A Deep Study of Fuzzy Implications

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Chapter 1

Introduction

1.1 Background

After Zadeh introduced the concept of *fuzzy sets* in his pioneering work ([99], Zadeh 1965), a huge amount of works in fuzzy set theory and *fuzzy logic* appeared, both theoretical and applied. There are two main branches in the study of fuzzy logic, *fuzzy logic in the narrow sense* and *fuzzy logic in the broad sense* [30], [66]. Fuzzy logic in the narrow sense is a form of many-valued logic [71] constructed in the spirit of classical binary logic. It is symbolic logic concerned with syntax, semantics, axiomatization, soundness, completeness, etc. [29], [30]. Fuzzy logic in the broad sense can be seen as an extension of fuzzy logic in the narrow sense. It is a way of interpreting natuaral language to model human reasoning [66].

A very important part of research in fuzzy logic (both in the narrow sense and in the broad sense) focuses on extending the classical binary logic operators *negation* (denoted as $\neg$), *conjunction* (denoted as $\wedge$), *disjunction* (denoted as $\vee$) and *implication* (denoted as $\rightarrow$) to fuzzy logic operators. This thesis contributes a deep study on the extension of the classical binary implication to fuzzy logic in the broad sense. These extensions are called *fuzzy implications* .

Table 1.1 gives the truth table of the classical binary implication '$\rightarrow$'.

Table 1.1: Truth table of the classical binary implication

p	q	$p \rightarrow q$
0	0	1
0	1	1
1	0	0
1	1	1

A fuzzy implication I is defined as a $[0,1]^2 \rightharpoonup [0,1]$ mapping that at least satisfies the boundary conditions

$$I(0,0) = I(0,1) = I(1,1) = 1 \quad \text{and} \quad \mathrm{I}(1,0) = 0. \tag{1.1}$$

Let us first have an overview of the research in the literature on fuzzy implications in fuzzy logic in the broad sense.

1. Fundamental requirements of fuzzy implications.
   The implication in classical binary logic works only on two truth values 0 and 1 while a fuzzy implication is a $[0,1]^2 \rightharpoonup [0,1]$ mapping. So besides the boundary condition (1.1), the first step to work on fuzzy implications is naturally to determine which fundamental requirements a fuzzy implication should fulfill. Most considerations are taken either from the point of view that a fuzzy implication is a generalization of the implication in classical binary logic (e.g., [19], [26], [35], [86], [86], [96]), or from the point of view of fulfilling different requirements from specific applications, especially approximate reasoning (e.g., [7], [20], [21], [87], [88], [94], [95]). Among these works, an axiomatic system of 13 axioms for fuzzy implications were determined (for details see Section 1.2).
2. Generate basic classes of fuzzy implications.
   In the earlier literature, different authors have proposed many individual definitions of fuzzy implications. See in Table 1.2. Besides these individual definitions of fuzzy implications, Trillas et al. proposed in ([86], Trillas 1981) and ([85], Trillas 1985) two classes of fuzzy implications generated from the fuzzy logic operators negation, conjunction and disjunction. They are *strong implications* (S-implications for short) and *residuated implications* (R-implications for short). S-implications are defined based on
   $$p \rightarrow q = \neg p \vee q \tag{1.2}$$
   in classical binary logic, where p and q are two propositions. R-implications are defined based on the fact that the implication is residuated with *and* in the classical binary logic. S- and R- implications are widely used in the early works about approximate reasoning (e.g., [20], [87]). Besides S- and R- implications, there is another class of fuzzy implications generated from the fuzzy logic operators negation, conjunction and disjunction coming from *quantum logic*. So they are called quantum logic implications (QL-implications for short). S-, R- and QL- implications are the most important classes of fuzzy implications which are widely studied in different aspects from the beginning until now. Examples of very recent works are: ([4], Baczyński 2006) and ([5], Baczyński 2007) work on the properties of S-implications generated from non-strong negations. Chapter 2 of ([6], Baczyński 2008) and ([53], Mas 2007) work on the interrelationship between S-, R- and QL-implications. ([51], Mas 2006), ([83], Trillas 2000) and ([78], Shi 2008) work on the properties of a group of QL-implications. ([54], Mesiar 2004) and ([67], Pei 2002) work on the properties of R-implications generated from left-continuous t-norms. ([60], Morsi 2002), ([84], Trillas 2008) and ([91], Whalen 2007) work on how fuzzy rules are represented by S-, R- and QL-implications.

Table 1.2: Individual Definitions of Fuzzy Implications

Name and symbol	Definition
Kleene-Dienes [42] [17]	$I_b(x,y) = \max(1-x, y)$
Reichenbach [70]	$I_r(x,y) = 1 - x + xy$
Most Strict [44]	$I_M(x,y) = \begin{cases} 1, & \text{if } x = 0 \\ y, & \text{if } x > 0 \end{cases}$
Largest [44]	$I_{LS}(x,y) = \begin{cases} y, & \text{if } x = 1 \\ 1-x, & \text{if } y = 0 \\ 1, & \text{if } x < 1 \text{ and } y > 0 \end{cases}$
Least Strict [44]	$I_{LR}(x,y) = \begin{cases} y, & \text{if } x = 1 \\ 1, & \text{if } x < 1 \end{cases}$
Łukasiewicz [45]	$I_{\mathbf{L}}(x,y) = \min(1-x+y, 1)$
R_0 [23]	$(I_{(\min_0)})_N(x,y) = \begin{cases} 1, & \text{if } x \leq y \\ \max(N(x), y), & \text{if } x > y \end{cases}$
Gödel [27]	$I_g(x,y) = \begin{cases} 1, & \text{if } x \leq y \\ y, & \text{if } x > y \end{cases}$
Weber [90]	$I_{WB}(x,y) = \begin{cases} 1, & \text{if } x < 1 \\ y, & \text{if } x = 1 \end{cases}$
Goguen [28]	$I_{\Delta}(x,y) = \begin{cases} 1, & \text{if } x \leq y \\ y/x, & \text{if } x > y \end{cases}$
Early Zadeh [101]	$I_m(x,y) = \max(1-x, \min(x,y))$
Standard Strict-Star [59]	$I_{sg}(x,y) = \min(I_s(x,y), I_g(1-x, 1-y))$
Standard Star-Strict [59]	$I_{gs}(x,y) = \min(I_g(x,y), I_s(1-x, 1-y))$
Standard Star-Star [59]	$I_{gg}(x,y) = \min(I_g(x,y), I_g(1-x, 1-y))$
Standard Strict-Strict [59]	$I_{ss}(x,y) = \min(I_s(x,y), I_s(1-x, 1-y))$
Willmott [92]	$I_\sharp(x,y) = \min(\max(1-x, y), \max(x, 1-x), \max(y, 1-y))$
Standard Sharp [59]	$I_\square(x,y) = \begin{cases} 1, & \text{if } x < 1 \text{ or } y = 1 \\ 0, & \text{if } x = 1 \text{ and } y < 1 \end{cases}$
Wu1 [93]	$I_{1b}(x,y) = \begin{cases} 1, & \text{if } x \leq y \\ \min(1-x, y), & \text{if } x > y \end{cases}$
Wu2 [93]	$I_{1e}(x,y) = \begin{cases} 0, & \text{if } x < y \\ y, & \text{if } x \geq y \end{cases}$
Yager [94]	$I_E(x,y) = y^x$

3. Other definitions of fuzzy implications with different motivations.
   Besides S-, R- and QL- implications, there are many other classes of fuzzy implications which are not generated from the fuzzy logic operators negation, conjunction and disjunction. For example, ([96], Yager 1999) and ([97], Yager 2004) define two parameterized classes of fuzzy implications generated from additive generating functions. Chapter 5 of ([6], Baczyński 2008) defines fuzzy implications generated from uninorms. ([32], Hatzimichailidis 2006) also defines a new class of fuzzy im-

plications.
Moreover, besides the fuzzy implications defined on the real $[0,1]$, there are discrete fuzzy implications [50], [52], and fuzzy implications defined on a finite chain [64]. They have their meaning and applications in approximate reasoning, fuzzy morphology, etc..

4. Compare different effects of fuzzy implications in approximate reasoning and other applications.

Definition 1.1. [38] Let U be an ordinary non-void set (the universe of discouse). A mapping $A : U \rightharpoonup [0,1]$ is called a *fuzzy set*.
The support of A is

$$\text{supp}A = \{x \in U | A(x) > 0\}.$$

The kernel of A is

$$\text{ker}A = \{x \in U | A(x) = 1\}$$

A is called *normal* if $\text{ker}A \neq \emptyset$.

In a fuzzy rule-based system, the approximate reasoning procedure is realized through using generalized modus ponens, generalized modus tollens, generalized fuzzy method of cases, etc., which are finally interpreted in Zadeh's compositional rule of inference [100]. For example, let X and Y be two *linguistic variables* on the *universes of discourse* U and V, respectively, A, $A^{'}$ be fuzzy sets on U, and B, $B^{'}$ be fuzzy sets on V. The generalized modus ponens based on a single-input-single-output IF-THEN rule is represented by

$$\begin{array}{r} \text{IF } X \text{ is } A \text{ THEN } Y \text{ is } B \\ X \text{ is } A^{'} \\ \hline Y \text{ is } B^{'}. \end{array}$$

We obtain $B^{'}$ through Zadeh's compositional rule of inference

$$(\forall y \in V)(B^{'}(y) = \sup_{x \in U} T(A^{'}(x), R(A(x), B(y)))), \tag{1.3}$$

where T denotes a t-norm and R denotes a fuzzy relation on $U \times V$.
One of the most important applications of fuzzy implications is to represent the fuzzy relation R in (1.3). Different rule-based system may have different requirements for reasoning results. That is the motivation to compare different effects of fuzzy implications in approximate reasoning [16], [58], [57], [56], [55] [59], [60], [61], [84].
Besides approximate reasoning, a fuzzy implication has its applications in different aspects as a $[0,1]^2 \rightharpoonup [0,1]$ mapping with the boundary condition (1.1), and sometimes, with the first place antitonicity and second place isotonicity. For example, ([36], Jayaram 2008) uses fuzzy implications to define similarity measures. ([31],

Hajek 1996) and ([37], Jayaram 2009) uses fuzzy implications to define generalized quantifiers. ([68], Pei 2008) uses fuzzy implications to define a new reasoning algorithm. ([98], Yan 2005) uses fuzzy implications to define implication operators in data mining.

5. Fuzzy implications used to define *fuzzy set inclusions*.
   There are several methods to define a fuzzy set inclusion [15], [79], [40] among which a fuzzy set inclusion based on a fuzzy implication is a very widely used one [8], [9], [11], [15]. Fuzzy set inclusions can be used to define fuzzy morphological operators in image processing, which is also one part of this thesis.

In the next sections we introduce various topics in this thesis.

1.2 Fuzzy Implication Axioms

The axiomatic system of a fuzzy implication I includes 13 axioms. They are:

- FI1. the first place antitonicity (FA):
  $(\forall(x_1, x_2, y) \in [0,1]^3)(x_1 < x_2 \Rightarrow I(x_1, y) \geq I(x_2, y))$;
- FI2. the second place isotonicity (SI):
  $(\forall(x, y_1, y_2) \in [0,1]^3)(y_1 < y_2 \Rightarrow I(x, y_1) \leq I(x, y_2))$;
- FI3. dominance of falsity of antecedent (DF): $(\forall x \in [0,1])(I(0, x) = 1)$;
- FI4. dominance of truth of consequent (DT): $(\forall x \in [0,1])(I(x, 1) = 1)$;
- FI5. boundary condition (BC): $I(1, 0) = 0$;
- FI6. neutrality of truth (NT): $(\forall x \in [0,1])(I(1, x) = x)$;
- FI7. exchange principle (EP): $(\forall(x, y, z) \in [0,1]^3)(I(x, I(y, z)) = I(y, I(x, z)))$;
- FI8. ordering principle (OP): $(\forall(x, y) \in [0,1]^2)(I(x, y) = 1 \Leftrightarrow x \leq y)$;
- FI9. the mapping N' defined as $(\forall x \in [0,1])(N'(x) = I(x, 0))$, is a strong fuzzy negation (SN);
- FI10. consequent boundary (CB): $(\forall(x, y) \in [0,1]^2)(I(x, y) \geq y)$;
- FI11. identity (ID): $(\forall x \in [0,1])(I(x, x) = 1)$;
- FI12. contrapositive principle (CP): $(\forall(x, y) \in [0,1]^2)(I(x, y) = I(N(y), N(x)))$, where N is a strong fuzzy negation;
- FI13. continuity (CO): I is a continuous mapping.

These axioms are determined because of their substantial meanings. For example, let U and V be two universes of discourse. A and B are fuzzy sets on U and V, respectively. $I(A(u), B(v))$ represents a fuzzy relation on $U \times V$. FI3 means that if we consider a point $u \in U$ such that $A(u) = 0$, then the corresponding value of v is completely not determined [20]. We will state the meanings of the other fuzzy implication axioms in Chapter 3. There are two main topics about fuzzy implication axioms in the literature. One is to investigate if a class of fuzzy implications satisfies these axioms, or under which conditions does the class of fuzzy implications satisfy these axioms (e.g., [4], [5], [23], [34],

[51], [82], [97]). The other is to investigate the interrelationships among these axioms (e.g., [2], [10],[24]), which helps to characterize some classes of fuzzy implications or to solve some functional equations about fuzzy implications [35]. The study of fuzzy implication axioms is still of great importance. For example, which QL-implications satisfy the axiom FI1 remains undetermined, while FI1 does has its significance in applications. Here are three examples:

1. **In Zadeh's compositional rule of inference**
Let A and B be two normal fuzzy sets on the universes of discourse U and V, respectively (that is, there exists a $u_0 \in U$ such that $A(u_0) = 1$ and there exists a $v_0 \in V$ such that $B(v_0) = 1$). A and B represent the antecedent and the consequent of the generalized modus ponens, respectively. Given a fact $A^{'}$, which is also a normal fuzzy set on U, we obtain the inference result $B^{'}$ through

$$(\forall v \in V)(B^{'}(v) = \sup_{u \in U} T(A^{'}(u), I(A(u), B(v)))),$$

where T denotes a t-norm and I denotes a fuzzy implication. Consider the situation where two given facts are precise i.e., represented as *singletons*

$$A_1^{'}(u) = \begin{cases} 1, & \text{if } u = u_1 \\ 0, & \text{otherwise} \end{cases}$$

and

$$A_2^{'}(u) = \begin{cases} 1, & \text{if } u = u_2 \\ 0, & \text{otherwise} \end{cases}$$

Thus the two inference results are obtained by:

$$(\forall v \in V)(B_1^{'}(v) = I(A(u_1), B(v)))$$

and

$$(\forall v \in V)(B_2^{'}(v) = I(A(u_2), B(v))).$$

According to ([97], Observation 2), if $B_1^{'}$ and $B_2^{'}$ are normal and $B_1^{'}(v) \geq B_2^{'}(v)$, for all $v \in V$, then $B_2^{'}$ is a stronger inference result than $B_1^{'}$. If $A(u_1) < A(u_2)$ which means the given fact $A_2^{'}$ is more similar to the antecedent A than the given fact $A_1^{'}$, then it is reasonable to require that $B_2^{'}$ be a stronger inference result than $B_1^{'}$. If I satisfies FI1, then I also satisfies FI4. We will have

$$(\forall (u_1, u_2) \in U \times V)(A(u_1) < A(u_2) \Rightarrow I(A(u_1), B(v)) \geq I(A(u_2), B(v))).$$

Since there exists a $v_0 \in V$ such that $B(v_0) = 1$ we obtain

$$I(A(u_1), B(v_0)) = I(A(u_2), B(v_0)) = 1.$$

Thus $B_1^{'}$ and $B_2^{'}$ are normal. Hence if I satisfies FI1, then the more similar the given fact to the antecedent of the generalized modus ponens is, the stronger the inference result is.

2. **In determining the association rules**
   In the study of fuzzy quantitative association rules, a fuzzy implication I (denoted as FIO in [98]) is used to determine the degree of implication (denoted as DImp in [98]) of a rule. According to [98], I should satisfy FI1 (see the proof of Theorem 1 in [98]) and it should satisfy the additional constraint:

$$(\forall(x,y) \in [0,1]^2)(1 + T(x,y) - x = I(x,y)) \tag{1.4}$$

   where T is a t-norm. In this case the degree of implication can be generated from the degree of support ((denoted as Dsupp in [98])) so that there is no need to scan the database again to obtain the degree of implication. Therefore it avoids database scanning in the process of calculating rules' DImps.
3. **In fuzzy DI-subsethood measure**
   In [11], a fuzzy DI-subsethood measure is defined to indicate the degree to which a fuzzy set A is contained in another fuzzy set B. The authors proved in Theorem 1 of [11] that the operator σ_{DI} which is generated by an aggregation operator M and a fuzzy implication I is a DI-subsethood measure iff I satisfies FI1, FI6, FI8 and FI9.

In this thesis, we will study the fuzzy implication axioms for S-, R- and QL-implications, especially for QL-implications in Chapter 3, and the complete interrelationships among the fuzzy implications axioms FI6-FI13 in Chapter 4.

1.3 Approximate Reasoning in Fuzzy Rule-based Systems

A fuzzy rule-based system can be applied to fuzzy control or to fuzzy decision making. There are four procedures of a fuzzy rule-based system: fuzzification, fuzzy rule base, approximate reasoning, and defuzzification. Fuzzy implications play significant roles in the approximate reasoning procedure in a fuzzy rule-based system. We give below an overview of the four procedures of a fuzzy rule-based system, and state the role fuzzy implications play in the approximate reasoning procedure.

1.3.1 Fuzzification

Let A be a fuzzy set on the universe of discourse U. The value $A(x)$, $x \in U$ expresses the degree of membership of x in A. In a fuzzy rule-based system, a linguistic variable X takes values which are lingustic terms characterized by fuzzy sets A_i, $i = 1, 2, \cdots, n$, $n \in \mathbb{N}$. In this case, these fuzzy sets are called the membership functions of X. For example, for the linguistic variable *pressure*, the membership functions can be *very low*, *low*, *middle*, *high* and *very high*. Normally membership functions have the form of triangular, trapezoidal, Gaussian functions or generalized Bell functions.
The fuzzification aims to turn input and output numerical variables of the rules of the system into linguistic variables, and to determine the membership functions. The experts'

opinions or training from existing data (e.g., neural network, genetic algorithms) can determine the supports and kernels of these membership functions [72].
Once the membership functions are determined in the rules, the real inputs in practice can also be fuzzy sets, or numerical values that called singletons. A singleton $x_0 \in U$ can be expressed as a fuzzy set of the form:

$$A(x) = \begin{cases} 1, & \text{if } x = x_0 \\ 0, & \text{if } x \neq x_0 \end{cases}, \quad x \in U. \tag{1.5}$$

Singletons inputs are widely used in early fuzzy control rule-based systems.

1.3.2 Fuzzy Rule Base

Let a fuzzy rule-based system have m inputs (m linguistic variables on the universe of discourse $U_1, U_2, \cdots, U_m$) and n outputs (n linguistic variables on the universe of discourse $V_1, V_2, \cdots, V_n$). Suppose the ith input has a_i membership functions $A_{i1}, A_{i2}, \cdots, A_{ia_i}$, and the jth output has b_j membership functions $B_{j1}, B_{j2}, \cdots, B_{jb_j}$. Then generally an IF-THEN rule of this fuzzy system has the form:

$$\begin{aligned} &\mathrm{IF} \quad \mathrm{X_1} \quad \mathrm{is} \quad \mathrm{A_{1i_1}} \quad \mathrm{AND} \quad \mathrm{X_2} \quad \mathrm{is} \quad \mathrm{A_{2i_2}} \quad \mathrm{AND} \cdots \mathrm{AND} \quad \mathrm{X_m} \quad \mathrm{is} \quad \mathrm{A_{mi_m}} \\ &\mathrm{THEN} \quad \mathrm{Y_1} \quad \mathrm{is} \quad \mathrm{B_{1j_1}} \quad \mathrm{AND} \quad \mathrm{Y_2} \quad \mathrm{is} \quad \mathrm{B_{2j_2}} \quad \mathrm{AND} \cdots \mathrm{AND} \quad \mathrm{Y_n} \quad \mathrm{is} \quad \mathrm{B_{nj_n}}, \end{aligned} \tag{1.6}$$

where $i_k = 1, 2, \cdots, a_i$, $i = 1, 2, \cdots, m$, and $j_l = 1, 2, \cdots, b_j$, $j = 1, 2, \cdots, n$.
In this case, there are in total $\prod_{i=1}^{m} a_i$ possible rules and each rule contains n outputs. To have more efficient reasoning that less computing resource costs, we can split the multi-input-multi-output (MIMO) rules represented by (1.6) into single-input-single-output (SISO) rules, and then use an *aggregation operator* to obtain the final n output fuzzy sets. A SISO rule has the form:

$$\mathrm{IF} \quad \mathrm{X_i} \quad \mathrm{is} \quad \mathrm{A_{ii_k}} \quad \mathrm{THEN} \quad \mathrm{Y_j} \quad \mathrm{is} \quad \mathrm{B_{jj_l}} \tag{1.7}$$

$$i_k = 1, 2, \cdots, a_i, i = 1, 2, \cdots, m, \text{ and } j_l = 1, 2, \cdots, b_j, j = 1, 2, \cdots, n.$$

Hence $\prod_{i=1}^{m} a_i$ rules are reduced to $\sum_{i=1}^{m} a_i$ rules and each rule contains one output. Each SISO rule represented by (1.7) can be viewed as a fuzzy relation on

$$U \times V = (U_1 \times U_2 \times \cdots \times U_m) \times (V_1 \times V_2 \times \cdots \times V_n),$$

[73].

1.3.3 Approximate Reasoning

The approximate reasoning procedure is based on the generalized modus ponens, generalized modus tollens, generalized fuzzy method of case , etc., and realized through Zadeh's compositional rule of inference (1.3). Recall that the generalized modus ponens of a SISO rule has the form

$$\begin{array}{r} \text{IF } X \text{ is } A \text{ THEN } Y \text{ is } B \\ X \text{ is } A^{'} \\ \hline Y \text{ is } B^{'}, \end{array}$$

where X and Y are linguistic variables on the universe of discourse U and V, respectively. A and $A^{'}$ are fuzzy sets on U, and B and $B^{'}$ are fuzzy sets on V.
The output fuzzy set $B^{'}$ is determined by Zadeh's compositional rule of inference (1.3).
The generalized modus tollens of an SISO rule has the form

$$\begin{array}{r} \text{IF } X \text{ is } A \text{ THEN } Y \text{ is } B \\ Y \text{ is } B^{'} \\ \hline X \text{ is } A^{'}, \end{array}$$

where X and Y are linguistic variables on the universe of discourse U and V, respectively. A and $A^{'}$ are fuzzy sets on U, and B and $B^{'}$ are fuzzy sets on V.
The output fuzzy set $A^{'}$ is determined by Zadeh's compositional rule of inference:

$$(\forall x \in U)(A^{'}(x) = \sup_{y \in V} T(B^{'}(y), R(A(x), B(y)))), \tag{1.8}$$

where T denotes a triangular norm (for details about triangular norms see in Chapter 2) and R denotes a fuzzy relation on $U \times V$.
There are two different types of fuzzy systems w.r.t. R. In a *conjunctive-type* fuzzy system [49], R is a t-norm. In an *implicative-type* fuzzy system [14], R is a fuzzy implication. The different choices of t-norms and fuzzy implications in (1.3) and (1.8) make the approximate reasoning procedure flexible. If the input $A^{'}$ in the generalized modus ponens is a singleton defined in (1.5), then (1.3) becomes

$$(\forall y \in V)(B^{'}(y) = R(A(x_0), B(y))). \tag{1.9}$$

If the input $B^{'}$ in the generalized modus tollens is a singleton defined in (1.5), then (1.8) becomes

$$(\forall x \in U)(A^{'}(x) = R(A(x), B(x_0))). \tag{1.10}$$

In a MIMO fuzzy rule-based system with m inputs and n outputs, there are $\sum_{i=1}^{m} a_i$ SISO rules for each output. We have two ways to aggregate these rules to obtain the final fuzzy set for the jth output, first aggregate then inference (FAFI in short) and first inference then aggregate (FIFA in short) . For FAFI we have for the jth output in generalized modus ponens:

$$(\forall y \in V)(B_j^{'}(y) = \sup_{x \in U} T(A^{'}(x), Agg_k I(A_k(x), B_j(y)))), \tag{1.11}$$

where $k = 1, 2, \cdots, \sum_{i=1}^{m} a_i$, and Agg denotes an aggregation operator.
For FIFA we have for the jth output in generalized modus ponens:

$$(\forall y \in V)(B_j^{'}(y) = Agg_k \sup_{x \in U} T(A^{'}(x), I(A_k(x), B_j(y)))), \tag{1.12}$$

where $k = 1, 2, \cdots, \sum_{i=1}^{m} a_i$, and Agg denotes an aggregation operator.
In FAFI, to obtain $B_j^{'}$, we need to calculate the implication operation $\sum_{i=1}^{m} a_i$ times and Zadeh's compositional rule of inference one time, while in FIFA, we need to calculate both the implication operation and Zadeh's compositional rule of inference $\sum_{i=1}^{m} a_i$ times. That is to say, FAFI is more efficient. Observe that if the inputs are singletons, then FAFI is equivalent to FIFA. The result for generalized modus tollens is similar.
Once the t-norm T and the aggregation operator Agg are determined in (1.9), (1.10) or (1.11), the flexibility of the approximate reasoning procedure depends on the choice of fuzzy implications. There are different criteria that we can consider to choose a demanded fuzzy implication in the approximate reasoning procedure. For example, in [97], the chosen fuzzy implications determine the strictness of the approximate reasoning result. In [47] and [48], the chosen fuzzy implications determine whether or not the approximate reasoning procedure can approximate any continuous mappings on a compact set to an arbitrary degree of accuracy. In [12], the chosen fuzzy implications determine the robustness which is called the δ-equalities of fuzzy sets of the approximate reasoning result. In Chapter 5 and Chapter 6, we will choose demanded fuzzy implications for the case that there are repeated information in the antecedent in an IF-THEN rule, and for the case that there are perturbations in the antecedent and the consequent in a determined IF-THEN rule in the rule base, respectively.

1.3.4 Defuzzification

In a fuzzy control rule-base system, when we obtain a final fuzzy set B_j for the jth output, $j = 1, 2, \cdots, n$, we need to defuzzify B_j to get a numerical output as the control signal. The most common defuzzifier is *the center of area* (COA for short), or the *mean of maxima* (MOM for short). For further information we refer to [89].

1.4 Fuzzy Morphology

Another important application of fuzzy implications is to generate fuzzy morphological operations. Mathematical morphology is an important theory developed in image processing to analyze the geometric features of n-dimensional images [74] while fuzzy morphology is developed to process gray-scale images [63]. Fuzzy morphological operations are basic tools in fuzzy morphology. In the fuzzy morphology based on fuzzy set inclusion (see [63]), fuzzy morphological operations are generated from fuzzy implications and conjunctions on the unit interval. The algebraic properties of these fuzzy implications and conjunctions on the unit interval are worthy of investigating. We will have deep study of this topic in Chapter 7.

1.5 Outline

This thesis targets to develop the classes and axioms of fuzzy implications and to compare the effects of different fuzzy implications under different requirements in approximate reasoning as well as in fuzzy morphology.

- Chapter 2 gives preliminaries for the fuzzy logic operators, S-, R- and QL- implications, and parameterized fuzzy implications.
- Chapter 3 investigates the characterizations of S- and R-implications, and the axioms of QL-implications.
- Chapter 4 gives a complete determination of the interrelationship between the fuzzy implication axioms FI6-FI13.
- Chapter 5 investigates a tautology in the approximate reasoning and solves the corresponding functional equations for S-, R- and QL- implications.
- Chapter 6 compares the robustness against perturbations of different fuzzy logic operators in approximate reasoning.
- Chapter 7 works on the fuzzy morphological operations generated from fuzzy implications and conjunctions on the unit interval.
- The last chapter concludes the research conducted in this thesis.

Chapter 2

Preliminaries and Classifications of Fuzzy Implications

2.1 Introduction

In fuzzy logic, the classical binary negation, conjunction, disjunction and implication are extended to mappings that take values in the unit interval respectively. A fuzzy negation operator is normally modelled as a *fuzzy negation*. A fuzzy conjunction operator is normally modelled as a *conjunction on the unit interval* or (more usually) as a *triangular norm* (*t-norm* for short). A fuzzy disjunction operator is normally modelled as a *triangular conorm* (*t-conorm* for short). There are many approaches to model a fuzzy implication operator. It can be constructed from the other three fuzzy logic operators, or it can be constructed from some parameterized generating functions. In this chapter we give the definitions and basic properties of the negation operators, conjunction operators, disjunction operators and implication operators in fuzzy logic.
The concepts of automorphism and conjugate will be useful in the whole chapter.

Definition 2.1. *([10], Definition 0)* A mapping $\varphi : [a,b] \rightharpoonup [a,b]$ ($[a,b] \subset \mathbb{R}$) is *an automorphism of the interval* $[a,b]$ if it is continuous and strictly increasing and satisfies the boundary conditions: $\varphi(a) = a$ and $\varphi(b) = b$.

Lemma 2.2. *If φ is an automorphism of the unit interval, then φ^{-1} is also an automorphism of the unit interval.*

PROOF. This proof is followed directly by Definition 2.1. □

Lemma 2.3. *(The chain rule) The composition of two automorphisms of the interval $[a,b]$ is again an automorphism of the interval $[a,b]$.*

Table 2.1: Truth table of the classical binary negation

p	$\neg p$
0	1
1	0

PROOF. Let $\gamma = \varphi \circ \phi$ (i.e., $\gamma(x) = \varphi(\phi(x))$, for all $x \in [0, 1]$), where φ and ϕ are two automorphisms of the interval $[a, b]$. We have

$$\gamma(a) = \varphi \circ \phi(a) = \varphi(a) = a,$$
$$\gamma(b) = \varphi \circ \phi(b) = \varphi(b) = b.$$

Moreover, γ is continuous strictly increasing because φ and ϕ are continuous strictly increasing. So according to Definition 2.1, γ is an automorphisms of the interval $[a, b]$. □

Definition 2.4. ([2], Definition 2) Two mappings F, G: $[0, 1]^n \rightharpoonup [0, 1]$, $n \in \mathcal{N}$, are *conjugate*, if there exists an automorphism φ of the unit interval such that $G = F_\varphi$, where

$$F_\varphi(x_1, x_2, \cdots, x_n) = \varphi^{-1}(F(\varphi(x_1), \varphi(x_2), \cdots, \varphi(x_n))). \tag{2.1}$$

2.2 Negations

The truth table of the classical binary negation $\neg$ is given in Table 2.1. In many-valued logic we extend the classical binary negation to the unit interval as a $[0, 1] \rightharpoonup [0, 1]$ mapping as follows:

Definition 2.5. A mapping N: $[0, 1] \rightharpoonup [0, 1]$ is a *fuzzy negation* if it satisfies:

N1. boundary conditions: $N(0) = 1$ and $N(1) = 0$,

N2. monotonicity: $(\forall (x, y) \in [0, 1]^2)(x \le y \Rightarrow N(x) \ge N(y))$.

Moreover, a fuzzy negation N is said to be *strict* if N is a continuous and strictly decreasing mapping.
A fuzzy negation N is said to be *strong* if $N(N(x)) = x$, for all $x \in [0, 1]$.
For any continuous fuzzy negation N, there exists a unique *equilibrium point* $e \in]0, 1[$ such that $N(e) = e$ and for all $x < e$, $N(x) > e > x$, for all $x > e$, $N(x) < e < x$ [[10], Section 1.1].

Notice that a strong fuzzy negation is strict and it is a continuous mapping. We give examples of non-continuous fuzzy negations, a continuous but non-strict fuzzy negation, a strict but non-strong fuzzy negation and a strong fuzzy negation, respectively, as follows.

Example 2.1 1. The fuzzy negation N_{1a}:

$$N_{1a}(x) = \begin{cases} 0 & \text{if } x = 1, \\ 1 & \text{otherwise} \end{cases}, \quad x \in [0,1] \tag{2.2}$$

and N_{1b}:

$$N_{1b}(x) = \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{otherwise} \end{cases}, \quad x \in [0,1] \tag{2.3}$$

are not continuous.

2. The fuzzy negation N_2:

$$N_2(x) = \begin{cases} 1 - x & \text{if } x \in [0, 0.4], \\ 0.6 & \text{if } x \in [0.4, 0.6] \\ -1.5x + 1.5 & \text{otherwise} \end{cases}, \quad x \in [0,1] \tag{2.4}$$

is continuous but not strict.

3. The fuzzy negation N_3:

$$N_3(x) = 1 - x^2, \quad x \in [0,1] \tag{2.5}$$

is strict but not strong.

4. The fuzzy negation N_0:

$$N_0(x) = 1 - x, \quad x \in [0,1] \tag{2.6}$$

is strong.
N_0 is the standard strong fuzzy negation. We have the next theorem for strong fuzzy negations.

Theorem 2.6. *[81] A fuzzy negation N is strong iff there exists an automorphism φ of the unit interval such that*

$$N(x) = \varphi^{-1}(1 - \varphi(x)), \quad x \in [0,1]. \tag{2.7}$$

2.3 Conjunctions

The truth table of the classical binary conjunction $\wedge$ is given in Table 2.2. In many-valued logic we extend the classical binary conjunction to the unit interval as a $[0,1]^2 \rightharpoonup [0,1]$ mapping as follows:

Definition 2.7. A mapping $\mathcal{C}$: $[0,1]^2 \rightharpoonup [0,1]$ is a *conjunction on the unit interval* if it satisfies:

$\mathcal{C}1$. boundary conditions: $\mathcal{C}(0,0) = \mathcal{C}(0,1) = \mathcal{C}(1,0) = 0$ and $\mathcal{C}(1,1) = 1$,

Table 2.2: Truth table of the classical binary conjunction

p	q	$p \wedge q$
0	0	0
0	1	0
1	0	0
1	1	1

$\mathcal{C}2$. monotonicity: $(\forall(x, y, z) \in [0,1]^3)(x \leq y \Rightarrow \mathcal{C}(x,z) \leq \mathcal{C}(y,z)$ and $\mathcal{C}(z,x) \leq \mathcal{C}(z,y))$.

Definition 2.8. A mapping T: $[0,1]^2 \rightharpoonup [0,1]$ is a *triangular norm* (*t-norm* for short) if for all $x, y, z \in [0,1]$ it satisfies:

T1. boundary condition: $T(x, 1) = x$,

T2. monotonicity: $y \leq z$ implies $T(x,y) \leq T(x,z)$,

T3. commutativity: $T(x,y) = T(y,x)$,

T4. associativity: $T(x, T(y,z)) = T(T(x,y), z)$.

A t-norm T always satisfies $T(x,x) \leq x$, for all $x \in [0,1]$.
Every t-norm is a conjunction on the unit interval. In fuzzy logic, t-norms are widely used to model conjunction operators. Besides T1-T4, there are also some additional requirements of t-norms. Here we mention seven of them as examples:

T5. continuity: T is continuous,

T6. left-continuity: the partial mappings of T are left-continuous,

T7. idempotency: $T(x,x) = x$, for all $x \in [0,1]$,

T8. subidempotency: $T(x,x) < x$, for all $x \in]0,1[$,

T9. Archimedean property: for all $(x,y) \in]0,1[^2$, there exists an $n \in \mathbb{N}$ such that $T(\underbrace{x, x, ..., x}_{n \text{ times}}) < y$ [[43], Definition 2.9 (iv)],

T10. strictness: T is continuous and $0 < y < z < 1 \Rightarrow T(x,y) < T(x,z)$, for all $x, y, z \in]0,1]$,

T11. nilpotency: T is continuous and for all $x \in]0,1[$, there exists a $y \in]0,1[$ such that $T(x,y) = 0$.

Proposition 2.9. *A t-norm T satisfies $T(x,x) \leq x$, for all $x \in [0,1]$.*

Remark 2.10. According to ([43], Definition 2.1 (i)), an element $x \in [0,1]$ such that $T(x,x) = x$ is called an *idempotent element* of T. The numbers 0 and 1, which are idempotent elements for each t-norm T, are called *trivial* idempotent elements of T. Each idempotent element in $]0,1[$ will be called a *non-trivial* idempotent element of T. Thus T satisfying subidempotency is equivalent to T having only trivial idempotent elements. Hence according to ([43], Fig. 2.2), the subidempotency is equivalent to the Archimedean property only for continuous t-norms.

Five very important t-norms ([43], Example 1.2, [23]) are commonly used:

1. $T_{\mathbf{M}}(x,y) = \min(x,y)$, (minimum)
2. $T_{\mathbf{P}}(x,y) = xy$, (product)
3. $T_{\mathbf{L}}(x,y) = \max(x+y-1,0)$, (Łukasiewicz t-norm)
4. $T_{\mathbf{D}}(x,y) = \begin{cases} \min(x,y) & \text{if } x=1 \text{ or } y=1 \\ 0 & \text{otherwise,} \end{cases}$ (drastic product)
5. $(T_{\min_0})_N(x,y) = \begin{cases} \min(x,y) & \text{if } y > N(x) \\ 0 & \text{otherwise} \end{cases}$ (nilpotent minimum), where N denotes a strong fuzzy negation.

The minimum $T_{\mathbf{M}}$ is the only idempotent t-norm ([44], Theorem 3.9). The nilpotent minimum $T_{\min_0}$ was first introduced by Fodor in [23]. It is a left-continuous t-norm. The drastic product T_D is not left-continuous (continuous).
Klement et al. state in their book [43] a collection of parameterized families of t-norms which are interesting from different points of view. They are:

1. Family of Schweizer-Sklar t-norms

$$T_s^{SS}(x,y) = \begin{cases} T_{\mathbf{M}}(x,y), & \text{if } s = -\infty \\ T_{\mathbf{P}}(x,y), & \text{if } s = 0 \\ T_{\mathbf{D}}(x,y), & \text{if } s = +\infty \\ (\max(x^s + y^s - 1, 0))^{\frac{1}{s}}, & \text{if } s \in]-\infty, 0[\cup]0, +\infty[\end{cases}, \quad (2.8)$$

2. Family of Hamacher t-norms

$$T_s^{H}(x,y) = \begin{cases} T_{\mathbf{D}}(x,y), & \text{if } s = +\infty \\ 0, & \text{if } s = x = y = 0 \\ \frac{xy}{s+(1-s)(x+y-xy)}, & \text{if } s \in [0,+\infty[\text{ and } (s,x,y) \neq (0,0,0) \end{cases}, \quad (2.9)$$

3. Family of Yager t-norms

$$T_s^{Y}(x,y) = \begin{cases} T_{\mathbf{D}}(x,y), & \text{if } s = 0 \\ T_{\mathbf{M}}(x,y), & \text{if } s = +\infty \\ \max(1 - ((1-x)^s + (1-y)^s)^{\frac{1}{s}}, 0), & \text{if } s \in]0,+\infty[\end{cases}, \quad (2.10)$$

4. Family of Dombi t-norms

$$T_s^D(x,y) = \begin{cases} T_{\mathbf{D}}(x,y), & \text{if } s=0 \\ T_{\mathbf{M}}(x,y), & \text{if } s=+\infty \\ \frac{1}{1+((\frac{1-x}{x})^s+(\frac{1-y}{y})^s)^{\frac{1}{s}}}, & \text{if } s\in\,]0,+\infty[\end{cases}, \tag{2.11}$$

5. Family of Sugeno-Weber t-norms

$$T_s^{SW}(x,y) = \begin{cases} T_{\mathbf{D}}(x,y), & \text{if } s=-1 \\ T_{\mathbf{P}}(x,y), & \text{if } s=+\infty \\ (\max(\frac{x+y-1+sxy}{1+s},0), & \text{if } s\in\,]-1,+\infty[\end{cases}, \tag{2.12}$$

6. Family of Aczél-Alsina t-norms

$$T_s^{AA}(x,y) = \begin{cases} T_{\mathbf{D}}(x,y), & \text{if } s=0 \\ T_{\mathbf{M}}(x,y), & \text{if } s=+\infty \\ (e^{-((-\log x)^s+(-\log y)^s)^{\frac{1}{s}}}, & \text{if } s\in\,]0,+\infty[\end{cases} \tag{2.13}$$

7. Family of Frank t-norms

$$T_s^F(x,y) = \begin{cases} T_{\mathbf{M}}(x,y), & \text{if } s=0 \\ T_{\mathbf{P}}(x,y), & \text{if } s=1 \\ T_{\mathbf{L}}(x,y), & \text{if } s=+\infty \\ \log_s(1+\frac{(s^x-1)(s^y-1)}{s-1}), & \text{otherwise} \end{cases}, \tag{2.14}$$

8. Family of Mayor-Torrens t-norms

$$T_s^{MT}(x,y) = \begin{cases} \max(x+y-s,0), & \text{if } s\in\,]0,1] \text{ and } (x,y)\in[0,s]^2 \\ \min(x,y), & \text{if } s=0 \text{ or } x>s \text{ or } y>s \end{cases}. \tag{2.15}$$

Theorem 2.11. *([44], Theorem 3.11) T is a continuous Archimedean t-norm iff there exists a decreasing generator f such that*

$$T(x,y) = f^{(-1)}(f(x)+f(y)), \quad \forall x,y\in[0,1], \tag{2.16}$$

where a decreasing generator f is defined as a continuous and strictly decreasing mapping from $[0,1]$ to $[0,+\infty[$ such that $f(1)=0$. The pseudo-inverse of f, $f^{(-1)}$: $[0,+\infty[$ $\rightharpoonup [0,1]$ is defined as

$$f^{(-1)}(x) = \begin{cases} f^{-1}(x) & \text{if } x\in[0,f(0)] \\ 0 & \text{if } x\in\,]f(0),+\infty[\end{cases}, \tag{2.17}$$

where f^{-1} denotes the ordinary inverse of f.

It has been proved that a continuous Archimedean t-norm is either strict or nilpotent [[43], p.33].

Theorem 2.12. *([43], Corollary 5.7)*

i) A t-norm T is strict iff it is conjugate with the product $T_{\mathbf{P}}$, i.e., there exists an automorphism φ of the unit interval such that

$$T(x,y) = \varphi^{-1}(\varphi(x)\varphi(y)), \quad x,y \in [0,1]. \tag{2.18}$$

ii) A t-norm T is nilpotent iff it is conjugate with the Łukasiewicz t-norm $T_{\mathbf{L}}$, i.e., there exists an automorphism φ of the unit interval such that

$$T(x,y) = \varphi^{-1}(\max(\varphi(x)+\varphi(y)-1,0)), \quad x,y \in [0,1]. \tag{2.19}$$

We can generate a fuzzy negation from a t-norm [6].

Definition 2.13. Let T be a t-norm. The *natural negation N_T of T* is defined as

$$(\forall x \in [0,1])(N_T(x) = \sup\{t \in [0,1] | T(x,t) = 0\}). \tag{2.20}$$

We give here examples of natural negations of the aforementioned five important t-norms.

Example 2.2 1. The natural negation of the minimum T_M is:

$$N_{T_{\mathbf{M}}}(x) = \sup\{t \in [0,1] | \min(x,t) = 0\} = N_{1b}(x). \tag{2.21}$$

2. The natural negation of the product T_P is:

$$N_{T_{\mathbf{P}}}(x) = \sup\{t \in [0,1] | xt = 0\} = N_{1b}(x). \tag{2.22}$$

3. The natural negation of the Łukasiewicz t-norm $T_{\mathbf{L}}$ is:

$$N_{T_{\mathbf{L}}}(x) = \sup\{t \in [0,1] | \max(x+t-1,0) = 0\} = N_0(x). \tag{2.23}$$

4. The natural negation of the drastic product $T_{\mathbf{D}}$ is:

$$N_{T_{\mathbf{D}}}(x) = \sup\{t \in [0,1] | T_{\mathbf{D}}(x,y) = 0\} = N_{1b}(x). \tag{2.24}$$

5. The natural negation of the nilpotent minimum $(T_{\min_0})_N$ is:

$$N_{T_{\mathbf{D}}}(x) = \sup\{t \in [0,1] | (T_{\min_0})_N(x,y) = 0\} = N(x). \tag{2.25}$$

The natural negation of a t-norm plays an important role in the *law of excluded middle* (*LEM* for short).

Theorem 2.14. *A left-continuous t-norm T and a fuzzy negation N satisfy*

$$(\forall x \in [0,1])(T(x, N(x)) = 0) \tag{2.26}$$

iff

$$(\forall x \in [0,1])(N(x) \leq N_T(x)). \tag{2.27}$$

PROOF. $\Longleftarrow$: Because T is left-continuous, equation (2.20) can be rewritten as

$$(\forall x \in [0,1])(N_T(x) = \max\{t \in [0,1] | T(x,t) = 0\}). \tag{2.28}$$

Then $T(x, N_T(x)) = 0$. Because of T2, if $N(x) \leq N_T(x)$, then $T(x, N(x)) = 0$.
$\Longrightarrow$: Because $T(x, N(x)) = 0$,

$$N(x) \in \{t \in [0,1] | T(x,t) = 0\}.$$

So

$$N(x) \leq \max\{t \in [0,1] | T(x,t) = 0\} = N_T(x).$$

□

Remark 2.15. For a non left-continuous t-norm T and a fuzzy negation N, condition (2.27) is necessary but not sufficient for them to satisfy equation (2.26).

Theorem 2.16. *([10], Theorem 2) A continuous t-norm T and a strict fuzzy negation N satisfy equation (2.26) iff T is conjugate with the Łukasiewicz t-norm $T_{\mathbf{L}}$, and*

$$(\forall x \in [0,1])(N(x) \leq \varphi^{-1}(1 - \varphi(x))). \tag{2.29}$$

Continuous t-norms have been well studied. We have the next definition and theorem.

Definition 2.17. ([43], Theorem 3.44) Let $\{T_m\}_{m \in M}$ be a family of t-norms and $\{[a_m, b_m]\}_{m \in M}$ be a non-empty family of non-overlapping, closed, proper subintervals of $[0,1]$, where M is a finite or countable index set. Then a t-norm T_o is called the *ordinal sum of* $\{[a_m, b_m], T_m\}_{m \in M}$ if

$$T_o(x,y) = \begin{cases} a_m + (b_m - a_m) T_m\left(\frac{x - a_m}{b_m - a_m}, \frac{y - a_m}{b_m - a_m}\right) & \text{if } (x,y) \in [a_m, b_m]^2 \\ T_{\mathbf{M}}(x,y) & \text{otherwise} \end{cases}. \tag{2.30}$$

Remark 2.18. If there exists only one subinterval $[a_1, b_1]$ of $[0,1]$ with $a_1 = 0$, $b_1 = 1$, then $T_o = T_1$. Henceforth we always assume for the ordinal sum defined in (2.30) that there exists at least one subinterval $[a_k, b_k]$ such that $a_k \neq 0$ or $b_k \neq 1$.

Theorem 2.19. *([24], Section 1.3.4, [43], Theorem 5.11) T is a continuous t-norm iff at least one of the following conditions holds:*

i) $T = T_{\mathbf{M}}$

ii) T is continuous Archimedean

iii) there exists a family $\{[a_m, b_m], T_m\}$ *where* $\{[a_m, b_m]\}$ *is a countable family of non-overlapping, closed, proper subintervals of* $[0, 1]$ *with each* T_m *being a continuous Archimedean t-norm such that* T *is the ordinal sum of this family.*

Table 2.3 shows for which parameters the aforementioned parameterized families of t-norms are continuous, Archimedean, strict, and nilpotent.
For Frank t-norms we have the following proposition:

Table 2.3: Parameterized families of t-norms and their properties

Family	Continuous	Archimedean	Strict	Nilpotent
T_s^{SS}	$s \in [-\infty, +\infty[$	$s \in]-\infty, +\infty]$	$s \in]-\infty, 0]$	$s \in]0, +\infty[$
T_s^{H}	$s \in [0, +\infty[$	$s \in [0, +\infty]$	$s \in [0, +\infty[$	none
T_s^{Y}	$s \in]0, +\infty]$	$s \in [0, +\infty[$	none	$s \in]0, +\infty[$
T_s^{D}	$s \in]0, +\infty]$	$s \in [0, +\infty[$	$s \in]0, +\infty[$	none
T_s^{SW}	$s \in]-1, +\infty]$	$s \in [-1, +\infty]$	$s = +\infty$	$s \in]-1, +\infty[$
T_s^{AA}	$s \in]0, +\infty]$	$s \in [0, +\infty[$	$s \in]0, +\infty[$	none
T_s^{F}	$s \in [0, +\infty]$	$s \in]0, +\infty]$	$s \in]0, +\infty[$	$s = +\infty$
T_s^{MT}	$s \in [0, 1]$	$s = 1$	none	$s = 1$

Proposition 2.20. *A Frank t-norm or an ordinal sum* T *defined in (2.30) where each* T_m*,* $m \in M$*, is a Frank t-norm different from* $T_{\mathbf{M}}$*, satisfies the 1-Lipschitz property:*

$$(\forall (x_1, x_2, y) \in [0, 1]^3)(x_1 \leq x_2 \Rightarrow T(x_2, y) - T(x_1, y) \leq x_2 - x_1) \qquad (2.31)$$

There is another important conjunction on the unit interval:

Definition 2.21. ([43], Definition 9.4) A mapping $C : [0, 1]^2 \rightharpoonup [0, 1]$ is a *(two dimensional) copula* if for all $x, x^*, y, y^* \in [0, 1]$ with $x \leq x^*$ and $y \leq y^*$, it satisfies:

C1. $C(x, y) + C(x^*, y^*) \geq C(x, y^*) + C(x^*, y)$

C2. $C(x, 0) = C(0, x) = 0$

C3. $C(x, 1) = C(1, x) = x$

Theorem 2.22. *([43], Theorem 9.10) A t-norm* T *is a copula iff* T *satisfies the 1-Lipschitz property (2.31).*

For more results and applications of t-norms, we refer to [43].

2.4 Disjunctions

The truth table of the classical binary disjunction $\vee$ is given in Table 2.4. The disjunction

Table 2.4: Truth table of the classical binary disjunction

p	q	$p \vee q$
0	0	0
0	1	1
1	0	1
1	1	1

in fuzzy logic is often modeled as follows:

Definition 2.23. A mapping S: $[0,1]^2 \rightharpoonup [0,1]$ is a *triangular conorm* (*t-conorm* for short) if for all $x, y, z \in [0,1]$ it satisfies:

S1. boundary condition: $S(x,0) = x$,

S2. monotonicity: $y \leq z$ implies $S(x,y) \leq S(x,z)$,

S3. commutativity: $S(x,y) = S(y,x)$,

S4. associativity: $S(x, S(y,z)) = S(S(x,y),z)$.

A t-conorm S always satisfies $S(x,x) \geq x$, for all $x \in [0,1]$.
Similar to t-norms, we mention here six more properties of t-conorms:

S5. continuity: S is continuous,

S6. idempotency: $S(x,x) = x$, for all $x \in [0,1]$,

S7. subidempotency: $S(x,x) > x$, for all $x \in \,]0,1[\,$,

S8. Archimedean property: for all $(x,y) \in \,]0,1[\,^2$, there exists an $n \in \mathbb{N}$ such that $S(\underbrace{x, x, ..., x}_{n \text{ times}}) > y$ [[43], Remark 2.20 (AP*)],

S9. strictness: S is continuous and $0 < y < z < 1 \Rightarrow S(x,y) < S(x,z)$, for all $x, y, z \in [0,1[\,$,

S10. nilpotency: S is continuous and for all $x \in \,]0,1[\,$, there exists a $y \in \,]0,1[$ such that $S(x,y) = 1$.

Proposition 2.24. *A t-conorm S satisfies $S(x,x) \geq x$, for all $x \in [0,1]$.*

Remark 2.25. ([43], Remark 2.20, Fig. 2.2) Similar to that four t-norms, the subidempotency is equivalent to the Archimedean property only for continuous t-conorms.

Four important t-conorms [[43], Example 1.14] are commonly used:

1. $S_{\mathbf{M}}(x,y) = \max(x,y)$, (maximum)
2. $S_{\mathbf{P}}(x,y) = x + y - xy$, (probabilistic sum)
3. $S_{\mathbf{L}}(x,y) = \min(x+y,1)$, (Łukasiewicz t-conorm, bounded sum)
4. $S_{\mathbf{D}}(x,y) = \begin{cases} \max(x,y) & \text{if } x = 0 \text{ or } y = 0 \\ 1 & \text{otherwise.} \end{cases}$ (drastic sum)

The maximum $S_{\mathbf{M}}$ is the only idempotent t-conorm [11, Theorem 3.14].

Theorem 2.26. *([44], Theorem 3.16) S is a continuous Archimedean t-conorm iff there exists an increasing generator g such that*

$$S(x,y) = g^{(-1)}(g(x) + g(y)), \quad \forall x,y \in [0,1], \tag{2.32}$$

where an increasing generator g is defined as a continuous and strictly increasing mapping from $[0,1]$ to $[0,+\infty[$ such that $g(0) = 0$. The pseudo-inverse of g, $g^{(-1)}$: $[0,+\infty[\rightharpoonup [0,1]$ is defined as

$$g^{(-1)}(x) = \begin{cases} g^{-1}(x) & \text{if } x \in [0, g(1)] \\ 1 & \text{if } x \in]g(1), +\infty[\end{cases}, \tag{2.33}$$

where g^{-1} denotes the ordinary inverse of g.

Similar to t-norms, it has been concluded that a continuous Archimedean t-conorm is either strict or nilpotent ([43], Remark 2.20).

Theorem 2.27. *([24], Theorem 1.8, Theorem 1.9)*

i) A t-conorm S is strict iff it is conjugate with the probabilistic sum $S_{\mathbf{P}}$, i.e., there exists an automorphism φ of the unit interval such that

$$S(x,y) = \varphi^{-1}(\varphi(x) + \varphi(y) - \varphi(x)\varphi(y)), \quad x,y \in [0,1]. \tag{2.34}$$

ii) A t-conorm S is nilpotent iff it is conjugate with the Łukasiewicz t-conorm $S_{\mathbf{L}}$, i.e., there exists an automorphism φ of the unit interval such that

$$S(x,y) = \varphi^{-1}(\min(\varphi(x) + \varphi(y), 1)), \quad x,y \in [0,1]. \tag{2.35}$$

Definition 2.28. The *dual t-conorm S of a t-norm T w.r.t. a strong fuzzy negation N* is defined as

$$(\forall (x,y) \in [0,1]^2)(S(x,y) = N(T(N(x), N(y))). \tag{2.36}$$

Example 2.3 Let N be a strong fuzzy negation. The dual t-conorm of the nilpotent minimum $(T_{\min_0})_N$ *nilpotent maximum* is represented as:

$$(S_{\max_0})_N(x,y) = \begin{cases} \max(x,y) & \text{if } N(y) > x \\ 1 \quad \text{else} & \end{cases}. \tag{2.37}$$

The Frank t-norms proposed by Frank [25] are to solve the Frank equation of a t-norm T and a t-conorm S:

$$(\forall (x,y) \in [0,1]^2)(S(x,y) + T(x,y) = x + y). \tag{2.38}$$

The next theorem presents the solution of equation (2.38).

Theorem 2.29. *([43], Theorem 5.14) A t-norm T and a t-conorm S satisfy the Frank equation (2.38) iff T is a Frank t-norm or an ordinal sum defined in (2.30) where each T_m, $m \in M$, is a Frank t-norm different from $T_{\mathbf{M}}$, and S is the dual t-conorm of T defined in (2.36) with $N = N_0$.*

We can generate a fuzzy negation from a t-conorm.

Definition 2.30. Let S be a t-conorm. The *natural negation N_S of S* is defined as

$$(\forall x \in [0,1])(N_S(x) = \inf\{t \in [0,1] | S(x,t) = 1\}). \tag{2.39}$$

We give here examples of natural negations of the aforementioned four important t-conorms and the nilpotent maximum.

Example 2.4 1. The natural negation of the maximum $S_{\mathbf{M}}$ is:

$$N_{S_{\mathbf{M}}}(x) = \inf\{t \in [0,1] | \max(x,t) = 1\} = N_{1a}(x). \tag{2.40}$$

2. The natural negation of the probabilistic sum $S_{\mathbf{P}}$ is :

$$N_{S_{\mathbf{P}}}(x) = \inf\{t \in [0,1] | x + t - xt = 1\} = N_{1a}(x). \tag{2.41}$$

3. The natural negation of the Łukasiewicz t-conorm $S_{\mathbf{L}}$ is:

$$N_{S_{\mathbf{L}}}(x) = \inf\{t \in [0,1] | \min(x+t,1) = 1\} = 1 - x = N_0(x). \tag{2.42}$$

4. The natural negation of the drastic sum $S_{\mathbf{D}}$ is:

$$N_{S_{\mathbf{D}}}(x) = \inf\{t \in [0,1] | S_{\mathbf{D}}(x,t) = 1\} = N_{1b}(x). \tag{2.43}$$

5. The natural negation of the nilpotent maximum $(S_{\max_0})_N$ is:

$$N_{(S_{\max_0})_N}(x) = \inf\{t \in [0,1] | (S_{\max_0})_N(x,t) = 1\} = N(x). \tag{2.44}$$

The natural negation of a t-conorm plays an important role in the LEM.

Theorem 2.31. *A right-continuous t-conorm S and a fuzzy negation N satisfy*

$$(\forall x \in [0,1])(S(x, N(x)) = 1) \tag{2.45}$$

iff

$$(\forall x \in [0,1])(N(x) \geq N_S(x)). \tag{2.46}$$

PROOF. $\Longleftarrow$: Because S is right-continuous, equation (2.39) can be rewritten as

$$(\forall x \in [0,1])(N_S(x) = \min\{t \in [0,1] | S(x,t) = 1\}). \tag{2.47}$$

Then $S(x, N_S(x)) = 1$. Because of S2, if $N(x) \geq N_S(x)$, then $S(x, N(x)) = 1$.
$\Longrightarrow$: Because $S(x, N(x)) = 1$, $N(x) \in \{t \in [0,1] | S(x,t) = 1\}$. So

$$N(x) \leq \min\{t \in [0,1] | S(x,t) = 1\} = N_S(x).$$

□

Remark 2.32. For a non right-continuous t-conorm S and a fuzzy negation N, condition (2.46) is necessary but not sufficient for them to satisfy equation (2.45).

Theorem 2.33. *([10], Theorem 1) A continuous t-conorm S and a strict fuzzy negation N satisfy equation (2.45) iff S is conjugate with the Łukasiewicz t-conorm $S_{\mathbf{L}}$, and $N(x) \geq \varphi^{-1}(1 - \varphi(x))$.*

Similar to continuous t-norms, we have the next definition and theorem for continuous t-conorms.

Definition 2.34. Let $\{S_m\}_{m \in M}$ be a family of t-conorms and $\{[a_m, b_m]\}_{m \in M}$ be a non-empty family of non-overlapping, closed, proper subintervals of $[0,1]$, where M is a finite or countable index set. Then a t-conorm S_o is called the *ordinal sum of* $\{[a_m, b_m], S_m\}_{m \in M}$ if

$$S_o(x,y) = \begin{cases} a_m + (b_m - a_m) S_m\left(\frac{x - a_m}{b_m - a_m}, \frac{y - a_m}{b_m - a_m}\right) & \text{if } (x,y) \in [a_m, b_m]^2 \\ S_{\mathbf{M}}(x,y) & \text{otherwise} \end{cases}. \tag{2.48}$$

Remark 2.35. If there exists only one subinterval $[a_1, b_1]$ of $[0,1]$ with $a_1 = 0$, $b_1 = 1$, then $S_o = S_1$. Henceforth we always assume for the ordinal sum defined in (2.48) that there exists at least one subinterval $[a_k, b_k]$ such that $a_k \neq 0$ or $b_k \neq 1$.

Theorem 2.36. *([24], Section 1.4.4)([43], Corollary 3.58) S is a continuous t-conorm iff one of the following conditions holds:*

i) $S = S_{\mathbf{M}}$

ii) S is continuous Archimedean

iii) there exists a family $\{[a_m, b_m], S_m\}$ where $\{[a_m, b_m]\}$ is a countable family of non-overlapping, closed, proper subintervals of $[0,1]$ with each S_m being a continuous Archimedean t-conorm such that S is the ordinal sum of this family.

Definition 2.37. If

$$(\forall (x,y,z) \in [0,1]^3)(S(x, T(y,z)) = T(S(x,y), S(x,z))), \tag{2.49}$$

then we say that *the t-conorm S is distributive over the t-norm T* ([43], Proposition 2.22). Similarly, if

$$(\forall (x,y,z) \in [0,1]^3)(T(x, S(y,z)) = S(T(x,y), T(x,z))), \tag{2.50}$$

then we say that the t-norm T is distributive over the t-conorm S.

The only distributive pair is $T_{\mathbf{M}}$ and $S_{\mathbf{M}}$ ([43], Proposition 2.22).

2.5 Fuzzy Implications Generated from Other Fuzzy Logic Operators

The truth table of the classical binary implication $\rightarrow$ is given in Table 2.5. In many-valued

Table 2.5: Truth table of the classical binary implication

p	q	$p \rightarrow q$
0	0	1
0	1	1
1	0	0
1	1	1

logic we extend the classical binary implication to the unit interval as a $[0,1]^2 \rightharpoonup [0,1]$ mapping as follows:

Definition 2.38. A mapping I: $[0,1]^2 \rightharpoonup [0,1]$ is a *fuzzy implication* if it satisfies the boundary conditions:

I1. $I(0,0) = I(0,1) = I(1,1) = 1$ and $I(1,0) = 0$.

As mentioned in [[10], Section 2] that all fuzzy implications are obtained by generalizing the implication of classical binary logic. There are three important ways to generate fuzzy implications through classical binary logic: S-implications, R-implications, and QL-implications.

2.5.1 S-implications

S-implications are the short for strong implications. An S-implication is generated from a fuzzy negation and a t-conorm, getting idea from the proposition in classical binary logic:

$$p \rightarrow q \Leftrightarrow \neg p \vee q.$$

Definition 2.39. Let S be a t-conorm and N be a fuzzy negation. An S-implication is defined as

$$I(x,y) = S(N(x),y), \quad \forall x,y \in [0,1]. \tag{2.51}$$

Remark 2.40. N is assumed to be a strong fuzzy negation (e.g., in [[10], Section 4] [[24], Definition 1.16]). However, N is not necessary supposed to be strong, even not necessary to be continuous [[43], Definition 11.5], [4] (where the authors call the S-implications generated from a fuzzy negation N and a t-conorm S (S,N)-implications).

2.5.2 R-implications

R-implications are the short for residual implications. An R-implication is generated from a conjunction on the unit interval (usually, a second place left-continuous mapping), getting idea from the equality in classical set theory:

$$A^c \cup B = (A - B)^c = \cup\{Z | A \cap Z \subseteq B\}.$$

where c denotes a set-complement operator and $-$ denotes a set-difference operator. An R-implication can be generated from a conjunction $\mathcal{C}$ on the unit interval:

Definition 2.41. Let $\mathcal{C}$ be a conjunction on the unit interval. An R-implication is defined as

$$I(x, y) = \sup\{t \in [0, 1] | \mathcal{C}(x, t) \leq y\}, \quad \forall x, y \in [0, 1]. \tag{2.52}$$

In Definition 2.41, if $\mathcal{C}$ is second place left-continuous, then (2.52) can be rewritten as

$$I(x, y) = \max\{t \in [0, 1] | \mathcal{C}(x, t) \leq y\}, \quad \forall x, y \in [0, 1]. \tag{2.53}$$

R-implications generated from t-norms play an important role in the literature. An R-implication generated from a t-norm T is represented by [29]

$$I(x, y) = \sup\{t \in [0, 1] | T(x, t) \leq y\}, \quad \forall x, y \in [0, 1]. \tag{2.54}$$

If T is a left-continuous t-norm, then (2.54) can be rewritten as

$$I(x, y) = \max\{t \in [0, 1] | T(x, t) \leq y\}, \quad \forall x, y \in [0, 1]. \tag{2.55}$$

2.5.3 QL-implications

QL-implications are the short for quantum logic implications. A QL-implication is generated from a strong fuzzy negation, a t-conorm and a t-norm, getting idea from the equivalency in classical binary logic:

$$p \to q \Leftrightarrow (\neg p \vee (p \wedge q)). \tag{2.56}$$

Definition 2.42. Let S be a t-conorm, N be a strong fuzzy negation and T be a t-norm. A QL-implication is defined by:

$$I(x, y) = S(N(x), T(x, y)), \quad \forall x, y \in [0, 1]. \tag{2.57}$$

Table 2.6 lists the popular fuzzy implications in the literature, which belong to the classes S-, R- or QL-implications. For the intersections of S-, R- and QL-implications, we refer to [6].

2.6 Other Methods to Generate Fuzzy Implications

There are several approaches to define a fuzzy implication besides to define it from the other fuzzy logic operators.

2.6.1 Fuzzy Implications Generated from Additive Generating Functions

Yager [97] introduced a class of fuzzy implications generated from additive generating functions, and analysed their roles in approximate reasoning. They are fuzzy implications generated from f-generators .

Definition 2.43. A *generator* f is a continuous $[0,1] \rightharpoonup [0,+\infty[$ mapping which is strictly decreasing and $f(1)=0$. Moreover the pseudo-inverse of f, $f^{(-1)}$ is defined as

$$f^{(-1)}(x) = \begin{cases} f^{-1}(x), & \text{if } x \leq f(0) \\ 0, & \text{otherwise} \end{cases}. \tag{2.58}$$

Definition 2.44. A f*-generated* implication I_f is defined as

$$(\forall (x,y) \in [0,1]^2)(I_f(x,y) = f^{(-1)}(xf(y))). \tag{2.59}$$

Remark 2.45. Observe that if the generator f is defined as $f(x) = -\log x$, then the f-generated implication is the widely-known Yager implication I_Y (see in [94]):

$$(\forall (x,y) \in [0,1]^2)(I_Y(x,y) = y^x). \tag{2.60}$$

2.6.2 Parameterized Fuzzy Implications Used in Fuzzy Morphology

Mathematical morphology is an important theory developed in image processing to analyze the geometric features of n-dimensional images. These images can be binary images which are represented as subsets of $\mathbb{R}^n$ or gray-scale images which are represented as $\mathbb{R}^n \rightharpoonup [0,1]$ mappings [62]. Morphological operations are the basic tools in mathematical morphology. They transform an image A by using another image B which is called the structuring element. Fuzzy morphological operations can be constructed from conjunctions on the unit interval and fuzzy implications. Besides the aforementioned fuzzy implications, we have still the following definitions.

Definition 2.46. The *Zadeh implication* [102] is defined as

$$(\forall (x,y) \in [0,1]^2)(I_Z(x,y) = \begin{cases} 1, & \text{if } x \leq y \\ 0, & \text{if } x > y \end{cases}). \tag{2.61}$$

Other two are parameterized fuzzy implications:

Definition 2.47. ([39]) A *generalized Łukasiewicz implication* is defined as:

$$(\forall (x,y) \in [0,1]^2)(I_\lambda(x,y) = \min(\lambda(x) + \lambda(1-y), 1)), \tag{2.62}$$

where λ is a $[0,1] \rightharpoonup [0,1]$ mapping that is decreasing with $\lambda(0)=1$ and $\lambda(1)=0$ and satisfies the following conditions:

$\Lambda 1.$ $\lambda(x) = 0$ has a unique solution,

$\Lambda 2.$ $(\forall \alpha \in [0.5, 1])(\lambda(x) = \alpha$ has a unique solution),

$\Lambda 3.$ $(\forall x \in [0, 1])(\lambda(x) + \lambda(1 - x) \geq 1)$.

Example 2.5 Two examples of λ-mappings with a parameter n are given as follows:

(i)
$$(\forall x \in [0, 1])(\lambda_n(x) = 1 - x^n), \quad n \geq 1. \tag{2.63}$$

Notice that if $n = 1$ in this case, then $\lambda_n(x) = 1 - x$, which is the standard strong fuzzy negation. Moreover, the corresponding implication via (2.62) is the famous Lukasiewicz implication $I_{\mathbf{L}}$.

(ii)
$$(\forall x \in [0, 1])(\lambda_n(x) = \frac{1 - x}{1 + nx}), \quad n \in \ \ ] - 1, 0]. \tag{2.64}$$

Definition 2.48. ([41]) A *Kitainik's implication* is defined as:

$$(\forall (x, y) \in [0, 1]^2)(I_\varphi(x, y) = \varphi(\max(x, 1 - y), \min(x, 1 - y))), \tag{2.65}$$

where φ is a $T_\tau \rightharpoonup [0, 1]$ mapping that is decreasing with $\varphi(0, 0) = \varphi(1, 0 = 1)$ and $\varphi(1, 1) = 0$. T_τ is the triangle defined as:

$$T_\tau = \{(x, y) | (x, y) \in [0, 1]^2 \quad \text{and} \quad \text{x} \geq \text{y}\}. \tag{2.66}$$

2.7 Summary

In this chapter we gave preliminaries of fuzzy negations, fuzzy conjunctions, fuzzy disjunctions, and fuzzy implications generated from these fuzzy logic operators as well as from some generated functions. We listed in Table 2.6 for the most famous fuzzy implications in the literature if they are generated from the other fuzzy logic operators, and which class they belong to.

Table 2.6: Popular fuzzy implications and their classes

Name and symbol	$I(x,y) =$	S-	R-	QL-
Kleene-Dienes I_b	$\max(1-x,y)$	$S = S_{\mathbf{M}}$ $N = N_0$	–	$S = S_{\mathbf{L}}$ $N = N_0$ $T = T_{\mathbf{L}}$
Reichenbach I_r	$1-x+xy$	$S = S_{\mathbf{P}}$ $N = N_0$	–	$S = S_{\mathbf{L}}$ $N = N_0$ $T = T_{\mathbf{P}}$
Most Strict I_M	$\begin{cases} 1, & x = 0 \\ y, & \text{otherwise} \end{cases}$	$S = S_{\mathbf{M}}$ $N = N_{1b}$	–	–
Largest I_{LS}	$\begin{cases} y, & x = 1 \\ 1-x, & y = 0 \\ 1, & \text{otherwise} \end{cases}$	$S = S_{\mathbf{D}}$ $N = N_0$	–	–
Least Strict I_{LR}	$\begin{cases} y, & x = 1 \\ 1, & \text{otherwise} \end{cases}$	$S = S_{\mathbf{M}}$ $N = N_{1a}$	$T = T_{\mathbf{D}}$	$S = S_{\mathbf{M}}$ $N = N_{1a}$ $T = T_{\mathbf{M}}$
Łukasiewicz $I_{\mathbf{L}}$	$\min(1-x+y,1)$	$S = S_{\mathbf{L}}$ $N = N_0$	$T = T_{\mathbf{L}}$	$S = S_{\mathbf{L}}$ $N = N_0$ $T = T_{\mathbf{M}}$
R_0 $(I_{(\min_0)})_N$	$\begin{cases} 1, & x \leq y \\ \max(N(x),y), & \text{otherwise} \end{cases}$	$S = S_{\max_0}$ N any strong fuzzy negation	$T = (T_{\min_0})_N$	–
Gödel I_g	$\begin{cases} 1, & x \leq y \\ y, & \text{otherwise} \end{cases}$	–	$T = T_{\mathbf{M}}$	–
Goguen I_Δ	$\begin{cases} 1, & x \leq y \\ y/x, & \text{otherwise} \end{cases}$	–	$T = T_{\mathbf{P}}$	
Early Zadeh I_m	$\max(1-x,\min(x,y))$	–	–	$S = S_{\mathbf{M}}$ $N = N_0$ $T = T_{\mathbf{M}}$
Klir and Yuan 1 I_p	$1-x+x^2y$	–	–	$S = S_{\mathbf{P}}$ $N = N_0$ $T = T_{\mathbf{P}}$
Klir and Yuan 2 I_q	$\begin{cases} y, & x = 1 \\ 1-x, & x \neq 1, y \neq 1 \\ 1, & x \neq 1, y = 1 \end{cases}$	–	–	$S = S_{\mathbf{D}}$ $N = N_0$ $T = T_{\mathbf{D}}$

Chapter 3

Fuzzy Implication Axioms

3.1 Introduction

A fuzzy implication I is an extension of the implication operator in the classical binary logic. So I must satisfy at least the boundary conditions

I1. $I(0,0) = I(0,1) = I(1,1) = 1$ and $I(1,0) = 0$.

Besides I1, there are several other potential axioms for I to satisfy in different theories and applications, among which the most important ones are (notice that FI5 is a part of I1):

FI1. the first place antitonicity (FA):
$(\forall(x_1, x_2, y) \in [0,1]^3)(x_1 < x_2 \Rightarrow I(x_1, y) \geq I(x_2, y))$;

FI2. the second place isotonicity (SI):
$(\forall(x, y_1, y_2) \in [0,1]^3)(y_1 < y_2 \Rightarrow I(x, y_1) \leq I(x, y_2))$;

FI3. dominance of falsity of antecedent (DF): $(\forall x \in [0,1])(I(0,x) = 1)$;

FI4. dominance of truth of consequent (DT): $(\forall x \in [0,1])(I(x,1) = 1)$;

FI5. boundary condition (BC): $I(1,0) = 0$;

FI6. neutrality of truth (NT): $(\forall x \in [0,1])(I(1,x) = x)$;

FI7. exchange principle (EP): $(\forall(x,y,z) \in [0,1]^3)(I(x, I(y,z)) = I(y, I(x,z)))$;

FI8. ordering principle (OP): $(\forall(x,y) \in [0,1]^2)(I(x,y) = 1 \Leftrightarrow x \leq y)$;

FI9. the mapping $N^{'}$ defined as $(\forall x \in [0,1])(N^{'}(x) = I(x,0))$, is a strong fuzzy negation (SN);

FI10. consequent boundary (CB): $(\forall(x,y) \in [0,1]^2)(I(x,y) \geq y)$;

FI11. identity (ID): $(\forall x \in [0,1])(I(x,x) = 1)$;

FI12. contrapositive principle (CP): $(\forall(x,y) \in [0,1]^2)(I(x,y) = I(N(y), N(x)))$, where N is a strong fuzzy negation;

FI13. continuity (CO): I is a continuous mapping.

FI1 and FI2 mean that the decreasing of the antecedent and/or the increasing of the consequent cause the non-decreasing of I. FI3 means that falsity implies everything. FI4 means that anything implies tautology. FI6 means that if the antecedent is a tautology, then the value of the implication is equal to the consequent. FI7 comes from the proposition in the binary logic:

$$\text{if } P_1 \text{ then (if } P_2 \text{ then } P_3) \Leftrightarrow \text{if } P_2 \text{ then (if } P_1 \text{ then } P_3).$$

FI8 means that I determines an ordering. FI9 means that if the consequent is false, then the implication is equal to the complement of the antecedent. FI10 comes from the proposition in the binary logic:

$$P \to (Q \to P).$$

FI11 comes from that $P \to P$ is always true. FI12 reperesents a relationship betweem modus ponens and modus tollens. F13 means that a small change of the antecedent or the consequent will not cause a chaotic change in the implication.
In this chapter we investigate for the three classes of fuzzy implications generated from fuzzy logical operators, namely the S-, R- and QL-implications, how many of the 13 axioms they satisfy, in general, or under some extra requirements. S- and R-implications are widely studied while QL-implications not. So we focus especially on QL-implications satisfying these 13 axioms.

3.2 Axioms of S-implications

First we observe the axioms of S-implications generated from t-conorms and any fuzzy negations.

Proposition 3.1. *([6], Proposition 2.4.3) An S-implication I generated from a t-conorm S and a fuzzy negation N satisfies FI1-FI7 and FI10.*

Remark 3.2. If the fuzzy negation N is not strong, then the converse of Proposition 3.1 is not true. To see this we first suppose that I is a $[0,1]^2 \rightharpoonup [0,1]$ mapping that satisfies FI1-FI7 and FI10 but not FI12, and there exists a proper t-conorm S and a non-strong fuzzy negation N such that for all x, $y \in [0,1]$, $I(x,y) = S(N(x),y)$. Then $S(y,N(x)) = I(N(y),N(x))$. Because I does not satisfy FI12, there exist $x_1, y_1 \in [0,1]$ such that $I(N(y_1),N(x_1)) \neq I(x_1,y_1)$. So

$$S(y_1,N(x_1)) = I(N(y_1),N(x_1)) \neq I(x_1,y_1) = S(N(x_1),y_1).$$

Thus S does not satisfy commutativity (S3), i.e., S is not a t-conorm, which is a contradiction.

For an S-implication generated from a t-conorm and a strong fuzzy negation, we have the following characterization:

Theorem 3.3. *A* $[0,1]^2 \rightharpoonup [0,1]$ *mapping* I *is an S-implication generated from a t-conorm* S *and a strong fuzzy negation* N *iff it satisfies FI1-FI7, FI9, FI10 and FI12.*

PROOF. $\Longrightarrow$: According to ([24], Theorem 1.13) and ([5], Theorem 1.6), if I is an S-implication generated from a t-conorm S and a strong fuzzy negation N, then I satisfies FI1-FI7 and FI12. Moreover, we have for all $x \in [0,1]$,

$$N^{'}(x) = I(x,0) = S(N(x),0) = N(x) \quad \text{by} \quad \text{S1}.$$

So I satisfies FI9. According to Proposition 2.24, for all $x \in [0,1]$ we have:

$$I(x,y) = S(N(x),y) \geq y.$$

So I satisfies FI10.
$\Longleftarrow$: Straightforward from the 'if' part of ([24], Theorem 1.13) and ([5], Theorem 1.6). □

The '$\Longleftarrow$' part of Theorem 3.3 shows that if a $[0,1]^2 \rightharpoonup [0,1]$ mapping satisfies FI1-FI7, FI9, FI10 and FI12, then there exists a proper t-conorm S and a proper strong fuzzy negation N such that I is the S-implication generated from S and N.
Besides FI1-FI7 and FI10, S-implications generated from t-conorms and any fuzzy negation satisfy the other axioms under certain additional conditions.

Theorem 3.4. *Let* I *be an S-implication generated from a t-conorm* S *and a fuzzy negation* N*. Then the following three conditions are equivalent:*

(i). I *satisfies FI9;*
(ii). N *is a strong fuzzy negation;*
(iii). I *satisfies FI12.*

PROOF. (i) $\Longrightarrow$ (ii): Straightforward.
(ii) $\Longrightarrow$ (iii): Because N is a strong fuzzy negation, we have for all $x, y \in [0,1]$:

$$\begin{aligned} I(N(y),N(x)) = S(N(N(y)),N(x)) &= S(N(x),y) \quad \text{by S3} \\ &= I(x,y). \end{aligned}$$

So I satisfies FI12.
(iii) $\Longrightarrow$ (i): We have for all $x, y \in [0,1]$,

$$\begin{aligned} I(x,0) &= I(N(0),N(x)) \quad \text{w.r.t. the strong fuzzy negation } N \\ &= I(1,N(x)) = N(x) \quad \text{by FI6.} \end{aligned}$$

So I satisfies FI9. □

Theorem 3.5. *([6] An S-implication generated by a t-conorm* S *and a fuzzy negation* N *satisfies FI8 iff* $N = N_S$ *is a strong fuzzy negation and the pair* (S, N_S) *satisfies LEM.*

Theorem 3.6. *Let I be an S-implication generated by a continuous t-conorm S and a fuzzy negation N. Then the following three conditions are equivalent:*

(i) I satisfies FI8,

(ii) $S = (S_{\mathbf{L}})_\varphi$ and $N = (N_0)_\varphi$,

(iii) $I = (I_{\mathbf{L}})_\varphi$.

Remark 3.7. If an S-implication I satisfies the conditions (i) and (ii) stated in Theorem 3.6, then it is also an R-implication generated from the t-norm $(T_{\mathbf{L}})_\varphi$ and a QL-implication generated from the t-conorm $(S_{\mathbf{L}})_\varphi$, the fuzzy negation N_φ and the t-norm $T_{\mathbf{M}}$. Namely I is an fuzzy implication that is conjugated to the Łukasiewicz implication $I_{\mathbf{L}}$.

An S-implication I generated by a t-conorm S and a fuzzy negation N satisfies FI11 iff S and N satisfy the law of excluded middle . For further results, see Theorems 2.14 and 2.16 in Chapter 2.

Theorem 3.8. *An S-implication I generated by a t-conorm S and a fuzzy negation N is continuous iff both S and N are continuous.*

PROOF. $\Longleftarrow$: Straightforward.

$\Longrightarrow$: If S is not continuous, then it is straightforward that I is also not continuous. Suppose N is not continuous. We have $I(x,0) = S(N(x),0) = N(x)$. Then the $[0,1] \rightharpoonup [0,1]$ mapping $F(x) = I(x,0)$, $\forall x \in [0,1]$ is not continuous. So if I is continuous, then S and N must be continuous. □

3.3 Axioms of R-implications Generated by T-norms

First we observe the axioms of R-implications generated from any t-norm.

Proposition 3.9. *([6], Theorem 2.5.4) An R-implication generated from a t-norm T satisfies FI1-FI6, FI10 and FI11.*

Remark 3.10. The converse of Proposition 3.9 is not true. To see this we assume T to be a left-continuous t-norm, and I is an R-implication generated from T satisfying FI1-FI6, FI10 and FI11 but not FI8. Then according to Equation (2.55), we have

$$(\forall (x,y) \in [0,1]^2)(T(x,y) = \min\{t | I(x,t) \geq y\}). \tag{3.1}$$

Because I satisfies FI2 and FI11 but not FI8, there exists $x_0, y_0 \in [0,1]$ such that $x_0 > y_0$ and $I(x_0, y_0) = 1$. Then we have

$$T(x_0, 1) = \min\{t | I(x_0, t) = 1\} < x_0,$$

which means T does not satisfy T1. Thus T is not a t-norm, which is a contradiction.

For an R-implication generated from a left-continuous t-norm, we have the following characterization:

Theorem 3.11. *([24], Theorem 1.14) A* $[0,1]^2 \rightharpoonup [0,1]$ *mapping is an R-implication generated by a left-continuous t-norm iff it satisfies FI1-FI8, FI10-FI11, and it is right-continuous w.r.t. the second variable.*

We see from the next corollary that a t-norm can also be generated by an R-implication.

Corollary 3.12. *([2], Corollary 10) A mapping* $T : [0,1]^2 \rightharpoonup [0,1]$ *is a left-continuous t-norm iff* T *can be represented by*

$$T(x,y) = \min\{t \in [0,1] | I(x,t) \geq y\} \tag{3.2}$$

for some mapping $I : [0,1]^2 \rightharpoonup [0,1]$ *which satisfies FI2, FI7, FI8 and right-continuity in the second argument.*

Besides FI1-FI8 and FI10-FI11, R-implications generated from t-norms also satisfy the other axioms under certain additional conditions.

Theorem 3.13. *([23], Corollary 2) An R-implication* I *generated from a continuous t-norm* T *satisfies FI12 w.r.t. a strong fuzzy negation* N *iff there exists an automorphism* φ *of the unit interval such that* $T = (T_{\boldsymbol{L}})_\varphi$ *and* $N = (N_0)_\varphi$*. In this case* $I = (I_{\boldsymbol{L}})_\varphi$*.*

Proposition 3.14. *([23], Corollary 1) If an R-implication* I *generated from a left-continuous t-norm satisfies FI12, then* I *satisfies FI9.*

Proposition 3.15. *If an R-implication generated by a t-norm* T *is continuous, then* T *is not necessary to be continuous.*

3.4 Axioms of QL-implications

QL-implications satisfy less axioms than S-implications and R-implications generated from t-norms. First we see which axioms a QL-implication satisfies.

Proposition 3.16. *([6], Proposition 2.6.2) A QL-implication* I *generated from a t-conorm* S*, a strong fuzzy negation* N *and a t-norm* T *satisfies FI2, FI3, FI5, FI6 and FI9.*

Remark 3.17. The converse of Proposition 3.16 is not true. The Most Strict implication I_M given in Tabel 2.6 also satisfies FI2, FI3, FI5, FI6 and FI9 (because it is an S-implication). We prove that I_M is not a QL-implication:
Assume I_M to be a QL-implication. Then there exist a t-conorm S, a strong fuzzy negation N and a t-norm T such that

$$I_M(x,y) = S(N(x), T(x,y)) = S_M(N_{1b}(x), y).$$

Take $y = 0$, we have $N(x) = N_{1b}(x)$. So

$$I_M(x,y) = S(N_{1b}(x), T(x,y))$$

$$= \begin{cases} 1, & \text{if } x = 0 \\ T(x,y), & \text{otherwise} \end{cases}$$
$$= \begin{cases} 1, & x = 0 \\ y, & \text{otherwise.} \end{cases}$$

So we obtain $T(x,y) = y$, for all $x > 0$. If we take $x_0 = 0.1$ and $y_0 = 0.2$, then $T(x_0, y_0) = 0.2$ while $T(y_0, x_0) = 0.1$, which means that T does not satisfy T3. So T is not a t-norm, which is a contradiction with the assumption. Thus I_M is not a QL-implication. Hence a $[0,1]^2 \rightharpoonup [0,1]$ mapping satisfying FI2, FI3, FI5, FI6 and FI9 is not always a QL-implication.

In the rest of this section we investigate under which conditions a QL-implication satisfies the other axioms.
Both S-implicatons and R-implications generated from t-norms satisfy FI1-FI5 which are used to define a fuzzy implication in some papers in the literature [6], [10], [11], [24], [36], [97]. However, not every QL-implication satisfies FI1 or FI4. We first study the QL-implications satisfying the other axioms as well as the relationship between their satisfying them and FI1. We then obtain the conditions under which a QL-implication satisfies FI1.

3.4.1 Interrelationship between QL-implications Satisfying the Axioms

Proposition 3.18. *Let S be a t-conorm, T be a t-norm and N be a strong fuzzy negation. The QL-implication $I(x,y) = S(N(x), T(x,y))$ satisfies FI4 iff S and N satisfy the LEM (2.45).*

PROOF. By straightforward verification. □

Proposition 3.19. *Let S be a t-conorm, T be a t-norm and N be a strong fuzzy negation. Then the LEM (2.45) is a necessary condition for the QL-implication $I(x,y) = S(N(x), T(x,y))$ to satisfy FI1, FI7, FI8 or FI12.*

PROOF. We have $I(x,1) = S(N(x), x)$. Moreover, according to Proposition 3.16, I satisfies FI3, i.e., $I(0,y) = 1$, for all $y \in [0,1]$.
If I satisfies FI1, then for all $x \in [0,1]$, $I(x,1) \geq I(1,1) = 1$, which means $S(N(x), x) = 1$.
If I satisfies FI7, then for all $x \in [0,1]$, $I(x,1) = I(x, I(0,1)) = I(0, I(x,1)) = 1$, which means $S(N(x), x) = 1$.
If I satisfies FI8, then for all $x \in [0,1]$, $x \leq 1 \Leftrightarrow I(x,1) = 1$, which means $S(N(x), x) = 1$.
If I satisfies FI12, then for all $x \in [0,1]$, $I(x,1) = I(N(1), N(x)) = I(0, N(x)) = 1$, which means $S(N(x), x) = 1$. □

Example 3.1 Condition (2.45) is not sufficient for a QL-implication to satisfy FI1, FI7, FI8 or FI12 as can be seen from the following examples:

(i) Set $\varphi(x) = x^2$, for all $x \in [0,1]$. Consider the t-norm

$$T(x,y) = (T_{\mathbf{L}})_\varphi(x,y) = \sqrt{\max(x^2 + y^2 - 1, 0)}$$

and the QL-implication

$$\begin{aligned} I(x,y) &= S_{\mathbf{L}}(N_0(x), T(x,y)) \\ &= \min(1 - x + \sqrt{\max(x^2 + y^2 - 1, 0)}, 1). \end{aligned}$$

$S_{\mathbf{L}}$ and N_0 satisfy condition (2.45). However, for $x_1 = 0.7$, $x_2 = 0.8$ and $y = 0.8$, we have $I(x_1, y) \approx 0.661$ while $I(x_2, y) \approx 0.729$, which is against FI1.

(ii) Consider the strong fuzzy negation $N(x) = \sqrt{1 - x^2}$ and the QL-implication

$$I(x,y) = S_{\mathbf{L}}(N(x), T_{\mathbf{P}}(x,y)) = \min(\sqrt{1 - x^2} + xy, 1). \tag{3.3}$$

Observe that $S_{\mathbf{L}}$ and N satisfy condition (2.45). Take $x = 0.6$, $y = \sqrt{0.99}$ and $z = 0.1$. Then $I(x, I(y,z)) \approx 0.9197$ while $I(y, I(x,z)) \approx 0.9557$, which is against FI7.

(iii) Let the QL-implication I be stated in (3.3). Take $x = 0.8$ and $y = 0.5$. Then $I(x,y) = 1$ while $x > y$, which is against FI8.

(iv) Again, consider the QL-implication I stated in (3.3). Notice that according to ([82], Lemma 1), if a QL-implication generated by a t-conorm S, a t-norm T and a strong fuzzy negation N satisfies FI12 w.r.t. a strong fuzzy negation N', then $N' = N$. Take $x = 0.8$ and $y = 0.49$. Then $I(x,y) = 0.992$ while $I(N(y), N(x)) = 1$, which is against FI12.

Remark 3.20. According to Propositions 3.18 and 3.19, once a QL-implication satisfies FI1, FI7, FI8 or FI12, it also satisfies FI4. From Example 3.1 we know that the converse is not true.

For the case that the t-conorm S is continuous, Propositions 3.18 and 3.19 can be further refined by the next proposition, according to ([10], Theorem 1) and ([51], Section 3.1).

Proposition 3.21. *Let S be a continuous t-conorm, T be a t-norm and N be a strong fuzzy negation. A sufficient and necessary condition for the QL-implication $I(x,y) = S(N(x), T(x,y))$ to satisfy FI4 (and therefore a necessary condition for I to satisfy FI1, FI7, FI8 or FI12) is that there exists an automorphism φ of the unit interval such that*

$$S = (S_{\mathbf{L}})_\varphi \quad \textit{and} \quad (\forall x \in [0,1])(N(x) \geq (N_0)_\varphi(x)). \tag{3.4}$$

Proposition 3.22. *([51], Remark 2) If a QL-implication satisfies FI7, then it also satisfies FI12.*

Proposition 3.23. *([10], Lemma 1 (ii)) If a $[0,1]^2 \rightharpoonup [0,1]$ mapping satisfies FI2 and FI12, then it also satisfies FI1.*

Theorem 3.24. *A QL-implication I satisfies FI7 iff it is also an S-implication.*

PROOF. $\Longleftarrow$: Directly from Theorem 3.3.
$\Longrightarrow$: According to Propositions 3.22 and 3.23, I satisfies FI12 and FI1. Thus according to Remark 3.20, I satisfies FI4. Moreover according to Proposition 3.16, I also satisfies FI2, FI3, I5 and I6. Thus according to Theorem 3.3, it is also an S-implication. □

Next theorems and corollaries treat the case that the t-conorm S which constructs the QL-implication I is continuous and I satisfies FI1, FI7 or FI12, i.e., according to Proposition 3.21, there exists an automorphism φ of the unit interval such that condition (3.4) is fulfilled. Thus the QL-implication is expressed by the formula

$$(\forall(x,y) \in [0,1]^2)(I(x,y) = (S_{\mathbf{L}})_\varphi((N_0)_\varphi(x), T(x,y))), \tag{3.5}$$

where T is a t-norm.
Observe that $I(x,y)$ stated in (3.5) is equal to $\varphi^{-1}(1-\varphi(x)+\varphi(T(x,y)))$, which means that I depends only on T and φ.
The authors of [51] have worked out the condition of a t-norm T under which the QL-implication stated in (3.5) satisfies FI1.

Theorem 3.25. *([51], Proposition 9) The QL-implication stated in (3.5) satisfies FI1 iff $T_{\varphi^{-1}}$ satisfies the 1-Lipschitz property (2.31).*

According to Theorem 2.29, we immediately have the next corollary.

Corollary 3.26. *The QL-implication stated in (3.5) satisfies FI1 iff $T_{\varphi^{-1}}$ is a copula.*

Theorem 3.27. *([51], Corollary 1) The QL-implication stated in (3.5) with $T_{\varphi^{-1}}$ satisfying the 1-Lipschitz property (2.31) satisfies FI7 iff $T_{\varphi^{-1}}$ is a Frank t-norm.*

Corollary 3.28. *The QL-implication stated in (3.5) satisfies FI7 iff $T_{\varphi^{-1}}$ is a Frank t-norm.*

PROOF. $\Longleftarrow$: If $T_{\varphi^{-1}}$ is a Frank t-norm, then according to Proposition 2.20, $T_{\varphi^{-1}}$ satisfies the 1-Lipschitz property (2.31). Thus according to Theorem 3.27, I satisfies FI7.
$\Longrightarrow$: If I satisfies FI7, then according to Remark 3.20, it also satisfies FI1. Thus according to Theorem 3.25, $T_{\varphi^{-1}}$ satisfies the 1-Lipschitz property (2.31). Hence according to Theorem 3.27, $T_{\varphi^{-1}}$ is a Frank t-norm. □

Theorem 3.29. *([51], Proposition 11) The QL-implication stated in (3.5) satisfies FI12 iff $T_{\varphi^{-1}}$ and its dual t-conorm satisfy the Frank equation (2.38).*

Corollary 3.30. *The QL-implication stated in (3.5) satisfies FI12 iff $T_{\varphi^{-1}}$ is a Frank t-norm or an ordinal sum defined in (2.30) where T_m, $m \in M$, is a Frank t-norm different from $T_{\mathbf{M}}$.*

PROOF. Straightforward from Theorems 3.29 and 2.29. □

Remark 3.31. Comparing Corollary 3.30 with Corollary 3.28, we can see that there exist QL-implications that satisfy FI12 but not FI7. For example, let

$$T_1(x,y) = \begin{cases} a_1 + (b_1 - a_1)T_P(\frac{x-a_1}{b_1-a_1}, \frac{y-a_1}{b_1-a_1}), & \text{if } (x,y) \in [a_1,b_1]^2 \\ T_{\mathbf{M}}(x,y), & \text{otherwise} \end{cases}$$

where $a_1 = \frac{1}{2}$ and $b_1 = 1$. Then T_1 is an ordinal sum of the Frank t-norm T_P, but T_1 itself is not a Frank t-norm because it is not strict and different from $T_{\mathbf{M}}$ and $T_{\mathbf{L}}$. Thus the QL-implication defined by $S_{\mathbf{L}}(N_0(x), T_1(x,y))$ satisfies FI12 but not FI7.

Remark 3.32. Because I always satisfies FI2, according to Proposition 3.23, if a QL-implication I satisfies FI12, then it also satisfies FI1. But a QL-implication satisfying FI1 does not necessarily satisfy FI12. We will give a counterexample in Remark 3.45. Moreover, according to Proposition 3.22, if I satisfies FI7, then it also satisfies FI1. According to Remark 3.31, there exist QL-implications that satisfy FI1 but not FI7.

Now we consider the condition under which a QL-implication satisfies FI8. From Proposition 3.22 and Remark 3.32 we see that if a QL-implication satisfies FI7, then it immediately satisfies FI12 and FI1. Hereafter we give an example to show that there exists a QL-implication which satisfies FI7 (and hence also FI12) but not FI8.

Example 3.2 Let I be the QL-implication generated by the t-conorm $S_{\mathbf{L}}$, the t-norm $T_{\mathbf{P}}$ and the strong fuzzy negation N_0. Then $I(x,y) = 1 - x + xy$. I is the S-implication $S_{\mathbf{P}}(N_0(x), y)$ which is called the Reichenbach implication ([24], Table 1.1). Thus I satisfies FI7, according to Theorem 3.3. But $1 - x + xy = 1 \Leftrightarrow x = 0$ or $y = 1$, which means I does not satisfy FI8.

Remark 3.33. One question is still open here: does a QL-implication satisfying FI1, FI8 and FI12 also satisfies FI7?

Theorem 3.34 gives a sufficient and necessary condition for the QL-implication generated by the t-conorm $(S_{\mathbf{L}})_\varphi$, a continuous t-norm and a strong fuzzy negation to satisfy FI8.

Theorem 3.34. *Let φ be an automorphism of the unit interval, T be a continuous t-norm and N be a strong fuzzy negation. The QL-implication*

$$(\forall (x,y) \in [0,1]^2)(I(x,y) = (S_{\mathbf{L}})_\varphi(N(x), T(x,y))) \tag{3.6}$$

satisfies FI8 iff

$$(\forall x \in [0,1])(T(x,x) = (N_0)_\varphi(N(x))) \tag{3.7}$$

and

$$(\forall x \in]0,1])(\forall y \in [0,x[)(T(x,y) < T(x,x)). \tag{3.8}$$

PROOF. $\Longrightarrow$: If I satisfies FI8, then for a fixed $x \in\,]0,1]$,

$$y \in [0, x[\quad \Rightarrow I(x,y) < 1 \Leftrightarrow \varphi(N(x)) + \varphi(T(x,y)) < 1,$$

and

$$y \in [x, 1] \Rightarrow I(x,y) \geq 1 \Leftrightarrow \varphi(N(x)) + \varphi(T(x,y)) \geq 1.$$

Since $y \mapsto \varphi(N(x)) + \varphi(T(x,y))$ is continuous, taking $y = x$, we get $\varphi(N(x)) + \varphi(T(x,x)) = 1$. Moreover, because $\varphi(N(0)) + \varphi(T(0,0)) = 1$, we get

$$\varphi(N(x)) + \varphi(T(x,x)) = 1 \Leftrightarrow T(x,x) = (N_0)_\varphi(N(x)),$$

for all $x \in [0,1]$. And for all $x \in\,]0,1]$ and $y \in [0, x[$, $T(x,y) < (N_0)_\varphi(N(x))$ (because $\varphi(N(x)) + \varphi(T(x,y)) < 1$, i.e., $T(x,y) < T(x,x)$).
$\Longleftarrow$: If T satisfies (3.7), then $(N_0)_\varphi(N(x)) \leq T(x,y)$, for all $y \in [x,1]$, which means $I(x,y) = 1$ when $x \leq y$. Moreover, if T satisfies (3.8), then $(N_0)_\varphi(N(x)) > T(x,y)$, for all $x \in\,]0,1]$ and $y \in [0,x[$, which means $I(x,y) < 1$ when $x > y$. Hence I satisfies FI8. □

Corollary 3.35. *If the QL-implication stated in (3.6) satisfies FI8, then T can neither be nilpotent nor an ordinal sum defined in (2.30) where there exists a nilpotent T_m, $m \in M$.*

PROOF. According to Theorem 3.34, if I satisfies FI8, then T must satisfy condition (3.8). However, if T is nilpotent, then there always exist $x_0 \in\,]0,1[$ and $y_0 \in [0, x_0[$ such that $T(x_0, y_0) = T(x_0, x_0) = 0$, which is against the condition (3.8). Moreover, if T is an ordinal sum defined in (2.30) where there exists a nilpotent T_m, $m \in M$, then there always exist $x_0 \in\,]a_m, b_m[\, \subseteq\,]0, b_m[$ and $y_0 \in [a_m, x_0[\, \subseteq [0, x_0[$ such that $T(x_0, y_0) = T(x_0, x_0) = a_m$, which is against the condition (3.8). □

According to Proposition 3.21, Theorem 3.34 and Corollary 3.35, we conclude that if the QL-implication generated by a continuous t-conorm, a continuous t-norm and a strong fuzzy negation satisfies FI8, then the t-norm can only be $T_{\mathbf{M}}$, or strict or an ordinal sum defined in (2.30) where each T_m, $m \in M$, is strict. Now we give examples of QL-implications that satisfy FI8.

Example 3.3

(i) Let φ be an automorphism of the unit interval, $T = T_{\mathbf{M}}$ and $N = (N_0)_\varphi$. Then T and N satisfy conditions (3.7) and (3.8). Thus according to Theorem 3.34, the QL-implication

$$\begin{aligned} I(x,y) &= (S_{\mathbf{L}})_\varphi((N_0)_\varphi(x), T_{\mathbf{M}}(x,y)) \\ &= \varphi^{-1}(\min(1 - \varphi(x) + \varphi(y), 1)) \end{aligned}$$

for all $(x,y) \in [0,1]^2$, satisfies FI8. Furthermore, I is an R-implication generated by the t-norm $(T_{\mathbf{L}})_\varphi$ as well as an S-implication generated by the t-conorm $(S_{\mathbf{L}})_\varphi$ and the strong fuzzy negation $(N_0)_\varphi$. Actually I is the fuzzy implication conjugated to the Łukasiewicz implication.

(ii) Let $\varphi(x) = x$, for all $x \in [0,1]$ and $\phi(x) = \begin{cases} e^{-(\frac{1-x}{x})}, & \text{if } x \in]0,1] \\ 0, & \text{if } x = 0 \end{cases}$. The strict t-norm T is defined as

$$T(x,y) = (T_{\mathbf{P}})_\phi(x,y) = \phi^{-1}(\phi(x)\phi(y)) = \begin{cases} \frac{xy}{x+y-xy}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0) \end{cases}.$$

Actually T is one of the Dombi t-norms, with the parameter $\lambda = 1$ ([43], Example 4.11). Moreover let $N(x) = \frac{2-2x}{2-x}$. Then N is a strong fuzzy negation which satisfies $N(x) \geq N_0(x)$, for all $x \in [0,1]$. Thus T and N satisfy conditions (3.7) and (3.8). Hence according to Theorem 3.34, the QL-implication

$$\begin{aligned} I(x,y) &= S_{\mathbf{L}}(N(x), T(x,y)) \\ &= \begin{cases} \min(\frac{2x+2y-2xy-2x^2+x^2y}{2x+2y-3xy-x^2+x^2y}, 1), & \text{if } (x,y) \neq (0,0) \\ 1, & \text{if } (x,y) = (0,0) \end{cases} \end{aligned}$$

for all $(x,y) \in [0,1]^2$, satisfies FI8. Furthermore, taking $x_0 = 0.4$ and $y_0 = 0.2$, we have $N(y_0) = \frac{8}{9}$ and $N(x_0) = \frac{3}{4}$. Thus $I(x_0,y_0) = \frac{47}{52}$ while $I(N(y_0), N(x_0)) = \frac{31}{35} \neq I(x_0,y_0)$. Therefore I does not satisfy FI12. According to Proposition 3.22, I does not satisfy FI7 either. Hence according to Theorems 3.3 and 3.11, I is neither an S-implication nor an R-implication generated by a left-continuous t-norm.

Remark 3.36. From Example 3.3 (ii) a QL-implication satisfying FI8 does not necessarily satisfy FI7 or FI12. In Corollary 3.58 we will investigate when QL-implications satisfying FI8 also satisfy FI1, and in particular the QL-implication presented in Example 3.3 (ii) satisfies FI1.

Proposition 3.37. *If a QL-implication I satisfies FI1, then I satisfies FI10.*

PROOF. According to Proposition 3.16, I satisfies FI6. Then because I satisfies FI1, we obtain

$$I(x,y) \geq I(1,y) = y.$$

Thus I satisfies FI10.

Example 3.4 The converse of Proposition 3.37 is not true. Let N be any strong fuzzy negation. Consider the QL-implication

$$\begin{aligned} I(x,y) &= S_{\mathbf{P}}(N(x), T_{\mathbf{L}}(x,y)) \\ &= \begin{cases} 1, & \text{if } x+y \leq 1 \\ x - xN(x) + N(x) + (1-N(x))y, & \text{if } x+y > 1 \end{cases} \end{aligned}$$

Because for all $x \in [0,1]$, $x(1-N(x)) \geq 0$, we obtain for all $y \in [0,1]$, $x - xN(x) + N(x) + (1-N(x))y \geq y$. So for all x, $y \in [0,1]$, $I(x,y) \geq y$. Thus I satisfies FI10. However, according to Theorem 2.33, S_P and N does not satisfy the LEM 2.45. So according to Proposition 3.19, I does not satisfy FI1. Thus I satisfying FI10 does not imply I satisfying FI10.

3.4.2 QL-implications and the First Place Antitonicity

Some work on whether a QL-implication satisfies FI1 or not has been done in [22], [83] and [51]. In [22], the conditions under which a QL-implication I and a t-norm T_* satisfy the residuation property:

$$(\forall x, y, z \in [0,1]) \quad (T_*(x,z) \leq y \Leftrightarrow z \leq I(x,y))$$

are found. This means that I is an R-implication as well ([22], Example 4.5). Hence I satisfies FI1 provided these conditions are fulfilled. However, being an R-implication is sufficient but not necessary for a QL-implication to satisfy FI1 (see Remarks 3.42, 3.47, 3.51 in [22]). In [83], the authors show how a QL-implication satisfies FI1 as well as the exchange principle FI7 ([83], Definition 1, Theorem 7, Theorem 11). It is proved that such a QL-implication is an S-implication as well. Again, being an S-implication is sufficient but not necessary for a QL-implication to satisfy FI1 (see Remarks 3.42, 3.47 in [83]). And in [51], the authors illustrate for a group of QL-implications the conditions under which they satisfy FI1. They restrict the relationship between the continuous t-conorm S and the strong fuzzy negation N which construct the QL-implications ([51], Proposition 14, Corollary 2).
In this section, we study the QL-implications generated by a t-conorm S, a t-norm T and a strong fuzzy negation N that satisfy FI1, especially for the cases that both S and T are continuous. We also indicate whether a QL-implication satisfying FI1 is also an S-implication or an R-implication.
First we give a lemma that will be useful in the following work.

Lemma 3.38. *Let φ be an automorphism of the unit interval. Then a QL-implication I satisfies FI1 iff $(I)_\varphi$ satisfies FI1.*

For the t-norm $T_{\mathbf{M}}$, we have the next theorem.

Theorem 3.39. *([51], Proposition 6) Let S be a t-conorm and N be a strong fuzzy negation. The QL-implication*

$$I(x,y) = S(N(x), T_{\mathbf{M}}(x,y)) = S(N(x), \min(x,y))$$

satisfies FI1 iff the condition (2.45) holds. In this case the QL-implication has the following form

$$I(x,y) = \begin{cases} 1, & \textit{if } x \leq y \\ S(N(x), y), & \textit{if } x > y \end{cases}$$

According to Proposition 3.21, the next corollary is for the special case that S is a continuous t-conorm.

Corollary 3.40. *Let S be a continuous t-conorm and N be a strong fuzzy negation. The QL-implication*

$$(\forall(x,y)\in[0,1]^2)(I(x,y)=S(N(x),T_{\mathbf{M}}(x,y)))$$

satisfies FI1 iff there exists an automorphism of the unit interval φ such that the condition (3.4) holds.

Remark 3.41. The QL-implication I defined in Theorem 3.39 with the condition (2.45) being fulfilled is also an S-implication that is expressed by $I(x,y)=S(N(x),y)$. Moreover, if there exists an automorphism φ of the unit interval such that $S=(S_{\mathbf{L}})_\varphi$ and $N=(N_0)_\varphi$, then I is also an R-implication generated by a left-continuous t-norm as well as an S-implication:

$$I(x,y)=\sup\{t\in[0,1]|(T_{\mathbf{L}})_\varphi(x,t)\leq y\}=(S_{\mathbf{L}})_\varphi((N_0)_\varphi(x),y)$$

for all $(x,y)\in[0,1]^2$. It is the fuzzy implication, which is conjugated to the Łukasiewicz implication.

In the rest of this chapter, we focus on the case that the t-conorm that constructs a QL-implication I is continuous. According to Proposition 3.21, if I satisfies FI1, then there exists an automorphism φ of the unit interval such that the condition (3.4) holds. Recall that the authors of [51] have done the work for the special case that $N=(N_0)_\varphi$, see in Theorem 3.25. In Section 4.1 we give some examples and remarks for Theorem 3.25 and Corollary 3.26.

QL-implications generated by $(S_{\mathbf{L}})_\varphi$, a t-norm T and $(N_0)_\varphi$

Next examples will give the t-norms that satisfy the 1-Lipschitz property and therefore have been identified as copulas. According to Theorem 3.25 and Corollary 3.26, the QL-implication stated in (3.5) satisfies FI1 if $T_{\varphi^{-1}}$ being one of the t-norms.

Example 3.5

(i) According to Proposition 2.20, the Frank t-norm T^s defined in (2.14) and the t-norm T_o which is an ordinal sum defined in (2.30) where each T_m, $m\in M$, is a Frank t-norm different from $T_{\mathbf{M}}$, satisfy the 1-Lipschitz property (2.31).

(ii) Some subfamilies of the parameterized families of t-norms given from equation (2.8) to equation (2.15) are copilas ([43], Example 9.13). They are:

(ii.1) Subfamily of family of Schweizer-Sklar t-norms

$$(T_s^{SS})(x,y)=\begin{cases}T_{\mathbf{M}}(x,y), & \text{if } s=-\infty\\ T_{\mathbf{P}}(x,y), & \text{if } s=0\\ (\max(x^s+y^s-1,0))^{\frac{1}{s}}, & \text{if } s\in\,]-\infty,0[\,\cup\,]0,1]\end{cases}$$

(ii.2) Subfamily of family of Hamacher t-norms

$$(T_s^H)(x,y) = \begin{cases} 0, & \text{if } s = x = y = 0 \\ \frac{xy}{s+(1-s)(x+y-xy)}, & \text{if } s \in [0,2] \text{ and } (s,x,y) \neq (0,0,0) \end{cases}$$

(ii.3) Subfamily of family of Yager t-norms

$$(T_s^Y)(x,y) = \begin{cases} T_{\mathbf{M}}(x,y), & \text{if } s = +\infty \\ \max(1 - ((1-x)^s + (1-y)^s)^{\frac{1}{s}}, 0), & \text{if } s \in [1, +\infty[\end{cases}$$

(ii.4) Subfamily of family of Dombi t-norms

$$(T_s^D)(x,y) = \begin{cases} T_{\mathbf{M}}(x,y), & \text{if } s = +\infty \\ \frac{1}{1+((\frac{1-x}{x})^s + (\frac{1-y}{y})^s)^{\frac{1}{s}}}, & \text{if } s \in [1, +\infty[\end{cases}$$

(ii.5) Subfamily of family of Sugeno-Weber t-norms

$$(T_s^{SW})(x,y) = \begin{cases} T_{\mathbf{P}}(x,y), & \text{if } s = +\infty \\ (\max(\frac{x+y-1+sxy}{1+s}, 0), & \text{if } s \in [0, +\infty[\end{cases}$$

(ii.6) Subfamily of family of Aczél-Alsina t-norms

$$(T_s^{AA})(x,y) = \begin{cases} T_{\mathbf{M}}(x,y), & \text{if } s = +\infty \\ (e^{-((-\log x)^s + (-\log y)^s)^{\frac{1}{s}}}, & \text{if } s \in [1, +\infty[\end{cases}.$$

Remark 3.42. Let I be the QL-implication stated in (3.5) with $T_{\varphi^{-1}}$ satisfying the 1-Lipschitz property (2.31). If $T_{\varphi^{-1}}$ is a Frank t-norm, then according to Theorem 3.27, I satisfies FI7 and according to Theorem 3.24 it is also an S-implication. If on the contrary $T_{\varphi^{-1}}$ is not a Frank t-norm, e.g., an ordinal sum defined in (2.30) where each T_m, $m \in M$, is a Frank t-norm different from $T_{\mathbf{M}}$, then according to Theorem 3.27, I does not satisfy FI7. Thus according to Theorem 3.3 and Theorem 3.11, I is neither an S-implication nor an R-implication generated by a left-continuous t-norm.

Remark 3.43. In the study of fuzzy quantitative association rules, a fuzzy implication I (denoted as FIO in [98]) is used to determine the degree of implication (denote as DImp in [98]) of a rule. According to [98], I should satisfy I1 (see the proof of Theorem 1 in [98]) and it should satisfy the additional constraint:

$$(\forall (x,y) \in [0,1]^2)(1 + T(x,y) - x = I(x,y)), \tag{3.9}$$

where T is a t-norm. This avoids database scanning in the process of calculating rules' Dimps.
Let T be $T_{\mathbf{M}}$ or $T_{\mathbf{P}}$ or $T_{\mathbf{L}}$ or an ordinal sum defined in (2.30) where each T_m, $m \in M$, is $T_{\mathbf{P}}$ or $T_{\mathbf{L}}$. It is easy to see that T always satisfies the 1-Lipschitz property (2.31). Thus according to Theorem 3.25, the QL-implication

$$(\forall (x,y) \in [0,1]^2)(I(x,y) = S_{\mathbf{L}}(N_0(x), T(x,y)))$$

satisfies FI1. Moreover, I satisfies (3.9). Therefore it can be a candidate for the fuzzy implication used to construct the association rules' Dimps in [98].

Besides the QL-implications generated by the t-conorm $(S_{\mathbf{L}})_\varphi$, the t-norm T and the strong fuzzy negation $(N_0)_\varphi$, which we discussed above, there exist other combinations of a continuous t-conorm S, a t-norm T and a strong fuzzy negation N to generate a QL-implication I which satisfies FI1. Next we discuss the case that both S and T are continuous, and examine which conditions N should fulfill to make I satisfy FI1. We will consider the cases where T is either strict (i.e., conjugated to $T_{\mathbf{P}}$) or nilpotent (i.e., conjugated to $T_{\mathbf{L}}$) or an ordinal sum defined in (2.30) where each T_m, $m \in M$, is strict or nilpotent.

QL-implications generated by a nilpotent t-conorm $(S_{\mathbf{L}})_\varphi$, a strict t-norm $(T_{\mathbf{P}})_\varphi$ and a strong fuzzy negation

Theorem 3.44. *Let N be a strong fuzzy negation and define a mapping f as $f(x) = \frac{1-N(x)}{x}$, for all $x \in \,]0,1]$. Then the QL-implication*

$$I(x,y) = S_{\mathbf{L}}(N(x), T_{\mathbf{P}}(x,y)) = \min(N(x) + xy, 1), \qquad (3.10)$$

for all $(x,y) \in [0,1]^2$, satisfies FI1 iff f is increasing.

PROOF. $\Longrightarrow$: Assume, that the QL-implication generated by the t-conorm $S_{\mathbf{L}}$, t-norm $T_{\mathbf{P}}$ and a strong fuzzy negation N satisfies FI1. According to Proposition 3.21, we have $N(x) \geq 1 - x$, for all $x \in [0,1]$. So $f(x) \in \,]0,1]$, for all $x \in \,]0,1]$. Let us fix arbitrarily x_1, $x_2 \in \,]0,1]$ such that $x_1 < x_2$ and take $y_0 = f(x_2)$. Then $I(x_2, y_0) = 1$. In order for I to satisfy FI1, it is necessary that

$$I(x_2, y_0) = 1 \Rightarrow I(x_1, y_0) = 1,$$

i.e., $f(x_1) \leq y_0$. Thus $f(x_1) \leq f(x_2)$. Hence f is increasing in $]0,1]$.
$\Longleftarrow$: Assume now that the mapping f defined above is increasing. Let us fix arbitrarily x_1, $x_2 \in \,]0,1]$ and $y \in [0,1]$ such that $x_1 < x_2$. We will consider several cases now.

(i) If $I(x_1, y) = 1$, then it is always greater than, or equal to $I(x_2, y)$, so I satisfies FI1 in this case.
(ii) If $I(x_2, y) = 1$, then

$$N(x_2) + x_2 y \geq 1 \Rightarrow y \geq \frac{1 - N(x_2)}{x_2}.$$

Since f is increasing, we have

$$\begin{aligned} &\frac{1 - N(x_2)}{x_2} \geq \frac{1 - N(x_1)}{x_1} \\ &\Rightarrow y \geq \frac{1 - N(x_1)}{x_1} \\ &\Rightarrow N(x_1) + x_1 y \geq 1 \end{aligned}$$

$$\Rightarrow I(x_1, y) = 1.$$

So I satisfies FI1 in this case also.

(iii) Assume now that $I(x_1, y) < 1$ and $I(x_2, y) < 1$. Then

$$N(x_1) + x_1 y < 1 \quad \text{and} \quad N(x_2) + x_2 y < 1$$
$$\Rightarrow x_1 > 0, \quad x_2 > 0, \quad y < \frac{1 - N(x_1)}{x_1} \quad \text{and} \quad y < \frac{1 - N(x_2)}{x_2}.$$

Since f is increasing, we have

$$\frac{1 - N(x_2)}{x_2} \geq \frac{1 - N(x_1)}{x_1}$$
$$\Rightarrow N(x_1)x_1 - N(x_2)x_1 \geq x_2 - x_2 N(x_1) - x_1 + N(x_1)x_1$$
$$\Rightarrow \frac{N(x_1) - N(x_2)}{x_2 - x_1} \geq \frac{1 - N(x_1)}{x_1} > y$$
$$\Rightarrow N(x_1) + x_1 y > N(x_2) + x_2 y$$
$$\Rightarrow I(x_1, y) \geq I(x_2, y).$$

Hence I satisfies FI1.

□

Example 3.6 Consider the QL-implication I stated in (3.3). Observe that I is the QL-implication stated in (3.10) where $N(x) = \sqrt{1 - x^2}$. Thus the mapping f defined in Theorem 3.44 for N is: $f(x) = \frac{1-N(x)}{x} = \frac{1-\sqrt{1-x^2}}{x}$, for all $x \in]0, 1]$. Since $\frac{df(x)}{dx} = \frac{1-\sqrt{1-x^2}}{\sqrt{1-x^2}x^2} \geq 0$, f is increasing. Hence I satisfies FI1.

Remark 3.45. According to Example 3.1 (iv), the QL-implication mentioned in Example 3.6 does not satisfy FI12. Thus a QL-implication satisfying FI1 does not necessarily satisfy FI12.

We now give a negative example, i.e., the example of a strong fuzzy negation N satisfying $N(x) \geq 1 - x$, for all $x \in [0, 1]$, but the mapping f defined in Theorem 3.44 for this N is not increasing.

Example 3.7 Let N be defined by

$$N(x) = \begin{cases} \frac{-1}{4}(x - \frac{1}{2}) + \frac{3}{4}, & \text{if } x \in [\frac{1}{6}, \frac{7}{10}] \\ -4(x - \frac{7}{10}) + \frac{7}{10}, & \text{if } x \in [\frac{7}{10}, \frac{5}{6}] \\ 1 - x, & \text{otherwise} \end{cases}$$

Because $N(N(x)) = x$, for all $x \in [0, 1]$, N is a strong fuzzy negation. Moreover, we have $N(x) \geq N_0(x)$, for all $x \in [0, 1]$. The mapping defined in Theorem 3.44 for N is $f(x) = \frac{1-N(x)}{x}$. We have $f(\frac{1}{2}) = \frac{1}{2}$ and $f(\frac{7}{10}) = \frac{3}{7}$. Thus f is not increasing. Hence the QL-implication I defined by $I(x, y) = S_{\mathbf{L}}(N(x), T_P(x, y))$ does not satisfy FI1.

Corollary 3.46. *Let φ and ϕ be two automorphisms of the unit interval. Define a mapping f as $f(x) = \frac{1-(N_0)_\gamma(x)}{x}$, for all $x \in]0,1]$ with $\gamma = \phi \circ \varphi^{-1}$. Then the QL-implication*

$$\begin{aligned} I(x,y) &= (S_{\mathbf{L}})_\varphi((N_0)_\phi(x), (T_{\mathbf{P}})_\varphi(x,y)) \\ &= \varphi^{-1}(\min(\varphi((N_0)_\phi(x)) + \varphi(x)\varphi(y), 1)) \end{aligned} \tag{3.11}$$

for all $(x,y) \in [0,1]^2$, satisfies FI1 iff f is increasing.

PROOF. Since $\gamma = \phi \circ \varphi^{-1}$,

$$I(x,y) = \varphi^{-1}(S_{\mathbf{L}}((N_0)_\gamma(\varphi(x)), T_{\mathbf{P}}(\varphi(x), \varphi(y)))).$$

According to Lemma 2.2 and Lemma 2.3, γ is also an automorphism of the unit interval. So $(N_0)_\gamma$ is a strong fuzzy negation. Thus
$I(x,y) = \varphi^{-1}(I^{'}(\varphi(x), \varphi(y)))$, where $I^{'}(x,y) = S_{\mathbf{L}}((N_0)_\gamma(x), T_{\mathbf{P}}(x,y))$.
According to Theorem 3.44, $I^{'}$ satisfies FI1 iff f is increasing. And according to Lemma 3.38, I satisfies FI1 iff $I^{'}$ satisfies FI1. Thus I satisfies FI1 iff f is increasing. □

In general the automorphisms to conjugate $S_{\mathbf{L}}$ and $T_{\mathbf{P}}$ need not be the same. This case will be studied in Theorem 3.56.

Remark 3.47. Let I be the QL-implication stated in (3.10) and f be the mapping defined in Theorem 3.44 with f being increasing. Then I is an S-implication iff $N = N_0$. Moreover I is not an R-implication generated by a left-continuous t-norm. The proof is given below:
Because $f(x) \leq f(1) = 1$, for all $x \in]0,1]$, $N(x) \geq N_0(x)$, for all $x \in]0,1]$. Since $N(0) = N_0(0)$, we have $N(x) \geq N_0(x)$, for all $x \in [0,1]$. Now we will consider two cases:

(i) If $N = N_0$, then

$$I(x,y) = S_{\mathbf{L}}(N_0(x), T_{\mathbf{P}}(x,y)) = 1 - x + xy$$

which is also an S-implication $S_{\mathbf{P}}(N_0(x), y)$, i.e., the Reichenbach implication. Reichenbach implication is not an R-implication generated by a left-continuous t-norm.

(ii) If $N \neq N_0$, then consider:

$$\begin{aligned} I(x, I(y,z)) &= \min(N(x) + x \cdot \min(N(y) + yz, 1), 1) \\ &= \begin{cases} N(x) + x(N(y) + yz), & \text{if } N(y) + yz < 1 \text{ and} \\ 1, & \text{otherwise} \end{cases} \end{aligned}$$

and

$$I(y, I(x,z)) = \min(N(y) + y \cdot \min(N(x) + xz, 1), 1)$$

$$= \begin{cases} N(y) + y(N(x) + xz), & \text{if } N(x) + xz < 1 \text{ and} \\ 1, & \text{otherwise} \end{cases}$$

Because $N \neq N_0$, there exists $x_1 \in]0,1[$ such that $\frac{1-x_1}{N(x_1)} < 1$. As we also have $\frac{1-0}{N(0)} = 1$, the continuity of $\frac{1-x}{N(x)}$ in $[0,1[$ implies that there exists $x_2 \in]0,x_1[$ such that $\frac{1-x_2}{N(x_2)} = \frac{1+\frac{1-x_1}{N(x_1)}}{2} < 1$. Thus $\frac{1-x_1}{N(x_1)} \neq \frac{1-x_2}{N(x_2)}$. Take $y_0 \in]N(\min(\frac{1-N(x_1)}{x_1}, \frac{1-N(x_2)}{x_2})), 1[$, we have

$$N(x_1) + x_1 N(y_0) < 1 \quad \text{and} \quad N(x_2) + x_2 N(y_0) < 1$$
$$\text{and} \quad \frac{1-y_0}{N(y_0)} \neq \frac{1-x_1}{N(x_1)} \quad \text{or} \quad \frac{1-y_0}{N(y_0)} \neq \frac{1-x_2}{N(x_2)}.$$

If $\frac{1-y_0}{N(y_0)} \neq \frac{1-x_1}{N(x_1)}$, then let $x_0 = x_1$. If on the contrary $\frac{1-y_0}{N(y_0)} \neq \frac{1-x_2}{N(x_2)}$, then let $x_0 = x_2$. Therefore we have

$$\frac{\frac{1-N(x_0)}{x_0} - N(y_0)}{y_0} > 0 \quad \text{and} \quad \frac{\frac{1-N(x_0)}{x_0} - N(y_0)}{y_0} \neq \frac{\frac{1-N(y_0)}{y_0} - N(x_0)}{x_0}.$$

We now consider two cases.

(ii.1) If $\frac{\frac{1-N(x_0)}{x_0} - N(y_0)}{y_0} > \frac{\frac{1-N(y_0)}{y_0} - N(x_0)}{x_0}$, then there exists z_0 such that

$$\frac{\frac{1-N(y_0)}{y_0} - N(x_0)}{x_0} < z_0 < \frac{\frac{1-N(x_0)}{x_0} - N(y_0)}{y_0} \leq \frac{1-N(y_0)}{y_0}.$$

In this case,

$$N(y_0) + y_0 z_0 < 1 \quad \text{and} \quad N(x_0) + x_0(N(y_0) + y_0 z_0) < 1$$
$$\text{and} \quad N(y_0) + y_0(N(x_0) + x_0 z_0) \geq 1,$$

which means $I(y_0, I(x_0, z_0)) = 1$ while $I(x_0, I(y_0, z_0)) < 1$.

(ii.2) If $\frac{\frac{1-N(x_0)}{x_0} - N(y_0)}{y_0} < \frac{\frac{1-N(y_0)}{y_0} - N(x_0)}{x_0}$, then there exists z_0 such that

$$\frac{\frac{1-N(x_0)}{x_0} - N(y_0)}{y_0} < z_0 < \frac{\frac{1-N(y_0)}{y_0} - N(x_0)}{x_0} \leq \frac{1-N(x_0)}{x_0}.$$

In this case,

$$N(x_0) + x_0 z_0 < 1 \quad \text{and} \quad N(y_0) + y_0(N(x_0) + x_0 z_0) < 1$$
$$\text{and} \quad N(x_0) + x_0(N(y_0) + y_0 z_0) \geq 1,$$

which means $I(x_0, I(y_0, z_0)) = 1$ while $I(y_0, I(x_0, z_0)) < 1$.

Thus I does not satisfy FI7. According to Theorem 3.3 and Theorem 3.11, I is neither an S-implication nor an R-implication generated by a left-continuous t-norm.

Analogously, let I be the QL-implication stated in (3.11) and f be the mapping defined in Corollary 3.46 with f being increasing. Then we have $(N_0)_\phi(x) \geq (N_0)_\varphi(x)$, for all $x \in [0,1]$. If $(N_0)_\phi = (N_0)_\varphi$, then I is also an S-implication which is conjugated to the Reichenbach implication. On the contrary, if $(N_0)_\phi \neq (N_0)_\varphi$, then I is neither an S-implication nor an R-implication generated by a left-continuous t-norm.

QL-implications generated by a nilpotent t-conorm $(S_{\mathbf{L}})_\varphi$, a nilpotent t-norm $(T_{\mathbf{L}})_\varphi$ and a strong fuzzy negation

Theorem 3.48. *Let N be a strong fuzzy negation. The QL-implication*

$$I(x,y) = S_{\mathbf{L}}(N(x), T_{\mathbf{L}}(x,y)) = \min(N(x) + \max(x+y-1, 0), 1)$$

for all $(x,y) \in [0,1]^2$, satisfies FI1 iff $N = N_0$.

PROOF. $\Longleftarrow$: Straightforward from Remark 3.43. Observe that in this case we obtain the well-known Kleene-Dienes implication: $I(x,y) = \max(1-x, y)$ ([24], Table 1.1).
$\Longrightarrow$: First notice that $N(0) = N_0(0) = 1$. Now take any $x \in \,]0,1]$. For all $z \in \,]0,x[$ and all $y \in \,]1-z, 2-N(z)-z[$, we have

$$\min(N(x) + x + y - 1, 1) = I(x,y) \leq I(z,y) = N(z) + z + y - 1 < 1$$

So $N(x) + x + y - 1 < 1$ for , and therefore

$$N(x) + x + (2 - N(z) - z) - 1 \leq 1.$$

Thus for any $z \in \,]0,x[$, $N(x) + x \leq N(z) + z$. The continuity of N now implies that

$$N(x) + x \leq N(0) + 0 = 1,$$

in other words $N(x) \leq 1 - x$. In combination with $1 - x \leq N(x)$, we conclude that $N(x) = 1 - x$. So also for $x > 0$, $N(x) = N_0(x)$. □

Corollary 3.49. *Let φ and ϕ denote two automorphisms of the unit interval. Then the QL-implication*

$$\begin{aligned} I(x,y) &= (S_{\mathbf{L}})_\varphi((N_0)_\phi(x), (T_{\mathbf{L}})_\varphi(x,y)) \\ &= \varphi^{-1}(\min(\varphi((N_0)_\phi(x)) + \max(\varphi(x) + \varphi(y) - 1, 0), 1)) \end{aligned}$$

for all $(x,y) \in [0,1]^2$, satisfies FI1 iff $(N_0)_\phi = (N_0)_\varphi$.

PROOF. Putting $\gamma = \phi \circ \varphi^{-1}$, then $\varphi((N_0)_\phi(x)) = (N_0)_\gamma(\varphi(x))$. According to Lemma 2.2 and Lemma 2.3, γ is also an automorphism of the unit interval. So $(N_0)_\gamma$ is a strong fuzzy negation. Thus

$$\begin{aligned} I(x,y) &= \varphi^{-1}(S_{\mathbf{L}}((N_0)_\gamma(\varphi(x)), T_{\mathbf{L}}(\varphi(x), \varphi(y)))) \\ &= \varphi^{-1}(I'(\varphi(x), \varphi(y))) \end{aligned}$$

where $I'(x,y) = S_{\mathbf{L}}((N_0)_\gamma(x), T_{\mathbf{L}}(x,y))$. According to Theorem 3.48, I' satisfies FI1 iff $(N_0)_\gamma = N_0$. And according to Lemma 3.38, I satisfies FI1 iff I' satisfies FI1. Thus I satisfies FI1 iff $(N_0)_\gamma = N_0$. So for all $x \in [0,1]$,

$$\begin{aligned} &(N_0)_\gamma(x) = 1 - x \Rightarrow \\ &(N_0)_\gamma(\varphi(x)) = 1 - \varphi(x) \Rightarrow \\ &\varphi((N_0)_\phi(x)) = 1 - \varphi(x) \Rightarrow \\ &(N_0)_\phi(x) = \varphi^{-1}(1 - \varphi(x)). \end{aligned}$$

Hence $(N_0)_\phi = (N_0)_\varphi$. □

Remark 3.50. A characterization of automorphisms which satisfy the equality $(N_0)_\phi = (N_0)_\varphi$ is given in ([24], Proposition 1.1).

In general the automorphisms conjugated to $S_{\mathbf{L}}$ and $T_{\mathbf{L}}$ need not be the same. This case will be studied in Corollary 3.57.

Remark 3.51. Notice that $T_{\mathbf{L}}$ is a Frank t-norm. Thus according to Remark 3.42, the QL-implication defined in Theorem 3.48 with $N = N_0$ and the QL-implication defined in Corollary 3.49 with $(N_0)_\phi = (N_0)_\varphi$ are both S-implications. And according to Theorem 3.11 and Corollary 3.35, they are not R-implications generated by left-continuous t-norms.

Next we consider the t-norm T which is an ordinal sum defined in (2.30) where each T_m, $m \in M$, is $T_{\mathbf{P}}$ or $T_{\mathbf{L}}$. The t-norms conjugated to T are also taken into account.

QL-implications generated by a nilpotent t-conorm $(S_{\mathbf{L}})_\varphi$, a t-norm which is an ordinal sum of continuous Archimedean t-norms and a strong fuzzy negation

Theorem 3.52. *Let T_o be the ordinal sum defined in (2.30) where each T_m, $m \in M$, is $T_{\mathbf{P}}$, and N be a strong fuzzy negation. Define $f_m(x) = \frac{1-N(x)-a_m}{x-a_m}$, for all $m \in M$ and $x \in]a_m, b_m]$. Then the QL-implication*

$$(\forall (x,y) \in [0,1]^2)(I(x,y) = S_{\mathbf{L}}(N(x), T_o(x,y)))$$

satisfies FI1 iff $N(x) \geq N_0(x)$, for all $x \in [0,1]$ and f_m is increasing, for all $m \in M$.

PROOF. $\Longrightarrow$: According to Proposition 3.21, if I satisfies FI1, then $N(x) \geq N_0(x)$, for all $x \in [0,1]$. Thus $f_m(x) \leq 1$. Let $m \in M$ and let us fix arbitrarily x_1, $x_2 \in]a_m, b_m]$ such that $x_1 < x_2$. We will consider two cases.

(i) If $f_m(x_2) \leq 0$, then $1 - N(x_2) - a_m \leq 0$. Since $N(x_1) > N(x_2)$, we get $1 - N(x_1) - a_m \leq 0$. But $x_2 - a_m > x_1 - a_m > 0$, so $f_m(x_2) \geq f_m(x_1)$, i.e., f_m is increasing in this case.

(ii) If $f_m(x_2) > 0$, then $f_m(x_2) \in]0, 1]$. There exists a $y_0 \in]a_m, b_m]$ such that $f_m(x_2) = \frac{y_0 - a_m}{b_m - a_m}$. Thus $I(x_2, y_0) = 1$. In order for I to satisfy FI1, it is necessary that

$$I(x_2, y_0) = 1 \Rightarrow I(x_1, y_0) = 1,$$

i.e., $f_m(x_1) \leq \frac{y_0 - a_m}{b_m - a_m}$. Thus $f_m(x_1) \leq f_m(x_2)$.

Hence f_m is increasing in $]a_m, b_m]$.
$\Longleftarrow$: Fix arbitrarily x_1, x_2 and $y \in [0, 1]$ and assume $x_1 < x_2$. We will consider three cases w.r.t. the positions of x_1, x_2 and y:

(i) For all $m \in M$, $(x_1, y) \notin [a_m, b_m]^2$. Then

$$\begin{aligned} I(x_1, y) &= \min(N(x_1) + \min(x_1, y), 1) \\ &= \begin{cases} 1, & \text{if } x_1 \leq y \\ \min(N(x_1) + y, 1), & \text{otherwise} \end{cases} \end{aligned}$$

Thus we need only to consider the situation that $x_2 > x_1 > y$. In this case $(x_2, y) \notin [a_m, b_m]^2$. Therefore $I(x_1, y) = \min(N(x_1)+y, 1)$ and $I(x_2, y) = \min(N(x_2) + y, 1)$. Since N is a decreasing mapping,
$I(x_1, y) \geq I(x_2, y)$.

(ii) There exists an $m \in M$ such that $(x_1, y) \in [a_m, b_m]^2$ and $x_2 \notin [a_m, b_m]$. This situation happens only if there exists $[a_m, b_m]$ such that $b_m < 1$. In this case,

$$\begin{aligned} I(x_1, y) &= \min(N(x_1) + a_m + \frac{(x_1 - a_m)(y - a_m)}{b_m - a_m}, 1) \\ I(x_2, y) &= \min(N(x_2) + y, 1) \end{aligned}$$

and $x_2 > b_m$. Thus we have $\frac{1-N(x_2)-a_m}{b_m-a_m} > \frac{1-N(b_m)-a_m}{b_m-a_m}$.
We now consider two cases.

(ii.1) If $x_1 \in]a_m, b_m]$, then since f_m is increasing,

$$\begin{aligned} \frac{1 - N(x_2) - a_m}{b_m - a_m} &> \frac{1 - N(x_1) - a_m}{x_1 - a_m} \Leftrightarrow \\ \frac{N(x_1) - N(x_2)}{b_m - x_1} &> \frac{1 - N(x_1) - a_m}{x_1 - a_m}. \end{aligned}$$

Assume $\frac{y-a_m}{b_m-a_m} < \frac{1-N(x_1)-a_m}{x_1-a_m}$. Then we have

$$\frac{y - a_m}{b_m - a_m} < \frac{N(x_1) - N(x_2)}{b_m - x_1}$$

$$\Leftrightarrow N(x_1) + a_m + \frac{(x_1 - a_m)(y - a_m)}{b_m - a_m} > N(x_2) + y,$$

which means that if both $I(x_1, y) < 1$ and $I(x_2, y) < 1$, then $I(x_1, y) > I(x_2, y)$.
Now assume $y \geq 1 - N(x_2)$. Then we have

$$y \geq \frac{(1 - N(x_1) - a_m)(b_m - a_m)}{x_1 - a_m} + a_m,$$

which means that if $I(x_2, y) = 1$, then $I(x_1, y) = 1$. Hence $I(x_1, y) \geq I(x_2, y)$ always holds.

(ii.2) If $x_1 = a_m$, then $I(x_1, y) = \min(N(a_m) + a_m, 1) = 1 \geq I(x_2, y)$.

(iii) There exists an $m \in M$ such that x_1, x_2, $y \in [a_m, b_m]$. In this case,

$$I(x_1, y) = \min(N(x_1) + a_m + (x_1 - a_m) \cdot \frac{y - a_m}{b_m - a_m}, 1)$$
$$I(x_2, y) = \min(N(x_2) + a_m + (x_2 - a_m) \cdot \frac{y - a_m}{b_m - a_m}, 1)$$

We now consider two cases.

(iii.1) If $x_1 \in]a_m, b_m]$, then since f_m is increasing,

$$\frac{1 - N(x_2) - a_m}{x_2 - a_m} \geq \frac{1 - N(x_1) - a_m}{x_1 - a_m} \Leftrightarrow$$
$$\frac{N(x_1) - N(x_2)}{x_2 - x_1} \geq \frac{1 - N(x_1) - a_m}{b_m - a_m}.$$

Assume $\frac{y-a_m}{b_m-a_m} < \frac{1-N(x_1)-a_m}{x_1-a_m}$. Then we have

$$\frac{y - a_m}{b_m - a_m} < \frac{N(x_1) - N(x_2)}{x_2 - x_1} \Leftrightarrow$$
$$N(x_1) + a_m + (x_1 - a_m) \cdot \frac{y - a_m}{b_m - a_m}$$
$$\geq N(x_2) + a_m + (x_2 - a_m) \cdot \frac{y - a_m}{b_m - a_m},$$

which means that if both $I(x_1, y) < 1$ and $I(x_2, y) < 1$, then $I(x_1, y) \geq I(x_2, y)$.
Now assume $\frac{y-a_m}{b_m-a_m} \geq \frac{1-N(x_2)-a_m}{x_2-a_m}$. Then since f_m is increasing, $\frac{y-a_m}{b_m-a_m} \geq \frac{1-N(x_1)-a_m}{x_1-a_m}$, which means that if $I(x_2, y) = 1$, then $I(x_1, y) = 1$. Hence $I(x_1, y) \geq I(x_2, y)$ always holds.

(iii.2) If $x_1 = a_m$, then $I(x_1, y) = 1 \geq I(x_2, y)$.

□

Example 3.8 Let T_o be the ordinal sum defined in (2.30) where each T_m, $m \in M$, is $T_{\mathbf{P}}$, and N be defined by: $N(x) = \sqrt{1-x^2}$. Note that $N(x) \geq N_0(x)$, for all $x \in [0,1]$. Then for each $m \in M$, the mapping f_m defined in Theorem 3.52 for N is $f_m(x) = \frac{1-\sqrt{1-x^2}-a_m}{x-a_m}$. We have

$$(\forall x \in \quad]a_m, b_m])(\frac{df_m(x)}{dx} = \frac{1 - a_m x + (a_m - 1)\sqrt{1-x^2}}{(x-a_m)^2\sqrt{1-x^2}}).$$

Since $a_m + (1-a_m) = 1$ and $x \leq 1$, $\sqrt{1-x^2} \leq 1$, $a_m x + (1-a_m)\sqrt{1-x^2} \leq 1$, $\frac{df_m(x)}{dx} \geq 0$, for all $x \in]a_m, b_m]$. Hence f_m is increasing. Therefore the QL-implication I defined by $I(x,y) = S_{\mathbf{L}}(N(x), T_o(x,y))$, for all $(x,y) \in [0,1]^2$ satisfies FI1.

Theorem 3.53. *Let T_o be the ordinal sum defined in (2.30) where each T_m, $m \in M$, is $T_{\mathbf{L}}$, and N be a strong fuzzy negation. Then the QL-implication*

$$(\forall (x,y) \in [0,1]^2)(I(x,y) = S_{\mathbf{L}}(N(x), T_o(x,y)))$$

satisfies FI1 iff

i) for all $x \in [0,1]$, $N(x) \geq N_0(x)$, and

ii) for all $x \in [a_m, b_m]$, if $N(b_m) + a_m < 1$, then the mapping $x \mapsto N(x) + x$ is decreasing in $[N(1-a_m), b_m]$.

PROOF. $\Longrightarrow$: According to Proposition 3.21, if I satisfies FI1, then $N(x) \geq N_0(x)$, for all $x \in [0,1]$.

Moreover, if $N(b_m) + a_m < 1$, then there exists an $x_0 \in [a_m, b_m[$ such that $N(x_0) + a_m = 1$, i.e., $x_0 = N(1-a_m)$. Since N is strictly decreasing, $N(x) + a_m < 1$, for all $x \in]x_0, b_m]$. Let $m \in M$ and let us fix arbitrarily $x_1, x_2 \in]x_0, b_m]$ and assume $x_1 < x_2$. Then there exists a $y_0 \in]a_m, b_m[$ such that

$$0 \leq N(x_1) + x_1 - 1 < b_m - y_0 < x_1 - a_m < x_2 - a_m.$$

Thus

$$I(x_1, y_0) = N(x_1) + x_1 + y_0 - b_m < 1$$
$$I(x_2, y_0) = \min(N(x_2) + x_2 + y_0 - b_m, 1).$$

If I satisfies FI1, then it is necessary that $I(x_2, y_0) \leq I(x_1, y_0)$, i.e.,

$$N(x_2) + x_2 + y_0 - b_m \leq N(x_1) + x_1 + y_0 - b_m.$$

Hence the mapping $x \mapsto N(x) + x$ must be decreasing in $]x_0, b_m]$. Because the mapping $x \mapsto N(x)+x$ is continuous, it is decreasing in $[x_0, b_m]$, i.e., the mapping $x \mapsto N(x) + x$ is decreasing in $[N(1-a_m), b_m]$.

$\Longleftarrow$: Fix arbitrarily $x_1, x_2 \in [0,1]$ and assume $x_1 < x_2$, then for all $y \in [0,1]$,

$$I(x_1, y) = \min(N(x_1) + T_o(x_1, y), 1))$$

$$I(x_2, y) = \min(N(x_2) + T_o(x_2, y), 1)).$$

We consider three cases according to the positions of x_1, x_2 and y w.r.t. a_m and b_m:

(i) For all $m \in M$, $(x_1, y) \notin [a_m, b_m]^2$. In this case,

$$\begin{aligned} I(x_1, y) &= \min(N(x_1) + \min(x_1, y), 1) \\ &= \begin{cases} 1, & \text{if } x_1 \leq y \\ \min(N(x_1) + y, 1), & \text{otherwise} \end{cases} \end{aligned}$$

Thus we need only to consider the situation that $x_2 > x_1 > y$. In this case $(x_2, y) \notin [a_m, b_m]^2$. Therefore $I(x_1, y) = \min(N(x_1)+y, 1)$ and $I(x_2, y) = \min(N(x_2) + y, 1)$. Since N is a decreasing mapping, we get $I(x_1, y) \geq I(x_2, y)$.

(ii) There exists an $m \in M$ such that $(x_1, y) \in [a_m, b_m]^2$ and $x_2 \notin [a_m, b_m]$. This situation happens only if there exists $[a_m, b_m]$ such that $b_m < 1$. In this case, $x_2 > b_m$ and

$$\begin{aligned} I(x_1, y) &= \min(N(x_1) + a_m + \max(x_1 + y - a_m - b_m, 0), 1) \\ I(x_2, y) &= \min(N(x_2) + y, 1). \end{aligned}$$

We consider two subcases now.

(ii.1) If $N(b_m)+a_m < 1$, then there exists an $x_0 \in [a_m, b_m[$ such that $N(x_0)+a_m = 1$, i.e., $x_0 = N(1 - a_m)$. Since N is strictly decreasing, we have $N(x) + a_m > 1$, for all $x \in [a_m, x_0[$, and $N(x) + a_m < 1$, for all $x \in]x_0, b_m]$. If $x_1 \in [a_m, x_0]$, then $I(x_1, y) = 1 \geq I(x_2, y)$. If $x_1 \in]x_0, b_m]$, then since the mapping $x \mapsto N(x) + x$ is decreasing,

$$N(x_2) + b_m < N(b_m) + b_m \leq N(x_1) + x_1.$$

Thus $a_m + b_m - x_1 < 1 - N(x_2)$. We will consider three cases according to the position of y w.r.t. $a_m + b_m - x_1$ and $1 - N(x_2)$.

(ii.1.1) $y \geq 1 - N(x_2)$. In this case,

$$N(x_1) + x_1 + y - b_m > N(x_2) + y \geq 1.$$

Thus $I(x_1, y) = I(x_2, y) = 1$.

(ii.1.2) $a_m + b_m - x_1 < y < 1 - N(x_2)$. In this case,

$$N(x_2) + y < 1 \quad \text{and} \quad N(x_1) + x_1 + y - b_m > N(x_2) + y.$$

Thus $I(x_1, y) > I(x_2, y)$.

(ii.1.3) $y \le a_m + b_m - x_1$. In this case,

$$y + N(x_2) \le a_m + b_m - x_1 + N(x_2) < N(x_1) + a_m.$$

Thus $I(x_1, y) > I(x_2, y)$

(ii.2) If $N(b_m) + a_m \ge 1$, then $N(x) + a_m > 1$, for all $x \in [a_m, b_m[$. Thus $I(x_1, y) = 1 \ge I(x_2, y)$.

Hence $I(x_1, y) \ge I(x_2, y)$ provided $N(b_m) + a_m < 1$.

(iii) There exists an $m \in M$ such that x_1, x_2, $y \in [a_m, b_m]$. In this case,

$$I(x_1, y) = \min(N(x_1) + a_m + \max(x_1 + y - a_m - b_m, 0), 1)$$
$$I(x_2, y) = \min(N(x_2) + a_m + \max(x_2 + y - a_m - b_m, 0), 1).$$

We consider two subcases now.

(iii.1) If $N(b_m)+a_m < 1$, then there exists an $x_0 \in [a_m, b_m[$ such that $N(x_0)+a_m = 1$, i.e., $x_0 = N(1 - a_m)$. Since N is strictly decreasing, we have $N(x) + a_m > 1$, for all $x \in [a_m, x_0[$, and $N(x) + a_m < 1$, for all $x \in]x_0, b_m]$. If $x_1 \in [a_m, x_0]$, then $I(x_1, y) = 1 \ge I(x_2, y)$. If x_1, $x_2 \in]x_0, b_m]$, then we consider three cases according to the position of y w.r.t. $a_m+b_m-x_2$ and $a_m+b_m-x_1$. Notice that $a_m+b_m-x_2 < a_m+b_m-x_1$.

(iii.1.1) $y \le a_m + b_m - x_2$. In this case,

$$I(x_1, y) = N(x_1) + a_m \quad \text{and} \quad I(x_2, y) = N(x_2) + a_m.$$

Since N is strictly decreasing, $I(x_1, y) > I(x_2, y)$.

(iii.1.2) $a_m + b_m - x_2 < y \le a_m + b_m - x_1$. In this case,

$$I(x_1, y) = N(x_1) + a_m$$
$$I(x_2, y) = \min(N(x_2) + x_2 + y - b_m, 1).$$

Since the mapping $x \mapsto N(x) + x$ is decreasing in $[x_0, b_m]$,

$$N(x_2) + x_2 + y - b_m \le N(x_1) + x_1 + y - b_m \le N(x_1) + a_m.$$

Thus $I(x_1, y) \ge I(x_2, y)$.

(iii.1.3) $y > a_m + b_m - x_1$. In this case,

$$I(x_1, y) = \min(N(x_1) + x_1 + y - b_m, 1)$$
$$I(x_2, y) = \min(N(x_2) + x_2 + y - b_m, 1).$$

Since the mapping $x \mapsto N(x) + x$ is decreasing in $[x_0, b_m]$, we get $I(x_1, y) \ge I(x_2, y)$.

Hence $I(x_1, y) \ge I(x_2, y)$ provided $N(b_m) + a_m < 1$.

(iii.2) If $N(b_m) + a_m \ge 1$, then $N(x) + a_m > 1$, for all $x \in [a_m, b_m[$. Thus $I(x_1, y) = 1 \ge I(x_2, y)$.

□

Example 3.9 Let T_o be the ordinal sum defined by:

$$T_o(x,y) = \begin{cases} a_1 + (b_1 - a_1)T_{\mathbf{L}}(\frac{x-a_1}{b_1-a_1}, \frac{y-a_1}{b_1-a_1}), & \text{if } (x,y) \in [a_1,b_1]^2 \\ T_{\mathbf{M}}(x,y), & \text{otherwise} \end{cases}$$

where $a_1 = 0.4$ and $b_1 = 0.9$, and N be defined by $N(x) = \sqrt{1-x^2}$. Then $N(x) \geq N_0(x)$, for all $x \in [0,1]$ and $N(0.9) + 0.4 < 1$. We have

$$(\forall x \in [N(1-0.4), 0.9])(\frac{d(N(x)+x)}{dx} = -\frac{x}{\sqrt{1-x^2}} + 1).$$

Since $x \geq N(1-0.4) = 0.8 > \frac{1}{\sqrt{2}}$, $\frac{d(N(x)+x)}{dx} < 0$ in $[N(1-0.4), 0.9]$, $x \mapsto N(x) + x$ is decreasing in $[N(1-0.4), 0.9]$. Therefore the QL-implication I defined by $I(x,y) = S_{\mathbf{L}}(N(x), T_o(x,y))$, for all $(x,y) \in [0,1]^2$ satisfies FI1.

Combining Theorems 3.52 and 3.53, we have the next corollary.

Corollary 3.54. *Let $\{[a_j,b_j]\}_{j\in J}$ and $\{[a_k,b_k]\}_{k\in K}$ be two non-empty families of non-overlapping, closed, proper subintervals of $[0,1]$ and*

$$\{[a_m,b_m]\}_{m\in M} = \{[a_j,b_j]\}_{j\in J} \cup \{[a_k,b_k]\}_{k\in K},$$

where J, K and M are finite or countable index sets. T_o is the ordinal sum of $\{[a_m,b_m],T_m\}_{m\in M}$, which is expressed as:

$$T_o(x,y) = \begin{cases} a_j + (b_j - a_j)T_{\mathbf{P}}(\frac{x-a_j}{b_j-a_j}, \frac{y-a_j}{b_j-a_j}), & \text{if } (x,y) \in [a_j,b_j]^2 \\ a_k + (b_k - a_k)T_{\mathbf{L}}(\frac{x-a_k}{b_k-a_k}, \frac{y-a_k}{b_k-a_k}), & \text{if } (x,y) \in [a_k,b_k]^2 \\ T_{\mathbf{M}}(x,y), & \text{otherwise} \end{cases} . \tag{3.12}$$

Moreover, let N be a strong fuzzy negation and define $f_j(x) = \frac{1-N(x)-a_j}{x-a_j}$, for all j and $x \in]a_j,b_j]$. Then the QL-implication

$$(\forall (x,y) \in [0,1]^2)(I(x,y) = S_{\mathbf{L}}(N(x), T_o(x,y)))$$

satisfies FI1 iff

i) for all $x \in [0,1]$, $N(x) \geq N_0(x)$, and

ii) f_j is increasing, for all $j \in J$, and

iii) for all $x \in [a_k,b_k]$, if $N(b_k) + a_k < 1$, then the mapping $x \mapsto N(x) + x$ is decreasing in $[N(1-a_k), b_k]$.

Example 3.10 Let T_o be the ordinal sum defined by:

$$T_o(x,y) = \begin{cases} a_1 + (b_1 - a_1)T_{\mathbf{P}}(\frac{x-a_1}{b_1-a_1}, \frac{y-a_1}{b_1-a_1}), & \text{if } (x,y) \in [a_1,b_1]^2 \\ a_2 + (b_2 - a_2)T_{\mathbf{L}}(\frac{x-a_2}{b_2-a_2}, \frac{y-a_2}{b_2-a_2}), & \text{if } (x,y) \in [a_2,b_2]^2 \\ T_{\mathbf{M}}(x,y), & \text{otherwise} \end{cases} ,$$

where $a_1 = 0.1$, $b_1 = 0.2$, $a_2 = 0.4$ and $b_2 = 0.9$, and N be defined by $N(x) = \sqrt{1-x^2}$, for all $x \in [0,1]$. Then $N(x) \geq N_0(x)$, for all $x \in [0,1]$. The mapping f_1 defined in Corollary 3.54 for N is $f_1(x) = \frac{1-N(x)-0.1}{x-0.1}$, for all $x \in]0.1, 0.2]$. According to Example 3.8, f_1 is increasing. Moreover, according to Example 3.9, we have $N(b_2) + a_2 < 1$ and the mapping $x \mapsto N(x) + x$ is decreasing in $[N(1-a_2), b_2]$. Thus the QL-implication I defined by $I(x,y) = S_{\mathbf{L}}(N(x), T_o(x,y))$ for all $(x,y) \in [0,1]^2$ satisfies FI1.

For the QL-implications conjugated to the one defined in Corollary 3.54, we have the next corollary.

Corollary 3.55. *Let T_o be a t-norm defined as (3.12), N be a strong fuzzy negation and φ be an automorphism of the unit interval. Define $f_j(x) = \frac{1-\varphi(N(\varphi^{-1}(x)))-a_j}{x-a_j}$, for all j and $x \in]a_j, b_j]$. Then the QL-implication*

$$(\forall (x,y) \in [0,1]^2)(I(x,y) = (S_{\mathbf{L}})_\varphi(N(x), (T_o)_\varphi(x,y)))$$

satisfies FI1 iff

i) for all $x \in [0,1]$, $N(x) \geq (N_0)_\varphi(x)$, and
ii) f_j is increasing, for all $j \in J$, and
iii) for all $x \in [a_k, b_k]$, if $\varphi(N(\varphi^{-1}(b_k)))+a_k < 1$, then the mapping $x \mapsto \varphi(N(\varphi^{-1}(x)))+x$ is decreasing in $[\varphi(N(\varphi^{-1}(1-a_k))), b_k]$.

PROOF. Let $N'(x) = \varphi(N(\varphi^{-1}(x)))$. According to Lemma 3.38, I satisfies FI1 iff $(I)_{\varphi^{-1}}$, which is expressed as

$$(\forall (x,y) \in [0,1]^2)((I)_{\varphi^{-1}}(x,y) = S_{\mathbf{L}}(N'(x), T_o(x,y)))$$

satisfies FI1. According to Corollary 3.54, $(I)_{\varphi^{-1}}$ satisfies FI1 iff the following three conditions are fulfilled:

i) for all $x \in [0,1]$, $N'(x) \geq N_0(x)$, and
ii) f'_j is increasing, for all $j \in J$, and
iii) for all $x \in [a_k, b_k]$, if $N'(b_k) + a_k < 1$, then the mapping $x \mapsto N'(x) + x$ is decreasing in $[N'(1-a_k), b_k]$,

where $f'_j(x) = \frac{1-N'(x)-a_j}{x-a_j}$ for all $x \in]a_j, b_j]$. They are equivalent to

i) for all $x \in [0,1]$, $N(x) \geq (N_0)_\varphi(x)$, and
ii) f_j is increasing, for all $j \in J$, and
iii) for all $x \in [a_k, b_k]$, if $\varphi(N(\varphi^{-1}(b_k))) + a_k < 1$, then the mapping $x \mapsto \varphi(N(\varphi^{-1}(x))) + x$ is decreasing in $[\varphi(N(\varphi^{-1}(1-a_k))), b_k]$.

□

Until now we have investigated the fulfillment of property FI1 for a QL-implica- tion I generated by particular combinations of a t-conorm S, a t-norm T and a strong fuzzy negation N. Indeed cases of the following two types have been considered:

(i) $S = (S_{\mathbf{L}})_\varphi$ (with φ being an automorphism of the unit interval), $N = (N_0)_\varphi$, T arbitrary;

(ii) $S = (S_{\mathbf{L}})_\varphi$, T fixed and continuous, N arbitrary.

QL-implications generated by a nilpotent t-conorm $(S_{\mathbf{L}})_\varphi$, a continuous t-norm and a strong fuzzy negation (general case)

Theorem 3.56. *Let φ be an automorphism of the unit interval, T be a continuous t-norm and N be a strong fuzzy negation. Define for each $y \in [0,1]$, $F_y(x) = \varphi(N(x)) + \varphi(T(x,y))$, for all $x \in [0,1]$. Then the QL-implication*

$$(\forall (x,y) \in [0,1]^2)(I(x,y) = (S_{\mathbf{L}})_\varphi(N(x), T(x,y)))$$

satisfies FI1 iff for all y, there exists an $x_0 \in [0,1]$ such that $F_y(x_0) = 1$ with $F_y(x) \geq 1$, for all $x \in [0, x_0]$ and F_y decreasing in $[x_0, 1]$.

PROOF. From the definition of F_y we get: $I(x,y) = \varphi^{-1}(\min(F_y(x), 1))$, for all $(x,y) \in [0,1]^2$.
$\Longrightarrow$:

(i) First we consider $y = 1$. In this case $F_1(x) = \varphi(N(x)) + \varphi(x)$, for all $x \in [0,1]$. If I satisfies FI1, then according to Proposition 3.21, $N(x) \geq (N_0)_\varphi(x)$, for all $x \in [0,1]$. Since $F_1(1) = 1$, we have $F_1(x) \geq 1$, for all $x \in [0,1]$. So we take $x_0 = 1$.

(ii) Second we consider $y \in [0,1[$. In this case $F_y(0) = 1$ and $F_y(1) = \varphi(y) < 1$. We now consider two cases.

(ii.1) If there exists an $x' \in]0,1[$ such that $F_y(x') \geq 1$, then since F_y is continuous, there exists $x_0 \in [x', 1[$ such that $F_y(x_0) = 1$ and $F_y(x) < 1$, for all $x \in]x_0, 1]$. Since I satisfies FI1, once $I(x,y) < 1$, F_y should be decreasing. Thus F_y is decreasing in $]x_0, 1]$. Since F_y is continuous, F_y is decreasing in $[x_0, 1]$. Moreover, since $I(x_0, y) = 1$, it is necessary that $I(x,y) = 1$, for all $x \in [0, x_0]$. Thus $F_y(x) \geq 1$, for all $x \in [0, x_0]$.

(ii.2) If $F_y(x) < 1$, for all $x \in]0,1[$, then $F_y(x) \leq 1$, for all $x \in [0,1]$. So

$$I(x,y) = \varphi^{-1}(\min(F_y(x), 1)) = \varphi^{-1}(F_y(x)),$$

for all $x \in [0,1]$. Since I satisfies FI1, once $I(x,y) < 1$, F_y should be decreasing. Thus F_y is decreasing in $]0,1]$. Since F_y is continuous, F_y is decreasing in $[0,1]$.

$\Longleftarrow$: Take a fixed $y \in [0,1]$. Let x_0 be the point for which $F_y(x_0) = 1$ and $F_y(x) \geq 1$, for all $x \in [0, x_0]$ and F_y decreasing in $[x_0, 1]$ and x_1, $x_2 \in [0,1]$

with $x_1 < x_2$. Suppose $0 \leq x_1 \leq x_0$, then $I(x_1, y) = 1 \geq I(x_2, y)$. Suppose $0 \leq x_0 \leq x_1 < x_2$, then since F_y is decreasing in $[x_0, 1]$, $I(x_1, y) \geq I(x_2, y)$. Thus I satisfies FI1. □

Notice that the result of Theorem 3.56 is valid for all the QL-implications that are generated by the t-conorm $S_{\mathbf{L}}$, a continuous t-norm T and a strong fuzzy negation N. We give some examples below to illustrate the mapping F_y for several QL-implications studied above that have been proved to satisfy FI1.

Example 3.11

(i) Let I be the QL-implication defined in Corollary 3.40 with $S = S_{\mathbf{L}}$ and $N(x) = \sqrt{1 - x^2}$. Then the corresponding $F_y(x)$:

$$F_y(x) = \sqrt{1 - x^2} + T_{\mathbf{M}}(x, y) = \begin{cases} \sqrt{1 - x^2} + x, & \text{if } x \leq y \\ \sqrt{1 - x^2} + y, & \text{if } x > y \end{cases}$$

for all $x \in [0, 1]$. For a fixed $y_0 \in [0, 1]$, $F_{y_0}(x) \geq 1$, for all $x \in [0, \sqrt{2y_0 - y_0^2}]$. If $x > \sqrt{2y_0 - y_0^2} \geq y_0$, then $F_{y_0}(x) = \sqrt{1 - x^2} + y_0 < 1$. Since the mapping $x \mapsto \sqrt{1 - x^2}$ is decreasing, F_{y_0} is decreasing in $[\sqrt{2y_0 - y_0^2}, 1]$. Thus F_{y_0} fulfills the conditions required in Theorem 3.56.

(ii) Let I be the QL-implication defined in Example 3.6. Then the corresponding $F_y(x)$:

$$F_y(x) = \sqrt{1 - x^2} + xy.$$

for all $x \in [0, 1]$. For a fixed $y_0 \in\,]0, 1[$, one can verify that $F_{y_0}(\frac{2y_0}{1+y_0^2}) = 1$ with $F_{y_0}(x) \geq 1$ for all $x \in [0, \frac{2y_0}{1+y_0^2}]$, and $\frac{x}{\sqrt{1-x^2}} \geq \frac{2y_0}{1+y_0^2} > y_0$ for all $x \in [\frac{2y_0}{1+y_0^2}, 1]$. Thus

$$\frac{d(\sqrt{1 - x^2} + x \cdot y_0)}{dx} = y_0 - \frac{x}{\sqrt{1 - x^2}} < 0,$$

for all $x \in [\frac{2y_0}{1+y_0^2}, 1]$, which means F_{y_0} is decreasing in $[\frac{2y_0}{1+y_0^2}, 1]$. Moreover, we have $F_0(0) = 1$ with F_0 decreasing in $[0, 1]$ and $F_1(1) = 1$ with $F_1(x) = \sqrt{1 - x^2} + x \geq 1$, for all $x \in [0, 1]$. Thus F_y fulfills the conditions required in Theorem 3.56 with $x_0 = \frac{2y}{1+y^2}$, for all $y \in [0, 1]$.

Following Theorem 3.56, if we take the t-norm T as conjugated to $T_{\mathbf{L}}$, then we obtain the following corollary:

Corollary 3.57. *Let φ and ϕ be two automorphisms of the unit interval and N be a strong fuzzy negation. Define for each $y \in [0, 1]$,*

$$(\forall x \in [0, 1])(F_y(x) = \varphi(N(x)) + \varphi((T_{\mathbf{L}})_\phi(x, y))).$$

Then the QL-implication

$$(\forall(x,y)\in[0,1]^2)(I(x,y)=(S_{\mathbf{L}})_{\varphi}(N(x),(T_{\mathbf{L}})_{\phi}(x,y)))$$

satisfies FI1 iff $F_1(x)\geq 1$, *for all* $x\in[0,1]$ *and* F_y *is decreasing in* $[0,1]$, *for all* $y\in[0,1[$.

PROOF. (i) First we consider $y=1$. In this case $F_1(x)=\varphi(N(x))+\varphi(x)$. We now consider two cases.

(i.1) If there exists $x^{'}\in\,]0,1[$ such that $F_1(x^{'})<1$, then since $F_1(1)=1$, there exists no $x_0\in[0,1]$ such that $F_1(x_0)=1$ with $F_1(x)\geq 1$ for all $x\in[0,x_0]$ and F_1 decreasing in $[x_0,1]$.

(ii.2) If $F_1(x)\geq 1$, for all $x\in[0,1]$, then take $x_0=1$, we have $F_1(x_0)=1$ with $F_1(x)\geq 1$, for all $x\in[0,1]$. Thus the sufficient and necessary condition in Theorem 3.56 is fulfilled iff $F_1(x)\geq 1$, for all $x\in[0,1]$.

(ii) Second we consider $y\in[0,1[$. In this case,

$$\begin{aligned}F_y(x)&=\varphi(N(x))+\varphi((T_{\mathbf{L}})_{\phi}(x,y))\\&=\varphi(N(x))+\varphi(\phi^{-1}(\max(\phi(x)+\phi(y)-1,0))).\end{aligned}$$

If $x\in[0,(N_0)_{\phi}(y)]$, then $F_y(x)=\varphi(N(x))$. So $F_y(x)<1$, for all $x\in\,]0,(N_0)_{\phi}(y)]$. Thus there exists no $x_0\in\,]0,1]$ such that $F_y(x_0)=1$ with $F_y(x)\geq 1$ for all $x\in[0,x_0]$. Therefore the sufficient and necessary condition in Theorem 3.56 is fulfilled iff we take $x_0=0$ and F_y being decreasing in $[0,1]$.

□

In Theorem 3.34 and Corollary 3.35 we have studied how the QL-implication generated by the t-conorm $(S_{\mathbf{L}})\varphi$, a continuous t-norm T and a strong fuzzy negation N satisfies FI8. Using the result of Theorem 3.56, we can see how a QL-implication satisfying FI8 also satisfies FI1.

Corollary 3.58. *Let* φ *be an automorphism of the unit interval,* T *be a continuous t-norm and* N *be a strong fuzzy negation. The QL-implication*

$$(\forall(x,y)\in[0,1]^2)(I(x,y)=(S_{\mathbf{L}})_{\varphi}(N(x),T(x,y)))$$

satisfies FI1 and FI8 iff conditions (3.7) and (3.8) are fulfilled and for all $0\leq y<x_1<x_2\leq 1$,

$$\varphi(T(x_2,x_2))-\varphi(T(x_1,x_1))\geq\varphi(T(x_2,y))-\varphi(T(x_1,y)) \tag{3.13}$$

PROOF. $\Longrightarrow$: According to Theorem 3.34, if I satisfies FI8, then T and N fulfill conditions (3.7) and (3.8). Thus defining for each $y\in[0,1[$,

$$(\forall x\in[0,1])(F_y(x)=\varphi(N(x))+\varphi(T(x,y))),$$

we get $F_y(x) = 1 - \varphi(T(x,x)) + \varphi(T(x,y))$. According to Theorem 3.56, if I satisfies FI1, then F_y is decreasing in $]y,1]$. Therefore we have

$$\varphi(T(x_2,x_2)) - \varphi(T(x_1,x_1)) \geq \varphi(T(x_2,y)) - \varphi(T(x_1,y)),$$

for all $y < x_1 < x_2$.
$\Longleftarrow$: If T and N fulfill conditions (3.7) and (3.8), then according to Theorem 3.34, I satisfies FI8. Moreover, if T and N fulfill conditions (3.7), (3.8) and (3.13), then

$$F_y(x) = \varphi(N(x)) + \varphi(T(x,y)) = 1 - \varphi(T(x,x)) + \varphi(T(y,y)),$$

for all x, $y \in [0,1]$ and we have $F_y(y) = 1$ with $F_y(x) \geq 1$, for all $x \in [0,y]$ and F_y decreasing in $]y,1]$. Since F_y is continuous, F_y is decreasing in $[y,1]$. Thus according to Theorem 3.56, I satisfies FI1. □

Example 3.12 Let the automorphism φ, the t-norm T, the strong fuzzy negation N and the QL-implication I be defined as those in Example 3.3 (ii). Consider any $0 \leq y < x_1 < x_2 \leq 1$, we have

$$\varphi(T(x_2,x_2)) - \varphi(T(x_1,x_1)) = \frac{x_2 - x_1}{(1-\frac{x_2}{2})(1-\frac{x_1}{2})}$$
$$\varphi(T(x_2,y)) - \varphi(T(x_1,y)) = \frac{x_2 - x_1}{\frac{x_2}{y^2} + \frac{1}{y} - \frac{x_2}{y}}.$$

Since $1 - \frac{x_2}{2} \leq 1 \leq \frac{x_2}{y^2} + \frac{1}{y} - \frac{x_2}{y}$, we obtain $\frac{x_2 - x_1}{(1-\frac{x_2}{2})(1-\frac{x_1}{2})} \geq \frac{x_2 - x_1}{\frac{x_2}{y^2} + \frac{1}{y} - \frac{x_2}{y}}$. Thus φ and T fulfill the condition (3.13). Recall that T and N fulfill conditions (3.7) and (3.8). Hence I is a QL-implication that satisfies FI1 and FI8. According to Example 3.3 (ii), I is neither an S-implication nor an R-implication generated by a left-continuous t-norm.

3.5 Summary

In this chapter, we have characterized S- and R- implications and have extensively studied QL-implications. We investigated under which conditions QL-implications satisfy the axioms required to obtain a suitable conclusion in fuzzy inference. Especially fuzzy implication axiom FI1 of QL-implications has been studied. Propositions, Theorems and Corollaries in Section 3.4.1 and Corollary 3.58 in Section 3.4.2 give general conditions under which a QL-implication can satisfy different commonly required axioms. Also the relationship between these axioms and FI1 is given. Moreover, Theorems and Corollaries in Section 3.4.2 state sufficient and necessary conditions for QL-implications generated by different combinations of a t-conorm, a t-norm and a strong fuzzy negation to satisfy FI1. Whether the QL-implications satisfying FI1 are equivalent to S- or R- implications has been illustrated in Remarks 3.41, 3.42, 3.47 and 3.51.
We still have an open question: does a QL-implication satisfying I1, I8 and I9 also satisfies I7?

Chapter 4

Dependence versus Independence of the Fuzzy Implication Axioms

4.1 Introduction

In the previous chapter we investigated for the three classes of fuzzy implications generated from fuzzy logical operators under which conditions they satisfy the axioms FI1-FI13. Observe that in many papers or books a fuzzy implication I is defined as a $[0,1]^2 \rightharpoonup [0,1]$ mapping that satisfies FI1-FI5. This is because in many applications of fuzzy implications FI1-FI5 are strongly required [6], [10], [11], [24], [36], [97], [98].

Remark 4.1. Notice that if a $[0,1]^2 \rightharpoonup [0,1]$ mapping satisfies FI3, FI4 and FI5, it also satisfies the boundary conditions I1 stated in Definition 2.38. There are several equivalent definitions for a fuzzy implication to satisfy FI1 to FI5. For example, according to ([3], Lemma 1), the assumption that I satisfies I1, FI1 and FI2 is an equivalent definition.

In this chapter we investigate for any fuzzy implication satisfying FI1-FI5 the dependence and the independence between the axioms. The motivation of this investigation includes the following three aspects:

1. Łukasiewicz implication $I_{\mathbf{L}}$ is the well-known fuzzy implication that satisfies all the axioms from FI1 to FI13. Through this investigation we can obtain different examples of fuzzy implications that satisfy different combinations of the axioms which fulfill different requirements in different applications.
2. Through this investigation we want to improve some existing theorems. For example, Baczyński (2004) improves the theorem of Smets-Magrez (1987) through proving that one of the four axioms (FI2) in the theory depends on the other three (FI7, FI8 and FI13).

3. To solve some functional equations with fuzzy implications we need the help of this investigation.

Existing works have mainly studied a few interrelationships between axioms FI6-FI13 [2], [6], [10], [24]. But the results are not complete. This chapter aims to determine a full view of the interrelationships between FI6-FI13.

4.2 Interrelationships between 8 Fuzzy Implication Axioms

4.2.1 Getting FI6 from the Other Axioms

Theorem 4.2. *([6], Lemma 1.54(v), Corollary 1.57 (iii)) A fuzzy implication I satisfying FI9 and FI12 satisfies FI6 iff $N = N^{'}$.*

In the rest of this section we consider the condition that $N^{'} \neq N$.

Proposition 4.3. *([2], Lemma 6) A fuzzy implication I satisfying FI7 and FI8 satisfies FI6.*

Proposition 4.4. *([6], Lemma 1.56(ii)) A fuzzy implication I satisfying FI7 and FI9 satisfies FI6.*

Proposition 4.5. *A fuzzy implication I satisfying FI7 and FI13 satisfies FI6.*

PROOF. Because I satisfies FI7, we have for all $x \in [0,1]$,

$$I(1, N^{'}(x)) = I(1, I(x,0)) = I(x, I(1,0)) = I(x,0) = N^{'}(x). \qquad (4.1)$$

Because I is a continuous mapping, $N^{'}$ is a continuous mapping. Thus expression (4.1) is equivalent to $I(1,a) = a$, for all $a \in [0,1]$. Hence I satisfies FI6. □

Remark 4.6. In Proposition 4.3, Proposition 4.4 and Proposition 4.5 we considered the following 3 cases:

FI7 $\wedge$ FI8 $\Rightarrow$ FI6

FI7 $\wedge$ FI9 $\Rightarrow$ FI6

FI7 $\wedge$ FI13 $\Rightarrow$ FI6

So we still need to consider the following 2 cases:

FI7 $\wedge$ FI10 $\wedge$ FI11 $\wedge$ FI12 $\overset{?}{\Rightarrow}$ FI6

FI8 $\wedge$ FI9 $\wedge$ FI10 $\wedge$ FI11 $\wedge$ FI12 $\wedge$ FI13 $\overset{?}{\Rightarrow}$ FI6

Proposition 4.7. *A fuzzy implication I satisfying FI7, FI10, FI11 and FI12 does not necessarily satisfy FI6.*

Example 4.1 The fuzzy implication I_1 stated in [18] is represented by

$$I_1(x,y) = \begin{cases} 0 & \text{if } x = 1 \text{ and } y = 0 \\ 1 & \text{else} \end{cases}, \quad x, y \in [0,1].$$

we have

$$\begin{aligned} I_1(x, I_1(y,z)) &= \begin{cases} 1, & \text{if } x < 1 \text{ or } y < 1 \text{ or } z > 0 \\ 0, & \text{else} \end{cases}, \quad x,y,z \in [0,1]. \\ &= I_1(y, I_1(x,z)) \end{aligned}$$

So I_1 satisfies FI7. Moreover, for all x, $y \in [0,1]$,

$I_1(x,y) \geq y$.

$I_1(x,x) = 1$.

$I_1(N(y), N(x)) = I_1(x,y)$, for any strong fuzzy negation N.

So I_1 satisfies FI10, FI11 and FI12 w.r.t. any strong fuzzy negation N. However, in case that $x \neq 1$, $I_1(1,x) = 1 \neq x$. So I_1 does not satisfy FI6.

Proposition 4.8. *A fuzzy implication satisfying FI8, FI9, FI10, FI11, FI12 and FI13 does not necessarily satisfy FI6.*

Example 4.2 Let a fuzzy implication I_2 be represented by

$$I_2(x,y) = \begin{cases} 1 & \text{if } x \leq y \\ \sqrt{1-(x-y)^2} & \text{if } x > y \end{cases}, \quad x, y \in [0,1].$$

For all x, $y \in [0,1]$,

$I_2(x,y) = 1$ iff $x \leq y$.

$N^{'}(x) = I_2(x,0) = \sqrt{1-x^2} = \varphi^{-1}(1-\varphi(x))$, where $\varphi(x) = x^2$ is an automorphism of the unit interval. So $N^{'}$ is a strong fuzzy negation.

$I_2(x,y) \geq y$.

$I_2(x,x) = 1$.

$I_2(1-y, 1-x) = I_2(x,y)$.

I_2 is a continuous mapping.

So I_2 satisfies FI8, FI9, FI10, FI11, FI12 w.r.t. the standard strong fuzzy negation N_0, and FI13. However, in case that $x \neq 1$, $I_2(1,x) = \sqrt{2x-x^2} \neq x$. So I_2 does not satisfy FI6.

So we considered all the possibilities that the fuzzy implication axiom FI6 can be implied from the other 7 axioms. Moreover we stated for each independent case a counterexample. We summary the results of this section in Table 4.1.

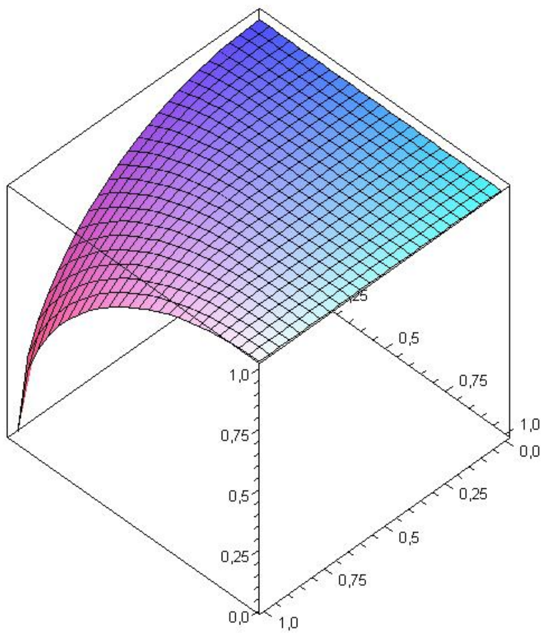

Figure 4.1: Example 4.2

Table 4.1: Getting FI6 from the other axioms

FI7$\wedge$ FI8 $\Rightarrow$ FI6
FI7$\wedge$ FI9 $\Rightarrow$ FI6
FI7$\wedge$ FI13 $\Rightarrow$ FI6
FI7$\wedge$ FI10$\wedge$ FI11 $\wedge$ FI12 $\not\Rightarrow$ FI6
FI8$\wedge$ FI9$\wedge$ FI10$\wedge$ FI11$\wedge$ FI12$\wedge$ FI13 $\not\Rightarrow$ FI6

4.2.2 Getting FI7 from the Other Axioms

Proposition 4.9. *A fuzzy implication I satisfying FI6, FI8, FI9, FI10, FI11, FI12 and FI13 does not necessarily satisfy FI7.*

Example 4.3 Let a fuzzy implication I_3 be represented by

$$I_3(x,y) = \begin{cases} 1 & \text{if } x \le y \\ (x-y)(y(1-x)-1)+1 & \text{if } x > y \end{cases}, \quad x,y \in [0,1].$$

For all x, $y \in [0,1]$,

$I_3(1,x) = x.$

$I_3(x, y) = 1$ iff $x \leq y$.

$N^{'}(x) = I_3(x, 0) = 1 - x$.

$I_3(x, y) \geq y$.

$I_3(x, x) = 1$.

$I_3(1 - y, 1 - x) = I_3(x, y)$.

I_3 is a continuous mapping.

So I_3 satisfies FI6, FI8, FI9, FI10, FI11, FI12 w.r.t. the standard strong fuzzy negation N_0, and FI13. However, take $x_0 = 0.3$, $y_0 = 0.9$ and $z_0 = 0.1$, we obtain $I(x_0, I(y_0, z_0)) = 0.9214$ and $I(y_0, I(x_0, z_0)) = 0.9210$. So I_3 does not satisfy FI7.

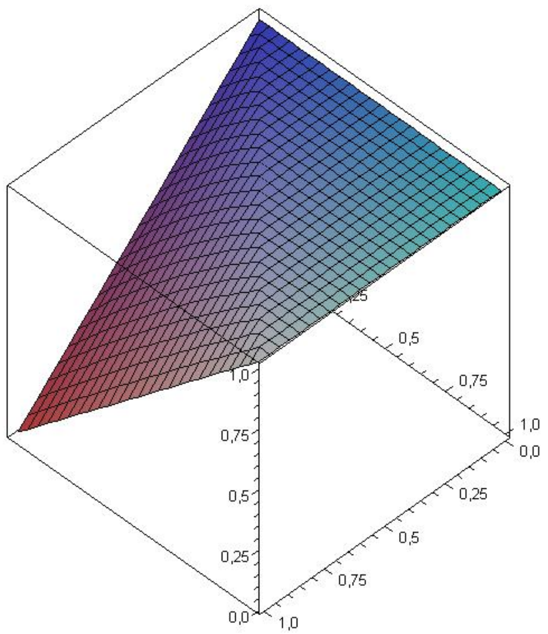

Figure 4.2: Example 4.3

Remark 4.10. The fuzzy implication I_{MM} presented in ([6], Table 1.5) is also an example that satisfies FI6, FI8, FI9, FI10, FI11, FI12 w.r.t. the standard strong fuzzy negation N_0, and FI13 but not FI7.

So FI7 is independent with any of the other 7 axioms.

4.2.3 Getting FI8 from the Other Axioms

Proposition 4.11. *A fuzzy implication I satisfying FI6, FI7, FI9, FI10, FI11, FI12 and FI13 does not necessarily satisfy FI8.*

Example 4.4 Given the strong fuzzy negation $N(x) = \sqrt{1-x^2}$, for all $x \in [0,1]$. The S-implication I_4 generated by the t-conorm $S_{\mathbf{L}}$ and the strong fuzzy negaton N is represented by

$$I_4(x,y) = S_{\mathbf{L}}(N(x),y) = \min(\sqrt{1-x^2}+y,1), x,y \in [0,1].$$

Because I_4 is an S-implication generated from a continuous t-conorm and a strong fuzzy negation, it satisfies FI6, FI7, FI9, FI10, FI12 and FI13 [24]. Moreover, for all x, $y \in [0,1]$, $I_4(x,x) = x$. So I_4 also satisfies FI11. However, take $x_0 = 0.5$ and $y_0 = 0.4$, we obtain $I(x_0,y_0) = 1$ while $x_0 > y_0$. So I_4 does not satisfy FI8.

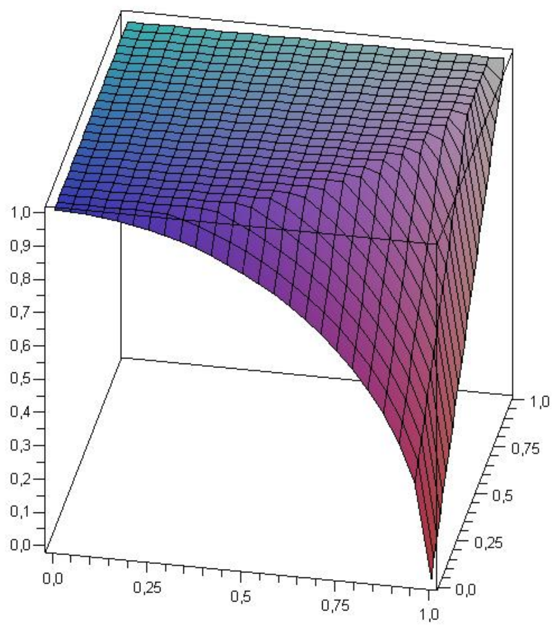

Figure 4.3: Example 4.4

So FI8 is independent with any of the other 7 axioms.

4.2.4 Getting FI9 from the Other Axioms

Proposition 4.12. *([6], Lemma 1.5.4(v)) A fuzzy implication I satisfying FI6 and FI12 satisfies FI9. Moreover, $N = N^{'}$.*

Corollary 4.13. *A fuzzy implication I satisfying FI7, FI8 and FI12 satisfies FI9. Moreover, $N = N^{'}$.*

PROOF. Straightforward from Propositions 4.3 and 4.12. □

Corollary 4.14. *A fuzzy implication I satisfying FI7, FI12 and FI13 satisfies FI9. Moreover, $N = N^{'}$.*

PROOF. Straightforward from Propositions 4.5 and 4.12. □

Proposition 4.15. *([2], Lemma 14)([24], Corollary 1.1) A fuzzy implication I satisfying FI7, FI8 and FI13 satisfies FI9.*

Remark 4.16. In Proposition 4.12, Corollary 4.13, Corollary 4.14 and Proposition 4.15 we considered the following 4 cases:

$$\text{FI6} \wedge \text{FI12} \Rightarrow \text{FI9}$$
$$\text{FI7} \wedge \text{FI8} \wedge \text{FI12} \Rightarrow \text{FI9}$$
$$\text{FI7} \wedge \text{FI8} \wedge \text{FI13} \Rightarrow \text{FI9}$$
$$\text{FI7} \wedge \text{FI12} \wedge \text{FI13} \Rightarrow \text{FI9}$$

So we still need to consider the following 5 cases:

$$\text{FI6} \wedge \text{FI7} \wedge \text{FI8} \wedge \text{FI10} \wedge \text{FI11} \overset{?}{\Rightarrow} \text{FI9}$$
$$\text{FI6} \wedge \text{FI7} \wedge \text{FI10} \wedge \text{FI11} \wedge \text{FI13} \overset{?}{\Rightarrow} \text{FI9}$$
$$\text{FI6} \wedge \text{FI8} \wedge \text{FI10} \wedge \text{FI11} \wedge \text{FI13} \overset{?}{\Rightarrow} \text{FI9}$$
$$\text{FI7} \wedge \text{FI10} \wedge \text{FI11} \wedge \text{FI12} \overset{?}{\Rightarrow} \text{FI9}$$
$$\text{FI8} \wedge \text{FI10} \wedge \text{FI11} \wedge \text{FI12} \wedge \text{FI13} \overset{?}{\Rightarrow} \text{FI9}$$

Proposition 4.17. *([24], Table 1.1) A fuzzy implication I satisfying FI6, FI7, FI8, FI10 and FI11 does not necessarily satisfy FI9.*

Example 4.5 The Gödel implication

$$I_g(x,y) = \begin{cases} 1, & \text{if } x \leq y \\ y, & \text{if } x > y \end{cases}, \quad x,y \in [0,1]. \tag{4.2}$$

is an R-implication generated by the continuous t-norm $T_{\mathbf{M}}$. So I_g satisfies FI6, FI7, FI8, FI10 and FI11 [24]. However we have for all $x \in [0,1]$,

$$N^{'}(x) = I_g(x,0) = \begin{cases} 1, & \text{if } x = 0 \\ 0, & \text{if } x > 0 \end{cases}.$$

So I_g does not satisfy FI9.

Proposition 4.18. *A fuzzy implication I satisfying FI6, FI7, FI10, FI11 and FI13 does not necessarily satisfy FI9.*

Example 4.6 Let a fuzzy implication I_5 be represented by

$$I_5(x,y) = \begin{cases} 1, & \text{if } x^2 \leq y \\ 1 - x^2 + y, & \text{if } x^2 > y \end{cases}, \quad x, y \in [0,1].$$

For all x, $y \in [0,1]$,

$I_5(1,x) = x.$

$I_5(x,y) \geq y.$

$I_5(x,x) = 1.$

I_5 is a continuous mapping.

So I_5 satisfies FI6, FI10, FI11 and FI13. Now we check axiom FI7 for I_5.
First consider the case that $y^2 \leq z$, then because I_5 satisfies FI10, $y^2 \leq I_5(x,z)$. So we obtain $I_5(x, I_5(y,z)) = I_5(x,1) = 1$ and $I_5(y, I_5(x,z)) = 1$. This is the same for the case that $x^2 \leq z$. Next we consider the case that $y^2 > z$ and $x^2 > z$. We have

$$I_5(x, I_5(y,z)) = \begin{cases} 1, & \text{if } x^2 \leq 1 - y^2 + z \\ 2 - x^2 - y^2 + z, & \text{else} \end{cases}, \quad x, y, z \in [0,1].$$

and

$$I_5(y, I(x,z)) = \begin{cases} 1, & \text{if } y^2 \leq 1 - x^2 + z \\ 2 - x^2 - y^2 + z, & \text{else} \end{cases}, \quad x, y, z \in [0,1].$$

Thus $I_5(x, I(y,z)) = I(y, I(x,z))$, for all x, y, $z \in [0,1]$, i.e., I_5 satisfies FI7. However, we have for all $x \in [0,1]$

$$N^{'}(x) = I_5(x,0) = 1 - x^2$$

which is not a strong fuzzy negation. So I_5 does not satisfy FI9

Proposition 4.19. *A fuzzy implication I satisfying FI6, FI8, FI10, FI11 and FI13 does not necessarily satisfy FI9.*

Example 4.7 Let a fuzzy implication I_6 be represented by

$$I_6(x,y) = \begin{cases} 1, & \text{if } x \leq y \\ \frac{(1-\sqrt{1-x})y}{x} + \sqrt{1-x}, & \text{if } x > y \end{cases}, \quad x, y \in [0,1].$$

For x, $y \in [0,1]$

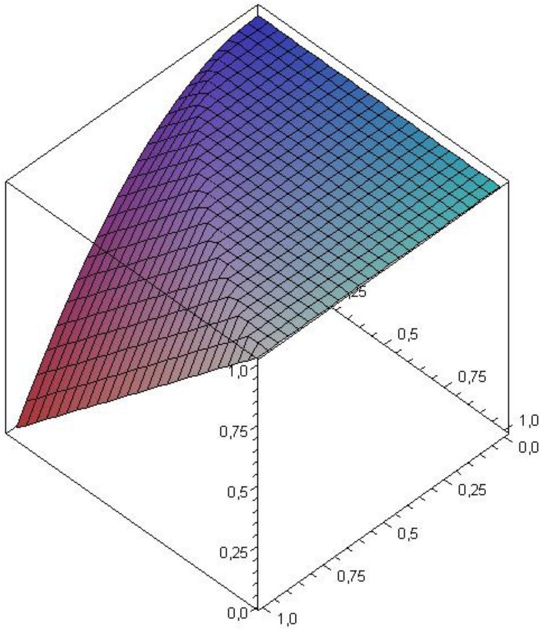

Figure 4.4: Example 4.6

$I_6(1, x) = x$.
$I_6(x, y) = 1$ iff $x \leq y$.
$I_6(x, y) \geq y$.
I_6 is a continuous mapping.

So I_6 satisfies FI6, FI8, FI10 and FI13. However, we have for all $x \in [0, 1]$

$$N'(x) = I_6(x, 0) = \sqrt{1 - x}$$

which is not a strong fuzzy negation. So I_6 does not satisfy FI9

Proposition 4.20. *A fuzzy implication I satisfying FI7, FI10, FI11 and FI12 does not necessarily satisfy FI9.*

The fuzzy implication I_1 stated in Example 4.1 satisfies FI7, FI10, FI11 and FI12. However, we have

$$N'(x) = I_1(x, 0) = \begin{cases} 1, & \text{if } x < 1 \\ 0, & \text{if } x = 1 \end{cases}, \quad x \in [0, 1],$$

So I_1 does not satisfy FI9.

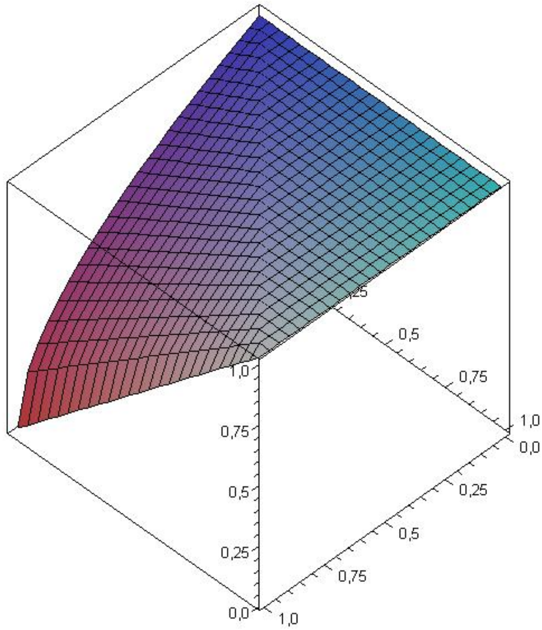

Figure 4.5: Example 4.7

Proposition 4.21. *A fuzzy implication satisfying FI8, FI10, FI11, FI12 and FI13 does not necessarily satisfy FI9.*

Example 4.8 Let a fuzzy implication I_7 be represented by

$$I_7(x,y) = \begin{cases} 1, & \text{if } x \le y \\ \sqrt{1-(x-y)}, & \text{if } x > y \end{cases}, \quad x, y \in [0,1].$$

For all x, $y \in [0,1]$,

$I_7(x,y) = 1$ iff $x \le y$.

$I_7(x,y) \ge y$.

$I_7(x,x) = 1$.

$I_7(1-y, 1-x) = I_7(x,y)$.

I_7 is a continuous mapping.

So I_7 satisfies FI8, FI10, FI11, FI12 w.r.t. the standard strong fuzzy negation N_0, and FI13. However, we have for all $x \in [0,1]$

$$N^{'}(x) = I_5(x,0) = \sqrt{1-x}$$

which is not a strong fuzzy negation. So I_7 does not satisfy FI9.

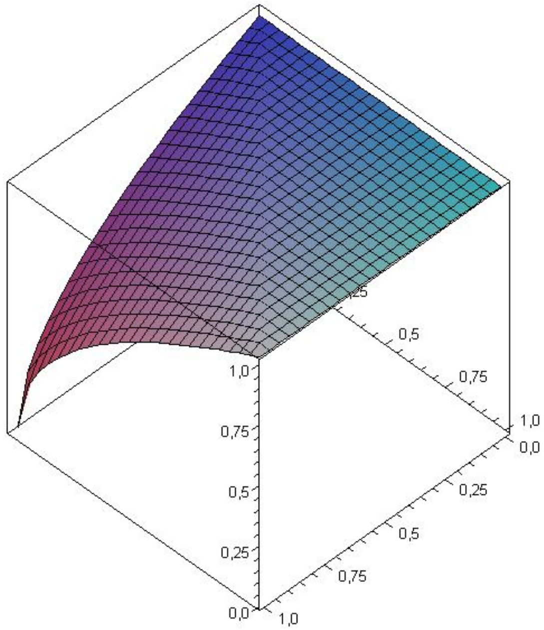

Figure 4.6: Example 4.8

Remark 4.22. The fuzzy implication I_{BZ} presented in ([6], Example 1.5.10(iv)) is also an example that satisfies FI8, FI10, FI11, FI12 w.r.t. the standard strong fuzzy negation N_0, and FI13 but not FI9.

So we considered all the possibilities that the fuzzy implication axiom FI9 can be implied from the other 7 axioms. Moreover we stated for each independent case a counter-example. We summary the results of this section in Table 4.2.

4.2.5 Getting FI10 from the Other Axioms

Proposition 4.23. *([10],Lemma 1 (viii)) A fuzzy implication I satisfying FI6 satisfies FI10.*

Corollary 4.24. *A fuzzy implication I satisfying FI7 and FI9 satisfies FI10.*

PROOF. Straightforward from Propositions 4.4 and 4.23. □

Corollary 4.25. *A fuzzy implication I satisfying FI7 and FI13 satisfies FI10.*

PROOF. Straightforward from Propositions 4.5 and 4.23. □

Table 4.2: Getting FI9 from the other axioms

FI6$\wedge$ FI12 $\Rightarrow$ FI9
FI7$\wedge$ FI8$\wedge$ FI12 $\Rightarrow$ FI9
FI7$\wedge$ FI12$\wedge$ FI13 $\Rightarrow$ FI9
FI7$\wedge$ FI8$\wedge$ FI13 $\Rightarrow$ FI9
FI6$\wedge$ FI7$\wedge$ FI8$\wedge$ FI10$\wedge$ FI11 $\not\Rightarrow$ FI9
FI7$\wedge$ FI10$\wedge$ FI11$\wedge$ FI12 $\not\Rightarrow$ FI9
FI6$\wedge$ FI7$\wedge$ FI10$\wedge$ FI11$\wedge$ FI13 $\not\Rightarrow$ FI9
FI6$\wedge$ FI8$\wedge$ FI10$\wedge$ FI11$\wedge$ FI13 $\not\Rightarrow$ FI9
FI8$\wedge$ FI10$\wedge$ FI11$\wedge$ FI12$\wedge$ FI13 $\not\Rightarrow$ FI9

Proposition 4.26. *([2],Lemma 6) A fuzzy implication I satisfying FI7 and FI8 satisfies FI10.*

Remark 4.27. In Proposition 4.23, Corollary 4.24, Corollary 4.25 and Proposition 4.26 we considered the following 4 cases:

FI6 $\Rightarrow$ FI10

FI7 $\wedge$ FI8 $\Rightarrow$ FI10

FI7 $\wedge$ FI9 $\Rightarrow$ FI10

FI7 $\wedge$ FI13 $\Rightarrow$ FI10

So we still need to consider the following 2 cases:

FI7 $\wedge$ FI11 $\wedge$ FI12 $\overset{?}{\Rightarrow}$ FI10

FI8 $\wedge$ FI9 $\wedge$ FI11 $\wedge$ FI12 $\wedge$ FI13 $\overset{?}{\Rightarrow}$ FI10

Proposition 4.28. *A fuzzy implication I satisfying FI7, FI11 and FI12 does not necessarily satisfy FI10.*

Example 4.9 Let a fuzzy implication I_8 be represented by

$$I_8(x,y) = \begin{cases} 1, & \text{if } x \leq 0.5 \text{ or } y \geq 0.5 \\ 0, & \text{else} \end{cases}, \quad x, y \in [0,1].$$

We obtain

$$I_8(x, I_8(y,z)) = \begin{cases} 1, & \text{if } x \leq 0.5 \text{ or } y \leq 0.5 \text{ or } z \geq 0.5 \\ 0, & \text{else} \end{cases}, \quad x, y \in [0,1].$$
$$= I_8(y, I_8(x,z))$$

So I_8 satisfies FI7. Moreover, for all $x, y \in [0,1]$,

$I_8(x,x) = 1.$

$I_8(1-y, 1-x) = I_8(x,y).$

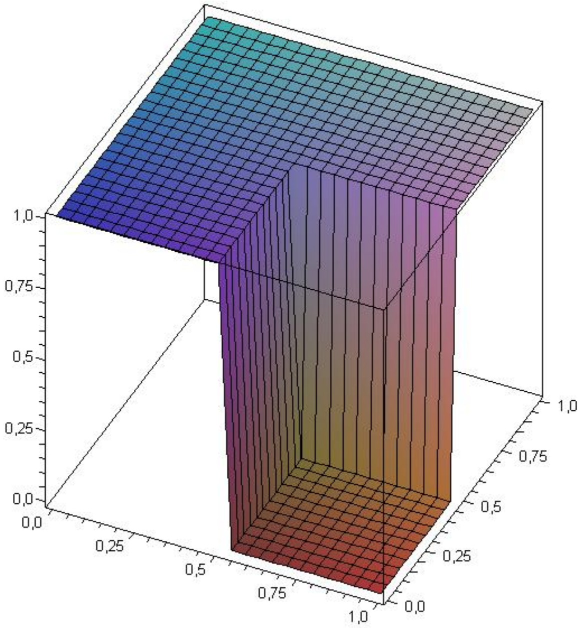

Figure 4.7: Example 4.9

So I_8 satisfies FI11 and FI12 w.r.t. the standard strong fuzzy negation N_0. However, take $x_0 = 1$ and $y_0 = 0.1$, we obtain $I_8(x_0, y_0) = 0 < y_0$. So I_8 does not satisfy FI10.

Proposition 4.29. *A fuzzy implication I satisfying FI8, FI9, FI11, FI12 and FI13 does not necessarily satisfy FI10.*

Example 4.10 Let a fuzzy implication I_9 be represented as

$$I_9(x,y) = \begin{cases} 1, & \text{if } x \le y \\ (1-\sqrt{x-y})^2, & \text{if } x > y \end{cases}, \quad x, y \in [0,1].$$

For all x, $y \in [0,1]$,

$I_9(x,y) = 1$ iff $x \le y$.

$N^{'}(x) = I_9(x,0) = (1-\sqrt{x})^2 = \varphi^{-1}(1-\varphi(x))$, where $\varphi(x) = \sqrt{x}$ is an automorphism of the unit interval.

$I_9(x,x) = 1$.

$I_9(1-y, 1-x) = I_9(x,y)$.

I_9 is a continuous mapping.

So I_9 satisfies FI8, FI9, FI11, FI12 w.r.t. the standard strong fuzzy negation N_0, and FI13. However, take $x_0 = 1$ and $y_0 = 0.64$, we obtain $I_9(x_0, y_0) = 0.16 < y_0$. So I_9 does not satisfy FI10.

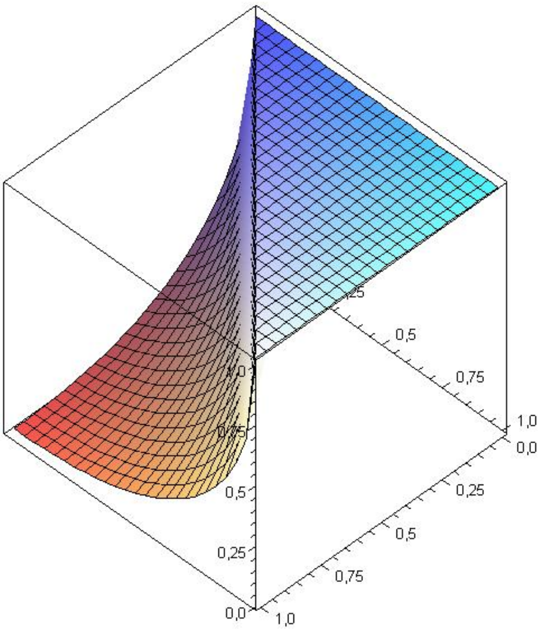

Figure 4.8: Example 4.10

So we considered all the possibilities that the fuzzy implication axiom FI10 can be implied from the other 7 axioms. Moreover we stated for each independent case a counter-example. We summary the results of this section in Table 4.3.

Table 4.3: Getting FI10 from the other axioms

FI6 $\Rightarrow$ FI10
FI7$\wedge$ FI9 $\Rightarrow$ FI10
FI7$\wedge$ FI13 $\Rightarrow$ FI10
FI7$\wedge$ FI8 $\Rightarrow$ FI10
FI7$\wedge$ FI11 $\wedge$ FI12 $\not\Rightarrow$ FI10
FI8$\wedge$ FI9$\wedge$ FI11$\wedge$ FI12 $\wedge$ FI13 $\not\Rightarrow$ FI10

4.2.6 Getting FI11 from the Other Axioms

Proposition 4.30. *A fuzzy implication I satisfying FI8 satisfies FI11.*

PROOF. Straightforward. □

Remark 4.31. In Proposition 4.30 we considered the following case:

FI8 ⇒ FI11.

So we still need to consider the following case:

FI6 ∧ FI7 ∧ FI9 ∧ FI10 ∧ FI12 ∧ FI13 $\overset{?}{\Rightarrow}$ FI11

Proposition 4.32. *A fuzzy implication I satisfying FI6, FI7, FI9, FI10, FI12 and FI13 does not necessarily satisfy FI11.*

Example 4.11 The Kleene-Dienes implication $I_b(x,y) = \max(1-x,y)$, for all $(x,y) \in [0,1]^2$ is an S-implication generated from the t-conorm $S_{\mathbf{M}}$ and the standard strong fuzzy negation N_0. So I_b satisfies FI6, FI7, FI9, FI10, FI12 w.r.t. the standard strong fuzzy negation N_0, and FI13. However, take $x_0 = 0.1$, we obtain $I_b(x_0,x_0) = 0.9 \neq 1$. So I_b does not satisfy FI11.

So we considered all the possibilities that the fuzzy implication axiom FI11 can be implied from the other 7 axioms, and stated for the independent case a counter-example.

4.2.7 Getting FI12 from the Other Axioms

Proposition 4.33. *([10],Lemma 1(ix)) A fuzzy implication I satisfying FI7 and FI9 satisfies FI12 w.r.t. the strong fuzzy negation N'.*

Proposition 4.34. *([2]) A fuzzy implication I satisfying FI7, FI8 and FI13 satisfies FI12.*

Remark 4.35. In Proposition 4.33 and Proposition 4.34 we considered the following 2 cases:

FI7 ∧ FI9 ⇒ FI12

FI7 ∧ FI8 ∧ FI13 ⇒ FI12

So we still need to consider the following 3 cases:

FI6 ∧ FI7 ∧ FI8 ∧ FI10 ∧ FI11 $\overset{?}{\Rightarrow}$ FI12

FI6 ∧ FI7 ∧ FI10 ∧ FI11 ∧ FI13 $\overset{?}{\Rightarrow}$ FI12

FI6 ∧ FI8 ∧ FI9 ∧ FI10 ∧ FI11 ∧ FI13 $\overset{?}{\Rightarrow}$ FI12

Proposition 4.36. *A fuzzy implication I satisfying FI6, FI7, FI8, FI10 and FI11 does not necessarily satisfy FI12.*

According to Example 4.5, the Gödel implication I_g satisfies FI6, FI7, FI8, FI10 and FI11. However, for any strong fuzzy negation N we obtain

$$I_g(N(y), N(x)) = \begin{cases} 1, & \text{if } x \le y \\ N(x), & \text{if } x > y \end{cases}.$$

In case that $N(x) \neq y$, $I_g(N(y), N(x)) \neq I_g(x, y)$. So I_g does not satisfy FI12 w.r.t. any strong fuzzy negation.

Proposition 4.37. *A fuzzy implication satisfying FI6, FI7, FI10, FI11 and FI13 does not necessarily satisfy FI12.*

The fuzzy implication I_5 stated in Example 4.6 satisfies FI6, FI7, FI10, FI11 and FI13. However, for any strong fuzzy negation N we obtain

$$I_5(N(y), N(x)) = \begin{cases} 1, & \text{if } (N(y))^2 \le N(x) \\ 1 - (N(y))^2 + N(x), & \text{if } (N(y))^2 > N(x) \end{cases}, \quad x, y \in [0, 1].$$

In case that $y \in]0, 1[$ and $(N(y))^2 < N(x) < N(y)$, $I_5(N(y), N(x)) = 1$ while $I_5(x, y) < 1$. So I_5 does not satisfy FI12 w.r.t. any strong fuzzy negation.

Proposition 4.38. *A fuzzy implication satisfying FI6, FI8, FI9, FI10, FI11 and FI13 does not necessarily satisfy FI12.*

Example 4.12 Let a fuzzy implication I_{10} be represented by

$$I_{10}(x, y) = \begin{cases} 1, & \text{if } x \le y \\ \frac{y + (x-y)\sqrt{1-x^2}}{x}, & \text{if } x > y \end{cases}, \quad x, y \in [0, 1].$$

For all x, $y \in [0, 1]$,

$I_{10}(1, x) = x$.

$I_{10}(x, y) = 1$ iff $x \le y$.

$N^{'}(x) = I_{10}(x, 0) = \sqrt{1 - x^2} = \varphi^{-1}(1 - \varphi(x))$, where $\varphi(x) = x^2$ is an automorphism of the unit interval.

$I_{10}(x, x) = 1$.

I_{10} is a continuous mapping.

So I_{10} satisfies FI6, FI8, FI9, FI11 and FI13. If I_{10} satisfies FI12 w.r.t. a strong fuzzy negation N, then for all $x \in [0, 1]$, we obtain

$$N(x) = I_{10}(1, N(x)) = I_{10}(x, 0) = N^{'}(x) = \sqrt{1 - x^2}.$$

However, take $x_0 = 0.8$ and $y_0 = 0.1$, we obtain $I_{10}(x_0, y_0) = 0.65$ and $I_{10}(N(y_0), N(x_0)) \approx 0.643$. So I_{10} does not satisfy FI12 w.r.t. any strong fuzzy negation N.

So we considered all the possibilities that the fuzzy implication axiom FI12 can be implied from the other 7 axioms. Moreover we stated for each independent case a counter-example. We summary the results of this section in Table 4.4.

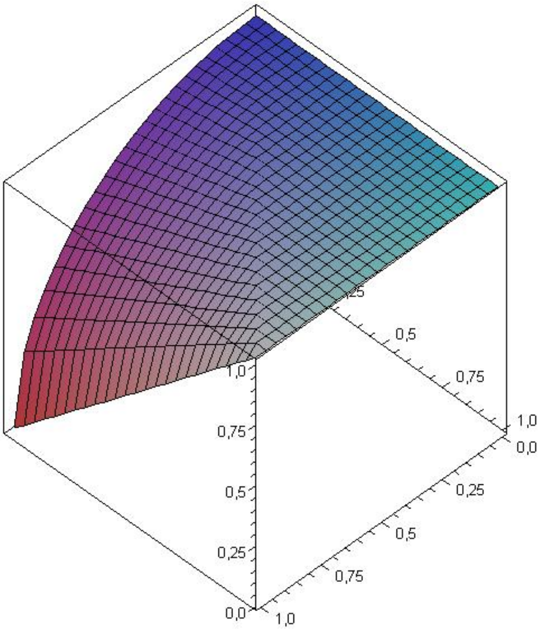

Figure 4.9: Example 4.12

Table 4.4: Getting FI12 from the other axioms

FI7 $\wedge$ FI9 $\Rightarrow$ FI12
FI7 $\wedge$ FI8 $\wedge$ FI13 $\Rightarrow$ FI12
FI6 $\wedge$ FI7 $\wedge$ FI8 $\wedge$ FI10 $\wedge$ FI11 $\not\Rightarrow$ FI12
FI6 $\wedge$ FI7 $\wedge$ FI10 $\wedge$ FI11 $\wedge$ FI13 $\not\Rightarrow$ FI12
FI6 $\wedge$ FI8 $\wedge$ FI9 $\wedge$ FI10 $\wedge$ FI11 $\wedge$ FI13 $\not\Rightarrow$ FI12

4.2.8 Getting FI13 from the Other Axioms

Proposition 4.39. *A fuzzy implication I satisfying FI6, FI7, FI8, FI9, FI10, FI11 and FI12 does not necessarily satisfy FI13.*

Example 4.13 Let N be a strong fuzzy negation. Recall the R_0-implication stated in [67] which is represented by

$$(I_{\min_0})_N(x,y) = \begin{cases} 1, & \text{if } x \le y \\ \max(N(x), y), & \text{if } x > y \end{cases}, \quad x, y \in [0,1].$$

$(I_{\min_0})_N$ is the R-implication generated by the left-continuous t-norm, nilpotent minimum [23]:

$$(T_{\min_0})_N(x,y) = \begin{cases} \min(x,y), & \text{if } y > N(x) \\ 0, & \text{if } y \le N(x) \end{cases}, \quad x,y \in [0,1].$$

$(I_{\min_0})_N$ satisfies FI6, FI7, FI8, FI9, FI10, FI11 and FI12 w.r.t. N, and is right-continuous in the second place [67] but it is not continuous.

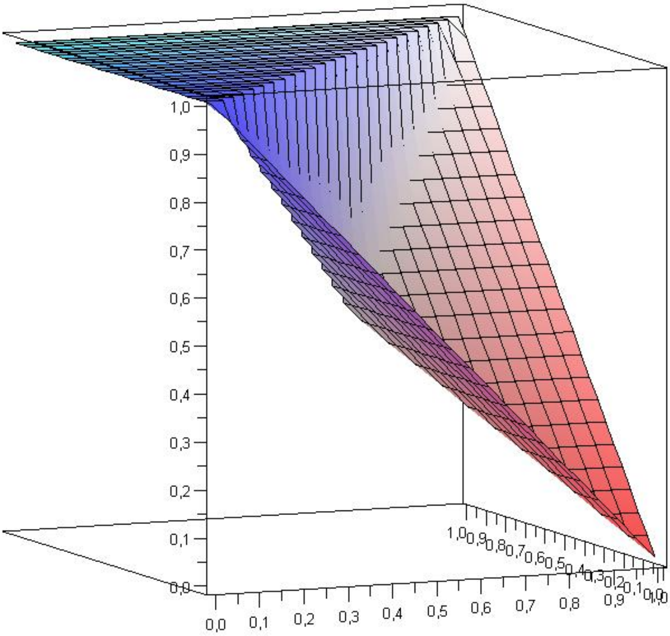

Figure 4.10: Example 4.13

So FI13 is independent with any of the other 7 axioms.

4.3 Summary

From Sections 4.2.2, 4.2.3 and 4.2.8 we see that axioms FI7, FI8 and FI13 are quite essential because they are totally independent from the other axioms. On the other hand, these three axioms are really important because the combination of them can imply all the other five axioms. From Section 4.2.6 we see that axiom FI11 is relatively essential because only FI8 can imply it. The combination of the other six axioms cannot imply FI11. However, none of the other axioms are dependent on FI11.

Table 4.5: Summary of the interrelationships between the eight axioms

FI7$\wedge$ FI10$\wedge$ FI11 $\wedge$ FI12 $\not\Rightarrow$ FI6 FI8$\wedge$ FI9$\wedge$ FI10$\wedge$ FI11$\wedge$ FI12$\wedge$ FI13 $\not\Rightarrow$ FI6 FI7$\wedge$ FI8 $\Rightarrow$ FI6 FI7$\wedge$ FI9 $\Rightarrow$ FI6 FI7$\wedge$ FI13 $\Rightarrow$ FI6
FI6$\wedge$ FI8$\wedge$ FI9$\wedge$ FI10$\wedge$ FI11$\wedge$ FI12$\wedge$ FI13 $\not\Rightarrow$ FI7
FI6$\wedge$ FI7$\wedge$ FI9$\wedge$ FI10$\wedge$ FI11$\wedge$ FI12$\wedge$ FI13 $\not\Rightarrow$ FI8
FI6$\wedge$ FI7$\wedge$ FI8$\wedge$ FI10$\wedge$ FI11 $\not\Rightarrow$ FI9 FI6$\wedge$ FI12 $\Rightarrow$ FI9 FI7$\wedge$ FI8$\wedge$ FI12 $\Rightarrow$ FI9 FI7$\wedge$ FI12$\wedge$ FI13 $\Rightarrow$ FI9 FI7$\wedge$ FI10$\wedge$ FI11$\wedge$ FI12 $\not\Rightarrow$ FI9 FI7$\wedge$ FI8$\wedge$ FI13 $\Rightarrow$ FI9 FI6$\wedge$ FI7$\wedge$ FI10$\wedge$ FI11$\wedge$ FI13 $\not\Rightarrow$ FI9 FI6$\wedge$ FI8$\wedge$ FI10$\wedge$ FI11$\wedge$ FI13 $\not\Rightarrow$ FI9 FI8$\wedge$ FI10$\wedge$ FI11$\wedge$ FI12$\wedge$ FI13 $\not\Rightarrow$ FI9
FI6 $\Rightarrow$ FI10 FI7$\wedge$ FI9 $\Rightarrow$ FI10 FI7$\wedge$ FI13 $\Rightarrow$ FI10 FI7$\wedge$ FI8 $\Rightarrow$ FI10 FI7$\wedge$ FI11$\wedge$ FI12 $\not\Rightarrow$ FI10 FI8$\wedge$ FI9$\wedge$ FI11$\wedge$ FI12$\wedge$ FI13 $\not\Rightarrow$ FI10
FI8 $\Rightarrow$ FI11 FI6$\wedge$ FI7$\wedge$ FI9$\wedge$ FI10$\wedge$ FI12$\wedge$ FI13 $\not\Rightarrow$ FI11
FI6$\wedge$ FI7$\wedge$ FI8$\wedge$ FI10$\wedge$ FI11 $\not\Rightarrow$ FI12 FI7$\wedge$ FI9 $\Rightarrow$ FI12 FI7$\wedge$ FI8$\wedge$ FI13 $\Rightarrow$ FI12 FI6$\wedge$ FI7$\wedge$ FI10$\wedge$ FI11$\wedge$ FI13 $\not\Rightarrow$ FI12 FI6$\wedge$ FI8$\wedge$ FI9$\wedge$ FI10$\wedge$ FI11$\wedge$ FI13 $\not\Rightarrow$ FI12
FI6$\wedge$ FI7$\wedge$ FI8$\wedge$ FI9$\wedge$ FI10$\wedge$ FI11$\wedge$ FI12 $\not\Rightarrow$ FI13

Table 4.5 summarizes the results we obtained in Sections 4.2.1-4.2.8.
Let S_1 denote a subset of

$$A = \{\mathrm{FI6}, \mathrm{FI7}, \mathrm{FI8}, \mathrm{FI9}, \mathrm{FI10}, \mathrm{FI11}, \mathrm{FI12}, \mathrm{FI13}\},$$

and

$$S_2 = A - S_1.$$

Then from Table 4.5 we can judge if a fuzzy implication satisfies all the axioms in S_1 then it also satisfies the axioms of S_2. For example, let

$$S_1 = \{\mathrm{FI7}, \mathrm{FI9}, \mathrm{FI11}\},$$

then

$$S_2 = \{\mathrm{FI6}, \mathrm{FI8}, \mathrm{FI10}, \mathrm{FI12}, \mathrm{FI13}\}.$$

According to rows 4, 7, 18, 26 and 30 of Table 4.5,

$$S_1 \Rightarrow \{\mathrm{FI6}, \mathrm{FI10}, \mathrm{FI12}\}.$$

Finally we summarize all the examples in this chapter in Table 4.6, where ‘Y’ denotes ‘yes’ and ‘N’ denotes ‘no’.

Table 4.6: Summary of the examples of fuzzy implications satisfying the indicated axioms

FI6	FI7	FI8	FI9	FI10	FI11	FI12	FI13	examples
N	Y	N	N	Y	Y	Y	N	I_1
N	N	Y	Y	Y	Y	Y	Y	I_2
Y	N	Y	Y	Y	Y	Y	Y	I_3
Y	Y	N	Y	Y	Y	Y	Y	I_4
Y	Y	N	N	Y	Y	N	Y	I_5
Y	N	Y	N	Y	Y	N	Y	I_6
N	N	Y	N	Y	Y	Y	Y	I_7
N	Y	N	N	N	Y	Y	N	I_8
N	N	Y	Y	N	Y	Y	Y	I_9
Y	N	Y	Y	Y	Y	N	Y	I_{10}
Y	Y	Y	N	Y	Y	N	N	I_g
Y	Y	N	Y	Y	N	Y	Y	I_b
Y	Y	Y	Y	Y	Y	Y	N	$I_{\min_0}$
Y	Y	Y	Y	Y	Y	Y	Y	$I_{\mathbf{L}}$

Chapter 5

Tautologies and a Functional Equation

In this Chapter and the next we will investigate for the three classes of fuzzy implications S-, R- and QL- implications, as well as the other fuzzy logic operators, their properties under different requirements in approximate reasoning.

5.1 Introduction

In [[69], Section 4], the authors analyzed some non-standard aspects in the construction of fuzzy set theory and dealt with the *derived boolean properties* of fuzzy operations, among which the *iterative boolean-like laws* [1] are considered as derived boolean laws not valid in any standard fuzzy set theories. In [1], the authors studied a class of *functional equations* ([[1], Definition 4.1]) with *the boolean background* ([1, Definition 4.2]), named as iterative boolean-like laws which are formulated in fuzzy logic where some variables appear several times because they come from boolean identities where no simplifications such as the application of idempotency or distributivity, absorption, etc. have been made. In [[1], Section 5], the authors analyzed some standard iterative boolean-like laws in fuzzy logic such as $(A \cup A) \cap co(A \cap A) = \emptyset$, $A \cup B = (A \cap B) \cup [(A \cup B) \cap co(A \cap B)]$, $(A \cup B \cup B) \cup (A \cap B \cap B) = A \cup B$, and solved the functional equations derived from them. But only laws containing fuzzy conjunctions, fuzzy disjunctions and fuzzy negations were considered. Since fuzzy implications are also fuzzy operations that play an important role in fuzzy logic, we will analyze some iterative boolean-like laws with fuzzy implications.
In classical binary logic, let $\wedge$ denote 'AND' and $\rightarrow$ denote IMPLY. Then because

$$\underbrace{(p \wedge p \wedge \cdots \wedge p)}_{n \quad \text{times} \quad \text{p}} \rightarrow q \equiv p \rightarrow q, \quad n = 1, 2, 3, \cdots,$$

and $\rightarrow$ and $\wedge$ are residuated,

$$\underbrace{p \rightarrow (p \rightarrow \cdots \rightarrow (p}_{n \quad \text{times} \quad \text{p}} \rightarrow q) \cdots) \equiv p \rightarrow q, \quad n = 1, 2, 3, \cdots \tag{5.1}$$

always holds. The inference scheme (5.1) means that repeating antecedents n times, the inference will remain the same. However, this equivalence does not hold for every fuzzy implication derived from classical binary logic. In order to judge if a fuzzy implication I has this feature in fuzzy inference, we need to investigate if I satisfies

$$I(x, y) = I(x, I(x, y)), \quad \forall x, y \in [0, 1]^2. \tag{5.2}$$

Authours of [10] have solved the functional equation (5.2) for the special case when $y = 0$. In this chapter we solve the functional equation (5.2) for fuzzy implications generated from fuzzy negations, t-norms, and t-conorms, for all $x, y \in [0, 1]$.

5.2 Solutions of the Functional Equation for S-, R- and QL-implications

Now we analyze under which conditions the three classes of fuzzy implications satisfy (5.2).

5.2.1 Solutions for S-implications

Theorem 5.1. *An S-implication I generated by a t-conorm S and a fuzzy negation N satisfies (5.2) iff the range of N is a subset of the set of idempotent elements of S.*

PROOF. $\Longleftarrow$: We obtain by S4

$$I(x, I(x, y)) = S(N(x), S(N(x), y)) = S(S(N(x), N(x)), y).$$

If the range of N is a subset of the set of idempotent elements of S, then for all $x \in [0, 1]$, $S(N(x), N(x)) = N(x)$ and hence I satisfies (5.2).
$\Longrightarrow$: By S1, for all $x \in [0, 1]$, $I(x, 0) = S(N(x), 0) = N(x)$ and

$$I(x, I(x, 0)) = S(N(x), S(N(x), 0)) = S(N(x), N(x)).$$

If I satisfies (5.2), then $I(x, 0) = I(x, I(x, 0))$, and hence
$N(x) = S(N(x), N(x))$, for all $x \in [0, 1]$. So for all $y \in rng(N)$, $S(y, y) = y$. □

The next corollary is a strong result for the condition that N refers to a continuous fuzzy negation in the above theorem.

Corollary 5.2. *An S-implication I generated by a t-conorm S and a continuous fuzzy negation N satisfies (5.2) iff $S = S_{\boldsymbol{M}}$.*

PROOF. $\Longleftarrow$: Straightforward.
$\Longrightarrow$: If N is continuous, then $rng(N) = [0,1]$. According to Theorem 5.1, I satisfies (5.2) iff the subset of the set of idempotent elements of S is $[0,1]$ i.e., S is an idempotent t-conorm. Since $\max$ is the only idempotent t-conorm, $S = S_{\mathbf{M}}$. $\square$

Example 5.1 1. Consider the fuzzy negation N_{1b}. Because the range of N_{1b} is $\{0,1\}$ and for every t-conorm, 0 and 1 are idempotent elements, when the S-implication I is generated by N_{1b} and any t-conorm, $I = I_{\mathbf{M}}$ and (5.2) holds.

2. The set of the idempotent elements of the nilpotent maximum $(S_{\max_0})_{N_0}$ is $[0,0.5[\cup \{1\}$. Consider the fuzzy negation
$N(x) = \begin{cases} 1 & x = 0 \\ f(x) & x \in]0,0.5] \\ 0 & x \in]0.5,1], \end{cases}$ where f denotes a strictly decreasing mapping satisfying $f(0) = 0.5$, $f(0.5) = 0$, for all $x \in [0,0.5]$. The range of N is equal to the set of the idempotent elements of S. So the S-implication generated by S and N satisfies (5.2).

5.2.2 Solutions for R-implications

Theorem 5.3. *A mapping $I : [0,1]^2 \rightharpoonup [0,1]$ satisfies the fuzzy implication axioms FI2, FI7, FI8, right-continuity in the second argument and (5.2) iff I is the Gödel implication defined in Table 2.6.*

PROOF. $\Longleftarrow$: By definition, the Gödel implication I_g satisfies FI2, FI7, FI8, right-continuity in the second argument and (5.2).
$\Longrightarrow$: Since I satisfies FI2, FI7, FI8 and right-continuity in the second argument, according to Corollary 3.12, a left-continuous t-norm T can be generated through equation 3.2. And if we denote an R-implication generated from T as I_T, then according to the proof of Theorem 1.14 in [24], $I_T = I$.
We will first show that $T(x,x) = x$, for all $x \in [0,1]$.
Indeed, from the formula (3.2), we have for all $x \in [0,1]$,

$$T(x,x) = \min\{t \in [0,1] | I(x,t) \geq x\}.$$

Obviously, when $t \in [x,1]$, $I(x,t) = 1 \geq x$ holds by FI8. Assume that there exists a $t_0 \in [0,x[$ such that $I(x,t_0) \geq x$. Then by FI8, $I(x,I(x,t_0)) = 1$. Since we have for all $(x,y) \in [0,1]^2$, $I(x,y) = I(x,I(x,y))$, $I(x,t_0) = I(x,I(x,t_0)) = 1$. And by FI8, $x \leq t_0$. This is a contradiction with the assumption that $t_0 < x$. So if $t^{'} \in \{t \in [0,1] | I(x,t) \geq x\}$, then $t^{'} \geq x$, i.e.,

$$T(x,x) = \min\{t \in [0,1] | I(x,t) \geq x\} = x, \forall x \in [0,1].$$

Hence, $T = T_{\mathbf{M}}$. Hence,

$$I(x,y) = \sup\{t \in [0,1] | \min(x,t) \leq y\}$$

$$= \begin{cases} 1 & x \le y \\ y & \text{otherwise} \end{cases}$$
$$= I_g(x, y).$$

□

Corollary 5.4. *An R-implication I generated by a left-continuous t-norm T satisfies (5.2) iff $T = T_{\mathbf{M}}$.*

PROOF. $\Longleftarrow$: The R-implication generated by $T = T_{\mathbf{M}}$ is the Gödel implication I_g and hence it satisfies (5.2).
$\Longrightarrow$: According to Theorem 3.11, an R-implication I satisfying FI2, FI7, FI8 and right-continuity in the second argument is generated by a left-continuous t-norm T. From Theorem 5.3, if I satisfies (5.2) additionally, it is the Gödel implication I_g. Denote T_{I_g} as the t-norm generated by I_g via (3.2) and $I_{T_{I_g}}$ as the R-implication generated by T_{I_g} via equation (2.55), then according to Theorem 3.11 and its corollary, $I_g = I_{T_{I_g}}$ i.e., I_g is generated by T_{I_g} through (2.55) where T_{I_g} is the t-norm generated by I_g through (3.2), that is $T_{I_g} = T_{\mathbf{M}}$. □

Remark 5.5. A continuous t-norm is also left-continuous. So Theorem 5.3 and its corollary are strong results for R-implications generated by a left-continuous t-norm. They are also proper for those R-implications generated by a continuous t-norm.

Remark 5.6. In [2], Theorem 1, which was first proposed by Smets and Magrez [80], shows that a mapping I: $[0, 1]^2 \rightharpoonup [0, 1]$ is continuous and satisfies FI2, FI7 and FI8 iff I is conjugate with I_a, which means the fuzzy implications being conjugate with I_a are the ones and only the ones to be continuous and to satisfy FI2, FI7 and FI8. Since $T_{\mathbf{M}}$ is conjugate only with itself, by Proposition 12 in [2], we know that I_g is also conjugate with itself. Thus Theorem 5.3 shows that the fuzzy implications being conjugate with I_g are the ones and only the ones to be right-continuous in the second argument and to satisfy FI2, FI7 and FI8. Thus Theorem 5.3 is an analogous result to Theorem 1 in [2].

5.2.3 Solutions for QL-implications

According to (2.57), a QL-implication I is generated by a t-conorm S, a t-norm T and a strong fuzzy negation N. In this section, let N denote any strong fuzzy negation. We suppose both S and T are continuous and intend to investigate all possible combinations of S and T and find the sufficient and necessary conditions for I to satisfy (5.2). In Table 5.1 we list all possible combinations of S and T.

Theorem 5.7. *A QL-implication I generated by the t-conorm $S_{\mathbf{M}}$, a continuous t-norm T and a strong fuzzy negation N satisfies (5.2) iff*

i) $T = T_{\mathbf{M}}$ *or*

Table 5.1: Nine possible combinations of S and T to generate a QL-implication

	$S = S_{\mathbf{M}}$	S is continuous Archimedean	S is an ordinal sum
$T = T_{\mathbf{M}}$	1	4	7
T is continuous Archimedean	2	5	8
T is an ordinal sum	3	6	9

ii) T is the ordinal sum of a family $\{[a_m, b_m], T_m\}$ where $\{[a_m, b_m]\}$ is a countable family of non-overlapping, closed, proper subintervals of $[0,1]$ with each T_m being a continuous Archimedean t-norm, and for all $[a_m, b_m]$, $b_m \leq e$, where e denotes the equilibrium point of N.

PROOF. Here we have

$$(\forall (x,y) \in [0,1]^2)(I(x,y) = \max(N(x), T(x,y)))$$

and

$$I(x, I(x,y)) = \max(N(x), T(x, \max(N(x), T(x,y)))).$$

$\Longleftarrow$:

i) Because $S_{\mathbf{M}}$ is distributive over $T_{\mathbf{M}}$, we obtain

$$\begin{aligned} I(x, I(x,y)) &= \max(N(x), \min(x, \max(N(x), \min(x,y)))) \\ &= \min(\max(N(x), x), \max(N(x), \max(N(x), \min(x,y)))) \\ &= \min(\max(N(x), x), \max(N(x), \min(x,y))) \\ &\quad \text{by S4 and the idempotency of } \max, \\ &= \min(\max(N(x), x), \min(\max(N(x), x), \max(N(x), y)) \\ &= \min(\max(N(x), x), \max(N(x), y)) \\ &\quad \text{by T4 and the idempotency of } \min \\ &= \max(N(x), \min(x,y)) = I(x,y). \end{aligned}$$

ii) Suppose T is the ordinal sum of the corresponding family $\{[a_m, b_m], T_m\}$ with each $b_m \leq e$. For an arbitrary pair $(x_0, y_0) \in [0,1]^2$, if for all $[a_m, b_m]$, $(x_0, y_0) \notin [a_m, b_m]^2$, then according to (2.30), $T(x_0, y_0) = \min(x_0, y_0)$, from the proof above we can see that

$$I(x_0, y_0) = I(x_0, I(x_0, y_0)).$$

If on the contrary there exists an interval $[a_m, b_m]$ such that $(x_0, y_0) \in [a_m, b_m]^2$, then since for $x_0 \leq b_m \leq e$, $N(x_0) \geq x_0$ holds, we have

$$N(x_0) \geq x_0 \geq T(x_0, y_0)$$

and

$$N(x_0) \geq x_0 \geq T(x_0, \max(N(x_0), T(x_0, y_0))).$$

Thus

$$I(x_0, I(x_0, y_0)) = N(x_0) = I(x_0, y_0).$$

Hence (5.2) holds for I.

$\Longrightarrow$: According to Theorem 2.19, if T is a continuous t-norm, then T is either $T_{\mathbf{M}}$, or continuous Archimedean, or the ordinal sum of a family $\{[a_m, b_m], T_m\}$. Thus we prove this part through proving that if T is a continuous Archimedean t-norm or the ordinal sum of a family $\{[a_m, b_m], T_m\}$ and there exists an interval $[a_m, b_m]$ such that $b_m > e$, then (5.2) does not hold.

i) Let T be continuous Archimedean. Since N is continuous, there always exists an $x_0 \in \,]0, 1[$ such that $N(x_0) < x_0$. For $y = 1$, we have

$$I(x_0, 1) = \max(N(x_0), x_0) = x_0$$

and

$$I(x_0, I(x_0, 1)) = \max(N(x_0), T(x_0, x_0)).$$

Since T is continuous Archimedean, $T(x_0, x_0) < x_0$. Thus (5.2) does not hold.

ii) Let T be the ordinal sum of a family $\{[a_m, b_m], T_m\}$ and suppose there exists $[a_m, b_m]$ such that $b_m > e$. Take $y = b_m$ and $x_0 \in \,]\max(a_m, e), b_m[$, then $N(x_0) < e < x_0$. Thus according to (2.30),

$$\begin{aligned} I(x_0, b_m) &= \max(N(x_0), T(x_0, b_m)) \\ &= \max(N(x_0), a_m + (b_m - a_m)T_m(\frac{x_0 - a_m}{b_m - a_m}, \frac{b_m - a_m}{b_m - a_m})) \\ &= \max(N(x_0), x_0) = x_0. \end{aligned}$$

Since T_m is continuous Archimedean and according to (2.30),

$$\begin{aligned} T(x_0, x_0) &= a_m + (b_m - a_m)T_m(\frac{x_0 - a_m}{b_m - a_m}, \frac{x_0 - a_m}{b_m - a_m}) \\ &< a_m + (b_m - a_m)\frac{x_0 - a_m}{b_m - a_m} = x_0. \end{aligned}$$

Thus

$$I(x_0, I(x_0, b_m)) = \max(N(x_0), T(x_0, x_0)) < x_0.$$

Hence (5.2) does not hold.

□

Remark 5.8. Theorem 5.7 treats the three possible combinations in the first column of Table 4.1:
CASE 1: a QL-implication generated by $S_{\mathbf{M}}$ and $T_{\mathbf{M}}$ satisfies (5.2);
CASE 2: a QL-implication generated by $S_{\mathbf{M}}$ and a continuous Archimedean T does not satisfy (5.2);
CASE 3: a QL-implication generated by $S_{\mathbf{M}}$ and an ordinal sum T satisfies (5.2) iff T satisfies the extra condition as mentioned in Theorem 5.7.

Theorem 5.9. *A QL-implication I generated by a continuous t-conorm S, the t-norm $T_{\mathbf{M}}$ and a strong fuzzy negation N satisfies (5.2) iff*

i) $S = S_{\mathbf{M}}$ *or*

ii) *S is the ordinal sum of a family $\{[a_m, b_m], S_m\}$ where $\{[a_m, b_m]\}$ is a countable family of non-overlapping, closed, proper subintervals of $[0,1]$ with each S_m being a continuous Archimedean t-conorm, and for all $[a_m, b_m]$, $a_m \geq e$, where e denotes the equilibrium point of N.*

PROOF. Here we have

$$(\forall (x,y) \in [0,1]^2)(I(x,y) = S(N(x), \min(x,y)))$$

and

$$I(x, I(x,y)) = S(N(x), \min(x, S(N(x), \min(x,y)))).$$

$\Longleftarrow$:

i) Straightforward from the proof of Theorem 5.7.

ii) Suppose S is the ordinal sum of the corresponding family $\{[a_m, b_m], S_m\}$ with each $a_m \geq e$. For an arbitrary $N(x_0) \in [0,1]$, if for all $[a_m, b_m]$, $N(x_0) \notin [a_m, b_m]$, then according to (2.48), for all $y \in [0,1]$, we have

$$I(x_0, y) = \max(N(x_0), \min(x_0, y))$$

and

$$I(x_0, I(x_0, y)) = \max(N(x_0), \min(x_0, \max(N(x_0), \min(x_0, y)))).$$

Thus according to the proof of Theorem 5.7,

$$I(x_0, y) = I(x_0, I(x_0, y)).$$

If on the contrary there exists $[a_m, b_m]$ such that $N(x_0) \in [a_m, b_m]$, then we have for all $y \in [0,1]$,

$$\min(x_0, y) \leq x_0 \leq e \leq a_m \leq N(x_0).$$

a) If $\min(x_0, y) = e$, then $x_0 = e = N(x_0) = a_m$. Thus according to (2.48),

$$I(x_0, y) = S(e, e) = e$$

and

$$I(x_0, I(x_0, y)) = S(e, \min(x_0, e)) = S(e, e) = e = I(x_0, y).$$

b) If $\min(x_0, y) < e$, then $\min(x_0, y) < a_m$. According to (2.48),

$$I(x_0, y) = \max(N(x_0), \min(x_0, y)) = N(x_0)$$

and

$$I(x_0, I(x_0, y)) = S(N(x_0), \min(x_0, N(x_0))) = S(N(x_0), x_0).$$

If $x_0 = e$, then $a_m = e = N(x_0)$. Thus according to (2.48),

$$I(x_0, I(x_0, y)) = S(e, e) = e = N(x_0) = I(x_0, y).$$

If $x_0 < e$, then $x_0 < a_m$. Thus according to (2.48),

$$I(x_0, I(x_0, y)) = \max(N(x_0), x_0) = N(x_0) = I(x_0, y).$$

Hence I satisfies (5.2). $\Longrightarrow$: According to Theorem 2.36, if S is a continuous t-conorm, then S is either $S_{\mathbf{M}}$, or continuous Archimedean, or the ordinal sum of a family $\{[a_m, b_m], S_m\}$. Thus we prove this part through proving that if S is continuous Archimedean or the ordinal sum of a family $\{[a_m, b_m], S_m\}$ and there exists an interval $[a_m, b_m]$ such that $a_m < e$, then (5.2) does not hold.

i) Let S be continuous Archimedean. Since N is continuous, there always exists $N(x_0) \in]0, 1[$ such that $N(x_0) < x_0$. For $y = 0$, we have $I(x_0, 0) = N(x_0)$ and

$$I(x_0, I(x_0, 0)) = S(N(x_0), \min(x_0, N(x_0))) = S(N(x_0), N(x_0)).$$

Since S is continuous Archimedean, $S(N(x_0), N(x_0)) > N(x_0)$. Thus (5.2) does not hold.

ii) Let S be the ordinal sum of a family $\{[a_m, b_m], S_m\}$ and suppose there exists $[a_m, b_m]$ such that $a_m < e$. Since N is continuous, there always exist $N(x_0) \in]a_m, \min(b_m, e)[$ and $x_0 > e > N(x_0)$. Take $y = 0$, since S_m is continuous Archimedean and according to (2.48),

$$\begin{aligned} I(x_0, I(x_0, 0)) &= S(N(x_0), \min(x_0, N(x_0))) = S(N(x_0), N(x_0)) \\ &= a_m + (b_m - a_m) S_m\left(\frac{N(x_0) - a_m}{b_m - a_m}, \frac{N(x_0) - a_m}{b_m - a_m}\right) \end{aligned}$$

$$> a_m + (b_m - a_m)\frac{N(x_0) - a_m}{b_m - a_m} = N(x_0).$$

Since $I(x_0, 0) = N(x_0)$, (5.2) does not hold.

□

Remark 5.10. Theorem 5.9 treats the three possible choices in the first row of Table 4.1:
CASE 1 a QL-implication generated by $S_{\mathbf{M}}$ and $T_{\mathbf{M}}$ satisfies (5.2);
CASE 4: a QL-implication generated by a continuous Archimedean S and $T_{\mathbf{M}}$ does not satisfy (5.2);
CASE 7: a QL-implication generated by an ordinal sum S and $T_{\mathbf{M}}$ satisfies (5.2) iff S satisfies some extra condition as mentioned in Theorem 5.9.

Lemma 5.11. *A necessary condition for a QL-implication I generated by a continuous Archimedean t-conorm S, a continuous t-norm T and a strong fuzzy negation N to satisfy (5.2) is the LEM (2.26).*

PROOF. Suppose there exists $x_0 \in [0, 1]$ such that $T(x_0, N(x_0)) > 0$, then $x_0 > 0$ and $N(x_0) > 0$. Since N is strong, $x_0 > 0$ implies $N(x_0) < 1$. According to (2.32) and (2.33), there exists a strictly increasing generator g and its pseudo-inverse $g^{(-1)}$ such that

$$S(N(x_0), T(x_0, N(x_0))) = g^{(-1)}(g(N(x_0)) + g(T(x_0, N(x_0)))) =$$
$$\begin{cases} g^{-1}(g(N(x_0)) + g(T(x_0, N(x_0)))) & g(N(x_0)) + g(T(x_0, N(x_0))) \in [0, g(1)] \\ 1 & \text{otherwise} \end{cases}$$

If $S(N(x_0), T(x_0, N(x_0))) = 1$, then $S(N(x_0), T(x_0, N(x_0))) > N(x_0)$.
If $S(N(x_0), T(x_0, N(x_0))) < 1$, then

$$S(N(x_0), T(x_0, N(x_0))) = g^{-1}(g(N(x_0)) + g(T(x_0, N(x_0)))).$$

Since g and g^{-1} are strictly increasing, $T(x_0, N(x_0)) > 0$ implies

$$g^{-1}(g(N(x_0)) + g(T(x_0, N(x_0)))) > g^{-1}(g(N(x_0))) = N(x_0)$$
$$\Rightarrow S(N(x_0), T(x_0, N(x_0))) > N(x_0).$$

According to (2.57), $I(x_0, 0) = N(x_0)$ and

$$I(x_0, I(x_0, 0)) = S(N(x_0), T(x_0, N(x_0))).$$

That is to say, $I(x_0, 0) < I(x_0, I(x_0, 0))$, and hence, I does not satisfy (5.2). □

Remark 5.12. According to Theorem 2.16, a continuous t-norm T satisfying $T(x, N(x)) = 0$, for all $x \in [0, 1]$ iff T is conjugate with the Łukasiewicz t-norm $T_{\mathbf{L}}$. Thus Lemma 5.11 treats the case 6 of Table 4.1: a QL-implication generated by a continuous Archimedean S and an ordinal sum T does not satisfy (5.2).

Lemma 5.13. *A necessary condition for a QL-implication I generated by a continuous Archimedean t-conorm S, a continuous Archimedean t-norm T and a strong fuzzy negation N to satisfy (5.2) is $S(x, N(x)) = 1$, for all $x \in [0, 1]$.*

PROOF. Suppose there exists $x_0 \in [0, 1]$ such that $S(x_0, N(x_0)) < 1$, then $x_0 \in \,]0, 1[$. According to (2.16) and (2.17),

$$\begin{aligned} &T(x_0, S(N(x_0), x_0)) = f^{(-1)}(f(x_0) + f(S(N(x_0), x_0))) \\ &= \begin{cases} f^{-1}(f(x_0) + f(S(N(x_0), x_0))) & f(x_0) + f(S(N(x_0), x_0)) \in [0, f(0)] \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Since f is strictly decreasing and $f(1) = 0$,

$$\begin{aligned} &S(N(x_0), x_0) < 1 \\ &\Longrightarrow f(S(N(x_0), x_0)) > 0. \end{aligned}$$

If $f(x_0) + f(S(N(x_0), x_0)) \in [0, f(0)]$, then

$$\begin{aligned} &T(x_0, S(N(x_0), x_0)) \\ &= f^{-1}(f(x_0) + f(S(N(x_0), x_0))) \\ &< f^{-1}(f(x_0)) = x_0. \end{aligned}$$

If

$$f(x_0) + f(S(N(x_0), x_0)) \in \quad]f(0), \infty],$$

then

$$T(x_0, S(N(x_0), x_0)) = 0 < x_0.$$

Thus $T(x_0, S(N(x_0), x_0)) < x_0$ always holds.
According to (2.32) and (2.33),

$$S(N(x_0), x_0) < 1 \Longrightarrow S(N(x_0), x_0) = g^{-1}(g(N(x_0)) + g(x_0)),$$

and we have $g(N(x_0)) + g(x_0) \in [0, g(1)[$.
Because $T(x_0, S(N(x_0), x_0)) < x_0$ and g is strictly increasing,

$$g(T(x_0, S(N(x_0), x_0))) < g(x_0).$$

Thus

$$\begin{aligned} &g(N(x_0)) + g(T(x_0, S(N(x_0), x_0))) < g(N(x_0)) + g(x_0) \\ \Longrightarrow &g(N(x_0)) + g(T(x_0, S(N(x_0), x_0))) \in [0, g(1)[. \end{aligned}$$

Thus we obtain

$$\begin{aligned}&S(N(x_0), T(x_0, S(N(x_0), x_0)))\\&= g^{-1}(g(N(x_0)) + g(T(x_0, S(N(x_0), x_0)))).\end{aligned}$$

Since g^{-1} is also strictly increasing and $g(N(x_0)) + g(x_0) \in [0, g(1)[$,

$$\begin{aligned}&g^{-1}(g(N(x_0)) + g(x_0)) > g^{-1}(g(N(x_0)) + g(T(x_0, S(N(x_0), x_0))))\\&\Longrightarrow S(N(x_0), x_0) > S(N(x_0), T(x_0, S(N(x_0), x_0))).\end{aligned}$$

According to (2.57), $I(x_0, 1) = S(N(x_0), x_0)$ and

$$I(x_0, I(x_0, 1)) = S(N(x_0), T(x_0, S(N(x_0), x_0))).$$

That is to say, $I(x_0, 1) > I(x_0, I(x_0, 1))$, and hence, I does not satisfy (5.2). □

Corollary 5.14. *If a QL-implication I generated by a continuous Archimedean t-conorm S, a continuous Archimedean t-norm T and a strong fuzzy negation N satisfies (5.2), then neither S nor T can be strict.*

PROOF. Straightforward from Lemma 5.11, Lemma 5.13, Theorem 2.16 and Theorem 2.33. □

Lemma 5.15. *If a QL-implication I generated by a continuous t-conorm S, a continuous Archimedean t-norm T and a strong fuzzy negation N satisfies (5.2), then there does not exist a countable family $\{[a_m, b_m]\}$ of non-overlapping, closed, proper subintervals of $[0, 1]$ such that S can be the ordinal sum of the corresponding family $\{[a_m, b_m], S_m\}$ with each S_m being a continuous Archimedean t-norm.*

PROOF. According to (2.57), take $y = 1$, then $I(x, 1) = S(N(x), x)$ and

$$I(x, I(x, 1)) = S(N(x), T(x, S(N(x), x))).$$

We will prove this lemma by contraposition.
Let e denote the equilibrium point of N and suppose there exists a countable family $\{[a_m, b_m]\}$ such that S can be the ordinal sum of the corresponding family $\{[a_m, b_m], S_m\}$ with each S_m being a continuous Archimedean t-norm.
If for all $[a_m, b_m]$, $[0, e] \not\subseteq [a_m, b_m]$, then since N is continuous, there always exists $N(x_0) \in]0, e[$ such that
$N(x_0) \notin [a_m, b_m]$ and $x_0 > e > N(x_0)$. Thus

$$I(x_0, 1) = S(N(x_0), x_0) = \max(N(x_0), x_0) = x_0.$$

And since T is continuous Archimedean and $x_0 \in]e, 1[$, $T(x_0, x_0) < x_0$. Thus by (2.57) and (2.48),

$$I(x_0, I(x_0, 1)) = S(N(x_0), T(x_0, x_0)) = \max(N(x_0), T(x_0, x_0)) < x_0.$$

Hence (5.2) does not hold.
If on the contrary there exists $[a_m, b_m]$ such that $[0, e] \subseteq [a_m, b_m]$, then we have $a_m = 0$ and $b_m \in [e, 1[$. Take $x_1 \in]b_m, 1[$, then $N(x_1) < N(b_m) \leq b_m < x_1$.
And since T is continuous Archimedean, $T(x_1, x_1) < x_1$.
Thus by (2.57) and (2.48), we have

$$I(x_1, 1) = S(N(x_1), x_1) = \max(N(x_1), x_1) = x_1,$$

and

$$\begin{aligned} I(x_1, I(x_1, 1)) &= I(x_1, x_1) = S(N(x_1), T(x_1, x_1)) \\ &= \begin{cases} a_m + (b_m - a_m) S_m\left(\frac{N(x_1) - a_m}{b_m - a_m}, \frac{T(x_1, x_1) - a_m}{b_m - a_m}\right) & T(x_1, x_1) \in [a_m, b_m] \\ \max(N(x_1), T(x_1, x_1)) & \text{otherwise} \end{cases} \end{aligned}$$

If $T(x_1, x_1) \in [a_m, b_m]$, then $I(x_1, I(x_1, 1)) \in [a_m, b_m]$, i.e., $I(x_1, I(x_1, 1)) < x_1$.
If $T(x_1, x_1) \notin [a_m, b_m]$, then

$$I(x_1, I(x_1, 1)) = \max(N(x_1), T(x_1, x_1)) < x_1.$$

Therefore we always have $I(x_1, I(x_1, 1)) < I(x_1, 1)$. Hence (5.2) does not hold. □

Remark 5.16. Lemma 5.15 treats the case 8 of Table 4.1: a QL-implication generated by an ordinal sum S and a continuous Archimedean T does not satisfy (5.2).

Hence besides the theorems and lemmas above, there are only two combinations of S and T to be investigated: the case 5 and the case 9. And from Corollary 5.14, we know that for the case 5, it is necessary that both S and T are nilpotent.

Theorem 5.17. *Let φ be an automorphism of the unit interval. A QL-implication I generated by the t-conorm $(S_\mathbf{L})_\varphi$, the t-norm $(T_\mathbf{L})_\varphi$ and a strong fuzzy negation N satisfies (5.2) iff $N = (N_0)_\varphi$.*

PROOF. According to Lemma 5.11 and Lemma 5.13, I satisfies (5.2) iff for all $x \in [0, 1]$, $T(x, N(x)) = 0$ and $S(N(x), x) = 1$. And according to Theorem 2.16 and Theorem 2.33, $T(x, N(x)) = 0$ and $S(N(x), x) = 1$ iff $N(x) = \varphi^{-1}(1 - \varphi(x))$. □

Remark 5.18. Actually, if there exists an automorphism φ of the unit interval such that the QL-implication I is generated by the t-conorm $(S_\mathbf{L})_\varphi$, the t-norm $(T_\mathbf{L})_\varphi$ and the strong fuzzy negation $(N_0)_\varphi$, then

$$\begin{aligned} I(x, y) &= (S_\mathbf{L})_\varphi((N_0)_\varphi(x), (T_\mathbf{L})_\varphi(x, y)) \\ &= \max(\varphi^{-1}(1 - \varphi(x)), y) = \max((N_0)_\varphi(x), y), \end{aligned}$$

i.e., I is an S-implication generated by $S_\mathbf{M}$ and the strong fuzzy negation $(N_0)_\varphi$, and hence according to Corollary 5.2, it satisfies (5.2).

Theorem 5.19. *Let φ and ϕ denote two different automorphisms of the unit interval. The QL-implication I generated by $(S_{\mathbf{L}})_\varphi$, the t-norm $(T_{\mathbf{L}})_\phi$ and a strong fuzzy negation N satisfies (5.2) iff N satisfies*

$$(\forall x \in [0,1])(\varphi^{-1}(1-\varphi(x)) \leq N(x) \leq \phi^{-1}(1-\phi(x))), \tag{5.3}$$

and I is expressed by

$$I(x,y) = \begin{cases} N(x) & T(x,y) = 0 \\ y & 0 < T(x,y) < \varphi^{-1}(1-\varphi(N(x))) \\ 1 & T(x,y) \geq \varphi^{-1}(1-\varphi(N(x))) \end{cases} . \tag{5.4}$$

PROOF. Here we have for all $(x,y) \in [0,1]^2$, $I(x,y) = S(N(x), T(x,y))$ and

$$I(x, I(x,y)) = S(N(x), T(x, S(N(x), T(x,y)))). \tag{5.5}$$

$\Longrightarrow$: Suppose that I satisfies (5.2), then according to Lemma 5.11, Lemma 5.13, Theorem 2.16 and Theorem 2.33, we have

$$\varphi^{-1}(1-\varphi(x)) \leq N(x) \leq \phi^{-1}(1-\phi(x)).$$

Hence (5.3) holds. Next we will prove that I satisfy (5.4).
According to (2.57), $T(x,y) = 0$ implies $I(x,y) = N(x)$. And

$$T(x,y) \geq \varphi^{-1}(1-\varphi(N(x)))$$

implies $I(x,y) = 1$. Thus in order to prove that I satisfies (5.4), we only need to prove:

$$0 < T(x,y) < \varphi^{-1}(1-\varphi(N(x))) \Rightarrow I(x,y) = y.$$

Indeed, if $T(x,y) < \varphi^{-1}(1-\varphi(N(x)))$, then

$$I(x,y) = \varphi^{-1}(\varphi(N(x)) + \varphi(T(x,y))) < 1.$$

If I satisfies (5.2), then $I(x, I(x,y)) < 1$. Thus according to (5.5), we have

$$I(x, I(x,y)) = \varphi^{-1}(\varphi(N(x)) + \varphi(T(x, \varphi^{-1}(\varphi(N(x)) + \varphi(T(x,y)))))).$$

Since $I(x,y) = I(x, I(x,y))$ and both φ and φ^{-1} are strictly increasing mappings, we have

$$T(x,y) = T(x, \varphi^{-1}(\varphi(N(x)) + \varphi(T(x,y)))).$$

Because $T(x,y) > 0$, $T(x,y) = \phi^{-1}(\phi(x) + \phi(y) - 1)$. And

$$\begin{aligned} &T(x, \varphi^{-1}(\varphi(N(x)) + \varphi(T(x,y)))) \\ &= \phi^{-1}(\phi(x) + \phi(\varphi^{-1}(\varphi(N(x)) + \varphi(T(x,y)))) - 1). \end{aligned}$$

Since both ϕ and ϕ^{-1} are strictly increasing mappings, we have

$$\begin{aligned} &y = \varphi^{-1}(\varphi(N(x)) + \varphi(T(x,y))) \\ &\Longrightarrow I(x,y) = y. \end{aligned}$$

Hence (5.4) holds.
$\Longleftarrow$: Suppose N and I satisfy (5.3) and (5.4), then we will prove that I satisfies (5.2). According to (5.4), if $T(x,y) = 0$, then $I(x,y) = N(x)$. And

$$I(x, I(x,y)) = I(x, N(x)) = S(N(x), T(x, N(x))).$$

Since N satisfies (5.3), $T(x, N(x)) = 0$, for all $x \in [0,1]$. Thus

$$S(N(x), T(x, N(x))) = S(N(x), 0) = N(x) = I(x,y).$$

If $T(x,y) \geq \varphi^{-1}(1 - \varphi(N(x)))$, then $I(x,y) = 1$. Thus

$$I(x, I(x,y)) = I(x,1) = S(N(x), x).$$

Since N satisfies (5.3), $S(N(x), x) = 1$ for all $x \in [0,1]$. Thus $I(x, I(x,y)) = 1 = I(x,y)$.
If $0 < T(x,y) < \varphi^{-1}(1 - \varphi(N(x)))$, then $I(x,y) = y$. Thus $I(x, I(x,y)) = I(x,y)$. Hence I satisfies (5.2). □

Remark 5.20. Corollary 5.14, Theorem 5.17 and Theorem 5.19 treat the case 5 of Table 4.1: a QL-implication generated by a continuous Archimedean S and a continuous Archimedean T satisfies (5.2) providing both are nilpotent and satisfy some extra conditions as stated.

Theorem 5.21. *Let $\{[a_m, b_m]\}$ and $\{[c_m, d_m]\}$ denote two countable families of non-overlapping, closed, proper subintervals of $[0,1]$. And let T be a t-norm which is the ordinal sum of the corresponding family $\{[a_m, b_m], T_m\}$ with each T_m being a continuous Archimedean t-norm and S be a t-conorm which is the ordinal sum of the corresponding family $\{[c_m, d_m], S_m\}$ with each S_m being a continuous Archimedean t-conorm. Then the QL-implication I generated by S, T and a strong fuzzy negation N satisfies (5.2) iff for all $[a_m, b_m]$, $b_m \leq e$ and for all $[c_m, d_m]$, $c_m \geq e$, where e denotes the equilibrium point of N.*

PROOF. We will first derive some special instances of $I(x, I(x,y))$. According to (2.57) and (5.5), for $y = 1$, we obtain $I(x,1) = S(N(x), x)$ and

$$I(x, I(x,1)) = S(N(x), T(x, S(N(x), x))).$$

And for $y = 0$, we obtain $I(x,0) = N(x)$ and

$$I(x, I(x,0)) = S(N(x), T(x, N(x))).$$

$\Longleftarrow$: In order to prove the '$\Longleftarrow$' part, we will consider three cases according to the position of x w.r.t. e:

i) For all $x < e$, we have $N(x) > e$, $x < N(x)$, and $x < c_m$, for all c_m. Since for all $y \in [0,1]$, $T(x,y) \le x < c_m$ and

$$T(x, S(N(x), T(x,y))) \le x < c_m,$$

according to (2.48), we have:

$$I(x,y) = S(N(x), T(x,y)) = \max(N(x), T(x,y)) = N(x)$$

and

$$\begin{aligned} &I(x, I(x,y)) = S(N(x), T(x, S(N(x), T(x,y)))) \\ &= \max(N(x), T(x, S(N(x), T(x,y)))) = N(x). \\ &\Rightarrow I(x,y) = I(x, I(x,y)). \end{aligned}$$

ii) For all $x > e$, we have $N(x) < e$, $x > N(x)$,

$$(\forall b_m)(x > b_m) \quad \text{and} \quad (\forall c_m)(N(x) < c_m).$$

Then by (2.48) and (2.30), for all $y \in [0,1]$,

$$I(x,y) = \max(N(x), \min(x,y))$$

and

$$I(x, I(x,y)) = \max(N(x), \min(x, \max(N(x), \min(x,y)))).$$

Thus according to the proof of Theorem 5.7, $I(x,y) = I(x, I(x,y))$.

iii) For $x = e$, we have $N(x) = e$ and

$$I(x,y) = I(e,y) = S(e, T(e,y)).$$

Two subcases will be considered depending now on the position of the variable y:

(1) $y \le e$.
If for all $[a_m, b_m]$, $(e,y) \notin [a_m, b_m]^2$, then according to (2.30), $T(e,y) = \min(e,y) = y$.
If on the contrary there exists an interval $[a_m, b_m]$ such that $(e,y) \in [a_m, b_m]^2$, then $b_m = e$. According to (2.30),

$$T(e,y) = a_m + (e - a_m) T_m\left(\frac{e - a_m}{e - a_m}, \frac{y - a_m}{e - a_m}\right) = y.$$

That is to say, $T(e,y) = y$, for all $y \le e$. Thus $I(x,y) = S(e,y)$.
And if for all $[c_m, d_m]$, $(e,y) \notin [c_m, d_m]^2$, then according to (2.48), $S(e,y) = \max(e,y) = e$.

If on the contrary there exists an interval $[c_m, d_m]$ such that $(e, y) \in [c_m, d_m]^2$, then $c_m = y = e$. According to (2.48),

$$S(e, y) = e + (d_m - e)S_m(\frac{e-e}{d_m - e}, \frac{y-e}{d_m - e}) = y = e.$$

That is to say, $I(x, y) = I(e, y) = e$, for all $y \leq e$.

(2) $y > e$.
In this case $y > b_m$. Thus according to (2.30), $T(e, y) = \min(e, y) = e$ and hence

$$I(x, y) = S(e, T(e, y)) = S(e, e).$$

Now we look at the position of c_m ($\geq e$) w.r.t. e.
If $c_m > e$, then according to (2.44), $S(e, e) = \max(e, e) = e$.
If $c_m = e$, then according to (2.48),

$$S(e, e) = e + (d_m - e)S_m(\frac{e-e}{d_m - e}, \frac{e-e}{d_m - e}) = e.$$

That is to say, $I(e, y) = e$, for all $y > e$. Hence $I(e, y) = e$, for all $y \in [0, 1]$. Therefore we have

$$I(x, I(x, y)) = I(e, I(e, y)) = S(e, T(e, I(e, y))) = S(e, T(e, e)).$$

Finally we consider the position of the upper bound b_m ($\leq e$) w.r.t. e.
If $b_m < e$, then according to (2.30), $T(e, e) = \min(e, e) = e$.
If $b_m = e$, then according to (2.30),

$$T(e, e) = a_m + (e - a_m)T_m(\frac{e - a_m}{e - a_m}, \frac{e - a_m}{e - a_m}) = e.$$

Thus $S(e, T(e, e)) = S(e, e) = e$, according to the proof above. Therefore

$$I(x, I(x, y)) = e = I(e, y) = I(x, y).$$

Hence we can conclude that I satisfies (5.2).
$\Longrightarrow$: The reverse implication (‘$\Longrightarrow$’) will be proved by contraposition.

i) First assume there exists an interval $[a_m, b_m]$ such that $b_m > e$.
If for all $[c_m, d_m]$, $c_m > N(b_m)$, then since N is continuous, there exists an x_0 such that $N(x_0) \in \,]N(b_m), \min(c_m, e)[$ and $x_0 \in \,]\max(a_m, e), b_m[$. Thus

$$I(x_0, 1) = S(N(x_0), x_0) = \max(N(x_0), x_0) = x_0$$

and

$$\begin{aligned}I(x_0, I(x_0,1)) &= S(N(x_0), T(x_0, S(N(x_0), x_0)))\\&= S(N(x_0), T(x_0, x_0))\\&= \max(N(x_0), T(x_0, x_0)).\end{aligned}$$

Since T_m is continuous Archimedean and according to (2.30),

$$T(x_0, x_0) = a_m + (b_m - a_m)T_m(\frac{x_0 - a_m}{b_m - a_m}, \frac{x_0 - a_m}{b_m - a_m}) < x_0.$$

Thus $I(x_0, I(x_0,1)) < x_0$. Hence (5.2) does not hold.
If on the contrary there exists an interval $[c_m, d_m]$ such that $c_m \leq N(b_m)$, then $c_m < e$. Since N is continuous, there exists an x_1 such that $N(x_1) \in]c_m, \min(d_m, e)[$ and $x_1 \in]b_m, 1[$. Thus

$$\begin{aligned}I(x_1, I(x_1,0)) &= S(N(x_1), T(x_1, N(x_1)))\\&= S(N(x_1), \min(x_1, N(x_1)))\\&= S(N(x_1), N(x_1)).\end{aligned}$$

Since S_m is continuous Archimedean and according to (2.48),

$$\begin{aligned}&S(N(x_1), N(x_1))\\&= c_m + (d_m - c_m)S_m(\frac{N(x_1) - c_m}{d_m - c_m}, \frac{N(x_1) - c_m}{d_m - c_m})\\&> N(x_1).\end{aligned}$$

And

$$I(x_1, 0) = N(x_1) < I(x_1, I(x_1, 0)).$$

Thus (5.2) does not hold.
So we have proved that if there exists an interval $[a_m, b_m]$ such that $b_m > e$, then (5.2) does not hold.

ii) Second assume that there exists an interval $[c_m, d_m]$ such that $c_m < e$ and for all $[a_m, b_m]$, $b_m \leq e$.
Since N is continuous, there exists an x_0 such that $N(x_0) \in]c_m, \min(d_m, e)[$ and $x_0 > b_m$. Then $I(x_0, 0) = N(x_0)$ and according to (2.30),

$$\begin{aligned}I(x_0, I(x_0,0)) &= S(N(x_0), T(x_0, N(x_0)))\\&= S(N(x_0), \min(x_0, N(x_0))) = S(N(x_0), N(x_0))\end{aligned}$$

Since S_m is continuous Archimedean and according to (2.48),

$$S(N(x_0), N(x_0))$$

$$= c_m + (d_m - c_m)S_m(\frac{N(x_0) - c_m}{d_m - c_m}, \frac{N(x_0) - c_m}{d_m - c_m})$$
$$> N(x_0).$$

Thus (5.2) does not hold.

□

Remark 5.22. Theorem 5.21 treats the case 9 of Table 4.1: a QL-implication generated by an ordinal sum S and an ordinal sum T satisfies (5.2) providing both satisfy some extra conditions as stated.

5.3 Summary

In this chapter, we studied an iterative boolean-like law in which fuzzy implications are concerned, namely whether $I(x, I(x, y)) = I(x, y)$ holds for all $(x, y) \in [0, 1]^2$ or not where I denotes a fuzzy implication derived from classical logic, which is generated by t-norms, t-conorms and fuzzy negations.
In Section 5.2 we gave sufficient and necessary conditions for an S-implication generated by any t-conorm and any fuzzy negation, an R-implication generated by a left continuous t-norm, and for a QL-implication generated by a continuous t-conorm, a continuous t-norm and a strong fuzzy negation to satisfy (5.2).
The standard t-norm T_M and the standard t-conorm S_M play important roles in the results of the investigated equation (5.2). An S-implication generated by a t-conorm S and a continuous fuzzy negation N satisfies equation (5.2) iff $S = S_M$. And an R-implication generated by a left-continuous t-norm T satisfies equation (5.2) iff $T = T_M$. But for a QL-implication generated by a fuzzy negation N, a t-norm T and a t-conorm S to satisfy equation (5.2), it is sufficient but not necessary for T to be T_M and S to be S_M. Instead, the equilibrium point of the strong fuzzy negation plays an important role in the solution of (5.2) for a QL-implication.

Chapter 6

Robustness of Fuzzy Logic Operators in Fuzzy Rule-based Systems

6.1 Introduction

Let X be a linguistic variable on the universe of discourse U, A_i and $A_i^{'}$ be fuzzy sets on U, Y be a linguistic variable on the universe of discourse V, B_i and $B_i^{'}$ be fuzzy sets on V, $i = 1, 2, \cdots, n$, $n \in \mathbb{N}$, respectively. In the approximate reasoning process of a fuzzy rule-based system, the ith single-input-single-output (SISO) IF-THEN rule in the rule base is represented as:

IF X is A_i THEN Y is B_i

and the input fuzzy set is $A_i^{'}$. We obtain the output fuzzy set $B_i^{'}$ through the generalized fuzzy modus ponens:

$$\begin{array}{c} \text{IF } X \text{ is } A_i \text{ THEN } Y \text{ is } B_i \\ X \text{ is } A_i^{'} \\ \hline Y \text{ is } B_i^{'}, \end{array}$$

This scheme is realized through using Zadeh's compositional rule of inference [100]

$$(\forall y \in V)(B_i^{'}(y) = \sup_{x \in U} T(A_i^{'}(x), R(A_i(x), B_i(y)))), \tag{6.1}$$

where T denotes a t-norm and R denotes a fuzzy relation on $U \times V$. If $A_i^{'}$ are singletons, i.e.,

$$A_i^{'}(x) = \begin{cases} 1, & \text{if } x = x_0 \\ 0, & \text{if } x \neq x_0, \end{cases},$$

then (6.1) becomes

$$(\forall y \in V)(B_i^{'}(y) = R(A_i(x_0), B_i(y))). \tag{6.2}$$

We then use

$$(\forall y \in V)(B^{'}(y) = Agg_i B_i^{'}(y)) \tag{6.3}$$

to obtain the final output fuzzy set B on V, where Agg denotes an aggregation operator. In a conjunction-based system, R is a t-norm and Agg is a t-conorm, and in an implication-based system, R is a fuzzy implication and Agg is a t-norm. Many authors considered the robustness of the approximate reasoning process of a fuzzy system (eg.,[12], [13], [46]), namely the capability to against the perturbations on the input and output variables or on the definition of the rules. In [46], the authors defined a *point-wise sensitivity*, a *maximum δ-sensitivity* and an *average δ-sensitivity* to estimate the robustness of the fuzzy connectives. They considered the perturbations of the input values as bounded values and mainly analyzed the maximum perturbations of the output value. We intend to regard the perturbations of the input values not only as bounded unknown values, but also as values being uniform distributions.

If the input fuzzy sets are singletons and there are small perturbations on the input variable, i.e., there are small perturbations on the first variable of R in (6.2), then we require that $B_i^{'}$ in (6.2) does not change much. If there are small perturbations on $B_i^{'}$ then we require that $B^{'}$ obtained in (6.3) does not change much. If there are small perturbations on the definition of the rules, i.e., there are small perturbations on the two variables of R in (6.1), then we require that $B_i^{'}$ in (6.1) does not change much. That is our motivation to investigate the capability of a t-norm, a t-conorm or a fuzzy implication against perturbations either on its first variable or on its second variable or on both variables. We investigate in Sections 6.2 and 6.3 the robustness of the most important continuous t-norms, t-conorms and fuzzy implications against unknown bounded perturbation and uniformly distributed perturbation, respectively.

6.2 Robustness of Fuzzy Logic Operators against Bounded Unknown Perturbation

6.2.1 Robustness of Fuzzy Logic Operators against Bounded Unknown Perturbation on One Variable

Definition 6.1. Let F be a $[0,1]^2 \rightharpoonup [0,1]$ mapping, and δ be a real number that takes values in $[0, \frac{1}{2}]$ which is the maximal perturbation. Then the *supreme aberration* of F at point $(x, y) \in [0,1]^2$ against the bounded unknown perturbation on x is defined as:

$$SAB^1(F, \delta, x, y) = \sup_{x^{'} \in [\max(x-\delta,0), \min(x+\delta,1)]} |F(x^{'}, y) - F(x, y)|, \tag{6.4}$$

and the supreme aberration of F at point $(x,y) \in [0,1]^2$ against the bounded unknown perturbation on y is defined as:

$$SAB^2(F,\delta,x,y) = \sup_{y' \in [\max(y-\delta,0),\min(y+\delta,1)]} |F(x,y') - F(x,y)|.$$

Moreover, the *robustness measure* of F against the bounded unknown perturbation on the first variable of F is defined as:

$$RMB^1(F,\delta) = \int_0^1 \int_0^1 SAB^1(F,\delta,x,y)dxdy, \tag{6.5}$$

and the robustness measure of F against the bounded unknown perturbation on the second variable of F is defined as:

$$RMB^2(F,\delta) = \int_0^1 \int_0^1 SAB^2(F,\delta,x,y)dxdy. \tag{6.6}$$

Now we investigate the robustness measure against the bounded unknown perturbation on the first variable of the three most important continuous t-norms: the minimum $T_{\mathbf{M}}$, the product $T_{\mathbf{P}}$ and the Łukasiewicz t-norm $T_{\mathbf{L}}$. Assume the perturbation bound is always $\delta \in [0, \frac{1}{2}]$. In this case $\delta \leq 1 - \delta$.

1. The supreme aberration of $T_{\mathbf{M}}$ at $(x,y) \in [0,1]^2$ against the bounded unknown perturbation on x is:

   If $x \in [0,\delta]$ then

$$SAB^1(T_{\mathbf{M}},\delta,x,y) = \begin{cases} y & \text{if } y \in [0,x] \\ x & \text{if } y \in]x,2x] \\ y-x & \text{if } y \in]2x,x+\delta] \\ \delta & \text{if } y \in]x+\delta,1] \end{cases}$$

   If $x \in]\delta,1]$ then

$$SAB^1(T_{\mathbf{M}},\delta,x,y) = \begin{cases} 0 & \text{if } y \in [0,x-\delta] \\ y-x+\delta & \text{if } y \in]x-\delta,x] \\ \delta & \text{if } y \in]x,1] \end{cases}$$

   So

   If $x \in [0,\delta]$ then

$$\begin{aligned} &\int_0^1 SAB^1(T_{\mathbf{M}},\delta,x,y)dx \\ &= \int_0^x ydy + \int_x^{2x} xdy + \int_{2x}^{x+\delta} (y-x)dy + \int_{x+\delta}^1 \delta dy \\ &= x^2 - x\delta + \delta - \frac{1}{2}\delta^2. \end{aligned}$$

If $x \in]\delta, 1]$ then

$$\int_{x-\delta}^{x} y - x + \delta dy + \int_{x}^{1} \delta dy$$
$$= -x\delta + \delta + \frac{1}{2}\delta^2.$$

So the robustness measure of $T_{\mathbf{M}}$ against the bounded unknown perturbation on the first variable of $T_{\mathbf{M}}$ is:

$$RMB^1(T_{\mathbf{M}}, \delta)$$
$$= \int_0^{\delta} x^2 - x\delta + \delta - \frac{1}{2}\delta^2 dx + \int_{\delta}^{1} -x\delta + \delta + \frac{1}{2}\delta^2 dx$$
$$= -\frac{2}{3}\delta^3 + \frac{1}{2}\delta^2 + \frac{1}{2}\delta.$$

2. The supreme aberration of $T_{\mathbf{P}}$ at $(x, y) \in [0, 1]^2$ against the bounded unknown perturbation on x is:

$$RMB^1(T_{\mathbf{P}}, \delta)(\forall x \in [0, 1])(SAB^1(T_{\mathbf{P}}, \delta, x, y) = \delta y).$$

So the robustness measure of $T_{\mathbf{P}}$ against the bounded unknown perturbation on the first variable of $T_{\mathbf{P}}$ is:

$$\int_0^1 \int_0^1 SAB^1(T_{\mathbf{P}}, \delta, x, y) dx dy = \int_0^1 \delta y dy = \frac{1}{2}\delta.$$

3. The supreme aberration of $T_{\mathbf{L}}$ at $(x, y) \in [0, 1]^2$ against the bounded unknown perturbation on x is:

If $x \in [0, 1 - \delta]$ then

$$SAB^1(T_{\mathbf{L}}, \delta, x, y) = \begin{cases} 0 & \text{if } y \in [0, 1 - x - \delta] \\ x + y + \delta - 1 & \text{if } y \in]1 - x - \delta, 1 - x] \\ \delta & \text{if } y \in]1 - x, 1] \end{cases}$$

If $x \in]1 - \delta, 1]$ then

$$SAB^1(T_{\mathbf{L}}, \delta, x, y) = \begin{cases} y & \text{if } y \in [0, 1 - x] \\ 1 - x & \text{if } y \in]1 - x, 2 - 2x] \\ x + y - 1 & \text{if } y \in]2 - 2x, 1 - x + \delta] \\ \delta & \text{if } y \in]1 - x + \delta, 1] \end{cases}$$

So

If $x \in [0, 1-\delta]$ then

$$\begin{aligned}
&\int_0^1 SAB^1(T_{\mathbf{L}}, \delta, x, y)dx \\
&= \int_{1-x-\delta}^{1-x} (x+y+\delta-1)dy + \int_{1-x}^{1} \delta dy \\
&= \delta x + \frac{1}{2}\delta^2.
\end{aligned}$$

If $x \in]1-\delta, 1]$ then

$$\begin{aligned}
&\int_0^1 SAB^1(T_{\mathbf{L}}, \delta, x, y)dx \\
&= \int_0^{1-x} ydy + \int_{1-x}^{2-2x} (1-x)dy + \int_{2-2x}^{1-x+\delta} (x+y-1)dy + \int_{1-x+\delta}^{1} \delta dy \\
&= x^2 - 2x + \delta x - \frac{1}{2}\delta^2 + 1.
\end{aligned}$$

So the robustness measure of $T_{\mathbf{L}}$ against the bounded unknown perturbation on the first variable of $T_{\mathbf{L}}$ is:

$$\begin{aligned}
&RMB^1(T_{\mathbf{L}}, \delta) \\
&= \int_0^{1-\delta} (\delta x + \frac{1}{2}\delta^2)dx + \int_{1-\delta}^{1} (x^2 - 2x + \delta x - \frac{1}{2}\delta^2 + 1)dx \\
&= \frac{-2}{3}\delta^3 + \frac{1}{2}\delta^2 + \frac{1}{2}\delta.
\end{aligned}$$

We summarize the robustness measure of the three most important continuous t-norms against the bounded unknown perturbation on the first variable of them in Table 6.1 Because $-\frac{2}{3}\delta^3 + \frac{1}{2}\delta^2 > 0$, we obtain

Table 6.1: Robustness measure of the three most important continuous t-norms against the bounded unknown perturbation on the first variable of them

$RMB^1(T_{\mathbf{M}}, \delta) = -\frac{2}{3}\delta^3 + \frac{1}{2}\delta^2 + \frac{1}{2}\delta$
$RMB^1(T_{\mathbf{P}}, \delta) = \frac{1}{2}\delta$
$RMB^1(T_{\mathbf{L}}, \delta) = -\frac{2}{3}\delta^3 + \frac{1}{2}\delta^2 + \frac{1}{2}\delta$

$$(\forall \delta \in]0, \frac{1}{2}])(RMB^1(T_{\mathbf{P}}, \delta) < RMB^1(T_{\mathbf{M}}, \delta) = RMB^1(T_{\mathbf{L}}, \delta)).$$

This means that among the three t-norms $T_{\mathbf{M}}$, $T_{\mathbf{P}}$ and $T_{\mathbf{L}}$, $T_{\mathbf{P}}$ has the best robustness against unknown bounded perturbations on its first variable. $T_{\mathbf{M}}$ and $T_{\mathbf{L}}$ have the same robustness against unknown bounded perturbations on their first variable.
For the robustness measure of a t-norm against the bounded unknown perturbation on the second variable we have the following theorem.

Theorem 6.2. *Let T be a t-norm. Then*

$$(\forall \delta \in [0, \frac{1}{2}])(RMB^2(T, \delta) = RMB^1(T, \delta)).$$

PROOF. Because T is communicative, we obtain

$$RMB^2(T, \delta) = \int_0^1 \int_0^1 \sup_{y' \in [\max(y-\delta,0),\min(y+\delta,1)]} |T(x, y') - T(x, y)| dxdy$$
$$= \int_0^1 \int_0^1 \sup_{y' \in [\max(y-\delta,0),\min(y+\delta,1)]} |T(y', x) - T(y, x)| dxdy$$
$$= RMB^1(T, \delta).$$

□

According to Theorem 6.2 we obtain

$$(\forall \delta \in]0, \frac{1}{2}])(RMB^2(T_{\mathbf{P}}, \delta) < RMB^2(T_{\mathbf{M}}, \delta) = RMB^2(T_{\mathbf{L}}, \delta)).$$

This means that among the three t-norms $T_{\mathbf{M}}$, $T_{\mathbf{P}}$ and $T_{\mathbf{L}}$, $T_{\mathbf{P}}$ has the best robustness against unknown bounded perturbations on its second variable. $T_{\mathbf{M}}$ and $T_{\mathbf{L}}$ have the same robustness against unknown bounded perturbations on their second variable.
For the robustness measure of a t-conorm against the bounded unknown perturbation on the first or second variable we have the following theorem.

Theorem 6.3. *Let T be a t-norm and S be the dual t-conorm of T w.r.t. the standard fuzzy negation N_0. Then*

$$(\forall \delta \in [0, \frac{1}{2}])(RMB^1(S, \delta) = RMB^1(T, \delta)),$$

and

$$(\forall \delta \in [0, \frac{1}{2}])(RMB^2(S, \delta) = RMB^2(T, \delta)).$$

PROOF.

$$RMB^1(S, \delta) = \int_0^1 \int_0^1 \sup_{x' \in [\max(x-\delta,0),\min(x+\delta,1)]} |S(x', y) - S(x, y)| dxdy$$
$$= \int_0^1 \int_0^1 \sup_{x' \in [\max(x-\delta,0),\min(x+\delta,1)]} |1 - T(1 - x', 1 - y) - (1 - T(1 - x, 1 - y))| dxdy$$
$$= \int_0^1 \int_0^1 \sup_{x' \in [\max(x-\delta,0),\min(x+\delta,1)]} |T(1 - x', 1 - y) - T(1 - x, 1 - y)| dxdy$$

$$= \int_1^0 \int_1^0 \sup_{1-x' \in [\max(1-x-\delta,0),\min(1-x+\delta,1)]} |T(x^{'}, y) - T(x, y)| d(1-x) d(1-y)$$
$$= \int_0^1 \int_0^1 \sup_{x' \in [\max(x-\delta,0),\min(x+\delta,1)]} |T(x^{'}, y) - T(x, y)| dx dy$$
$$= RMB^1(T, \delta).$$

The proof for $RMB^2(S, \delta) = RMB^2(T, \delta)$ is similar. □

According to Theorem 6.3, we obtain

$$(\forall \delta \in]0, \frac{1}{2}])(RMB^1(S_{\mathbf{P}}, \delta) < RMB^1(S_{\mathbf{M}}, \delta) = RMB^1(S_{\mathbf{L}}, \delta)),$$

and

$$(\forall \delta \in]0, \frac{1}{2}])(RMB^2(RMB^2(S_{\mathbf{P}}, \delta) < S_{\mathbf{M}}, \delta) = RMB^2(S_{\mathbf{L}}, \delta)).$$

This means that among the three t-conorms $S_{\mathbf{M}}$, $S_{\mathbf{P}}$ and $S_{\mathbf{L}}$, $S_{\mathbf{P}}$ has the best robustness against unknown bounded perturbations on its first or second variable, while $S_{\mathbf{M}}$ and $S_{\mathbf{L}}$ have the same robustness against unknown bounded perturbations on their first or second variable.

S-implications generated by N_0 and the aforementioned t-norms are all continuous while R-implications generated by $T_{\mathbf{M}}$ and $T_{\mathbf{P}}$ are not continuous. The R-implication generated by $T_{\mathbf{L}}$ is the same as the S-implication generated by N_0 and $S_{\mathbf{L}}$. We now investigate the robustness measure of an S-implication against the bounded unknown perturbation on the first or second variable. We have the following theorem.

Theorem 6.4. *Let I be an S-implication generated by a t-conorm S and the standard fuzzy negation N_0. Then*

$$(\forall \delta \in [0, \frac{1}{2}])(RMB^1(I, \delta) = RMB^1(S, \delta)),$$

and

$$(\forall \delta \in [0, \frac{1}{2}])(RMB^2(I, \delta) = RMB^2(S, \delta)).$$

PROOF.

$$RMB^1(I, \delta) = \int_0^1 \int_0^1 \sup_{x' \in [\max(x-\delta,0),\min(x+\delta,1)]} |I(x^{'}, y) - I(x, y)| dx dy$$
$$= \int_0^1 \int_0^1 \sup_{x' \in [\max(x-\delta,0),\min(x+\delta,1)]} |S(1-x^{'}, y) - S(1-x, y)| dx dy$$
$$= \int_0^1 \int_1^0 \sup_{1-x' \in [\max(1-x-\delta,0),\min(1-x+\delta,1)]} |S(x^{'}, y) - S(x, y)| d(1-x) dy$$

$$= \int_0^1 \int_0^1 \sup_{x' \in [\max(x-\delta,0),\min(x+\delta,1)]} |S(x', y) - S(x, y)| dxdy$$
$$= RMB^1(S, \delta).$$

The proof for $RMB^2(I, \delta) = RMB^2(S, \delta)$ is similar. □

According to Theorem 6.4, we obtain

$$(\forall \delta \in]0, \frac{1}{2}])(RMB^1(I_r, \delta) < RMB^1(I_b, \delta) = RMB^1(I_{\mathbf{L}}, \delta)),$$

and

$$(\forall \delta \in]0, \frac{1}{2}])(RMB^2(I_r, \delta) < RMB^2(I_b, \delta) = RMB^2(I_{\mathbf{L}}, \delta)).$$

This means that among the three S-implications I_b, I_r and $I_{\mathbf{L}}$, I_r has the best robustness against unknown bounded perturbations on its first or second variable. I_b and $I_{\mathbf{L}}$ have the same robustness against unknown bounded perturbations on their first or second variable.

6.2.2 Robustness of Fuzzy Logic Operators against Bounded Unknown Perturbation on Two Variables

Definition 6.5. Let F be a $[0, 1]^2 \rightharpoonup [0, 1]$ mapping, and δ be a real number that takes values in $[0, \frac{1}{2}]$ which is the maximal perturbation. Then the supreme aberration of F at point $(x, y) \in [0, 1]^2$ against the bounded unknown perturbation is defined as:

$$SAB^{1,2}(F, \delta, x, y) = \sup_{x' \in [\max(x-\delta,0),\min(x+\delta,1)]} \sup_{y' \in [\max(y-\delta,0),\min(y+\delta,1)]} |F(x', y') - F(x, y)|. \tag{6.7}$$

Moreover, the robustness measure of F against the bounded unknown perturbation is defined as:

$$RMB^{1,2}(F, \delta) = \int_0^1 \int_0^1 SAB^{1,2}(F, \delta, x, y) dxdy. \tag{6.8}$$

For the robustness measure of t-norms against the unknown bounded perturbation we first have the following theorem.

Theorem 6.6. *Let T denote a continuous t-norm. Then*

$$(\forall \delta \in [0, \frac{1}{2}])(\forall (x, y) \in [0, 1]^2)(SAB^{1,2}(T, \delta, x, y) = \delta) \tag{6.9}$$

iff $T = T_M$.

PROOF. $\Longleftarrow$: First we have

$$SAB^{1,2}(T,\delta,x,y) = \max(T(x,y) - T(\max(x-\delta,0),\max(y-\delta,0)),$$
$$T(\min(x+\delta,1),\min(y+\delta,1)) - T(x,y)). \quad (6.10)$$

In (6.10) if $T = T_M$ then we obtain

$$\min(x,y) - \min(\max(x-\delta,0),\max(y-\delta,0))$$
$$= \begin{cases} \min(x,y)(<\delta), & \text{if } x-\delta<0 \text{ or } y-\delta<0 \\ \delta, & \text{otherwise} \end{cases}$$

and

$$\min(\min(x+\delta,1),\min(y+\delta,1)) - \min(x,y)$$
$$= \begin{cases} \max(1-x,1-y)(<\delta), & \text{if } x+\delta>1 \text{ and } y+\delta>1 \\ \delta, & \text{otherwise} \end{cases}.$$

Because $\delta \in [0,\frac{1}{2}]$, if (x,y) satisfies $x-\delta<0$ or $y-\delta<0$, then $x+\delta\leq 1$ or $y+\delta\leq 1$. Similarly, if (x,y) satisfies $x+\delta>1$ and $y+\delta>1$, then $x-\delta>0$ and $y-\delta>0$. Thus (6.10) is always equal to δ, i.e.,

$$(\forall\delta\in[0,\frac{1}{2}])(\forall((x,y)\in[0,1]^2)(SAB^{1,2}(T_M,\delta,x,y)=\delta).$$

$\Longrightarrow$: We have always

$$SAB^{1,2}(T,\delta,0,1) = \delta$$
$$SAB^{1,2}(T,\delta,0,0) = T(\delta,\delta)$$
$$SAB^{1,2}(T,\delta,1,1) = 1 - T(\delta,\delta).$$

Recall that a continuous t-norm is either T_M or a continuous Archimedean t-norm or an ordinal sum of the family $\{[a_m,b_m],T_m\}$ where $\{[a_m,b_m]\}$ is a countable family of non-overlapping, closed, proper subintervals of $[0,1]$ and T_m is a continuous Archimedean t-norm associated with each $[a_m,b_m]$ (see Theorem 2.19 in Chapter 2).
If T is Archimedean, then

$$(\forall\delta\in[0,\frac{1}{2}])(T(\delta,\delta)<\delta),$$

and

$$1 - T(1-\delta,1-\delta) > 1-(1-\delta) = \delta.$$

So

$$SAB^{1,2}(T,\delta,0,1,) \neq RM_T^{uk}(0,0,\delta) \neq RM_T^{uk}(1,1,\delta).$$

Thus $\exists(x,y) \in [0,1]^2$ such that $SAB^{1,2}(T,\delta,x,y) \neq \delta$, for all $\delta \in]0, \frac{1}{2}]$.
If T is the ordinal sum of the family $\{[a_m, b_m], T_m\}$, then there at least exists one subinterval $[a,b]$ of $[0,1]$, such that

$$(\forall (x,y) \in [a,b]^2)(T(x,y) = a + (b-a)T_A(\frac{x-a}{b-a}, \frac{y-a}{b-a})),$$

where T_A denotes a continuous Archimedean t-norm. Because $\delta \in [0, \frac{1}{2}]$, $1-\delta \in [0,5,1]$. So $\forall [a,b] \subset [0,1]$, there exists a $\delta \in [0, \frac{1}{2}]$ such that $\delta \in]a,b[$ or $1-\delta \in]a,b[$. Thus $SAB^{1,2}(T,\delta,0,0) < \delta$ or $SAB^{1,2}(T,\delta,1,1) > \delta$. So

$$SAB^{1,2}(T,\delta,0,1) \neq SAB^{1,2}(T,\delta,0,0)$$

or

$$SAB^{1,2}(T,\delta,0,1) \neq SAB^{1,2}(T,\delta,1,1).$$

Thus there exists $(x,y) \in [0,1]^2$ that $SAB^{1,2}(T,\delta,x,y) \neq \delta$, for all $\delta \in [0, \frac{1}{2}]$.
Hence if (6.9) holds, then $T = T_M$.

□

From Theorem 6.6, we obtain

$$(\forall \delta \in [0, \frac{1}{2}])(RMB^{1,2}(T_M, \delta) = \delta).$$

We use Matlab to calculate $RMB^{1,2}(T,\delta)$ for $\delta \in [0, \frac{1}{2}]$, where $T = T_M$, $T = T_p$ and $T = T_L$, and illustrate the result in Figure 6.1. We see that

$$(\forall \delta \in [0, \frac{1}{2}])(RMB^{1,2}(T_M,\delta) \leq RMB^{1,2}(T_P,\delta) \leq RMB^{1,2}(T_L,\delta)).$$

This means that among the three t-norms $T_{\mathbf{M}}$, $T_{\mathbf{P}}$ and $T_{\mathbf{L}}$, $T_{\mathbf{M}}$ has the best robustness against unknown bounded perturbations on its two variables, while $T_{\mathbf{L}}$ has the worst robustness against unknown bounded perturbations on its two variables.
For the robustness measure of a t-conorm against the bounded unknown perturbation we have the following theorem.

Theorem 6.7. *Let T be a t-norm and S be the dual t-conorm of T w.r.t. the standard fuzzy negation N_0. Then*

$$(\forall \delta \in [0, \frac{1}{2}])(RMB^{1,2}(S,\delta) = RMB^{1,2}(T,\delta)).$$

PROOF.

$$RMB^{1,2}(S,\delta) = \int_0^1 \int_0^1 \sup_{x' \in [\max(x-\delta,0), \min(x+\delta,1)]} \sup_{y' \in [\max(y-\delta,0), \min(y+\delta,1)]}$$

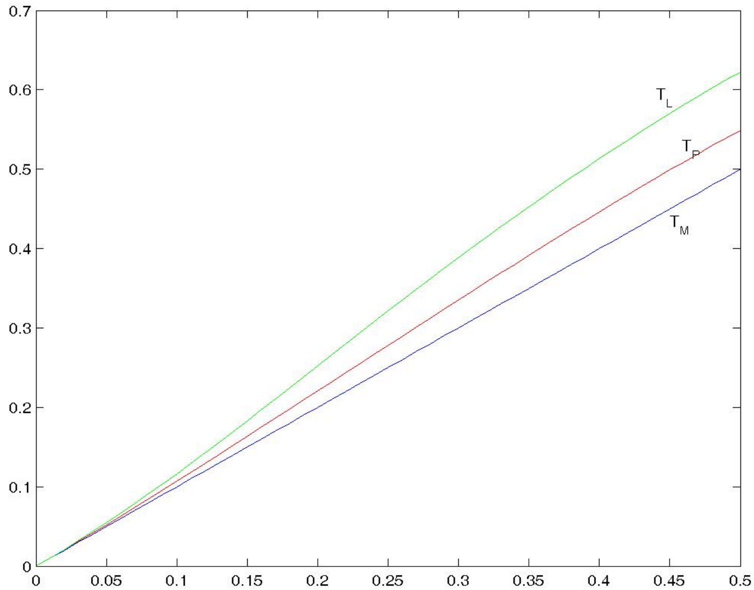

Figure 6.1: Robustness of t-norms against unknown bounded perturbation
x-axis: δ, y-axis: $RMB^{1,2}(T,\delta)$

$$|S(x^{'},y^{'})-S(x,y)|dxdy$$

$$=\int_0^1\int_0^1 \sup_{x^{'}\in[\max(x-\delta,0),\min(x+\delta,1)]}\ \sup_{y^{'}\in[\max(y-\delta,0),\min(y+\delta,1)]}$$

$$|1-T(1-x^{'},1-y^{'})-(1-T(1-x,1-y))|dxdy$$

$$=\int_0^1\int_0^1 \sup_{x^{'}\in[\max(x-\delta,0),\min(x+\delta,1)]}\ \sup_{y^{'}\in[\max(y-\delta,0),\min(y+\delta,1)]}$$

$$|T(1-x^{'},1-y^{'})-T(1-x,1-y)|dxdy$$

$$=\int_1^0\int_1^0 \sup_{1-x^{'}\in[\max(1-x-\delta,0),\min(1-x+\delta,1)]}\ \sup_{1-y^{'}\in[\max(1-y-\delta,0),\min(1-y+\delta,1)]}$$

$$|T(x^{'},y^{'})-T(x,y)|d(1-x)d(1-y)$$

$$=\int_0^1\int_0^1 \sup_{x^{'}\in[\max(x-\delta,0),\min(x+\delta,1)]}\ \sup_{y^{'}\in[\max(y-\delta,0),\min(y+\delta,1)]}$$

$$|T(x^{'},y^{'})-T(x,y)|dxdy$$

$$= RMB^{1,2}(T, \delta).$$

□

According to Theorem 6.7 we obtain

$$(\forall \delta \in [0, \frac{1}{2}])(RMB^{1,2}(S_M, \delta) \leq RMB^{1,2}(S_P, \delta) \leq RMB^{1,2}(S_L, \delta)).$$

This means that among the three t-conorms $S_{\mathbf{M}}$, $S_{\mathbf{P}}$ and $S_{\mathbf{L}}$, $S_{\mathbf{M}}$ has the best robustness against unknown bounded perturbations on its two variables, while $S_{\mathbf{L}}$ has the worst robustness against unknown bounded perturbations on its two variables.
For the robustness measure of an S-implication against the bounded unknown perturbation we have the following theorem.

Theorem 6.8. *Let I be an S-implication generated by a t-conorm S and the standard fuzzy negation N_0. Then*

$$(\forall \delta \in [0, \frac{1}{2}])(RMB^{1,2}(I, \delta) = RMB^{1,2}(S, \delta)).$$

PROOF.

$$RMB^{1,2}(I, \delta) = \int_0^1 \int_0^1 \sup_{x' \in [\max(x-\delta,0),\min(x+\delta,1)]} \sup_{y' \in [\max(y-\delta,0),\min(y+\delta,1)]}$$
$$|I(x', y') - I(x, y)|dxdy$$
$$= \int_0^1 \int_0^1 \sup_{x' \in [\max(x-\delta,0),\min(x+\delta,1)]} \sup_{y' \in [\max(y-\delta,0),\min(y+\delta,1)]}$$
$$|S(1 - x', y') - S(1 - x, y)|dxdy$$
$$= \int_1^0 \int_0^1 \sup_{1-x' \in [\max(1-x-\delta,0),\min(1+x+\delta,1)]} \sup_{y' \in [\max(y-\delta,0),\min(y+\delta,1)]}$$
$$|S(x', y') - S(x, y)|d(1 - x)dy$$
$$= \int_0^1 \int_0^1 \sup_{x' \in [\max(x-\delta,0),\min(x+\delta,1)]} \sup_{y' \in [\max(y-\delta,0),\min(y+\delta,1)]} |S(x', y') - S(x, y)|dxdy$$
$$= RMB^{1,2}(S, \delta).$$

□

According to Theorem 6.8 we obtain

$$(\forall \delta \in [0, \frac{1}{2}])(RMB^{1,2}(I_b, \delta) \leq RMB^{1,2}(I_r, \delta) \leq RMB^{1,2}(I_{\mathbf{L}}, \delta)).$$

This means that among the three S-implications I_b, I_r and $I_{\mathbf{L}}$, I_b has the best robustness against unknown bounded perturbations on its two variables, while $I_{\mathbf{L}}$ has the worst robustness against unknown bounded perturbations on its two variables.

6.3 Robustness of Fuzzy Logic Operators against Uniformly Distributed Perturbation

6.3.1 Robustness of Fuzzy Logic Operators against Uniformly Distributed Perturbation on One Variable

Definition 6.9. Let F be a $[0,1]^2 \rightharpoonup [0,1]$ mapping, and δ be a real number that takes values in $[0, \frac{1}{2}]$ which is the maximal perturbation. Then an *average aberration* of F at point $(x,y) \in [0,1]^2$ against the uniformly distributed perturbation on x is defined as:

$$AAU^1(F,\delta,x,y) = \frac{1}{\min(x+\delta,1) - \max(x-\delta,0)} \int_{\max(x-\delta,0)}^{\min(x+\delta,1)} |F(x^{'},y) - F(x,y)| dx^{'}, \tag{6.11}$$

and an average aberration of F at point $(x,y) \in [0,1]^2$ against the uniformly distributed perturbation on y is defined as:

$$AAU^2(F,\delta,x,y) = \frac{1}{\min(y+\delta,1) - \max(y-\delta,0)} \int_{\max(y-\delta,0)}^{\min(y+\delta,1)} |F(x,y^{'}) - F(x,y)| dy^{'}.$$

Moreover, a robustness measure of F against the uniformly distributed perturbation on the first variable of F is defined as:

$$RMU^1(F,\delta) = \int_0^1 \int_0^1 AAU^1(F,\delta,x,y) dx dy, \tag{6.12}$$

and a robustness measure of F against the uniformly distributed perturbation on the second variable of F is defined as:

$$RMU^2(F,\delta) = \int_0^1 \int_0^1 AAU^2(F,\delta,x,y) dx dy. \tag{6.13}$$

Here we investigate the robustness measure against the uniformly distributed perturbation on the first variable of the three most important continuous t-norms: the minimum $T_{\mathbf{M}}$, the product $T_{\mathbf{P}}$ and the Łukasiewicz t-norm $T_{\mathbf{L}}$. Assume the perturbation bound is always $\delta \in [0, \frac{1}{2}]$. In this case $\delta \leq 1 - \delta$.

1. The average aberration of $T_{\mathbf{M}}$ at $(x,y) \in [0,1]^2$ against the uniformly distributed perturbation on x is:

   If $x \in [0,\delta]$ then

$$AAU^1(T_{\mathbf{M}},\delta,x,y) = \begin{cases} 0 & \text{if } y \in [0, x-\delta] \\ \frac{1}{2\delta}(\frac{1}{2}x^2 + \frac{1}{2}y^2 - xy + x\delta - y\delta + \frac{1}{2}\delta^2) & \\ & \text{if } y \in \,]x-\delta, x] \\ \frac{1}{2\delta}(-\frac{1}{2}x^2 - \frac{1}{2}y^2 + xy - \delta x + \delta y + \frac{1}{2}\delta^2) & \\ & \text{if } y \in [x, x+\delta] \\ \frac{\delta}{2} & \text{if } y \in \,]x+\delta, 1]. \end{cases}$$

If $x \in]\delta, 1-\delta[$ then

$$AAU^1(T_{\mathbf{M}}, \delta, x, y) = \begin{cases} \frac{y^2}{2(x+\delta)} & \text{if } y \in [0, x] \\ \frac{1}{x+\delta}(-\frac{1}{2}y^2 + xy - x\delta + y\delta) & \text{if } y \in]x, x+\delta] \\ \frac{1}{x+\delta}(\frac{1}{2}x^2 + \frac{1}{2}\delta^2) & \text{if } y \in]x+\delta, 1]. \end{cases}$$

If $x \in [1-\delta, 1]$ then

$$AAU^1(T_{\mathbf{M}}, \delta, x, y) = \begin{cases} 0 & \text{if } y \in [0, x-\delta] \\ \frac{1}{1-x+\delta}(\frac{1}{2}x^2 + \frac{1}{2}y^2 - xy - x\delta + y\delta + \frac{1}{2}\delta^2) & \\ & \text{if } y \in]x-\delta, x] \\ \frac{1}{1-x+\delta}(\frac{1}{2}x^2 - \frac{1}{2}y^2 - x + y + \frac{1}{2}\delta^2) & \\ & \text{if } y \in]x, 1] \end{cases}.$$

So the robustness measure of $T_{\mathbf{M}}$ against the uniformly distributed perturbation on the first variable of $T_{\mathbf{M}}$ is:

$$\begin{aligned} RMU^1(T_{\mathbf{M}}, \delta) &= \int_0^1 \int_0^1 AAU^1(T_{\mathbf{M}}, \delta, x, y)dxdy \\ &= -\delta^3 - \frac{1}{4}\delta^2 + (ln2)\delta^2 + \frac{1}{4}\delta. \end{aligned}$$

2. The average aberration of $T_{\mathbf{P}}$ at $(x, y) \in [0,1]^2$ against the uniformly distributed perturbation on x is:

$$AAU^1(T_{\mathbf{P},x,y}, \delta) = \begin{cases} \frac{y}{x+\delta}(\frac{1}{2}x^2 + \frac{1}{2}\delta^2) & \text{if } x \in [0, \delta] \\ \frac{y}{2\delta} & \text{if } x \in]\delta, 1-\delta[\\ \frac{y}{1-x+\delta}(\frac{1}{2}x^2 - x + \frac{1}{2}\delta^2 + \frac{1}{2}) & \text{if } x \in [1-\delta, 1] \end{cases}.$$

So the robustness measure of $T_{\mathbf{P}}$ against the uniformly distributed perturbation on the first variable of $T_{\mathbf{P}}$ is:

$$\begin{aligned} RMU^1(T_{\mathbf{P}}, \delta) &= \int_0^1 \int_0^1 AAU^1(T_{\mathbf{M}}, \delta, x, y)dxdy \\ &= -\frac{3}{4}\delta^2 + (ln2)\delta^2 + \frac{1}{4}\delta. \end{aligned}$$

3. The average aberration of $T_{\mathbf{L}}$ at $(x, y) \in [0,1]^2$ against the uniformly distributed perturbation on x is:

If $x \in [0, \delta]$ then

$$AAU^1(T_{\mathbf{L}}, \delta, x, y) = \begin{cases} 0 & \text{if } y \in [0, 1-x-\delta] \\ \frac{1}{x+\delta}(\frac{1}{2}x^2 + \frac{1}{2}y^2 + xy - x - y + \delta x + \delta y + \frac{1}{2}\delta^2 & \\ -\delta + \frac{1}{2}) & \\ & \text{if } y \in]1-x-\delta, 1-x] \\ \frac{1}{x+\delta}(\frac{1}{2}x^2 - \frac{1}{2}y^2 + y + \frac{1}{2}\delta^2 - \frac{1}{2}) & \\ & \text{if } y \in]1-x, 1 \end{cases}.$$

If $x \in [\delta, 1-\delta]$ then

$$AAU^1(T_{\mathbf{L}}, \delta, x, y) = \begin{cases} 0 & \text{if } y \in [0, 1-x-\delta] \\ \frac{1}{2\delta}(\frac{1}{2}x^2 + \frac{1}{2}y^2 + xy - x - y + \delta x + \delta y + \frac{1}{2}\delta^2 \\ -\delta + \frac{1}{2}) & \text{if } y \in]1-x-\delta, 1-x] \\ \frac{1}{2\delta}(-\frac{1}{2}x^2 - \frac{1}{2}y^2 + x + y - xy + \delta x + \delta y - \delta - \frac{1}{2}) & \text{if } y \in]1-x, 1-x+\delta] \\ \frac{\delta}{2} & \text{if } y \in]1-x+\delta, 1] \end{cases}.$$

If $x \in]1-\delta, 1]$ then

$$AAU^1(T_{\mathbf{L}}, \delta, x, y) = \begin{cases} \frac{1}{2(1-x+\delta)}y^2 & \text{if } y \in [0, 1-x] \\ \frac{1}{2(1-x+\delta)}((1-x)^2 + \delta^2) & \text{if } y \in]1-x, 1] \end{cases}.$$

So the robustness measure of $T_{\mathbf{L}}$ against the uniformly distributed perturbation on the first variable of $T_{\mathbf{L}}$ is:

$$RMU^1(T_{\mathbf{L}}, \delta) = \int_0^1 \int_0^1 AAU^1(T_{\mathbf{L}}, \delta, x, y)dxdy$$
$$= \frac{1}{6}(ln2)\delta^3 - \frac{3}{4}\delta^2 + (ln2)\delta^2 + \frac{1}{4}\delta.$$

We summarize the robustness measure of the three most important continuous t-norms against the uniformly distributed perturbation on the first variable of them in Table 6.2.
Because $\delta \leq \frac{1}{2}$, $RMU^1(T_{\mathbf{P}}, \delta) \leq RMU^1(T_{\mathbf{M}}, \delta)$. Moreover, because $1 + \frac{1}{6}(ln2) > \frac{1}{2}$,

Table 6.2: Robustness measure of the three most important continuous t-norms against the uniformly distributed perturbation on the first variable of them

$RMU^1(T_{\mathbf{M}}, \delta) = -\delta^3 - \frac{1}{4}\delta^2 + (ln2)\delta^2 + \frac{1}{4}\delta$
$RMU^1(T_{\mathbf{P}}, \delta) = -\frac{3}{4}\delta^2 + (ln2)\delta^2 + \frac{1}{4}\delta$
$RMU^1(T_{\mathbf{L}}, \delta) = \frac{1}{6}(ln2)\delta^3 - \frac{3}{4}\delta^2 + (ln2)\delta^2 + \frac{1}{4}\delta$

$RMU^1(T_{\mathbf{M}}, \delta) < RMU^1(T_{\mathbf{L}}, \delta)$. Thus we obtain

$$(\forall \delta \in \ ]0, \frac{1}{2}[)(RMB^1(T_{\mathbf{M}}, \delta) < RMB^1(T_{\mathbf{P}}, \delta) < RMB^1(T_{\mathbf{L}}, \delta)).$$

This means that among the three t-norms $T_{\mathbf{M}}$, $T_{\mathbf{P}}$ and $T_{\mathbf{L}}$, $T_{\mathbf{P}}$ has the best robustness against uniformly distributed perturbations on its first variable, while $T_{\mathbf{L}}$ has the worst robustness against uniformly distributed perturbations on its first variable.
For the robustness measure of a t-norm against the uniformly distributed perturbation on the second variable we have the following theorem.

Theorem 6.10. *Let T be a t-norm. Then*

$$(\forall \delta \in [0, \frac{1}{2}])(RMU^2(T, \delta) = RMU^1(T, \delta)).$$

PROOF. The proof is similar to that of Theorem 6.2. □

According to Theorem 6.10 we obtain

$$(\forall \delta \in]0, \frac{1}{2}[)(RMU^2(T_{\mathbf{P}}, \delta) < RMU^2(T_{\mathbf{M}}, \delta) < RMU^2(T_{\mathbf{L}}, \delta)).$$

This means that among the three t-norms $T_{\mathbf{M}}$, $T_{\mathbf{P}}$ and $T_{\mathbf{L}}$, $T_{\mathbf{P}}$ has the best robustness against uniformly distributed perturbations on its second variable, while $T_{\mathbf{L}}$ has the worst robustness against uniformly distributed perturbations on its second variable.
For the robustness measure of a t-conorm against the uniformly distributed perturbation on the first or second variable we have the following theorem.

Theorem 6.11. *Let T be a t-norm and S be the dual t-conorm of T w.r.t. the standard fuzzy negation N_0. Then*

$$(\forall \delta \in [0, \frac{1}{2}])(RMU^1(S, \delta) = RMU^1(T, \delta)),$$

and

$$(\forall \delta \in [0, \frac{1}{2}])(RMU^2(S, \delta) = RMU^2(T, \delta)).$$

PROOF. The proof is similar to that of Theorem 6.3. □

According to Theorem 6.11, we obtain

$$(\forall \delta \in]0, \frac{1}{2}[)(RMU^1(S_{\mathbf{P}}, \delta) < RMU^1(S_{\mathbf{M}}, \delta) < RMU^1(S_{\mathbf{L}}, \delta)),$$

and

$$(\forall \delta \in]0, \frac{1}{2}[)(RMU^2(S_{\mathbf{P}}, \delta) < RMU^2(S_{\mathbf{M}}, \delta) < RMU^2(S_{\mathbf{L}}, \delta)).$$

This means that among the three t-conorms $S_{\mathbf{M}}$, $S_{\mathbf{P}}$ and $S_{\mathbf{L}}$, $S_{\mathbf{P}}$ has the best robustness against uniformly distributed perturbations on its first or second variable, while $S_{\mathbf{L}}$ has the worst robustness against uniformly distributed perturbations on its first or second variable.
For the robustness measure of an S-implication against the uniformly distributed perturbation on the first or second variable we have the following theorem.

Theorem 6.12. *Let I be an S-implication generated by a t-conorm S and the standard fuzzy negation N_0. Then*

$$(\forall \delta \in [0, \frac{1}{2}])(RMU^1(I, \delta) = RMU^1(S, \delta)),$$

and

$$(\forall \delta \in [0, \frac{1}{2}])(RMU^2(I, \delta) = RMU^2(S, \delta)).$$

PROOF. The proof is similar to that of Theorem 6.4. □

According to Theorem 6.12, we obtain

$$(\forall \delta \in]0, \frac{1}{2}[)(RMU^1(I_r, \delta) < RMU^1(I_b, \delta) < RMU^1(I_{\mathbf{L}}, \delta)),$$

and

$$(\forall \delta \in]0, \frac{1}{2}[)(RMU^2(I_r, \delta) < RMU^2(I_b, \delta) < RMU^2(I_{\mathbf{L}}, \delta)).$$

This means that among the three S-implications I_b, I_r and $I_{\mathbf{L}}$, I_r has the best robustness against uniformly distributed perturbations on its first or second variable, while $I_{\mathbf{L}}$ has the worst robustness against uniformly distributed perturbations on its first or second variable.

6.3.2 Robustness of Fuzzy Logic Operators against Uniformly Distributed Perturbation on Two Variables

Definition 6.13. Let F be a $[0,1]^2 \rightharpoonup [0,1]$ mapping, and δ be a real number that takes values in $[0, \frac{1}{2}]$ which is the maximal perturbation. Then an average aberration of F at point $(x, y) \in [0,1]^2$ against the uniformly distributed perturbation is defined as:

$$AAU^{1,2}(F, \delta, x, y) = \frac{1}{\min(x+\delta, 1) - \max(x-\delta, 0)} \frac{1}{\min(y+\delta, 1) - \max(y-\delta, 0)}$$

$$\int_{\max(x-\delta,0)}^{\min(x+\delta,1)} \int_{\max(y-\delta,0)}^{\min(y+\delta,1)} |F(x', y') - F(x, y)| dx' dy'. \quad (6.14)$$

Moreover, a robustness measure of F against the uniformly distributed perturbation of F is defined as:

$$RMU^{1,2}(F, \delta) = \int_0^1 \int_0^1 AAU^{1,2}(F, \delta, x, y) dx dy, \quad (6.15)$$

First we use Matlab to calculate $RMB^{1,2}(T, \delta)$ for $\delta \in [0, \frac{1}{2}]$, where $T = T_M$, $T = T_p$ and $T = T_L$, and illustrate the result in Figure 6.2. We see that

$$(\forall \delta \in [0, \frac{1}{2}])(RMB^{1,2}(T_L, \delta) \leq RMB^{1,2}(T_P, \delta) \leq RMB^{1,2}(T_M, \delta)).$$

This means that among the three t-norms $T_{\mathbf{M}}$, $T_{\mathbf{P}}$ and $T_{\mathbf{L}}$, $T_{\mathbf{L}}$ has the best robustness against uniformly distributed perturbations on its two variables, while $T_{\mathbf{M}}$ has the worst robustness against uniformly distributed perturbations on its two variables.
For the robustness measure of a t-conorm against the uniformly distributed perturbation we have the following theorem.

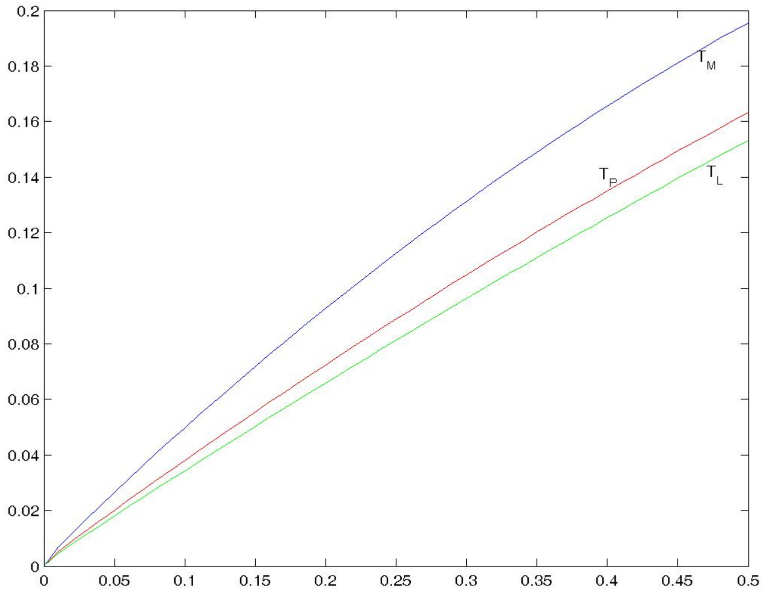

Figure 6.2: Robustness of t-norms against uniformly distributed perturbation
x-axis: δ, y-axis: $RMB^{1,2}(T, \delta)$

Theorem 6.14. *Let T be a t-norm and S be the dual t-conorm of T w.r.t. the standard fuzzy negation N_0. Then*

$$(\forall \delta \in [0, \frac{1}{2}])(RMU^{1,2}(S, \delta) = RMU^{1,2}(T, \delta)).$$

PROOF. The proof is similar to that of Theorem 6.7. □

According to Theorem 6.14, we obtain

$$(\forall \delta \in [0, \frac{1}{2}])(RMU^{1,2}(S_{\mathbf{L}}, \delta) \leq RMU^{1,2}(S_{\mathbf{P}}, \delta) \leq RMU^{1,2}(S_{\mathbf{M}}, \delta)).$$

This means that among the three t-conorms $S_{\mathbf{M}}$, $S_{\mathbf{P}}$ and $S_{\mathbf{L}}$, $S_{\mathbf{L}}$ has the best robustness against uniformly distributed perturbations on its two variables, while $S_{\mathbf{M}}$ has the worst robustness against uniformly distributed perturbations on its two variables.
For the robustness measure of an S-implication against the uniformly distributed perturbation we have the following theorem.

Theorem 6.15. *Let I be an S-implication generated by a t-conorm S and the standard fuzzy negation N_0. Then*

$$(\forall \delta \in [0, \frac{1}{2}])(RMU^{1,2}(I, \delta) = RMU^{1,2}(S, \delta)).$$

PROOF. The proof is similar to that of Theorem 6.8. □

According to Theorem 6.15, we obtain

$$(\forall \delta \in [0, \frac{1}{2}])(RMU^{1,2}(I_{\mathbf{L}}, \delta) \leq RMU^{1,2}(I_r, \delta) \leq RMU^{1,2}(I_b, \delta)).$$

This means that among the three S-implications I_b, I_r and $I_{\mathbf{L}}$, $I_{\mathbf{L}}$ has the best robustness against uniformly distributed perturbations on its two variables, while I_b has the worst robustness against uniformly distributed perturbations on its two variables.

6.4 Summary

In this chapter we defined robustness measures for a t-norm, a t-conorm and a fuzzy implication against the unknown bounded perturbation and against the uniformly distributed perturbation on the first or second variables or on both variables, respectively, and have compared the robustness of the most important continuous t-norms, t-conorms and fuzzy implications. The minimum $T_{\mathbf{M}}$, the maximum $S_{\mathbf{M}}$ and the Kleene-Dienes implication I_b have the best robustness against the unknown bounded perturbation on the first variable, the second variable and both variables among the investigated t-norms, t-conorms and fuzzy implications, respectively. The product $T_{\mathbf{P}}$, the probabilistic sum $S_{\mathbf{P}}$ and the Reichenbach implication I_r have the best robustness against the uniformly distributed perturbation on the first or second variable among the investigated t-norms, t-conorms and fuzzy implications, respectively. The Łukasiewicz t-norm $T_{\mathbf{L}}$, the Łukasiewicz t-conorm $S_{\mathbf{L}}$ and the Łukasiewicz implication have the best robustness against the uniformly distributed perturbation on both variables among the investigated t-norms, t-conorms and fuzzy implications, respectively.

Chapter 7

Fuzzy Adjunctions and Fuzzy Morphological Operations Based on Fuzzy Implications

7.1 Introduction

Mathematical morphology is an important theory developed in image processing to analyze the geometric features of n-dimensional images. These images can be binary images which are represented as subsets of $\mathbb{R}^n$ or gray-scale images which are represented as $\mathbb{R}^n \rightharpoonup [0, 1]$ mappings [62]. morphological operations are the basic tools in mathematical morphology. They transform an image A by using another image B which is called the structuring element. Dilation and erosion are the two basic morphological operations. Another two important morphological operations, closing and opening, can be constructed via dilation and erosion. If a dilation and an erosion form an adjunction in a complete lattice, then a group of properties of them will be fulfilled, as well for the closing and opening constructed by them [33].

When extending binary morphology to gray-scale morphology, fuzzy sets in $\mathbb{R}^n$ are proper representations of gray-scale images. Fuzzy dilation and fuzzy erosion can be defined via a conjunction on the unit interval and a fuzzy implication, respectively [65]. Moreover, a fuzzy implication can be defined as a fuzzy adjunction in the complete lattice $\mathcal{F}(\mathbb{R}^n)$, which is the set of all fuzzy sets on $\mathbb{R}^n$. In this chapter, we analyze for different fuzzy dilations defined by a conjunction on the unit interval and fuzzy erosions defined by a fuzzy implication when they form adjunctions defined by a fuzzy implication. In Section 7.2 we give some basic notions and results of conjunctions on the unit interval and fuzzy implications. And then we define fuzzy morphological operations via conjunctions on the unit interval and fuzzy implications. In Section 7.3 we extend a classical adjunction to a fuzzy adjunction which is generated by a fuzzy implication. We analyze under which conditions a fuzzy dilation which is generated by a conjunction on the unit interval and a fuzzy erosion which is generated by a fuzzy implication form an adjunction and then outline the importance of the adjointness between the conjunctions on the unit interval and the fuzzy implications.

7.2 Fuzzy Morphological Operations Defined by Conjunctions on the Unit Interval and Fuzzy Implications

7.2.1 Conjunctions on the Unit Interval and Fuzzy Implications

Based on the definitions of a conjunction on the unit interval and a fuzzy implication given in Chapter 2, we have further the following definitions.

Definition 7.1. A conjunction on the unit interval $\mathcal{C}$ and a fuzzy implication I are called *adjoint* if

$$(\forall(x,y,z)\in[0,1]^3)(\mathcal{C}(x,z)\le y \Leftrightarrow z\le I(x,y)). \tag{7.1}$$

According to Definition 7.1 we can define a fuzzy implication I via a conjunction on the unit interval $\mathcal{C}$ by

$$(\forall(x,y)\in[0,1]^2)(I(x,y)=\sup\{t\in[0,1]|\mathcal{C}(x,t)\le y\}). \tag{7.2}$$

And we define a conjunction on the unit interval $\mathcal{C}$ via a fuzzy implication I by

$$(\forall(x,y)\in[0,1]^2)(\mathcal{C}(x,y)=\inf\{t\in[0,1]|I(x,t)\ge y\}). \tag{7.3}$$

Theorem 7.2. *([29], Proposition 5.4.2) Let $\mathcal{C}$ be a conjunction on the unit interval and I be a fuzzy implication defined by (7.2). Then $\mathcal{C}$ and I are adjoint iff $\mathcal{C}$ is left-continuous w.r.t. the second variable.*

According to the proof of Theorem 7.2, we have the following corollary.

Corollary 7.3. *Let I be a fuzzy implication and $\mathcal{C}$ be a conjunction on the unit interval defined by (7.3). Then I and $\mathcal{C}$ are adjoint iff I is right-continuous w.r.t. the second variable.*

From Theorem 7.2 and Corollary 7.3, we obtain the following corollaries.

Corollary 7.4. *([29], Corollary 5.4.1) Let $\mathcal{C}$ be a conjunction on the unit interval. If $\mathcal{C}$ is left-continuous w.r.t. the second variable, then (7.2) becomes*

$$(\forall(x,y)\in[0,1]^2)(I(x,y)=\max\{t\in[0,1]|\mathcal{C}(x,t)\le y\}). \tag{7.4}$$

Corollary 7.5. *Let I be a fuzzy implication. If I is right-continuous w.r.t. the second variable, then (7.3) becomes*

$$(\forall(x,y)\in[0,1]^2)(\mathcal{C}(x,y)=\min\{t\in[0,1]|I(x,t)\ge y\}). \tag{7.5}$$

We will see later that adjointness of a conjunction on the unit interval and a fuzzy implication plays an important role in defining a fuzzy dilation and a fuzzy erosion which can form a fuzzy adjunction in $\mathcal{F}(\mathbb{R}^n)$.

7.2.2 Fuzzy Morphological Operations

We define binary morphological operations according to [74].

Definition 7.6. Let A, $B \subseteq \mathbb{R}^n$. The binary dilation $D(A, B)$ is defined by

$$D(A, B) = \{y \in \mathbb{R}^n | T_y(B) \cap A \neq \emptyset\}.$$

And the binary erosion $E(A, B)$ is defined by

$$E(A, B) = \{y \in \mathbb{R}^n | T_y(B) \subseteq A\},$$

where $T_y(B) = \{x \in \mathbb{R}^n | x - y \in B\}$.

Define for all $x \in \mathbb{R}^n$, $-B(x) = B(-x)$. A binary closing $C(A, B)$ is defined by

$$C(A, B) = E(D(A, B), -B).$$

And a binary opening $O(A, B)$ is defined by

$$O(A, B) = D(E(A, B), -B),$$

where D and E are binary dilation and binary erosion, respectively.
When extending binary morphology to gray-scale morphology, fuzzy sets in $\mathbb{R}^n$ are proper representations of gray-scale images. Using a conjunction on the unit interval to express the intersection of two fuzzy sets and a fuzzy implication to express the fuzzy set inclusion, we can fuzzify the binary dilation and the binary erosion into a fuzzy dilation and a fuzzy erosion.

Definition 7.7. ([65]) Let A, $B \in \mathcal{F}(\mathbb{R}^n)$ and let $\mathcal{C}$ be a conjunction on the unit interval and I be a fuzzy implication. The fuzzy dilation $D_{\mathcal{C}}(A, B)$ is defined by

$$(\forall y \in \mathbb{R}^n)(D_{\mathcal{C}}(A, B)(y) = \sup_{x \in \mathbb{R}^n} \mathcal{C}(B(x - y), A(x)))$$

and the fuzzy erosion $E_I(A, B)$ is defined by

$$(\forall y \in \mathbb{R}^n)(E_I(A, B)(y) = \inf_{x \in \mathbb{R}^n} I(B(x - y), A(x))).$$

Observe that according to [62] and [65], several fuzzy approaches to gray-scale morphology are embedded in the approach stated in Definition 7.7. For example, the approach of fuzzy set inclusion of Zadeh uses the Zadeh implication to define a fuzzy erosion via

Definition 7.7. The approach of fuzzy set inclusion of Sinha and Dougherty uses the generalized Łukasiewicz implication to define a fuzzy erosion via Definition 7.7. However both of the aforementioned approaches use conjunctions on the unit interval generated by $(\forall(x,y) \in [0,1]^2)(\mathcal{C}(x,y) = N(I(x,N(y))))$, where N is a strong fuzzy negation and $I = I_Z$ or $I = I_\lambda$ to define the fuzzy dilation. In the next section, we will see that it is more proper to define the conjunction on the unit interval C to be adjoint with the implication I.
Similar to the case in binary morphology, a fuzzy closing $C_{\mathcal{C},I}(A,B)$ is defined by

$$C_{\mathcal{C},I}(A,B) = E_I(D_{\mathcal{C}}(A,B), -B)$$

and a opening $O_{\mathcal{C},I}(A,B)$ is defined by

$$O_{\mathcal{C},I}(A,B) = D_{\mathcal{C}}(E_I(A,B), -B),$$

where $D_{\mathcal{C}}$ and E_I are fuzzy dilation and binary erosion respectively.

7.3 Fuzzy I-Adjunctions in $\mathcal{F}(\mathbb{R}^n)$

7.3.1 Classical Adjunctions and Fuzzy Adjunctions

In this chapter we discuss adjunctions in a *complete lattice*. Recall that a lattice $(\mathcal{L}, \preceq)$ is a poset such that any two unary elements L_1, $L_2 \in \mathcal{L}$ have a supremum and an infimum. A complete lattice $(\mathcal{L}, \preceq)$ is a lattice such that any subset of $\mathcal{L}$ has a supremum and an infimum in $\mathcal{L}$. First we define the classical adjunctions.

Definition 7.8. [33] Let δ and ϵ be two operations on a complete lattice $(\mathcal{L}, \preceq)$. The pair (δ, ϵ) is an adjunction in $\mathcal{L}$ iff

$$(\forall(L_1, L_2) \in \mathcal{L}^2)(\delta(L_1) \preceq L_2 \Leftrightarrow L_1 \preceq \epsilon(L_2)).$$

Define the ordering $A \subseteq B \Leftrightarrow (\forall x \in \mathbb{R}^n)(A(x) \leq B(x))$ resulting in the complete lattice $(\mathcal{F}(\mathbb{R}^n), \subseteq)$. For any A, $B \in \mathcal{F}(\mathbb{R}^n)$, let $\delta(A) = D_{\mathcal{C}}(A,B)$ which is a fuzzy dilation on A by means of B and $\epsilon(A) = E_I(A, -B)$ which is a fuzzy erosion on A by means of $-B$. If (δ, ϵ) is a proper adjunction in $\mathcal{F}(\mathbb{R}^n)$, then the fuzzy dilation $D_{\mathcal{C}}$, the fuzzy erosion E_I and the fuzzy closing and fuzzy opening defined by them will fulfill the properties below:

(1) $\epsilon(\bigcap_{j \in J} A_j) = \bigcap_{j \in J} \epsilon(A_j)$
(2) $\delta(\bigcup_{j \in J} A_j) = \bigcup_{j \in J} \delta(A_j)$
(3) $A \preceq \epsilon(\delta(A))$
(4) $\delta(\epsilon(A)) \preceq A$
(5) $\epsilon(\delta(\epsilon(A))) = \epsilon(A)$
(6) $\delta(\epsilon(\delta(A))) = \delta(A)$
(7) $\epsilon(A) = \bigcup\{B \in \mathcal{L} | \delta(B) \preceq A\}$, $\bigcup$ modeled by $\sup$

(8) $\delta(A) = \bigcap\{B \in \mathcal{L} | \epsilon(B) \preceq A\}$, $\bigcap$ modeled by inf

Properties (1) and (2) state that ϵ is an algebraic erosion on $\mathcal{L}$ and δ is an algebraic dilation on $\mathcal{L}$, respectively. Properties (3) and (4) state that $\epsilon\delta$ is extensive and $\delta\epsilon$ is restrictive, respectively. Properties (5) and (6) state that both $\epsilon\delta$ and $\delta\epsilon$ are idempotent. Properties (7) and (8) state that one operator from a given adjunction can be generated by the other one [62].
Authors of [62] defined a fuzzy adjunction by a fuzzy implication I, which extends the properties of classical adjunctions. The fuzzy implication I needs to fulfill the ordering principle.

Definition 7.9. Let I be a fuzzy implication that satisfies the ordering principle. Moreover, let δ and ϵ be two $\mathcal{F}(\mathbb{R}^n) \to \mathcal{F}(\mathbb{R}^n)$ mappings. Then the pair (δ, ϵ) is a fuzzy I-adjunction in $\mathcal{F}(\mathbb{R}^n)$ iff

$$(\forall (A_1, A_2) \in \mathcal{F}(\mathbb{R}^n) \times \mathcal{F}(\mathbb{R}^n)(\inf_{x \in \mathbb{R}^n} I(\delta(A_1)(x), A_2(x)) = \inf_{x \in \mathbb{R}^n} I(A_1(x), \epsilon(A_2)(x)))$$

Let $\mathcal{C}$ be a conjunction on the unit interval, I and $I^{'}$ be two fuzzy implications. Next theorem and propositions will state when (δ, ϵ) is a fuzzy I-adjunction, where $\delta(A) = D_{\mathcal{C}}(A, B)$ and $\epsilon(A) = E_{I'}(A, -B)$.

Theorem 7.10. *Let $\mathcal{C}$ be a conjunction on the unit interval and I be a fuzzy implication. Moreover, let $\delta(A) = D_{\mathcal{C}}(A, B)$ and $\epsilon(A) = E_I(A, -B)$. Then (δ, ϵ) is a fuzzy I_Z-adjunction iff $\mathcal{C}$ and I are adjoint, where I_Z is the Zadeh implication.*

Proof. According to Definition 7.7, we have

$$(\forall y \in \mathbb{R}^n)(\delta(A_1)(y) = \sup_{x \in \mathbb{R}^n} \mathcal{C}(B(x - y), A_1(x)))$$

and

$$(\forall y \in \mathbb{R}^n)(\epsilon(A_2)(y) = \inf_{x \in \mathbb{R}^n} I(B(y - x), A_2(x))).$$

According to Definition 7.9, (δ, ϵ) is a fuzzy I_Z-adjunction, which means that

$$(\forall x \in \mathbb{R}^n)(\delta(A_1(x)) \leq A_2(x)) \Leftrightarrow (\forall x \in \mathbb{R}^n)(A_1(x) \leq \epsilon(A_2(x))).$$

Because (δ, ϵ) is a fuzzy I_Z-adjunction, we have

$$\begin{aligned}
&(\forall y \in \mathbb{R}^n)(\sup_{x \in \mathbb{R}^n} \mathcal{C}(B(x - y), A_1(x)) \leq A_2(y)) \\
&\Leftrightarrow (\forall y \in \mathbb{R}^n)(A_1(y) \leq \inf_{x \in \mathbb{R}^n} I(B(y - x), A_2(x))), i.e., \\
&(\forall (x, y) \in \mathbb{R}^n \times \mathbb{R}^n)(\mathcal{C}(B(x - y), A_1(x)) \leq A_2(y)) \\
&\Leftrightarrow (\forall (x, y) \in \mathbb{R}^n \times \mathbb{R}^n)(A_1(y) \leq I(B(y - x), A_2(x))), i.e., \\
&(\forall (x, y) \in \mathbb{R}^n \times \mathbb{R}^n)(\mathcal{C}(B(x - y), A_1(x)) \leq A_2(y))
\end{aligned}$$

$$\Leftrightarrow (\forall(x,y) \in \mathbb{R}^n \times \mathbb{R}^n)(A_1(x) \leq I(B(x-y), A_2(y)))$$

which is equivalent to

$$(\forall(a,b,c) \in [0,1]^3)(\mathcal{C}(a,b) \leq c \Leftrightarrow b \leq I(a,c)),$$

which means that $\mathcal{C}$ and I are adjoint. □

Lemma 7.11. *([29]) Let f be a $[0,1] \rightharpoonup [0,1]$ mapping and $(a_i)_{i\in I}$ an arbitrary family in $[0,1]$. Then the following statements hold:*

- *if f is left-continuous and decreasing then:* $\inf_{i\in I} f(a_i) = f(\sup_{i\in I} a_i)$;
- *if f is left-continuous and increasing then:* $\sup_{i\in I} f(a_i) = f(\sup_{i\in I} a_i)$;
- *if f is right-continuous and decreasing then:* $\sup_{i\in I} f(a_i) = f(\inf_{i\in I} a_i)$;
- *if f is right-continuous and increasing then:* $\inf_{i\in I} f(a_i) = f(\inf_{i\in I} a_i)$.

Proposition 7.12. *Let $\mathcal{C}$ be an associative conjunction on the unit interval being left-continuous w.r.t. the second variable and I be a fuzzy implication which is left-continuous w.r.t. the first variable and right-continuous w.r.t. the second variable and that satisfies the ordering principle. Moreover, let $\delta(A) = D_{\mathcal{C}}(A,B)$ and $\epsilon(A) = E_I(A,-B)$. Then (δ, ϵ) is a fuzzy I-adjunction if $\mathcal{C}$ and I are adjoint.*

Proof.

$$I(\mathcal{C}(a,b),c) = \max\{t|\mathcal{C}(\mathcal{C}(a,b),t) \leq c\}.$$

Because $\mathcal{C}$ and I are adjoint,

$$\mathcal{C}(a,\mathcal{C}(b,t)) \leq c \Leftrightarrow \mathcal{C}(b,t) \leq I(a,c).$$

So

$$I(b,I(a,c)) = \max\{t|\mathcal{C}(b,t) \leq I(a,c)\}$$
$$= \max\{t|\mathcal{C}(b,t) \leq \max\{s|\mathcal{C}(a,s) \leq c\}\}.$$

Since $\mathcal{C}$ is associative, we have

$$I(b,I(a,c)) = \max\{t|\mathcal{C}(a,\mathcal{C}(b,t)) \leq c\}$$
$$= \max\{t|\mathcal{C}(\mathcal{C}(a,b),t)) \leq c\}$$
$$= I(\mathcal{C}(a,b),c).$$

Thus for all A_1, A_2, $B \in \mathcal{F}(\mathbb{R}^n)$, we get by applying the above formula:

$$\inf_{y\in\mathbb{R}^n} \inf_{x\in\mathbb{R}^n} I(\mathcal{C}(B(x-y), A_1(x)), A_2(y))$$

$$= \inf_{y\in\mathbb{R}^n} \inf_{x\in\mathbb{R}^n} I(A_1(x), I(B(x-y), A_2(y))),$$

or equivalently,

$$\inf_{y\in\mathbb{R}^n} \inf_{x\in\mathbb{R}^n} I(\mathcal{C}(B(x-y), A_1(x)), A_2(y))$$
$$= \inf_{y\in\mathbb{R}^n} \inf_{x\in\mathbb{R}^n} I(A_1(y), I(B(y-x), A_2(x))).$$

Since I is left-continuous w.r.t. the first variable and right-continuous w.r.t. the second variable, according to Lemma 7.11, we have

$$\inf_{y\in\mathbb{R}^n} I(\sup_{x\in\mathbb{R}^n} \mathcal{C}(B(x-y), A_1(x)), A_2(y))$$
$$= \inf_{y\in\mathbb{R}^n} I(A_1(y), \inf_{x\in\mathbb{R}^n} I(B(y-x), A_2(x)))$$
$$\Leftrightarrow \inf_{y\in\mathbb{R}^n} I(\delta(A_1)(y), A_2(y)) = \inf_{y\in\mathbb{R}^n} I(A_1(y), \epsilon(A_2)(y)).$$

Thus according to Definition 7.9, (δ, ϵ) is an I-adjunction. □

Proposition 7.13. *Let $\mathcal{C}$ be a conjunction on the unit interval, I be a fuzzy implication being left-continuous w.r.t. the first variable and right-continuous w.r.t. the second variable and which satisfies the ordering principle, and I' a fuzzy implication. Moreover, let $\delta(A) = D_{\mathcal{C}}(A, B)$ and $\epsilon(A) = E_{I'}(A, -B)$. If ($\delta$, ϵ) is a fuzzy I-adjunction, then $\mathcal{C}$ and I' are adjoint.*

Proof. If (δ, ϵ) is a fuzzy I-adjunction, then for all A_1, A_2, $B \in \mathcal{F}(\mathbb{R}^n)$,

$$\inf_{y\in\mathbb{R}^n} I(\delta(A_1)(y), A_2(y)) = \inf_{y\in\mathbb{R}^n} I(A_1(y), \epsilon(A_2)(y)),$$

or equivalently,

$$\inf_{y\in\mathbb{R}^n} I(\sup_{x\in\mathbb{R}^n} \mathcal{C}(B(x-y), A_1(x)), A_2(y))$$
$$= \inf_{y\in\mathbb{R}^n} I(A_1(y), \inf_{x\in\mathbb{R}^n} I'(B(y-x), A_2(x))).$$

Since I is left-continuous w.r.t. the first variable and right-continuous w.r.t. the second variable, according to Lemma 7.11 we have

$$\inf_{y\in\mathbb{R}^n} \inf_{x\in\mathbb{R}^n} I(\mathcal{C}(B(x-y), A_1(x)), A_2(y))$$
$$= \inf_{y\in\mathbb{R}^n} \inf_{x\in\mathbb{R}^n} I(A_1(y), I'(B(y-x), A_2(x))),$$

or equivalently,

$$\inf_{y\in\mathbb{R}^n} \inf_{x\in\mathbb{R}^n} I(\mathcal{C}(B(x-y), A_1(x)), A_2(y))$$

$$= \inf_{y\in\mathbb{R}^n} \inf_{x\in\mathbb{R}^n} I(A_1(x), I^{'}(B(x-y), A_2(y))).$$

Thus we have

$$(\forall (a,b,c) \in [0,1]^3)(I(\mathcal{C}(a,b),c) = I(b, I^{'}(a,c))),$$

and hence, taking into account that I satisfies the ordering principle, we get:

$$(\forall (a,b,c) \in [0,1]^3)(\mathcal{C}(a,b) \leq c \Leftrightarrow b \leq I^{'}(a,c)).$$

Hence $\mathcal{C}$ and $I^{'}$ are adjoint. □

Thus we can see that in the framework of fuzzy adjunctions, it is very important for a conjunction on the unit interval $\mathcal{C}$ and a fuzzy implication I to be adjoint. Let us take a closer look at this adjointness.

7.3.2 Adjointness between Conjunctions on the Unit Interval and Fuzzy Implications

For R-implications generated by left-continuous t-norms, the following theorem and corollary hold.

Theorem 7.14. *([24], Theorem 1.14) A $[0,1]^2 \rightharpoonup [0,1]$ mapping I is an R-implication generated by a left-continuous t-norm via (2.55) iff it has increasing second partial mappings, satisfies the exchange principle and the ordering principle and it is right-continuous w.r.t. the second variable.*

Corollary 7.15. *([2], Corollary 10) A $[0,1]^2 \rightharpoonup [0,1]$ mapping T is a left-continuous t-norm iff T can be represented by*

$$T(x,y) = \inf\{t \in [0,1] | I(x,t) \geq y\} \tag{7.6}$$

for some $[0,1]^2 \rightharpoonup [0,1]$ mapping I which has increasing second partial mappings, satisfies the exchange principle and the ordering principle and it is right-continuous w.r.t. the second variable.

We see from Corollary 7.15 that we cannot always generate a left-continuous t-norm from a fuzzy implication. For example, we cannot generate a t-norm from an S-implication which is not an R-implication. Using the Reichenbach implication I_r in (7.3), we have

$$\begin{aligned} &\inf\{t \in [0,1] | I_r(x,t) \geq y\} \\ &= \inf\{t \in [0,1] | 1 - x + xt \geq y\} \\ &= \max(\frac{x+y-1}{x}, 0), \end{aligned}$$

which is not a t-norm, because of the lack of commutativity.

Nevertheless, the corollary following the next theorem shows that we can generate conjunctions on the unit interval with fuzzy implications which are not only R-implications.

Theorem 7.16. *A* $[0,1]^2 \rightharpoonup [0,1]$ *mapping* I *is a fuzzy implication being right-continuous w.r.t. the second variable and satisfies*

$$(\forall x \in [0,1])(I(1,x) = 1 \Leftrightarrow x = 1) \tag{7.7}$$

iff I *is generated by a conjunction on the unit interval* $\mathcal{C}$ *via ((7.2)), where* $\mathcal{C}$ *is left-continuous w.r.t. the second variable and satisfies*

$$(\forall x \in [0,1])(\mathcal{C}(1,x) = 0 \Leftrightarrow x = 0). \tag{7.8}$$

Proof. To prove the if-part, assume $0 \leq x_1 < x_2 \leq 1$. Then according to Corollary 7.4,

$$(\forall y \in [0,1])(I(x_1,y) = \max\{t|\mathcal{C}(x_1,t) \leq y\}),$$

and

$$(\forall y \in [0,1])(I(x_2,y) = \max\{t|\mathcal{C}(x_2,t) \leq y\}).$$

Because $\mathcal{C}$ has increasing first partial mappings, then

$$(\forall t \in [0,1])(\mathcal{C}(x_2,t) \leq y \Rightarrow \mathcal{C}(x_1,t) \leq y).$$

Thus $\max\{t|\mathcal{C}(x_2,t) \leq y\} \in \{t|\mathcal{C}(x_1,t) \leq y\}$. Therefore

$$\max\{t|\mathcal{C}(x_2,t) \leq y\} \leq \max\{t|\mathcal{C}(x_1,t) \leq y\},$$

i.e., $I(x_2,y) \leq I(x_1,y)$. Hence I has decreasing first partial mappings.
Now assume $0 \leq y_1 < y_2 \leq 1$. Then

$$(\forall x \in [0,1])(I(x,y_1) = \max\{t|\mathcal{C}(x,t) \leq y_1\}),$$

and

$$(\forall x \in [0,1])(I(x,y_2) = \max\{t|\mathcal{C}(x,t) \leq y_2\}).$$

We have

$$(\forall t \in [0,1])(\mathcal{C}(x,t) \leq y_1 \Rightarrow \mathcal{C}(x,t) \leq y_2).$$

Thus $\max\{t|\mathcal{C}(x,t) \leq y_1\} \in \{t|\mathcal{C}(x,t) \leq y_2\}$. Therefore

$$\max\{t|\mathcal{C}(x,t) \leq y_1\} \leq \max\{t|\mathcal{C}(x,t) \leq y_2\},$$

i.e., $I(x,y_1) \leq I(x,y_2)$. Hence I has increasing second partial mappings.
Now we will verify the boundary conditions for I. We obtain:

$$I(0,0) = \max\{t|\mathcal{C}(0,t) = 0\} = 1,$$
$$I(0,1) = \max\{t|\mathcal{C}(0,t) \leq 1\} = 1,$$

$$I(1,1) = \max\{t|\mathcal{C}(1,t) \le 1\} = 1.$$

Moreover, because $\mathcal{C}$ satisfies (7.8), we get $I(1,0) = \max\{t|\mathcal{C}(1,t) = 0\} = 0$. Therefore I is a fuzzy implication. Because $\mathcal{C}$ is left-continuous w.r.t. the second variable, according to Theorem 7.2, $\mathcal{C}$ and I are adjoint. Thus we have

$$\mathcal{C}(x,y) \le z \Leftrightarrow y \le I(x,z).$$

Hence I can be represented by (7.4). Therefore according to Corollary 7.3, I is right-continuous w.r.t. the second variable. If $I(1,x) = \max\{t|\mathcal{C}(1,t) \le x\} = 1$, then $\mathcal{C}(1,1) \le x$, which implies that $x = 1$. If $x = 1$, then we already know $I(1,1) = 1$. Thus $I(1,x) = 1$ iff $x = 1$.
Hence I is a fuzzy implication being right-continuous w.r.t. the second variable and satisfies (7.7).
To prove the 'only if' part, define a $[0,1]^2 \rightharpoonup [0,1]$ mapping $\mathcal{C}$ by (7.3). Assume $0 \le x_1 < x_2 \le 1$. Then

$$(\forall y \in [0,1])(\mathcal{C}(x_1,y) = \min\{t|I(x_1,t) \ge y\}),$$

and

$$(\forall y \in [0,1])(\mathcal{C}(x_2,y) = \min\{t|I(x_2,t) \ge y\}).$$

Because I has decreasing first partial mappings, then

$$(\forall t \in [0,1])(I(x_2,t) \ge y \Rightarrow I(x_1,t) \ge y).$$

Thus $\min\{t|I(x_2,t) \ge y\} \in \{t|I(x_1,t) \ge y\}$. Therefore

$$\min\{t|I(x_2,t) \ge y\} \ge \min\{t|I(x_1,t) \ge y\},$$

i.e., $\mathcal{C}(x_2,y) \ge \mathcal{C}(x_1,y)$. Hence $\mathcal{C}$ has increasing first partial mappings.
Now assume $0 \le y_1 < y_2 \le 1$. Then

$$(\forall x \in [0,1])(\mathcal{C}(x,y_1) = \min\{t|I(x,t) \ge y_1\}),$$

and

$$(\forall x \in [0,1])(\mathcal{C}(x,y_2) = \min\{t|I(x,t) \ge y_2\}).$$

We obtain

$$(\forall t \in [0,1])(I(x,t) \ge y_2 \Rightarrow I(x,t) \ge y_1).$$

Thus $\min\{t|I(x,t) \ge y_2\} \in \{t|I(x,t) \ge y_1\}$. Therefore

$$\min\{t|I(x,t) \ge y_2\} \ge \min\{t|I(x,t) \ge y_1\}$$

i.e., $\mathcal{C}(x, y_2) \geq \mathcal{C}(x, y_1)$. Hence $\mathcal{C}$ has increasing second partial mappings.
Now we will consider the boundary conditions of $\mathcal{C}$. We obtain:

$$\mathcal{C}(0,0) = \min\{t|I(0,t) \geq 0\} = 0,$$
$$\mathcal{C}(0,1) = \min\{t|I(0,t) = 1\} = 0,$$
$$\mathcal{C}(1,0) = \min\{t|I(1,t) \geq 0\} = 0.$$

Moreover, because I satisfies (7.7), we get $\mathcal{C}(1,1) = \min\{t|I(1,t) = 1\} = 1$.
Therefore $\mathcal{C}$ is a conjunction on the unit interval. Because I is left-continuous w.r.t. the second variable, according to Corollary 7.3, $\mathcal{C}$ and I are adjoint. Thus we have

$$\mathcal{C}(x,y) \leq z \Leftrightarrow y \leq I(x,z).$$

Hence I can be represented by (7.5). Therefore according to Corollary 7.2, $\mathcal{C}$ is left-continuous w.r.t. the second variable.
If $\mathcal{C}(1,x) = \min\{t|I(1,t) \geq x\} = 0$, then $I(1,0) \geq x$, which implies that $x = 0$. If $x = 0$, then we already know $\mathcal{C}(1,0) = 0$. Thus $\mathcal{C}(1,x) = 0$ iff $x = 0$.
Hence $\mathcal{C}$ is a conjunction on the unit interval being left-continuous w.r.t. the second variable and satisfying (7.8). □

From the proof of Theorem 7.16, we have the next corollary.

Corollary 7.17. *A $[0,1]^2 \rightharpoonup [0,1]$ mapping $\mathcal{C}$ is a conjunction on the unit interval being left-continuous w.r.t. the second variable and satisfying condition (7.8) iff $\mathcal{C}$ is generated by a fuzzy implication I via (7.3), where I is right-continuous w.r.t. the second variable and satisfies (7.7).*

Below we give several examples of adjoint couples consisting of right-continuous non-R-implications given in Chapter 2 (Sections 2.5-2.6) and the conjunctions on the unit interval generated by them via 7.2.

Example 7.1 (i) Observe that S-implications always fulfill (7.7). Consider the Kleene-Dienes implication $I_b(x,y) = \max(1-x, y)$ which is an S-implication generated by the t-conorm $S_{\mathbf{M}}$ and the standard fuzzy negation, and the Reichenbach implication $I_r(x,y) = 1 - x + xy$ which is an S-implication generated by the t-conorm $S_{\mathbf{P}}$. I_b satisfies

$$(\forall (x,y) \in [0,1]^2)(I_b(x,y) = 1 \Leftrightarrow x = 0 \quad \text{or} \quad \text{y} = 1),$$

and I_r satisfies

$$(\forall (x,y) \in [0,1]^2)(I_r(x,y) = 1 \Leftrightarrow x = 0 \quad \text{or} \quad \text{y} = 1).$$

Thus I_b and I_r are not residuated with any left-continuous t-norm but they are right-continuous w.r.t. the second variable. Then

$$\mathcal{C}_{I_b}(x,y) = \inf\{t \in [0,1] | I_b(x,t) \geq y\} = \begin{cases} 0, & \text{if } x \leq y \\ y, & \text{if } x > y \end{cases}$$

and

$$\mathcal{C}_{I_r}(x,y) = \inf\{t \in [0,1] | I_r(x,t) \geq y\} = \begin{cases} \max(\frac{x+y-1}{x}, 0), & \text{if } x \geq 0 \\ 0, \quad \text{if } x = 0 \end{cases}.$$

I_b and I_r are right continuous non R-implications that are adjoint to the conjunctions on the unit interval $\mathcal{C}_{I_b}$ generated by I_b and $\mathcal{C}_{I_r}$ generated by I_r, respectively.

(ii) A f-generated implication I_f also fulfills (7.7). Moreover I_f satisfies

$$(\forall (x,y) \in [0,1]^2)(I_f(x,y) = 1 \Leftrightarrow x = 0 \quad \text{or} \quad \text{y} = 1).$$

Thus I_f is not residuated to any left-continuous t-norm. Because both f and $f^{(-1)}$ are continuous mappings, I_f is continuous w.r.t. the second variable. Then $\mathcal{C}_{I_f}(x,y) = \inf\{t \in [0,1] | I_f(x,t) \geq y\} = \begin{cases} f^{(-1)}(\frac{f(y)}{x}), & \text{if } x > 0 \\ 0, \quad \text{if } x = 0 \end{cases}$. I_f is a right continuous non R-implication that is adjoint to the conjunctions on the unit interval $\mathcal{C}_{I_f}$ generated by I_f.

(iii) Recall the two aforementioned λ-functions.

(iii,1) If $\lambda_n(x) = 1 - x^n$ $(n \geq 1)$, then

$$\begin{aligned} I_{\lambda_n}(x,y) &= \min(\lambda_n(x) + \lambda_n(1-y), 1) \\ &= \min(1 - x^n + 1 - (1-y)^n, 1) \\ &= \min(2 - x^n - (1-y)^n, 1). \end{aligned}$$

Thus

$$\begin{aligned} \mathcal{C}_{I_{\lambda_n}}(x,y) &= \inf\{t | \min(2 - x^n - (1-t)^n, 1) \geq y\} \\ &= \max(1 - (2 - x^n - y)^{\frac{1}{n}}, 0). \end{aligned}$$

(iii,2) If $\lambda_n(x) = \frac{1-x}{1+nx}$ $(n \in]-1, 0])$, then

$$\begin{aligned} I_{\lambda_n}(x,y) &= \min(\lambda_n(x) + \lambda_n(1-y), 1) \\ &= \min(\frac{1-x}{1+nx} + \frac{y}{1+n-ny}, 1). \end{aligned}$$

Thus

$$\begin{aligned} \mathcal{C}_{I_{\lambda_n}}(x,y) &= \inf\{t | \min(\frac{1-x}{1+nx} + \frac{t}{1+n(1-t)}, 1) \geq y\} \\ &= \max(\frac{(1+n)(y - \frac{1-x}{1+nx})}{1 + n(y - \frac{1-x}{1+nx})}, 0). \end{aligned}$$

I_{λ_n} is a right continuous non R-implication that is adjoint to the conjunctions on the unit interval $\mathcal{C}_{I_{\lambda_n}}$ generated by I_{λ_n}.

7.4 Summary

In this chapter we extended classical adjunctions to fuzzy adjunctions. We used a fuzzy implication I to define a fuzzy I-adjunction in $\mathcal{F}(\mathbb{R}^n)$. And then we studied the conditions under which a fuzzy dilation which is defined by a conjunction on the unit interval $\mathcal{C}$ and a fuzzy erosion which is defined by a fuzzy implication $I^{'}$ form a fuzzy I-adjunction. These conditions are essential in order that the fuzzification of the morphological operation of dilation, erosion, opening and closing obey similar properties as their algebraic counterparts. We found out that the adjointness between the conjunction on the unit interval $\mathcal{C}$ and the implication I or the adjointness between the conjunction on the unit interval $\mathcal{C}$ and the implication $I^{'}$ play important roles in such conditions. We then worked out Theorem 7.16 and Corollary 7.17 on the adjointness between a conjunction on the unit interval and a fuzzy implication. Based on Theorem 7.16 and Corollary 7.17 we can generate a conjunction on the unit interval from a fuzzy implication so that they are adjoint. Examples of generating conjunctions on the unit interval from famous existing fuzzy implications were given at the end.

Chapter 8

Conclusions

8.1 Main Contributions

Four main contributions of this thesis are given below:

1. A deep study of the axioms of QL-implications. Investigating under which conditions a QL-implication satisfies the 13 axioms and the inter-relationship among the axioms of QL-implications.
2. The interrelationship between the 8 axioms in the axiomatic system of fuzzy implications for the fuzzy implications satisfying the first 5 axioms.
3. Comparing different effects of fuzzy implications in approximate reasoning from both a logic point of view and a practice view.
4. Using fuzzy implications and conjunctions on the unit interval to generate fuzzy morphological operations.

We describe the details of these 4 countibutions below:

1. **A deep study of the axioms of QL-implications. Investigating under which conditions a QL-implication satisfies the 13 axioms and the inter-relationship among the axioms of QL-implications.**
   Among the three most important classes of fuzzy implications generated from the other fuzzy logic operators, S- and R-implications are very widely studied in the literature of fuzzy set theory while QL-implications not because they do not always satisfy the first place antitonicity. But sometimes a QL-implication is a good candidate for some specific applications. So it is meaningful to find the QL-implications that satisfy the first place antitonicity. As a result they will also satisfy dominance of truth of consequent. These QL-implications are then included in the most important fuzzy implications which satisfy the first 5 axioms in the axiomatic system of fuzzy implications. Therefore we investigated for the strong fuzzy negation, the t-norm and the t-conorm that generate a QL-implication I under which conditions

I satisfies the first place antitonicity and found out a group of QL-implications that satisfy this axiom. This work is represented in [78].

2. **The interrelationship between the 8 axioms in the axiomatic system of fuzzy implications for the implications satisfying the first 5 axioms.**
Different works before worked out some of the interrelationship between the axioms of fuzzy implications, e.g., the interrelationship between the axioms FI6, FI7, FI8 and FI11 in [[6], Baczyński 2008]. But the full interrelationship between the axioms remained not complete. It is meaningful to have a further study of the complete interrelationship between the axioms to replenish the gaps among the existed work. We worked out the complete interrelationship between the 8 axioms for the fuzzy implications satisfying the first 5, and stated for each independence case a counter-example. These examples are potential candidates for specific requirements. Through these dependence and independence we obtained a partition of the fuzzy implications. Moreover, given a fuzzy implication satisfying some axioms, we can immediately determine which other axioms it satisfies, which has a significant meaning in selecting fuzzy implications under different requirements. This work is represented in [77].

3. **Comparing different effects of fuzzy implications in approximate reasoning from both a logic point of view and a practice view.**
We first compared for the three classes of fuzzy implications generated from the other fuzzy logic operators, namely, S-, R- and QL-implications, if they satisfy the proposition

$$p \to (p \to q) = p \to q$$

in fuzzy logic. This proposition means that if we repeat the antecedent in the generalized modus ponens n times, the reasoning result remains the same. To find the answers, we solved the functional equation

$$(\forall (x, y) \in [0, 1]^2)(I(x, I(x, y)) = I(x, y))$$

for all the S-, R- and QL- implications I. The results w.r.t. QL-implications are most interesting. This work is represented in [76]
We then compared for the most important continuous t-norms, their dual t-conorms, the S-implication generated from the standard fuzzy negation and these dual t-conorms, the R-implications generated from these continuous t-norms their ability against the perturbations in the process of approximate reasoning in a fuzzy rule-based system. We considered the perturbations with some specific probability distributions, which is a further step of the existing works, e.g., [[12], Cai 2001], [[13], Cordon 2000] and [[46], Li 2005]. The results show that the compared fuzzy logic operators have their superior under different probability distributions respectively.
These works are just small parts of investigating the roles fuzzy implications play in approximate reasoning. But they are indeed good examples of comparing and

selecting fuzzy implications under different requirements. A part of this work is represented in [75].

4. **Using fuzzy implications and conjunctions on the unit interval to generate fuzzy morphological operations.**
This is the work where we compared different effects of fuzzy implications in an application other than approximate reasoning. To have the desired algebraic properties of the fuzzy morphological operations, namely, the fuzzy dilation, the fuzzy erosion, the fuzzy opening and the fuzzy closing, we obtained that the adjointness between the fuzzy implication and the conjunction on the unit interval used to generate a fuzzy dilation and a fuzzy erosion, respectively, is of great importance. An R-implication and the t-norm from which it is generated constitute a well-known adjoint couple in the literature. Our contribution resulted in obtaining the adjoint couples of S-implications and the parameterized fuzzy implications and the corresponding conjunctions on the unit interval. This resulted in many other fuzzy implication candidates than R-implications in fuzzy morphology. This work is represented in [77].

8.2 Future Work

The future work will be in the below three aspects:

1. We obtained the complete interrelationship between 8 axioms for the most important group of fuzzy implications that satisfy the other 5 axioms. Our future work is to determine the complete interrelationship between the 13 axioms for a $[0,1]^2 \to [0,1]$ mapping that only satisfies
$$I(0,0) = I(0,1) = I(1,1) = 1 \quad \text{and} \quad I(1,0) = 0.$$
2. To compare and select fuzzy implications under different requirements will remain the most interesting future work of us. First, we will use different fuzzy implications and t-norms to model the approximate reasoning process in a fuzzy rule-based system, for example, a fuzzy rule-based system for decision making, to obtain best results under different requirements. Second we will further study the fuzzy morphological operations generated from fuzzy implications and conjunctions on the unit interval, and use them in real applications in image processing.
3. We will extend our work with fuzzy morphological operations generated from fuzzy implications and conjunctions on the uint interval to a lattice $(\mathcal{L}, \preceq)$ in general, investigating their algebraic properties and applying them in image processing.

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