,a) \in \mathbb{H}^{n+1}$ and $k, l \in \mathbb{N}$, then $\left(a^l + \frac{1}{a^l}\right) \asymp \left(a + \frac{1}{a}\right)^l$ and $\langle b\rangle^k \asymp |b|^k$ when $|b|$ is large enough, see also Remark 1 below.

We introduce spaces of Gelfand-Shilov type by analogy to $\mathcal{S}(\mathbb{R}^n)$, $\mathcal{S}_0(\mathbb{R}^n)$ and $\mathcal{S}(\mathbb{H}^{n+1})$. The family of spaces $\mathcal{S}^{\rho_1}_{\rho_2}(\mathbb{R}^n)$ introduced by I. M. Gelfand and G. E. Shilov in the study of Cauchy problems was systematically studied in [12], see [26] for a recent survey.

Recall that $\varphi \in \mathcal{S}(\mathbb{R}^n)$ belongs to the Gelfand-Shilov space $\mathcal{S}^{\rho_1}_{\rho_2}(\mathbb{R}^n)$, $\rho_1, \rho_2 > 0$, if there exists a constant $h > 0$ such that

$$|x^\alpha \varphi^{(\beta)}(x)| \lesssim h^{-|\alpha+\beta|}\, \alpha!^{\rho_2}\beta!^{\rho_1}, \quad x \in \mathbb{R}^n,\, \alpha, \beta \in \mathbb{N}^n.$$

The space $\mathcal{S}^{\rho_1}_{\rho_2}(\mathbb{R}^n)$ is nontrivial if and only if $\rho_1 + \rho_2 \geq 1$. The family of norms

$$p^{\rho_1,\rho_2}_h(\varphi) = \sup_{x\in\mathbb{R}^n, \alpha,\beta\in\mathbb{N}^n} \frac{h^{|\alpha+\beta|}}{\alpha!^{\rho_2}\beta!^{\rho_1}} |x^\alpha \varphi^{(\beta)}(x)|, \quad h > 0, \tag{2}$$

defines the canonical inductive topology of $\mathcal{S}^{\rho_1}_{\rho_2}(\mathbb{R}^n)$.

It is well known [6] that $\varphi \in \mathcal{S}^{\rho_1}_{\rho_2}(\mathbb{R}^n)$ if and only if there exists $h > 0$ such that

$$\sup_{x\in\mathbb{R}^n} e^{h|x|^{1/\rho_2}}|\varphi(x)| < \infty \quad \text{and} \quad \sup_{\xi\in\mathbb{R}^n} e^{h|\xi|^{1/\rho_1}}|\hat{\varphi}(\xi)| < \infty.$$

Hence, the Fourier transform is an isomorphism between $\mathcal{S}^{\rho_1}_{\rho_2}(\mathbb{R}^n)$ and $\mathcal{S}^{\rho_2}_{\rho_1}(\mathbb{R}^n)$.

The space $\mathcal{S}^{\rho_1}_{\rho_2}(\mathbb{R}^n)$ is non-quasianalytic, namely, it contains compactly supported functions, if and only if $\rho_1 > 1$. Then it consists of Gevrey ultra-differentiable functions, cf. [36]. If $\rho_1 = 1$, then $\varphi \in \mathcal{S}^{\rho_1}_{\rho_2}(\mathbb{R}^n)$ is a real analytic function, and if $0 < \rho_1 < 1$, then it is an entire function.

Remark 1. We will often use an equivalent family of norms where in (2) (and in other similar situations) $x^\alpha \varphi^{(\beta)}(x)$ is replaced by $\langle x\rangle^\alpha \varphi^{(\beta)}(x)$, $(\langle x\rangle^\alpha \varphi(x))^{(\beta)}$ or $(x^\alpha \varphi(x))^{(\beta)}$. Moreover, instead of the supremum norm any L^p norm $(1 \le p < \infty)$ gives rise to an equivalent topology on $\mathcal{S}^{\rho_1}_{\rho_2}(\mathbb{R}^n)$ (cf. [5, Ch 2.5]).

We denote by $(\mathcal{S}^{\rho_1}_{\rho_2})_0(\mathbb{R}^n)$ the closed subspace of $\mathcal{S}^{\rho_1}_{\rho_2}(\mathbb{R}^n)$ given by

$$(\mathcal{S}^{\rho_1}_{\rho_2})_0(\mathbb{R}^n) = \left\{\varphi \in \mathcal{S}^{\rho_1}_{\rho_2}(\mathbb{R}^n) : \ \mu_m(\varphi) = 0, \ \forall m \in \mathbb{N}^n\right\}.$$

One can show that $(\mathcal{S}^{\rho_1}_{\rho_2})_0(\mathbb{R}^n)$, $\rho_1, \rho_2 > 0$, is nontrivial if and only if $\rho_2 > 1$ (cf. [12]).

2.1. **Gelfand-Shilov type spaces on the upper half-space.** In this subsection we introduce a new scale of function spaces which describes sharp subexponential/super-exponential localization over the upper half-space.

To this end, we employ parameters which measure the decay properties of a function with respect to the scaling variable $a > 0$ at zero and at infinity, as well as their Gevrey ultra-differentiability and decay properties in the localization variable b. While the seminorms in (1) measure polynomial decay of a certain order with respect to the scaling parameter $a > 0$ at zero and at infinity, the seminorms in (3) may detect (super- and sub-) exponential decay of different orders at zero and at infinity.

Definition 1. Let $s, t, \tau_1, \tau_2 > 0$. A smooth function Φ belongs to $\mathcal{S}^s_{t,\tau_1,\tau_2}(\mathbb{H}^{n+1})$, if for every $\alpha \in \mathbb{N}$ there exists a constant $h > 0$ such that

$$p^{s,t,\tau_1,\tau_2}_{\alpha,h}(\Phi) = \sup \frac{h^{|\beta|+k+l_1+l_2}}{\beta!^s k!^t l_1!^{\tau_1} l_2!^{\tau_2}} \left(a^{l_1} + \frac{1}{a^{l_2}}\right) \langle b\rangle^k \left|\partial^\alpha_a \partial^\beta_b \Phi(b,a)\right| < \infty, \tag{3}$$

where the supremum is taken over

$$((b,a),(k,l_1,l_2),\beta) \in \Lambda = \mathbb{H}^{n+1} \times \mathbb{N}^3 \times \mathbb{N}^n.$$

The topology of $\mathcal{S}^s_{t,\tau_1,\tau_2}(\mathbb{H}^{n+1})$ is defined via the family of seminorms (3), as inductive limit with respect to h and projective limit with respect to α.

The space $\mathcal{S}^s_{t,\tau_1,\tau_2}(\mathbb{H}^{n+1})$ is nontrivial if and only if $s + t \ge 1$, which can be proved as follows.

Consider the set of smooth functions in $\mathbb{H}^{n+1}$ of the form $\Phi(b,a) = g(b)f(a)$, $b \in \mathbb{R}^n$, $a \in \mathbb{R}_+$. Then $p^{s,t,\tau_1,\tau_2}_{\alpha,h}(\Phi) < \infty$ is equivalent to $p^{s,t}_h(g) < \infty$ and

$$\sup_{a>0, l_1, l_2 \in \mathbb{N}} \frac{h^{l_1+l_2}}{l_1!^{\tau_1} l_2!^{\tau_2}} \left(a^{l_1} + \frac{1}{a^{l_2}}\right) |\partial^\alpha_a f(a)| < \infty. \tag{4}$$

Thus, if $s + t \ge 1$, then $\mathcal{S}^s_{t,\tau_1,\tau_2}(\mathbb{H}^{n+1})$ is non-trivial, $\tau_1, \tau_2 > 0$. For example, if $g \in \mathcal{S}^s_t(\mathbb{R}^n)$, then $\mathbb{H}^{n+1} \ni (b,a) \mapsto e^{-a^{1/\tau_1} - a^{-1/\tau_2}} g(b) \in \mathcal{S}^s_{t,\tau_1,\tau_2}(\mathbb{H}^{n+1})$.

Since for fixed $a \in \mathbb{R}_+$ and $l_1 = l_2 = \alpha = 0$, it follows from (3) that $\Phi(\cdot, a) \in \mathcal{S}^s_t(\mathbb{R}^n)$, we see that the condition $s + t \geq 1$ is also necessary for the non-triviality of $\mathcal{S}^s_{t,\tau_1,\tau_2}(\mathbb{H}^{n+1})$.

Obviously, the family $\mathcal{S}^s_{t,\tau_1,\tau_2}(\mathbb{H}^{n+1})$ is increasing with respect to parameters s, t, τ_1, τ_2. The parameters τ_1 and τ_2 measure the behavior of $\Phi \in \mathcal{S}^s_{t,\tau_1,\tau_2}(\mathbb{H}^{n+1})$, with respect to $a > 0$ at infinity and at zero, respectively.

It can be shown that all these spaces of test functions are closed under multiplication by (ultra-)polynomials, partial differentiation (or more generally ultra-differential operators), translation and dilation, cf. [12] for $\mathcal{S}^{\rho_1}_{\rho_2}(\mathbb{R}^n)$. The following lemma can be proved in the same way as it is done in [12, Chapter IV 6.2] for $\mathcal{S}^{\rho_1}_{\rho_2}(\mathbb{R}^n)$, we therefore omit its proof.

Lemma 1. *Let $\Phi \in C^\infty(\mathbb{H}^{n+1})$ and let $\mathcal{F}_1\Phi$ denote its Fourier transform with respect to the first variable:*

$$\mathcal{F}_1\Phi(\xi, a) = \int_{\mathbb{R}^n} e^{-ib\xi}\, \Phi(b, a)\, db, \quad (\xi, a) \in \mathbb{H}^{n+1}.$$

Then $\Phi \in \mathcal{S}^s_{t,\tau_1,\tau_2}(\mathbb{H}^{n+1})$ if and only if $\mathcal{F}_1\Phi \in \mathcal{S}^t_{s,\tau_1,\tau_2}(\mathbb{H}^{n+1})$. Furthermore, $\mathcal{F}_1$ is a topological isomorphism between $\mathcal{S}^s_{t,\tau_1,\tau_2}(\mathbb{H}^{n+1})$ and $\mathcal{S}^t_{s,\tau_1,\tau_2}(\mathbb{H}^{n+1})$.

Next, we show that (3) precisely describes the rate of decay of the derivatives of Φ.

Proposition 1. *Let $\Phi \in \mathcal{S}^s_{t,\tau_1,\tau_2}(\mathbb{H}^{n+1})$ and $\alpha \in \mathbb{N}$. Set*

$$q^{s,t,\tau_1,\tau_2}_{\alpha,h}(\Phi) := \sup_{((b,a),\beta)\in\mathbb{H}^{n+1}\times\mathbb{N}^n} \frac{h^{|\beta|}}{\beta!^s}\, e^{h\left(a^{1/\tau_1}+a^{-1/\tau_2}+|b|^{1/t}\right)} \left|\partial^\alpha_a \partial^\beta_b \Phi(b, a)\right|.$$

Then $p^{s,t,\tau_1,\tau_2}_{\alpha,h}(\Phi) < \infty$ for some $h > 0$, if and only if $q^{s,t,\tau_1,\tau_2}_{\alpha,h}(\Phi) < \infty$ for some $h > 0$.

Proof. Assume that $p^{s,t,\tau_1,\tau_2}_{\alpha,h}(\Phi) < \infty$ for some $h > 0$. Then, for any given $l_1, l_2, k \in \mathbb{N}$,

$$\frac{h^{l_1+l_2+k}}{l_1!^{\tau_1} l_2!^{\tau_2} k!^t} \left(a^{l_1} + \frac{1}{a^{l_2}}\right) \langle b\rangle^k |\partial^\alpha_a \Phi(b, a)|$$

is uniformly bounded on $\mathbb{H}^{n+1}$. This implies that appropriate summations over l_1, l_2 and k are also uniformly bounded. Indeed, the estimate

$$C^{-1}\, e^{(r-\varepsilon)\eta^{1/r}} \leq \sum_{j=0}^{\infty} \frac{\eta^j}{(j!)^r} \leq C\, e^{(r+\varepsilon)\eta^{1/r}}, \ \forall \eta \geq 0,$$

which holds for every $r, \varepsilon > 0$ and for some $C = C(r, \varepsilon) > 0$, yields

$$|\partial^\alpha_a \partial^\beta_b \Phi(b, a)| \lesssim \tilde{h}^{|\beta|} \beta!^s e^{-\tilde{h}\left(a^{\frac{1}{\tau_1}} + (\frac{1}{a})^{\frac{1}{\tau_2}} + |b|^{\frac{1}{t}}\right)}, \quad (b, a) \in \mathbb{H}^{n+1}, \beta \in \mathbb{N}^n,$$

for some $\tilde{h} > 0$. By taking the corresponding supremum, we obtain that $q^{s,t,\tau_1,\tau_2}_{\alpha,h}(\Phi)$ is finite for some $h > 0$.

Conversely, assume that $q^{s,t,\tau_1,\tau_2}_{\alpha,h}(\Phi) < \infty$ for some $h > 0$. Employing the same estimate as above, we conclude that

$$(1 + a^{l_1})|\partial^\alpha_a \partial^\beta_b \Phi(b, a)| \lesssim h^{|\beta|+l_1} \beta!^s l_1!^{\tau_1},$$

$$\left(1+\frac{1}{a^{l_2}}\right)|\partial_a^\alpha\partial_b^\beta\Phi(b,a)|\lesssim h^{|\beta|+l_2}\beta!^s l_2!^{\tau_2}$$

and

$$\langle b\rangle^k|\partial_a^\alpha\partial_b^\beta\Phi(b,a)|\lesssim h^{|\beta|+k}\beta!^s k!^t,$$

for every $((b,a),(k,l_1,l_2),\beta)\in\Lambda$. Hence,

$$\left(a^{l_1}+\frac{1}{a^{l_2}}\right)\langle b\rangle^k|\partial_a^\alpha\partial_b^\beta\Phi(b,a)|^3\lesssim h^{3|\beta|+l_1+l_2+k}\beta!^{3s}l_1!^{\tau_1}l_2!^{\tau_2}k!^t,$$

i.e.

$$\frac{\tilde{h}^{|\beta|+l_1+l_2+k}}{\beta!^s l_1!^{\tau_1}l_2!^{\tau_2}k!^t}\left(a^{l_1}+\frac{1}{a^{l_2}}\right)\langle b\rangle^k|\partial_a^\alpha\partial_b^\beta\Phi(b,a)|<C$$

for some $\tilde{h},C>0$. By taking the supremum over Λ, we obtain that $p_{\alpha,h}^{s,t,\tau_1,\tau_2}(\Phi)$ is finite for some $h>0$.

□

3. Wavelet transform of ultradifferentiable functions and ultradistributions

In this section we study continuity properties of wavelet transforms on Gelfand-Shilov spaces of ultradifferentiable functions and their duals. In particular, we derive the resolution of identity formula in a class of tempered ultradistributions. As mentioned in the introduction, the most technical proofs are postponed to Section 4.

3.1. **Continuity theorems.** A function $\psi\in\mathcal{S}_{\rho_2}^{\rho_1}(\mathbb{R}^n)$ is called a *wavelet* if $\mu_0(\psi)=0$. The wavelet transform of a tempered ultradistribution $f\in(\mathcal{S}_{\rho_2}^{\rho_1}(\mathbb{R}^n))'$ with respect to the wavelet $\psi\in\mathcal{S}_{\rho_2}^{\rho_1}(\mathbb{R}^n)$ is defined via

$$\mathcal{W}_\psi f(b,a)=\left\langle f(x),\frac{1}{a^n}\bar{\psi}\left(\frac{x-b}{a}\right)\right\rangle=\frac{1}{a^n}\int_{\mathbb{R}^n}f(x)\bar{\psi}\left(\frac{x-b}{a}\right)\mathrm{d}x, \tag{5}$$

where $(b,a)\in\mathbb{H}^{n+1}$. In fact, if ψ is a test function and the dual pairing in (5) makes sense, then we call $\mathcal{W}_\psi f$ the wavelet transform of f with respect to ψ.

We first investigate continuity properties of the wavelet transform when the analyzing function belongs to a space of ultradifferentiable functions.

Theorem 1. *Let $\rho_1>0$, $\rho_2>1$ and let $s>0$, $t>\rho_1+\rho_2$, $\tau_1>\rho_1$ and $\tau_2>\rho_2-1$. Then the mapping*

$$\mathcal{W}:(\mathcal{S}_{\rho_2}^{\rho_1})_0(\mathbb{R}^n)\times(\mathcal{S}_{1-\rho_1+\min\{t-\rho_2,\tau_1\}}^{\min\{s,\tau_2-\rho_2+1\}})_0(\mathbb{R}^n)\to\mathcal{S}_{t,\tau_1,\tau_2}^s(\mathbb{H}^{n+1}),$$

given by $\mathcal{W}:(\psi,\varphi)\mapsto\mathcal{W}_\psi\varphi$, is continuous.

Remark 2. In the sequel we will use the continuity of

$$\mathcal{W}:(\mathcal{S}_{\rho_2}^{\rho_1})_0(\mathbb{R}^n)\times(\mathcal{S}_{t+1-\rho_1-\rho_2}^{s})_0(\mathbb{R}^n)\to\mathcal{S}_{t,t-\rho_2,s+\rho_2-1}^s(\mathbb{H}^{n+1})$$

which follows from Theorem 1, when $\tau_1=t-\rho_2$ and $\tau_2=s+\rho_2-1$.

The following simple facts are useful in calculations:
By the Plancherel theorem, we have

$$\mathcal{W}_\psi\varphi(b,a)=\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}e^{ib\xi}\overline{\hat{\psi}}(a\xi)\hat{\varphi}(\xi)d\xi,\quad (b,a)\in\mathbb{H}^{n+1}.$$

Hence $\mathcal{F}_1\mathcal{W}_\psi\varphi(\xi,a)=\overline{\hat{\psi}}(a\xi)\hat{\varphi}(\xi)$, $(\xi,a)\in\mathbb{H}^{n+1}$. Moreover, for $(b,a)\in\mathbb{H}^{n+1}$,

$$\partial_b^\beta W_\psi\varphi(b,a)=\int_{\mathbb{R}^n}\varphi^{(\beta)}(ax+b)\overline{\psi}(x)dx=i^{|\beta|}\int_{\mathbb{R}^n}e^{i\xi b}\xi^\beta\hat{\varphi}(\xi)\overline{\hat{\psi}}(a\xi)d\xi,$$

and, if $\psi\in(\mathcal{S}^{\rho_1}_{\rho_2})_0(\mathbb{R}^n)$ then $\int b^\gamma\mathcal{W}_\psi\varphi(b,a)db=0$, $\gamma\in\mathbb{N}^n$.

In order to construct the left-inverse for the wavelet transform, we proceed as follows. The wavelet synthesis transform of $\Phi\in\mathcal{S}^s_{t,\tau_1,\tau_2}(\mathbb{H}^{n+1})$, $s,t,\tau_1,\tau_2>0$, $s+t\geq 1$, with respect to $\phi\in(\mathcal{S}^{\rho_1}_{\rho_2})_0(\mathbb{R}^n)$, $\rho_1>0$, $\rho_2>1$, is defined by

$$\mathcal{M}_\phi\Phi(x)=\int_0^\infty\left(\int_{\mathbb{R}^n}\Phi(b,a)\frac{1}{a^n}\phi\left(\frac{x-b}{a}\right)\mathrm{d}b\right)\frac{\mathrm{d}a}{a},\quad x\in\mathbb{R}^n.$$

Theorem 2. *Let $\rho_1>0$, $\rho_2>1$ and let $s>0$, $t>\rho_2$ and $\tau>0$. Then the bilinear mappings*

a) $\mathcal{M}:(\mathcal{S}^{\rho_1}_{\rho_2})_0(\mathbb{R}^n)\times\mathcal{S}^\tau_{t,t-\rho_2,s-\rho_1}(\mathbb{H}^{n+1})\to(\mathcal{S}^s_t)_0(\mathbb{R}^n)$, *when* $s>\rho_1$*;*
b) $\mathcal{M}:(\mathcal{S}^{\rho_1}_{\rho_2})_0(\mathbb{R}^n)\times\mathcal{S}^s_{t,t-\rho_2,\tau}(\mathbb{H}^{n+1})\to(\mathcal{S}^s_t)_0(\mathbb{R}^n)$,

given by $\mathcal{M}:(\phi,\Phi)\mapsto\mathcal{M}_\phi\Phi$, are continuous.

Remark 3. *1.* It will be seen from the proof of Theorem 2 that a more general statement holds true. In fact, $\mathcal{M}$ can actually be extended to a continuous mapping from $\mathcal{S}^{\rho_1}_{\rho_2}(\mathbb{R}^n)\times\mathcal{S}^\tau_{t,t-\rho_2,s-\rho_1}(\mathbb{H}^{n+1})$ or from $\mathcal{S}^{\rho_1}_{\rho_2}(\mathbb{R}^n)\times\mathcal{S}^s_{t,t-\rho_2,\tau}(\mathbb{H}^{n+1})$ to $\mathcal{S}^s_t(\mathbb{R}^n)$. However, we will only use wavelets with all vanishing moments in the rest of this article.

2. The continuity properties from Theorem 2 a) and b) provide information about high regularity and the decay properties of $\mathcal{M}_\phi\Phi$. In the notation of Gelfand-Shilov spaces the upper index is related to Gevrey ultra-differentiability while the lower index is related to the decay of a function. Note that when $s=1$, the function $\mathcal{M}_\phi\Phi$ is real analytic and if $0<s<1$, it extends to an entire function on $\mathbb{C}^n$. The index t gives subexponential decay at rate $e^{-h|x|^{1/t}}$, for some $h>0$. In Theorem 2 a) the regularity of the image $\mathcal{M}_\phi\Phi$ is measured in terms of the regularity of the wavelet ϕ and the decay of Φ when $a>0$ tends to zero, while Theorem 2 b) shows that the regularity of Φ is preserved under the action of the synthesis operator. Similarly, the decay of $\mathcal{M}_\phi\Phi$ at infinity is related to the corresponding decays of ϕ and Φ in both Theorem 2 a) and b).

The importance of the wavelet synthesis operator follows from the fact that it can be used to construct a left inverse for the wavelet transform, whenever the wavelet possesses nice reconstruction properties. We end this subsection with a necessary and sufficient condition for such property to hold in the context of Gelfand-Shilov spaces.

We start with some terminology. We say that a wavelet $\psi \in \mathcal{S}_0(\mathbb{R}^n)$ admits a reconstruction wavelet $\phi \in \mathcal{S}_0(\mathbb{R}^n)$ if

$$c_{\psi,\phi}(\omega) = \int_0^\infty \overline{\hat{\psi}}(r\omega)\hat{\phi}(r\omega)\frac{\mathrm{d}r}{r}, \ \ \omega \in \mathbb{S}^{n-1},$$

is finite, non-zero, and independent of the direction $\omega \in \mathbb{S}^{n-1}$. In such a case we write $c_{\psi,\phi} := c_{\psi,\phi}(\omega)$.

For example, if $\psi \in \mathcal{S}_0(\mathbb{R}^n)$ is non-trivial and rotation invariant, then it is its own reconstruction wavelet. In fact, the existence of a reconstruction wavelet is equivalent to *non-degenerateness* in the sense of the following definition (see [31, Proposition 5.1]).

Definition 2. ([30, 31]) A test function $\varphi \in \mathcal{S}(\mathbb{R}^n)$ is said to be *non-degenerate* if for any $\omega \in \mathbb{S}^{n-1}$ the function $R_\omega(r) = \hat{\varphi}(r\omega)$, $r \in [0,\infty)$ is not identically zero, that is, $\operatorname{supp} R_\omega \neq \emptyset$, for each $\omega \in \mathbb{S}^{n-1}$. If in addition $\mu_0(\varphi) = 0$, then φ is called a non-degenerate wavelet.

We can now state the reconstruction formula for the wavelet transform (cf. [17, Theorem 14.0.2]). If $\psi \in \mathcal{S}_0(\mathbb{R}^n)$ is non-degenerate and $\phi \in \mathcal{S}_0(\mathbb{R}^n)$ is a reconstruction wavelet for it, then

$$\varphi = \frac{1}{c_{\psi,\phi}}\mathcal{M}_\phi \mathcal{W}_\psi \varphi, \ \ \forall \varphi \in \mathcal{S}(\mathbb{R}^n). \tag{6}$$

We are interested in wavelets in Gelfand-Shilov spaces. The ensuing proposition shows that if the non-degenerate wavelet ψ possesses higher regularity properties, then it is possible to choose a reconstruction wavelet with the same regularity as ψ.

Proposition 2. *Let $\psi \in (\mathcal{S}^{\rho_1}_{\rho_2})_0(\mathbb{R}^n)$, $\rho_1 > 0$, $\rho_2 > 1$, be non-degenerate. Then, it admits a reconstruction wavelet ϕ such that $\phi \in (\mathcal{S}^{\tau}_{\rho_2})_0(\mathbb{R}^n)$, $\forall \tau > 0$.*

Proof. The proof is similar to that of [31, Proposition 5.1]. However, here the reconstruction wavelet should satisfy additional regularity properties.

Since ψ is non-degenerate, then, by Definition 2, there exist $0 < r_1 < r_2$ such that $\operatorname{supp} \hat{\psi}(r\omega) \cap [r_1, r_2] \neq \emptyset$, $\forall \omega \in \mathbb{S}^{n-1}$.

The condition $\rho_2 > 1$ implies that the space $\mathcal{S}^{\rho_2}_{\tau}(\mathbb{R})$ is non-quasianalytic for every $\tau > 0$. Let $\eta \in \bigcap_{\tau>0} \mathcal{S}^{\rho_2}_{\tau}(\mathbb{R})$ be a compactly supported nonnegative rotation-invariant function with $0 \notin \operatorname{supp} \eta$ and $\eta(\xi) = 1$ for $r_1 \leq |\xi| \leq r_2$.

Consider the auxiliary function

$$g(\omega) = \int_0^\infty \eta(r)|\hat{\psi}(r\omega)|^2 \frac{\mathrm{d}r}{r} > 0, \quad \omega \in \mathbb{S}^{n-1}.$$

A straightforward computation shows that

$$\sup_{\omega \in \mathbb{S}^{n-1}} |\partial^\alpha g(\omega)| \leq C h^{|\alpha|}(\alpha!)^{\rho_2}, \quad \alpha \in \mathbb{N}^n.$$

In fact, $g \in \mathcal{E}^{\{(\alpha!)^{\rho_2}\}}(\mathbb{S}^{n-1})$, the Gevrey class of $\{(\alpha!)^{\rho_2}\}$-ultra-differentiable functions on the unit sphere $\mathbb{S}^{n-1}$, see [20] for the definition. Then, by employing an atlas on $\mathbb{S}^{n-1}$ consisting of functions from $\mathcal{E}^{\{(\alpha!)^{\rho_2}\}}(\mathbb{S}^{n-1})$ and [21, Lemma 1], once concludes that $1/g \in \mathcal{E}^{\{(\alpha!)^{\rho_2}\}}(\mathbb{S}^{n-1})$, i.e., the partial derivatives of $1/g$ satisfy the same decay properties as those of g. Moreover, by [19, Theorem 8.2.4] and since the function $\omega : \xi \mapsto \xi/|\xi|$ is analytic off the origin, it follows that $1/g(\xi/|\xi|) \in \mathcal{E}^{\{(\alpha!)^{\rho_2}\}}$ away from the origin.

Finally we define the reconstruction wavelet via its Fourier transform as follows. Set $\hat{\phi}(\xi) := \eta(\xi)\hat{\psi}(\xi)/g(\xi/|\xi|)$. It is a compactly supported function, all of its partial derivatives vanish at the origin and $\hat{\phi} \in \cap_{\tau>0}\mathcal{S}^{\rho_2}_{\tau}(\mathbb{R}^n)$. Therefore $\phi \in (\mathcal{S}^{\tau}_{\rho_2})_0(\mathbb{R}^n)$, $\forall \tau > 0$. Moreover, by construction $c_{\psi,\phi} = c_{\psi,\phi}(\omega) = 1$. This completes the proof.

□

Next we give an example of a non-degenerate wavelet from $(\mathcal{S}^{\rho_1}_{\rho_2})_0(\mathbb{R}^n)$.

Example 1. Assume that $\rho_1 > 0$ and $\rho_2 > 1$. Let $e_j = (0, 0, \dots, 1, \dots, 0)$, with 1 at the $j-$th coordinate, and let $B_{\pm j} = B(\pm\frac{1}{2}e_j, \frac{1}{2})$, $j = 1, 2, \dots, n$ denote the closed balls centered at $\pm\frac{1}{2}e_j$ with radius $\frac{1}{2}$. Since the class $\mathcal{S}^{\rho_2}_{\rho_1}(\mathbb{R}^n)$ is non-quasiananalytic, it contains compactly supported functions. Set $\hat{\psi} = \sum_{\substack{j=-n \\ j\neq 0}}^{n} \hat{\phi}_j$, where the $\hat{\phi}_{\pm j} \in \mathcal{S}^{\rho_2}_{\rho_1}(\mathbb{R}^n)$ are functions supported by $B_{\pm j}$, $j = 1, 2, \dots, n$, respectively, and positive in its interior. Then the function ψ, the inverse Fourier transform of $\hat{\psi}$, is an example of non-trivial non-degenerate wavelet from $(\mathcal{S}^{\rho_1}_{\rho_2})_0(\mathbb{R}^n)$.

3.2. **The wavelet transform of tempered ultradistributions.** We start with a useful growth estimate for the wavelet transform of an ultradistribution. Recall, the wavelet transform of an ultradistribution f with respect to the test function ψ is given by (5) whenever the dual pairing is well defined.

Proposition 3. *Let $\rho_1, \rho_2 > 0$, $\rho_1 + \rho_2 \geq 1$, $s > \rho_1$ and $t > \rho_2$. If $\psi \in \mathcal{S}^{\rho_1}_{\rho_2}(\mathbb{R}^n)$ and $f \in (\mathcal{S}^s_t(\mathbb{R}^n))'$, then for every $k > 0$,*

$$|\mathcal{W}_\psi f(b,a)| \lesssim e^{k\left(a^{\frac{1}{t-\rho_2}} + (\frac{1}{a})^{\frac{1}{s-\rho_1}} + |b|^{\frac{1}{t}}\right)}, \quad (b,a) \in \mathbb{H}^{n+1}.$$

Proof. Since $\psi \in \mathcal{S}^{\rho_1}_{\rho_2}(\mathbb{R}^n)$,

$$|\partial^\beta \psi(x)| \lesssim \tilde{h}^{-|\beta|}(\beta!)^{\rho_1} e^{-A|x|^{1/\rho_2}}, \quad x \in \mathbb{R}^n, \beta \in \mathbb{N}^n,$$

for some $\tilde{h}, A > 0$. For every $h > 0$ there exists $C_h > 0$ such that:

$$|\mathcal{W}_\psi f(b,a)| \leq C_h\, p^{s,t}_h \left(\frac{1}{a^n}\bar{\psi}\Big(\frac{\cdot - b}{a}\Big)\right), \quad (b,a) \in \mathbb{H}^{n+1}.$$

So we have (for every $c > 0$):

$$|\mathcal{W}_\psi f(b,a)| \lesssim \frac{1}{a^n} \sup_{x\in\mathbb{R}^n, \beta\in\mathbb{N}^n} \frac{h^{|\beta|}}{\beta!^s} e^{c|ax+b|^{\frac{1}{t}}} \frac{1}{a^{|\beta|}} |\psi^{(\beta)}(x)|$$

$$= \frac{1}{a^n} \sup_{x\in\mathbb{R}^n, \beta\in\mathbb{N}^n} \frac{\tilde{h}^{|\beta|}|\psi^{(\beta)}(x)|}{\beta!^{\rho_1}} e^{A|x|^{\frac{1}{\rho_2}}} \left(\frac{h}{\tilde{h}a}\right)^{|\beta|} \frac{1}{\beta!^{s-\rho_1}} e^{c|ax+b|^{\frac{1}{t}} - A|x|^{\frac{1}{\rho_2}}},$$

where $\tilde{h}, A > 0$. Therefore,

$$|\mathcal{W}_\psi f(b,a)| \lesssim p^{\rho_1,\rho_2}_l(\psi)\, e^{(s-\rho_1)(\frac{h}{\tilde{h}a})^{\frac{1}{s-\rho_1}}}\, e^{c|b|^{\frac{1}{t}}} \sup_{x\in\mathbb{R}^n} (e^{c|ax|^{\frac{1}{t}} - A|x|^{\frac{1}{\rho_2}}}),$$

for some $l>0$. Since $g(r):=c(a\cdot r)^{\frac{1}{t}}-Ar^{\frac{1}{\rho_2}}$, $r>0$ attains its maximal value at $(\frac{\rho_2 c}{tA})^{\frac{t\rho_2}{t-\rho_2}}a^{\frac{\rho_2}{t-\rho_2}}$, and since we may chose arbitrary $h>0$ and $c>0$, it follows that

$$|\mathcal{W}_\psi f(b,a)|\lesssim e^{k\left((\frac{1}{a})^{\frac{1}{s-\rho_1}}+a^{\frac{1}{t-\rho_2}}+|b|^{\frac{1}{t}}\right)},$$

for every $k>0$. □

Remark 4. Naturally, if $\psi\in(\mathcal{S}^{\rho_1}_{\rho_2})_0(\mathbb{R}^n)$, then Proposition 3 remains valid for $f\in((\mathcal{S}^s_t)_0(\mathbb{R}^n))'$. Furthermore, if $\mathcal{B}'\subset(\mathcal{S}^s_t(\mathbb{R}^n))'$ is a bounded set (resp. $\mathcal{B}'\subset((\mathcal{S}^s_t)_0(\mathbb{R}^n))'$ when $\psi\in(\mathcal{S}^{\rho_1}_{\rho_2})_0(\mathbb{R}^n)$)), then the conclusion of Proposition 3 holds uniformly for $f\in\mathcal{B}'$, as follows from the Banach-Steinhaus theorem.

Next, we give an alternative definition of the wavelet transform of an ultradistribution via duality.

Definition 3. Let $\rho_1>0$, $t>\rho_2>1$, $s>0$ and $\tau>0$. If $\psi\in(\mathcal{S}^{\rho_1}_{\rho_2})_0(\mathbb{R}^n)$ and $f\in((\mathcal{S}^s_t)_0(\mathbb{R}^n))'$ then the wavelet transform $\mathcal{W}_\psi f$ of f with respect to the wavelet ψ is defined as

$$\langle\mathcal{W}_\psi f(b,a),\Phi(b,a)\rangle:=\langle f(x),\mathcal{M}_{\bar{\psi}}\Phi(x)\rangle,\ \ \Phi\in\mathcal{S}^s_{t,t-\rho_2,\tau}(\mathbb{H}^{n+1}).\tag{7}$$

Thus, $\mathcal{W}_\psi:((\mathcal{S}^s_t)_0(\mathbb{R}^n))'\to(\mathcal{S}^s_{t,t-\rho_2,\tau}(\mathbb{H}^{n+1}))'$ is continuous for the strong dual topologies.

By Theorem 2 b), the transposition in (7) is well defined. Note that we have freedom of the choice of τ. This fact will be crucial below. If we assume that $s>\rho_1$, then the choice $\tau=s-\rho_1$ leads to the continuous mapping $\mathcal{W}_\psi:((\mathcal{S}^s_t)_0(\mathbb{R}^n))'\to(\mathcal{S}^s_{t,t-\rho_2,s-\rho_1}(\mathbb{H}^{n+1}))'$. The next result shows the consistency between Definition 3 and (5) for this choice of τ.

Proposition 4. *Assume that $s>\rho_1>0$ and $t>\rho_2>1$. Let $f\in((\mathcal{S}^s_t)_0(\mathbb{R}^n))'$ and $\psi\in(\mathcal{S}^{\rho_1}_{\rho_2})_0(\mathbb{R}^n)$. Then, for every $\Phi\in\mathcal{S}^s_{t,t-\rho_2,s-\rho_1}(\mathbb{H}^{n+1})$,*

$$\langle f(x),\mathcal{M}_{\bar{\psi}}\Phi(x)\rangle=\int_0^\infty\int_{\mathbb{R}^n}\mathcal{W}_\psi f(b,a)\Phi(b,a)\,\frac{dbda}{a}.\tag{8}$$

Proof. Fix $\Phi\in\mathcal{S}^s_{t,t-\rho_2,s-\rho_1}(\mathbb{H}^{n+1})$. Proposition 1 implies that there is $h>0$ such that

$$|\Phi(b,a)|\lesssim e^{-h\left(a^{\frac{1}{t-\rho_2}}+a^{-\frac{1}{s-\rho_1}}+|b|^{\frac{1}{t}}\right)},\quad (b,a)\in\mathbb{H}^{n+1}.\tag{9}$$

Let $\{f_j\}_{j=0}^\infty$ be a sequence such that $f_j\to f$ in $((\mathcal{S}^s_t)_0(\mathbb{R}^n))'$ and $f_j\in\mathcal{S}_0(\mathbb{R}^n)$, for every $j\in\mathbb{N}$. In view of Proposition 3 (cf. Remark 4),

$$|\mathcal{W}_\psi f_j(b,a)|\lesssim e^{\frac{h}{2}\left(a^{\frac{1}{t-\rho_2}}+(\frac{1}{a})^{\frac{1}{s-\rho_1}}+|b|^{\frac{1}{t}}\right)},\quad (b,a)\in\mathbb{H}^{n+1},\tag{10}$$

uniformly in $j\in\mathbb{N}$. Fubini's theorem and the regularity of f_j imply

$$\int_{\mathbb{R}^n}f_j(x)\mathcal{M}_{\bar{\psi}}\Phi(x)dx=\int_0^\infty\int_{\mathbb{R}^n}\mathcal{W}_\psi f_j(b,a)\Phi(b,a)\,\frac{dbda}{a}\tag{11}$$

Noticing that $\mathcal{W}_\psi f_j(b,a)\to\mathcal{W}_\psi f(b,a)$ pointwisely, the estimates (9) and (10) allow us to use the Lebesgue dominated convergence theorem in (11) to conclude (8). □

We also introduce the the wavelet synthesis transform of an ultradistribution on $\mathbb{H}^{n+1}$ via a duality approach. The consistency of the following definition is ensured by Theorem 1 (cf. Remark 2).

Definition 4. Let $\rho_1 > 0$, $\rho_2 > 1$, $s > 0$ and $t > \rho_1+\rho_2$. Let $F \in (\mathcal{S}^s_{t,t-\rho_2,s+\rho_2-1}(\mathbb{H}^{n+1}))'$ and $\phi \in (\mathcal{S}^{\rho_1}_{\rho_2})_0(\mathbb{R}^n)$. The wavelet synthesis transform $\mathcal{M}_\phi F$ of F with respect to the wavelet ϕ is defined by

$$\langle \mathcal{M}_\phi F(x), \varphi(x)\rangle := \langle F(b,a), \mathcal{W}_{\bar{\phi}}\varphi(b,a)\rangle, \quad \varphi \in (\mathcal{S}^s_{t+1-\rho_1-\rho_2})_0(\mathbb{R}^n).$$

Thus, $\mathcal{M}_\phi : (\mathcal{S}^s_{t,t-\rho_2,s+\rho_2-1}(\mathbb{H}^{n+1}))' \to ((\mathcal{S}^s_{t+1-\rho_1-\rho_2})_0(\mathbb{R}^n))'$ is continuous.

We derive the following resolution of the identity mapping Id as an easy consequence of our previous results. In the next theorem we implicitly use the choice $\tau = s+\rho_2-1$ in Definition 3.

Theorem 3. *Let $\rho_1 > 0$, $\rho_2 > 1$, $s > 0$ and $t > \rho_1+\rho_2$. Let $\psi \in (\mathcal{S}^{\rho_1}_{\rho_2})_0(\mathbb{R}^n)$ be a non-degenerate wavelet and let $\phi \in (\mathcal{S}^{\rho_1}_{\rho_2})_0(\mathbb{R}^n)$ be a reconstruction wavelet for it. Then the Calderón reproducing formula*

$$\mathrm{Id} = \frac{1}{c_{\psi,\phi}} \mathcal{M}_\phi \mathcal{W}_\psi$$

holds in $((\mathcal{S}^s_t)_0(\mathbb{R}^n))'$.

Proof. Let $f \in ((\mathcal{S}^s_t)_0(\mathbb{R}^n))'$. Since $(\mathcal{S}^s_{t+1-\rho_1-\rho_2})_0(\mathbb{R}^n)$ is dense in the space $(\mathcal{S}^s_t)_0(\mathbb{R}^n)$, it is enough to prove the identity for test functions $\varphi \in (\mathcal{S}^s_{t+1-\rho_1-\rho_2})_0(\mathbb{R}^n)$. Then, by Definitions 4 and 3, and the reconstruction formula (6), it follows that

$$\langle \mathcal{M}_\phi \mathcal{W}_\psi f, \varphi\rangle = \langle \mathcal{W}_\psi f, \mathcal{W}_{\bar{\phi}}\varphi\rangle = \langle f, \mathcal{M}_{\bar{\psi}} \mathcal{W}_{\bar{\phi}}\varphi\rangle = c_{\bar{\phi},\bar{\psi}}\langle f,\varphi\rangle. \tag{12}$$

□

Combining Remark 2, Proposition 4, and the relation (12), we obtain an extension of the *desingularization* formula now in the context of ultradistributions (cf. [17, 31] for the case of distributions).

Corollary 1. *In addition to the assumptions of Theorem 3 suppose that $\sigma := \rho_1+\rho_2-1 < s$. Then,*

$$\langle f,\varphi\rangle = \frac{1}{c_{\psi,\phi}} \int_0^\infty \int_{\mathbb{R}^n} \mathcal{W}_\psi f(b,a) \mathcal{W}_{\bar{\phi}}\varphi(b,a) \frac{dbda}{a}, \ \forall \varphi \in (\mathcal{S}^{s-\sigma}_{t-\sigma})_0(\mathbb{R}^n).$$

As an immediate consequence of Theorem 3 and Theorem 2 b) we have the following regularity theorem for ultradistributions.

Corollary 2. *Let $\rho_1 > 0$, $\rho_2 > 1$, $s > 0$ and $t > \rho_1+\rho_2$. Let $\psi \in (\mathcal{S}^{\rho_1}_{\rho_2})_0(\mathbb{R}^n)$ be non-degenerate and let $f \in ((\mathcal{S}^s_t)_0(\mathbb{R}^n))'$. If $\mathcal{W}_\psi f \in \mathcal{S}^s_{t,t-\rho_2,\tau}(\mathbb{H}^{n+1})$ for some $\tau > 0$ then $f \in (\mathcal{S}^s_t)_0(\mathbb{R}^n)$.*

4. Proofs of main results

This section collects the proofs of Theorems 1 and 2.

Remark 5. On several occasions we will use the following facts. If $\varphi \in \mathcal{S}^{\rho_1}_{\rho_2}(\mathbb{R}^n)$, $\rho_1+\rho_2 \geq 1$, so that $p^{\rho_1,\rho_2}_{h_0}(\varphi) < \infty$ for some $h_0 > 0$, then there exists $h_1 > 0$ such that

$$\sup_{\alpha,\beta\in\mathbb{N}^n} \frac{h_1^{|\alpha+\beta|}}{\alpha!^{\rho_2}\beta!^{\rho_1}} \int |x|^{|\alpha|}|\varphi^{(\beta)}(x)|dx \lesssim p^{\rho_1,\rho_2}_{h_0}(\varphi). \tag{13}$$

In addition, if (13) holds, then there exists $h_2 > 0$ such that

$$\sup_{\alpha,\beta\in\mathbb{N}^n} \frac{h_2^{|\alpha+\beta|}}{\alpha!^{\rho_2}\beta!^{\rho_1}} |x|^{|\alpha|}|\varphi^{(\beta)}(x)| < \infty.$$

We will omit the parts of the proofs where these arguments appear. We will often make use of the fact that multiplication by $|\cdot|^{|\alpha|}$ (or by $\langle\cdot\rangle^{|\alpha|}$) simply enlarges the corresponding constants $h > 0$. We will also use, without explicit reference, the estimate

$$\sup_{\beta\in\mathbb{N}^n,x\in\mathbb{R}^n} \frac{h_0^{|\beta|}(fg)^{(\beta+\alpha)}(x)}{\beta!^s} \lesssim \sup_{\beta\in\mathbb{N}^n,x\in\mathbb{R}^n} \frac{h_1^{|\beta|}f^{(\beta)}(x)}{\beta!^s} \sup_{\beta\in\mathbb{N}^n,x\in\mathbb{R}^n} \frac{h_1^{|\beta|}g^{(\beta)}(x)}{\beta!^s}$$

for some $h_1 = h_1(\alpha) > 0$. Finally, we shall need the following form of the reminder term in the Taylor formula

$$(R_{\alpha,m}f)(x,y) = \sum_{|\alpha|=m} \frac{m(y-x)^\alpha}{\alpha!} \int_0^1 (1-\theta)^{m-1} f^{(\alpha)}(x+\theta(y-x))d\theta.$$

4.1. **Proof of Theorem 1.** Let φ and ψ satisfy

$$p_h^{\min\{s,\tau_2-\rho_2+1\},1-\rho_1+\min\{t-\rho_2,\tau_1\}}(\varphi)p_h^{\rho_1,\rho_2}(\psi) < \infty. \tag{14}$$

We will show that there exists $h_0 > 0$ such that $p^{s,t,\tau_1,\tau_2}_{\alpha,h_0}(\mathcal{W}_\psi\varphi) < \infty$, that is, the supremum of

$$J = \frac{h_0^{|\beta|+k+l_1+l_2}(a^{l_1}+a^{-l_2})\langle b\rangle^k |\partial_a^\alpha \partial_b^\beta \mathcal{W}_\psi\varphi(b,a)|}{\beta!^s k!^t l_1!^{\tau_1} l_2!^{\tau_2}}$$

over

$$((b,a),(k,l_1,l_2),\beta) \in \Lambda = \mathbb{H}^{n+1}\times\mathbb{N}^3\times\mathbb{N}^n$$

is finite for some $h_0 > 0$. Without loss of generality we assume from now on that $\alpha = 0$.

Remark 6. In the following steps of the proof we start with $h_0 > 0$ and in every step determine a new constant $h_1 \leq h_2 \leq \cdots \leq h_7$ which successively depends on the previous one, that is, h_m depends on h_{m-1}, $m = 1,2,\ldots,7$ and h_7 should be equal to $h > 0$ in (14). Then, by going in the opposite direction, we determine h_6 from h_7, h_5 from h_6, …, and h_0 from h_1. In such way for the constant $h > 0$ given in (14) we find $h_0 > 0$ so that

$$p^{s,t,\tau_1,\tau_2}_{0,h_0}(\mathcal{W}_\psi\varphi) = \sup_\Lambda J \lesssim p_h^{\min\{s,\tau_2-\rho_2+1\},1-\rho_1+\min\{t-\rho_2,\tau_1\}}(\varphi)p_h^{\rho_1,\rho_2}(\psi),$$

which will prove the Theorem.

We will estimate

$$J_1 = \frac{h_0^{|\beta|+2k+2l_1}(1+a^{2l_1})\langle b\rangle^{2k}|\partial_b^\beta \mathcal{W}_\psi\varphi(b,a)|}{\beta!^s k!^{2t} l_1!^{2\tau_1}}$$

over $((b,a),(k,l_1),\beta)\in\Lambda_1=\mathbb{H}^{n+1}\times\mathbb{N}^2\times\mathbb{N}^n$, and

$$J_2 = \frac{h_0^{|\beta|+2l_2}(1+a^{-2l_2})|\partial_b^\beta \mathcal{W}_\psi\varphi(b,a)|}{\beta!^s l_2!^{2\tau_2}}$$

over $((b,a),(k,l_2),\beta)\in\Lambda_2=\mathbb{H}^{n+1}\times\mathbb{N}^2\times\mathbb{N}^n$.

Since

$$\frac{h_0^{2(|\beta|+k+l_1+l_2)}(a^{l_1}+a^{-l_2})^2\langle b\rangle^{2k}|\partial_b^\beta \mathcal{W}_\psi\varphi(b,a)|^2}{\beta!^{2s}k!^{2t}l_1!^{2\tau_1}l_2!^{2\tau_2}} \lesssim \sup_{\Lambda_1} J_1 \sup_{\Lambda_2} J_2$$

we would have

$$p_{0,h_0}^{s,t,\tau_1,\tau_2}(\mathcal{W}_\psi\varphi) \lesssim \sqrt{\sup_{\Lambda_1} J_1 \sup_{\Lambda_2} J_2}.$$

We will show that there exists $h_7>0$ which depends on $h_0>0$ such that

$$\sup_{\Lambda_1} J_1 \lesssim p_{h_7}^{s,1-\rho_1+\min\{t-\rho_2,\tau_1\}}(\varphi)\, p_{h_7}^{\rho_1,\rho_2}(\psi)$$

and

$$\sup_{\Lambda_2} J_2 \lesssim p_{h_7}^{\min\{s,\tau_2-\rho_2+1\},t}(\varphi)\, p_{h_7}^{\rho_1,\rho_2}(\psi).$$

We first estimate $\sup_{\Lambda_1} J_1$. There exists $h_1=h_1(h_0)$ such that

$$J_1 \lesssim \frac{h_1^{|\beta|+2k+2l_1}(1+a^{2l_1})\langle b\rangle^{2k}}{\beta!^s(2k)!^t(2l_1)!^{\tau_1}\langle b\rangle^{2k}}\left|\int_{\mathbb{R}^n} e^{i\xi b}(1-\triangle_\xi)^k(\xi^\beta\hat{\varphi}(\xi)\overline{\hat{\psi}}(a\xi))d\xi\right|$$

$$= \frac{h_1^{|\beta|+2k+2l_1}}{\beta!^s(2k)!^t(2l_1)!^{\tau_1}}\left|\int_{\mathbb{R}^n} e^{i\xi b}\sum_{|\gamma|\le 2k} c_\gamma\partial_\xi^\gamma(\xi^\beta\hat{\varphi}(\xi)(1+a^{2l_1})\overline{\hat{\psi}}(a\xi))d\xi\right|$$

$$\lesssim \frac{h_1^{|\beta|+2k+2l_1}}{\beta!^s(2k)!^t(2l_1)!^{\tau_1}}\sum_{|\gamma|\le 2k}|c_\gamma|\sum_{i+j\le\gamma}|\tilde{c}_{i,j}|\int_{\mathbb{R}^n}|\partial^i(\xi^\beta\hat{\varphi}(\xi))|a^{|j|}(1+a^{2l_1})|\hat{\psi}^{(j)}(a\xi)|d\xi,$$

where c_γ and $\tilde{c}_{i,j}$ are correspondent binomial coefficients. As already noticed, by the use of Leibniz rule, the binomial coefficients simply increase the constant h_1 so that

$$J_1 \lesssim \sup \frac{h_2^{|\beta|+2k+2l_1}}{\beta!^s(2k)!^t(2l_1)!^{\tau_1}}\sum_{|i+j|\le 2k}(I_1+I_2),$$

for some $h_2=h_2(h_1)>0$ which does not depend on β, k and l_1, where

$$I_1 = \int_{|\xi|\le 1}|\partial^i(\xi^\beta\hat{\varphi}(\xi))|a^{|j|}(1+a^{2l_1})|\hat{\psi}^{(j)}(a\xi)|d\xi,$$

and

$$I_2 = \int_{|\xi|\ge 1}|\partial^i(\xi^\beta\hat{\varphi}(\xi))|a^{|j|}(1+a^{2l_1})|\hat{\psi}^{(j)}(a\xi)|d\xi,$$

and the supremum is taken over β, k and l_1.

By Remarks 1 and 5 it follows that there exists $h_3 = h_3(h_2) > 0$, which does not depend on β, i, j and l_1, such that

$$\frac{h_2^{|\beta|+|i|+|j|+2l_1}}{\beta!^s|i|!^t|j|!^t(2l_1)!^{\tau_1}} I_2$$

$$\lesssim \frac{h_2^{|\beta|+|i|+|j|+2l_1}}{\beta!^s|i|!^t|j|!^t(2l_1)!^{\tau_1}} \int_{|\xi|\geq 1} |\partial^i(\xi^\beta \hat{\varphi}(\xi))||\xi|^n a^n |a\xi|^{|j|-n}(1+|a\xi|^{2l_1})|\hat{\psi}^{(j)}(a\xi)|d\xi$$

$$\lesssim \sup \frac{h_3^{|\beta|+|i|}}{\beta!^s|i|!^t}\langle\xi\rangle^n|\partial^i(\xi^\beta\hat{\varphi}(\xi))| \sup \frac{h_3^{|j|+2l_1}}{|j|!^t(2l_1)!^{\tau_1}} \int_{\mathbb{R}^n} |x|^{|j|-n}(1+|x|^{2l_1})|\hat{\psi}^{(j)}(x)|dx$$

$$\lesssim p_{h_3}^{s,t}(\varphi) p_{h_3}^{\min\{t-\rho_2,\tau_1\},\rho_2}(\psi),$$

where the suprema are taken over β, i, j, l_1 and ξ, and we have splited $|j|!^t$ into $|j|!^{t-\rho_2}$ and $|j|!^{\rho_2}$.

From the above calculations we conclude that there exists $h_3 > 0$ such that

$$\sup_{\Lambda_1} \frac{h_2^{|\beta|+2k+2l_1}}{\beta!^s(2k)!^t(2l_1)!^{\tau_1}} I_2 \lesssim p_{h_3}^{s,t}(\varphi) p_{h_3}^{\min\{t-\rho_2,\tau_1\},\rho_2}(\psi).$$

Next, we estimate the term with I_1:

$$I_1 \lesssim \sum_{p\leq i} \binom{i}{p} \frac{\beta!}{p!} \left| \int_{|\xi|\leq 1} \xi^{\beta-p}\hat{\varphi}^{(p)}(\xi) a^{|j|}(1+a^{2l_1})\overline{\hat{\psi}}^{(j)}(a\xi)d\xi \right|$$

$$= \sum_{p\leq i} \binom{i}{p} \frac{\beta!}{p!} \Big| \int_{|\xi|\leq 1} \xi^{\beta-p} \sum_{|r|=2l_1+|j|-n} \frac{(2l_1+|j|)\xi^r}{r!} \cdot I \cdot a^{|j|}(1+a^{2l_1})\overline{\hat{\psi}}^{(j)}(a\xi)d\xi \Big|$$

where $I = \int_0^1 (1-\theta)^{2l_1+|j|-n-1}\hat{\varphi}^{(p+r)}(\theta\xi)d\theta$, and we have used Taylor's formula for $\hat{\varphi}$ and the vanishing moments of φ.

Since $\displaystyle\sum_{|r|=2l_1+|j|-n} \frac{1}{r!} \lesssim \frac{c^{2l_1+|j|}}{(2l_1)!|j|!}$ for some $c > 0$ and binomial coefficients just increase the constant h_2, we obtain

$$\frac{h_2^{|\beta|+2k+2l_1}}{\beta!^s(2k)!^t(2l_1)!^{\tau_1}} I_1$$

$$\lesssim \sup \frac{h_4^{|\beta|+|i|+|j|+2l_1}}{\beta!^s|i|!^t|j|!^{t+1}(2l_1)!^{\tau_1+1}} |\hat{\varphi}^{(i+r)}(x)| \int_{|\xi|\leq 1} a^n(1+|a\xi|^{|j|+2l_1-n})|\hat{\psi}^{(j)}(a\xi)|d\xi$$

$$\lesssim \sup h_4^{|\beta|+|i|+|j|+2l_1} \frac{|\hat{\varphi}^{(i+r)}(x)|}{\beta!^s|i|!^t|j|!^{t-\rho_1-\rho_2+1}(2l_1)!^{\tau_1+1-\rho_1}} \int_{\mathbb{R}^n} \frac{\langle\xi\rangle^{|j|+2l_1-n}|\hat{\psi}^{(j)}(\xi)|}{|j|!^{\rho_1}(2l_1)!^{\rho_1}|j|!^{\rho_2}} d\xi$$

with $|r| = 2l_1 + |j|$, the suprema taken over β, i, j, l_1 and x, and where $h_4 > 0$ does not depend on β, i, j, l_1. Moreover, we may choose $h_4 \geq h_3$.

By Remarks 1 and 5 and similar arguments to those used in the estimates of I_2 it follows that there exists $h_5 = h_5(h_4) > 0$ (which does not depend on β, k and l_1) such that

$$\frac{h_2^{|\beta|+2k+2l_1}}{\beta!^s(2k)!^t(2l_1)!^{\tau_1}} I_1 \lesssim p_{h_5}^{s,1-\rho_1+\min\{t-\rho_2,\tau_1\}}(\varphi)\, p_{h_5}^{\rho_1,\rho_2}(\psi).$$

Since the sequence $h_0, h_1, \dots, h_5$ is non-decreasing, we conclude that

$$\sup_{\Lambda_1} J_1 \lesssim p_{h_5}^{s,1-\rho_1+\min\{t-\rho_2,\tau_1\}}(\varphi)\, p_{h_5}^{\rho_1,\rho_2}(\psi).$$

It remains to estimate $\sup_{\Lambda_2} J_2$. We now use Taylor's formula for φ and the vanishing moments of ψ to obtain, for $a < 1$,

$$J_2 = \frac{h_0^{|\beta|+2l_2}(1+a^{-2l_2})}{\beta!^s(2l_2)!^{\tau_2}} \times$$

$$\times \left| \int_{\mathbb{R}^n} \left(\sum_{|r|=2l_2} \frac{2l_2}{r!} \left(\int_0^1 (1-\theta)^{2l_2-1} \varphi^{(\beta+r)}(b+\theta a x) d\theta \right) a^{2l_2} x^r \overline{\psi}(x) \right) dx \right|$$

$$\lesssim \frac{h_6^{|\beta|+2l_2}}{\beta!^s(2l_2)!^{\tau_2+1}} \sup_{x\in\mathbb{R}^n} \max_{|r|=2l_2} |\varphi^{(\beta+r)}(x)| \sup_{x\in\mathbb{R}^n} |\langle x\rangle^{r+n+1}\overline{\psi}(x)|$$

$$\lesssim p_{h_7}^{\min\{s,\tau_2-\rho_2+1\},t}(\varphi)\, p_{h_7}^{\rho_1,\rho_2}(\psi),$$

for some $h_6 \geq h_5$ and $h_7 = h_7(h_6)$. When $a \geq 1$, we employ a similar argument (Taylor's formula is not needed for this case).

Thus, we choose $h_7 = h > 0$ for which

$$p_h^{\min\{s,\tau_2-\rho_2+1\},1-\rho_1+\min\{t-\rho_2,\tau_1\}}(\varphi)\, p_h^{\rho_1,\rho_2}(\psi) < \infty.$$

Now, reasoning as in Remark 6, we determine for given $h_7 = h$ the corresponding $h_0 > 0$ so that

$$p_{0,h_0}^{s,t,\tau_1,\tau_2}(\mathcal{W}_\psi\varphi) \lesssim p_{h_7}^{\min\{s,\tau_2-\rho_2+1\},1-\rho_1+\min\{t-\rho_2,\tau_1\}}(\varphi)\, p_{h_7}^{\rho_1,\rho_2}(\psi) < \infty,$$

which proves the Theorem.

4.2. **Proof of Theorem 2.** We may again assume that $\alpha = 0$.

a) Let $h > 0$ be chosen so that

$$p_{0,h}^{\tau,t,t-\rho_2,s-\rho_1}(\Phi)\, p_h^{\rho_1,\rho_2}(\phi) < \infty.$$

By Lemma 1 it is enough to prove that there exists $h_0 > 0$ such that

$$p_{h_0}^{s,t}(\mathcal{M}_\phi\Phi) \lesssim p_{0,h}^{t,\tau,t-\rho_2,s-\rho_1}(\mathcal{F}_1\Phi)\, p_h^{\rho_1,\rho_2}(\phi). \tag{15}$$

Let $\beta \in \mathbb{N}^n$ and $k \in \mathbb{N}$. We may assume that k is even. Then

$$\langle x\rangle^k \left|\partial_x^\beta\big(\mathcal{M}_\phi\Phi(x)\big)\right| = \langle x\rangle^k \left| \partial_x^\beta \int_{\mathbb{R}_+}\int_{\mathbb{R}^n} \Phi(b,a)\frac{1}{a^n}\phi\left(\frac{x-b}{a}\right) db\,\frac{da}{a} \right|$$

$$\lesssim \langle x\rangle^k \left| \int_{\mathbb{R}_+}\int_{\mathbb{R}^n} \partial_b^\beta \Phi(x-b,a)\frac{1}{a^n}\phi\left(\frac{b}{a}\right) db\,\frac{da}{a} \right|$$

$$\lesssim \langle x\rangle^k \left|\int_{\mathbb{R}_+}\int_{\mathbb{R}^n} e^{-ix\xi}\xi^\beta \hat{\Phi}(-\xi,a)\hat{\phi}(a\xi)\,d\xi\,\frac{da}{a}\right|$$

$$\lesssim \langle x\rangle^k \left|\int_{\mathbb{R}_+}\int_{\mathbb{R}^n} \frac{(1-\Delta_\xi)^{\frac{k}{2}}e^{-ix\xi}}{\langle x\rangle^k}\xi^\beta \hat{\Phi}(-\xi,a)\hat{\phi}(a\xi)\,d\xi\,\frac{da}{a}\right|$$

$$\lesssim \sum_{|r|+|q|\leq k}\int_{\mathbb{R}_+}\int_{\mathbb{R}^n} \partial_\xi^r(\xi^\beta\hat{\Phi}(-\xi,a))\partial_\xi^q(\hat{\phi}(a\xi))|\,d\xi\,\frac{da}{a} \lesssim \sum_{|r|+|q|\leq k} I,$$

where

$$I=\int_{\mathbb{R}_+}\int_{\mathbb{R}^n} a^{|q|}\,|\xi|^{|\beta|}\,|\partial_\xi^r\hat{\Phi}(-\xi,a)||\hat{\phi}^{(q)}(a\xi)|\,d\xi\,\frac{da}{a}.$$

We use again Remark 5 in a similar way as it was done in the proof of Theorem 1. In the corresponding steps of the proof we enlarge $h_0>0$ and regroup the integrands in an appropriate way. In fact, by taking the corresponding suprema, one can show that there exist $h_1=h_1(h_0)>0$ and $h_2=h_2(h_1)>0$, which do not depend on β,q and r, such that

$$\frac{h_0^{|\beta|+k}}{\beta!^s k!^t}\langle x\rangle^k|\partial_x^\beta\big(\mathcal{M}_\phi\Phi(x)\big)| \lesssim \frac{h_0^{|\beta|+k}}{\beta!^s k!^t}\sum_{|r|+|q|\leq k} I \tag{16}$$

$$\lesssim \sup h_1^{|\beta|+|r|+|q|}\frac{\big(a^{|q|}+\frac{1}{a^{|\beta|+1}}\big)\,|\partial_\xi^r\hat{\Phi}(-\xi,a)|}{\beta!^{s-\rho_1}q!^{t-\rho_2}r!^t}\frac{|a\xi|^{|\beta|}|\hat{\phi}^{(q)}(a\xi)|}{\beta!^{\rho_1}q!^{\rho_2}}$$

$$\lesssim p_{0,h_2}^{t,\tau,t-\rho_2,s-\rho_1}(\mathcal{F}_1\Phi)\,p_{h_2}^{\rho_1,\rho_2}(\phi),$$

where the supremum is taken over ξ,a,β,r and q (we have also used $a^{|q|-|\beta|-1}\leq \big(a^{|q|}+\frac{1}{a^{|\beta|+1}}\big)$, $a>0$).

By Remark 6 for $h_2=h$, there exists $h_0>0$ such that

$$\sup_{\beta\in\mathbb{N}^n,k\in\mathbb{N}}\frac{h_0^{|\beta|+k}}{\beta!^s k!^t}\sum_{|r|+|q|\leq k} I \lesssim p_{0,h}^{t,\tau,t-\rho_2,s-\rho_1}(\mathcal{F}_1\Phi)\,p_h^{\rho_1,\rho_2}(\phi),$$

which implies (15).

b) Here we bound (16) by

$$\sup h_3^{|\beta|+|r|+|q|}\frac{\big(a^{|q|}+\frac{1}{a}\big)\,|\xi|^{|\beta|}|\partial_\xi^r\hat{\Phi}(-\xi,a)|}{\beta!^s q!^{t-\rho_2}r!^t}\frac{|\hat{\phi}^{(q)}(a\xi)|}{q!^{\rho_2}}$$

$$\lesssim p_{0,h_4}^{t,s,t-\rho_2,\tau}(\mathcal{F}_1\Phi)\,p_{h_4}^{\rho_1,\rho_2}(\phi),$$

for some $h_3=h_3(h_0)$, and $h_4=h_4(h_3)$, where the supremum is taken over ξ,a,β,r and q. Once again by Remark 6 for $h_4=h$, it follows that there exists $h_0>0$ such that

$$\sup_{\beta\in\mathbb{N}^n,k\in\mathbb{N}}\frac{h_0^{|\beta|+k}}{\beta!^s k!^t}\sum_{|r|+|q|\leq k} I \lesssim p_{0,h}^{t,s,t-\rho_2,\tau}(\mathcal{F}_1\Phi)\,p_h^{\rho_1,\rho_2}(\phi)<\infty,$$

which completes the proof.

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S. Pilipović, University of Novi Sad, Faculty of Sciences, Department of Mathematics and Informatics, Trg Dositeja Obradovića 4, 21000 Novi Sad, Serbia
E-mail address: pilipovics@yahoo.com

D. Rakić, University of Novi Sad, Faculty of Technology, Bul. cara Lazara 1, 21000 Novi Sad, Serbia
E-mail address: drakic@tf.uns.ac.rs

N. Teofanov, University of Novi Sad, Faculty of Sciences, Department of Mathematics and Informatics, Trg Dositeja Obradovića 4, 21000 Novi Sad, Serbia
E-mail address: nenad.teofanov@dmi.uns.ac.rs

J. Vindas, Department of Mathematics, Ghent University, Krijgslaan 281 Gebouw S22, B 9000 Gent, Belgium
E-mail address: jvindas@cage.UGent.be