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Point-line spaces related to Jordan pairs

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Contents

Introduction

Jordan pairs In the present work we introduce a class of incidence geometries, more precisely, a class of point-line spaces equipped with an opposition relation, that is related to a class of algebraic structures, called Jordan pairs. The theory of Jordan pairs generalises the concept of a Jordan algebra, a commutative, not necessarily associative algebra over a commutative unital ring. Jordan algebras go back to Pascual Jordan, a German physicist of the 20th century, who introduced them to formulate quantum mechanical processes as abstract and general as possible.

A Jordan pair $V = (V^+, V^-)$ is a pair of modules over a commutative unital ring k together with a pair (Q_+, Q_-) of quadratic maps $Q_\sigma \colon V^\sigma \to \mathrm{Hom}(V^{-\sigma}, V^\sigma)$, for $\sigma \in \{+,-\}$, such that the identities

(JP1) $D_\sigma(x,y) \circ Q_\sigma(x) = Q_\sigma(x) \circ D_{-\sigma}(y,x)$

(JP2) $D_\sigma(Q_\sigma(x)y, y) = D_\sigma(x, Q_{-\sigma}(y)x)$

(JP3) $Q_\sigma(Q_\sigma(x)y) = Q_\sigma(x) \circ Q_{-\sigma}(y) \circ Q_\sigma(x)$

hold in all scalar extensions of V, where $D_\sigma(x,y)(z) := (Q_\sigma(x+z) - Q_\sigma(x) - Q_\sigma(z))y$. If the two k-modules V^+ and V^- coincide, one obtains a Jordan algebra by identifying the two k-modules.

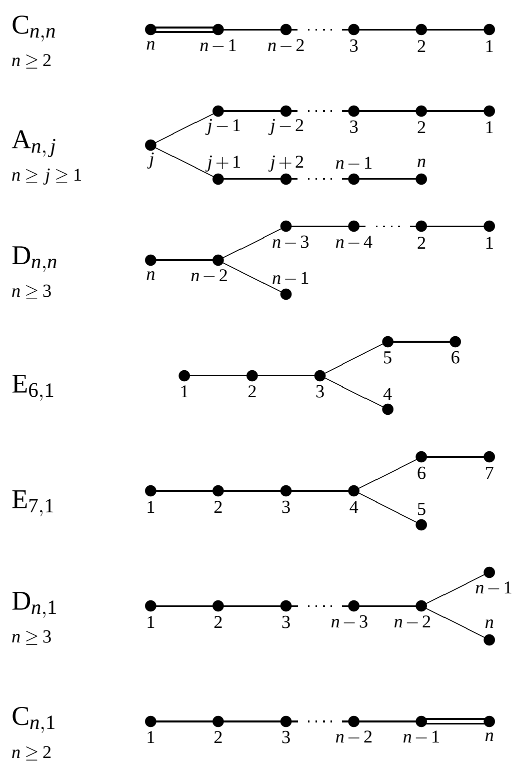
Buildings Jacques Tits, a contemporary Belgian mathematician, introduced the theory of buildings, i. e. particular combinatorial structures that provide a geometrical interpretation for semisimple isotropic linear algebraic groups; see [Tit74]. For each type of buildings there exists a Coxeter diagram which is attached to it. Furthermore, to each type of buildings there is a class of incidence geometries that is related to this type and hence as well to the attached Coxeter diagram.

In [Loo75], O. Loos classified the Jordan pairs of finite dimension. The types listed there match in a certain way a part of the list that results from the classification of buildings of finite rank. This fact motivates the conjecture that there is a connection between Jordan pairs and incidence geometries. The present work is a part of the approach to find such a connection. More precisely, we give a rather simple axiomatisation for geometries and prove that this axiomatisation holds exactly for those geometries that we expect to be the ones that are related to the Jordan pairs.

Jordan pairs and geometries Concerning a connection between Jordan pairs and geometries some earlier results have already been obtained. W. Bertram established in [Ber00] a geometric interpretation of Jordan structures by showing a strong correspondence to symmetric spaces. Furthermore, he introduced in [Ber02] generalisations of projective geometries. These generalisations are based on what he calls affine pair geometries, i. e. a pair of sets (X^+, X^-) together with a relation $M \subset X^+ \times X^-$ such that for all $a \in X^+$, the set $V_a := \{y \in X^- \mid (a,y) \in M\}$ has the structure of an affine space, and dually. The elements of M are called remote pairs. In this context, an affine space is meant in the algebraic sense which means that V_a is a module over a commutative ring. Based on the scalar multiplication and the module structure of V_a, Bertram defines ternary product maps from a subset of $X^\sigma \times X^{-\sigma} \times X^\sigma$ to X^σ, where $\sigma \in \{+,-\}$. He gives a list of certain "fundamental identities" that are satisfied if X^+ is a projective space and X^- is its dual. In this context, an affine pair geometry that fulfils these identities is called a generalised projective geometry. Further work concerning the correspondence of Jordan pairs and geometries such as symmetric spaces and generalised projective geometries is done by W. Bertram and K.-H. Neeb; see [BN04] and [BN05]. However, all these geometries are based on algebraic laws.

In the present work we use a completely different sight of geometries that is based on incidence axioms. There are some apparent similarities: Instead of remote pairs we use a relation that we call opposition relation. Furthermore, as in the work of Bertram, in the case that we have a pair of projective spaces, the set of opposite points to a given point forms an affine space. Despite these analogies we expect a direct connection between the introduced class of incidence geometries and Jordan pairs. More precisely, we think that it should be possible to construct geometries from Jordan pairs that satisfy our axioms and conversely, to construct Jordan pairs out of our geometries. Such a connection would provide the possibility to apply geometric and combinatorial methods to study Jordan pairs. Applying these methods may lead to some new results for Jordan pairs of arbitrary dimension and, eventually, to a classification of them.

The relevant diagrams The Coxeter diagrams that correspond to the list of Jordan pairs are listed below. Some of the diagrams are drawn in an unusual way. The motivation for doing so is to highlight one vertex in each diagram. This vertex is depicted as the leftmost one and represents the objects that are considered to be the points of the incidence geometries. Also the order in which we list the diagrams is not the usual, namely the alphabetical one. Instead, we order the diagrams by the symplectic rank of the geometries that are represented by the diagram together with the leftmost vertex. This symplectic rank can be easily read out of the diagram: It is the natural number r such that one can obtain a diagram of type $C_{r,1}$ or $D_{r,1}$ by repeatedly erasing the rightmost vertices.



The incidence geometries that are related to buildings of the listed types are known (for a short overview see Appendix B). The most common ones are the ones of type $A_{n,1}$, which are projective spaces, and those of the types $C_{n,1}$ and $D_{n,1}$, which are polar spaces; see Appendix A for an introduction.

Point-line spaces The rank of a building of type $X_{n,j}$, where $X_{n,j}$ is a type of the given list, and the rank of the corresponding geometry both equal n. The buildings that are related to the Jordan pairs of finite dimension are of finite rank. The aim of this work is to characterise geometries that are related to Jordan pairs of arbitrary dimension. Therefore we consider a class of geometries that contains the listed types related to the Jordan pairs of finite dimension as well as generalisation of each them that includes geometries of infinite rank.

Note that for all diagrams, n is a natural number. We do not give diagrams for geometries of infinite rank since this leads to serious problems; see Section 6.7 for a discussion. Also the geometries of infinite rank themselves provide discouraging properties of many kinds. A way to avoid some of these problems is to study point-line truncations of the given geometries, i. e. the subgeometries one obtains by considering only two kinds of objects (those that are called "points" and "lines") and forgetting about the rest. Geometries whose objects are just points and lines are also called point-line spaces.

Characterisations The first characterisation of point-line spaces is the one for projective spaces. It was published in 1965 and is due to O. Veblen and J. Young; see [VY65]. Almost ten years later, F. Buekenhout and E. Shult gave in [BS74] a characterisation for polar spaces. This characterisation is astonishingly nice since it needs solely one simple axiom. In the following years, motivated by this very nice characterisation there was put a lot of effort to characterise other types of point-line spaces that arise from buildings. In this context, one should mention among others the work of F. Buekenhout ([Bue82]), P. Cameron ([Cam82]), A. Cohen and B. Cooperstein ([CC83]), G. Hanssens ([Han86] and [Han88]), A. Kasikova ([KS02]) and E. Shult ([Shu89], [Shu94] and [Shu03]). For an overview of characterisations of point-line spaces see [Coh95] and the forthcoming book of Shult. The obtained characterisations include all the types of our list and many more. However, some of the characterisations provide a list of up to ten axioms including rather technical ones.

Point-line spaces of infinite rank For each of the types $A_{n,j}$, $C_{n,1}$, $D_{n,1}$, $C_{n,n}$ and $D_{n,n}$ there is a natural way to give a generalisation that includes point-line spaces of infinite rank. Thereby, the polar spaces (types $C_{n,1}$ and $D_{n,1}$) and the point-line spaces of type $A_{n,j}$, for any fixed $j \in \mathbb{N}$, play a special role. The polar spaces have all diameter 2 and the characterisation of Buekenhout and Shult still holds for polar spaces of infinite rank. The diameter of the point-line spaces of type $A_{n,j}$ (called Grassmannians), is the minimum of j and $n-j$. Hence, for the ones of infinite rank, we always obtain diameter j. Accordingly, if both j and $n-j$ increase, we obtain point-line spaces of any finite diameter. The same is true for point-line spaces of type $C_{n,n}$ (called dual polar spaces) that have diameter n and for those of type $D_{n,n}$ (called half-spin spaces) that have diameter $\lfloor \frac{n}{2} \rfloor$. A generalisation of all these types that allows the point-line spaces to have infinite rank leads to point-line spaces that are disconnected. More precisely, one obtains point-line spaces with infinitely many connected components. Thus, the known characterisations do not work anymore.

Using an idea of B. Mühlherr, we pick two of these connected components and equip the so obtained pair of point-line spaces with an opposition relation that relates points of the one component with points of the other one. This approach is in the spirit of the theory of twin buildings allows us to give a characterisation that is still valid for the infinite rank case. Additionally, one can state axioms that are less technical and thus, we are able to give a list of four quite nice axioms that characterises the point-line spaces in question. By the way, the geometrical objects we are now dealing with consist of two parts that are related to each other; just like Jordan pairs.

Setup In Chapter 1 we introduce point-line spaces. Moreover, we present the concept of an opposition relation to consider point-line spaces that are disconnected. At this point, the reader might familiarise himself with projective and polar spaces which are introduced in Appendix A since in the following chapters both of them will appear as well as some of the results stated there. We also will make some comments about point-line spaces arising from buildings which are considered in Appendix B. However, the results of Appendix B are needed in Chapter 5 at the latest.

The main matter of the present work starts in Chapter 2. Here we introduce SPO spaces, the class of point-line spaces that is the topic of our research. We state a list of axioms that characterises the SPO spaces. Moreover, we already state several strong results which deliver some deep insight into the subsequent classification.

Chapter 3 provides a first classification of connected subspaces that live in SPO spaces. Thereby we demand the connected subspaces to have a certain regularity. We call the subspaces with this regularity rigid subspaces. We will see that the list of connected rigid subspaces we consider in this chapter coincides with the types of finite rank that are listed above. The only exceptions are the point-line spaces of the types $A_{n,1}$, $C_{n,1}$ and $D_{n,1}$ since for these cases we also obtain their generalisations to point-line spaces of arbitrary rank.

In Chapter 4 we show that each SPO space can be decomposed into subspaces that are all rigid SPO spaces. Conversely, each composition of rigid SPO spaces is again an SPO space. This allows us to restrain our study to rigid SPO spaces. Thus, we may use the classification results of Chapter 3 for the classification of arbitrary SPO spaces.

Before we give the full classification, we discuss in Chapter 5 the point-line spaces of the types listed above. Moreover, we give generalisations of the distinct types that allows the point-line spaces to be of arbitrary rank. Thus, the class of subspaces we obtain is exactly class of the point-line spaces that we wanted to characterise.

Chapter 6 provides the main result of this thesis. We give the classification of rigid SPO spaces and prove that this classification matches exactly the point-line spaces presented in Chapter 5.

1 Preliminaries and notations

1.1 Point-line spaces

A *point-line space* $\mathscr{S} = (\mathscr{P}, \mathscr{L})$ is a pair consisting of a set $\mathscr{P}$, whose elements are called *points* and a set $\mathscr{L} \subset \mathfrak{P}(\mathscr{P})$ of subsets of $\mathscr{P}$ with cardinality at least 2, which are called *lines* (By $\mathfrak{P}(M)$ we denote the power set of a set M). If all points are subsets of a common set, we sometimes regard a line as the union of its points.

Points on a common line are called *collinear* . We write $p_0 \perp p_1$ to denote that p_0 and p_1 are collinear. The relation $\perp$ induces a graph on the point set $\mathscr{P}$ that we call the *collinearity graph*. If $p_0 \perp p_1$, then we call p_1 a *neighbour* of p_1. By $p^\perp$ we denote the set of all neighbours of a point p, called the *perp* of p. For a set of points X we denote by $X^\perp := \bigcap_{p \in X} p^\perp$ the perp of X, i. e. the set of all common neighbours.

We give a list of some elementary rules that are valid in arbitrary point-line spaces:

Lemma 1.1.1. *Let M and N be sets of points of a point-line space with $N \subseteq M$. Then:*

(i) $N^\perp \supseteq M^\perp$

(ii) $M \subseteq M^{\perp\perp}$

(iii) $M^\perp = M^{\perp\perp\perp}$

Proof. $N \subseteq M$ implies $M^\perp = \bigcap_{p \in M} p^\perp = (\bigcap_{p \in N} p^\perp) \cap (\bigcap_{p \in M \setminus N} p^\perp) \subseteq N^\perp$. Since every point of M is collinear to every point of $M^\perp$, we obtain $M \subseteq M^{\perp\perp}$. This implies $M^\perp \subseteq (M^\perp)^{\perp\perp}$ and $M^\perp \supseteq (M^{\perp\perp})^\perp$. □

A *subspace* of a point-line space $\mathscr{S} = (\mathscr{P}, \mathscr{L})$ is a point-line space $\mathscr{S}' = (\mathscr{P}', \mathscr{L}')$ with $\mathscr{P}' \subseteq \mathscr{P}$ and $\mathscr{L}' \subseteq \mathscr{L}$ such that every line in $\mathscr{L} \setminus \mathscr{L}'$ has at most one point with $\mathscr{P}'$ in common and every line in $\mathscr{L}'$ is contained in $\mathscr{P}'$. We write $\mathscr{S}' \leq \mathscr{S}$, if $\mathscr{S}'$ is a subspace of $\mathscr{S}$ and $\mathscr{S}' < \mathscr{S}$ if $\mathscr{S}'$ is properly contained. Since $\mathscr{S}'$ is determined by its point set, we call $\mathscr{P}'$ itself a subspace. Correspondingly, we treat $\mathscr{S}$ sometimes as its own point set. A proper subspace is called a *hyperplane* if it intersects every line. For a set of points M, we denote by $\langle M \rangle$ the smallest subspace which contains M, called the *span* of M. For a family of points $p_0, \dots p_s$ and a family of sets of points $M_0, \dots, M_r$ we will write $\langle p_0, \dots, p_s, M_0, \dots, M_r \rangle$ rather than $\langle \{p_0, \dots, p_s\} \cup M_0 \cup \dots \cup M_r \rangle$.

A *partially linear space* is a point-line space such that no two different lines have two different points in common. Clearly, subspaces of partially linear spaces are again partially linear. For two distinct collinear points p and q of a partially linear space, the unique line joining p and q is denoted by pq. A space that contains exactly one point is called a *singleton*.

A point-line space where every two points are collinear is called *singular*. Singular partially linear spaces are called *linear*. The *rank* of a singular space $\mathscr{S}$ is denoted by $\mathrm{rk}(\mathscr{S})$ and equals $\alpha - 2$, where α is the maximal possible cardinality of a well-ordered chain of subspaces of $\mathscr{S}$. Hence, the rank of the empty space is -1 and the rank of a singleton is 0. Note that there might exist well-ordered chains that are maximal but not of maximal possible cardinality. For a point-line space $\mathscr{S}$ let $\mathfrak{S}(\mathscr{S}) := \{X \leq \mathscr{S} \mid X \subseteq X^{\perp}\}$ denote the set of all singular subspaces of $\mathscr{S}$. The *singular rank* of $\mathscr{S}$ is defined as $\mathrm{srk}(\mathscr{S}) := \sup\{\mathrm{rk}(X) \mid X \in \mathfrak{S}(\mathscr{S})\}$.

We take for point-line spaces some terminology over from the underlying collinearity graph: A *path* (of length k) between two points p_0 and p_k is a finite sequence $(p_i)_{0 \leq i \leq k}$ of points such that $p_i \perp p_{i+1}$ for every $i < k$. We define the *distance* $\mathrm{dist}(p, q)$ between two points p and q as the length of a shortest path between them. If no such path exists, the distance between p and q is set to be ∞. We call two points p and q *connected*, if their distance is finite and *disconnected* otherwise. A point-line space is called connected if every pair of its points is connected. A maximal connected subspace is called a *connected component*. Let X be a set of points. Then the *diameter* of X is the supremum of all distances between two points of X and is denoted by $\mathrm{diam}(X)$.

A shortest path between two points is called a *geodesic* . A set of points is called *convex* if it contains for every pair of points all geodesics. For a set of points M, we denote by $\langle M \rangle_{\mathrm{g}}$ the smallest convex subspace which contains M, called the *convex span* of M.

A *gamma space* is a point-line space with the property that for each point p and each line l, the set $p^{\perp} \cap l$ is either empty, a singleton or equals l. In other words a point-line space is a gamma space if and only if for every point p, the set $p^{\perp}$ is a subspace. This property yields some useful applications. The first one is that the perp of a subspace equals the perp of any set of points spanning it:

Lemma 1.1.2. *Let M be a set of points of a gamma space. Then $\langle M^\perp\rangle = M^\perp = \langle M\rangle^\perp$.*

Proof. Since $M^\perp$ is a subspace, the first equation is trivial. Since $M^{\perp\perp}$ is a subspace containing M, we obtain $M \subseteq \langle M\rangle \leq M^{\perp\perp}$. By Lemma 1.1.1 we conclude $M^\perp \geq \langle M\rangle^\perp \geq M^{\perp\perp\perp} = M^\perp$. □

The second property concerns singular subspaces. More precisely, the span of a set of points with diameter 1 has again diameter 1.

Lemma 1.1.3. *Let M be a set of mutually collinear points of a gamma space. Then the subspace $\langle M\rangle$ is singular.*

Proof. Since $M \subseteq M^\perp$, we obtain $M^\perp \geq M^{\perp\perp}$. Since $M^\perp = M^{\perp\perp\perp}$ by Lemma 1.1.1, this implies that $M^{\perp\perp}$ has to be singular. Since $M \subseteq M^{\perp\perp}$, we obtain $\langle M\rangle \leq M^{\perp\perp}$. Thus, $\langle M\rangle$ is singular. □

A *morphism* $\varphi\colon (\mathscr{P}_0,\mathscr{L}_0) \to (\mathscr{P}_1,\mathscr{L}_1)$ of point-line spaces is a map from $\mathscr{P}_0$ to $\mathscr{P}_1$ such that the image of every line in $\mathscr{L}_0$ is contained in some line of $\mathscr{L}_1$. If for every line in $\mathscr{L}_0$, the image under the morphism φ is an element of $\mathscr{L}_1$, then φ is called a *homomorphism*. An *isomorphism* is a bijective morphism φ, such that the inverse map φ^{-1} is again a morphism.

Let I be an index set and let $(\mathscr{S}_i)_{i\in I}$ be a family of point-line spaces. For $\mathscr{S}_i$, we denote by $\mathscr{P}_i$ the set of point and by $\mathscr{L}_i$ the set of lines of $\mathscr{S}_i$. We define the *grid product* of the point-line spaces $(\mathscr{S}_i)_{i\in I}$ as

$$\bigotimes_{i\in I}\mathscr{S}_i := \left(\prod_{i\in I}\mathscr{P}_i, \bigcup_{i\in I}\left\{\prod_{j\in I} S_j \;\middle|\; \begin{array}{ll} S_j \in \mathscr{L}_j & \text{if } j = i \\ S_j = \{p_j\} \text{ with } p_j \in \mathscr{P}_j & \text{if } j \neq i \end{array}\right\}\right).$$

Even if for every $i \in I$, the point-line space $\mathscr{S}_i$ is connected, it might happen that $\bigotimes_{i\in I}\mathscr{S}_i$ is a disconnected point-line space. This is the case if I is infinite and every point-line space contains at least two points. Therefore we introduce a concept that is similar to the grid product and preserves connectedness. For this we require that $\mathscr{P}_i$ is non-empty for every $i \in I$. We choose a point $p_i \in \mathscr{P}_i$ for every $i \in I$. Now we define $\bigodot_{i\in I}(\mathscr{S}_i, p_i) \leq \bigotimes_{i\in I}\mathscr{S}_i$ to be the subspace that consists of all points $(q_i)_{i\in I} \in \prod_{i\in I}\mathscr{P}_i$ such that the set $\{i \in I \mid p_i \neq q_i\}$ is finite. We call $\bigodot_{i\in I}(\mathscr{S}_i, p_i)$ the *grid sum* of $(\mathscr{S}_i)_{i\in I}$ with *origin* $(p_i)_{i\in I}$. By definition of the lines of $\bigotimes_{i\in I}\mathscr{S}_i$ it is clear that $\bigodot_{i\in I}(\mathscr{S}_i, p_i)$ is indeed a subspace of $\bigotimes_{i\in I}\mathscr{S}_i$. If there is a point p such that $\mathscr{P}_i \cap \mathscr{P}_j = \{p\}$ for every two distinct indices i and j of I, we write $\bigodot_{i\in I}\mathscr{S}_i$ instead of $\bigodot_{i\in I}(\mathscr{S}_i, p)$.

1.2 Point-line spaces with a codistance

In a point-line space that is disconnected there is a priori no link at all between two distinct connected components. In this section we introduce a method to relate them to each other.

Recall that for a set M a relation $R \subset M \times M$ is called *left-total*, if $M = \{x \mid \exists y \in M\colon (x,y) \in R\}$. *Right-total* is defined in the analogous way. A relation that is left-total and right-total is called *total*. For a symmetric relation these three terms are obviously equivalent.

Definition 1.2.1. Let $(\mathscr{P},\mathscr{L})$ be a point-line space with a symmetric, total point-relation $R \subset \mathscr{P} \times \mathscr{P}$. Then we call $\mathrm{cod}_R(x,y) := \min\{\mathrm{dist}(z,y) \mid (x,z) \in R\}$ the *R-codistance* from x to y.

Note that this definition does not imply $\mathrm{cod}(x,y) = \mathrm{cod}(y,x)$. Nevertheless, in the following we will always consider a symmetric, total point-relation R such that the derived R-codistance is symmetric.

Since we introduce a codistance function to study point-line spaces that are disconnected, in most of the cases the underlying symmetric, total point-relation R will contain only pairs of disconnected points. Thereby the codistance function is some kind of refined distance function for points at infinite distance. More precisely, the pairs contained in R can be understood as pairs of points at maximal distance. Therefore, the greater the codistance between two points is, the closer these points are in a certain sense. For a natural number n, it is helpful to visualise "codistance n" as "distance $\infty - n$", where ∞ should be seen as a symbol that stands for the diameter of the point-line space. Note that finite codistance does not always imply infinite distance since the concept of the codistance also works for point-line spaces with a finite diameter. In the following, whenever we consider point-line spaces of finite diameter with a codistance function, the mentioned symbol ∞ can be substituted by the diameter of the point-line space and we obtain the actual distance.

This point of view motivates to define the R-codistance for two sets of points X and Y by $\mathrm{cod}_R(X,Y) := \sup\{\mathrm{cod}_R(x,y) \mid x \in X \wedge y \in Y\}$. Correspondingly, the *R-codiameter* for a set of points X is defined by $\mathrm{codm}_R(X) := \min\{\mathrm{cod}_R(x,y) \mid \{x,y\} \subseteq X\}$.

Definition 1.2.2. Let U be a subspace and let p be a point of a point-line space $\mathscr{S}$. If $\mathrm{dist}(p,U) < \infty$, we call the set $\mathrm{pr}_U(p) := \{u \in U \mid \mathrm{dist}(p,u) = \mathrm{dist}(p,U)\}$ the *projection* of p in U.
Let R be a symmetric, total point-relation. Then we call $\mathrm{copr}_{R,U}(p) := \{u \in U \mid \mathrm{cod}_R(p,u) = \mathrm{cod}_R(p,U)\}$ the *R-coprojection* of p in U if $\mathrm{cod}_R(p,U) < \infty$.

Definition 1.2.3. Let U and V be two subspaces of a point-line space $\mathscr{S}$. Further let $\operatorname{dist}(U,V) < \infty$. Then we call U *one-parallel* to V if for every point $u \in U$, $\operatorname{dist}(u,V) = \operatorname{dist}(U,V)$ and $\operatorname{pr}_V(u)$ is a singleton.
Let R be a symmetric, total point-relation. Further let $\operatorname{cod}_R(U,V) < \infty$. Then we call U *R-one-coparallel* to V if for every point $u \in U$, $\operatorname{cod}(u,V) = \operatorname{cod}(U,V)$ and $\operatorname{copr}_{R,V}(u)$ is a singleton.

Note that our definitions of one-parallel and R-one-coparallel are not symmetric. In most cases, the disconnected point-line spaces with a codistance that we consider consist of two connected components. Furthermore, they are of the following type:

Definition 1.2.4. Let $\mathscr{S}^+ = (\mathscr{P}^+, \mathscr{L}^+)$ and $\mathscr{S}^- = (\mathscr{P}^-, \mathscr{L}^-)$ be two disjoint partially linear spaces. Further let $R \subseteq (\mathscr{P}^+ \times \mathscr{P}^-) \cup (\mathscr{P}^- \times \mathscr{P}^+)$ be a symmetric, total relation on $\mathscr{P}^+ \cup \mathscr{P}^-$ such that for every pair $(p,l) \in \mathscr{P}^+ \times \mathscr{L}^- \cup \mathscr{P}^- \times \mathscr{L}^+$, the following holds:

(OP) If $(\{p\} \times l) \cap R$ is non-empty, there is a point $q \in l$ such that $(\{p\} \times l) \cap R = \{p\} \times (l \setminus \{q\})$.

Then we call the pair $(\mathscr{S}^+, \mathscr{S}^-)$ a *twin space* and R the *opposition relation* of $(\mathscr{S}^+, \mathscr{S}^-)$.

Let (p,q) be a pair of points of a twin space that is contained in the opposition relation. Then we say p and q are *opposite* points or p is opposite q and denote it by $p \leftrightarrow q$. With this way of speaking we can reformulate (OP) as follows: Each point is non-opposite to either all or exactly one point of a given line.

If we talk about a codistance in a twin space, it always refers to the opposition relation of the twin space.

A *morphism* $\varphi\colon (\mathscr{S}_0^+, \mathscr{S}_0^-) \to (\mathscr{S}_1^+, \mathscr{S}_1^-)$ of point-line spaces is a mapping of the union of the underlying point sets that preserves opposition and for $\sigma \in \{+,-\}$, the restriction $\varphi|_{\mathscr{S}_0^\sigma}$ is a morphism of point-line spaces from $\mathscr{S}_0^\sigma$ to either $\mathscr{S}_1^+$ or $\mathscr{S}_1^-$. The morphism φ is called a *homomorphism* (resp. an *isomorphism*) if for $\sigma \in \{+,-\}$, the restriction $\varphi|_{\mathscr{S}_0^\sigma}$ is a homomorphism into (resp. an isomorphism onto) either $\mathscr{S}_1^+$ or $\mathscr{S}_1^-$.

2 SPO spaces

In this chapter we introduce a class of point-line spaces that play the main role in this work. These point-line spaces are equipped with a symmetric, total point-relation, called "opposition relation", that gives rise to a codistance. Since in the majority of the cases there is no doubt about the point-relation we refer to, we talk about "codistance", "coprojection" and "one-coparallel" without mentioning the underlying point-relation in these terms.

We shall classify these point-line spaces in the present work. Therefore we discuss some extra assumptions each one of which yields nice extra conditions that facilitate the classification. We will justify why these assumptions can be made. However, some of them will be motivated in the subsequent chapters. Furthermore, we prove some first properties concerning the structure of the lattice of subspaces.

2.1 Main Definition and fundamental properties

We start by defining the point-line spaces that will be the objects of interest in all the present work. For the independence of the four axioms of the following definition, see Appendix C. Since at first sight these axioms look rather technical, we give subsequent to the definition a brief discussion about their intention as well as a motivation how they should be visualised.

Definition 2.1.1. Let $\mathscr{S}$ be a point-line space and let R be a symmetric, total point-relation that induces a codistance on $\mathscr{S}$. Then we call $\mathscr{S}$ an *SPO space*[1] and R an *opposition relation* of $\mathscr{S}$ if the following conditions hold for all points x, y and z with $\mathrm{dist}(y,z) < \infty$ and $\mathrm{cod}(x,y) < \infty$. We set $n := \mathrm{dist}(y,z)$ and $V := \langle y,z \rangle_{\mathrm{g}}$.

[1]SPO stands for "**s**trongly **p**arapolar with an **o**pposition relation". This is because later on we will see that each non-singular connected component of an SPO space is strongly parapolar.

(A1) If $(x,v) \in R$ for some $v \in V$, then $\mathrm{cod}(x,V) = n$.

(A2) If $(x,v) \in R$ for some $v \in V$, then $\mathrm{copr}_V(x)$ is a singleton.

(A3) If $z \in \mathrm{copr}_V(x)$ and $w \perp x$ with $\mathrm{cod}(x,y) > \mathrm{cod}(w,y)$, then

(a) $\mathrm{cod}(x,V) \geq \mathrm{cod}(w,V)$ and $\mathrm{copr}_V(x) \supsetneq \mathrm{copr}_V(w)$ or
(b) $\mathrm{cod}(x,V) > \mathrm{cod}(w,V)$ and $\mathrm{copr}_V(x) \supseteq \mathrm{copr}_V(w)$.

(A4) If $y \perp z$ and $(x,y) \in R$, then there is a point $w \perp x$ with $(w,z) \in R$.

Mostly, we do not mention any opposition relation explicitly. In this case, the opposition relation will be denoted by $\leftrightarrow$. As for twin spaces, we call two points *opposite* if they form a pair of the opposition relation.

We state some immediate consequences of the given axioms and a short motivation how they can be interpreted.

The axiom (A4) is equivalent to the assertion that the codistance of an SPO space is symmetric as the first claim of the following proposition implies. The second claim relates the distance and the codistance function. More precisely, it can be seen as extension of the triangle inequality to the case of infinite distances.

Proposition 2.1.2. *Let x, y and z be points of an SPO space such that* $\mathrm{cod}(x,y) < \infty$ *and* $\mathrm{dist}(y,z) < \infty$. *Then*

(i) $\mathrm{cod}(x,y) = \mathrm{cod}(y,x)$ *and*

(ii) $\mathrm{dist}(y,z) \geq \mathrm{cod}(x,y) - \mathrm{cod}(x,z)$.

Proof. (i) Let $\mathrm{cod}(x,y) = n$ and let $(y_i)_{0 \leq i \leq n}$ be a geodesic with $y_n = y$ and $y_0 \leftrightarrow x$. Set $x_0 := x$. By (A4) there is for every $i < n$ a point x_{i+1} collinear to x_i and opposite y_{i+1}. We conclude $\mathrm{cod}(y,x) \leq n$. Equality follows by exchanging x and y.
(ii) Let w be a point with $w \leftrightarrow x$ and $\mathrm{dist}(z,w) = \mathrm{cod}(x,z)$. Then $\mathrm{dist}(y,w) \leq \mathrm{cod}(x,z) + \mathrm{dist}(y,z)$. □

Since this proposition is just what one would usually expect of a refinement of the distance function, we will use these conditions in the following without referring to them.

Axiom (A1) controls the size of the convex span of two points at finite distance as the following proposition shows. Note we do not make use of any axiom other than (A1).

Proposition 2.1.3. *Let y and z be two points of an SPO space at distance n. Then* $\mathrm{diam}(\langle y,z\rangle_{\mathrm{g}}) = n$.

Proof. Let u and v be two points of $\langle y,z\rangle_{\mathrm{g}}$ at distance k. Let p be a point opposite u. Then $\mathrm{cod}(p,\langle u,v\rangle_{\mathrm{g}}) = k$ and $\mathrm{cod}(p,\langle y,z\rangle_{\mathrm{g}}) = n$ by (A1). Thus, $k \leq n$. □

The consequences of (A2) and (A3) are less obvious. Axiom (A2) is a kind of generalisation of (OP) for twin spaces. Note that the axiom (BS) for polar spaces is also similar to (A2) if we understand non-collinear points in a polar space as opposite points. Furthermore, as we will see in the following subsection, (A1) and (A2) imply that an SPO space can be treated as a partially linear space.

The Axiom (A3) is the least intuitive of the four axioms. Let all notation be like in Definition 2.1.1. If we understand opposite points to be points at maximal distance, then the points in $\mathrm{copr}_V(x)$ are the points of V at minimal distance to x. We know already that the diameter of V equals n. Hence, y is a point of V with maximal possible distance to a point of $\mathrm{copr}_V(x)$. One would expect such a point to have minimal possible codistance to x, what is actually true as we will see later. Now we decrease this minimal possible codistance to y by stepping from x onto w and the claim of (A3) is now that either the codistance to V decreases or the codistance to V stays the same and the coprojection decreases. One can visualise this situation in the way that if we move away from y, we move away from V.

2.1.1 Simplifications

There are two extra assumptions we will make to simplify studying SPO spaces. We will motivate why these assumptions can be done and show that they do not affect the theory of SPO spaces too much. The first one concerns the lines of an SPO space. We consider the subspaces spanned by single line and show that they can be regarded as new lines.

Lemma 2.1.4. *Let g be a line of an SPO space. Then $\langle g\rangle = \langle g\rangle_{\mathrm{g}}$.*

Proof. Let p and q be distinct points on g. Then $\langle g\rangle_{\mathrm{g}} = \langle p,q\rangle_{\mathrm{g}}$ since $g \subseteq \langle p,q\rangle$ and therefore $\mathrm{diam}(\langle g\rangle_{\mathrm{g}}) = 1$ by Proposition 2.1.3. Since $\langle g\rangle \leq \langle g\rangle_{\mathrm{g}}$, this implies $\mathrm{diam}(\langle g\rangle) = 1$. Thus, $\langle g\rangle$ is convex and therefore $\langle g\rangle_{\mathrm{g}} = \langle g\rangle$. □

Lemma 2.1.5. *Let g and h be two lines of an SPO space. Then $|\langle g\rangle \cap \langle h\rangle| \geq 2$ implies $\langle g\rangle = \langle h\rangle$.*

Proof. Let y be an arbitrary point of $\langle g\rangle$ and let x be a point opposite y. Since $\langle g\rangle = \langle g\rangle_{\mathrm{g}}$, we obtain by (A1) and (A2) that there is a point $z \in \langle g\rangle$ such that $\mathrm{cod}(x,z) = 1$ and x is opposite to all points of $\langle g\rangle \setminus \{z\}$. Since $\mathrm{cod}(x,z) = 1$, there is a point $w \perp x$ with $w \leftrightarrow z$. Set $U := \langle w,x\rangle_{\mathrm{g}}$. Then $\mathrm{diam}(U) = 1$ by Proposition 2.1.3. Moreover, (A1) and (A2) imply that x is the unique point of U that is non-opposite to z. By (A1) and (A2) there is exactly one point in U that is not opposite y. Since this point is distinct to x, we may assume that w is the unique point in U not opposite y. Again by (A1) and (A2) all points of $\langle g\rangle \setminus \{y\}$ are opposite w.
Now let p and q be two distinct points of $\langle g\rangle$. We may assume $q \neq y$ and hence

$w \leftrightarrow q$. By (A1) we obtain $\mathrm{cod}(w, \langle p,q\rangle_{\mathrm{g}}) = 1$ and therefore $y \in \langle p,q\rangle$ since $\langle p,q\rangle_{\mathrm{g}} = \langle p,q\rangle \leq \langle g\rangle$. We conclude $\langle g\rangle \leq \langle p,q\rangle$ by the arbitrary choice of y. Thus, $\langle g\rangle \leq \langle h\rangle$ for every line h with $|\langle g\rangle \cap \langle h\rangle| \geq 2$. The claim follows by symmetric reasons. □

Proposition 2.1.6. *Let $\mathscr{S} = (\mathscr{P}, \mathscr{L})$ be an SPO space. Then $\mathscr{S}' := (\mathscr{P}, \{\langle g\rangle \mid g \in \mathscr{L}\})$ is again an SPO space with the same opposition relation. Moreover, the distance and the codistance in $\mathscr{S}$ and $\mathscr{S}'$ are the same and a set of points $U \subseteq \mathscr{P}$ is a subspace of $\mathscr{S}$ if and only if it is a subspace in $\mathscr{S}'$.*

Proof. Set $\mathscr{S} := (\mathscr{P}, \mathscr{L})$ and $\mathscr{S}' := (\mathscr{P}, \{\langle g\rangle \mid g \in \mathscr{L}\})$. Let U be a subspace of $\mathscr{S}$ and let p and q be two distinct points of U such that there is a line $g \in \mathscr{L}$ with $\{p,q\} \subseteq \langle g\rangle$. By Proposition 2.1.3 and Lemma 2.1.4 we know that $\langle g\rangle$ is a singular subspace of $\mathscr{S}$. Thus, there is a line $h \in \mathscr{L}$ that joins p and q. Therefore Lemma 2.1.5 implies $\langle g\rangle = \langle h\rangle \leq U$ and hence, $U \leq \mathscr{S}'$. Now let U be a subspace of $\mathscr{S}'$ and let p and q be two distinct points of U such that there is a line $g \in \mathscr{L}$ with $\{p,q\} \subseteq g$. Then $g \subseteq \langle g\rangle \leq U$ and hence, $U \leq \mathscr{S}$.
Since $\langle g\rangle$ is singular for ever line $g \in \mathscr{L}$, two points are collinear in $\mathscr{S}$ if and only if they are collinear in $\mathscr{S}'$. Therefore, in both spaces the distance between two certain points is the same. Consequently, using the same opposition relation in $\mathscr{S}'$ as in $\mathscr{S}$ implies that the codistance is maintained, too.
By the accordance of the distance, a subspace of $\mathscr{S}$ is convex if and only if it is a convex subspace of $\mathscr{S}'$. Now it is easy to check that all four conditions of Definition 2.1.1 hold in $\mathscr{S}$ if and only if they hold in $\mathscr{S}'$. □

Remark 2.1.7. For an arbitrary SPO space $\mathscr{S} = (\mathscr{P}, \mathscr{L})$ the SPO space $\mathscr{S}' := (\mathscr{P}, \{\langle g\rangle \mid g \in \mathscr{L}\})$ is partially linear by Lemma 2.1.5. Therefore we call $\mathscr{S}'$ the *associated partially linear SPO space*. By Proposition 2.1.6 the point-line spaces $\mathscr{S}$ and $\mathscr{S}'$ have the same lattice of subspaces. Singularity, convexity, distance and codistance coincide as well. The main difference between $\mathscr{S}$ and $\mathscr{S}'$ is that we have to exchange the term "line" by "span of a line". Obviously, this just makes the notation more complicated and takes the advantage of having unique lines away.
These facts allow us to restrict our studies to SPO spaces that are partially linear. All the results we obtain can be easily transformed into results for arbitrary SPO spaces. Thus, from now on we consider all SPO spaces to be partially linear. Note that a partially linear SPO space is still a SPO space if we substitute an arbitrary line l by a singular subspace S that contains the same points as l and coincides with the span of each of its lines. For example, if l contains more than 3 points, we may substitute l by any set of lines of the kind $\{g \subseteq l \mid |g| = \alpha\}$, where $3 \leq \alpha \leq |l|$.

The second simplification we will make concerns the opposition relation and the connected components. Let $\mathscr{S}$ be an SPO space. For a point $p \in \mathscr{S}$ we denote by $\mathscr{S}_p$ the connected component of $\mathscr{S}$ containing p. Now let x and y be opposite points of $\mathscr{S}$. Then each point of $\mathscr{S}_x$ has finite distance to x and consequently, each point of $\mathscr{S}_x$ has finite codistance to y. This implies that for each point $p \in \mathscr{S}_x$ there is a point in $\mathscr{S}_y$ that is opposite p. Conversely, to every point of $\mathscr{S}_y$ we find an opposite point in $\mathscr{S}_x$. This motivates us to call two connected components *opposite* if one of them contains a point that is opposite to a point of the other. Now we define the *connectivity graph* $\Gamma_C(\mathscr{S})$ of $\mathscr{S}$ as the graph whose vertex set consists of the set of connected components of $\mathscr{S}$ and whose edges are the pairs of opposite connected components.

If $\Gamma_C(\mathscr{S})$ is disconnected, then the union of the vertices of each connected component of $\Gamma_C(\mathscr{S})$ is an SPO space itself. These SPO spaces form a partition of $\mathscr{S}$. Moreover, each two of these SPO spaces do not interact in any way whatsoever. Since on the other hand every disjoint union of a family of SPO spaces is again an SPO space, we may constrain ourselves on SPO spaces whose connectivity graphs are connected.

It might happen that $\Gamma_C(\mathscr{S})$ has *loops*, i. e. edges that join a vertex with itself. Since $\leftrightarrow$ is total, every vertex of this graph is contained in at least one edge. We pick an edge of $\Gamma_C(\mathscr{S})$ and denote the vertices of this edge by $\mathscr{S}^+$ and $\mathscr{S}^-$. Then we delete all other edges and all vertices but $\mathscr{S}^+$ and $\mathscr{S}^-$. Now we consider the subspace $\mathscr{S}' := \mathscr{S}^+ \cup \mathscr{S}^- \leq \mathscr{S}$. Further we restrict the induced opposition relation $\leftrightarrow|_{\mathscr{S}'}$ to pairs of points that have a member in either of the connected components $\mathscr{S}^+$ and $\mathscr{S}^-$ and denote the so obtained point-relation by $\leftrightarrow'$. The subspace $\mathscr{S}'$ together with the relation $\leftrightarrow'$ is exactly the substructure that matches to the graph consisting of $\mathscr{S}^+$ and $\mathscr{S}^-$ and the edge joining them. For two points of $\mathscr{S}'$ the distance in $\mathscr{S}'$ between them is the same as their distance in $\mathscr{S}$. The codistance might differ as long as $\mathscr{S}^+ \neq \mathscr{S}^-$ and $\Gamma_C(\mathscr{S})$ has a loop at $\mathscr{S}^+$ or $\mathscr{S}^-$. However, the codistance between a point of $\mathscr{S}^+$ and a point of $\mathscr{S}^-$ in $\mathscr{S}'$ is the same as their codistance in $\mathscr{S}$ since two points $p \in \mathscr{S}^+$ and $q \in \mathscr{S}^-$ are opposite in $\mathscr{S}'$ if and only if they are opposite in $\mathscr{S}$. It is now easy to check that the four axioms of Definition 2.1.1 are still valid in $\mathscr{S}'$. Since the restricted opposition relation $\leftrightarrow'$ is a total relation in $\mathscr{S}'$, we conclude that $\mathscr{S}'$ is again an SPO space. Thus, every connected component of an SPO space is the connected component of an SPO space whose connectivity graph possesses one single edge.

Assume $\mathscr{S}$ is connected and consequently, $\Gamma_C(\mathscr{S})$ consists of one vertex and a loop on it. Let $\mathscr{S}^+$ and $\mathscr{S}^-$ be disjoint copies of $\mathscr{S}$. For $\sigma \in \{+,-\}$ let φ_σ be the canonical isomorphism from $\mathscr{S}^\sigma$ onto σ. We set $\mathscr{S}' := \mathscr{S}^+ \cup \mathscr{S}^-$. Since we do not add any additional lines to $\mathscr{S}'$ beside the ones of $\mathscr{S}^+$ and $\mathscr{S}^-$, we obtain for two points p and q of $\mathscr{S}'$ that p and q are connected if and only if they both belong to $\mathscr{S}^\sigma$ for $\sigma \in \{+,-\}$. Moreover, the distance of p and q in $\mathscr{S}'$ coincides with the distance of their images in $\mathscr{S}$ under φ_σ. Two points in $\mathscr{S}'$ are opposite if and only if one point belongs to $\mathscr{S}^+$ and the other one to $\mathscr{S}^-$ and their images under

φ_+ and φ_- are opposite in $\mathscr{S}$. We denote the so obtained relation by $\leftrightarrow'$ and the opposition relation of $\mathscr{S}$ by $\leftrightarrow$. By construction, $\leftrightarrow'$ is a symmetric, total point-relation that induces a codistance on $\mathscr{S}'$. Moreover, for a point $p \in \mathscr{S}^+$ all points of $\mathscr{S}^+$ are at infinite codistance and all points of $\mathscr{S}^-$ are at finite codistance since $\mathscr{S}^-$ is connected. Let q and r be points of $\mathscr{S}^-$. Then $p \leftrightarrow' r \Leftrightarrow p^{\varphi_+} \leftrightarrow r^{\varphi_-}$ and $\operatorname{dist}(q,r) = \operatorname{dist}(q^{\varphi_-}, r^{\varphi_-})$. We conclude $\operatorname{cod}(p,q) = \operatorname{cod}(p^{\varphi_+}, r^{\varphi_-})$. Again it is easy to check that the four axioms of Definition 2.1.1 are satisfied in $\mathscr{S}'$. Hence, $\mathscr{S}'$ is a SPO space and therefore, every connected component of an SPO space is a connected component of an SPO space whose connectivity graph possesses one single edge and two vertices. Therefore, it suffices to study SPO spaces of this type if one is interested in what connected components of SPO spaces look like. This motivates us to give them a special name:

Definition 2.1.8. Let $\mathscr{S}$ be a partially linear SPO space consisting of two connected components $\mathscr{S}^+$ and $\mathscr{S}^-$ such that two points have finite codistance if and only if they have infinite distance. Then we call $(\mathscr{S}^+, \mathscr{S}^-)$ a *twin SPO space*, where $(\mathscr{S}^+, \mathscr{S}^-)$ carries the same opposition relation as $\mathscr{S}$.

This definition is motivated by the following property.

Proposition 2.1.9. *Every twin SPO space is a twin space.*

Proof. By the definition of the opposition relation in a twin SPO space, it remains to check that (OP) is fulfilled. Since in a partially linear space every line coincides with the convex span of any two of its points, (OP) follows directly from (A1) and (A2). □

Although we restrain ourselves from now on to twin SPO spaces, there will still appear SPO spaces that are not twin SPO spaces, namely those kinds whose connectivity graphs consist of a single vertex and a loop. This is necessary since there are connected subspaces of a twin SPO space which are again an SPO space using a different opposition relation (cf. Proposition 2.1.23).

Since in a twin SPO space two points have either finite distance or finite codistance, we may understand the codistance as a completion of the ordinary distance where distance 0 is the smallest possible distance and codistance 0 is the biggest possible distance. In this sense in a twin SPO space there is an exact value for the distance of any two points.

2.1.2 Subspaces of finite diameter

Regarding the axioms (A1), (A2) and (A3), it is obvious that one of our main interests concerns the convex subspaces that are spanned by two points at a finite distance. Beside them we study the singular subspaces and explore some properties of the structure of SPO spaces that are based on these subspaces.

Definition 2.1.10. Let U be a connected subspace of a point-line space $\mathscr{S}$. Further let p be a point with $\operatorname{dist}(p,U) < \infty$. If there is a point $q \in U$ such that for every point $r \in U$ there is a geodesic from r to p passing q, we call q a *gate* for p in U. If every point r with $\operatorname{dist}(r,U) < \infty$ has a gate in U, we call U *gated*.
Let $\mathscr{S}$ be a point-line space with a codistance. Again let U be a connected subspace of $\mathscr{S}$ and let p be a point with $\operatorname{cod}(p,U) < \infty$. If there is a point $q \in U$ such that $\operatorname{cod}(p,q) = \operatorname{cod}(p,r) + \operatorname{dist}(q,r)$ for every point $r \in U$, we call q a *cogate* for p in U. If every point r with $\operatorname{cod}(r,U) < \infty$ has a cogate in U, we call U *cogated*.

In a point-line space with a codistance we can define a gate even for some disconnected subspaces as follows:

Definition 2.1.11. Let $\mathscr{S}$ be a point-line space with a codistance. Further let U be a subspace such that every two points of U have finite distance or finite codistance. Then for a point p with $\operatorname{dist}(p,U) < \infty$, we call $q \in U$ a *gate* for p in U if $\operatorname{cod}(p,r) = \operatorname{cod}(q,r) - \operatorname{dist}(p,q)$ for every point $r \in U$ with $\operatorname{cod}(q,r) < \infty$ and $\operatorname{dist}(p,r) = \operatorname{dist}(p,q) + \operatorname{dist}(q,r)$ for every point $r \in U$ with $\operatorname{dist}(q,r) < \infty$. As for connected subspaces we call U *gated* if every point at finite distance to U has a gate in U.

Proposition 2.1.12. *Let y and z be two points of an SPO space at finite distance n and set $V := \langle y,z\rangle_{\mathrm{g}}$. Further let x be a point at finite codistance to V. Then the following conditions hold:*

(i) *For every point $u \in V$, there is a point $v \in V$ with $\operatorname{dist}(u,v) = n$.*

(ii) *If $\operatorname{copr}_V(x)$ contains a single point v, then v is cogate for x in V.*

(iii) *For every two points u and v of V with $\operatorname{dist}(u,v) = n$, the convex span $\langle u,v\rangle_{\mathrm{g}}$ equals V.*

(iv) *If there is a point $v \in V$ with $\operatorname{cod}(x,V) = \operatorname{cod}(x,v) + n$, then x has a cogate in V.*

Proof. (i) Let p be a point opposite u. By (A1) there is a point $v \in V$ with $\operatorname{cod}(p,v) = n$. Hence, $\operatorname{dist}(u,v) \geq n$. Equality follows from Proposition 2.1.3.
(ii) Let $u \in V$ be an arbitrary point. Set $d := \operatorname{dist}(v,u)$, $k := \operatorname{cod}(x,u)$ and $U := \langle v,u\rangle_{\mathrm{g}}$. We prove $\operatorname{cod}(x,v) = k + d$ by induction over k. For $k = 0$ the claim follows by (A1). Now let $k > 0$. Then there is a point $w \perp x$ with $\operatorname{cod}(w,u) = k-1$. Since $\operatorname{copr}_U(x) = \{v\}$, we obtain $\operatorname{copr}_U(w) = \{v\}$ and $\operatorname{cod}(w,v) = \operatorname{cod}(x,v) - 1$ by (A3). By the induction hypothesis we obtain $\operatorname{cod}(w,v) = \operatorname{cod}(w,u) + \operatorname{dist}(v,u) = k + d - 1$ and hence, $\operatorname{cod}(x,v) = k + d$.
(iii) Let $p \in \langle u,v\rangle_{\mathrm{g}}$ and let r be a point opposite p. By (A1) and (A2) there is exactly one point $q \in V$ with $\operatorname{cod}(r,q) = n$ and for all other points of V the codistance to r is $< n$. Hence, $q \in \langle u,v\rangle_{\mathrm{g}}$ by (A1). By Proposition 2.1.3 we obtain

$\operatorname{dist}(p,q) = n$. Now let $p' \in V$ be a point collinear to p. We want to show $p' \in \langle u,v\rangle_{\mathrm{g}}$ and therefore we may assume $p' \neq p$. By (A1) there is a point q' on the line pp' with $\operatorname{cod}(r,q') = 1$. Thus, $\operatorname{dist}(q',q) = n-1$ by (ii) and hence, $q' \in \langle p,q\rangle_{\mathrm{g}} \leq \langle u,v\rangle_{\mathrm{g}}$. Thus, $l \leq \langle u,v\rangle_{\mathrm{g}}$ and the claim follows by the connectedness of V.
(iv) Let $x' \in V$ be a point with $\operatorname{cod}(x,V) = \operatorname{cod}(x,x')$. Then $\operatorname{dist}(x',v) = n$ and hence $V = \langle x',v\rangle_{\mathrm{g}}$ by (iii). Now let $v' \leftrightarrow x$ be a point with $\operatorname{dist}(v,v') = \operatorname{cod}(x,v)$. Then $\operatorname{dist}(x',v') = \operatorname{cod}(x,V)$ and hence, $\operatorname{copr}_{\langle x',v'\rangle_{\mathrm{g}}}(x) = \{x'\}$ by (A1) and (A2). Since $\langle x',v\rangle_{\mathrm{g}} \leq \langle x',v'\rangle_{\mathrm{g}}$ the claim follows by (ii). ☐

Let V be the convex span of two points of an SPO space at finite distance n. Further let x be a point that is opposite to some point of V. Then (A1) and (A2) imply that there is a point $z \in V$ at codistance n to x such that $\operatorname{copr}_V(x) = \{z\}$. Now Proposition 2.1.12(ii) implies that z is a cogate for x in V. Hence, the following condition holds for every SPO space:

(A12) If $x \leftrightarrow v$ for some $v \in V$, then x has a cogate at codistance n in V.

The labelling (A12) is motivated since both (A1) and (A2) are direct consequences of this condition. Note that (A12) is not just the unification of (A1) and (A2) since in the proof of Proposition 2.1.12(ii) we made use of (A3).

Lemma 2.1.13. *Let y and z be two points of an SPO space at distance n. Then there is a point x with $x \leftrightarrow y$ and $\operatorname{cod}(x,z) = n$.*

Proof. Set $V := \langle y,z\rangle_{\mathrm{g}}$. Let w be a point opposite z. By (A12) there is a point $y' \in V$ with $\operatorname{cod}(w,y') = n$ such that $\operatorname{copr}_V(w) = \{y'\}$. Take a point $x' \leftrightarrow y'$ with $\operatorname{dist}(w,x') = n$. Then again by (A12) there is a point $x \in \langle x',w\rangle_{\mathrm{g}} =: U$ with $\operatorname{cod}(x,z) = n$ such that $\operatorname{copr}_U(z) = \{x\}$. By Proposition 2.1.3 we obtain $\operatorname{dist}(x,w) = n$. Since $x' \leftrightarrow y'$, the point w is a cogate for y' in U by (A12) and therefore $x \leftrightarrow y'$. Hence again by (A12), z is a cogate for x in V and we conclude $y \leftrightarrow x$. ☐

Lemma 2.1.14. *Let y and z be two points of an SPO space at distance n. Set $V := \langle y,z\rangle_{\mathrm{g}}$ and let x be a point with $\operatorname{dist}(x,V) < \infty$. Further let $v \in V$ be a point with $\operatorname{dist}(x,V) = \operatorname{dist}(x,v) + n$. Then x has a gate in V.*

Proof. Set $k := \operatorname{dist}(x,V)$ and let $u \in V$ be a point with $\operatorname{dist}(x,u) = k$. Then $\operatorname{dist}(x,v) = k+n$ and hence, Lemma 2.1.13 implies that there is a point $w \leftrightarrow v$ with $\operatorname{cod}(w,x) = k+n$. By (A12) the point x is a cogate for w in $\langle x,v\rangle_{\mathrm{g}}$. Since $\operatorname{dist}(x,u) = k$, we obtain $\operatorname{cod}(w,u) \geq n$ and consequently, (A12) implies that u is a cogate for w in V with $\operatorname{cod}(u,w) = n$. Since $V \leq \langle x,v\rangle_{\mathrm{g}}$, we obtain $\operatorname{dist}(x,p) = k+n-\operatorname{cod}(w,p) = k+\operatorname{dist}(u,p)$. ☐

As a direct consequence of this lemma, we can state a first result concerning the structure of an SPO space.

Corollary 2.1.15. *Every SPO space is a gamma space.*

Proof. Let p be a point of an SPO space and let l be a line. Assume there are points q and r on l such that $q \perp p$ and $r \not\perp p$. Then $\operatorname{dist}(p,q) = 1$ and $\operatorname{dist}(p,r) = 2$ and therefore $\operatorname{pr}_l(p) = \{q\}$ by Lemma 2.1.14. □

We know concentrate our attention to the coprojection of a given point in the convex span of two points at finite distance.

Proposition 2.1.16. *Let y and z be two points of an SPO space at distance n and set $V := \langle y,z\rangle_{\mathrm{g}}$. Further let x be a point at finite codistance to V and set $U := \operatorname{copr}_V(x)$. Then*

(i) *U is a convex subspace of V,*

(ii) $\operatorname{dist}(v,U) + \operatorname{cod}(x,v) = \operatorname{cod}(x,V)$ *for every $v \in V$ and*

Proof. (i) Let $l \leq V$ be a line. Then by Proposition 2.1.12(iv) the set $\operatorname{copr}_l(x)$ is a singleton or the whole line. Hence, U is a subspace. Now, let u and v be two distinct points of U. We have to show that an arbitrary point $v' \perp v$ with $\operatorname{dist}(u,v') = \operatorname{dist}(u,v) - 1$ is contained in U.
Suppose $\operatorname{cod}(x,v') = \operatorname{cod}(x,v) - 1$. Take a point $w \leftrightarrow v$ at distance $\operatorname{cod}(x,v)$ to x and set $W := \langle w,x\rangle_{\mathrm{g}}$. Suppose $\operatorname{cod}(v',W) < \operatorname{dist}(x,w)$. Then $x \in \operatorname{copr}_W(v')$. Moreover, by (A1) there is no point in W opposite v' and hence, $\operatorname{cod}(w,v') = 1$. This is a contradiction to (A3) since $\operatorname{cod}(w,v) < \operatorname{cod}(w,v')$ but $\operatorname{cod}(W,v) > \operatorname{cod}(W,v')$. Thus, $\operatorname{cod}(v',W) \geq \operatorname{dist}(x,w)$. This implies $x \notin \operatorname{copr}_W(v')$. By (A12) we know that x is a cogate for v in W. Thus, $\operatorname{cod}(v,W \setminus \{x\}) = \operatorname{cod}(x,v) - 1 = \operatorname{dist}(w,x) - 1$ and we conclude $\operatorname{cod}(v',W) = \operatorname{cod}(v,W)$. Hence, for any point $x' \in \operatorname{copr}_W(v')$, we obtain $x' \perp x$ and $\operatorname{cod}(x',v') = \operatorname{cod}(x,v) = \operatorname{cod}(x',v) + 1$. Thus, we may apply (A3) to conclude $\operatorname{cod}(x,\langle u,v\rangle_{\mathrm{g}}) \geq \operatorname{cod}(x',\langle u,v\rangle_{\mathrm{g}})$ and therefore $v' \in \operatorname{copr}_{\langle u,v\rangle_{\mathrm{g}}}(x')$. This is a contradiction to $\operatorname{copr}_{\langle u,v\rangle_{\mathrm{g}}}(x') \leq \operatorname{copr}_{\langle u,v\rangle_{\mathrm{g}}}(x)$. Therefore v' has to be contained in U.
(ii) Let $u \in U$ be a point with $\operatorname{dist}(v,U) = \operatorname{dist}(v,u)$. Set $V' := \langle u,v\rangle_{\mathrm{g}}$. By Lemma 2.1.13 there is a point $w \leftrightarrow u$ with $\operatorname{cod}(w,v) = \operatorname{dist}(u,v)$. Since v is a cogate for w in V' by (A12) and on every line there is a point that is not opposite w, we conclude that every line of V' has at most distance $\operatorname{dist}(u,v) - 1$ to v. Hence $V' \cap U = \{u\}$ by (i) and consequently, Proposition 2.1.12(ii) implies that u is a cogate for x in V'. □

Proposition 2.1.17. *Let y and z be two points of an SPO space at distance n and set $V := \langle y,z\rangle_{\mathrm{g}}$.*

(i) *Let u and v be two points of V at distance k and set $U := \langle u,v\rangle_{\mathrm{g}}$. Then* $\mathrm{dist}(p,U) \leq n-k$ *for every point $p \in V$.*

(ii) *Let x be a point with* $\mathrm{cod}(x,V) < \infty$*. Then* $\mathrm{cod}(x,V) \geq n$*.*

Proof. Suppose one of the claims does not hold. Then we may assume that n is minimal under the condition that there exists a counterexample V.
If $k = n$, claim (i) follows from Proposition 2.1.12(iii). Hence, we may assume $k < n$. By Proposition 2.1.12(i) there is a point $q \in V$ with $\mathrm{dist}(p,q) = n$ for every point $p \in V$. By Lemma 2.1.13 there is a point r with $r \leftrightarrow q$ and $\mathrm{cod}(p,r) = n$. Thus by Proposition 2.1.12(iv), the point p is a cogate for r in V. Since $k < n$, we conclude $\mathrm{cod}(r,U) \geq k$ by (ii) and consequently, (i) holds for V.
Thus, V is a minimal counterexample for claim (ii). This implies that (ii) holds for the convex span of any two points at distance $n-1$, and therefore $\mathrm{cod}(x,V) = n-1$. We may assume that x is a point such that $\mathrm{diam}(\mathrm{copr}_V(x))$ is minimal. Set $m := \mathrm{diam}(\mathrm{copr}_V(x))$. By Proposition 2.1.12(iii) we may assume $z \in \mathrm{copr}_V(x)$. Let p and q be points of $\mathrm{copr}_V(x)$ at distance m. Then $\mathrm{dist}(y,\langle p,q\rangle_{\mathrm{g}}) \leq n-m$ by (i). Since $\mathrm{copr}_V(x)$ is a convex subspace by Proposition 2.1.16(i), this implies $\mathrm{dist}(y,\mathrm{copr}_V(x) \leq n-m$. Since $\mathrm{dist}(z,y) = n$ and $z \in \mathrm{copr}_V(x)$, we conclude $\mathrm{dist}(y,\mathrm{copr}_V(x)) = n-m$ and therefore $\mathrm{cod}(x,y) = m-1$ by Proposition 2.1.16(ii). Since $\mathrm{cod}(x,V) = n-1$, we obtain $m-1 > 0$ by (A1). Thus, there is a point $w \perp x$ with $\mathrm{cod}(w,y) = m-2$. By (A3) this implies $\mathrm{cod}(w,V) \leq n-1$ and $\mathrm{copr}_V(w) \leq \mathrm{copr}_V(x)$. Since V is a minimal counterexample, we conclude $\mathrm{cod}(w,V) = n-1$ as for x. Thus, $\mathrm{dist}(y,\mathrm{copr}_V(w)) \geq n-m+1$. Since $\mathrm{copr}_V(w)$ is a convex subspace, we conclude $\mathrm{diam}(\mathrm{copr}_V(w)) \leq m-1$ by (i). This is a contradiction to the choice of x and the claim follows. □

Corollary 2.1.18. *The convex span of two points at distance* 2 *of an SPO space is a non-degenerate polar space of rank* ≥ 2*.*

Proof. Let Y be the convex span of two points at distance 2. Let $l \leq Y$ be a line and let $p \in Y$ be a point. Then $\mathrm{dist}(p,l) \leq 1$ by Proposition 2.1.17(i). Thus, the Buekenhout-Shult Axiom (BS) follows from Lemma 2.1.14. By Proposition 2.1.12(i) V is non-degenerate. Since Y contains a line, we obtain $\mathrm{rk}(Y) \geq 2$. □

Remark 2.1.19. If we do not restrain ourselves to consider SPO spaces that are partially linear, this corollary does not hold anymore. Hence, we cannot apply Proposition A.2.7 at this point to prove that there are partially linear subspaces in an arbitrary SPO space. The reason for this is that in the axioms given in Definition 2.1.1 lines do not occur without their span. An additional axiom that for every line l, there is no point at codistance 0 to l would avoid this fact. Moreover, such an axiom would imply that every SPO space is partially linear.

We are now ready to prove the property that motivates the usage of the term SPO space. For the definition of parapolar and strongly parapolar spaces, see Definition B.3.2.

Theorem 2.1.20. *Let V be a connected convex subspace of an SPO space with* $\operatorname{diam}(V) \geq 2$. *Then V is a strongly parapolar space.*

Proof. We know already that V is a convex partially linear gamma space. As symplecta we take the subspaces of V that are convex spans of two points at distance 2. By Corollary 2.1.18 each symplecton is a non-degenerate polar space of rank ≥ 2.
Now let p and q be two points of V at distance 2. Then every quadrangle that contains p and q is contained in $\langle p,q\rangle_{\mathrm{g}}$. Moreover, Proposition 2.1.12(iii) implies that every convex span of two points at distance 2 that contains p and q coincides with $\langle p,q\rangle_{\mathrm{g}}$.
It remains to check that every line $l \leq V$ is contained in a symplecton. Since $\operatorname{diam}(V) \geq 2$, there is a symplecton $Y \leq V$. We may assume $l \nleq Y$ since otherwise we are done. First we consider the case $l \cap Y = \varnothing$. Let p and q be distinct points of l such that $\operatorname{dist}(p,Y) = \operatorname{dist}(l,Y)$. Then there is a point $y \in Y$ with $\operatorname{dist}(p,y) \geq 2$ since otherwise p would be contained in Y. Since V is convex, there is a point $z \in V$ with $\operatorname{dist}(p,z) = 2$. Since $Y' := \langle p,z\rangle_{\mathrm{g}}$ is a symplecton of V, it remains to check the case $l \cap Y = \{p\}$. By Lemma A.2.3(i) there are points y and z in $V \cap p^{\perp}$ with $y \not\perp z$. Since $q \notin Y$, we conclude $y \not\perp q$ or $z \not\perp q$. Thus, $\langle q,z\rangle_{\mathrm{g}}$ or $\langle q,y\rangle_{\mathrm{g}}$ is a symplecton that contains l. □

A symplecton is said to be of rank r if it is a polar space of rank r. Let $\mathscr{S}$ be a parapolar space such that every symplecton has rank r. Then we call $\mathscr{S}$ a parapolar space of *symplectic rank* r, denoted by $\operatorname{yrk}(\mathscr{S}) = r$. If every symplecton of a parapolar space $\mathscr{S}$ has rank $\geq r$, we say that $\mathscr{S}$ is of symplectic rank $\geq r$.

According to the term symplecton we call the convex span of two points of an SPO space that have finite distance to each other a *metaplecton*. By this definition, singletons, lines and symplecta are the three smallest kinds of metaplecta.

The next subspaces we study are the singular subspaces. Our goal is to show that SPO spaces are paraprojective; see Definition B.3.1. It is known that every parapolar space is already paraprojective; see [Bue82] or [BCar]. For SPO spaces, this is not sufficient since there are connected components of SPO spaces that are singular and hence they are not parapolar.

Lemma 2.1.21. *For $n \in \mathbb{N}$ let $M := \{y_i \mid 0 \leq i < n\}$ be a set of mutually collinear points of an SPO space.*

(i) *Let x be a point with $x \not\leftrightarrow y_i$ for all $0 \leq i < n$. Then $x \not\leftrightarrow p$ for every point $p \in \langle M\rangle$.*

(ii) *Let* $\mathrm{cod}(x,\langle M\rangle)=1$ *and* $\mathrm{cod}(x,y_i)=1$ *for* $0\le i<n$*. Then there is a point* $y_n \leftrightarrow x$ *such that* $y_n \perp y_i$ *for* $0 \le i < n$*.*

(iii) *Let* $y_i \notin \langle y_j \mid 0\le j<i\rangle$ *for* $y\in M$*. Then there is a set* $\{x_i \mid 0\le i<n\}$ *of mutually collinear points such that* $x_i \leftrightarrow y_j \Leftrightarrow i=j$ *for* $0\le i<n$ *and* $0\le j<n$*.*

Proof. (i) Let y and z be two distinct collinear points with $y \not\leftrightarrow x \not\leftrightarrow z$. Then $yz=\langle y,z\rangle_{\mathrm{g}}$ and hence by (A2), there is no point on yz opposite x. Set $M_0 := M$. For $i\in\mathbb{N}$, we set recursively $M_{i+1} := \bigcup_{(y,z)\in M_i\times M_i} yz$. Since $x\not\leftrightarrow p$ for every point $p\in M$, we apply induction to conclude for $i\in\mathbb{N}$ that $x\not\leftrightarrow p$ holds for every point $p\in M_i$.

Since the points of M are mutually collinear, we know that $\langle M\rangle$ is singular. Since by the definition we obtain $M_i\subseteq\langle M\rangle$ for every $i\in\mathbb{N}$, we obtain $\bigcup_{i\in Nat} M_i\subseteq\langle M\rangle$. Moreover, the points of M_i are mutually collinear. Let $i\le j$ and take two points $p\in M_i$ and $q\in M_j$. Then $p\in M_j$ and the line pq is contained in M_{j+1}. Thus $\langle M\rangle=\bigcup_{i\in Nat} M_i$ and the claim follows.

(ii) We proceed by induction over n. Since $\mathrm{cod}(x,y_0)=1$, the claim holds for $n\le 1$. Now let $n>1$ and assume that there is a point $y'_n\leftrightarrow x$ such that $y'_n\perp y_i$ for $0\le i<n-1$. If $y'_n\perp y_{n-1}$, we are done. Therefore we may assume $\mathrm{dist}(y'_n,y_{n-1})=2$. Then $Y:=\langle y'_n,y_{n-1}\rangle_{\mathrm{g}}$ is a symplecton that contains M. By (A12) x has a cogate x' in Y with $\mathrm{cod}(x,x')=2$. Therefore x' is collinear to every point of M and $S:=\langle x',M\rangle$ is a singular subspace of Y.

Since $\mathrm{rk}(S)<\infty$, we conclude by Lemma A.2.17 and induction that there is a generator G of Y that is disjoint to S. Since $\mathrm{cod}(x,\langle M\rangle)=1$, we know $\langle M\rangle<S$. Thus, Proposition A.2.20 and Lemma A.2.22(ii) imply that $G':=\langle M\rangle\oplus G$ is a generator of Y with $\mathrm{crk}_G(G\cap G')=\mathrm{rk}(\langle M\rangle)+1$. Since $S\cap G=\varnothing$ and $\mathrm{rk}(S)>\mathrm{rk}(\langle M\rangle)$, we conclude $S\nsubseteq G'$ and consequently, $x'\notin G'$. By the maximality of G' there is a point $y_n\in G'$ that is not collinear to x'. Thus, $x\leftrightarrow y_n$. The claim follows since $M\subseteq G'$.

(iii) We proceed by induction over n. For $n\le 1$ the claim follows from (A4). Now assume there is a set of mutually collinear points $\{w_i\mid 0\le i<n\}$ such that $w_i\leftrightarrow y_j \Leftrightarrow i=j$ for $0\le i<n$ and $0\le j<n$. Further let y_n be a point with $y_n\perp y_i$ for $0\le i<n$ and $y_n\notin\langle y_i\mid 0\le i<n\rangle$.

Set $z_0:=y_n$ and for $0\le i<n$, let z_{i+1} be the unique point on the line y_iz_i not opposite w_i. Since $z_{i+1}\in\langle y_i\mid 0\le i\le n\rangle$ we obtain $y_j\perp z_{i+1}$ for $j<n$. Furthermore we obtain $z_{i+1}\notin\langle y_j\mid 0\le j<n\rangle$ since $z_{i+1}\ne y_i$ by $z_{i+1}\not\leftrightarrow w_i\leftrightarrow y_i$ and $z_i\notin\langle y_j\mid 0\le j<n\rangle$. Finally, $z_{i+1}\not\leftrightarrow w_j$ whenever $i\le j$ since this is true by definition for $i=j$ and follows recursively by $w_j\not\leftrightarrow z_i$ and $w_j\not\leftrightarrow y_i$ if $i>j$. Thus, $z_n\not\leftrightarrow w_j$ for $0\le i<n$ and hence $\mathrm{cod}(z_n,w_j)=1$ since $z_n\perp y_j$.

By (ii) there is a point $w_n\leftrightarrow z_n$ with $w_n\perp w_i$ for $0\le i<n$. Now set $u_0:=w_n$ and for $i<n$ define recursively u_{i+1} to be the unique point on the line w_iu_i non-

opposite to y_i. Since $w_i \leftrightarrow y_i \not\leftrightarrow u_{i+1}$ we obtain $w_i \neq u_{i+1}$ and hence $z_n \leftrightarrow u_{i+1}$ by $z_n \leftrightarrow u_i$. Furthermore, $y_j \not\leftrightarrow u_{i+1}$ for $0 \leq j < i$ since $y_j \not\leftrightarrow u_i$ and $y_j \not\leftrightarrow w_i$. Hence, $y_i \not\leftrightarrow u_n$ for $0 \leq i < n$. With $u_n \in \langle w_i \mid 0 \leq i \leq n \rangle$ we obtain $w_j \perp u_n$ for $0 \leq j < n$. Set $x_n := u_n$. With $x_n \leftrightarrow z_n$ and $x_n \not\leftrightarrow y_{n-1}$ we obtain $x_n \leftrightarrow z_{n-1}$ and hence analogously $x_n \leftrightarrow z_i$ with $x_n \not\leftrightarrow y_i$ for all $0 \leq i < n$. Hence, x_n has already the demanded conditions since $z_0 = y_n$. Now let x_i for $0 \leq i < n$ be the point on the line $w_i x_n$ that is not opposite y_n. Since x_n is the unique point on $w_i x_n$ not opposite y_i and $x_n \neq x_i$ because of $x_n \leftrightarrow y_n \not\leftrightarrow x_i$ we conclude $x_i \leftrightarrow y_i$. Furthermore $x_i \not\leftrightarrow y_j$ if $j < n$ and $j \neq i$ since $y_j \not\leftrightarrow w_i$ and $y_j \not\leftrightarrow x_n$. Finally, since $x_i \in \langle w_j \mid 0 \leq j \leq n \rangle$ for $i \leq n$ the set $\{x_i \mid 0 \leq i < n\}$ consists of mutually collinear points. □

Theorem 2.1.22. *Every SPO space is paraprojective.*

Proof. Let S be a singular subspace of an SPO space. Let g and h be two lines of S intersecting in a point p. For $i \in \{0,1\}$ let l_i be a line intersecting g in a point $a_i \neq p$ and h in a point $b_i \neq p$. By Definition A.1.1 we have to show that l_0 and l_1 intersect. Therefore we may assume $a_0 \neq a_1$ and $b_0 \neq b_1$. By Lemma 2.1.21(iii) there is a point q opposite p with $q \not\leftrightarrow a_1$ and $q \not\leftrightarrow b_1$. Since $q \leftrightarrow p$ we conclude by (A2) that a_1 is the unique point on the line g that is non-opposite q. Hence, $a_0 \leftrightarrow q$ and analogously, $b_0 \leftrightarrow q$. Thus by (A1), there has to be a third point c on the line l_0 with $c \not\leftrightarrow q$.
Since $\{c, a_1, b_1\} \subseteq \langle a_0, b_0, p \rangle \leq S$ the points c, p, a_0 and b_0 are pairwise collinear. By Lemma 2.1.21(i) there is no point in $\langle c, a_1, b_1 \rangle$ opposite q. Therefore, $p \notin \langle c, a_1, b_1 \rangle$. Suppose $c \notin l_1$. Then by Lemma 2.1.21(iii) we find a point r opposite c with $r \not\leftrightarrow p$, $r \not\leftrightarrow a_1$ and $r \not\leftrightarrow b_1$. This is a contradiction to Lemma 2.1.21(i) since $c \in \langle a_1, b_1, p \rangle$. Hence, l_0 and l_1 intersect in c. □

We conclude this section by studying the metaplecta of an SPO space. Our first result is that metaplecta are again SPO spaces:

Proposition 2.1.23. *Let y and z be two points of an SPO space at distance n and set $V := \langle y, z \rangle_{\mathrm{g}}$. Set $R := \{(u,v) \in V \times V \mid \mathrm{dist}(u,v) = n\}$. Then V is an SPO space with opposition relation R. Furthermore, $\mathrm{cod}_R(u,v) + \mathrm{dist}(u,v) = n$ for every pair of points $(u,v) \in V \times V$.*

Proof. The relation R is symmetric and by Proposition 2.1.12(i) total. Now let u and v be two points of V. Further let $w \leftrightarrow v$ be a point with $\mathrm{cod}(w,u) = \mathrm{dist}(u,v)$. Then w has a cogate w' in V at codistance n. Hence, $\mathrm{dist}(w',u) = n - \mathrm{dist}(u,v)$ and therefore (A4) holds for R in V. For an arbitrary point $x' \in V$, we find a point x with $\mathrm{cod}(x,x') = n$ and $\mathrm{copr}_V(x) = \{x'\}$ by Proposition 2.1.12(i) and Lemma 2.1.13. Since x' is the cogate of x in V, we obtain $\mathrm{cod}(x,u) = \mathrm{cod}_R(x',u)$ for every $u \in V$ and hence, we may carry over the axioms (A1), (A2) and (A3). □

Lemma 2.1.24. *Let y and z be two points of an SPO space at distance n and set $V := \langle y,z\rangle_{\mathrm{g}}$. Further let x be a point at finite codistance to V such that $z \in \mathrm{copr}_V(x)$. Then $\mathrm{cod}(x,y) = \min\{\mathrm{cod}(x,v) \mid v \in V\}$.*

Proof. Set $U := \mathrm{copr}_V(x)$. By Lemma 2.1.13 there is a point w opposite z with $\mathrm{cod}(w,y) = n$. Since $\mathrm{dist}(y,z) = n$, we obtain $\mathrm{dist}(y,U) \geq n - \mathrm{diam}(U)$. By Proposition 2.1.17(i) and Proposition 2.1.16(i) we obtain $\mathrm{dist}(v,U) \leq n - \mathrm{diam}(U)$ for every $v \in V$. Now the claim follows by Proposition 2.1.16(ii). □

Proposition 2.1.25. *Let y and z be two points of an SPO space at distance n and set $V := \langle y,z\rangle_{\mathrm{g}}$. Let x be a point at finite distance k to V and set $U := \mathrm{pr}_V(x)$.*

(i) *Let U be a singleton. Then the point of U is a gate for x in V.*

(ii) *U is a convex subspace of V.*

(iii) $\mathrm{dist}(v,U) + \mathrm{dist}(x,V) = \mathrm{dist}(x,v)$ *for every $v \in V$.*

(iv) *Let $z \in U$. Then* $\mathrm{dist}(x,y) = \max\{\mathrm{dist}(x,v) \mid v \in V\}$.

Proof. (i) Let $u \in V$ such that $U = \{u\}$. Further let $v \in V$ be an arbitrary point. We prove the claim by induction over $m := \mathrm{dist}(u,v)$. For $m \leq 1$, there is nothing to prove. We assume that the claim holds for $m-1$. Let $v' \perp v$ be a point with $\mathrm{dist}(u,v') = m-1$. Then $\mathrm{dist}(x,v') = k+m-1$. Let $w \leftrightarrow v'$ be a point with $\mathrm{cod}(w,x) = k+m-1$. Then $\mathrm{cod}(w,u) = m-1$. Since $v' \in \langle u,v\rangle_{\mathrm{g}}$, the point w has a cogate w' in $\langle u,v\rangle_{\mathrm{g}}$ with $\mathrm{cod}(w,w') = m$ by (A12). This implies $\mathrm{dist}(w',v') = m$ and $w' \perp u$ and hence, $\mathrm{dist}(x,w') = k+1$ since $w' \in V \setminus U$. Thus, $u \in \langle x,w'\rangle_{\mathrm{g}}$ and therefore $x \notin \mathrm{copr}_{\langle x,w'\rangle_{\mathrm{g}}}(w)$ by Lemma 2.1.24. Hence, there is a point $x' \in \langle x,w'\rangle_{\mathrm{g}}$ with $\mathrm{cod}(w,x') = k+m$. By Proposition 2.1.12(iv) the point x' is a cogate for w in $\langle x,w'\rangle_{\mathrm{g}}$ since $\mathrm{cod}(w,u) = \mathrm{cod}(w,x') - k - 1$. This implies $x' \perp x$, $\mathrm{dist}(x',w) = k$ and $\mathrm{dist}(x',u) = k+1$. Thus, $\mathrm{dist}(x',v') = k+m$ since $w \leftrightarrow v'$.
Set $W := \langle x',v'\rangle_{\mathrm{g}}$. Now x, w' and u are all contained in W since they all lie on geodesics from x' to v'. Consequently, $v \in W$ since $\langle u,v\rangle_{\mathrm{g}} = \langle w',v'\rangle_{\mathrm{g}}$. By Proposition 2.1.12(i) and Lemma 2.1.13 there is a point s that is opposite to some point in W such that $\mathrm{cod}(s,x) = k+m$. Then x is the cogate for s in W and therefore $\mathrm{cod}(s,u) = m$. Since $\langle u,v\rangle_{\mathrm{g}} \leq V$ and $\mathrm{dist}(x, V \setminus U) = k+1$ there are no other points in $\langle u,v\rangle_{\mathrm{g}}$ at codistance $\geq m$ to s. Hence, u is a cogate for s in $\langle u,v\rangle_{\mathrm{g}}$ and therefore $s \leftrightarrow v$. The claim follows.
(ii) By Lemma 2.1.14 the set U is a subspace. Now assume U is not connected and let u and v be two points of different connected components of U such that $\mathrm{dist}(u,v)$ is minimal. Then $U' := \mathrm{pr}_{\langle u,v\rangle_{\mathrm{g}}}(x)$ does not contain any line, since otherwise U would have connected components at lower distance than $\mathrm{dist}(u,v)$ by Proposition 2.1.17(i). Hence U' is a union of singletons which are pairwise at distance $\mathrm{dist}(u,v)$ to each other. Let u' be a point collinear to u with $\mathrm{dist}(u',v) =$

$\mathrm{dist}(u,v)-1$. Then $\mathrm{pr}_{\langle u',v\rangle_{\mathrm{g}}}(x)=\{v\}$ and hence, $\mathrm{dist}(x,u')=k+\mathrm{dist}(u,v)-1$ by (i). Since $u\perp u'$, this yields $\mathrm{dist}(u,v)=2$. Thus, $\{u,v\}\subset\langle x,u'\rangle_{\mathrm{g}}$ and hence, $\mathrm{dist}(x,\langle u,v\rangle_{\mathrm{g}})\leq k-1$ by Proposition 2.1.17(i), a contradiction. Therefore U is connected.

To show that U is convex it suffices to show that for two points u and v of U at distance m, every point $u'\perp u$ with $\mathrm{dist}(u',v)=m-1$ is again in U. If $m=1$ then $u'=v$, hence let $m>1$ and assume the claim holds for $m-1$. Since $\mathrm{pr}_{\langle u,v\rangle_{\mathrm{g}}}(x)$ is connected, there is a point $v'\in U$ with $v'\perp u$ and $\mathrm{dist}(v',v)=m-1$. Hence, $U':=\langle v,v'\rangle_{\mathrm{g}}\leq U$. By Lemma 2.1.14 we conclude that v' is a gate for u in U'. Hence, v' is the only point of U' collinear to u. Since for $u'=v'$ there is nothing to prove, we may assume $u'\notin U'$. Then $\mathrm{dist}(u',U')=1$ by Proposition 2.1.17(i) since $U'\leq\langle u,v\rangle_{\mathrm{g}}$. If $\mathrm{pr}_{U'}(u')$ is a singleton $\{u''\}$, then u'' is a gate for u' in U' by (i) and hence, $\mathrm{dist}(u'',v)=m-2$. If $\mathrm{pr}_{U'}(u')$ contains a line, then there is by Proposition 2.1.17(i) a point u'' on this line with $\mathrm{dist}(u'',v)=m-2$. Hence, in both cases we obtain $u\perp u'\perp u''$ and therefore $\mathrm{dist}(u,u'')=2$. Suppose $u'\notin U$. Since both u and u'' are contained in U, we conclude $\{u,u''\}\subset\langle x,u'\rangle_{\mathrm{g}}$ and thus, $\mathrm{dist}(x,\langle u,u''\rangle_{\mathrm{g}})=m-1$ by Proposition 2.1.17(i), a contradiction.

(iii) Let $u\in U$ be a point with $\mathrm{dist}(v,U)=\mathrm{dist}(v,u)$. Set $V':=\langle u,v\rangle_{\mathrm{g}}$. By Proposition 2.1.17(i) there is no line in $V'\cap U$, since otherwise we would obtain $\mathrm{dist}(v,U)<\mathrm{dist}(v,u)$. Thus, $V'\cap U=\{u\}$ and the claim follows from (i).

(iv) Since $\mathrm{dist}(y,z)=n$, we obtain $\mathrm{dist}(y,U)\geq n-\mathrm{diam}(U)$. By Proposition 2.1.17(i) and (ii) we obtain $\mathrm{dist}(v,U)\leq n-\mathrm{diam}(U)$ for every $v\in V$. Hence, the claim follows with (iii). □

Lemma 2.1.26. *Let y and z be two points of an SPO space at distance $n\geq 2$ and set $V:=\langle y,z\rangle_{\mathrm{g}}$. Further let u and v be points of V that are collinear to y. Then there is a symplecton $Y\leq V$ containing y, u and v.*

Proof. Assume that y, u and v are on a common line l of V. Then $\mathrm{dist}(z,l)=n-1$ by Proposition 2.1.17(i). Hence, there is a point z' with $\mathrm{dist}(z',z)=n-2$ and $\mathrm{dist}(z',l)=1$. We obtain $l\leq V$. Thus, we may assume that uy and vy are distinct lines. By u' we denote the point on uy with $\mathrm{dist}(u',z)=n-1$ and by v' the point on vy with $\mathrm{dist}(v',z)=n-1$.

If $\mathrm{dist}(v',u')=2$, then $Y:=\langle u',v'\rangle_{\mathrm{g}}$ has the demanded properties. Hence, let $u'\perp v'$. Since $\mathrm{pr}_{\langle u',z\rangle_{\mathrm{g}}}(y)=\{u'\}$ by Lemma 2.1.14, we conclude $v'\notin\langle u',z\rangle_{\mathrm{g}}$ and therefore $u'\in\mathrm{pr}_{\langle u',z\rangle_{\mathrm{g}}}(v')$. Since $\mathrm{dist}(v',z)=n-1$, we know that u' is not a gate for v' in $\langle u',z\rangle_{\mathrm{g}}$ and therefore, Proposition 2.1.25(ii) implies that $\mathrm{pr}_{\langle u',z\rangle_{\mathrm{g}}}(v')$ contains a line l through u'. By Proposition 2.1.17(i) there is a point z' on l with $\mathrm{dist}(z',z)=n-2$. Now $Y:=\langle y,z'\rangle_{\mathrm{g}}$ has the demanded properties. □

Proposition 2.1.27. *Let V be a metaplecton of an SPO space with* $\mathrm{diam}(V)\geq 2$. *Further let x be a point at finite distance to V such that $U:=\mathrm{pr}_V(x)$ has diameter*

1. *Then U is a maximal singular subspace of V. Furthermore, U is contained in a singular subspace M with* $\operatorname{dist}(x,M) = \operatorname{dist}(x,V) - 1$.

Proof. Set $d := \operatorname{dist}(x,V)$. Let $S \leq V$ be a singular subspace with $U \leq S$. Assume there is a point $s \in S \setminus U$. Then by Lemma 2.1.13 there is a point $p \leftrightarrow s$ with $\operatorname{cod}(p,x) = d+1$. This implies $\operatorname{cod}(p,u) = 1$ for every point $u \in U$. Let $g \leq U$ be a line. Then by Lemma 2.1.26 there is a symplecton $Y \leq V$ containing g and s. By (A12) p has a cogate q in Y at codistance 2. Hence, q is collinear to all points of g and we conclude $g \leq \langle q,x\rangle_{\mathrm{g}}$. Therefore $\operatorname{cod}(p,\langle q,x\rangle_{\mathrm{g}}) < d+2$ by Proposition 2.1.17(i). Thus, $x \in \operatorname{copr}_{\langle q,x\rangle_{\mathrm{g}}}(p)$. This is a contradiction to Lemma 2.1.24 and we conclude $S = U$.
Now let u and v be distinct points of U and set $W := \langle u,x\rangle_{\mathrm{g}}$. By Proposition 2.1.17(i) we obtain $uv \nleq W$ and hence, $u \in \operatorname{pr}_W(v)$. Furthermore, by Proposition 2.1.25(iii) there is a line l through u in $\operatorname{pr}_W(v)$. Let $w \in l$ be the point with $\operatorname{dist}(x,w) = d-1$. Then $uv \leq \operatorname{pr}_V(w)$ and hence, $\operatorname{pr}_V(w)$ is a maximal singular subspace of V. Since $\operatorname{pr}_V(w) \leq U$, the claim follows. ☐

Lemma 2.1.28. *Let V be a metaplecton of an SPO space and set $n :=$* $\operatorname{diam}(V)$. *Further let x be a point at finite codistance to V. Set $m :=$* $\min\{\operatorname{cod}(x,p) \mid p \in V\}$. *Then for every point $u \in V$, there is a point $v \in V$ with* $\operatorname{cod}(x,v) = m$ *and* $\operatorname{dist}(u,v) = \operatorname{cod}(x,u) - d$.

Proof. It suffices to show that for any point $u \in V$ with $\operatorname{cod}(x,u) > m$, there is a point $v \in V$ with $v \perp u$ and $\operatorname{cod}(x,v) = \operatorname{cod}(x,u) - 1$. Suppose there is a point u such that this claim does not hold. We may assume that n is minimal under the condition that there exists a counterexample.
Set $U := \operatorname{copr}_V(x)$ and let $z \in U$ such that $\operatorname{dist}(u,z)$ is maximal. Since by Proposition 2.1.17(ii) for every point opposite to u there is a point at codistance $\geq \operatorname{diam}(U)$ in U, we obtain $\operatorname{dist}(u,z) \geq \operatorname{diam}(U)$. By Proposition 2.1.23 the metaplecton V is a SPO space and hence by Lemma 2.1.13, there is a point $y \in V$ with $\operatorname{dist}(y,z) = n$ such that u is on a geodesic from y to z. By Lemma 2.1.24 we conclude $\operatorname{cod}(x,y) = m$. Thus, $y \neq u$.
For $u \in U$, we obtain $\operatorname{dist}(u,z) = \operatorname{diam}(U)$. Therefore, every point $v \perp u$ with $\operatorname{dist}(v,y) = \operatorname{dist}(u,y) - 1$ has distance $\operatorname{diam}(U)+1$ to z and hence, $v \notin U$. This is a contradiction to the assumption that no neighbour of u in V has codistance $\operatorname{cod}(x,u) - 1$ to x. Thus, we may assume $u \notin U$ and consequently, $\operatorname{dist}(u,y) < n$. Since $\langle u,y\rangle_{\mathrm{g}} \leq V$, this leads to a contradiction to the minimality of n. ☐

Proposition 2.1.29. *Let y and z be two points with* $\operatorname{dist}(y,z) = n < \infty$. *Set $V := \langle y,z\rangle_{\mathrm{g}}$ and let x be a point with* $\operatorname{dist}(x,V) < \infty$ *and* $\operatorname{pr}_V(x) = \{z\}$. *Then there is a point w with* $\operatorname{dist}(w,x) = n$ *and* $\operatorname{pr}_V(w) = \{y\}$. *For every such point w, the metaplecta $U := \langle w,x\rangle_{\mathrm{g}}$ and V are one-coparallel to each other. Moreover, the bijective map $\varphi\colon U \to V$ with $\{u^{\varphi}\} = \operatorname{pr}_V(u)$ for all $u \in U$ is an isomorphism.*

Proof. Set $d := \mathrm{dist}(x,V)$. By Proposition 2.1.25(i) z is a gate for x in V. Hence, $\mathrm{dist}(x,y) = d+n$ and the metaplecton $\langle x,y\rangle_{\mathrm{g}}$ contains z and therefore $V \leq \langle x,y\rangle_{\mathrm{g}}$. By Proposition 2.1.23 $\langle x,y\rangle_{\mathrm{g}}$ is an SPO space. Hence, there is a point $w' \in \langle x,y\rangle_{\mathrm{g}}$ with $\mathrm{dist}(w',x) = n$ and $\mathrm{dist}(w',z) = d+n$. By (A1) and Proposition 2.1.12(iv) y has a gate w in $\langle w',x\rangle_{\mathrm{g}}$ with $\mathrm{dist}(y,w) = d$. Hence, $\mathrm{dist}(w,x) = n$ and $\mathrm{pr}_V(w) = \{y\}$. Now let w be an arbitrary point with $\mathrm{dist}(w,x) = n$ and $\mathrm{pr}_V(w) = \{y\}$. Then Proposition 2.1.25(i) implies that y is a gate for w in V and hence, $\mathrm{dist}(w,z) = \mathrm{dist}(w,y)+n$. Since $\mathrm{dist}(w,z) \leq \mathrm{dist}(x,z)+\mathrm{dist}(w,x) = d+n$ and $\mathrm{dist}(w,y) \geq \mathrm{dist}(x,y)-\mathrm{dist}(w,x) = d$, we conclude $\mathrm{dist}(w,y) = d$ and hence, $w \in \langle x,y\rangle_{\mathrm{g}}$. Thus, we stay in the SPO space $\langle x,y\rangle_{\mathrm{g}}$.
Let $u \in U \setminus \{x\}$ with $u \perp x$. Then $\mathrm{dist}(u,V) \leq d$ by Proposition 2.1.17(ii). Since by Proposition 2.1.25(i) x is a gate for z in U, we obtain $\mathrm{dist}(u,z) = d+1$. Since z is a gate for x in V, we obtain $\mathrm{dist}(w,v) \geq \mathrm{dist}(x,v)-1 = d+\mathrm{dist}(v,z)-1 \geq d$ for all $v \in V \setminus \{z\}$. Thus, $\mathrm{dist}(u,V) = d$ and $\mathrm{pr}_V(u) \leq z^{\perp}$. By Proposition 2.1.25(ii) we conclude $\mathrm{diam}(\mathrm{pr}_V(u)) < 2$ since otherwise $z \in \mathrm{pr}_V(u)$. This implies that $\langle z, \mathrm{pr}_V(u)\rangle$ is a singular subspace. Therefore, $\mathrm{pr}_V(u)$ has to be a singleton by Proposition 2.1.27. Hence by Proposition 2.1.25(i), u has a gate v in V. Since $v \perp z$, we obtain by symmetric reasons that u is the gate of v in U. Thus, we may repeat this argument to prove that all points of U that are collinear to U have a gate in V that is at distance d. Since U is connected, U is one-parallel to V. Analogously, V is one-parallel to U.
Since every point $q \in V$ has a unique gate p in U, we conclude that φ is bijective. Since $z = x^{\varphi}$, $v = u^{\varphi}$ and $z \perp v$, we already know that φ preserves collinearity. It remains to check whether $p^{\varphi} \in zv$ for every point $p \in xu$. Suppose $p^{\varphi} \notin zv$. Then Lemma 2.1.21(iii) implies that there is a point $s \in V$ with $\mathrm{dist}(s,z) = \mathrm{dist}(s,v) = n-1$ and $\mathrm{dist}(s,p^{\varphi}) = n$ since by Proposition 2.1.23 V is a SPO space. Thus, $\mathrm{dist}(p,s) = d+n$ and $\mathrm{dist}(x,s) = \mathrm{dist}(u,s) = d+n-1$, a contradiction to Lemma 2.1.14. ☐

The corresponding assertion for a point x at finite codistance to a metaplecton V with $\mathrm{copr}_V(x) = \{z\}$ also holds; see Corollary 4.2.8. Anyhow, we do not prove this claim at this point, since we will use for the proof the classification of rigid subspaces of finite diameter. These subspace will be introduced in the following section.

2.2 Rigid subspaces

To prove further conditions for the structure of SPO spaces we study rigid subspaces, i. e. convex subspaces that fulfil an additional property. We will see in this section that there are some regularities that are valid in rigid subspaces. Even though for a classification of SPO spaces there is still a long way to go, we get already at this stage some insight into the list of diagrams attached to the SPO spaces. In the introduction we mentioned how one can read out of the diagram the symplectic rank of the associated point-line space. We give some more facts one can read from a diagram without any proof. Observing the diagrams should only motivate the significance of some of the following propositions.

Given a diagram with one branching (where the leftmost vertex of $\mathrm{A}_{n,j}$ with $1 < j < n$ counts as a branching) one can obtain a diagram of type $\mathrm{D}_{r,1}$ by repeatedly erasing the rightmost vertices. The symplecta of the associated point-line space are all of type $\mathrm{D}_{r,1}$. By erasing either the upper or the lower branch that goes to the right starting from the branching point one obtains a diagram of type $\mathrm{A}_{s,1}$ or $\mathrm{A}_{t,1}$, respectively. This means for the point-line space that the maximal singular subspaces are projective spaces of the types $\mathrm{A}_{s,1}$ and $\mathrm{A}_{t,1}$. In this spirit, starting at the leftmost vertex and ending at a vertex immediately right to the branching point one obtains a diagram of type $\mathrm{A}_{r-1,1}$ that corresponds to a generator of a symplecton.

Definition 2.2.1. We call a symplecton Y *rigid* if Y contains a point that is contained in at least three lines of Y. A subspace is called *rigid* if it is convex and all its symplecta are rigid.

Let Y be a symplecton of an SPO space and let $p \in Y$ be a point. By Corollary 2.1.18 every symplecton is a non-degenerate polar space of rank ≥ 2. Hence, there is a generator $G \leq Y$ with $p \in G$ and $\mathrm{rk}(G) \geq 1$. Let $q \in G \setminus \{p\}$. Since by Proposition 2.1.23 Y is an SPO space, Lemma 2.1.21(iii) implies that there is a point $r \in Y$ with $p \perp r \not\perp q$. Hence, rp is a line not contained in G. Assume $\mathrm{rk}(Y) \geq 3$. Then $G > g$ and hence, there is a line in G through p that is distinct to g. This implies that every symplecton of rank ≥ 3 is rigid. Thus, every symplecton that is not rigid is of rank 2.

Lemma 2.2.2. *Let Y be a rigid symplecton of rank 2 and let $l \leq Y$ be a line. Then there is a point $p \in l$ that is contained in three lines of Y. Furthermore, let p and q be non-collinear points of Y. Then p is contained in three lines of Y if and only if q is contained in three lines of Y.*

Proof. Since Y is rigid, there is a point $q \in Y$ that is contained in distinct lines g_0, g_1 and g_2. We may assume $q \notin l$ since otherwise we are done. Let $p' \in l$ be a point collinear to q. Since $\mathrm{rk}(Y) = 2$, we know $l \not\leq q^{\perp}$ since otherwise $\langle q, l \rangle$ would be a singular subspace of rank 2. Let $p \in l \setminus \{p'\}$. Then $p \not\perp q$ and hence by (BS), on every line through q there is a point collinear to p. For $i \in \{0,1,2\}$, let $q_i \in g_i \cap p^{\perp}$. Since $q_i \neq p$ and Y does not contain a singular subspace of rank 2, we obtain $q_i \not\perp q_j$ for $0 \leq i < j \leq 2$. Thus, pq_0, pq_1 and pq_2 are three distinct lines. □

Every non-rigid symplecton is a *grid*; see [vM98, Theorem 1.6.2].

Lemma 2.2.3. *Let Y be a symplecton of an SPO space with* $\mathrm{rk}(Y) \geq 3$. *Further let V be a metaplecton such that $S := Y \cap V$ is a singular subspace.*

(i) *Let* $\mathrm{rk}(S) \geq 2$. *Then S is a generator of Y.*

(ii) *Let S be a line. Then $v \in V$ has a gate z in V if and only if* $\mathrm{pr}_S(v) = \{z\}$.

Proof. (i) Since $S \leq V$, we know that V is a metaplecton of diameter ≥ 2. By Lemma 2.1.26 there is a symplecton $Z \leq V$ that contains three points of S that are not collinear. Hence, $\mathrm{rk}(Z \cap S) \geq 2$. Thus, it suffices to show that already $S' := Z \cap Y$ is a generator of Y and consequently, $S = S'$.
Let $s \in S'$ and $p \in Z$ be non-collinear points. Since Z is a polar space, the subspace $p^\perp \cap S'$ contains a line g. Hence, $\mathrm{pr}_Y(p)$ is a generator of Y by Proposition 2.1.27. Since Y is a polar space, $s^\perp$ contains a hyperplane H of $\mathrm{pr}_Y(p)$. Now $H \leq \langle p, s\rangle_{\mathrm{g}} = Z$ and hence, $H \leq S$. Since $s \notin p^\perp$, we conclude $s \notin H$ and therefore $H < S$. This implies by Lemma A.2.13 that $\langle s, H\rangle$ is again a generator of Y. The claim follows with $\langle s, H\rangle \leq S' \leq S \leq Y$ and the maximality of $\langle s, H\rangle$.
(ii) Set $n := \mathrm{dist}(v, z)$. Let $v \in V$ be a point with $\mathrm{pr}_S(v) = \{z\}$ for a point $z \in S$. Then $\mathrm{pr}_Y(v) < Y$ and hence, $\mathrm{pr}_Y(v)$ is singular by Proposition 2.1.25(ii). Let $y \in S \setminus \{z\}$. Then by Proposition 2.1.25(iii) there is a geodesic from y to v containing a point of $\mathrm{pr}_Y(v)$. Since $\langle y, v\rangle_{\mathrm{g}} \leq V$, we obtain $\mathrm{pr}_Y(v) \cap V \neq \varnothing$ and therefore $z \in \mathrm{pr}_Y(v)$. Moreover, since Y is a polar space, $y^\perp \cap \mathrm{pr}_Y(v)$ contains a hyperplane of $\mathrm{pr}_Y(v)$. Since $y^\perp \cap \mathrm{pr}_Y(v) \leq \langle y, v\rangle_{\mathrm{g}} \leq V$, we obtain $y^\perp \cap \mathrm{pr}_Y(v) \leq S$ and hence, $y^\perp \cap \mathrm{pr}_Y(v) = \{z\}$. Thus, $\mathrm{rk}(\mathrm{pr}_Y(v)) \leq 1$ and with Proposition 2.1.27 this implies $\mathrm{pr}_Y(v) = \{z\}$. Now the claim follows from Proposition 2.1.25(i). ☐

The following Proposition shows that whenever a symplecton of rank r has a generator that is not a maximal singular subspace then this symplecton is of type D_r, see Theorem B.2.3.

Proposition 2.2.4. *Let Y be a symplecton of an SPO space $\mathscr{S}$. Further let M be a generator of Y that is not a maximal singular subspace of $\mathscr{S}$. Then the following assertions hold:*

(i) *Every hyperplane of M is contained in at most two generators of Y.*

(ii) *Let* $\mathrm{rk}(Y) \geq 3$. *Then every hyperplane of M is contained in at most two maximal singular subspaces of $\mathscr{S}$.*

Proof. (i) First let $\mathrm{rk}(Y) \geq 3$. Suppose there are generators N and N' of Y such that M, N and N' are pairwise different and intersect in a common hyperplane H. Let $p \in N' \setminus H$. By Lemma 2.1.13 and (A12) there is a point s at codistance 2 to p such that p is a cogate for s in Y. Let $x \in M \setminus H$ and $y \in N \setminus H$. Then p, x and

y are pairwise non-collinear and hence, $x \leftrightarrow s \leftrightarrow y$.
Let $M' \leq \mathscr{S}$ be a singular subspace containing M properly. Let H' be the hyperplane of M' that contains all points that are non-opposite s. Then H' contains H properly since $H < M < M'$. Let $z \in H' \setminus H$. Then $z \not\perp y$ since $z \notin \langle x, y\rangle_{\mathrm{g}}$. Hence $Z := \langle y, z\rangle_{\mathrm{g}}$ is a symplecton. Since y and H are contained in Z, the singular space $\langle y, H\rangle = N$ is a generator of Z by Lemma 2.2.3(i). Thus, $\langle z, H\rangle$ is a generator of Z. Since $\langle z, H\rangle \leq H'$, all points of this generator have codistance 1 to s. Since $y \in Z$ there is a cogate s' for s in Z with $\mathrm{cod}(s, s') = 2$. Thus, all points of $\langle z, H\rangle$ are collinear to s'. We conclude that $\langle s', z, H\rangle$ is a singular subspace containing $\langle z, H\rangle$ properly, a contradiction.
Now let $\mathrm{rk}(Y) = 2$. Then M is a line. Let S be a singular space that contains M properly. Suppose there is a point $y \in M$ that is contained in three lines of Y. Let $x \in M \setminus \{y\}$ and let s be a point with $\mathrm{cod}(s, y) = 1$ and $s \leftrightarrow x$. Then s has a cogate s' in Y with $\mathrm{cod}(s, s') = 2$. Hence, $s'y$ is a line. Furthermore, since $\mathrm{rk}(S) \geq 2$, there is a line $g \leq S$ containing y such that all points on g are at codistance 1 to s. Let $h \leq Y$ be a line through y distinct to M and $s'y$. Take a point $z \in h \setminus \{y\}$. Then we obtain $s \leftrightarrow z$ and $x \not\perp z$. Let $w \in g \setminus \{y\}$. Since $w \notin Y = \langle x, z\rangle_{\mathrm{g}}$ and $x \perp w$, we obtain $w \not\perp z$ and hence, $Z := \langle w, z\rangle_{\mathrm{g}}$ is a symplecton. By (A12) we conclude that s has a cogate at codistance 2 in Z. Since this cogate is collinear to all points of g, we conclude $\mathrm{rk}(Z) \geq 3$. Since $\mathrm{rk}(Y) = 2$ and $Y \neq Z$, we obtain $Z \cap Y = h$. Thus, we may apply Lemma 2.2.3(ii) to conclude that y is a gate for x in Z. This implies $\mathrm{dist}(x, w) = 2$, a contradiction.
(ii) Let H be a hyperplane of M and let M' be a maximal singular subspace of $\mathscr{S}$ containing M. Let N' be a maximal singular subspace of $\mathscr{S}$ with $N' \neq M'$ and $H \leq N'$. Let $p \in N' \setminus M'$. Then there is a point $q \in M'$ with $q \not\perp p$ by the maximality of M'. Suppose $M \leq p^{\perp}$. Then M is contained in the symplecton $\langle p, q\rangle_{\mathrm{g}}$. Thus, Lemma 2.2.3(i) implies that M is a generator of $\langle p, q\rangle_{\mathrm{g}}$, a contradiction to $M < \langle p, M\rangle \leq \langle p, q\rangle_{\mathrm{g}}$. Therefore we may assume $q \in M$. Furthermore, we conclude that M' is the unique maximal singular subspace of $\mathscr{S}'$ containing M.
Assume it is not possible to choose p such that $p \notin Y$. Then $\langle p, H\rangle$ is a generator of Y by Lemma A.2.13. Hence by (i), M and $\langle p, H\rangle$ are the only generators of Y containing H. The claim follows. Now let $p \notin Y$. Then by Proposition 2.1.27 $N := \mathrm{pr}_Y(p)$ is a generator of Y. Since $q \notin N$, we know $N \neq M$. By (i) M and N are the only generators of Y containing H. Thus, for every point $r \in N' \setminus H$, we obtain $\mathrm{pr}_Y(r) = N$. Since by Lemma A.2.13 H is a hyperplane of N, there is a point $s \in N$ such that $N = \langle s, H\rangle$. Since $s \perp r$ for every point $r \in N' \setminus H$, we conclude $s \in N'$ and hence, $N \leq N'$ by the maximality of N'. Analogously to M, N' is the only maximal singular subspace containing N.
For a third maximal singular subspace L of $\mathscr{S}$ with $H \leq L$, we conclude again $M \not\leq Y$ and that $L \cap Y$ contains a generator of Y. Since $N \not\leq Y$ by analogous reasons, this leads to a contradiction to (i). $\square$

The following proposition implies that the diagrams attached to SPO spaces have at most one branching. We know that the generators of a symplecton appear in the diagram as a subdiagram of type A starting at the leftmost vertex and ending one vertex after the first branching. With a second branching one would find generators that are contained in different maximal singular subspaces.

Proposition 2.2.5. *Let Y be a rigid symplecton of an SPO space $\mathscr{S}$. Then every generator of Y is contained in a unique maximal singular subspace of $\mathscr{S}$.*

Proof. Let G be a generator of Y. Suppose there are two distinct maximal singular subspaces M and N of $\mathscr{S}$ with $G \leq M \cap N$. Then there are non-collinear points $p \in M$ and $q \in N$. Hence, $\langle p,q \rangle_{\mathrm{g}}$ is a symplecton containing G. Since $Y \neq \langle p,q \rangle_{\mathrm{g}}$, we obtain $Y \cap \langle p,q \rangle_{\mathrm{g}} = G$. Since $\langle p,G \rangle \leq \langle p,q \rangle_{\mathrm{g}}$ is a singular subspace containing G properly, we conclude $\mathrm{rk}(G) < 2$ by Lemma 2.2.3(i). Hence, $\mathrm{rk}(Y) = 2$ by Corollary 2.1.18 and consequently, G is a line. Since Y is rigid, G contains a point y that is contained in three lines of Y by Lemma 2.2.2. By Proposition 2.2.4(i) this implies that G is a maximal singular subspace of $\mathscr{S}$, a contradiction to $\langle p,G \rangle > G$. □

Our next goal is to show that in connected rigid subspaces all symplecta are of the same rank and therefore, connected rigid subspaces of diameter ≥ 2 are strongly parapolar spaces with symplectic rank r for a cardinal r.

Lemma 2.2.6. *Let Y and Z be two rigid symplecta having a line in common. Then* $\mathrm{rk}(Y) = \mathrm{rk}(Z)$ *or Y and Z are both of infinite rank.*

Proof. Let g be a common line of Y and Z. If $Y \cap Z > g$, the claim follows from Lemma 2.2.3(i). Hence, we may assume $Y \cap Z = g$. First let $\mathrm{rk}(Y) = 2$. Then g is a generator of Y and since Y is rigid, there is a point $y \in g$ that is contained in three generators of Y. Thus, g is already a maximal singular subspace by Proposition 2.2.4(i). We conclude that g is a generator of Z and therefore $\mathrm{rk}(Z) = 2$.
Now assume that Y and Z both have rank ≥ 3. Let $M \leq Y$ be a generator containing g and choose a point $p \in M \setminus g$. Analogously, let $q \in N \setminus g$ for a generator N of Z with $g \leq N$. By Proposition 2.1.23 and Lemma 2.1.21(iii) there is a point $r \in Y$ with $\mathrm{dist}(p,r) = 2$ and $g \leq r^{\perp}$. Since $q \notin Y$, it cannot happen that q is collinear to both p and r. Hence, we may assume $\mathrm{dist}(p,q) = 2$. Now $\langle p,q \rangle_{\mathrm{g}}$ is a symplecton that intersects both Y and Z in a generator by Lemma 2.2.3(i). The claim follows. □

We will see later on that the case where both Y and Z have infinite rank only occurs for the trivial case $Y = Z$ and therefore $\mathrm{rk}(Y) = \mathrm{rk}(Z)$ holds for all cases.

Corollary 2.2.7. *Let Y and Z be two symplecta of a connected rigid subspace. Then* $\mathrm{rk}(Y) = \mathrm{rk}(Z)$ *or Y and Z are both of infinite rank.*

Proof. Since Y and Z are contained in a connected rigid subspace, we find a finite sequence $(Y_i)_{0\leq i\leq n}$ of rigid symplecta such that $Y = Y_0$, $Z = Y_n$ and $Y_i \cap Y_{i+1} \neq \varnothing$ for $0 \leq i < n$. Hence, we may restrain ourselves to the case $Y \cap Z \neq \varnothing$
If $Y \cap Z$ contains a line, we obtain $\mathrm{rk}(Y) = \mathrm{rk}(Z)$ by Lemma 2.2.6. Hence, let $Y \cap Z$ contain a single point s. Let $p \in Y \setminus \{s\}$ be a point collinear to s. Since $p \notin Z$, there is a point $q \in Z$ with $q \perp s$ and $\mathrm{dist}(p,q) = 2$. Now $\langle p,q\rangle_{\mathrm{g}}$ is rigid since p and q are contained in a common rigid subspace. Since $ps \leq \langle p,q\rangle_{\mathrm{g}} \cap Y$ and $qs \leq \langle p,q\rangle_{\mathrm{g}} \cap Z$, the claim follows from Lemma 2.2.6. □

Again, as we will see later, the case that both Y and Z are of infinite rank only occurs if $Y = Z$. In other words, a rigid subspace that contains a symplecton Y of infinite rank already equals Y.

The following proposition considers polar spaces of type $\mathrm{D}_{r,1}$, see Theorem B.2.3. In terms of diagrams, the two different subdiagrams of type $\mathrm{A}_{r-1,1}$ corresponds to the two different classes of generators.

Proposition 2.2.8. *Let Y be a polar space of finite rank r such that every singular space of Y of rank $r-2$ is contained in exactly two generators of Y. Further let M, N and L be generators of Y. Then $\mathrm{crk}_M(M\cap N) + \mathrm{crk}_L(L\cap M) + \mathrm{crk}_L(L\cap N)$ is even. Equivalently, the dual polar graph of Y is bipartite.*

Proof. If $M = N$, there is nothing to prove. Hence, we may assume that M, N and L are pairwise disjoint. Assume M and N intersect in a common hyperplane H. Suppose there are points $p \in M \setminus H$ and $q \in N \setminus H$ that are both contained in L. Then $p \perp q$ and hence, $M = \langle p,H\rangle \leq q^{\perp}$. Since M is a generator, this implies $q \in M$, a contradiction. Therefore we may assume $N \cap L \leq H$.
Let B be a basis of H such that $B\cap L$ is a basis of $H\cap L$. Set $r' := \mathrm{rk}(H\cap L)$. Then $|B| = r-1$ and $|B\cap L| = r'+1$. Since $b^{\perp} \cap L$ is a hyperplane of L for every $b \in B\setminus L$, we conclude $\mathrm{rk}(L\cap H^{\perp}) = \mathrm{rk}(L\cap(B\setminus L)^{\perp}) \geq (r-1)-(r-r'-2) = r'+1$. Thus, there is a point $s \in (L\cap H^{\perp})\setminus H$. We conclude that $\langle s,H\rangle$ is a generator of Y and since M and N are the only generators containing H, this implies $M = \langle s,H\rangle$. Thus, $\mathrm{crk}_L(L\cap N) = \mathrm{crk}_L(L\cap M)+1$. The claim follows since $\mathrm{crk}_M(M\cap N) = 1$.
Let $\mathfrak{G}$ be the set of generators of Y. Further set $\mathfrak{G}_0 := \{G \in \mathfrak{G} \mid \mathrm{crk}_L(L\cap G) \in 2\cdot\mathbb{N}\}$ and $\mathfrak{G}_1 := \mathfrak{G}\setminus\mathfrak{G}_0$. We conclude that the dual polar graph of Y is bipartite since every edge has one vertex in $\mathfrak{G}_0$ and one in $\mathfrak{G}_1$. Now the claim follows since $\mathrm{crk}_M(M\cap N)$ equals the distance of M and N in the dual polar graph. □

Translating the following proposition into the language of diagrams provides a list of strong restrictions to the possible diagrams with one branching. We will call the branches of the diagram the left, the upper and the lower branch, always excluding the branching point. We may assume that the upper branch is at least as long as the lower one. Claim (iii) states that the given symplecton is of type

$\mathrm{D}_{r,1}$. Note that this symplecton has a genenerator that is not a maximal singular subspace and thus, the upper branch has length at least 2. Claim (iv) says that if the left branch possesses at least one vertex, then the lower branch has exactly one vertex. Moreover, if the left branch possesses at least two vertices, the upper branch possesses exactly two vertices by claim (vii). Finally, the left branch has at most three vertices by claim (viii). This provides exactly the list of diagrams with a branch given in the introduction.

Proposition 2.2.9. *Let Y be a rigid symplecton of an SPO space $\mathscr{S}$ and let x be a point with $\operatorname{dist}(x,Y)=1$ such that $X:=\operatorname{pr}_Y(x)$ is a generator of Y. Further let $\mathfrak{G}_i$ be the set of all generators $W \leq Y$ with $\operatorname{crk}_X(X\cap W)=2n+i$ where $n\in\mathbb{N}$ and $i\in\{0,1\}$.*

(i) $\operatorname{rk}(Y)\geq 3$.

(ii) *Let $W\in\mathfrak{G}_0$. Then there is a point $w\in\langle x,Y\rangle_{\mathrm{g}}$ with $\operatorname{dist}(w,Y)=1$ and $\operatorname{pr}_Y(w)=W$.*

(iii) *Let $W\in\mathfrak{G}_i$ and $W'\in\mathfrak{G}_j$. Then $\operatorname{crk}_W(W\cap W')\in 2\mathbb{N}$ if and only if $i=j$.*

(iv) *Let $\operatorname{rk}(Y)\geq 4$ and let $W\in\mathfrak{G}_1$. Then W is a maximal singular subspace.*

(v) *Let $\operatorname{rk}(Y)\geq 4$ and let $W\in\mathfrak{G}_i$ such that $W=Y\cap Z$ for some symplecton Z. Then $i=1$.*

(vi) *Let $\operatorname{rk}(Y)\geq 4$, $W\in\mathfrak{G}_0\setminus\{X\}$ and $w\notin Y$ such that $\langle w,W\rangle$ is singular. Then $x\perp w$ implies $\operatorname{crk}_X(X\cap W)=2$.*

(vii) *Let $\operatorname{rk}(Y)\geq 5$. Then $\langle x,X\rangle$ is a maximal singular subspace.*

(viii) $\operatorname{rk}(Y)\leq 6$.

Proof. (i) Suppose $\operatorname{rk}(Y)=2$. Then X is a line. Since Y is rigid, there is a point on X that is contained in two other lines of Y. Thus, Proposition 2.2.4(i) implies that X is a maximal singular subspace of $\mathscr{S}$, a contradiction to the existence of x. Hence, $\operatorname{rk}(Y)\geq 3$.
(ii) Set $S:=X\cap W$. First assume $\operatorname{crk}_X(S)=2$. Then $S\neq\varnothing$ by (i). Take a point $p\in W\setminus S$. Then $\langle p,x\rangle_{\mathrm{g}}$ is a symplecton that contains a hyperplane $H:=\operatorname{pr}_X(p)$ of X. Thus, $\langle p,H\rangle$ is a common generator of Y and $\langle p,x\rangle_{\mathrm{g}}$. Now let $q\in W\setminus\langle p,H\rangle$. Then $M:=\operatorname{pr}_{\langle p,x\rangle_{\mathrm{g}}}(q)$ contains $\langle p,S\rangle=W\cap\langle p,H\rangle$ and hence M is a generator of $\langle p,x\rangle_{\mathrm{g}}$ by Proposition 2.1.27. This implies $M>\langle p,S\rangle$. Thus, for any point $w\in M\setminus Y=M\setminus\langle p,S\rangle$, we obtain $\operatorname{pr}_Y(w)\geq\langle q,p,S\rangle=W$, $\operatorname{dist}(w,Y)=1$ and $w\in\langle p,x\rangle_{\mathrm{g}}\leq\langle x,Y\rangle_{\mathrm{g}}$.
For a generator W_n with $\operatorname{crk}_X(X\cap W_n)=2n$ there is a sequence $(W_i)_{0\leq i\leq n}$ of generators of Y with $W_0=X$ and $\operatorname{crk}_{W_i}(W_i\cap W_{i+1})=2$ for $0\leq i<n$. By induction there are points w_i with $\operatorname{dist}(w_i,Y)=1$ such that $\operatorname{pr}_Y(w_i)=W_i$ and $w_i\in\langle w_{i-1},Y\rangle_{\mathrm{g}}\leq\langle x,Y\rangle_{\mathrm{g}}$ for $1\leq i\leq n$ and $w_0=x$.

(iii) Let M and N be generators of Y that intersect in a common hyperplane H. If $M \in \mathfrak{G}_0$, then M is not a maximal subspace in $\mathscr{S}$ by (ii). Thus, Proposition 2.2.4(i) implies that M and N are the only generators of Y containing H. If $M \in \mathfrak{G}_1$, then there is a point $p \in X \setminus M$ and we obtain $W := p \oplus M \in \mathfrak{G}_0$ by Lemma A.2.19. By Lemma A.2.16 the generators M and W intersect in a common hyperplane by H'. Hence as before, W and M are the only generators containing H'. If N and W have a hyperplane in common, then $H' \leq N$ by Proposition A.2.14 and hence $W = N$. If N and W have no hyperplane in common, H and H' are distinct and we obtain $\mathrm{crk}_M(H' \cap H) = 2$. This implies $\mathrm{crk}_W(W \cap N) = 2$. By (ii) there is a point w with $\mathrm{dist}(w,Y) = 1$ such that $\mathrm{pr}_Y(w) = W$. Hence, again by (ii), N is not a maximal singular subspace of $\mathscr{S}$ and we conclude by Proposition 2.2.4(i) that M and N are the only generators of Y containing H. Now Proposition 2.2.8 implies that the dual polar graph of Y is bipartite and the claim follows.
(iv) Let W' be a generator with $\mathrm{crk}_W(W \cap W') = 1$. Then $W' \in \mathfrak{G}_0$ by (iii) and hence by (ii), there is a point $w' \notin Y$ such that $\langle w', W' \rangle$ is a singular subspace. Assume there is a point w with $\mathrm{dist}(w,Y) = 1$ and $\mathrm{pr}_Y(w) = W$. Then $w' \not\perp w$ since otherwise the generator W would be contained properly in the singular subspace $\langle w, W \rangle$ of the symplecton $\langle p, w' \rangle_{\mathrm{g}}$, where $p \in W \setminus W'$, a contradiction to Lemma 2.2.3(i). Hence, $\langle w', w \rangle_{\mathrm{g}}$ is a symplecton that contains $W \cap W'$. Since $\mathrm{rk}(W \cap W') \geq 2$, there is a common generator W'' of Y and $\langle w, w' \rangle_{\mathrm{g}}$ by Lemma 2.2.3(i). Since $\langle w, W \rangle$ is singular, W cannot be a generator of $\langle w, w' \rangle_{\mathrm{g}}$ and thus, $W \neq W''$. Analogously $W' \neq W''$ and we conclude that W, W' and W'' are pairwise distinct generators of Y containing the common hyperplane $W \cap W'$, a contradiction to Proposition 2.2.4(i).
(v) Let Z be a symplecton such that $Y \cap Z = W$. Then there is a point $w' \in Z$ such that $\mathrm{pr}_W(w')$ is a hyperplane of W. Hence, $\mathrm{pr}_W(w')$ intersects W in a hyperplane. Since $\mathrm{rk}(W) \geq 3$, Proposition 2.1.27 implies that $\mathrm{pr}_Y(w')$ is a generator of Y. Thus, $\mathrm{pr}_Y(w') \in \mathfrak{G}_0$ by (iv) and therefore $W \in \mathfrak{G}_1$ by (iii).
(vi) Set $S := X \cap W$ and let $p \in X \setminus S$. Assume $w \perp x$. Then the symplecton $\langle p, w \rangle_{\mathrm{g}}$ contains a hyperplane H of W and S contains a hyperplane of H since $x \in \langle p, w \rangle_{\mathrm{g}}$. Thus, $\mathrm{crk}_X(S) = 2$ by (iv).
(vii) Assume there is a singular subspace M containing $\langle x, X \rangle$ properly. Let $y \in Y \setminus X$ and set $H := X \cap y^{\perp}$. Then $\langle x, y \rangle_{\mathrm{g}}$ is a symplecton containing H. Hence by Lemma 2.2.3(i), Y and $\langle x, y \rangle_{\mathrm{g}}$ have a generator in common since $\mathrm{rk}(H) \geq 3$. Since H is a hyperplane of this generator, $\langle x, H \rangle$ is a generator of $\langle x, y \rangle_{\mathrm{g}}$.
Now let $v \in \langle x, y \rangle_{\mathrm{g}} \setminus \langle x, H \rangle$ with $v \perp x$ and let $u \in M \setminus \langle x, X \rangle$. Then $v \not\perp u$ since otherwise $u \in \langle x, y \rangle_{\mathrm{g}}$ and $\langle x, H \rangle$ would be no generator of $\langle x, y \rangle_{\mathrm{g}}$. Thus, $Z := \langle u, v \rangle_{\mathrm{g}}$ is a symplecton containing a hyperplane S of H. Since $\mathrm{rk}(S) \geq 2$, the symplecta Y and Z have a generator G in common. Since $\mathrm{crk}_X(S) = 2$, we obtain $\mathrm{crk}_G(S) = 2$ and hence, $\langle u, x, S \rangle$ is a generator of Z. This implies $M \cap Z = \langle u, x, S \rangle$. Since $x \notin Y$, there is a point $s \in G \setminus M$. Let $z \in X$ with $z \not\perp s$. Then $Y = \langle s, z \rangle_{\mathrm{g}}$ and since s is

collinear to a point on xu this implies $Y \cap xu \neq \varnothing$, a contradiction since $Y \cap M = X$.
(viii) Let W be a generator of Y with $\mathrm{crk}_X(X \cap W) = 4$. Further let w be a point with $\mathrm{dist}(w,Y) = 1$ and $\mathrm{pr}_Y(w) = W$. Assume $\mathrm{yrk}(Y) \geq 5$. Then $w \not\perp x$ by (vi). Hence, $\langle w,x \rangle_{\mathrm{g}}$ is a symplecton containing $W \cap X$. Set $S := \langle w,x \rangle_{\mathrm{g}} \cap Y$. Then $\mathrm{crk}_S(S \cap w^{\perp} \cap x^{\perp}) = \mathrm{crk}_S(W \cap X) \leq 2$. Hence, S is not a generator of Y and therefore $\mathrm{rk}(W \cap X) \leq 1$ by Lemma 2.2.3(i). We conclude $\mathrm{rk}(X) \leq 5$ and consequently $\mathrm{rk}(Y) \leq 6$. □

We conclude this section by examining the case of symplecta with infinite rank and revisiting connected rigid subspaces.

Remark 2.2.10. Let Y be a symplecton with infinite rank of an SPO space. Further let Z be a symplecton that has a line l with Y in common. Then for every point $p \in Z$ with $l \leq p^{\perp}$, the subspace $p^{\perp}$ contains a generator of Y by Proposition 2.1.27. By Proposition 2.2.9(viii) this implies $p \in Y$ and hence, every generator of Z containing l is already contained in Y. We conclude that Z equals Y. In a connected rigid subspace that contains more than one symplecton there are always two symplecta that have a line in common. Hence by Corollary 2.2.7, a connected rigid subspace of diameter ≥ 2 has always a symplectic rank r. Moreover, this rank is either finite or the connected rigid subspace is a symplecton of infinite rank.

2.3 Twin SPO spaces

As already mentioned, in a twin SPO space every two points have either finite distance or finite codistance. Therefore, every two points of a twin SPO space are somehow related to each other. This fact has some consequences which we state in this section.

Definition 2.3.1. Let V be a convex subspace of an SPO space such that for any two points x and y of V with $\mathrm{cod}(x,y) < \infty$, every point $z \perp y$ with $\mathrm{cod}(x,z) = \mathrm{cod}(x,y) + 1$ is contained in V. Then we call V *coconvex*.

For a set of points M, we denote by $\langle M \rangle_{\mathrm{G}}$ the *coconvex span* of M, which is the smallest coconvex subspace containing M.

Lemma 2.3.2. *Let $\mathscr{S}$ be a twin SPO space and let x and y be points of $\mathscr{S}$ with $x \leftrightarrow y$. Then $\langle x,y \rangle_{\mathrm{G}} = \mathscr{S}$.*

Proof. For every point $z \perp y$, we obtain $\mathrm{cod}(x,yz) = 1$ by (A1). Thus, $y^{\perp} \leq \langle x,y \rangle_{\mathrm{G}}$. Moreover, there is a point $w \perp x$ with $w \leftrightarrow z$ by (A4). Since by symmetric reasons $w \in \langle x,y \rangle_{\mathrm{G}}$, we may repeat this argument to show, that every point collinear to z is contained in $\langle x,y \rangle_{\mathrm{G}}$ and consequently, every point connected with y is contained in $\langle x,y \rangle_{\mathrm{G}}$. Analogously, every point connected with x is contained in $\langle x,y \rangle_{\mathrm{G}}$. □

Let x, y and z be points of an SPO space that are pairwise at infinite distance and finite codistance > 0. Note that in this case x, y and z are contained in pairwise distinct connected components that we denote by $\mathscr{S}_x$, $\mathscr{S}_y$ and $\mathscr{S}_z$ and the union of every two of them is a twin SPO space. Then it might happen, that there is a point $z' \leftrightarrow y$ with $\operatorname{cod}(x,z') = \operatorname{cod}(x,z) + \operatorname{dist}(z,z')$. Since then $z' \in \langle x,y,z\rangle_G$, we conclude by the previous lemma that $\mathscr{S}_y$ and $\mathscr{S}_z$ are contained in $\langle x,y,z\rangle_G$. Consequently, there is a point opposite to x in $\langle x,y,z\rangle_G$ and we obtain $\mathscr{S}_x \leq \langle x,y,z\rangle_G$. In contrast, for twin SPO spaces, studying coconvex subspaces makes much more sense.

Lemma 2.3.3. *Let $\mathscr{S}$ be a twin SPO space that contains two points at distance n. Further let z be a point of $\mathscr{S}$.*

(i) *Let y be a point of $\mathscr{S}$ with $\operatorname{cod}(y,z) = k < n$. Then there is a point x with $\operatorname{cod}(y,x) = n$ and $\operatorname{dist}(x,z) = n-k$.*

(ii) *Let y be a point of $\mathscr{S}$ with $\operatorname{dist}(y,z) = k < n$. Then there is a point x with $\operatorname{dist}(x,y) + k = \operatorname{dist}(x,z) = n$.*

Proof. Let $\mathscr{S}^+$ and $\mathscr{S}^-$ be the two connected components of the twin SPO space $\mathscr{S}$. We may assume that z is contained in $\mathscr{S}^+$. Let p and q be two points at distance n. Then by Lemma 2.1.13 there is a point r with $r \leftrightarrow p$ and $\operatorname{cod}(r,q) = n$. Hence, there is a point $s \leftrightarrow q$ with $\operatorname{dist}(r,s) = n$. Since r and s are contained in the other connected component as p and q, we may assume that p and q are contained in $\mathscr{S}^+$.
(i) Since $z \in \mathscr{S}^+$, we know $y \in \mathscr{S}^-$. Since $\operatorname{cod}(y,\langle p,q\rangle_g) \geq n$ by Proposition 2.1.17(ii), there is a point $w \in \mathscr{S}^+$ with $\operatorname{cod}(y,w) = n$. By Proposition 2.1.16(ii) we conclude that there is a point $x \in \langle w,z\rangle_g$ with $\operatorname{cod}(y,x) = n$ and $\operatorname{dist}(x,z) = n-k$.
(ii) Since $z \in \mathscr{S}^+$, we know $y \in \mathscr{S}^+$. By Lemma 2.1.13 there is a point $w \in \mathscr{S}^-$ that is opposite to z with $\operatorname{cod}(w,y) = k$. By (i) there is a point $x \in \mathscr{S}^+$ with $\operatorname{cod}(w,x) = n$ and $\operatorname{dist}(x,y) = n-k$. This implies $\operatorname{dist}(x,z) = n$. □

As a consequence of this lemma the two connected components of a twin SPO space have the same diameter. Therefore whenever we speak in the following of the *diameter of a twin SPO space*, we mean the diameter of each of the two connected components.

Definition 2.3.4. Let U and V be two convex subspaces of an SPO space. Then we call U and V *opposite* if for every point of U there is an opposite point in V and for every point of V there is an opposite point in U.

Proposition 2.3.5. *Let S be a singular subspace of finite rank of an SPO space. Then there is a singular subspace T that is opposite S. Furthermore, every convex subspace T that is opposite S is singular and has the same rank as S.*

Proof. Let T be a convex subspace that is opposite S. Suppose there are points p and q in T that are not collinear. Then for a point $s \in S$, there is by Proposition 2.1.17(ii) a point $t \in \langle p,q\rangle_{\mathrm{g}} \leq T$ with $\mathrm{cod}(s,t) \geq 2$. Since S is singular, there is no point in S opposite t, a contradiction. Thus, T is singular.
There is a basis B of S with $|B| = \mathrm{rk}(S)+1$. Since S is opposite T, we conclude that $\mathrm{copr}_T(p)$ is a hyperplane of T for every point $p \in B$. Furthermore, Lemma 2.1.21(i) implies $\bigcap_{p\in B} p^{\perp} \cap T = \varnothing$. Therefore we conclude $\mathrm{rk}(T) \leq \mathrm{rk}(S)$. By symmetric reasons we obtain $\mathrm{rk}(S) = \mathrm{rk}(T)$.
We prove the existence of T by induction. For $\mathrm{rk}(S) = 0$, there is nothing to prove since $\leftrightarrow$ is total. Now let $\mathrm{rk}(S) = n$ and assume that the claim holds for every singular space of rank $n-1$. Let S' be a hyperplane of S and let T' be a singular subspace that is opposite S'. Then $\mathrm{rk}(S') = \mathrm{rk}(T') = n-1$. Moreover, T' is not opposite S since $\mathrm{rk}(T') < \mathrm{rk}(S)$ and hence, there is a point $q \in S$ such that $\mathrm{cod}(p,q) = 1$ for every point $p \in T'$. Since every point of T' is opposite to a point in S' and q is collinear to all points of S', we obtain $\mathrm{cod}(q,T') = 1$. Therefore we may apply Lemma 2.1.21(ii) to conclude that there is a point $r \leftrightarrow q$ with $T' \leq r^{\perp}$. Set $T := \langle r,T'\rangle$.
Take a point $s \in S$. If $s \in S'$, then there is a point in T' that is opposite s. If $s = q$, then $r \leftrightarrow s$. Finally, if $s \notin S' \cup \{q\}$, then the line sq intersects S' in a point s' since S' is a hyperplane of S. Let $t \in T'$ with $t \leftrightarrow s'$. Since $q \not\leftrightarrow t$, we conclude by (A2) that q is the only point on sq that is non-opposite t and therefore $t \leftrightarrow s$.
Now take a point $t \in T$. If $t \in T'$, then there is a point in S' that is opposite t. If $t = r$, then $q \leftrightarrow t$. Finally, if $t \notin T' \cup \{r\}$, then the line tr intersects T' in a point t' since T' is a hyperplane of T. Since $t' \not\leftrightarrow q$, we conclude by (A2) that t' is the only point on tr that is non-opposite q and therefore $q \leftrightarrow t$. Thus, S and T are opposite. ☐

A consequence of this proposition is that if a twin SPO space $\mathscr{S} = (\mathscr{S}^+, \mathscr{S}^-)$ has finite singular rank, we obtain $\mathrm{srk}(\mathscr{S}) = \mathrm{srk}(\mathscr{S}^+) = \mathrm{srk}(\mathscr{S}^-)$. Furthermore, if $\mathrm{srk}(\mathscr{S})$ is infinite then both $\mathrm{srk}(\mathscr{S}^+)$ and $\mathrm{srk}(\mathscr{S}^-)$ are infinite.

3 Connected rigid subspaces

In this chapter we consider connected rigid subspaces and give a rough classification of them. By Remark 2.2.10 every connected rigid subspace has a finite symplectic rank or it is a polar space of infinite rank. It turns out to be convenient to distinguish the connected rigid subspaces by their symplectic rank. By definition, a symplectic rank only occurs for subspace of diameter ≥ 2. Furthermore, the symplectic rank is at least 2.

3.1 Maximal singular subspaces

The union of a chain of singular subspaces is again a singular subspace. Thus, Zorn's Lemma implies that each SPO space and each of its subspaces contain maximal singular subspaces. Moreover, let V be a subspace of an SPO space $\mathscr{S}$. Then every maximal singular subspace of V is contained in a maximal singular subspace of $\mathscr{S}$. Conversely, there are maximal singular subspaces of $\mathscr{S}$ that intersect V in a maximal singular subspace. These subspaces play an important role by the classification of connected rigid subspaces.

Lemma 3.1.1. *Let V be a connected rigid subspace of an SPO space $\mathscr{S}$ such that* $\mathrm{diam}(V) \geq 2$*. Further let $\mathfrak{M}$ be the set of maximal singular subspaces of $\mathscr{S}$ that contain a maximal singular subspace of V and let $\mathfrak{G}$ be the set of subspaces that are a generator of a symplecton of V.*

(i) *Every singular subspace S with* $\mathrm{rk}(S) < \mathrm{yrk}(V)$ *is contained in an element of $\mathfrak{G}$.*

(ii) *For every subspace $M \in \mathfrak{M}$, there is a subspace $G \in \mathfrak{G}$ with $M \geq G$.*

(iii) *Every maximal singular subspace $M \leq \mathscr{S}$ with* $\mathrm{rk}(M \cap V) \geq \mathrm{yrk}(V) - 2 \geq 1$ *is an element of $\mathfrak{M}$.*

(iv) *Let M and N be distinct elements of $\mathfrak{M}$. Then $M \cap N$ is properly contained in an element of $\mathfrak{G}$.*

(v) *Let* $\mathrm{yrk}(V) < \infty$ *and let M and N be elements of $\mathfrak{M}$. Then there is a sequence $(M_i)_{0 \leq i \leq n} \in \mathfrak{M}^{n+1}$ with $M_0 = M$ and $M_n = N$ such that $M_i \cap M_{i+1}$ is a hyperplane of an element if $\mathfrak{G}$ for $i < n$.*

(vi) *Let M and N be elements of $\mathfrak{M}$. Then* $\mathrm{dist}(M,N) = \mathrm{dist}(M \cap V, N \cap V)$.

Proof. (i) Let $p \in S$. Since $\mathrm{diam}(V) = 2$, there is a symplecton $Y \leq V$ and hence, there is a point $q \in Y$ with $\mathrm{dist}(p,q) = 2$. Set $Y_0 := \langle p,q \rangle_{\mathrm{g}}$. If $S \leq Y_0$, there is nothing to prove. Thus, we assume that for $i \in \mathbb{N}$, we already defined Y_i and there is a point $p_i \in S$ with $p_i \notin Y_i$. If $\mathrm{pr}_{Y_i}(p_i)$ is a singleton, then there is a point $q_i \in Y_i$ with $\mathrm{dist}(p_i,q_i) = 2$ and we obtain $\langle p_i, Y_i \cap S \rangle \leq Y_{i+1}$, where $Y_{i+1} := \langle p_i, q_i \rangle_{\mathrm{g}}$. If $\mathrm{pr}_{Y_i}(p_i)$ contains a line, then $\mathrm{rk}(Y_i) \leq 6$ by Proposition 2.2.9(viii) and therefore $\mathrm{rk}(S) \leq 5$. This implies $\mathrm{rk}(S \cap Y_i) \leq 4$ and hence, there is a point $q_i \leq Y_i$ with $q_i \not\perp p_i$ and $S \cap Y_i \leq {q_i}^{\perp}$ since every singular subspace in a polar space of finite rank is the intersection of two generators. Thus, $\langle p_i, Y_i \cap S \rangle \leq Y_{i+1}$, where $Y_{i+1} := \langle p_i, q_i \rangle_{\mathrm{g}}$. By Proposition 2.2.9(viii) we obtain after finitely many steps a symplecton containing S.

(ii) First assume $\mathrm{yrk}(V)$ is finite. Then $\mathrm{rk}(M \cap V) \geq \mathrm{yrk}(V) - 1$ by (i) and the maximality of $M \cap V$. By (i) we obtain $G \in \mathfrak{G}$ for every subspace $G \leq M \cap V$ with $\mathrm{rk}(G) = \mathrm{yrk}(V) - 1$. Now assume that $\mathrm{yrk}(V)$ is infinite. Let $x \in V$ be a point with $\mathrm{dist}(x,M) = 1$. Then $\mathrm{pr}_M(x) \cap V < M \cap V$ since $M \cap V$ is a maximal singular subspace of V. Thus, there is a point $y \in M \cap V$ with $x \not\perp y$ and $Y := \langle x,y \rangle_{\mathrm{g}}$ is a symplecton of V. Suppose there is a point $z \in M \setminus Y$. Then $\langle y, \mathrm{pr}_M(x) \rangle \leq \mathrm{pr}_Y(z)$ and hence, $\mathrm{pr}_Y(z)$ is a generator of Y. This contradicts Proposition 2.2.9(viii). Thus, $M \leq Y$ and we conclude $M \in \mathfrak{G}$ by the maximality of M.

(iii) We assume $M \nleq V$ since otherwise there is nothing to prove. Let $S \leq M \cap V$ be a subspace with $\mathrm{rk}(S) = \mathrm{yrk}(V) - 2$ and let $x \in M$ be a point not contained in V. By (i) there is a symplecton $Y \leq V$ with $S \leq Y$. Since $x \notin Y$, we obtain $S \leq \mathrm{pr}_Y(x)$. Hence by Proposition 2.1.27, $\mathrm{pr}_Y(x)$ is a generator of Y. By Proposition 2.2.5 there is a unique maximal singular subspace N in $\mathscr{S}$ with $\mathrm{pr}_Y(x) \leq N$. This implies $x \in N$ and $N \in \mathfrak{M}$.

We may assume $N \neq M$ since otherwise we are done. Let $y \in M \setminus N$. Suppose $\mathrm{pr}_Y(x) \leq \mathrm{pr}_N(y)$. Then $\langle y, \mathrm{pr}_Y(x) \rangle$ is singular and therefore contained in N, a contradiction. Thus, there is a point $z \in \mathrm{pr}_Y(x)$ with $\mathrm{dist}(y,z) = 2$. Since $\langle z,S \rangle \leq Y \cap \langle y,z \rangle_{\mathrm{g}}$, Lemma 2.2.3(i) implies that Y and $\langle y,z \rangle_{\mathrm{g}}$ have a common generator. Since $\mathrm{rk}(\langle z,S \rangle) = \mathrm{yrk}(V) - 1$, we obtain $\langle z,S \rangle = \mathrm{pr}_Y(x)$. Hence, $\mathrm{pr}_Y(x)$ is a common generator of $\langle y,z \rangle_{\mathrm{g}}$ and Y. This is a contradiction since $x \in \langle y,z \rangle_{\mathrm{g}}$ and therefore $\langle x, \mathrm{pr}_Y(x) \rangle \leq \langle y,z \rangle_{\mathrm{g}}$. We conclude $N = M$.

(iv) We may assume $M \cap N \neq \varnothing$ since otherwise there is nothing to prove. By

(ii) there are subspaces $G \leq M$ and $H \leq N$ with $\{G,H\} \subseteq \mathfrak{G}$. There is no singular subspace containing G and H since otherwise there would be an element in $\mathfrak{M}$ containing G and H, a contradiction to Proposition 2.2.5 and $M \neq N$. Hence, there are points $x \in G$ and $y \in H$ with $x \not\perp y$. Thus, $M \cap N$ is properly contained in the singular subspace $\langle x, M \cap N \rangle$ of the symplecton $\langle x,y \rangle_{\mathrm{g}}$ of V.
(v) By (ii) there are symplecta Y and Z in V such that $Y \cap M$ is a generator of Y and $Z \cap N$ is a generator of Z. Since V is connected and convex, we find a sequence $(Y_i)_{0 \leq i \leq n}$ of symplecta in V such that $Y_i \cap Y_{i+1} \neq \varnothing$, where $Y_0 = Y$ and $Y_n = Z$. If for $i < n$, the intersection $Y_i \cap Y_{i+1}$ is not a generator, then there are points $y \in Y_i$ and $z \in Y_{i+1}$ such that $Y_i \cap Y_{i+1} \leq y^\perp \cap z^\perp$ and $y \not\perp z$. Hence, we may insert $\langle y,z \rangle_{\mathrm{g}}$ between Y_i and Y_{i+1} to obtain a sequence of symplecta with greater intersections. Since $\mathrm{yrk}(V) < \infty$, we may assume that $Y_i \cap Y_{i+1}$ is a generator.
Since $\mathrm{yrk}(Y_i) < \infty$ for $0 \leq i \leq n$, we conclude by Proposition A.2.20 that there is a finite sequence of generators $(G_{i,j})_{0 \leq j \leq n_i}$ in Y_i such that $G_{i,j}$ and $G_{i,j+1}$ intersect in a hyperplane for $j < n_i$, where $n_i \in \mathbb{N}$ and furthermore $G_{i+1,0} = G_{i,n_i} = Y_i \cap Y_{i+1}$ for $i < n$, $G_{0,0} = M \cap Y$ and $G_{i,n_n} = N \cap Z$. Now the claim follows from Proposition 2.2.5.
(vi) Set $d := \mathrm{dist}(M,N)$. For $d = 0$ this follows by (iv). Therefore we may assume $d > 0$. Let $p \in M \cap V$ and let $r \in N$ with $\mathrm{dist}(r,M) = d$. Assume $\mathrm{dist}(M \cap V, N) > d$. Then $r \notin V$ since otherwise $\mathrm{pr}_M(r) \leq \langle p,r \rangle_{\mathrm{g}} \leq V$. Let $q \in N \cap V$. Then $\mathrm{dist}(p,q) = d+1$, since otherwise $r \in \langle p,q \rangle_{\mathrm{g}} \leq V$. Since $U := \langle p,q \rangle_{\mathrm{g}} \leq V$, we obtain $r \notin U$ and therefore $q \in \mathrm{pr}_U(r)$. By Proposition 2.1.25(i) we obtain $\mathrm{pr}_U(r) > \{q\}$ since $\mathrm{dist}(r,p) = d+1$ and hence q is no gate for r in U. Thus by Proposition 2.1.27 $\mathrm{pr}_U(r)$ is a maximal singular subspace of U. Since $\mathrm{dist}(p,q) \geq 2$, we conclude by (ii) that $\mathrm{pr}_U(r)$ contains a subspace $G \in \mathfrak{G}$. By Proposition 2.2.5 there is a unique maximal subspace N' of $\mathscr{S}$ that contains G. This implies $\langle r, \mathrm{pr}_U(r) \rangle \leq N'$. Since $N \cap N' \not\leq V$, we obtain $N = N'$ by (iv). This is a contradiction, since $\mathrm{dist}(p, \mathrm{pr}_U(r)) = d$ by Proposition 2.1.17(i). Thus, there is a point $s \in M \cap V$ with $\mathrm{dist}(s,N) = d$. Since for every $t \in N \cap V$ with $\mathrm{dist}(s,t) = d+1$ we obtain $\mathrm{pr}_N(s) \leq \langle s,t \rangle_{\mathrm{g}} \leq V$, we conclude $\mathrm{dist}(s, N \cap V) = d$. □

3.2 Connected subspaces of symplectic rank 2

We start our case distinction with the lowest possible symplectic rank 2. Before we start we prove a condition for arbitrary SPO spaces which we will need in this section.

Lemma 3.2.1. *Let x be a point of an SPO space and let l be a line with* $\mathrm{dist}(x,l) =: d < \infty$ *and* $\mathrm{pr}_l(x) = l$. *Then there is a point y with* $\mathrm{dist}(x,y) = d-1$ *and* $l \leq y^\perp$.

Proof. Let p and q be distinct points of l. Then $l \cap \langle x,p \rangle_{\mathrm{g}} = \{p\}$ by Proposition

2.1.17(i). Thus, $\operatorname{dist}(q,\langle x,p\rangle_{\mathrm{g}})=1$. Since $\operatorname{dist}(q,x)=\operatorname{dist}(p,x)=d$, we conclude that p is not a gate for q in $\langle x,p\rangle_{\mathrm{g}}$. Hence, Proposition 2.1.25(i) implies that $\operatorname{pr}_{\langle x,p\rangle_{\mathrm{g}}}(q)$ contains a line g through p. By Proposition 2.1.17(i) there is a point $y\in g$ with $\operatorname{dist}(y,x)=d-1$. The claim follows since $y\notin l$ and $p\perp y\perp q$. □

Proposition 3.2.2. *Let V be a connected rigid subspace of an SPO space $\mathscr{S}$ with* $\operatorname{yrk}(V)=2$*. Then V is gated.*

Proof. Since V has a symplectic rank, we know $\operatorname{diam}(V)\geq 2$. Let $x\in\mathscr{S}$ be a point with $\operatorname{dist}(x,V)<\infty$. Suppose there is a line $l\leq\operatorname{pr}_V(x)$. Then by Lemma 3.2.1 there is a point y with $\operatorname{dist}(y,V)=1$ and $l\leq\operatorname{pr}_V(y)$. Since $\operatorname{diam}(V)\geq 2$, Lemma 3.1.1(i) implies that l is a generator of a symplecton of V. This is a contradiction to Proposition 2.2.9(i). Thus, $\operatorname{pr}_V(x)$ contains a single point y. Now let $p\in V$. Since $\operatorname{dist}(p,y)<\infty$, the point y is a gate for x in $\langle p,y\rangle_{\mathrm{g}}$ by Proposition 2.1.25(i). □

This proposition enables us to give a first classification of rigid subspaces of symplectic rank 2 and finite diameter.

Theorem 3.2.3. *Let V be a rigid subspace of an SPO space with* $\operatorname{yrk}(V)=2$ *and* $\operatorname{diam}(V)<\infty$*. Then V is a metaplecton.*

Proof. Set $n:=\operatorname{diam}(V)$. Let y and z be two point of V with $\operatorname{dist}(y,z)=n$. Set $U:=\langle y,z\rangle_{\mathrm{g}}$ and let x be a point of V. By Proposition 3.2.2 x has a gate w in U. Since there is a point in U at distance n to w, we obtain $x=w$ by the diameter of V. Hence, $U=V$. □

In the rest of this section we study connected subspaces that contain a rigid subspace of symplectic rank 2.

Lemma 3.2.4. *Let U be a metaplecton of an SPO space with* $\operatorname{diam}(U)=3$*. Further let $Y\leq U$ be a rigid symplecton of rank* 2*.*

(i) *Let $X\leq U$ be a symplecton with $X\cap Y=\varnothing$. Then X and Y are isomorphic.*

(ii) *Let $Z\leq U$ be a symplecton with $Y>Z\cap Y\neq\varnothing$. Then $Y\cap Z$ is a line and* $\operatorname{rk}(Z)=2$.

(iii) *For every line $g\leq Y$, there is a symplecton $Z\leq U$ with $Y\cap Z=g$.*

(iv) *Let X and Z be symplecta of U that are distinct but not disjoint to Y. Then $X\cong Z$.*

(v) *Let $Z\leq U$ be a rigid symplecton with $Y>Z\cap Y\neq\varnothing$. Then U is rigid and all symplecta of U are isomorphic.*

Proof. (i) Let u and v be points of X with $\langle u,v\rangle_g = X$. By Proposition 3.2.2 both u and v have a gate in Y. Thus by Proposition 2.1.29, X and Y are isomorphic.
(ii) Let $w \in Y \cap Z$ and let $z \in Z$ with $\mathrm{dist}(z,w) = 2$. Then $z \notin Y$ since $Y \neq Z$. By Proposition 2.1.17(i) there is a point $y \in Y$ with $y \perp z$. Thus, Proposition 3.2.2 implies that y is a gate for z in Y. With Proposition 2.1.25(iii) this implies $y \in \langle w,z\rangle_g = Z$. Since $Y \neq Z$ and $\mathrm{rk}(Y) = 2$, we conclude $Y \cap Z = wy$. Since every symplecton that is not rigid has rank 2, the claim follows by Lemma 2.2.6.
(iii) Let y be a point on g and let $z \in Y$ be a point with $\mathrm{dist}(y,z) = 2$. Since U is an SPO space by Proposition 2.1.23, there is a point $x \in U$ with $x \perp y$ and $\mathrm{dist}(x,z) = 3$. This implies $x \notin Y$ and y is a gate for x in Y by Lemma 2.1.14. Thus, $\langle x,g\rangle_g$ is a symplecton. By (ii) we obtain $\langle x,g\rangle_g \cap Y = g$.
(iv) By (ii) we know that both X and Z intersect Y in a line. Set $g := X \cap Y$ and $h := Z \cap Y$. Moreover, $\mathrm{rk}(X) = \mathrm{rk}(Z) = 2$.
First assume $g \cap h = \varnothing$. Let $x \in g$. Since Y contains no triangle, there is a unique point $y \in h$ that is collinear to x. Let $z \in Z$ with $\mathrm{dist}(y,z) = 2$. Then by Proposition 3.2.2 z has a gate in Y and consequently, this gate is on h and distinct to y. Hence, $\mathrm{dist}(z,x) = 3$. By Lemma 2.1.14 this implies that y is a gate for x in Z. By analogous reasons, x is a gate for y in X. Suppose there is a point $p \in X \cap Z$. Then $y \in \langle x,p\rangle_g \leq X$, a contradiction. Hence, X and Z are disjoint. Since $\mathrm{dist}(x,z) = 3$ and $\mathrm{dist}(z,X) \leq 1$ by Proposition 2.1.17(i), we conclude by Lemma 2.1.14 that z has a gate in X. Hence, Proposition 2.1.29 implies that X and Z are isomorphic.
Now assume that g and h intersect. Let $g' \leq Y$ be a line that is disjoint to g. By (iii) there is a symplecton $X' \leq U$ with $X' \cap Y = g'$. As above we obtain $X \cong X'$. If $g' \cap h = \varnothing$, we obtain further $X' \cong Z$ and hence, we are done. Thus, we may assume $g' \cap h \neq \varnothing$. Let x be the intersection point of g and h and let x' be the intersection point of g' and h. Assume g and h are the only lines of Y through x. Then there is a point $y \in g$ such that there are three lines of Y meeting in y. Since Y contains no triangles, we obtain $\mathrm{dist}(y,x') = 2$ and hence by Lemma 2.2.2, there are three lines through x' in Y. Since we want to show $X \cong Z$ or $X' \cong Z$, we may assume by symmetric reasons that there are three lines in Y through x. Let $y' \in g' \setminus \{x'\}$. Then $\mathrm{dist}(x,y') = 2$ as above. By Lemma 2.2.2 there are three lines through y' in Y. Since Y contains no triangle, g' is the unique line through y' that intersects h. Analogously, there is a unique line through y' that intersects g. Thus, there is a line h' through y' that is disjoint to both h and g. By (iii) there is a symplecton $Z' \leq U$ with $Z' \cap Y = h'$. Now we conclude $X \cong Z' \cong Z$ as above.
(v) By (ii) $l := Y \cap Z$ is a line. Let x be a point on l. Further let $y \in Y \setminus l$ and $z \in Z \setminus l$ be points with $y \perp x \perp z$. Then $\mathrm{dist}(y,z) = 2$ since by Proposition 3.2.2 x is a gate for y in Z. Thus, $X := \langle y,z\rangle_g$ is a symplecton of U. Since $z \in X \setminus Y$, we obtain $X \neq Y$. Hence, (iv) implies $X \cong Z$. Analogously, $X \cong Y$ and hence, $Y \cong Z$. Now let W be an arbitrary symplecton of U that is distinct to Y. If $Y \cap W = \varnothing$, then $W \cong Y$ by (i). Otherwise $W \cong Z$ by (ii). The claim follows. □

Proposition 3.2.5. *Let V be a rigid subspace of an SPO space $\mathscr{S}$ such that* $\mathrm{yrk}(V) = 2$ *and* $\mathrm{diam}(V) < \infty$. *Further let x be a point with* $\mathrm{dist}(x,V) = 1$. *Then* $\mathrm{diam}(\langle x,V\rangle_{\mathrm{g}}) = \mathrm{diam}(V)+1$. *Let $l \leq V$ be a line with $\mathrm{pr}_V(x) \leq l$. Then $\langle x,V\rangle_{\mathrm{g}}$ is rigid if and only if $\langle x,l\rangle_{\mathrm{g}}$ is rigid. Moreover, if $\langle x,V\rangle_{\mathrm{g}}$ is rigid, then all symplecta of $\langle x,V\rangle_{\mathrm{g}}$ are isomorphic.*

Proof. Set $d := \mathrm{diam}(V)$ and $X := \langle x,l\rangle_{\mathrm{g}}$. By Proposition 3.2.2 x has a gate y in V. Thus, X is a symplecton. By Theorem 3.2.3 there is a point $z \in V$ with $\langle y,z\rangle_{\mathrm{g}} = V$ and hence, $\langle x,V\rangle_{\mathrm{g}} = \langle x,y,z\rangle_{\mathrm{g}}$. Since y is a gate for x in V, this implies $\langle x,V\rangle_{\mathrm{g}} = \langle x,z\rangle_{\mathrm{g}}$ and $\mathrm{diam}(\langle x,z\rangle_{\mathrm{g}}) = d+1$. If $\langle x,V\rangle_{\mathrm{g}}$ is rigid, X is a rigid symplecton. Thus, it remains to prove that if X is rigid, every symplecton of $\langle x,V\rangle_{\mathrm{g}}$ is isomorphic to X.

Let Y be a symplecton of $\langle x,V\rangle_{\mathrm{g}}$ such that $Y \cap V \neq \varnothing$. Let $p \in Y \cap V$ and let $q \in Y$ such that $Y = \langle p,q\rangle_{\mathrm{g}}$. Since $\mathrm{dist}(q,V) \leq 1$ by Proposition 2.1.17(i) and q has a gate in V by Proposition 3.2.2, we conclude that $Y \cap V$ contains a line g. Now assume there is a metaplecton $\langle u,v\rangle_{\mathrm{g}} \leq \langle x,V\rangle_{\mathrm{g}}$ with $\mathrm{dist}(u,v) = 3$ such that $Y \leq \langle u,v\rangle_{\mathrm{g}}$. We may assume $p = u$. Again by Proposition 2.1.17(i) we obtain $\mathrm{dist}(v,V) \leq 1$. Since v has a gate in V by Proposition 3.2.2, we conclude that there is symplecton Z that is contained in $\langle p,q\rangle_{\mathrm{g}} \cap V$.

Since V has finite diameter, there is a finite sequence of lines starting with l and ending with g such that two consecutive lines intersect. Thus, it suffices to show $X \cong Y$ for the case $g \cap l \neq \varnothing$. We may assume $Y \neq X$ since otherwise there is nothing to prove. First assume $l = g$. Let $u \in Y \setminus g$. Then by Proposition 3.2.2 u has a gate in X that lies on g. Thus, there is a point $v \in X$ with $\mathrm{dist}(u,v) = 3$. We obtain $X \cup Y \subseteq \langle u,v\rangle_{\mathrm{g}}$ since $X = \langle \mathrm{pr}_X(u),v\rangle_{\mathrm{g}}$. Now $\langle u,v\rangle_{\mathrm{g}} \cap V$ contains a rigid symplecton Z. Since $l \leq Z$, we may apply Lemma 3.2.4 to conclude $X \cong Y$.

Now assume that g and l intersect in a single point u. Let $v \in g \setminus \{u\}$. Then u is the gate of v in X. Hence, $\langle v,X\rangle_{\mathrm{g}}$ is a metaplecton of diameter 3. Since $X \leq \langle v,X\rangle_{\mathrm{g}}$, we conclude that $\langle v,X\rangle_{\mathrm{g}} \cap V$ contains a symplecton Z. Since $X \not\leq V$, we conclude $\langle v,X\rangle_{\mathrm{g}} \cap V = Z$. Now Lemma 3.2.4 implies that there is a symplecton $X' \leq \langle v,X\rangle_{\mathrm{g}}$ with $X' \cap Z = g$ and $X \cong X'$. Since this implies $X' \cap V = g$, we obtain $X' \cong Y$ as above.

Finally, let $Y \leq \langle x,V\rangle_{\mathrm{g}}$ be a symplecton that is disjoint to V. Let u and v be points of Y with $\langle u,v\rangle_{\mathrm{g}} = Y$. By Proposition 2.1.17(i) and Proposition 3.2.2 u has a gate u' in V and v has a gate v' in V. Since $\mathrm{dist}(u,v') \leq 3$, we obtain $\mathrm{dist}(u',v') \leq 2$. Since $u' \notin Y$, we know $u' \neq v'$. Suppose $u' \perp v'$. Then $\langle u,v'\rangle_{\mathrm{g}}$ is a symplecton that intersects V and hence, $\langle u,v'\rangle_{\mathrm{g}}$ is rigid and of rank 2. Since $\langle u,v'\rangle_{\mathrm{g}} \neq Y$, we obtain $v \notin \langle u,v'\rangle_{\mathrm{g}}$. Now $v \perp v'$ and $\mathrm{dist}(v,u) = 2$ implies by Proposition 2.1.25(i) that $\mathrm{pr}_{\langle u,v'\rangle_{\mathrm{g}}}(v)$ contains a line. This is a contradiction to Proposition 2.2.9(i). Thus, $\mathrm{dist}(u',v') = 2$ and consequently, $\langle u',v'\rangle_{\mathrm{g}}$ is a rigid symplecton. Now the claim follows since $\langle u',v'\rangle_{\mathrm{g}} \cong Y$ by Proposition 2.1.29. □

3.3 Connected subspaces of symplectic rank ≥ 3

There are some properties for connected rigid subspaces of symplectic rank 3 that are also valid for connected rigid subspaces of higher rank. Therefore we first study these common properties before we continue with connected rigid subspaces of symplectic rank 3.

Lemma 3.3.1. *Let V be a connected rigid subspace of an SPO space.*

(i) *Let* $\mathrm{yrk}(V) \geq 3$ *and let Y be a convex subspace with* $\mathrm{diam}(Y) \geq 2$ *that is properly contained in V. Then there is a point $x \in V$ with* $\mathrm{dist}(x,Y) = 1$ *such that* $\mathrm{pr}_Y(x)$ *is a singular subspace of rank* $\geq \mathrm{yrk}(V) - 1$.

(ii) *Let* $\mathrm{yrk}(V) \geq 5$ *and let x be a point with* $\mathrm{dist}(x,V) = 1$ *such that* $\mathrm{pr}_V(x)$ *contains a line. Then V is a symplecton.*

Proof. (i) Let $x \in V \setminus Y$ with $\mathrm{dist}(x,Y) = 1$. Then $\mathrm{diam}(\mathrm{pr}_Y(x)) < 2$ since $x \notin Y$. Assume $\mathrm{pr}_Y(x) = \{z\}$ for a point $z \in Y$. Let $y \in Y$ with $\mathrm{dist}(y,z) = 1$. Then $\mathrm{dist}(x,y) = 2$ and $Z := \langle x,y\rangle_{\mathrm{g}}$ is a symplecton. Since $x^{\perp}$ contains a hyperplane of $Y \cap Z$, we conclude $Y \cap Z = yz$. Since $\mathrm{rk}(Z) \geq 3$, there is a point $x' \in Z \setminus Y$ such that $y \perp x' \perp z$. Thus, we may assume that $\mathrm{pr}_Y(x)$ contains a line l. By Lemma 3.1.1(i) there is a symplecton $X \leq Y$ with $l \leq X$. Hence by Proposition 2.1.27, $\mathrm{pr}_X(x)$ is a generator of X. The claim follows since $\mathrm{pr}_X(x) \leq \mathrm{pr}_Y(x)$.
(ii) Set $X := \mathrm{pr}_V(x)$. By Lemma 3.1.1(i) there is a symplecton $Y \leq V$ such that $X \cap Y$ contains a line. By Proposition 2.1.27 X contains a generator of Y. Suppose $Y < V$. Then by (i) there is a point $w \in V \setminus Y$ such that $\mathrm{pr}_Y(w)$ is a generator of Y. By Proposition 2.2.9(iv) we conclude $\mathrm{crk}_{X\cap Y}(\mathrm{pr}_Y(w) \cap X) \in 2 \cdot \mathbb{N}$. Thus, Proposition 2.2.9(ii) implies that there is a point $z \in \langle w,Y\rangle_{\mathrm{g}} \leq V$ with $\mathrm{pr}_Y(z) = X \cap Y$. By Proposition 2.2.5 and Proposition 2.2.9(vii) $\langle z, X \cap Y\rangle$ is the only singular subspace that contains $X \cap Y$ properly, a contradiction to $X \leq x^{\perp}$. □

The map we introduce in the following definition turns out to be a useful tool for some proofs. Therefore we give this map an own name.

Definition 3.3.2. Let V be a connected rigid subspace of an SPO space $\mathscr{S}$ with $\mathrm{yrk}(V) \geq 3$. Further let M and N be two maximal singular subspaces of $\mathscr{S}$ that contain a maximal singular subspace of V such that $\mathrm{rk}(M \cap N) = \mathrm{yrk}(V) - 3$. Then we set $\pi_{M,N} \colon \mathfrak{P}(M) \to \mathfrak{P}(N) \colon R \mapsto \bigcup_{p\in R} \mathrm{pr}_N(p)$, where $\mathfrak{P}(M)$ and $\mathfrak{P}(N)$ denote the power sets of the point sets of the subspaces M and N.

Lemma 3.3.3. *Let V be a connected rigid subspace of an SPO space $\mathscr{S}$ with* $\mathrm{yrk}(V) \geq 3$. *Further let $\mathfrak{M}$ be the set of maximal singular subspaces of $\mathscr{S}$ that contain a maximal singular subspace of V. Let M and N be elements of $\mathfrak{M}$ with* $\mathrm{rk}(M \cap N) = \mathrm{yrk}(V) - 3$.

(i) $\mathrm{rk}(\mathrm{pr}_M(p)) = \mathrm{yrk}(V) - 2$ *for every* $p \in N \setminus M$ *and* $\mathrm{pr}_M(p) \le V$ *if and only if* $p \in V$.

(ii) *Let* $M' \le M$ *and* $N' \le N$ *be subspaces such that* $N \cap M' = M \cap N' = \varnothing$, $\langle M \cap N, M' \rangle = M$ *and* $\langle M \cap N, N' \rangle = N$. *Then every point in* M' *is collinear to a unique point in* N'. *Moreover, the map* $\varphi \colon M' \to N'$ *that maps every point of* M' *to its unique neighbour in* N' *is an isomorphism.*

(iii) *Set* $\mathfrak{S}_M := \{R \le M \mid S \le R\}$ *and* $\mathfrak{S}_N := \{R \le N \mid S \le R\}$. *Then* $\pi_{M,N}$ *maps the lattice* $\mathfrak{S}_M$ *isomorphically onto the lattice* $\mathfrak{S}(N)$.

(iv) $\mathrm{rk}(M) = \mathrm{rk}(N)$, $\mathrm{rk}(M \cap V) = \mathrm{rk}(N \cap V)$ *and* $\mathrm{crk}_M(M \cap V) = \mathrm{crk}_N(N \cap V)$.

Proof. Set $S := M \cap N$. By Lemma 3.1.1(i) we know that S is not maximal and hence, $M \neq N$. Moreover, $\mathrm{rk}(M \cap V) \ge \mathrm{yrk}(V) - 1$ and $\mathrm{rk}(N \cap V) \ge \mathrm{yrk}(V) - 1$. Since V has a symplectic rank, we know $\mathrm{diam}(V) \ge 2$.

(i) Since $N \cap V$ is a maximal singular subspace of V, we obtain that for every point $p \in (M \cap V) \setminus S$, there is a point $q \in N \cap V$ with $\mathrm{dist}(p,q) = 2$. Set $Y := \langle p,q \rangle_{\mathrm{g}}$. Then $Y \le V$ and hence, $\mathrm{rk}(Y) = \mathrm{yrk}(V)$. Since $\mathrm{rk}(\langle p,S \rangle) = \mathrm{yrk}(V) - 2$, the subspace $\langle p,S \rangle$ is a hyperplane of a generator of Y. Assume there is a point $r \in M \setminus Y$. Then by Proposition 2.1.27 $\mathrm{pr}_Y(r)$ is a generator of Y. Since $\langle p,S \rangle \le \mathrm{pr}_Y(r)$ and $\mathrm{rk}(\langle r,p,S \rangle) = \mathrm{yrk}(V) - 1$, Lemma 3.1.1(iv) implies that there is at most one element in $\mathfrak{M}$ that contains $\mathrm{rk}(\langle r,p,S \rangle)$. Thus, $\langle r, \mathrm{pr}_Y(r) \rangle \le M$. If $M \le Y$, then M is a generator of Y since $\mathrm{rk}(M \cap V) \ge \mathrm{yrk}(V) - 1$. Thus in any case, M contains a generator of Y. Since $\mathrm{pr}_M(q) \le Y$ and $p \notin \mathrm{pr}_M(q)$, we conclude $\mathrm{rk}(\mathrm{pr}_M(q)) = \mathrm{yrk}(V) - 2$ and $\mathrm{pr}_M(q) \le V$. Analogously, the claim holds for p.

Now let $s \in N \setminus V$. Then $s \not\perp p$ since $\mathrm{pr}_N(p) \le V$. Set $Z := \langle p,s \rangle_{\mathrm{g}}$. Since $\langle p, \mathrm{pr}_N(p) \rangle \le Y \cap Z$ and $\mathrm{rk}(\langle p, \mathrm{pr}_N(p) \rangle) \ge 2$, we obtain $\mathrm{rk}(Z) = \mathrm{rk}(Y)$ by Lemma 2.2.3(i). Let $g \le M \cap V$ be a line with $g \cap S = \varnothing$. Then $s \notin \mathrm{pr}_N(r)$ for every $r \in g$, since $g \le V$ and $s \notin V$. Thus, $g \cap \mathrm{pr}_M(s) = \varnothing$ and hence, $g \not\le Z$. Let $r \in g \setminus Z$. By Proposition 2.1.27 $\mathrm{pr}_Z(r)$ is a generator of Z since $\langle p,S \rangle \le \mathrm{pr}_Z(r)$. Since $\mathrm{rk}(\langle r,p,S \rangle) = \mathrm{yrk}(V) - 1$, Lemma 3.1.1(iv) implies that M is the unique element of $\mathfrak{M}$ that contains $\mathrm{rk}(\langle r,p,S \rangle)$. Thus, $\langle r, \mathrm{pr}_Z(r) \rangle \le M$. Since $\mathrm{pr}_M(s) \le Z$ and $p \notin \mathrm{pr}_M(s)$, we conclude $\mathrm{rk}(\mathrm{pr}_M(s)) = \mathrm{yrk}(V) - 2$. Let $t \in \mathrm{pr}_M(s) \setminus S$. Then $\mathrm{pr}_N(t) \not\le V$ since $t \perp r$. Thus, $t \notin V$ and the claim follows.

(ii) By (i) we conclude that S is a hyperplane of $\mathrm{pr}_N(p)$ and hence, $\mathrm{pr}_N(p)$ intersects N' in a single point q. By symmetric reasons every point of N' has exactly one neighbour in M' and therefore φ is bijective.

Now let p and p' be two distinct points of M' and let q and q' be the points of N' with $q \perp p$ and $q' \perp p'$. Then $\langle p,q' \rangle_{\mathrm{g}}$ is a symplecton that contains the lines pp' and qq'. Hence, every point on pp' is collinear to a point on qq' and vice versa. Thus, $(pp')^{\varphi} = qq'$.

(iii) Let M', N' and φ like in (ii). Further let $p \in M'$ and $r \in \langle p,S \rangle \setminus S$. Set

$q := p^\varphi$. Then $q \perp r$ since we obtain $\mathrm{pr}_M(q) = \langle p, S\rangle$ by (i). Analogously, $\mathrm{pr}_N(r) = \langle q, S\rangle = \mathrm{pr}_N(p)$. This implies

$$\pi_{M,N}(R) = \bigcup_{r\in R\cap M'} \mathrm{pr}_N(r) = \langle S, \{r^\varphi \mid r \in R\cap M'\}\rangle$$

for every $R \in \mathfrak{S}_M$. The claim follows since by (ii) we know that $R' \mapsto \{r^\varphi \mid r \in R'\}$ yields an isomorphism from the lattice of subspaces of M' onto the lattice of subspaces of N'.

(iv) Since by (i) we obtain $\pi_{M,N}R \leq V \;\Leftrightarrow\; R \leq V$ for every $R \in \mathfrak{S}_M$, the claim follows with (iii). $\square$

Lemma 3.3.4. *Let V be a connected rigid subspace of an SPO space $\mathscr{S}$ with* $\mathrm{yrk}(V) \in \{3,4\}$. *Further let $\mathfrak{M}$ be the set of maximal singular subspaces of $\mathscr{S}$ that contain a maximal singular subspace of V. Let M and N be elements of $\mathfrak{M}$ with* $\mathrm{rk}(M\cap N) = \mathrm{yrk}(V) - 3$ *and* $\mathrm{rk}(M) \geq \mathrm{yrk}(V)$. *Then* $\pi_{M,N}(\langle S, \mathrm{pr}_M(x)\rangle) = \langle S, \mathrm{pr}_N(x)\rangle$ *for every point $x \in V$, where $S := M\cap N$.*

Proof. By Lemma 3.1.1(i) we know that S is not maximal and hence, $M \neq N$. Moreover, $\mathrm{rk}(M\cap V) \geq \mathrm{yrk}(V) - 1$ and $\mathrm{rk}(N\cap V) \geq \mathrm{yrk}(V) - 1$. Since V has a symplectic rank, we know $\mathrm{diam}(V) \geq 2$.

By the maximality of M there are points $p \in M$ and $q \in N$ with $\mathrm{dist}(p,q) = 2$. Set $Y := \langle p,q\rangle_{\mathrm{g}}$. Then $\mathrm{pr}_N(p) \leq Y$. On the other hand $p^\perp$ contains a hyperplane of $Y\cap N$. This implies $Y\cap N = \langle q, \mathrm{pr}_N(p)\rangle$. Thus, $\mathrm{rk}(Y\cap N) = \mathrm{yrk}(V) - 1$ by Lemma 3.3.3(i). Since $\mathrm{yrk}(V) - 1 > 1$, M intersects Y in a generator by Lemma 3.1.1(iii) and therefore $\mathrm{rk}(Y) = \mathrm{yrk}(V)$.

Set $\mathrm{dist}(x,M) = d$. Assume $\mathrm{dist}(x,N) = d+1$. Then every point of S is at distance $d+1$ to x. Moreover, for every point $p \in \mathrm{pr}_M(x)$ the projection $\mathrm{pr}_N(p)$ is contained in $\mathrm{pr}_N(x)$. Thus, $\pi_{M,N}(\langle S, \mathrm{pr}_M(x)\rangle) \leq \langle S, \mathrm{pr}_N(x)\rangle = \mathrm{pr}_N(x)$. Suppose there is a point $p \in \mathrm{pr}_N(x) \setminus \pi_{M,N}(\langle S, \mathrm{pr}_M(x)\rangle)$. Let $q \in \mathrm{pr}_M(x)$. Then $Y := \langle p,q\rangle_{\mathrm{g}}$ is a symplecton of V since by Lemma 3.3.3(i) both $\mathrm{pr}_M(x)$ and $\mathrm{pr}_N(x)$ are contained in V. Moreover, $\mathrm{rk}(\mathrm{pr}_N(q)) = \mathrm{yrk}(V) - 2$ and thus, $\langle p, \mathrm{pr}_N(q)\rangle$ is a generator of Y. Since $\langle p, \mathrm{pr}_N(q)\rangle \leq \mathrm{pr}_N(x)$, we conclude $\mathrm{dist}(x,Y) \geq d$. With $\mathrm{dist}(x,q) = d$ and $\mathrm{dist}(p,q) = 2$ this implies $\mathrm{dist}(x,Y) = 2$ and hence by Proposition 2.1.25(i) and Proposition 2.1.27 that $\mathrm{pr}_Y(x)$ is a generator of Y. Since $p \notin \pi_{M,N}(\langle S, \mathrm{pr}_M(x)\rangle)$, we obtain $\mathrm{pr}_M(p) \cap \mathrm{pr}_M(x) = \varnothing$ and hence, $\mathrm{pr}_Y(x) \cap (M\cap Y) = \{q\}$. With $\mathrm{pr}_Y(x) \cap (N\cap Y) = \varnothing$ and $M\cap N\cap Y = S$ this leads to a contradiction to Proposition 2.2.9(iii) since $\mathrm{rk}(M) \geq \mathrm{yrk}(V)$. Thus, $\pi_{M,N}(\langle S, \mathrm{pr}_M(x)\rangle) = \mathrm{pr}_N(x)$. By Lemma 3.3.3(iii) this implies $\pi_{N,M}(\mathrm{pr}_N(x)) = \langle S, \mathrm{pr}_N(x)\rangle$. Hence, it remains the case $\mathrm{dist}(x,N) = d$.

Let $p \in M \setminus S$ and $q \in N \setminus \mathrm{pr}_N(p)$; this is possible since $\mathrm{rk}(M\cap V) > \mathrm{rk}(\mathrm{pr}_N(p))$ by Lemma 3.3.3(i). Set $Y := \langle p,q\rangle_{\mathrm{g}}$. We have to show $p \in \langle S, \mathrm{pr}_M(x)\rangle \;\Leftrightarrow$

$\mathrm{pr}_N(p) \leq \langle S, \mathrm{pr}_N(x)\rangle$. Since by Lemma 3.1.1(iii) $M \cap Y$ is a generator of Y, we obtain $\mathrm{dist}(x,Y) \geq d-1$. Let $\mathrm{dist}(x,Y) = d-1$. Assume $\mathrm{pr}_Y(x)$ is a generator of Y. Then $p \in \mathrm{pr}_M(x)$ and $\mathrm{pr}_N(p) \leq \mathrm{pr}_N(x)$. Now assume x has a gate y in Y. Then $\mathrm{pr}_M(x)$ intersects $Y \cap M$ in a hyperplane H and $\langle y, H\rangle$ is a generator of Y. Analogously, $\mathrm{pr}_N(x)$ intersects $Y \cap N$ in a hyperplane H'. For the case $S \leq H$ we obtain by Proposition 2.2.9(iii) that $\langle y, H\rangle$ and $N \cap Y$ have a hyperplane in common and therefore $H' \leq \langle y, H\rangle$. Thus, $p \in \mathrm{pr}_M(x) \Leftrightarrow p \in H \Leftrightarrow \mathrm{pr}_N(p) = H' \Leftrightarrow \mathrm{pr}_N(p) \leq \mathrm{pr}_N(x)$. For the case $S \nleq H$ we obtain $M \cap Y = \langle S, H\rangle \leq \langle S, \mathrm{pr}_M(x)\rangle$. Hence, $p \in \langle S, \mathrm{pr}_M(x)\rangle$ and analogously, $\mathrm{pr}_N(p) \leq \langle S, \mathrm{pr}_N(x)\rangle$.
Now let $\mathrm{dist}(x,Y) = d+1$. Then we obtain $\langle p, S\rangle \cap \mathrm{pr}_M(x) = \varnothing$ and hence, $p \notin \langle S, \mathrm{pr}_M(x)\rangle$. Analogously, $\mathrm{pr}_N(p) \cap \mathrm{pr}_N(x) = \varnothing$ and hence, $\mathrm{pr}_N(p) \nleq \mathrm{pr}_N(x)$.
Finally let $\mathrm{dist}(x,Y) = d$. If $\mathrm{pr}_Y(x) = Y$, then $p \in \mathrm{pr}_M(x)$ and $\mathrm{pr}_N(p) \leq \mathrm{pr}_N(x)$. Hence, it remains the case $\mathrm{dist}(x,Y) = d$ and $\mathrm{pr}_Y(x) < Y$. First assume x has a gate y in Y. Then $y \in Y \cap M$, since otherwise there would be a point $y' \in Y \cap M$ with $\mathrm{dist}(y,y') = 2$ and hence, $\mathrm{dist}(x,y') = d+2$. Analogously, $y \in N$ and consequently, $y \in S$. Since py intersects both S and $\mathrm{pr}_M(x)$ only in y, we obtain $p \notin \langle S, \mathrm{pr}_M(x)\rangle$. Analogously, $p' \notin \langle S, \mathrm{pr}_N(x)\rangle$ for every $p' \in \mathrm{pr}_N(p) \setminus S$ and hence, $\mathrm{pr}_N(p) \nleq \langle S, \mathrm{pr}_N(x)\rangle$. By Proposition 2.1.27 it now remains the case that $\mathrm{pr}_Y(x)$ is a generator of Y. Set $G := \mathrm{pr}_Y(x) \cap M$ and $H := \mathrm{pr}_Y(x) \cap N$. Since $S \leq Y \cap M$, we obtain $\langle S, \mathrm{pr}_M(x)\rangle \cap Y = \langle S, G\rangle$. Analogously, $\langle S, \mathrm{pr}_N(x)\rangle \cap Y = \langle S, H\rangle$. Since $\mathrm{rk}(S) = \mathrm{yrk}(Y) - 3$, we know $\mathrm{crk}_G(G \cap S) \in \{0,1,2\}$. Analogously, $\mathrm{crk}_H(H \cap S) \in \{0,1,2\}$. Proposition 2.2.9(iii) implies that $\mathrm{rk}(G) = \mathrm{rk}(H)$ is even. Thus, $\mathrm{crk}_G(G \cap S) - \mathrm{crk}_H(H \cap S)$ is also even. Hence if S intersects G in a hyperplane, then S intersects H in a hyperplane. Since $\langle G, H\rangle \leq \mathrm{pr}_Y(x)$, this implies $\pi_{M,N}(\langle S, G\rangle) = \langle S, H\rangle$ and therefore, $\mathrm{pr}_N(p) = \langle S, H\rangle \Leftrightarrow p \in \langle S, \mathrm{pr}_M(x)\rangle$. If S contains both G and H, there is nothing to prove. Also if $\langle S, G\rangle = Y \cap M$ and $\langle S, H\rangle = Y \cap N$, there is nothing to prove.
By symmetric reasons and Lemma 3.3.3(iii) it remains the case where $H \leq S$ and $\langle S, G\rangle = Y \cap M$. Then for every point $s \in S$ the line ps contains a point of $G \setminus \{s\}$. Thus, we may assume $p \in \mathrm{pr}_M(x)$. Suppose $S \cap \mathrm{pr}_M(x) = \varnothing$. Then $H = \varnothing$ and hence, $\mathrm{pr}_N(p) \cap \mathrm{pr}_N(x) = \varnothing$. Let $r \in \mathrm{pr}_N(x)$. Then $\langle p, r\rangle_{\mathrm{g}}$ contains $\mathrm{pr}_N(p)$. Since $\mathrm{rk}(\mathrm{pr}_N(p)) \geq 1$ and $\mathrm{dist}(x, \mathrm{pr}_N(p)) = d+1$, we obtain $\mathrm{dist}(x, \langle p, r\rangle_{\mathrm{g}}) = d$. This is a contradiction to Proposition 2.1.25(ii) since $\mathrm{dist}(x,p) = \mathrm{dist}(x,r) = d$. Now suppose $S \leq \mathrm{pr}_M(x)$. Then we obtain $M \cap Y = \mathrm{pr}_Y(x)$, a contradiction to Proposition 2.1.27 and Proposition 2.2.5 since $\mathrm{dist}(x,M) = d$. Thus, $\varnothing < S \cap G < S$ and therefore $\mathrm{yrk}(V) = 4$ and $S \cap G$ is a singleton. This implies $\mathrm{rk}(G) = 2$. Since the generators $\mathrm{pr}_Y(x)$ and $(M \cap Y)$ intersect in the hyperplane G and $(M \cap Y) < M$, we conclude by Proposition 2.2.9(iv) that $\mathrm{pr}_Y(x)$ is a maximal singular subspace. This is a contradiction to Proposition 2.1.27. □

3.4 Connected subspaces of symplectic rank 3

Throughout this section we are dealing with a connected rigid subspace V of symplectic rank 3 that lives in an SPO space $\mathscr{S}$. Thereby, the set $\mathfrak{M}$ that contains all maximal singular subspaces of $\mathscr{S}$ that contain a maximal singular subspace of V plays an important role. For this, we introduce a distance function for the elements of $\mathfrak{M}$ that differs from dist. For two elements M and N of $\mathfrak{M}$ we write $M \sim N$ if and only if they have a line in common. By Lemma 3.1.1(iv) this implies for $M \sim N$ that either $M = N$ or $M \cap N$ is a line. By $\operatorname{dist}_{\mathfrak{M}}(M,N)$ we denote the distance of M and N in the graph on $\mathfrak{M}$ that is induced by the relation $\sim$.

Proposition 3.4.1. *For V, exactly one of the following assertions hold:*

(a) *The subspace V is a symplecton and each element of $\mathfrak{M}$ is a generator of V.*

(b) *There is a subspace $M \in \mathfrak{M}$ with $\operatorname{rk}(M) \geq 3$ and each line of V is contained in exactly two elements of $\mathfrak{M}$.*

Proof. We assume (a) does not hold. Let g be a line of V. By Lemma 3.1.1(i) there is symplecton $Y \leq V$ with $g \leq Y$. Since $\operatorname{rk}(Y) < \infty$, there are two generators of Y that intersect in g and hence, there are two elements N and N' of $\mathfrak{M}$ that contain g. If Y is properly contained in V, Lemma 3.3.1(i) implies that there is a generator G of Y that is properly contained in a singular subspace of V. If $Y = V$, then there is a generator G of Y that is properly contained in an element of $\mathfrak{M}$ by the assumption. By Propositions 2.2.9(iii) and 2.2.9(ii) we may assume $g \leq G$. Hence by Proposition 2.2.4(ii), N and N' are the only elements of $\mathfrak{M}$ containing g. □

Lemma 3.4.2. *Let $X \in \mathfrak{M}$ and set $\mathfrak{M}_i := \{M \in \mathfrak{M} \mid \operatorname{dist}_{\mathfrak{M}}(M,X) \in 2\mathbb{N}+i\}$ for $i \in \{0,1\}$. Let $M \in \mathfrak{M} \setminus \{X\}$ and let $i \in \{0,1\}$ such that $M \in \mathfrak{M}_i$. Set $d := \operatorname{dist}(M,X)$ and $S := \{p \in M \mid \operatorname{dist}(p,X) = d\}$. Then the following holds.*

(i) *The subspace S is contained in V and $\operatorname{rk}(S) = d + i$.*

(ii) *Let $x \in X$. Then $\operatorname{rk}(\operatorname{pr}_M(x)) = \operatorname{dist}(x,M)$ and $\operatorname{pr}_M(x) \not\leq V$ if and only if $x \notin V$ and $i = 0$.*

(iii) *If d is even, then $\operatorname{dist}_{\mathfrak{M}}(M,X) = d + 2 - i$. If d is odd, then $\operatorname{dist}_{\mathfrak{M}}(M,X) = d + 1 + i$.*

Proof. Assume that V is a symplecton and every element of $\mathfrak{M}$ is contained in V. Then (i) follows from Proposition A.2.20 and (ii) follows directly from (BS). Hence by Proposition 3.4.1, we may restrain ourselves to the case where every line of V is contained in exactly two elements of $\mathfrak{M}$.
For $M \sim X$, we obtain $S = M \cap X$ and hence, $\operatorname{rk}(S) = 1$. By Lemma 3.1.1(iv) we obtain $S \leq V$. Let $p \in M \cap V \setminus S$. Since $X \cap V$ is a maximal singular subspace

of V, there is a point $q \in X \cap V$ with $\operatorname{dist}(p,q) = 2$. Now $\langle p,q\rangle_{\mathrm{g}}$ is a symplecton of V and hence $\operatorname{rk}(\langle p,q\rangle_{\mathrm{g}}) = 3$. Therefore $\operatorname{pr}_X(p) = S$ since $\operatorname{pr}_X(p) \leq \langle p,q\rangle_{\mathrm{g}}$ and $q \notin \operatorname{pr}_X(p)$. Analogously, $\operatorname{pr}_M(x) = S$ if $x \in V \setminus S$. If $x \notin V$, then $\operatorname{dist}(p,x) = 2$. By Lemma 2.2.3(i) the symplecta $\langle p,q\rangle_{\mathrm{g}}$ and $\langle p,x\rangle_{\mathrm{g}}$ have the generator $\langle p,S\rangle$ in common. Again we conclude $\operatorname{pr}_M(x) = S$. Thus, the claim holds since $M \in \mathfrak{M}_1$ and $\operatorname{dist}_{\mathfrak{M}}(M,X) = 1$.

Now let $d = 0$ and $M \not\sim X$. Then $M \cap X$ is a singleton. Let $p \in M \cap V \setminus X$. Then $\operatorname{rk}(\operatorname{pr}_X(p)) = 1$ and $\operatorname{pr}_X(p) \leq V$ by Lemma 3.3.3(i). By Lemma 3.1.1(i) and Proposition 2.2.5 there is an element $N \in \mathfrak{M}$ with $\langle p, \operatorname{pr}_X(p)\rangle \leq N$. Since $M \cap X \leq \operatorname{pr}_X(p)$, this implies $M \sim N \sim X$. Thus, $M \in \mathfrak{M}_0$ and the claim follows from Lemma 3.3.3(i).

Let $d > 0$ and assume the claim holds for all subspaces $N \in \mathfrak{M}$ with $\operatorname{dist}(X,N) < d$. By Lemma 3.1.1(vi) there is a geodesic $(p_i)_{0 \leq i \leq d}$ in V such that $p_0 \in X$ and $p_d \in M$. Set $M_0 := X$. Recursively, let M_{i+1} be a maximal singular subspace of $\mathscr{S}$ containing p_{i+1} and $\operatorname{pr}_{M_i}(p_{i+1})$ for $i < d$. Since $p_i p_{i+1} \leq M_{i+1}$, we obtain $M_{i+1} \in \mathfrak{M}$ by Lemma 3.1.1(iii). Hence, $\operatorname{dist}(M_i, M_{i+1}) = 0$ implies $\operatorname{rk}(\operatorname{pr}_{M_i}(p_{i+1})) = 1$. Thus by Proposition 2.2.5, M_{i+1} is uniquely defined and $M_i \sim M_{i+1}$. If $M_d \cap M = \{p_d\}$, we set $N := M_d$. If $M_d \sim M$, let N be the unique element of $\mathfrak{M}$ with $N \cap M_d = p_{d-1}p_d$. Since $p_{d-1} \in N$ and $\operatorname{pr}_M(p_{d-1}) = M_d \cap M$, we obtain $N \cap M \leq M_d \cap M$. Since every line of V is contained in only two elements of $\mathfrak{M}$, we conclude $N \cap M = \{p_d\}$.

Since $p_{d-1} \in N$, we obtain $\operatorname{dist}(X,N) = d-1$ and therefore the claim holds for N. By Lemma 3.3.4 we know $\pi_{M,N}(\langle p_d, \operatorname{pr}_M(x)\rangle) = \langle p_d, \operatorname{pr}_N(x)\rangle$. This implies $\operatorname{rk}(\langle p_d, \operatorname{pr}_M(x)\rangle) = \operatorname{rk}(\langle p_d, \operatorname{pr}_N(x)\rangle)$ by Lemma 3.3.3(iii). In the case $\operatorname{dist}(x,M) = \operatorname{dist}(x,N)$ we have either $p_d \in \operatorname{pr}_N(x) \cap \operatorname{pr}_M(x)$ or $p_d \notin \operatorname{pr}_N(x) \cup \operatorname{pr}_M(x)$ and thus, $\operatorname{rk}(\operatorname{pr}_M(x)) = \operatorname{rk}(\operatorname{pr}_N(x)) = \operatorname{dist}(x,M)$. In the case $\operatorname{dist}(x,M) > \operatorname{dist}(x,N)$ we have $p_d \in \operatorname{pr}_M(x) \setminus \operatorname{pr}_N(x)$. This implies $\operatorname{rk}(\operatorname{pr}_M(x)) = \operatorname{rk}(\operatorname{pr}_N(x)) + 1 = \operatorname{dist}(x,M)$. The case $\operatorname{dist}(x,M) < \operatorname{dist}(x,N)$ is not possible, since $\operatorname{dist}(x,M) \geq d$ and $\operatorname{dist}(x,N) \leq \operatorname{dist}(X,M) + 1 = d$.

Set $T := \{p \in N \mid \operatorname{dist}(p,X) = d-1\}$ and $R := \{p \in X \mid \operatorname{dist}(p,N) = d-1\}$. Assume $\operatorname{dist}(x,N) = \operatorname{dist}(x,M) = d$. Then $T \leq \operatorname{pr}_N(x)$. For $N \in \mathfrak{M}_1$, we obtain $\operatorname{rk}(T) = d$. Thus, $T = \operatorname{pr}_N(x)$ and we obtain $\operatorname{pr}_M(x) \leq \pi_{N,M}(\langle p_d, T\rangle)$ by Lemma 3.3.4. For $N \in \mathfrak{M}_0$, we obtain $\operatorname{rk}(T) = d-1$.

Suppose $\operatorname{dist}(x, p_d) = d+1$. Since $\operatorname{rk}(\operatorname{pr}_N(x)) = d$, there is a point $y \in \operatorname{pr}_N(x) \setminus T$ such that $\operatorname{pr}_N(x) = \langle y, T\rangle$. Set $U := \langle x, p_d\rangle_{\mathrm{g}}$. Then $\operatorname{pr}_N(x)$ and $\operatorname{pr}_M(x)$ are both contained in U and hence by Lemma 3.1.1(iii), both M and N intersect U in maximal singular subspace of U. Since $p_d \in N$, Proposition 2.1.17(i) implies that $\operatorname{pr}_N(x)$ is a hyperplane of $U \cap N$ and hence $\operatorname{rk}(U \cap N) = d+1$. Since $p_d \notin \operatorname{pr}_N(x)$, we obtain $yp_d \cap T = \varnothing$. Hence, an arbitrary point $r \in R$ has distance d to every point of yp_d. By Proposition 2.1.23 we know that U is an SPO space. Hence, Lemma 2.1.21(ii) implies that there is a point $s \in U$ with $\operatorname{dist}(r,s) = d+1$ and $yp_d \leq s^{\perp}$.

Clearly, $s \notin N$ since $\mathrm{dist}(r,N) = d-1$. Let N' be a maximal singular subspace of $\mathscr{S}$ that contains $\langle s,y,p_d\rangle$. Then $N' \in \mathfrak{M}$ by Lemma 3.1.1(iii) and N' and N are the only maximal singular subspaces that contain the line yp_d. We know $\mathrm{pr}_X(p_d) \leq U$ since $\mathrm{dist}(p_d,X) = d$ and $\mathrm{dist}(p_d,x) = d+1$. Hence, Lemma 3.1.1(iii) implies that X intersect U in maximal singular subspace of U. Since every point on yp_d has distance d to r and $\mathrm{dist}(s,r) = d+1$, we obtain that r has distance $d+1$ to every point of $\langle s,y,p_d\rangle \setminus yp_d$. Hence by Proposition 2.1.17(i), every point of $\langle s,y,p_d\rangle$ has distance d to X. By Proposition 2.2.5 and Lemma 2.1.21(ii) we conclude that for every point in $X \cap U$, there is a point in $N' \cap U$ at distance $d+1$. Since $y \neq p_d$, we know $y \notin M$ and hence, $\mathrm{pr}_M(y)$ is a line. Since $\mathrm{pr}_M(y) \not\leq N$, we obtain $\langle y, \mathrm{pr}_M(y)\rangle \leq N'$ since $yp_d \leq \langle y, \mathrm{pr}_M(y)\rangle$. Thus M and N' intersect in the line $\mathrm{pr}_M(y)$. Let $y' \in \mathrm{pr}_M(y) \setminus \{p_d\}$. Since both $\mathrm{pr}_X(y')$ and $\mathrm{pr}_X(p_d)$ are hyperplanes of $X \cap U$ and $\mathrm{rk}(X \cap U) \geq 2$, there is a point $x' \in \mathrm{pr}_X(y') \cap \mathrm{pr}_X(p_d)$. By Lemma 3.1.1(i) there is a symplecton Z with $\langle y, \mathrm{pr}_M(y)\rangle \leq Z$. Now x' has distance d to every point of $\mathrm{pr}_M(y)$. Thus, $\mathrm{dist}(x',Z) = d-1$ by Proposition 2.1.17(i). Furthermore, by Propositions 2.1.27 and 2.1.25(i) x' has either a gate in Z or $\mathrm{pr}_Z(x')$ is a generator. In both cases, there is a generator $G \leq Z$ with $\mathrm{dist}(x',G) = d-1$ and $\mathrm{pr}_M(y) \leq G$. Hence, $G \leq M$ or $G \leq N'$. With $\mathrm{dist}(N',X) = \mathrm{dist}(M,X) = d$, this leads to a contradiction. Hence, the case $\mathrm{dist}(x,p_d) = d+1$ is not possible for $\mathrm{dist}(x,N) = \mathrm{dist}(x,M) = d$ and we obtain $\mathrm{dist}(x,p_d) = d$.
Since $p_d \notin T$ and $\mathrm{rk}(T) = \mathrm{rk}(\mathrm{pr}_N(x)) - 1$, we obtain $\mathrm{pr}_N(x) = \langle p_d, T\rangle$ and therefore $\mathrm{pr}_M(x) = \pi_{N,M}(\langle p_d,T\rangle)$ by Lemma 3.3.4. Thus, $\mathrm{pr}_M(x) \leq \pi_{N,M}(\langle p_d,T\rangle)$, whenever $\mathrm{dist}(x,M) = d$. We conclude $S = \pi_{N,M}(\langle p_d,T\rangle)$ and hence, $\mathrm{rk}(S) = \mathrm{rk}(T)+1$.
Since $\mathrm{dist}(X,L) \geq d-1$ for every $L \in \mathfrak{M}$ with $L \sim M$, we obtain $\mathrm{dist}_{\mathfrak{M}}(X,M) \geq d+1$. Thus, $\mathrm{dist}_{\mathfrak{M}}(X,M_d) = d$ and $\mathrm{dist}_{\mathfrak{M}}(M,M_d) \leq 2$ yield $\mathrm{dist}_{\mathfrak{M}}(X,M) \in \{d+1, d+2\}$. If $\mathrm{dist}_{\mathfrak{M}}(X,M) = d+1$, we may assume $M \sim M_d$ and hence, $M_d \neq N$. Thus, $N \cap M_{d-1}$ is a singleton since $p_d \in N$ and by Proposition 2.2.5 M_d is the unique element of $\mathfrak{M}$ that contains $\langle p_d, \mathrm{pr}_{M_{d-1}}(p_d)\rangle$. This implies $\mathrm{rk}(T) = \mathrm{rk}(S')+1$, where $S' := \{p \in M_{d-1} \mid \mathrm{dist}(p,X) = d-2\}$. Since the claim holds for M_{d-1} and for N, we conclude $\mathrm{dist}_{\mathfrak{M}}(X,N) = d+1$. If $\mathrm{dist}_{\mathfrak{M}}(X,M) = d+2$, we obtain $\mathrm{dist}_{\mathfrak{M}}(M,M_d) = 2$ and hence, $N = M_d$. Thus, $\mathrm{dist}_{\mathfrak{M}}(X,N) = d$. We conclude $N \in \mathfrak{M}_i$. With $\mathrm{rk}(T) = d-1+i$ we obtain $\mathrm{rk}(S) = d+i$. By Lemma 3.3.3(i) we obtain $\mathrm{pr}_M(x) \not\leq V$ if and only if $\mathrm{pr}_N(x) \not\leq V$, since $\mathrm{pr}_M(x) \leq \pi_{N,M}(\langle p_d, \mathrm{pr}_N(x)\rangle)$ and $\mathrm{pr}_N(x) \leq \pi_{M,N}(\langle p_d, \mathrm{pr}_M(x)\rangle)$ by Lemma 3.3.4. Since the claim holds for N, this implies that $\mathrm{pr}_M(x) \not\leq V$ if and only if $x \notin V$ and $i = 0$. □

Lemma 3.4.3. *Let V contain a singular subspace of rank 2 that is not maximal in $\mathscr{S}$. Further let $X \in \mathfrak{M}$. For $i \in \{0,1\}$, set $\mathfrak{M}_i := \{M \in \mathfrak{M} \mid \mathrm{dist}_{\mathfrak{M}}(M,X) \in 2\mathbb{N}+i\}$.*

(i) *Every line of V is contained in unique elements of $\mathfrak{M}_0$ and $\mathfrak{M}_1$.*

(ii) *For* $i \in \{0,1\}$ *let* M *and* N *be elements of* $\mathfrak{M}_i$. *Then there is a sequence* $(M_j)_{0\leq j\leq n} \in \mathfrak{M}_i{}^{n+1}$ *with* $M_0 = M$ *and* $M_n = N$ *such that* $M_j \cap M_{j+1} \neq \varnothing$ *for* $j < n$.

(iii) *For* $i \in \{0,1\}$ *let* M *and* N *be elements of* $\mathfrak{M}_i$. *Then* $\mathrm{rk}(M) = \mathrm{rk}(N)$, $\mathrm{rk}(M \cap V) = \mathrm{rk}(N \cap V)$ *and* $\mathrm{crk}_M(M \cap V) = \mathrm{crk}_N(N \cap V)$.

(iv) *Let* M *and* N *be elements of* $\mathfrak{M}_0$. *Further let* $x \in X \setminus V$ *and* $y \in \langle \mathrm{pr}_N(x), M \cap V\rangle \setminus V$. *Then* $\langle \mathrm{pr}_N(y), N \cap V\rangle = \langle \mathrm{pr}_N(x), N \cap V\rangle$.

Proof. (i) By Proposition 3.4.1 we know that every line is contained in exactly two elements of $\mathfrak{M}$.
Set $d := \mathrm{dist}(g,X)$. For $g \leq X$, we may choose $X = N$ and the claim follows. Now assume that g intersects X in a singleton. Let $p \in g \setminus X$. Then $h := \mathrm{pr}_X(p)$ is a line by Lemma 3.4.2. Thus, $h \in L$ for every $L \in \mathfrak{M}$ with $L \sim X$ and $g \leq L$. By Lemma 3.1.1(i) and Proposition 2.2.5 $\langle h,g\rangle$ is contained in a unique element N of $\mathfrak{M}$. Hence, for the other subspace $M \in \mathfrak{M}$ that contains g we obtain $\mathrm{dist}_{\mathfrak{M}}(X,M) = 2$.
Now let $d > 0$ and assume the claim holds for every line l with $\mathrm{dist}(l,X) < d$. Let p and q be two distinct points on g such that $\mathrm{dist}(p,X) = d$. Further let M and N be the two elements of $\mathfrak{M}$ containing g. By the convexity of V there is a point $x \in X \cap V$ with $\mathrm{dist}(x,p) = d$.
Assume $\mathrm{dist}(x,q) = d+1$. We suppose that the claim does not hold for M and N since otherwise we are done. Then there is a point $r \perp p$ with $\mathrm{dist}(x,r) = d-1$ and hence, $\mathrm{dist}(r,q) = 2$. Therefore $\langle r,q\rangle_{\mathrm{g}}$ is a symplecton. Let M' and N' be the elements of $\mathfrak{M}$ that contain pr. Since $\mathrm{dist}(x,r) = d-1$, the claim holds for pr and we may assume $M' \in \mathfrak{M}_0$ and $N' \in \mathfrak{M}_1$. By Proposition 2.2.8 we know that the dual polar graph of $\langle r,q\rangle_{\mathrm{g}}$ is bipartite. Hence by Proposition A.2.20, M intersect either M' or N' in a line, we may assume $M \sim N'$. This implies $N \not\sim N'$ and hence, $N \sim M'$ by analogous reasons. Since both lines $M \cap N'$ and $N \cap M'$ are contained in $\{r,q\}^{\perp}$, we conclude that each point of these lines has distance d to x. Since the claim does not hold for either $M \cap N'$ or $N \cap M'$, we may restrain ourselves to the case $\mathrm{dist}(x,q) = d$.
By Lemma 3.2.1 there is a point r with $g \leq r^{\perp}$ and $\mathrm{dist}(x,r) = d-1$. This implies $r \in V$ and hence the singular subspace $\langle r,g\rangle$ is either contained in M or in N. We may assume $r \in N$. Since $\mathrm{pr}_M(r) = g$, we obtain $\mathrm{dist}(r,s) = 2$ for every point $s \in M \setminus g$. Hence, $\langle r,s\rangle_{\mathrm{g}}$ is a symplecton containing g and therefore $r \in \mathrm{pr}_{\langle r,s\rangle_{\mathrm{g}}}(x)$. By Proposition 2.1.25(iv) this implies $\mathrm{dist}(x,s) \geq d$ and consequently, $\mathrm{dist}(x,M) = d$. Suppose there is a point $y \in X$ with $\mathrm{dist}(y,M) = d-1$. Then $\mathrm{pr}_M(x) \geq \langle g, \mathrm{pr}_M(y)\rangle$. By Lemma 3.4.2 this implies $\mathrm{pr}_M(s) \cap g \neq \varnothing$, a contradiction to $\mathrm{dist}(g,X) = d$. Thus, $\mathrm{dist}(X,M) = d$.
For $\{L,K\} \subseteq \mathfrak{M}$ set $L_K := \{s \in L \mid \mathrm{dist}(s,K) = \mathrm{dist}(L,K)\}$. By Lemma 3.4.2 we know $\mathrm{rk}(\mathrm{pr}_X(p)) = \mathrm{rk}(\mathrm{pr}_X(q)) = d$. Assume $N \in \mathfrak{M}_0$. Then $\mathrm{rk}(N_X) = \mathrm{rk}(X_N) =$

$d-1$ by Lemma 3.4.2. Hence, there is a point $p' \in \mathrm{pr}_X(p) \setminus X_N$. This implies $\mathrm{dist}(p',N) = d$ and therefore, $\mathrm{pr}_N(p') = \langle p, N_X \rangle$ by Lemma 3.4.2. Thus, $\mathrm{dist}(p',q) = d+1$ and we conclude $\mathrm{pr}_X(p) \neq \mathrm{pr}_X(q)$. Therefore $\langle \mathrm{pr}_X(p), \mathrm{pr}_X(q) \rangle$ has at least rank $d+1$. Since X_M is a subspace, we obtain $\langle \mathrm{pr}_X(p), \mathrm{pr}_X(q) \rangle \leq X_M$ and therefore we conclude $M \in \mathfrak{M}_1$ by Lemma 3.4.2. Assume $N \in \mathfrak{M}_1$. Then $\mathrm{rk}(N_X) = \mathrm{rk}(X_N) = d$ and therefore $\mathrm{pr}_X(q') = X_N$ for every point $q' \in g$. Let $y \in X_M$. Then $\mathrm{pr}_M(y)$ is at least a hyperplane of M_X by Lemma 3.4.2. Since $g \leq M_X$, there is a point $q' \in g$ with $\mathrm{dist}(y,q') = d$ and therefore $y \in X_N$. We conclude $X_M = X_N$ and consequently $M \in \mathfrak{M}_0$.
(ii) Set $d := \mathrm{dist}(M,N)$. By Lemma 3.1.1(vi) and since V is convex there is a geodesic $(p_i)_{0 \leq j \leq d}$ in V such that $p_0 \in M$ and $p_d \in N$. By (i) there is a subspace $M_j \in \mathfrak{M}_i$ with $p_{j-1}p_j \leq M_j$ for $1 \leq j \leq d$. The claim follows with $M_{d+1} := N$.
(iii) Let $i = 0$. By (ii) and Lemma 3.3.3(iv) the claim follows by induction over $\mathrm{dist}(M,N)$. By (i) the graph on $\mathfrak{M}$ induced by $\sim$ is bipartite with the partition $\{\mathfrak{M}_0, \mathfrak{M}_1\}$. Hence, choosing $X \in \mathfrak{M}_1$ interchanges $\mathfrak{M}_0$ and $\mathfrak{M}_1$.
(iv) Set $d := \mathrm{dist}(x,M)$. By Lemma 3.4.2 $\mathrm{dist}(X,M) = d-1$ and there is a subspace $S \leq M \cap V$ with $\mathrm{rk}(S) = d-1$ such that $\mathrm{dist}(p,X) = d-1$ for every $p \in S$. Since $\mathrm{rk}(\mathrm{pr}_M(x)) = d$ and $S \leq \mathrm{pr}_M(x) \cap V$, this implies by Lemma 3.4.2 that V intersects $\mathrm{pr}_M(x)$ in a hyperplane. Thus, $M \cap V$ is a hyperplane of $\langle \mathrm{pr}_M(x), M \cap V \rangle$.
We proceed by induction over $n := \mathrm{dist}(M,N)$. Since $\mathrm{pr}_M(y) = \{y\}$, the claim follows for $M = N$. For $L \in \mathfrak{M}_0$, we set $L_V := L \cap V$ and $L^p := \langle \mathrm{pr}_L(p), L_V \rangle$, where $p \in \{x,y\}$. Now let $N \neq M$. For $n = 0$, let $L = M$ and for $n > 0$, let $L \in \mathfrak{M}_0$ with $L \cap N \neq \varnothing$ and $\mathrm{dist}(M,L) = n-1$. Then $L \cap N$ is a singleton.
By Lemma 3.3.4 we obtain $\pi_{L,N}(\langle L \cap N, \mathrm{pr}_L(x) \rangle) = \langle L \cap N, \mathrm{pr}_N(x) \rangle$. Furthermore, Lemma 3.3.3(i) implies $\pi_{L,N}(L_V) = N_V$ and therefore $\pi_{L,N}(L^x) = N^x$. Analogously, $\pi_{L,N}(L^y) = N^y$. Since $L^x = L^y$ by the induction hypothesis, we conclude $N^x = N^y$. □

Theorem 3.4.4. *Let V be a rigid subspace that contains a symplecton properly. Further let* $\mathrm{diam}(V) < \infty$ *and* $\mathrm{yrk}(V) = 3$. *Then every line $g \leq V$ is contained in exactly two maximal singular subspaces M and N of V, where* $\mathrm{rk}(M) = \mathrm{diam}(V)$ *and* $\mathrm{rk}(N) = \mathrm{srk}(V)$. *The subspace V is a metaplecton if and only if* $\mathrm{diam}(V) = \mathrm{srk}(V)$.

Proof. By Proposition 3.4.1 every line g in contained in exactly two elements M and N of $\mathfrak{M}$. By Lemma 3.1.1(iv) we obtain $M \cap N = g$ and hence, $M \cap V \neq N \cap V$ by Lemma 3.1.1(i).
Let p and q be two points of V with $\mathrm{dist}(p,q) = \mathrm{diam}(V) =: d$. Set $U := \langle p,q \rangle_{\mathrm{g}}$ and let $g \leq U$ be a line with $q \in g$. Let M and N be the maximal singular subspaces of V that contain g. Then $\mathrm{dist}(p,M) = \mathrm{dist}(p,N) = d-1$ since $\mathrm{dist}(p,g) = d-1$ by Proposition 2.1.17(i). By Lemma 3.4.2 we obtain $\mathrm{pr}_M(p) = d-1$. Since $\mathrm{pr}_M(p)$ is a hyperplane of $M \cap U$ by Proposition 2.1.17(i), we obtain $\mathrm{rk}(M \cap U) = d$ and

analogously, $\mathrm{rk}(N \cap U) = d$. For $\mathrm{srk}(V) = d$ the claim follows. Thus, we may assume $\mathrm{srk}(V) > d$. By Lemmas 3.4.3(i) and 3.4.3(iii) we may assume $\mathrm{rk}(N) = \mathrm{srk}(V)$.
By Proposition 2.1.23 we know that U is an SPO space. Thus, Proposition 2.3.5 implies that there is singular subspace $L' \leq U$ with $\mathrm{rk}(L') = d$ and $\mathrm{dist}(L', N \cap U) = d-1$. Let $L \leq V$ be the maximal singular subspace of V with $L \geq L'$. Since $\mathrm{srk}(U) = d$, Lemma 3.1.1(vi) implies $\mathrm{dist}(L,N) = d-1$. By Proposition 2.1.17(i) we obtain $\mathrm{dist}(x,N) = d-1$ for every $x \in L'$. Since $\mathrm{rk}(L') = d$, this implies $\mathrm{dist}_{\mathfrak{M}}(L,N) \in 2\cdot\mathbb{N}+1$ by Lemma 3.4.2. Let $r \in N \setminus U$. Then $\mathrm{dist}(r,L) = d$ by Lemma 3.4.2. Since $\mathrm{diam}(V) = d$, we obtain $L = \mathrm{rk}(\mathrm{pr}_L(r))$ and therefore $\mathrm{rk}(L) = d$ by Lemma 3.4.2. The claim follows by Lemmas 3.4.3(i) and 3.4.3(iii). □

Proposition 3.4.5. *Let* $\mathrm{diam}(V) < \infty$ *and let* x *be a point with* $\mathrm{dist}(x,V) = 1$ *such that* $\mathrm{pr}_V(x)$ *contains a line. Then* $V' := \langle x, V\rangle_{\mathrm{g}}$ *is a rigid subspace and* $\mathrm{dist}(p,V) \leq 1$ *for every* $p \in V'$.

(i) *Let* $\mathrm{rk}(\mathrm{pr}_V(x)) = \mathrm{srk}(V)$. *Then* $\mathrm{diam}(V') = \mathrm{diam}(V)$ *and* $\mathrm{srk}(V') = \mathrm{srk}(V)+1$.

(ii) *Let* $\mathrm{rk}(\mathrm{pr}_V(x)) < \mathrm{srk}(V)$. *Then* $\mathrm{diam}(V') = \mathrm{diam}(V)+1$ *and* $\mathrm{srk}(V') = \mathrm{srk}(V)$.

Proof. By Proposition 2.1.27 $\mathrm{pr}_V(x)$ is a maximal singular subspace of V. Hence, there exists a subspace $X \in \mathfrak{M}$ with $\langle x, \mathrm{pr}_V(x)\rangle \leq X$. Set $\mathfrak{M}_0 := \{M \in \mathfrak{M} \mid \mathrm{dist}_{\mathfrak{M}}(X,M) \in 2\cdot\mathbb{N}\}$ and $\mathfrak{M}_1 := \mathfrak{M} \setminus \mathfrak{M}_0$. For every $M \in \mathfrak{M}$, we set $M_V = M \cap V$ and $M^x := \langle \mathrm{pr}_M(x), M_V\rangle$. We claim $V' = U := \bigcup_{M \in \mathfrak{M}} M^x$. For every $M \in \mathfrak{M}_0$ with $X \cap M \neq \varnothing$, we obtain $M^x \leq V'$ since $\mathrm{rk}(M) \geq 2$ by Theorem 3.4.4 and $\mathrm{rk}(\mathrm{pr}_M(x)) = 1$ by Lemma 3.4.2. Let $y \in M^x \setminus M_V$. By Lemma 3.4.3(iv) we know $N^x = N^y$ for every $N \in \mathfrak{M}_0$. Hence by Lemma 3.4.3(ii), we may apply induction to conclude $N^x \leq V'$ for every $N \in \mathfrak{M}_0$. For $N \in \mathfrak{M}_1$, we obtain $N^x \leq V$ by Lemma 3.4.2 and thus, $N^x \leq V'$. Conversely, $x \in X^x$ and for every $p \in V$, there is a subspace $M \in \mathfrak{M}$ with $p \in M_V \leq M^x$ by Lemma 3.4.3(i). Thus, to prove $U = V'$, it remains to show that U is a convex subspace.
Let p and q be two distinct points of U. We have to show $W := \langle p,q\rangle_{\mathrm{g}} \subseteq U$. Set $d := \mathrm{dist}(p,q)$. If p and q are both contained in V, there is nothing to prove. Thus, we may assume $p \notin V$. Let $M \in \mathfrak{M}$ with $p \in M$ and let $N \in \mathfrak{M}$ with $q \in N$. Since $L^x = L_V$ for every $L \in \mathfrak{M}_1$, this implies $M \in \mathfrak{M}_0$. For $M = N$ there is nothing to prove, so we may assume $M \neq N$. Since by Lemma 3.4.3(i) every point of V is contained in an element of $\mathfrak{M}_0$, we may assume $N \subseteq \mathfrak{M}_0$.
By Lemma 3.4.3(iv) we know $M^x = \langle p, M_V\rangle$ since $\mathrm{pr}_M(p) = \{p\}$. Moreover, M_V is a hyperplane of M^x. Analogously, N_V is a hyperplane of N^x. Assume $\mathrm{dist}(p,N) = d-1$. Then $\mathrm{rk}(\mathrm{pr}_N(p)) = d-1$ and $\mathrm{dist}(M,N) = d-2$ by Lemma

3.4.2. Consequently, $d \geq 2$. By Proposition 2.1.17(i) $\mathrm{pr}_N(p)$ intersects $N \cap W$ in a hyperplane. Since $\langle q, \mathrm{pr}_N(p)\rangle \leq W$, this implies $\langle q, \mathrm{pr}_N(p)\rangle = W \cap N$ and $\mathrm{rk}(N \cap W) = d$. Analogously, $\langle p, \mathrm{pr}_M(q)\rangle = W \cap M$ and $\mathrm{rk}(M \cap W) = d$. Since $\mathrm{pr}_N(p) \leq N^x$ by Lemma 3.4.3(iv) and $\mathrm{pr}_N(p) \nleq N_V$ by Lemma 3.4.2, this implies $\mathrm{rk}(N_V \cap W) = d-1$. Hence, since $N_V \cap W \neq \mathrm{pr}_N(p)$ there is a point $q' \in N_V \cap W$ with $\mathrm{dist}(p, q') = d$. Thus, we may assume $q \in N_V$. By Lemma 3.4.2 we obtain $\mathrm{pr}_M(q) = M_V \cap W$.
We know $N_V \cap W = N \cap (V \cap W)$. Since $\mathrm{rk}(N_V \cap W) \geq 1$ and $\mathrm{rk}(N \cap (V \cap W)) = d-1$, we conclude $\mathrm{diam}(W \cap V) \leq d-1$ by Lemma 3.4.3(i) and Theorem 3.4.4. Since $\mathrm{pr}_M(q) = M_V \cap W$ and $\mathrm{dist}(q, M) = d-1$, we conclude by Proposition 2.1.17(i) that $V \cap W$ is no metaplecton. Hence, $\mathrm{srk}(W \cap V) \geq d$ by Theorem 3.4.4 for $d \geq 3$. For $d = 2$, we obtain $\mathrm{srk}(V \cap W) = 2$ since $\mathrm{rk}(\langle q, \mathrm{pr}_M(q)\rangle) = 2$. By Lemmas 3.4.3(i) and 3.4.3(iii) this implies for $i \in \{0, 1\}$ that every element of $\mathfrak{M}_i$ that contains a line of $W \cap V$ intersects $W \cap V$ in a singular subspace of rank $d-1+i$.
Let $r \in W \setminus V$. By Proposition 2.1.17(i) there is a point $r' \in W \cap V$ with $r \perp r'$. Let $p' \in W \cap V$ with $\mathrm{dist}(p', r') = d-1$. By Lemma 3.4.3(i) there is a subspace $L \in \mathfrak{M}$ such that $r' \in L$ and $\mathrm{rk}(L \cap W \cap V) \geq d$. By Lemma 3.4.2 there is a line $g \leq L \cap W \cap V$ with $r' \in g$ and $\mathrm{dist}(p', g) = d-1$. Let $q' \in g \setminus \{r'\}$. Then $r' \notin \langle p', q'\rangle_{\mathrm{g}}$. With Proposition 2.1.17(i) this implies that $\mathrm{pr}_{W \cap V}(r)$ contains a line. Let K be a maximal singular subspace of $\mathscr{S}$ with $\langle r, \mathrm{pr}_{W \cap V}(r)\rangle \leq K$. Then $K \in \mathfrak{M}$ by Lemma 3.1.1(iii). Hence by Lemma 3.1.1(iv), K is uniquely determined.
Let $s \in W$ be a point with $\mathrm{dist}(s, r) = d$. As for r, there is a subspace $K' \in \mathfrak{M}$ with $s \in K'$ and $\mathrm{rk}(K'_V \cap W) \geq 1$. Since $\mathrm{rk}(K \cap W) \geq 2$, Proposition 2.1.17(i) implies $\mathrm{dist}(s, K) = d-1$ and hence $\mathrm{rk}(\mathrm{pr}_K(s)) = d-1$ by Lemma 3.4.2. Since $r \in K \cap W$, Proposition 2.1.17(i) implies that $\mathrm{pr}_K(s)$ is a hyperplane of $K \cap W$ and therefore $\mathrm{rk}(K \cap W) = d$. Since $r \notin V$, $K \cap (W \cap V)$ is properly contained in $K \cap W$ and therefore $K \in \mathfrak{M}_0$. Suppose $p \notin \langle \mathrm{pr}_M(r), (M_V \cap W)\rangle$. Then $\mathrm{pr}_M(r) \leq \langle r, p\rangle_{\mathrm{g}} \leq W$. Since $M_V \cap W$ is a hyperplane of $M \cap W$, this implies $\mathrm{pr}_M(r) \leq M_V$, a contradiction to Lemma 3.4.2. Thus, $p \in \langle \mathrm{pr}_M(r), (M_V \cap W)\rangle$. Hence, Lemma 3.4.3(iv) implies $\langle r, K_V\rangle = \langle \mathrm{pr}_K(p), K_V\rangle = K^x$ and therefore $r \in U$.
It remains to prove that every symplecton of U is rigid. Hence, we may assume $\mathrm{dist}(p, q) = 2$ and that W is a symplecton. Since for $W \leq V$ there is nothing to prove, we may again restrain to the case $p \notin V$ and $q \in V$. Thus as above, $\mathrm{srk}(W \cap V) = 2$. Moreover, since $p \in W \setminus V$, we conclude that $W \cap V$ is a singular subspace of rank 2. Since by Lemma 3.1.1(i) $W \cap V$ is a generator of a symplecton of V, we conclude by Lemma 2.2.3(i) that $W \cap V$ is a generator of W. The claim follows. □

3.5 Connected subspaces of symplectic rank 4

Throughout this section let V be a connected rigid subspaces of symplectic rank 4 that lives in an SPO space $\mathscr{S}$. By $\mathfrak{M}$ we denote the set of maximal singular subspaces of $\mathscr{S}$ that contain a maximal singular subspace of V. Furthermore, we set $\mathfrak{M}_0 := \{S \in \mathfrak{M} \mid \mathrm{rk}(S) = 3\}$ and $\mathfrak{M}_1 := \mathfrak{M} \setminus \mathfrak{M}_0$.

Lemma 3.5.1. *Let x be a point with $\mathrm{dist}(x,V) < \infty$. Then $\mathrm{pr}_V(x)$ is a singleton or there is a subspace $M \in \mathfrak{M}_1$ with $\mathrm{pr}_V(x) = M \cap V$.*

Proof. Suppose there are points y and z in $\mathrm{pr}_V(x)$ with $\mathrm{dist}(y,z) = 2$. Set $Y := \langle y,z\rangle_{\mathrm{g}}$ and $d := \mathrm{dist}(x,V)$. Then $\mathrm{dist}(x,y) = \mathrm{dist}(x,z) = \mathrm{dist}(x,Y) = d$. Since $x \notin Y$, this implies $d \geq 2$. Set $X := \langle x,y\rangle_{\mathrm{g}}$. Then $X \cap Y = \{y\}$ since otherwise $X \cap Y$ would contain a line and hence, $\mathrm{dist}(x,Y) \leq d-1$ by Proposition 2.1.17(i). Thus, $\mathrm{dist}(z,X) = 2$ since otherwise $X \cap Y$ would contain a line by Proposition 2.1.25(iii). Therefore, $y \in \mathrm{pr}_X(z)$ and since $\mathrm{dist}(z,x) = d$, there is a point $x' \in \mathrm{pr}_X(z)$ with $\mathrm{dist}(x',x) = \mathrm{dist}(z,x) - \mathrm{dist}(z,X) = d-2$. Hence, $\mathrm{dist}(y,x') \geq 2$. By Proposition 2.1.17(i) and Proposition 2.1.25(ii) we obtain $\mathrm{diam}(\mathrm{pr}_X(z)) \leq 2$ since otherwise $\mathrm{dist}(x, \mathrm{pr}_X(z)) < d-2$. Hence, $\mathrm{dist}(x',y) = \mathrm{dist}(x',z) = 2$. Moreover, $\mathrm{dist}(x,Y) = 2$ since $\mathrm{dist}(x,x') = d-2$. Thus, we may restrain ourselves to the case $d = 2$.
Let $w \in X$ with $x \perp w \perp y$. Then $\mathrm{dist}(w,z) = 2$ by Proposition 2.1.25(ii). Set $Y' := \langle w,z\rangle_{\mathrm{g}}$. Since $w \notin Y$, we obtain $y \in \mathrm{pr}_Y(w)$. Since $\mathrm{dist}(w,z) = 2$, we conclude by Proposition 2.1.25(i) and Proposition 2.1.27 that $\mathrm{pr}_Y(w)$ is a generator of Y. Since $z^\perp$ intersects $\mathrm{pr}_Y(w)$ in a hyperplane and $\mathrm{pr}_Y(w) \cap z^\perp \leq Y'$ the symplecta Y and Y' intersect in a common generator G by Lemma 2.2.3(i). Thus, $\mathrm{rk}(Y') = 4$. Analogously, we conclude $\mathrm{rk}(\langle x,z\rangle_{\mathrm{g}}) = 4$ and that $\langle x,z\rangle_{\mathrm{g}}$ and Y' intersect in common generator G'. Now $z \in G \cap G'$ implies $\mathrm{rk}(G \cap G') \in \{1,3\}$ by Propositions 2.2.9(iii) and 2.2.9(v). By Proposition 2.1.17(i) we obtain $\mathrm{dist}(x, G \cap G') \leq 1$, a contradiction to $\mathrm{dist}(x,Y) = 2$. Therefore we conclude $\mathrm{diam}(\mathrm{pr}_V(x)) \leq 1$ by Proposition 2.1.25(ii).
It remains to check the case $\mathrm{diam}(\mathrm{pr}_V(x)) = 1$. Let y and z be distinct points of $\mathrm{pr}_V(x)$ and set $X := \langle x,y\rangle_{\mathrm{g}}$. Then $z \notin X$ by Proposition 2.1.17(i) and thus by Proposition 2.1.25(i), $\mathrm{pr}_X(z)$ contains a line since $\mathrm{dist}(z,x) = \mathrm{dist}(y,x)$. Hence, there is a point $w \in \mathrm{pr}_X(z)$ with $\mathrm{dist}(w,V) = 1$ and $yz \leq \mathrm{pr}_V(w)$. We may assume that w is the point on the line yw with $\mathrm{dist}(w,x) = \mathrm{dist}(x,y) - 1$. By Lemma 3.1.1(i) there is symplecton $Y \leq V$ containing yz. Hence, $\mathrm{pr}_Y(w)$ is a generator by Proposition 2.1.27. By Proposition 2.2.5, there is a unique subspace $M \in \mathfrak{M}$ containing $\mathrm{pr}_Y(w)$. Since $\langle w, \mathrm{pr}_Y(w)\rangle$ is singular, we conclude $w \in M$ and hence, $M \in \mathfrak{M}_1$ since $\mathrm{rk}(\mathrm{pr}_Y(w)) = 3$. Moreover, $M \cap V = \mathrm{pr}_V(x)$ since $w \in M$ and $\mathrm{diam}(\mathrm{pr}_V(x)) = 1$. □

Proposition 3.5.2. *For V, exactly one of the following assertions hold:*

(a) *The subspace V is a symplecton, each element of $\mathfrak{M}_0$ is a generator of V and $\mathfrak{M}_1 = \varnothing$.*

(b) *Each subspace $S \leq V$ with $\mathrm{rk}(S) = 2$ is contained in exactly one element of $\mathfrak{M}_0$ and one element of $\mathfrak{M}_1$.*

Proof. We assume (a) does not hold. Let $S \leq V$ with $\mathrm{rk}(S) = 2$. By Lemma 3.1.1(i) there is symplecton $Y \leq V$ with $S \leq Y$. Since $\mathrm{rk}(Y) < \infty$, there are two generators of Y that intersect in S and hence, there are two elements N and N' of $\mathfrak{M}$ that contain S. If Y is properly contained in V, Lemma 3.3.1(i) implies that there is a generator G of Y that is properly contained in a singular subspace of V. If $Y = V$, then there is a generator G of Y that is properly contained in an element of $\mathfrak{M}_1$ since $\mathfrak{M}_1 \neq \varnothing$ by the assumption. By Propositions 2.2.9(iii) and 2.2.9(ii) we may assume $S \leq G$ and hence $N \in \mathfrak{M}_1$. By Proposition 2.2.4(ii), N and N' are the only elements of $\mathfrak{M}$ containing S. Moreover, $N' \in \mathfrak{M}_0$ by Propositions 2.2.9(iii) and 2.2.9(iv). □

Lemma 3.5.3. *Let $V \leq \mathscr{S}$ such that $\mathfrak{M}_1 \neq \varnothing$. Let $M \in \mathfrak{M}_1$.*

(i) *Let $N \in \mathfrak{M}_1$. Then $\mathrm{rk}(M) = \mathrm{rk}(N)$, $\mathrm{rk}(M \cap V) = \mathrm{rk}(N \cap V)$ and $\mathrm{crk}_M(M \cap V) = \mathrm{crk}_N(N \cap V)$.*

(ii) *Let $N \in \mathfrak{M}_1 \setminus \{M\}$ and let $x \in N$. Set $d := \mathrm{dist}(M,N)$, $S := \{p \in M \mid \mathrm{dist}(p,N) = d\}$ and $X := \mathrm{pr}_M(x)$. Then $S \leq V$, $\mathrm{rk}(S) = 2d+1$ and $\mathrm{rk}(X) = 2 \cdot \mathrm{dist}(x,M)$. Furthermore, $X \leq V$ if $x \in V$ and $\mathrm{rk}(X \cap V) = 2 \cdot \mathrm{dist}(x,M) - 1$ if $x \notin V$.*

(iii) *Let X and N be elements of $\mathfrak{M}_1$. Further let $x \in X \setminus V$ and $y \in \langle \mathrm{pr}_M(x), M \cap V\rangle \setminus V$. Then $\langle \mathrm{pr}_N(x), N \cap V\rangle = \langle \mathrm{pr}_N(y), N \cap V\rangle$.*

Proof. (i) By Lemma 3.1.1(v) we may confine ourselves to the case where there exists an subspace $G \in \mathfrak{M}_0$ with $\mathrm{rk}(M \cap G) = \mathrm{rk}(N \cap G) = 2$. Then Proposition 3.5.2 implies and $\mathrm{rk}(M \cap N) = 1$. The claim follows by Lemma 3.3.3(iv).
(ii) First let $M \cap N \neq \varnothing$. Then $S = M \cap N$. By the maximality of $M \cap V$ there are points $u \in M \cap V$ and $v \in N \cap V$ with $u \not\perp v$. Since $S \leq \langle u, v\rangle_{\mathrm{g}}$, Lemma 3.1.1(iii) implies that both M and N contain a generator of $\langle u, v\rangle_{\mathrm{g}}$. With Propositions 2.2.9(iv) and 2.2.9(iii) we obtain $\mathrm{rk}(S) = 1$. Since $S \leq V$, the claim follows with Lemma 3.3.3(iv).
Now let $d > 0$. By Lemma 3.1.1(vi) and the convexity of V there are points $p \in S \cap V$ and $q \in V$ with $p \perp q$ and $\mathrm{dist}(q,N) = d-1$. By Lemma 3.1.1(i) and Proposition 3.5.2, there is an element $M' \in \mathfrak{M}_1$ with $pq \leq M'$. We obtain $M \cap M' \neq \varnothing$ and $\mathrm{dist}(N,M') = d-1$. Set $X' := \mathrm{pr}_{M'}(x)$.
We assume that the claim holds for M' and N. We know that $g := M \cap M'$ is line.

Let $S' \leq M'$ be the set of points at distance $d-1$ to N. Further let $T' \leq M'$ with $g \cap T' = \varnothing$ such that $\langle g, T' \rangle = M'$ and $\langle g \cap X', T' \cap X' \rangle = X'$ and let $T \leq M$ with $g \cap T = \varnothing$ such that $\langle g, T \rangle = M$ and $\langle g \cap X, T \cap X \rangle = X$. Thus, every line in M that intersects X, g and T intersects $X \cap g$ or $X \cap T$. The analogous holds for X', g and T'. From Lemma 3.3.3(ii) we know that the map φ which maps every point of T' to its unique collinear point in T is an isomorphism from T' onto T.
First let $\operatorname{dist}(x, M') = \operatorname{dist}(x, M) = d$. Then $S' \leq X'$. Since $\operatorname{rk}(S') = 2d-1$ and $\operatorname{rk}(X') = 2d$ there is a point $v \in g$ with $\operatorname{dist}(x, v) = d+1$. Suppose $\operatorname{dist}(x, g) = d+1$. Then there is a point $p \in T'$ with $p^\varphi \in X$. Since $g \cap X' = \varnothing$, we obtain $X' \leq T'$. Since $\operatorname{rk}(X') = 2d$, $\operatorname{rk}(\operatorname{pr}_{M'}(p^\varphi)) = 2$ and $g \leq \operatorname{pr}_{M'}(p^\varphi)$, there is a point $q \in X' \cap T'$ with $q \not\perp p$. Since $\operatorname{dist}(x, p) = \operatorname{dist}(x, q) = d$, we conclude $\operatorname{dist}(x, \langle p, q \rangle_{\mathrm{g}}) \leq d-1$ by Lemma 3.5.1 and Proposition 2.1.25(ii), this is a contradiction to Proposition 2.1.17(i) since $\operatorname{dist}(x, g) = d+1$. Thus, there is a point $u \in g \cap X$. Again let $p \in X' \cap T'$. Further let $q \in T' \setminus \{p\}$ and set $Y := \langle p, q^\varphi \rangle_{\mathrm{g}}$. If $\operatorname{dist}(x, Y) = d$, then by Lemma 3.5.1 $\operatorname{pr}_Y(x)$ is a generator that is properly contained in a singular subspace of $\mathscr{S}$ since $up \leq \operatorname{pr}_Y(x)$. Now we conclude by Propositions 2.2.9(iv) and 2.2.9(iii) that $\langle p, p^\varphi, g \rangle$ and $\langle q, q^\varphi, g \rangle$ are both elements of $\mathfrak{M}_0$ since both intersect M in a singular subspace of rank 2. Consequently, they both intersect $\operatorname{pr}_Y(x)$ in a singleton or in a hyperplane. This implies $\operatorname{dist}(x, p) = \operatorname{dist}(x, p^\varphi) = d$ since $X' = \langle u, T' \cap X' \rangle$. Analogously, $\operatorname{dist}(x, q) = \operatorname{dist}(x, q^\varphi)$ for $\operatorname{dist}(x, q) = d$. For $\operatorname{dist}(x, q) = d-1$, we obtain $\operatorname{rk}(\langle q, q^\varphi, g \rangle \cap \operatorname{pr}_Y(x)) = 0$ and hence, again $\operatorname{dist}(x, q) = \operatorname{dist}(x, q^\varphi)$. If $\operatorname{dist}(x, Y) = d-1$, then $\operatorname{dist}(x, vq) = d$ and hence, $\operatorname{dist}(x, q) = d$. Analogously, $\operatorname{dist}(x, p^\varphi) = \operatorname{dist}(x, q^\varphi) = d$. Hence, we obtain in all cases $p^\varphi \in X$ and $q \in X' \Leftrightarrow q^\varphi \in X$. Thus, we conclude $X = \langle u, (S')^\varphi \rangle$ since $X' = \langle u, S' \rangle$. This implies $\operatorname{rk}(X) = 2d$. Moreover, $X \leq V$ by Lemma 3.3.3(i) since $\langle u, S' \rangle \leq V$. Since the claim holds for N and M', we conclude $x \in V$ in this case.
Now let $d' := \operatorname{dist}(x, M) = \operatorname{dist}(x, M') + 1$. Then $X' \leq T'$ and $g \leq X$. Let $p \in X'$ and $q \in T' \setminus \{p\}$. Set $Y := \langle p, q^\varphi \rangle_{\mathrm{g}}$. Since $\operatorname{dist}(x, p) = d'-1$ and $\operatorname{dist}(x, g) = d'$, we obtain $p \in \operatorname{pr}_Y(x)$. If $\operatorname{pr}_Y(x) = \{p\}$, then by Proposition 2.1.25(i) p is a gate for x in Y and hence, $\operatorname{dist}(x, p^\varphi) = \operatorname{dist}(x, q) = d'$ and $\operatorname{dist}(x, q^\varphi) = d'+1$. If $\operatorname{pr}_Y(x)$ is a generator of Y, then $\operatorname{pr}_Y(x)$ is properly contained in a singular subspace by Lemma 3.5.1. Thus by Propositions 2.2.9(iv) and 2.2.9(iii), $\operatorname{pr}_Y(x)$ intersects $\langle q, q^\varphi, g \rangle$ in a singleton or in a hyperplane. Analogously, $M' \cap Y$ intersects $\operatorname{pr}_Y(x)$ in a line since $g \cap \operatorname{pr}_Y(x) = \varnothing$ and $p \in \operatorname{pr}_Y(x)$. This implies $\operatorname{dist}(x, q) = \operatorname{dist}(x, q^\varphi) - 1 = d'-1$ and hence, $\operatorname{dist}(x, p^\varphi) = d'$. Thus, we obtain in both cases $\operatorname{dist}(x, p) = \operatorname{dist}(x, p^\varphi) - 1$ and $\operatorname{dist}(x, q) = \operatorname{dist}(x, q^\varphi) - 1$. We conclude $X = \langle g, (X')^\varphi \rangle$ and therefore $\operatorname{rk}(X) = 2d'$ since $\operatorname{rk}(X') = 2d'-2$. Furthermore, since $q \in V \Leftrightarrow q^\varphi \in V$ by Lemma 3.3.3(i), we conclude $\operatorname{rk}(X \cap V) = \operatorname{rk}(X' \cap V) + 2$. Hence, as regards X, the claim holds by the hypothesis that the claim holds for M' and N.
It remains to determine $\operatorname{rk}(S)$ and to prove $S \leq V$. For $\operatorname{dist}(x, M) = \operatorname{dist}(x, M') = d$,

we obtained $X \leq V$ and $X \leq \langle g, (S')^{\varphi} \rangle$. Note that $\langle g, (S')^{\varphi} \rangle$ is independent of the choice of T' and T. For $\operatorname{dist}(x, M) = d$ and $\operatorname{dist}(x, M') = d - 1$, we obtained $X = \langle g, (X')^{\varphi} \rangle$. Since $X' \leq S'$ in this case and $S' \leq V$, this implies again $X \leq V$ and $X \leq \langle g, (S')^{\varphi} \rangle$. Since $g \leq S \setminus S'$ and $(S')^{\varphi} \leq S$, we obtain $S = \langle g, (S')^{\varphi} \rangle$ and hence, $\operatorname{rk}(S) = \operatorname{rk}(S') + 2$.
(iii) Set $d := \operatorname{dist}(x, M)$. By (ii) $\operatorname{dist}(X, M) = d - 1$ and there is a subspace $S \leq M \cap V$ with $\operatorname{rk}(S) = 2d - 1$ such that $\operatorname{dist}(p, X) = d - 1$ for every $p \in S$. Since $\operatorname{rk}(\operatorname{pr}_M(x)) = 2d$ and $S \leq \operatorname{pr}_M(x) \cap V$, this implies by (ii) that V intersects $\operatorname{pr}_M(x)$ in a hyperplane. Thus, $M \cap V$ is a hyperplane of $\langle \operatorname{pr}_M(x), M \cap V \rangle$.
We proceed by induction over $n := \operatorname{dist}(M, N)$. Since $\operatorname{pr}_M(y) = \{y\}$, the claim follows for $M = N$. For $L \in \mathfrak{M}_1$, we set $L_V := L \cap V$ and $L^p := \langle \operatorname{pr}_L(p), L_V \rangle$, where $p \in \{x, y\}$. Now let $N \neq M$. For $n = 0$, let $L = M$ and for $n > 0$, let $L \in \mathfrak{M}_1$ with $L \cap N \neq \varnothing$ and $\operatorname{dist}(M, L) = n - 1$. Then $L \cap N$ is a line by (ii).
By Lemma 3.3.4 we obtain $\pi_{L,N}(\langle L \cap N, \operatorname{pr}_L(x) \rangle) = \langle L \cap N, \operatorname{pr}_N(x) \rangle$. By Lemma 3.3.3(i) we obtain $\pi_{L,N}(L_V) = N_V$ and therefore $\pi_{L,N}(L^x) = N^x$. Analogously, $\pi_{L,N}(L^y) = N^y$. Since $L^x = L^y$ by the induction hypothesis, we conclude $N^x = N^y$. ☐

Theorem 3.5.4. *Let V be a rigid subspace with $n := \operatorname{diam}(V) < \infty$ and $\operatorname{yrk}(V) = 4$. Further let x and y be points of V with $\operatorname{dist}(x, y) = n$. Then*

(a) $\operatorname{srk}(V) = 2n - 1$ *and* $V = \langle x, y \rangle_{\mathrm{g}}$ *or*

(b) $\operatorname{srk}(V) = 2n$ *and for every point $p \in V$ there is a subspace $M \in \mathfrak{M}_1$ that contains p and intersects $\langle x, y \rangle_{\mathrm{g}}$ in a maximal singular subspace.*

Proof. If V is a symplecton, there is nothing to prove and we are in situation (a). Hence by Proposition 3.5.2, we may assume $\mathfrak{M}_1 \neq \varnothing$. By Lemma 3.5.3(i) we know that every element of $\mathfrak{M}_1$ intersects V in a singular subspace of rank $r := \operatorname{srk}(V)$.
Set $U := \langle x, y \rangle_{\mathrm{g}}$. By Lemma 3.1.1(i) and Proposition 3.5.2 there is an element $M \in \mathfrak{M}_1$ with $y \in M$ such that $M \cap U$ is a maximal singular subspace of U. Hence, $\operatorname{rk}(M \cap U) = \operatorname{srk}(\langle x, y \rangle_{\mathrm{g}})$ as above. By Proposition 2.1.17(i) we obtain $\operatorname{dist}(x, M) = n - 1$ and $\operatorname{pr}_M(x)$ is a hyperplane of $M \cap U$. Thus, $\operatorname{rk}(M \cap U) = 2n - 1$ by Lemma 3.5.3(ii) and the claim holds if V is a metaplecton.
Suppose $2n < r$. Then there is a line $l \leq M \cap V$ disjoint to U. Let $z \in \operatorname{pr}_M(x)$. By Lemma 3.1.1(i) there is a symplecton $Y \leq V$ with $\langle y, z, l \rangle \leq Y$. Since $\operatorname{dist}(x, l) = n$ and $\operatorname{dist}(x, z) = n - 1$, we obtain $z \in \operatorname{pr}_Y(x)$. With $\operatorname{diam}(V) = n$ and Proposition 2.1.25(i) we obtain $\operatorname{pr}_Y(x) > \{z\}$. Thus, Lemma 3.5.1 implies that $\operatorname{pr}_Y(x)$ is a generator of Y that is contained in an element of $\mathfrak{M}_1$. This is a contradiction to Proposition 2.2.9(iv) since $\operatorname{pr}_Y(x) \cap (M \cap Y) = \{z\}$. Thus, $2n - 1 \leq r \leq 2n$.
Assume there is a point $p \in V \setminus U$. Since $\operatorname{diam}(V) = n$, we obtain by Proposition

2.1.25(i) $\mathrm{diam}(\mathrm{pr}_U(p)) \geq 1$. Thus, $\mathrm{pr}_U(p)$ is a singular subspace of rank $2n-1$ by Lemma 3.5.1. By Proposition 2.1.17(i) and Proposition 2.1.25(iii) this implies $\mathrm{dist}(p,U) = 1$ and hence, $\langle p, \mathrm{pr}_U(p)\rangle$ is a singular subspace of rank $2n$. Thus, $\mathrm{srk}(V) = 2n$. The claim follows since by Lemma 3.1.1(i) and Proposition 3.5.2 every point of U is contained in an element of $\mathfrak{M}_1$ that intersects U in a maximal singular subspace. ☐

Proposition 3.5.5. *Let* $\mathrm{diam}(V) < \infty$. *Further let* x *be a point at distance* 1 *to* V *such that* $\mathrm{pr}_V(x)$ *contains a line. Then* $\langle x, V\rangle_{\mathrm{g}}$ *is a rigid subspace with* $\mathrm{srk}(\langle x,V\rangle_{\mathrm{g}}) = \mathrm{srk}(V)+1$ *and* $\mathrm{diam}(\langle x,V\rangle_{\mathrm{g}}) < \infty$.

Proof. Set $n := \mathrm{diam}(V)$. By Lemma 3.5.1 $\mathrm{pr}_V(x)$ is a maximal singular subspace of V. Hence, by Proposition 3.5.2 and Theorem 3.5.4 there is a subspace $X \in \mathfrak{M}_1$ with $X \geq \langle x, \mathrm{pr}_V(x)\rangle$ and $\mathrm{rk}(\mathrm{pr}_V(x)) = \mathrm{srk}(V)$. For elements M and N of $\mathfrak{M}_1$ we set $M_V := M \cap V$ and $M^p = \langle \mathrm{pr}_M(p), M_V\rangle$, where $p \in N \setminus V$.
Set $U := \bigcup_{M\in\mathfrak{M}_1} M^x$. We claim $\langle x,V\rangle_{\mathrm{g}} = U$. Let $M \in \mathfrak{M}_1$. If there is a point $p \in M_V \setminus \mathrm{pr}_M(x)$, then $\mathrm{pr}_M(x) \leq \langle x,p\rangle_{\mathrm{g}} \leq \langle x,V\rangle_{\mathrm{g}}$ and therefore $M^x \leq \langle x,V\rangle_{\mathrm{g}}$. Hence, we may assume $M_V \leq \mathrm{pr}_M(x)$. Since $\mathrm{rk}(M_V) \geq 3$, this implies $\mathrm{dist}(x,M) = \mathrm{dist}(X,M)+1 \geq 2$ by Lemma 3.5.3(ii). Thus, there is a subspace $N \in \mathfrak{M}_1$ with $X \cap N \neq \varnothing$ and $\mathrm{dist}(N,M) < \mathrm{dist}(X,M)$. Let $y \in N^x \setminus V$. Then by Lemma 3.5.3(ii) $\mathrm{pr}_M(y) \cap V < \mathrm{pr}_M(x) \cap V$ and hence, there is a point $p \in M_V \setminus \mathrm{pr}_M(y)$. By Lemma 3.5.3(iii) we obtain $M^x = M^y \leq \langle x,V\rangle_{\mathrm{g}}$. Thus, $U \subseteq \langle x,V\rangle_{\mathrm{g}}$.
By Lemmas 3.1.1(i) and Proposition 3.5.2 there exists for every $p \in V$ a subspace $M \in \mathfrak{M}_1$ with $p \in M$ and therefore $p \in M^x$. Since $x \in X^x$, it remains to show, that U is convex. Let y and z be points of U. If both points are contained in V then $\langle y,z\rangle_{\mathrm{g}} \leq V \subseteq U$. Hence, we assume $z \notin V$. Then by Lemma 3.5.3(iii) $M^z = M^x$ for every $M \in \mathfrak{M}_1$ and $\mathrm{pr}_V(z) = N_V$, where $N \in \mathfrak{M}_1$ with $z \in N^x$. Thus, we may assume $z = x$. Set $d := \mathrm{dist}(x,y)$ and $W := \langle x,y\rangle_{\mathrm{g}}$.
First let $y \in V$. Then $\mathrm{pr}_X(y) \leq V$ by Lemma 3.5.3(ii) and hence, $\mathrm{dist}(y,X) = d-1$. Thus, $\mathrm{diam}(W \cap V) = d-1$ and therefore $\mathrm{dist}(p,V) \leq 1$ for every point $p \in W$ by Proposition 2.1.17(i). Furthermore, $\mathrm{rk}(\mathrm{pr}_X(y)) = 2d-2$ and hence, $\mathrm{rk}(X \cap W) = 2d-1$ since $\langle x, \mathrm{pr}_X(y)\rangle \leq W$ and $\mathrm{pr}_X(y)$ is a hyperplane of $X \cap W$ by Proposition 2.1.17(i). If $d = 1$, then $W = xy \leq X$ and thus, there is nothing to prove. Therefore we may assume $d \geq 2$. For $d = 2$, the subspace W is a symplecton. Since $\mathrm{pr}_X(y) \leq W \cap V$, there is by Lemma 3.1.1(i) a symplecton $Y \leq V$ with $\mathrm{pr}_X(y) \leq Y$. Hence, Lemma 2.2.3(i) implies that W and Y have a generator G in common and therefore $\mathrm{rk}(W) = 4$. Now for every point $p \in W \setminus V$, the subspace $H := G \cap p^{\perp}$ is contained in $\mathrm{pr}_V(p)$ and hence by Lemma 3.5.1 there is a subspace $M \in \mathfrak{M}_1$ with $\langle p,H\rangle \leq M$. If $M = X$, then $\langle p,H\rangle = X \cap W = \langle x, X \cap G\rangle \leq X^x \subseteq U$. If $M \neq X$, then by Lemma 3.5.3(ii) $X \cap M$ is a line since $X \cap H \neq \varnothing$ and $\mathrm{pr}_M(x)$ is a hyperplane of $M \cap W$ with $\mathrm{pr}_M(x) \nleq H$. Thus, $M \cap W = \langle \mathrm{pr}_M(x), H\rangle \leq M^x \subseteq U$.
Now we may assume $d > 2$. Since $\mathrm{pr}_X(y) \leq W \cap V$, we obtain $\mathrm{srk}(W \cap V) = 2d-2$

by Theorem 3.5.4. Let $p \in W \setminus V$. Then there are points q and r in $W \cap V$ with $p \perp q$ and $\operatorname{dist}(q,r) = d-1$. By Proposition 3.5.2 and 3.5.3(i) there is a subspace $M \in \mathfrak{M}_1$ with $r \in M$ and $\operatorname{rk}(M \cap (V \cap W)) = 2d-2$. By Lemma 3.5.3(ii) there is a line $h \leq M_V \cap W$ with $\operatorname{dist}(q,h) = d-1$. Since by Proposition 2.1.17(i) there is no line in W at distance d to p, we may assume that r is a point on h with $\operatorname{dist}(p,r) < d$. Now Proposition 2.1.25(i) implies that $p^\perp \cap \langle q,r\rangle_{\mathrm{g}}$ contains a line and hence by Lemma 3.5.1, there is a subspace $N \in \mathfrak{M}_1$ with $p \in N$ and $\operatorname{rk}(N \cap W \cap V) = 2d-2$. By Lemma 3.5.3(ii) $\operatorname{pr}_N(x) \not\leq V$. By Proposition 2.1.17(i) we know $\operatorname{dist}(x,N) < d$ and hence, $\operatorname{rk}(\operatorname{pr}_N(x)) \leq 2d-2$. Hence, $(N_V \cap W) \setminus \operatorname{pr}_N(x)) \neq \varnothing$ and therefore $\operatorname{pr}_N(x) \leq W$. If $x \notin \operatorname{pr}_X(p)$, then $\operatorname{pr}_X(p) \leq W$. Thus, Lemma 3.5.3(ii) implies that $X_V \cap W$ is a hyperplane of $\langle \operatorname{pr}_X(p), X_V \cap W\rangle$. Since $\operatorname{rk}(X_V \cap W) = 2d-2$ and $\operatorname{rk}(X \cap W) = 2d-1$, this implies $X \cap W \leq X^p$. If $x \notin \operatorname{pr}_X(p)$, then again $X \cap W \leq X^p$ since $X \cap W = \langle x, X_V \cap W\rangle$. Applying Lemma 3.5.3(iii) leads to $N^x = N^p$ and hence, $p \in U$.
Now let $y \notin V$ and let $M \in \mathfrak{M}_1$ with $y \in M^x$. Assume $y \notin \operatorname{pr}_M(x)$. Let $x' \in \operatorname{pr}_M(x) \setminus V$. Then $x'y$ intersects M_V in a point $y' \notin \operatorname{pr}_M(x)$. Hence, $W = \langle x, y'\rangle_{\mathrm{g}}$ and we obtain $W \subseteq U$ as above. Thus, we may assume $y \in \operatorname{pr}_M(x)$. If there is a point $y' \in M^x \notin \operatorname{pr}_M(x)$, then $W \leq \langle x,y'\rangle_{\mathrm{g}} \subseteq U$. Thus, we may assume $\operatorname{pr}_M(x) = M^x$. With Theorem 3.5.4 and Lemma 3.5.3(ii) this implies $\operatorname{dist}(x,y) = n$, $\operatorname{dist}(X,M) = n-1$ and $\operatorname{rk}(M_V) = 2n-1$. Let $u \in X_V$. Then by Lemma 3.5.3(ii) there is a point $v \in M_V$ with $\operatorname{dist}(u,v) = n$. We obtain $V = \langle u,v\rangle_{\mathrm{g}}$ by Theorem 3.5.4. By Proposition 2.1.12(iii) and since W has finite diameter, it suffices to show that every line $g \leq \langle x,y\rangle_{\mathrm{g}}$ with $x \in g$ is contained in U. Let $p \in g$ with $\operatorname{dist}(p,y) = n-1$. Then $\operatorname{dist}(p,u) \leq 2$ and $\operatorname{dist}(p,v) \leq n$. With Lemma 3.5.1 and Proposition 2.1.25(iii) this implies $\operatorname{dist}(p,V) \leq 1$. Suppose $p \in V$. Then $p \in X_V$ and hence, $g \in X^x$. This implies $\operatorname{dist}(y,X) < \operatorname{dist}(x, \operatorname{pr}_N(x))$ and hence, $y \in V$ by Lemma 3.5.3(ii), a contradiction. Thus, $\operatorname{dist}(p,V) = 1$. Let $q \in V$ with $p \perp q$. If $q \notin X$, then $\langle x,q\rangle_{\mathrm{g}}$ is a symplecton that contains p and $\operatorname{pr}_X(q)$. Since $\langle x,q\rangle_{\mathrm{g}} \in U$ as above, we obtain $g \subseteq U$. If $q \in X$, then by Lemma 3.5.3(ii) there is a point $r \in M_V$ with $\operatorname{dist}(q,r) = n$. Since $\operatorname{dist}(p,r) \leq n$ Proposition 2.1.25(i) implies that $p^\perp \cap \langle q,r\rangle_{\mathrm{g}}$ contains a line. Thus, $\operatorname{pr}_V(p)$ is a singular subspace of rank $2n-1$ by Lemma 3.5.1. Moreover, there is subspace $N \in \mathfrak{M}_1$ with $\langle p, \operatorname{pr}_V(p)\rangle \leq N$. Since $\operatorname{dist}(p,y) = n-1$, Lemma 3.5.3(ii) implies $\operatorname{dist}(N,M) \leq n-2$ and hence, $X \neq N$. Since $q \in X \cap N$, we obtain $\operatorname{rk}(\operatorname{pr}_N(x)) = 2$. Thus, there is a point $q' \in N_V \setminus \operatorname{pr}_N(x)$. We obtain $g \leq \langle x,q'\rangle_{\mathrm{g}} \subseteq U$ as above. We conclude $\langle x,V\rangle_{\mathrm{g}} = U$ and therefore $\operatorname{diam}(\langle x,V\rangle_{\mathrm{g}}) < \infty$.
To prove that U is rigid, it suffices to check that W is rigid if W is a symplecton. For $y \in V$, this is already done. Hence, by Proposition 2.1.12(iii) it remains the case $W \cap V = \varnothing$. Let M be the elements of $\mathfrak{M}_1$ with $y \in M$. Since $W \cap V = \varnothing$, we obtain $X \cap M = \varnothing$ by Lemma 3.1.1(iv) and therefore $\operatorname{dist}(X,M) = 1$ by Lemma 3.5.3(ii). Let $w \in W$ with $x \perp w \perp y$ and let $N \in \mathfrak{M}_1$ with $w \in N$. Then both X and M intersect N in a line by Lemma 3.5.3(ii). Let $q \in M \cap N$. Then $Y := \langle x,q\rangle_{\mathrm{g}}$

is a symplecton since $\mathrm{pr}_V(x) \leq M$. We obtain $\mathrm{rk}(Y) = 4$ as above since $q \in V$. With $w \in Y$ we obtain $wq \leq \mathrm{pr}_Y(y)$ and hence by Lemma 3.5.1, $\mathrm{pr}_Y(y)$ is a generator of Y. Since $x^\perp$ contains a hyperplane of $\mathrm{pr}_Y(y)$, this implies $\mathrm{rk}(W) = 4$ by Lemma 2.2.3(i). Thus, U is rigid. Since by Lemma 3.1.1(iv) $M \cap X \leq V$ for every $M \in \mathfrak{M}_1 \setminus \{X\}$, we obtain $X \cap U = X^x$ and hence, $\mathrm{srk}(U) = \mathrm{rk}(X^x) = \mathrm{srk}(V) + 1$ by Lemma 3.5.3(i). □

3.6 Connected subspaces of symplectic rank 5

Compared to connected rigid subspaces of symplectic rank 3 and 4, the maximal singular subspaces are less important for studying connected rigid subspaces of symplectic rank at least 5. This is because there is a rather low upper bound for the singular rank of subspaces of symplectic rank at least 5. Moreover, rigid subspace of symplectic rank at least 5 are very limited in their maximal diameter, as we will see. Therefore, the following quite technical lemmas concern all the intersection of symplecta of rank 5. More precisely, the assertions are mostly of the form that two given symplecta have non-empty intersection or even that they have at least one generator in common.

Lemma 3.6.1. *Let Y and Z be symplecta of rank 5. Further let p and q be two distinct points of Z.*

(i) *Let p and q be contained in Y. Then Y and Z have a generator in common.*

(ii) *Let $\mathrm{dist}(p,Y) = \mathrm{dist}(q,Y) = 1$ and let $\mathrm{pr}_Y(p)$ and $\mathrm{pr}_Y(q)$ be generators of Y. Then Y and Z intersect.*

Proof. (i) We assume $Y \neq Z$ since otherwise there is nothing to prove. Hence $p \perp q$. Let r and s in be points of Z such that $pq \leq r^\perp \cap s^\perp$ and $\mathrm{dist}(s,r) = 2$. We may assume that neither r nor s is contained in Y since otherwise we are done by Lemma 2.2.3(i). Then $\mathrm{pr}_Y(r)$ and $\mathrm{pr}_Y(s)$ are both generators by Proposition 2.1.27 since they both contain pq. Hence, $\mathrm{pr}_Y(r) \cap \mathrm{pr}_Y(s) \leq pq$ implies $\mathrm{rk}(\mathrm{pr}_Y(r) \cap \mathrm{pr}_Y(s)) \geq 2$ by Proposition 2.2.9(iv). Since $\mathrm{pr}_Y(r) \cap \mathrm{pr}_Y(s) \leq Y \cap Z$ the claim follows by Lemma 2.2.3(i).
(ii) Set $P := \mathrm{pr}_Y(p)$ and $Q := \mathrm{pr}_Y(q)$. By Proposition 2.2.9(iv) we obtain $P \cap Q \neq \varnothing$. Thus, the claim follows if $\mathrm{dist}(p,q) = 2$ and so we may assume $p \perp q$. If pq intersects Y there is nothing to prove. Hence, we may assume $pq \cap Y = \varnothing$ and therefore $P \neq Q$ by Proposition 2.2.9(vii). Let $r \in P \setminus Q$. Then $X := \langle r,q \rangle_{\mathrm{g}}$ is a symplecton that contains a hyperplane of Q. Hence, $X \cap Y$ is a generator by Lemma 2.2.3(i). On the other hand X contains pq and and hence $X \cap Z$ is a generator by (i). The generators $X \cap Y$ and $X \cap Z$ intersect by Propositions 2.2.9(v) and 2.2.9(iii). We conclude $Y \cap Z \neq \varnothing$. □

Lemma 3.6.2. *Let y and z be two points of an SPO space such that $X := \langle y,z\rangle_{\mathrm{g}}$ is a symplecton of rank* 5*. Further let x be a point with* $\mathrm{dist}(x,X)=2$ *and* $\mathrm{pr}_X(x)=X$. *Let $u \in \langle x,y\rangle_{\mathrm{g}}$ and $v \in \langle x,z\rangle_{\mathrm{g}}$.*

(i) $\mathrm{rk}(\langle x,y\rangle_{\mathrm{g}}) = \mathrm{rk}(\langle x,z\rangle_{\mathrm{g}}) = 5$.

(ii) *If* $\mathrm{dist}(u,X)=1$, *then* $\mathrm{pr}_X(u)$ *is a generator of* X.

(iii) $\mathrm{dist}(u,v)\le 2$.

(iv) *If* $\mathrm{dist}(u,v)=2$, *then* $\mathrm{rk}(\langle u,v\rangle_{\mathrm{g}})=5$ *and* $\langle u,v\rangle_{\mathrm{g}}\cap X \neq \varnothing$.

Proof. We set $Y := \langle x,z\rangle_{\mathrm{g}}$ and $Z := \langle x,y\rangle_{\mathrm{g}}$. Since $\mathrm{dist}(x,X)=2$ we obtain $X\cap Y = \{z\}$ and $X\cap Z=\{y\}$ by Proposition 2.1.17(i). This implies $\mathrm{dist}(y,Y)=2$ and $\mathrm{dist}(z,Z)=2$ by Proposition 2.1.25(iii). Since $\mathrm{dist}(y,x)=\mathrm{dist}(y,z)$, we obtain $\mathrm{pr}_Y(y)=Y$ by Proposition 2.1.25(ii) and analogously, $\mathrm{pr}_Z(z)=Z$. Thus, it remains to prove (i) to show that we are in a completely symmetric situation concerning x, y and z.

(i) Let z' be a point with $z\perp z'\perp y$. Since $z'\notin Z$ there is a point $x'\in Z$ with $y\perp x'\perp x$ such that $\mathrm{dist}(x',z')=2$. By Lemma 2.2.3(ii) we obtain $\langle z',x'\rangle_{\mathrm{g}}\cap X > z'y$ since $\mathrm{dist}(z,x')=2$ and $z'y\cap z^{\perp}=\{z\}$. Analogously, $\langle z',x'\rangle_{\mathrm{g}}\cap Z > x'y$. Thus by Lemma 2.2.3(i), the symplecton $\langle x',y'\rangle_{\mathrm{g}}$ intersects both X and Z in a common generator. We conclude $\mathrm{rk}(X)=\mathrm{rk}(Z)=5$ and analogously, $\mathrm{rk}(Y)=5$.

(ii) By Proposition 2.1.25(iii) we obtain $y\in\mathrm{pr}_X(u)$ since $\langle u,y\rangle_{\mathrm{g}}\cap X=\{y\}$. With $\mathrm{dist}(z,u)=2$ the claim follows by Proposition 2.1.25(i) and Proposition 2.1.27.

(iii) For $u\in Y$, there is nothing to prove. If $\mathrm{dist}(u,x)=1$, then $\mathrm{pr}_Y(u)$ is a generator of Y by (ii) and hence the claim follows. Now let $\mathrm{dist}(u,x)=2$. Then $x\in\mathrm{pr}_Y(u)$ by Proposition 2.1.25(iii) since $\langle u,x\rangle_{\mathrm{g}}\cap Y=\{x\}$. Since $\mathrm{dist}(z,u)=2$, we obtain $\mathrm{pr}_Y(u)=Y$ and therefore $\mathrm{dist}(u,v)=2$ by Proposition 2.1.25(ii).

(iv) Set $X' := \langle u,v\rangle_{\mathrm{g}}$. We may assume $v\notin Z$ since otherwise $X'=Z$ and there is nothing to prove. For $\mathrm{dist}(v,Z)=1$, we obtain $\mathrm{rk}(\mathrm{pr}_Z(v))=4$ by (ii). Thus, $\mathrm{rk}(X')=5$ by Lemma 2.2.3(i). For $\mathrm{dist}(v,Z)=2$, we obtain $\mathrm{pr}_Z(v)=Z$ by (iii) and hence $\mathrm{rk}(X')=5$ by (i).

Since $\mathrm{rk}(Z)=5$, the subspace $x^{\perp}\cap y^{\perp}$ contains a line. Hence, there is a point $u'\in Z\setminus\{u\}$ collinear to u, x and y. Set $Y' := \langle u',z\rangle_{\mathrm{g}}$. Since $\mathrm{dist}(u',X)=\mathrm{dist}(u',Y)$, we obtain $\mathrm{rk}(\mathrm{pr}_Y(u'))=\mathrm{rk}(\mathrm{pr}_X(u'))=4$ by (ii). Hence, we may apply Lemma 2.2.3(i) to conclude that Y' intersects both X and Y in a generator. Since $\mathrm{dist}(z,Z)=2$, we obtain $Z\cap Y'=\{u'\}$ by Proposition 2.1.17(i). Hence by Proposition 2.1.27, $\mathrm{pr}_{Y'}(u)$ is a generator since $u\perp u'$ and $\mathrm{dist}(u,z)=2$. For $v\in Y'$, let u^* be a point of $\mathrm{pr}_{Y'}(u)\cap v^{\perp}$. For $v\notin Y'$, the subspace $\mathrm{pr}_{Y'}(v)$ is a generator of Y' since $v^{\perp}$ contains a hyperplane of $Y'\cap Y$. Hence by Proposition 2.2.9(iv), there is a point $u^*\in\mathrm{pr}_{Y'}(u)\cap\mathrm{pr}_{Y'}(v)$. We may assume $u^*\notin X$ since otherwise we are done. Since $(u^*)^{\perp}$ contains a hyperplane of $X\cap Y'$, we conclude $\mathrm{dist}(u^*,X)=1$ and that $\mathrm{pr}_X(u^*)$ is a generator of X.

Since $\mathrm{pr}_X(u^*)$ cannot contain both y and z we may for symmetric reasons assume $z \notin \mathrm{pr}_X(u^*)$. Let $G \leq X$ be a generator with $G \cap \mathrm{pr}_X(u^*) = \varnothing$ and $z \in G$. Further let p and q be distinct points of $G \cap y^\perp$. By (ii) both $\mathrm{pr}_Y(p)$ and $\mathrm{pr}_Y(q)$ are generators. Moreover, $\mathrm{rk}(\mathrm{pr}_Y(p) \cap \mathrm{pr}_Y(q)) \geq 2$ by Proposition 2.2.9(vi). Thus, there is a line $g \leq \mathrm{pr}_Y(p) \cap \mathrm{pr}_Y(q) \cap x^\perp$. Let $v' \in g$ such that $v \perp v'$ and $v \neq v'$. We obtain $\mathrm{rk}(\mathrm{pr}_Z(v')) = \mathrm{rk}(\mathrm{pr}_X(v')) = 4$ by (ii). Thus, the symplecton $Z' := \langle v', y \rangle_{\mathrm{g}}$ intersects both X and Z in a generator by Lemma 2.2.3(i). Since $\mathrm{dist}(y, Y) = 2$, we obtain $Y \cap Z' = \{v'\}$ by Proposition 2.1.17(i). Hence by Proposition 2.1.27, $\mathrm{pr}_{Z'}(v)$ is a generator since $v \perp v'$ and $\mathrm{dist}(v, y) = 2$. For $u \in Z'$, let v^* be any point of $\mathrm{pr}_{Z'}(v) \cap u^\perp$. For $u \notin Z'$, the subspace $\mathrm{pr}_{Z'}(u)$ is a generator of Z' since $u^\perp$ contains a hyperplane of $Z' \cap Z$. Hence by Proposition 2.2.9(iv), there is a point $v^* \in \mathrm{pr}_{Z'}(u) \cap \mathrm{pr}_{Z'}(v)$. We may assume $v^* \notin X$ since otherwise we are done. Since $(v^*)^\perp$ contains a hyperplane of $X \cap Z'$, we conclude $\mathrm{dist}(v^*, X) = 1$ and that $\mathrm{pr}_X(v^*)$ is a generator of X. Since $pq \leq Z'$, we obtain $\mathrm{pr}_X(v^*) \cap G \neq \varnothing$. This implies $u^* \neq v^*$ and therefore the claim follows by Lemma 3.6.1(ii). □

The following lemma is in a certain way similar to (VY) if we exchange the terms "projective space" and "line" by "rigid subspace of symplectic rank 5" and "symplecton".

Lemma 3.6.3. *Let Y_0, Y_1 and Y_2 be symplecta of rank 5 that intersect pairwise. Set $S_i := Y_j \cap Y_k$ for $\{i, j, k\} = \{0, 1, 2\}$. Let $S_i \cup S_j$ contain two points for $i \neq j$. Then every symplecton $\langle x_0, x_1 \rangle_{\mathrm{g}}$ with $x_0 \in Y_0$ and $x_1 \in Y_1$ is of rank 5 and intersects Y_2.*

Proof. Set $Y := \langle x_0, x_1 \rangle_{\mathrm{g}}$. We may assume $x_0 \notin Y_1$ and $x_1 \notin Y_0$ since otherwise there is nothing to prove. This implies $Y_0 \neq Y_1$. For $Y_0 = Y_2$, we obtain $S_0 = S_2$ and hence, S_2 contains a line. Thus, Lemma 3.6.1(i) implies that S_2 is a generator of both Y_0 and Y_1. Since $\mathrm{rk}(S_2 \cap {x_0}^\perp \cap {x_1}^\perp) \geq 2$, the claim follows by Lemma 2.2.3(i). By symmetric reasons it remains the case where Y_0, Y_1 and Y_2 are pairwise distinct. Hence by Lemma 3.6.1(i), for $\{i, j, k\} = \{0, 1, 2\}$, the subspace S_i is a singleton or a common generator of Y_j and Y_k.
Let S_2 be a generator. Then $\mathrm{pr}_{Y_0}(x_1)$ contains a hyperplane of S_2 and therefore $\mathrm{pr}_{Y_0}(x_1)$ is a generator by Proposition 2.1.27. Analogously $\mathrm{pr}_{Y_1}(x_0)$ is a generator and hence by Lemma 2.2.3(i), Y intersects both Y_0 and Y_1 in a common generator. This implies $\mathrm{rk}(Y) = 5$. Since $S_0 \cup S_1$ contains more than one point, there are points $s_0 \in S_0$ and $s_1 \in S_1$ with $s_0 \neq s_1$. We may assume $s_0 \notin Y$ and $s_1 \notin Y$ since otherwise we are done. Then $\mathrm{pr}_Y(s_0)$ is a generator since it contains a hyperplane of $Y \cap Y_1$. Analogously, $\mathrm{pr}_Y(s_1)$ is a generator and the claim follows by Lemma 3.6.1(ii).
From now on we may assume that S_2 contains a single point s_2. Let S_1 be a generator. Since $S_2 \neq S_0$, there is a point $s_0 \in S_0 \setminus S_2$. Now $H := {s_0}^\perp \cap S_1$ is a hyperplane

of S_1. Since $H \leq Y_0$ and $S_2 = \{s_2\}$, we obtain $H \not\leq Y_1$ and therefore $s_0 \perp s_2$. Suppose there is a point $p \in H \setminus s_2{}^\perp$. Then $s_0 \in \langle s_2, p\rangle = Y_0$, a contradiction to $s_0 \notin S_2$. Thus, $H \leq s_2{}^\perp$. Now let $q \in Y_0$ be a point with $\mathrm{dist}(q,s_2) = 2$. Then by Proposition 2.1.25(iii) $\mathrm{dist}(q,Y_1) = 2$ since $S_2 = \{s_2\}$. Let p and p' be two distinct points of $H \cap q^\perp$. Then $pp' \cap Y_1 = \varnothing$. Since $s_0s_1 \leq \mathrm{pr}_{Y_1}(p) \cap \mathrm{pr}_{Y_1}(p')$, we conclude by Proposition 2.2.9(vii) that $\mathrm{pr}_{Y_1}(p)$ and $\mathrm{pr}_{Y_1}(p')$ are distinct generators. Since both generators are contained in $\mathrm{pr}_{Y_1}(q)$, we conclude $\mathrm{pr}_{Y_1}(q) = Y_1$. If $\mathrm{dist}(x_1,Y_0) = 1$, we obtain by Proposition 2.1.25(iii) $x_1 \perp s_2$ since $S_2 = \{s_2\}$. Hence, $\mathrm{pr}_{Y_0}(x_1)$ is a generator by Proposition 2.1.25(i) and Proposition 2.1.27. Hence, Lemma 2.2.3(i) implies that Y and Y_0 intersect in a common generator. Thus, $\mathrm{rk}(Y) = 5$ and the claim follows since $Y \cap Y_0$ intersects S_1 by Propositions 2.2.9(v) and 2.2.9(iii). If $\mathrm{dist}(x_1,Y_0) = 2$, then $\mathrm{pr}_{Y_0}(x_1) = Y_0$ since $\mathrm{dist}(x_1,s_2) = \mathrm{dist}(x_1,q) = 2$. Hence, $\mathrm{rk}(Y) = 5$ by Lemma 3.6.2(i) and $Y \cap Y_0 = \{x_0\}$ by Proposition 2.1.17(i). If $\mathrm{dist}(x_0,s_2) = 2$, the claim follows by Lemma 3.6.2(iv) since $s_0 \notin Y_0$ and hence there is a point $s \in S_1$ with $\langle s,s_0\rangle_{\mathrm{g}} = Y_2$. If $\mathrm{dist}(x_0,s_2) = 1$, then $\mathrm{pr}_{Y_1}(x_0)$ is a generator. This implies by Lemma 2.2.3(i) that $Y \cap Y_1$ is generator and consequently, $\mathrm{pr}_Y(s_0)$ is a generator. On the other hand there is a point $s \in S_1$ with $s \perp x_0$. Then $\mathrm{pr}_Y(s)$ is a generator since $\mathrm{dist}(s_1,x_1) = 2$. Since $s_0 \neq s_1$ the claim follows by Lemma 3.6.1(ii).

Since the case where S_0 is a generator is analogous, it remains the case where S_i contains a single point s_i for $0 \leq i \leq 2$. Then $s_i \notin Y_i$ for $0 \leq i \leq 2$ since otherwise $s_0 = s_1 = s_2$. Let $\mathrm{dist}(s_0,s_1) = 2$. Then $Y_2 = \langle s_0,s_1\rangle_{\mathrm{g}}$ and hence, $\mathrm{dist}(s_0,s_2) = 2$ or $\mathrm{dist}(s_1,s_2) = 2$ since $s_2 \notin Y_2$. We may assume $\mathrm{dist}(s_0,s_2) = 2$. Then $Y_1 = \langle s_0,s_2\rangle_{\mathrm{g}}$. Assume $\mathrm{dist}(s_1,s_2) = 1$. Then $\mathrm{pr}_{Y_1}(s_1)$ is a generator since $s_1 \perp s_2$ and $\mathrm{dist}(s_1,s_0) = 2$. Thus, $\langle s_0,s_1\rangle_{\mathrm{g}}$ contains a hyperplane of $\mathrm{pr}_{Y_1}(s_1)$, a contradiction to $S_0 = \{s_0\}$. Hence, $\mathrm{dist}(s_1,s_2) = 2$. Now $S_1 = \{s_1\}$ implies $\mathrm{dist}(s_2,Y_2) = 2$ and since $Y_2 = \langle s_0,s_1\rangle_{\mathrm{g}}$, this leads to $\mathrm{pr}_{Y_2}(s_2) = Y_2$. Hence, Lemma 3.6.2(iv) proves the claim.

By symmetric reasons it remains the case where the points s_0, s_1 and s_2 are pairwise collinear. Then $\mathrm{pr}_{Y_2}(s_2)$ is a generator since $s_0s_1 \leq \mathrm{pr}_{Y_2}(s_2)$. Hence, there are points $p_0 \in \mathrm{pr}_{Y_2}(s_2)$ and $p_1 \in Y_2$ such that $p_i \perp s_j$ for $\{i,j\} \leq \{0,1\}$ and $p_0 \not\perp p_1$. Then $Y_0 \cap p_0{}^\perp$ is a generator since it contains s_1s_2. Since $\mathrm{pr}_{Y_2}(s_2)$ is a generator of Y_2, we conclude by Lemma 2.2.3(i) that $\langle p_1,s_2\rangle_{\mathrm{g}}$ and Y_2 have a generator in common. Thus, Lemma 3.6.1(i) implies that $\langle p_1,s_2\rangle_{\mathrm{g}}$ and Y_0 have a generator in common and therefore $Y_0 \cap p_1{}^\perp$ is a generator. Thus, for $r \in Y_0$, we obtain $\mathrm{dist}(p_i,r) \leq 2$ where $i \in \{0,1\}$. Furthermore, $Y_0 \cap Y_2 = \{s_1\}$ implies $\mathrm{dist}(r,Y_2) = \mathrm{dist}(r,s_1)$ by Proposition 2.1.25(iii). Hence, $\mathrm{pr}_{Y_2}(r) = Y_2$ if $\mathrm{dist}(r,s_1) = 2$. Thus, we ma apply Lemma 3.6.2(ii) to conclude that $\mathrm{pr}_{Y_2}(r)$ is a generator if $\mathrm{dist}(r,s_1) = 1$. The analogous holds for s_i, Y_j and $r \in Y_k$ where $\{i,j,k\} = \{0,1,2\}$.

Let $x_0 \perp s_2$. Then $\mathrm{pr}_{Y_1}(x_0)$ is a generator. Hence by Lemma 2.2.3(i), Y intersects

Y_1 in a common generator and therefore $\mathrm{rk}(Y) = 5$. Now there are two points in $Y \cap Y_1 \cap s_0{}^{\perp} \setminus \{s_0\}$. Since for both points the projection in Y_2 is a generator, the claim follows by Lemma 3.6.1(ii). By symmetric reasons the claim follows if $x_1 \perp s_2$. Hence, we may assume $\mathrm{dist}(x_0,s_2) = \mathrm{dist}(x_1,s_2) = 2$. Then $\mathrm{dist}(x_0,Y_1) = 2$ and $\mathrm{pr}_{Y_1}(x_0) = Y_1$. Thus, $\mathrm{rk}(Y) = 5$ by Lemma 3.6.2(i). This implies $Y \cap Y_i = \{x_i\}$ for $i \in \{0,1\}$ and hence, $\mathrm{dist}(s_2,Y) = 2$ by Proposition 2.1.25(iii) and consequently, $\mathrm{pr}_Y(s_2) = Y$. If $\mathrm{dist}(s_0,Y) = \mathrm{dist}(s_1,Y) = 1$, then Lemma 3.6.2(ii) implies that $\mathrm{pr}_Y(s_0)$ and $\mathrm{pr}_Y(s_1)$ are both generators of Y and hence, the claim follows by Lemma 3.6.1(ii). Thus, we may assume $\mathrm{dist}(s_0,Y) = 2$.
For $i \in \{0,1\}$, let $q_i \in Y_i$ be a point with $s_{1-i} \perp q_i \not\perp s_2$. Then $s_i \not\perp q_i$ since $\mathrm{dist}(s_i,Y_i) = 1$ and $s_2 \in \mathrm{pr}_{Y_i}(s_i)$. Then $Z_i := \langle s_i,q_i\rangle_{\mathrm{g}}$ is a symplecton of rank 5 with $s_0s_1 \leq Z_i$. By Lemma 3.6.2(iv) Z_i intersects Y. Since $s_0 \in Z_i$ and $\mathrm{dist}(s_0,Y) = 2$, the symplecta Z_i and Y intersect in a single point r_i. Since $\mathrm{dist}(q_0,s_2) = 2$, we obtain $\mathrm{dist}(q_0,Y_1) = 2$ and hence, $Z_0 \cap Y_1 = \{s_0\}$. With $q_1s_0 \leq Z_1 \cap Y_1$ this implies $r_0 \neq r_1$ since $Z_i = \langle r_i,s_0\rangle_{\mathrm{g}}$. By Lemma 3.6.1(i) Z_i and Y_2 have a generator in common since they both contain s_0s_1. Hence, $r_i{}^{\perp} \cap Y_2$ contains a generator of Y_2 and the claim follows by Lemma 3.6.1(ii). □

Proposition 3.6.4. *Let Y be a symplecton of an SPO space with $\mathrm{rk}(Y) = 5$. Further let x be a point with $\mathrm{dist}(x,Y) = 1$ such that $\mathrm{pr}_Y(x)$ contains a line. Then $V := \langle x,Y\rangle_{\mathrm{g}}$ is a connected rigid subspace with $\mathrm{srk}(V) = 5$ and $\mathrm{diam}(V) = 2$. Moreover, there is a point $y \in V$ with $\mathrm{dist}(y,Y) = 2$ such that $V = \bigcup_{v\in Y}\langle y,v\rangle_{\mathrm{g}}$.*

Proof. Set $X := \mathrm{pr}_Y(x)$. By Proposition 2.1.27 X is a generator of Y. Let $W \leq Y$ be a generator with $\mathrm{crk}_X(W \cap X) = 4$ and let z be the unique point of $W \cap X$. By Proposition 2.2.9(ii) there is a point w at distance 1 to Y such that $\mathrm{pr}_Y(w) = W$ and $w \in V$. By Proposition 2.2.9(vi) we obtain $w \not\perp x$ and hence, $Z := \langle w,x\rangle_{\mathrm{g}}$ is a symplecton of V. Since both $w^{\perp}$ and $x^{\perp}$ contain a hyperplane of $Z \cap Y$, we conclude $\mathrm{rk}(Z \cap Y) \leq \mathrm{rk}(W \cap X) + 2 = 2$. With Lemma 2.2.3(i) this implies $\mathrm{rk}(Z \cap Y) \leq 1$. Since X is a generator of Y, there is no point in Y at distance 3 to x. Thus, Lemma 2.2.3(ii) implies $Z \cap Y \leq x^{\perp}$. Analogously, $Z \cap Y \leq z^{\perp}$ and therefore $Y \cap Z = \{z\}$. Let $y \in Z$ with $\mathrm{dist}(y,z) = 2$ and $x \perp y$. We may assume that y is the point on xy with $y \perp w$. Since $Z \cap Y = \{z\}$, Proposition 2.1.25(iii) implies $\mathrm{dist}(y,Y) = 2$. Thus, W and X are both contains in $\mathrm{pr}_Y(y)$. By Proposition 2.1.25(ii) this implies $\mathrm{pr}_Y(y) = Y$ since $Y = \langle W,X\rangle_{\mathrm{g}}$. Since $w \in V$ we obtain $y \in V$. On the other hand $x \in \langle y,z\rangle_{\mathrm{g}} \leq \langle y,Y\rangle_{\mathrm{g}}$ and therefore $\langle y,Y\rangle_{\mathrm{g}} = V$.
Set $U := \bigcup_{v\in Y}\langle y,v\rangle_{\mathrm{g}}$. Let u and v be points of Y and let $p \in \langle y,u\rangle_{\mathrm{g}}$ and $q \in \langle y,v\rangle_{\mathrm{g}}$. By Lemma 3.6.2(i) we obtain $\mathrm{rk}(\langle y,u\rangle_{\mathrm{g}}) = \mathrm{rk}(\langle y,v\rangle_{\mathrm{g}}) = 5$. If $\mathrm{dist}(u,v) = 2$, we obtain $\mathrm{dist}(p,q) \leq 2$ by Lemma 3.6.2(iii). For $\mathrm{dist}(u,v) < 2$, Proposition 2.1.25(i) implies that $v^{\perp}$ contains a generator of $\langle y,u\rangle_{\mathrm{g}}$. Thus by Lemma 2.2.3(i), $\langle y,u\rangle_{\mathrm{g}}$ and $\langle y,v\rangle_{\mathrm{g}}$ have a generator G in common. With $\mathrm{rk}(G \cap p^{\perp} \cap q^{\perp}) \geq 2$ we conclude $\mathrm{dist}(p,q) \leq 2$. We show $Z := \langle p,q\rangle_{\mathrm{g}} \subseteq U$ and if $\mathrm{dist}(p,q) = 2$, thenrk$(Z) = 5$.

For $u = v$ there is nothing to prove. If $\text{dist}(p,q) < 2$, there is a point $q' \in \langle y,v\rangle_g$ with $\text{dist}(p,q') = 2$ and $Z \leq \langle p,q'\rangle_g$. Thus, we may straiten to the case $u \neq v$ and $\text{dist}(p,q) = 2$. Since u, v and y are pairwise disjoint, we may apply Lemma 3.6.3 to conclude that Z is a symplecton of rank 5 and $Z \cap Y \neq \varnothing$. If $p = u$ and $q = v$, we obtain $Z = Y$ and there is nothing to prove. Hence, by symmetric reasons we may assume $q \neq v$. If $Z \cap Y = \{v\}$, then $u \notin Z$ since $u \neq v$. Thus, we obtain $p \neq u$ and $\{u\} \neq Z \cap Y$. Therefore, we may assume $q \neq v$ and $Z \cap Y \neq \{v\}$ by symmetric reasons.
Let $s \in Z$. If $\text{dist}(y,p) = 1$, then by Proposition 2.1.25(i) $\text{pr}_Z(y) \neq \{p\}$ since $\text{dist}(y,q) \leq 2$. Thus, $\text{dist}(y,s) \leq 2$ for $\text{dist}(y,p) \leq 1$. Analogously, $\text{dist}(y,s) \leq 2$ for $\text{dist}(y,q) \leq 1$. Now let $\text{dist}(y,p) = \text{dist}(y,q) = 2$. Assume $\text{dist}(y,Z) = 1$. Then $\text{pr}_Z(y) \cap \langle y,p\rangle_g \neq \varnothing$ by Proposition 2.1.25(iii). Since $p \in Z \cap \langle y,p\rangle_g$ and $p \notin \text{pr}_Z(y)$, Lemma 3.6.1(i) implies that Z and $\langle y,p\rangle_g$ have a generator in common. Thus, $\text{pr}_Z(y)$ is a generator by Proposition 2.1.27 and we obtain $\text{dist}(y,s) \leq 2$. For $\text{dist}(y,Z) = 2$, we obtain $\text{dist}(y,s) = 2$ by Proposition 2.1.25(ii). Thus, there is a point $t \in Z$ such that $\text{dist}(y,t) = 2$ and $\langle y,s\rangle_g \leq \langle y,t\rangle_g$. Since $Z \cap Y \neq \{v\}$, there is a point $z' \in Z \cap Y \setminus \{v\}$. Since $\text{dist}(y,Y) = 2$ and $v \neq q$, we obtain $q \notin Y$ and therefore $z' \neq q$. Thus, z', v and q, are pairwise disjoint and we may apply Lemma 3.6.3 to the symplecta $\langle y,v\rangle_g$, Z and Y to conclude that $\langle y,t\rangle_g$ and Y intersect. Hence, we may assume $t \in Y$ since $\text{dist}(y,Y) = 2$. We conclude $Z \subseteq U$. This implies $V = U$ and $\text{yrk}(U) = 5$. Since $\text{dist}(y,p) \leq 2$ for every $p \in U$, we obtain $\text{diam}(V) = 2$. Since $\langle x,X\rangle \leq V$, we obtain $\text{srk}(V) \geq 5$. By Lemma 3.1.1(i) and Proposition 2.2.9(vii) every singular subspace $S \leq \langle u,v\rangle_g$ with $\text{rk}(S) = 5$ is maximal. □

Theorem 3.6.5. *Let V be a connected rigid subspace of an SPO space with symplectic rank* 5 *and let $Y < V$ be a symplecton. Then* $\text{diam}(V) = 2$, $\text{yrk}(V) = \text{srk}(V) = 5$ *and there is a point $x \in V$ with* $\text{dist}(x,Y) = 2$ *such that* $V = \bigcup_{y \in Y} \langle x,y\rangle_g$.

Proof. By Lemma 3.3.1(i) there is a point $z \in V$ such that $\text{pr}_Y(z)$ is a generator of Y. Thus by Proposition 3.6.4, $\langle z,Y\rangle_g$ is a rigid subspace with $\text{diam}(V) = 2$ and $\text{yrk}(V) = \text{srk}(V) = 5$. Moreover, there is a point $x \in \langle z,Y\rangle_g$ with $\text{dist}(x,Y) = 2$ such that $\langle z,Y\rangle_g = \bigcup_{y \in Y} \langle x,y\rangle_g$. Suppose $V > \langle z,Y\rangle_g$. Then by Lemma 3.3.1(i) there is a point $y \in V \setminus \langle z,Y\rangle_g$ such that $\text{pr}_{\langle x,Y\rangle_g}(y)$ contains a line, a contradiction to Lemma 3.3.1(ii). □

3.7 Connected subspaces of symplectic rank ≥ 6

We conclude this chapter by considering the case of connected rigid subspace of symplectic rank ≥ 6. As for the case of symplectic rank 5, there exists only one type of such a subspace beside being a symplecton. This type turns out to be of symplectic rank 6. Therefore, we first consider subspaces of symplectic rank 6.

Proposition 3.7.1. *Let Y be a connected rigid subspace of an SPO space with symplectic rank* 6*. Further let x be a point with* $\mathrm{dist}(x,Y)=1$ *such that* $\mathrm{pr}_Y(x)$ *contains a line. Then Y is a symplecton of rank* 6*. Moreover,* $V := \langle x,Y\rangle_{\mathrm{g}}$ *is a connected rigid metaplecton with* $\mathrm{diam}(V)=3$ *and* $\mathrm{srk}(V)=6$.

Proof. Set $X := \mathrm{pr}_Y(x)$. By Lemma 3.3.1(ii) Y is a symplecton and by Proposition 2.1.27 X is a generator of Y. Let $W \leq Y$ be a generator with $\mathrm{crk}_X(W\cap X)=4$. Then $l := W\cap X$ is a line. By Proposition 2.2.9(ii) there is a point $w\in V$ at distance 1 to Y such that $\mathrm{pr}_Y(w)=W$. By Proposition 2.2.9(vi) we obtain $w \not\perp x$ and hence, $Z := \langle w,x\rangle_{\mathrm{g}}$ is a symplecton. Since $x\notin Y$, we obtain $\mathrm{diam}(Y\cap Z)\leq 1$. Since both $w^{\perp}$ and $x^{\perp}$ contain a hyperplane of $Y\cap Z$, we conclude $\mathrm{rk}(Y\cap Z)\leq \mathrm{rk}(W\cap X)+2=3$. With Lemma 2.2.3(i) this implies $\mathrm{rk}(Y\cap Z)\leq 1$ and thus, $l = Y\cap Z$.
Let p and q be distinct point of l. Further let $u\in Z$ with $u\perp p$ and $\mathrm{dist}(u,q)=2$ and $v\in Y$ with $v\perp q$ and $\mathrm{dist}(v,p)=2$. Then $\mathrm{dist}(u,v)=3$ by Lemma 2.2.3(ii). Since p and q are both contained in $\langle u,v\rangle_{\mathrm{g}}$, we conclude that $x\in Z=\langle u,q\rangle_{\mathrm{g}}$ and $Y=\langle v,p\rangle_{\mathrm{g}}$ are contained in $\langle u,v\rangle_{\mathrm{g}}$ and therefore, $V\leq\langle u,v\rangle_{\mathrm{g}}$. On the other hand $w\in V$ implies $u\in Z\leq V$ and hence, $\langle u,v\rangle_{\mathrm{g}}=V$.
Now let Z be an arbitrary symplecton of V with $Z\neq Y$. Let $y\in Z\setminus Y$. Then $\mathrm{dist}(y,Y)=1$ by Proposition 2.1.17(i). Thus, there is a point $z\in Y\setminus Z$ with $\mathrm{dist}(y,z)=2$. Again by Proposition 2.1.17(i) we obtain $\mathrm{dist}(z,Z)=1$. By Proposition 2.1.25(iii) this implies that both Y and Z have a common line with $\langle y,z\rangle_{\mathrm{g}}$. With Lemma 2.2.6 we conclude $\mathrm{rk}(\langle y,z\rangle_{\mathrm{g}})=\mathrm{rk}(Z)=6$. Thus, $\langle u,v\rangle_{\mathrm{g}}$ is rigid. Since $\langle x,X\rangle\leq V$, we obtain $\mathrm{srk}(V)\geq 6$. By Lemma 3.1.1(i) and Proposition 2.2.9(vii) every singular subspace $S\leq\langle u,v\rangle_{\mathrm{g}}$ with $\mathrm{rk}(S)=6$ is maximal. ☐

Theorem 3.7.2. *Let V be a connected rigid subspace of an SPO space with symplectic rank* ≥ 6*. Then V is either a symplecton or V is a metaplecton with* $\mathrm{diam}(V)=3$ *and* $\mathrm{yrk}(V)=\mathrm{srk}(V)=6$.

Proof. Let $Y\leq V$ be a symplecton. Assume $Y<V$. Then by Lemma 3.3.1(i) there is a point $x\in V$ such that $\mathrm{pr}_Y(x)$ is a generator of Y. By Proposition 2.2.9(viii) this implies $\mathrm{rk}(Y)=6$. Thus by Proposition 3.7.1, $\langle x,Y\rangle_{\mathrm{g}}$ is a rigid metaplecton with $\mathrm{diam}(V)=3$ and $\mathrm{yrk}(V)=\mathrm{srk}(V)=6$. Suppose $V>\langle x,Y\rangle_{\mathrm{g}}$. Then by Lemma 3.3.1(i) there is a point $y\in V\setminus\langle x,Y\rangle_{\mathrm{g}}$ such that $\mathrm{pr}_{\langle x,Y\rangle_{\mathrm{g}}}(y)$ contains a line, a contradiction to Proposition 3.7.1. ☐

4 Maximal rigid subspaces

In this chapter we study the maximal rigid subspaces of an SPO space. As we will see, SPO spaces are composed in a very nice way by maximal rigid subspaces. More precisely, the maximal connected rigid subspaces yield a decomposition of the set of lines. Moreover, each connected component of an SPO space is the grid sum of its maximal rigid subspaces through any given point. A corresponding property can also be found for twin SPO spaces. This fact justifies to restrain our studies to rigid SPO spaces.

4.1 Maximal connected rigid subspaces

Firstly, we consider maximal rigid subspaces of a given connected component. The aim of this chapter is to show that each line is contained in precisely one maximal connected rigid subspace. Furthermore, there exists a canonical equivalence relation on the set of maximal rigid subspace of a given connected component such that every two equivalent spaces are isomorphic and disjoint.

Let C be a chain of connected rigid subspaces of an SPO space. Then the union of the members of C is a again a connected rigid subspace. Hence, every chain of connected rigid subspaces has an upper bound and we may apply Zorn's Lemma to conclude that every connected rigid subspace is contained in a maximal connected rigid subspace. Of course all subspaces occurring in this section live in an SPO space.

Lemma 4.1.1. *Let V be a connected rigid subspace. Further let $x \notin V$ be a point and let $l \leq V$ be a line with $\operatorname{dist}(x,l) = 1$ such that $\langle x,l \rangle_{\mathrm{g}}$ is rigid. Then $\langle x,V \rangle_{\mathrm{g}}$ is connected and rigid.*

Proof. Let $\mathfrak{F}$ be the set of finite sets of points of V. We first show $\operatorname{diam}(\langle M \rangle_{\mathrm{g}}) < \infty$ for every $M \in \mathfrak{F}$ by induction over the cardinality of M. Since V is connected, we

obtain $\operatorname{diam}(\langle M\rangle_{\mathrm{g}}) < \infty$ for $|M| < 3$ by Proposition 2.1.3. Now let $|M| \geq 3$. If M consists of mutually collinear points, $\langle M\rangle_{\mathrm{g}}$ is singular. Thus, we may assume that there are two non-collinear points in M. Since $|M| \geq 3$, there is a point $p \in M$ such that $\operatorname{diam}(\langle M \setminus \{p\}\rangle_{\mathrm{g}}) \geq 2$. Set $U := \langle M \setminus \{p\}\rangle_{\mathrm{g}}$. By the induction hypothesis we may assume $\operatorname{diam}(U) < \infty$. Since $\operatorname{dist}(p,U) < \infty$, it suffices to consider the case $\operatorname{dist}(p,U) = 1$. Since $\operatorname{diam}(U) \geq 2$, we know by Remark 2.2.10 that V has a symplectic rank. If $\operatorname{yrk}(V) = 2$, then U is a metaplecton by Theorem 3.2.3. Moreover, p has a gate q in U by Proposition 3.2.2. Hence, for any point $p' \in U$ with $\operatorname{dist}(p'q) = \operatorname{diam}(U)$, we obtain $\operatorname{dist}(p,p') = \operatorname{diam}(U)+1$ and therefore $\langle M\rangle_{\mathrm{g}} = \langle p,U\rangle_{\mathrm{g}} = \langle p,p'\rangle_{\mathrm{g}}$ is again a metaplecton.
Now let $\operatorname{yrk}(V) \geq 3$. Assume $\operatorname{pr}_U(p)$ contains a single point q. Let $r \in U \setminus \{q\}$ with $q \perp r$. Since $\operatorname{rk}(\langle p,r\rangle_{\mathrm{g}}) \geq 3$, there is a point $s \in \langle p,r\rangle_{\mathrm{g}} \setminus qr$ that is collinear to all points of qr. We may assume that s is the point on the line sr that is collinear to p. This implies $s \notin U$. Hence, $\operatorname{pr}_U(s)$ contains a line. Since p is collinear to two points of $\langle s,U\rangle_{\mathrm{g}}$, we may constrain ourselves to the case where $\operatorname{pr}_U(p)$ contains a line g. Since by 3.1.1(i) there is a symplecton if U that contains g, we conclude $\operatorname{yrk}(U) \leq 6$ by Proposition 2.2.9(viii). Thus, $\operatorname{diam}(\langle M\rangle_{\mathrm{g}}) < \infty$ follows from Propositions 3.4.5, 3.5.5, 3.6.4 and 3.7.1.
Set $W := \bigcup_{M\in\mathfrak{F}}\langle x,l,M\rangle_{\mathrm{g}}$. Since $\langle x,l,M\rangle_{\mathrm{g}} \leq \langle x,V\rangle_{\mathrm{g}}$ for every $M \in \mathfrak{F}$ and $v \in \langle x,l,v\rangle_{\mathrm{g}} \subseteq W$ for every $v \in V$, we obtain $\langle W\rangle_{\mathrm{g}} = \langle x,V\rangle_{\mathrm{g}}$. Let u and v be two points of W. Further let M and N be the finite sets of points of V such that $u \in \langle x,l,M\rangle_{\mathrm{g}}$ and $v \in \langle x,l,N\rangle_{\mathrm{g}}$. Then $\langle u,v\rangle_{\mathrm{g}} \leq \langle x,l,M\cup N\rangle_{\mathrm{g}} \subseteq W$ yields $W = \langle W\rangle_{\mathrm{g}}$ and hence, $W = \langle x,V\rangle_{\mathrm{g}}$. Thus, it remains to show $\operatorname{dist}(u,v) < \infty$ and that $\langle u,v\rangle_{\mathrm{g}}$ is rigid. Since $\langle u,v\rangle_{\mathrm{g}} \leq \langle x,l,M\cup N\rangle_{\mathrm{g}}$ and $(M\cup N) \in \mathfrak{F}$, it suffices to show that $\langle x,l,M\rangle_{\mathrm{g}}$ is rigid and connected for $M \in \mathfrak{F}$.
Set $U := \langle l,M\rangle_{\mathrm{g}}$. Since for two distinct points p and q of l we obtain $\langle p,q,M\rangle_{\mathrm{g}} = U$ and $(M\cup\{p,q\}) \in \mathfrak{F}$, we know $\operatorname{diam}(U) < \infty$. First assume $\langle x,l\rangle_{\mathrm{g}}$ is a symplecton of rank 2. Then Proposition 2.2.4(i) implies that l is a maximal singular subspace since l is a generator of the rigid symplecton $\langle x,l\rangle_{\mathrm{g}}$ and hence, there are three lines of $\langle x,l\rangle_{\mathrm{g}}$ meeting in a point of l. Thus, we may assume $\operatorname{diam}(U) \geq 2$ since otherwise $U = l$ and there is nothing left to prove. By Lemma 3.1.1(i) we conclude $\operatorname{yrk}(U) = 2$. Hence, U is a metaplecton by Theorem 3.2.3. This implies $\operatorname{diam}(\langle x,U\rangle_{\mathrm{g}}) < \infty$ and $\langle x,U\rangle_{\mathrm{g}}$ is rigid and connected by Proposition 3.2.5.
Now assume $\langle x,l\rangle_{\mathrm{g}}$ is a singular subspace or a symplecton of rank ≥ 3. In the latter case there is a point $y \in \langle x,l\rangle_{\mathrm{g}} \setminus l$ with $y \perp x$ such that $\langle y,l\rangle_{\mathrm{g}}$ is singular. Then $U \cap y^{\perp}$ contains a line and $\langle y,U\rangle_{\mathrm{g}} \cap x^{\perp}$ contains a line. Since $\langle x,U\rangle_{\mathrm{g}} = \langle x,y,U\rangle_{\mathrm{g}}$, we may restrain ourselves to the case $l \leq x^{\perp}$.
For $\operatorname{diam}(U) \geq 2$, there is a symplecton $Y \leq U$ such that $l \leq \operatorname{pr}_Y(x)$ by Lemma 3.1.1(i). Thus, $3 \leq \operatorname{yrk}(V) \leq 6$ by Propositions 2.2.9(i) and 2.2.9(viii). Now the claim follows from Propositions 3.4.5, 3.5.5, 3.6.4 and 3.7.1. Hence, it remains the case that U is singular. If $U \leq \operatorname{pr}_V(x)$, then $\langle x,U\rangle_{\mathrm{g}}$ is singular and we are

done. Thus, we may assume that there is a point $y \in M$ with $\mathrm{dist}(x,y) = 2$. Since $\langle x,l\rangle \leq \langle x,y\rangle_{\mathrm{g}}$, the symplecton $Y := \langle x,y\rangle_{\mathrm{g}}$ is rigid. Now $\mathrm{pr}_Y(p)$ contains the line l for every point $p \in M$. Since $\mathrm{diam}(Y) = 2$ and M is finite, the claim follows by induction. □

Proposition 4.1.2. *Every line is contained in a unique maximal connected rigid subspace.*

Proof. Let l be a line. Since l is a rigid subspace, there is a maximal connected rigid subspace V that contains l. Now let U be an arbitrary connected rigid subspace with $l \leq U$. Suppose there is a point $x \in U \setminus V$. Since U is connected and $U \cap V \neq \varnothing$, we may assume $\mathrm{dist}(x,V) = 1$. Since $\mathrm{diam}(U \cap V) \geq 1$, there is a line $g \in U \cap V$ with $\mathrm{dist}(x,g) = 1$. Now $\langle x,g\rangle_{\mathrm{g}}$ is rigid since $\langle x,g\rangle_{\mathrm{g}} \leq U$. Thus, Lemma 4.1.1 implies that $\langle x,V\rangle_{\mathrm{g}}$ is rigid, a contradiction the maximality of V. Thus, $U \leq V$ and V is uniquely defined. □

Corollary 4.1.3. *Let U and V be two connected rigid subspaces with a common line. Then $\langle U,V\rangle_{\mathrm{g}}$ is a connected rigid subspace.*

Proof. Let $l \leq U \cap V$ be a line. By Proposition 4.1.2 there is a unique maximal connected rigid subspace W that contains l. This implies $U \leq W$ and $V \leq W$. Thus, the intersection of all subspaces of W that contain U and V is defined and equals $\langle U,V\rangle_{\mathrm{g}}$. □

Proposition 4.1.4. *Every maximal connected rigid subspace of an SPO space is gated.*

Proof. Let V be a maximal connected rigid subspace and let x be a point with $\mathrm{dist}(x,V) < \infty$. Suppose there is a line $l \leq \mathrm{pr}_V(x)$. Then by Lemma 3.2.1 there is a point y with $\mathrm{dist}(y,V) = 1$ and $l \leq \mathrm{pr}_V(y)$. By Lemma 4.1.1 this implies that $\langle y,V\rangle_{\mathrm{g}}$ is rigid and connected, a contradiction to the maximality of V. Thus, $\mathrm{pr}_V(x)$ contains a single point z. For any point $v \in V$, Proposition 2.1.25(i) implies that z is a gate for x in $\langle v,z\rangle_{\mathrm{g}}$. The claim follows. □

Our next goal is to show that the maximal rigid subspaces of a given connected component can be partitioned into equivalence classes such that any two subspaces of a given equivalence class are isomorphic and one-parallel to each other.

Lemma 4.1.5. *Let g and h be one-parallel lines of an SPO space. Further let U and V be maximal connected rigid subspaces with $h \leq U$ and $g \leq V$. Then U and V are one-parallel to each other.*

Proof. Since g and h are one-parallel, we obtain $d := \mathrm{dist}(g,h) < \infty$. Let w and x be two distinct points of h and let y and z be the points on g such that $\mathrm{dist}(w,y) = \mathrm{dist}(x,z) = d$. Set $n := \mathrm{dist}(w,V)$. By Proposition 4.1.4 w has a gate w' in V. This implies $\mathrm{dist}(w',y) = d-n$ and $\mathrm{dist}(w',z) = d-n+1$. Since $\langle w,z\rangle_{\mathrm{g}}$ contains x and w', we obtain $\mathrm{dist}(x,\langle w',z\rangle_{\mathrm{g}}) \leq n$ by Proposition 2.1.17(i). Since $y \in \langle w',z\rangle_{\mathrm{g}}$ and $\mathrm{dist}(x,y) = d+1$, Lemma 2.1.14 implies that x has a gate x' in $\langle w',z\rangle_{\mathrm{g}}$ with $\mathrm{dist}(x,x') = n$. Thus, $\mathrm{dist}(x',z) = d-n$ and hence, $w' \neq x'$. Since $\mathrm{dist}(x,w') \leq \mathrm{dist}(w,w')+1$, we obtain $w' \perp x'$. Therefore, $\mathrm{dist}(w,x') = \mathrm{dist}(x,w') = n+1$ and Proposition 2.1.29 implies that h and $w'x'$ are one-parallel to each other. Since $w'x' \leq V$, we may assume $\mathrm{pr}_V(w) = \{y\}$. Now if $pr_U(y) \neq \{w\}$, we repeat this argument to obtain a line $h' \leq U$ that is one-parallel to g with $\mathrm{dist}(h',g) < n$. Thus and since $\mathrm{dist}(U,V) < \infty$, we may assume that w is the gate of y in U and consequently, $\mathrm{dist}(w,V) = \mathrm{dist}(y,U) = d$.
Suppose $\mathrm{dist}(x,V) < d$. Then we may apply the same argument as above to obtain a line $g' \leq V$ that is one-parallel to h with $\mathrm{dist}(h,g') < d$, a contradiction to $\mathrm{dist}(w,V) = d$. Since $\mathrm{dist}(x,z) = d$, we obtain $\mathrm{dist}(x,V) = d$ and hence by Proposition 4.1.4, z is the gate for x in V. Analogously, x is the gate for z in U. Moreover, for every point $p \in U \setminus \{w\}$ with $p \perp w$, we obtain $\mathrm{dist}(p,V)$ if there is line in V through y that is one-parallel to pw.
Now let $p \in U \setminus h$ with $p \perp w$. Then $\mathrm{dist}(p,y) = d+1$, since w is a gate for y in wp. First assume $\mathrm{dist}(x,p) = 2$. Then $\mathrm{dist}(p,z) = d+2$ since x is a gate for z in U. Thus, $\langle p,z\rangle_{\mathrm{g}}$ contains x and w and consequently, $y \in \langle p,z\rangle_{\mathrm{g}}$. By Proposition 2.1.23 $\langle p,z\rangle_{\mathrm{g}}$ is an SPO space. Since $\mathrm{dist}(x,y) = d+1$, there is a point $q \in \langle p,z\rangle_{\mathrm{g}}$ with $\mathrm{dist}(x,q) = d+2$ and $y \perp q$. This implies $\mathrm{dist}(q,z) = 2$. Now x is a gate for z in $\langle p,x\rangle_{\mathrm{g}}$ since $\langle p,x\rangle_{\mathrm{g}} \leq U$. By (A12) we know that q has a gate in $\langle p,x\rangle_{\mathrm{g}}$ at distance d since q and x are opposite in $\langle p,z\rangle_{\mathrm{g}}$. Thus, Proposition 2.1.29 implies that $\langle x,p\rangle_{\mathrm{g}}$ and $\langle z,q\rangle_{\mathrm{g}}$ are one-parallel to each other and isomorphic. The gate of p in $\langle z,q\rangle_{\mathrm{g}}$ has distance 2 to z and hence distance $d+2$ to x. Moreover, the gate of p in $\langle z,q\rangle_{\mathrm{g}}$ is collinear to y since $\mathrm{dist}(p,y) = d+1$. Therefore we may assume that q is the gate for p in $\langle z,q\rangle_{\mathrm{g}}$. Now $\langle x,p\rangle_{\mathrm{g}}$ is rigid since it is contained in V. This implies that $\langle z,q\rangle_{\mathrm{g}}$ is rigid. Since $g \leq \langle z,q\rangle_{\mathrm{g}}$ and V is maximal, Proposition 4.1.2 implies $\langle z,q\rangle_{\mathrm{g}} \leq V$. Since wp and yq are one-parallel to each other, we obtain $\mathrm{dist}(p,V) = d$ and hence, $\mathrm{pr}_V(p) = \{q\}$.
Now assume $p \perp x$. Then $\mathrm{dist}(p,y) = \mathrm{dist}(p,z) = d+1$. Thus, Proposition 4.1.4 implies $\mathrm{dist}(p,V) \leq d$. Hence, $y \notin \mathrm{pr}_V(p)$. Now $d < \mathrm{dist}(w,\mathrm{pr}_V(p)) \leq \mathrm{dist}(p,\mathrm{pr}_V(p))+1$ yields $\mathrm{dist}(p,V) = d$. By Proposition 4.1.4 p has a gate q in V. Since $\mathrm{dist}(p,y) = d+1$, we obtain $y \neq q$ and $y \perp q$. Thus again, the lines wp and yq are one-parallel to each other. Since U is connected this implies that U is one-parallel to V. Analogously, V is one-parallel to U. ☐

Proposition 4.1.6. *Let V be a maximal connected rigid subspace of an SPO space.*

Further let x be a point with $\operatorname{dist}(x,V) < \infty$*. Then there is exactly one maximal rigid subspace U with $x \in U$ such that V and U are one-parallel to each other.*

Proof. Set $d := \operatorname{dist}(x,V)$. We may assume $d > 0$ since otherwise there is nothing to prove. By Proposition 4.1.4 x has a gate z in V. Since $d > 0$, there is a line through z and hence, $V > \{z\}$ by Proposition 4.1.2. Let $y \in V \setminus \{z\}$ be a point with $y \perp z$. Then $\operatorname{dist}(x,y) = d+1$. Thus, $z \in \langle x,y\rangle_{\mathrm{g}}$. Since $\langle x,y\rangle_{\mathrm{g}}$ is an SPO space by Proposition 2.1.23, there is a point $w \in \langle x,y\rangle_{\mathrm{g}}$ with $w \perp x$, $\operatorname{dist}(w,y) = d$ and $\operatorname{dist}(w,z) = d+1$. Let U be a maximal connected rigid subspace with $wx \leq U$. Since by Proposition 2.1.29 the lines wx and yz are one-parallel to each other, Lemma 4.1.5 implies that U and V are one-parallel to each other.
Now let W be a maximal connected rigid subspace with $x \in W$ such that V and W are one-parallel to each other. Then $\operatorname{dist}(y,W) = d$ since $\operatorname{dist}(x,V) = d$. By Proposition 4.1.4 the point $u \in W$ with $\operatorname{dist}(y,u) = d$ is a gate for y in W and therefore $\operatorname{dist}(u,x) = 1$. Suppose $u \neq w$. Then $u \notin U$ since $\operatorname{pr}_U(y) = \{w\}$. By Lemma 4.1.1 and the maximality of U this implies that $\langle u,xw\rangle_{\mathrm{g}}$ is not rigid and therefore $u \not\perp w$. Thus, $\langle u,w\rangle_{\mathrm{g}}$ is a symplecton and the only lines in $\langle u,w\rangle_{\mathrm{g}}$ through x are ux and wx. Since $\operatorname{pr}_U(z) = \operatorname{pr}_W(z) = \{x\}$, this implies that all points in $\langle u,w\rangle_{\mathrm{g}} \cap x^{\perp} \setminus \{x\}$ have distance $d+1$ to z. Thus, x is a gate for z in $\langle u,w\rangle_{\mathrm{g}}$ by Propositions 2.1.25(ii) and 2.1.25(i). Since $\langle u,w\rangle_{\mathrm{g}} \leq \langle y,x\rangle_{\mathrm{g}}$, this is a contradiction to Proposition 2.1.17(i). We conclude $u = w$ and hence, $U = W$ by Proposition 4.1.2. ☐

Proposition 4.1.7. *Let U and V be two maximal connected rigid subspaces that are one-parallel to each other. Then the map $\varphi\colon U \to V$ with $\operatorname{pr}_V(u) = \{u^{\varphi}\}$ for every point $u \in U$ yields an isomorphism from U onto V.*

Proof. Since U and V are one-parallel to each other, φ is a bijection. Set $d := \operatorname{dist}(U,V)$. By Proposition 4.1.4 φ maps every point of U onto its gate in V. Now let w and x be distinct collinear points of U. Further let y be the gate of w and let z be the gate of x in V. Since conversely, w is the gate for y in U, we obtain $\operatorname{dist}(y,x) = d+1$ and hence, $y \perp z$. Since $\operatorname{dist}(w,z) = d+1$ by analogous reasons, Proposition 2.1.29 implies that wx and yz are one-parallel lines. Thus, for every point u on wx the gate of u in V is contained in yz. By symmetric reasons, the preimage of every point of yz is contained in wx. Hence, φ is an isomorphism. ☐

4.2 Rigid subspaces at finite codistance

Now that we know something about maximal rigid subspaces of a given connected component of an SPO space, we proceed with rigid subspaces of distinct connected components that are adjacent in the connectivity graph. Therefore we first we study the coprojection of a point in convex subspace of finite diameter. Throughout this section $\mathscr{S}$ is always an SPO space.

Lemma 4.2.1. *Let $x \in \mathscr{S}$ be a point and let l be a line with $\operatorname{cod}(x,l) < \infty$ and $\operatorname{copr}_l(x) = l$. Then there is a point $y \leftrightarrow x$ with $\operatorname{dist}(y,l) = \operatorname{cod}(x,l)$ and $\operatorname{pr}_l(y) = l$.*

Proof. Set $d := \operatorname{cod}(x,l)$. Let z be a point with $z \leftrightarrow x$ and $\operatorname{dist}(z,l) = d$. We may assume that there is a point $q \in l$ with $\operatorname{dist}(z,q) = d+1$ since otherwise we are done. By (A12) we conclude that x has a cogate x' in $\langle q,z\rangle_{\mathrm{g}}$ with $\operatorname{cod}(x,x') = d+1$. Hence, $\langle x',l\rangle$ is a singular space of rank 2. By Proposition 2.1.23 we know that $\langle q,z\rangle_{\mathrm{g}}$ is an SPO space. Thus by Lemma 2.1.21(iii), there is a point $y \in \langle q,z\rangle_{\mathrm{g}}$ with $\operatorname{dist}(y,p) = d$ for every point $p \in l$ and $\operatorname{dist}(y,x') = d+1$. We obtain $x \leftrightarrow y$ since x' is a cogate for x. ☐

Lemma 4.2.2. *Let $V \leq \mathscr{S}$ be a metaplecton and let x be a point at finite codistance to V such that $\operatorname{copr}_V(x)$ contains a line. Then there is a point z with $\operatorname{dist}(z,V) = 1$ and $\operatorname{cod}(x,z) < \operatorname{cod}(x,v)$ for every $v \in V$ such that $\operatorname{pr}_V(z)$ contains a line.*

Proof. Set $d := \operatorname{cod}(x,V)$ and $n := \operatorname{diam}(V)$. By Proposition 2.1.17(ii) we obtain $d \geq n$. Let $g \leq \operatorname{copr}_V(x)$ be a line. By Proposition 2.1.23 V is an SPO space and hence, there is a line $h \leq V$ such that h and g are one-parallel to each other with $\operatorname{dist}(g,h) = n-1$. Hence by Lemma 2.1.24, we obtain $\operatorname{cod}(x,p) = m$ for every point $p \in h$, where $m := \min\{\operatorname{cod}(x,v) \mid v \in V\}$. By Lemma 4.2.1, there is a point $y \leftrightarrow x$ with $\operatorname{dist}(y,h) = m$ and $\operatorname{pr}_h(y) = h$. Hence by Lemma 3.2.1, there is a point z with $\operatorname{dist}(z,y) = m-1$ and $h \leq z^{\perp}$. We conclude $\operatorname{cod}(x,z) = m-1$ and consequently, $z \notin V$. ☐

Proposition 4.2.3. *Let $V \leq \mathscr{S}$ be a connected rigid subspace with $\operatorname{yrk}(V) = 2$ and $\operatorname{diam}(V) < \infty$. Then V is cogated.*

Proof. By Theorem 3.2.3 V is a metaplecton. Let x be a point with $\operatorname{cod}(x,V) < \infty$. Suppose $\operatorname{copr}_V(x)$ contains a line g. Then by Lemma 4.2.2 there is a point z with $\operatorname{dist}(z,V) = 1$ such that $\operatorname{pr}_V(z)$ contains a line. This is a contradiction to Proposition 3.2.2. Hence, $\operatorname{copr}_V(x)$ is a singleton and the claim follows from Proposition 2.1.12(ii). ☐

Proposition 4.2.4. *Let $V \leq \mathscr{S}$ be a connected rigid subspace with $\operatorname{yrk}(V) \geq 5$. Further let x be a point with $\operatorname{cod}(x,V) < \infty$. Then one of the following holds.*

(a) *V is a symplecton and x has a cogate in V.*

(b) *V is a symplecton of rank 5 or 6 and $\operatorname{copr}_V(x)$ is a generator of V.*

(c) *V is a symplecton of rank 5 and $\operatorname{copr}_V(x) = V$.*

(d) *V is a maximal connected rigid subspaces with* $\operatorname{yrk}(V) = \operatorname{srk}(V) = 5$ *and* $\operatorname{copr}_V(x)$ *is a symplecton.*

(e) *V is a metaplecton with* $\operatorname{diam}(V) = 3$ *and* $\operatorname{yrk}(V) = 6$ *and x has a cogate in V.*

Proof. First assume $\operatorname{yrk}(V) \geq 6$. Then by Theorem 3.7.2 V is either a symplecton or metaplecton with $\operatorname{diam}(V) = 3$ and $\operatorname{yrk}(V) = 6$. Hence, if $\operatorname{copr}_V(x)$ is a singleton, Proposition 2.1.12(ii) implies that we are either in case (a) or (e). Therefore we may assume $\operatorname{copr}_V(x)$ contains a line. Thus by Lemma 4.2.2, there is a point z with $\operatorname{dist}(z,V) = 1$ such that $\operatorname{pr}_V(z)$ contains a line. Hence by Lemma 3.3.1(ii), V is a symplecton. Moreover, Proposition 2.2.9(viii) implies $\operatorname{rk}(V) = 6$ and by Proposition 3.7.1 $W := \langle z,V\rangle_{\mathrm{g}}$ is a rigid metaplecton with $\operatorname{diam}(V) = 3$ and $\operatorname{yrk}(V) = 6$. Now $\operatorname{copr}_W(x)$ does not contain a line, since we already know that this would imply that W is a symplecton of rank 6. Thus, Proposition 2.1.12(ii) implies that x has a cogate x' in W. Since $\operatorname{copr}_V(x)$ contains a line, we obtain $x' \notin V$ and hence, $\operatorname{dist}(x',V) = 1$ by Proposition 2.1.17(i). Thus, $\operatorname{pr}_V(x')$ is singular and hence by Proposition 2.1.27, $\operatorname{pr}_V(x)$ is either a singleton or a generator of V. Since x' is a cogate for x in W and $V \leq W$, we conclude $\operatorname{copr}_W(x) = \operatorname{pr}_V(x')$ and therefore we are in case (b).

Now let $\operatorname{yrk}(V) = 5$. First assume that V contains a symplecton properly. Then $\operatorname{yrk}(V) = \operatorname{srk}(V) = 5$ and $\operatorname{diam}(V) = 2$ by Theorem 3.6.5. Set $d := \operatorname{cod}(x,V)$. Let $v \in V$ with $\operatorname{cod}(x,v) = \min\{\operatorname{cod}(x,p) \mid p \in V\}$ and let $u \in \operatorname{copr}_V(x)$. If $v \perp v$, then by 3.1.1(i) there is a symplecton in V that contains u and v. Hence by Lemma 2.1.24, we may assume $\operatorname{dist}(u,v) = 2$. Set $Y := \langle u,v\rangle_{\mathrm{g}}$. Suppose $\operatorname{cod}(x,v) \geq d-1$. Then by Proposition 2.1.12(ii) $\operatorname{copr}_Y(x)$ contains a line g. Thus by Lemma 4.2.2, there is a point z with $\operatorname{cod}(x,z) = \operatorname{cod}(x,v) - 1$ such that $z^\perp$ contains a line of Y. This implies $z \notin V$, a contradiction to Lemma 3.3.1(ii). Hence, $\operatorname{cod}(x,v) = d-2$ since $\operatorname{diam}(V) = 2$.

By Theorem 3.6.5 there is a point $w \in V$ with $\operatorname{dist}(w,Y) = 2$. Set $Z := \langle v,w\rangle_{\mathrm{g}}$. Then Z is a symplecton. Since $\operatorname{dist}(w,Y) = 2$, Proposition 2.1.17(i) implies that $Y \cap Z$ contains no line and hence, $Y \cap Z = \{v\}$. Suppose $\operatorname{cod}(x,Z) < d$. Then x has no cogate in Z since $Z \leq V$ and $\operatorname{cod}(x,p) \geq d-2$ for every point $p \in V$. Thus, Proposition 2.1.12(ii) implies that $\operatorname{copr}_Z(x)$ contains a line g. Now Lemma 4.2.2 implies that there is a point z with $\operatorname{cod}(x,z) = \operatorname{cod}(x,v) - 1$ such that $z^\perp$ contains a line of Z. This implies $z \notin V$, a contradiction to Lemma 3.3.1(ii). Hence, $\operatorname{cod}(x,Z) = d$ and Proposition 2.1.12(iv) implies that x has a gate w' in Z. Since $Z \cap Y = \{v\}$, Proposition 2.1.25(iii) implies $\operatorname{dist}(u,Z) = 2$. Hence, $X := \langle u,w'\rangle_{\mathrm{g}}$ is a symplecton that is contained in $\operatorname{copr}_Y(x)$. By Theorem 3.6.5 there is a point $y \in V$ such that $\operatorname{dist}(y,X) = 2$ and $V = \bigcup_{p\in X}\langle y,p\rangle_{\mathrm{g}}$. There is a point $p \in X$ such that $u \in \langle p,y\rangle_{\mathrm{g}}$. By Proposition 2.1.12(iv) p is a cogate for x in $\langle p,y\rangle_{\mathrm{g}}$ and hence, $\operatorname{cod}(x,y) = d-2$. This implies that for every point $q \in X$, q is a cogate for x in

$\langle q,y\rangle_{\mathrm{g}}$. Thus, $\mathrm{copr}_V(x) = X$ and we have case (d).
Finally let V be a symplecton of rank 5. If $\mathrm{copr}_V(x)$ is a singleton, (a) holds by Proposition 2.1.12(ii). Therefore we assume that $\mathrm{copr}_V(x)$ contains a line g. Then Lemma 4.2.2 implies that there is a point z with $\mathrm{cod}(x,z) = \mathrm{cod}(x,v) - 1$ such that $z^\perp$ contains a line of Z. By Proposition 3.6.4 $W := \langle x,V\rangle_{\mathrm{g}}$ is a rigid subspace with $\mathrm{diam}(W) = 2$ and $\mathrm{yrk}(W) = \mathrm{srk}(W) = 5$. Set $d := \mathrm{cod}(x,W)$. As before $X := \mathrm{copr}_W(x)$ is a symplecton and there is a point $y \in W$ with $\mathrm{cod}(x,y) = d-2$ such that $W = \bigcup_{p\in X}\langle y,p\rangle_{\mathrm{g}}$. Let $u \in g$ and let $p \in X$ such that $u \in \langle p,y\rangle_{\mathrm{g}}$. Let $v \in V$ such that $\langle u,v\rangle_{\mathrm{g}} = V$ and $q \in X$ such that $v \in \langle q,y\rangle_{\mathrm{g}}$. Since p is a cogate for x in $\langle p,y\rangle_{\mathrm{g}}$, we know $\langle p,y\rangle_{\mathrm{g}} \neq V$ and hence, $v \notin \langle p,y\rangle_{\mathrm{g}}$. This implies $q \neq p$. Thus, we may apply Lemma 3.6.3 to conclude that V and X intersect. Hence, $\mathrm{cod}(x,V) = d$ and $g \leq X$. Now Lemma 3.6.1(i) implies that V and X are either equal or intersect in a generator. In other words, either (c) or (b) holds. □

The following two assertions are the counterpart to Proposition 2.1.27. Note that we make use of the classification of rigid subspaces with finite diameter given in Chapter 3.

Proposition 4.2.5. *Let $V \leq \mathscr{S}$ be a rigid metaplecton and let x be a point with* $\mathrm{cod}(x,V) < \infty$ *such that* $\mathrm{diam}(\mathrm{copr}_V(x)) = 1$. *Then* $\mathrm{rk}(\mathrm{copr}_V(x)) = \mathrm{srk}(V)$.

Proof. Set $d := \mathrm{cod}(x,V)$ and $n := \mathrm{diam}(V)$. We may assume $n \geq 2$, since otherwise there is nothing to prove. By Proposition 4.2.3 we conclude $\mathrm{yrk}(V) \geq 3$. For $\mathrm{yrk}(V) \geq 5$, the claim follows from Proposition 4.2.4. Thus, we may assume $\mathrm{yrk}(V) \in \{3,4\}$.
Let z be a point of $\mathrm{copr}_V(x)$ and let $g \leq \mathrm{copr}_V(x)$ be a line through z. For every point $p \in V$ with $\mathrm{dist}(p,z) = n$, we obtain $\mathrm{cod}(x,p) \leq d-n+1$ by Proposition 2.1.16(ii) since $\mathrm{diam}(\mathrm{copr}_V(x)) = 1$. By Proposition 2.1.17(i) this implies $\mathrm{cod}(x,p) = d-n+1 = \min\{\mathrm{cod}(x,q) \mid q \in V\}$. Now we may apply Lemma 4.2.2 to conclude that there is a point y with $\mathrm{cod}(x,y) = d-n$ such that $y^\perp$ contains a line of V. Set $W := \langle y,V\rangle_{\mathrm{g}}$. Then Propositions 3.4.5 and 3.5.5 imply $\mathrm{diam}(W) = n$. Thus, $\mathrm{cod}(x,W) = d$ since W contains z and y. Moreover, if $\mathrm{yrk}(V) = 3$, then Theorem 3.4.4 and Proposition 3.4.5 imply $\mathrm{srk}(W) = n+1$. If $\mathrm{yrk}(V) = 4$, then Theorem 3.5.4 and Proposition 3.5.5 imply $\mathrm{srk}(W) = 2n$. Set $U := \langle z,y\rangle_{\mathrm{g}}$. Then we conclude by Proposition 3.4.5 and Theorem 3.5.4 $\mathrm{dist}(p,U) = 1$ for every point $p \in W \setminus U$.
Let $w \in \mathrm{copr}_W(x) \setminus \{z\}$. Suppose $\mathrm{dist}(w,z) \geq 2$. Then by Proposition 2.1.25(iii) there is a point $w' \in U$ with $w' \perp w$ and $w' \in \langle w,z\rangle_{\mathrm{g}}$. Since $\langle w,z\rangle_{\mathrm{g}} \leq \mathrm{copr}_W(x)$ by Proposition 2.1.16(i), this is a contradiction to $\mathrm{copr}_U(x) = \{z\}$. Thus, $\mathrm{copr}_W(x) \leq z^\perp$ and therefore $\mathrm{diam}(\mathrm{copr}_W(x)) = 1$ by Propositions 2.1.16(i) and 2.1.12(i).
Assume $\mathrm{yrk}(V) = 3$. Let y' be a point collinear to z with $\mathrm{dist}(y,y') = n-1$. Then

there is a symplecton $Y \leq W$ containing g and y'; if $\langle y', g\rangle_{\mathrm{g}}$ is singular, this follows from Lemma 3.1.1(i). We obtain $\mathrm{dist}(y,Y) = n-1$. Hence, $\mathrm{diam}(W) = n$ implies that y has no gate in Y. By Proposition 2.1.27 we conclude that $\mathrm{pr}_Y(y)$ is a generator of Y and moreover, $\mathrm{pr}_Y(y)$ is properly contained in singular subspace of $\mathscr{S}$. Hence by Lemma 3.4.3(i) and Proposition 2.2.4(ii), there are exactly two maximal singular subspaces M and N of $\mathscr{S}$ that contain g. By Lemma 3.1.1(iii) both M and N contain a generator of Y. Since both $M\cap Y$ and $N\cap Y$ contain g, Proposition 2.2.9(iii) implies that $\mathrm{pr}_Y(y)$ is disjoint to either $M\cap Y$ or $N\cap Y$. We may assume $N\cap Y\cap \mathrm{pr}_Y(y) = \varnothing$. Now $\mathrm{dist}(y,M) = n-1$ and hence, $\mathrm{rk}(\mathrm{pr}_M(y)) = n-1$ and $\mathrm{pr}_M(y) \leq W$ by Lemma 3.4.2. Since $\mathrm{srk}(W) = n+1$, this implies $M\cap W = \langle g, \mathrm{pr}_M(y)\rangle$ and $\mathrm{rk}(M\cap W) = n+1$. By Theorem 3.4.4 we conclude $\mathrm{rk}(N\cap W) = n$. Analogously to M, $\mathrm{dist}(y,N) = n-1$ would imply $\mathrm{rk}(N\cap W) \geq n+1$ and therefore $\mathrm{dist}(y,N) = n$. Since $M\cap W = \langle g, \mathrm{pr}_M(y)\rangle$ and $\mathrm{cod}(x, \mathrm{pr}_M(y)) \leq d-1$, we obtain $\mathrm{copr}_W(x)\cap M = g$ and consequently, $\mathrm{copr}_W(x)$ is contained in N.

Now assume $\mathrm{yrk}(V) = 4$. If $\mathrm{rk}(\mathrm{copr}_W(x)) \geq 3$, let $G \leq \mathrm{copr}_W(x)$ be a subspace with $g \leq G$ and $\mathrm{rk}(G) = 3$. Otherwise, Lemma 3.1.1(i) implies that there is a singular subspace $G \leq W$ with $\mathrm{copr}_W(x) < G$ and $\mathrm{rk}(G) = 3$. By Lemma 3.1.1(i) there is a symplecton $Y \leq W$ such that G is a generator of Y. Since $g \leq Y$, we obtain $\mathrm{dist}(y,Y) \geq n-1$. Since $\mathrm{diam}(W) = n$, Proposition 2.1.25(i) implies that $\mathrm{pr}_Y(y)$ contains a line. Hence by Lemma 3.5.1, $\mathrm{pr}_Y(y)$ is a generator of Y and $\mathrm{dist}(y,Y) = n-1$. Since $\mathrm{cod}(x, \mathrm{pr}_Y(y)) \leq d-1$, we obtain $\mathrm{copr}_W(x)\cap \mathrm{pr}_Y(y) = \varnothing$. Hence, $G\cap \mathrm{pr}_Y(y) = \varnothing$ if $G \leq \mathrm{copr}_W(x)$. If $G > \mathrm{copr}_W(x)$, then there is a generator of Y containing $\mathrm{copr}_W(x)$ and being disjoint to $\mathrm{pr}_Y(y)$. Hence, we may assume $G\cap \mathrm{pr}_Y(y) = \varnothing$. By Proposition 2.2.5 there is a unique maximal singular subspace $N \leq \mathscr{S}$ containing G. By Proposition 2.1.27 we know that $\mathrm{pr}_Y(y)$ is not a maximal singular subspace of W. Thus, $G < N$ by Proposition 2.2.9(ii). This implies $\mathrm{rk}(N\cap W) = 2n$ by Lemma 3.5.3(i) and Theorem 3.5.4. Suppose $\mathrm{dist}(y,N) = n-1$. Then $\mathrm{rk}(\mathrm{pr}_N(y)) = 2n-2$ by Lemma 3.5.3(ii). Since $G\cap \mathrm{pr}_N(y) = \varnothing$, this implies $\mathrm{rk}(N\cap W) \geq 2n+2$, a contradiction. Thus, $\mathrm{dist}(y,N) = n$.

Thus, for both possibilities of $\mathrm{yrk}(V)$, there is a maximal singular subspace N with $\mathrm{copr}_W(x) \leq N$ and $\mathrm{dist}(y,N) = n$ that intersects W in a maximal singular subspace of W. Suppose there is a point $w \in N \setminus \mathrm{copr}_W(x)$. Then $\mathrm{dist}(y,w) = n$ since $\mathrm{dist}(y,N) = n$ and $\mathrm{diam}(W) = n$. Hence, $\langle w,y\rangle_{\mathrm{g}} \cap N$ contains no line since otherwise Proposition 2.1.17(i) would imply $\mathrm{dist}(y,N) \leq n-1$. Hence, $\langle w,y\rangle_{\mathrm{g}} \cap N = \{w\}$ and therefore $\mathrm{copr}_W(x)\cap \langle w,y\rangle_{\mathrm{g}} = \varnothing$. This implies $\mathrm{cod}(x, \langle w,y\rangle_{\mathrm{g}}) = d-1$ and $w \in \mathrm{copr}_{\langle w,y\rangle_{\mathrm{g}}}(x)$. Since $\mathrm{cod}(x,y) = d-n$, the point w is not a cogate for x in $\langle w,y\rangle_{\mathrm{g}}$ and thus, $\mathrm{copr}_{\langle w,y\rangle_{\mathrm{g}}}(x)$ contains a line by Proposition 2.1.12(ii). Now Lemma 4.2.2 implies that there is a point y' with $\mathrm{cod}(x,y') < \mathrm{cod}(x,y) = d-n$ such that $(y')^{\perp}\cap \langle w,y\rangle_{\mathrm{g}}$ contains a line. Since $\mathrm{diam}(W) = n$ and $\mathrm{cod}(x,W) = d$, we conclude

$y' \notin W$. Since $\operatorname{cod}(x,W) - \operatorname{cod}(x,y') \geq n+1$, we obtain $\operatorname{diam}(\langle y',W\rangle_{\mathrm{g}}) \geq n+1$. For $\operatorname{yrk}(V)=3$, we obtain by Proposition 3.4.5 that $\langle y',W\rangle_{\mathrm{g}}$ has the same singular rank W and hence, $\langle y',W\rangle_{\mathrm{g}}$ is a metaplecton with diameter $n+1$ by Theorem 3.4.4. For $\operatorname{yrk}(V)=4$, we obtain $\operatorname{srk}(\langle y',W\rangle_{\mathrm{g}}) = \operatorname{srk}(W)+1 = 2n+1$ and hence, $\langle y',W\rangle_{\mathrm{g}}$ is a metaplecton with diameter $n+1$ by Theorem 3.5.4. Now Proposition 2.1.12(iv) implies that x has a cogate in $\langle y',W\rangle_{\mathrm{g}}$ which is at codistance d to x, a contradiction to $g \leq W$. Thus we conclude $\operatorname{copr}_W(x) = N \cap W$.
Since $g \leq N \cap V$ and $V \leq W$, we obtain $\operatorname{copr}_V(x) = N \cap V$. For $\operatorname{yrk}(V)=3$, Lemma 3.1.1(iii) implies that N contains a maximal singular subspace of V since $g \leq N \cap V$. Thus, $\operatorname{rk}(N\cap V) = \operatorname{srk}(V) = n$ by Theorem 3.4.4. For $\operatorname{yrk}(V)=4$, we know $\operatorname{rk}(N\cap W) = 2n$. Suppose $N \cap V = g$. Then there is a point $p \in N\cap W \setminus V$. By Lemma 3.1.1(i) there is symplecton $Z \leq V$ that contains g. Now Lemma 3.5.1 implies that $\operatorname{pr}_Z(p)$ is a generator of Z that contains g. Since by Proposition 3.5.2 the singular subspace $\langle p,g\rangle$ is contained in a unique maximal singular subspace of rank ≥ 4, we conclude $\langle p, \operatorname{pr}_Z(p)\rangle \leq N$, a contradiction to $N\cap V = g$. Hence, $\operatorname{rk}(N\cap V) \geq 2$ and consequently, N contains a maximal singular subspace of V by Lemma 3.1.1(iii). By Proposition 3.5.2 and Lemma 3.5.3(i) we conclude $\operatorname{rk}(N\cap V) = \operatorname{srk}(V)$. □

Corollary 4.2.6. *Let V be a metaplecton and let x be a point with* $\operatorname{cod}(x,V) < \infty$ *such that* $\operatorname{diam}(\operatorname{copr}_V(x)) = 1$. *Then* $\operatorname{copr}_V(x))$ *is a maximal singular subspace of V.*

Proof. Let $g \leq \operatorname{copr}_V(x)$ be a line. We may assume that g is not a maximal singular subspace of V, since otherwise $\operatorname{copr}_V(x) = g$ and we are done. Then there is a point $p \in V \setminus g$ with $g \leq p^{\perp}$. By Lemma 2.1.26 there is a symplecton $Y \leq V$ that contains $\langle p,g\rangle$. This implies $\operatorname{rk}(Y) \geq 3$ and therefore Y is rigid. Since $\operatorname{copr}_V(x)$ is singular and $g \leq \operatorname{copr}_V(x) \cap Y$, Proposition 4.2.5 implies that $\operatorname{copr}_Y(x)$ is a generator of Y.
Now let $q \in V \setminus g$ be another point with $g \leq q^{\perp}$. As before, there is a rigid symplecton Z that contains $\langle q,g\rangle$ and $\operatorname{copr}_Z(x)$ is a generator of Z. Hence, $q \in \operatorname{copr}_Z(x)$ or $\operatorname{copr}_Z(x) \nleq q^{\perp}$. Thus, $\operatorname{copr}_V(x)$ is a maximal singular subspace of V. □

Now we are ready to study how convex subspaces at finite codistance are related to each other. The following two assertions are the counterpart to Proposition 2.1.29.

Lemma 4.2.7. *Let x and z be two points of an SPO space with* $\operatorname{cod}(x,z) = n < \infty$. *Further let w and y be points with $w \leftrightarrow z$, $y \leftrightarrow x$ and* $\operatorname{dist}(w,x) = \operatorname{dist}(y,z) = n$. *Set* $U := \langle w,x\rangle_{\mathrm{g}}$ *and* $V := \langle y,z\rangle_{\mathrm{g}}$. *Then*

(i) *U and V are one-coparallel to each other with* $\operatorname{cod}(U,V) = n$ *and*

(ii) *the bijective map* $\varphi\colon U \to V$ *with* $\mathrm{copr}_V(u) = \{u^\varphi\}$ *for all* $u \in U$ *is an isomorphism.*

Proof. By (A12) w and x have a cogate at codistance n in V. Hence, the cogate for x in V is z. Let y' be the cogate for w in U. Then $\mathrm{dist}(y',z) = n$ since $w \leftrightarrow z$. Thus, $x \leftrightarrow y'$ and we may assume $y = y'$. Again by (A12) x is a cogate for w in U and z is a cogate for y in U.
Let $u \in U \setminus \{x\}$ with $u \perp x$. Then $\mathrm{cod}(u,V) \geq n$ by Proposition 2.1.17(ii). Since x is a cogate for z in U, we obtain $\mathrm{cod}(u,z) = n-1$. Since $u \perp x$ and z is a cogate for x in V, we obtain $\mathrm{cod}(u,v) \leq n - \mathrm{dist}(v,z) + 1$ for all $v \in V$. Hence, $\mathrm{cod}(u,V) = n$ and $\mathrm{copr}_V(u) \leq z^\perp$. Thus, $\mathrm{diam}(\mathrm{pr}_V(u)) < 2$ since otherwise $z \in \mathrm{copr}_V(u)$ by Proposition 2.1.16(i). This implies that $\langle z, \mathrm{copr}_V(u)\rangle$ is a singular subspace. Hence, $\mathrm{copr}_V(u)$ is no maximal singular subspace of V and therefore, $\mathrm{copr}_V(u)$ is a singleton by Corollary 4.2.6. Thus, Proposition 2.1.12(ii) implies that u has a cogate v in V with $z \perp v$. By symmetric reasons, u is the cogate for v in U.
Now (i) follows by induction. Since every point $p \in U$ has a cogate q in V and p is then the cogate for q in U, we conclude that φ is bijective. Since $z = x^\varphi$, $v = u^\varphi$ and $z \perp v$, we know already that φ preserves collinearity. It remains to check that $p^\varphi \in zv$ for every $p \in xu$. Suppose $p^\varphi \notin zv$. Since by Proposition 2.1.23 V is an SPO space, we may apply Lemma 2.1.21(iii) to conclude that there is a point $s \in V$ with $\mathrm{dist}(s,z) = \mathrm{dist}(s,v) = n-1$ and $\mathrm{dist}(s,p^\varphi) = n$. This implies $s \leftrightarrow p$ and $x \not\leftrightarrow s \not\leftrightarrow u$, a contradiction to (A2). □

Corollary 4.2.8. *Let x, y and z be points of an SPO space such that* $\mathrm{dist}(y,z) = n$ *and* $\mathrm{cod}(x,z) = n + \mathrm{cod}(x,y) < \infty$. *Then there is a point w at distance n to x such that* $\mathrm{cod}(w,y) = n + \mathrm{cod}(w,z)$. *For every such point, the metaplecta* $\langle w,x\rangle_{\mathrm{g}}$ *and* $\langle y,z\rangle_{\mathrm{g}}$ *are one-coparallel to each other. Moreover, the map* $\varphi\colon \langle w,x\rangle_{\mathrm{g}} \to \langle y,z\rangle_{\mathrm{g}}$ *that maps every point* $p \in \langle w,x\rangle_{\mathrm{g}}$ *to the unique point of* $\mathrm{copr}_{\langle y,z\rangle_{\mathrm{g}}}(p)$ *is an isomorphism.*

Proof. Let $y' \leftrightarrow x$ be a point with $\mathrm{dist}(y,y') = \mathrm{cod}(x,y)$ and let $w' \leftrightarrow z$ be a point with $\mathrm{dist}(x,w') = \mathrm{cod}(x,z)$. Since $x \leftrightarrow y'$ and $\mathrm{dist}(z,y) + \mathrm{dist}(y,y') = \mathrm{cod}(x,z)$, we obtain $\mathrm{dist}(z,y') = \mathrm{cod}(x,z)$ and hence $\mathrm{dist}(w',x) = \mathrm{dist}(y',z)$. Thus by Lemma 4.2.7, the metaplecta $\langle w',x\rangle_{\mathrm{g}}$ and $\langle y',z\rangle_{\mathrm{g}}$ are one-coparallel to each other with $\mathrm{cod}(\langle w',x\rangle_{\mathrm{g}}, \langle y',z\rangle_{\mathrm{g}}) = \mathrm{cod}(x,z)$. Let $w \in \langle w',x\rangle_{\mathrm{g}}$ be the point with $\mathrm{cod}(x,z) = \mathrm{cod}(w,y)$. Since $\mathrm{cod}(w,y) = n + \mathrm{cod}(x,y)$ and by Proposition 2.1.12(ii) w is the cogate for y in $\langle x,w'\rangle_{\mathrm{g}}$, we obtain $\mathrm{dist}(x,w) = n$. Since x is the cogate for z in $\langle w',x\rangle_{\mathrm{g}}$, we conclude $\mathrm{cod}(w,y) = \mathrm{cod}(x,z) = n + \mathrm{cod}(w,z)$.
Now let w be an arbitrary point with $\mathrm{cod}(w,y) = n + \mathrm{cod}(w,z)$ and $\mathrm{dist}(x,w) = n$. Then $\mathrm{cod}(x,z) \leq \mathrm{cod}(w,z) + n$ since $\mathrm{dist}(w,x) = n$ and hence, $\mathrm{cod}(x,z) \leq \mathrm{cod}(w,y)$. Analogously, $\mathrm{cod}(w,y) \leq \mathrm{cod}(x,z)$ and therefore equality holds. Let $y' \leftrightarrow x$ be as above and let $w' \leftrightarrow z$ be a point with $\mathrm{dist}(w,w') = \mathrm{cod}(w,z)$. Then

$\operatorname{dist}(y',z) = \operatorname{dist}(w',x) = \operatorname{cod}(x,z)$ and again $U := \langle w',x\rangle_{\mathrm{g}}$ and $V := \langle y',z\rangle_{\mathrm{g}}$ are one-coparallel to each other with $\operatorname{cod}(U,V) = \operatorname{cod}(x,z)$. Now let $\psi\colon U \to V$ be the unique isomorphism mapping every point of U to its cogate in V. Then $y = w^{\varphi}$ since $\operatorname{cod}(w,y) = \operatorname{cod}(x,z)$. Thus, since ψ is an isomorphism, we obtain $v \in \langle w,x\rangle_{\mathrm{g}}$ if and only if $v^{\psi} \in \langle y,z\rangle_{\mathrm{g}}$. Hence, $\varphi := \psi|_U$ yields an isomorphism from U onto V. ☐

Remark 4.2.9 (Opposite metaplecta)*.* Let U and V be metaplecta that are one-coparallel to each other with $\operatorname{cod}(U,V) = \operatorname{diam}(U) = \operatorname{diam}(V)$. Then every point $p \in U$ has a cogate in V by Proposition 2.1.12(ii) and hence, there is a point $q \in V$ with $q \leftrightarrow p$ by Proposition 2.1.12(i). Analogously, for every point $q \in V$, there is a point $p \in U$ with $p \leftrightarrow q$. Thus, U and V are subspaces that are opposite to each other.
Let V be an arbitrary metaplecton of an SPO space and let y and z be points with $\langle y,z\rangle_{\mathrm{g}} = V$. Since by Lemma 2.1.13 there is a point x opposite y with $\operatorname{cod}(x,z) = \operatorname{dist}(y,z)$, Corollary 4.2.8 implies that there is a metaplecton U that is opposite V. Hence, to every metaplecton there is an opposite metaplecton.
Finally, let U and V be metaplecta that are opposite to each other. Set $n := \operatorname{diam}(U)$ and let $v \in V$. Since there is a point in U that is opposite v, (A12) implies that v has a gate u in U with $\operatorname{cod}(u,v) = n$. This implies $\operatorname{cod}(U,V) = n$. Now u is opposite to a point of V and hence, $\operatorname{diam}(V) \geq n$ since $v \in V$. With (A1) and $\operatorname{cod}(U,V) = n$ we conclude $\operatorname{diam}(V) = n$. As above, this implies that every point of U has a cogate in V at codistance n. Hence, U and V are metaplecta that are one-coparallel to each other with $\operatorname{cod}(U,V) = \operatorname{diam}(U) = \operatorname{diam}(V)$.

Lemma 4.2.10. *Let V be a maximal connected rigid subspace. Further let x be a point such that* $\operatorname{cod}(x,y) = d < \infty$ *for a point $y \in V$ and* $\operatorname{cod}(x,v) \geq d$ *for every point $v \in V$. Then* $\operatorname{cod}(x,l) \geq d+1$ *for every line $l \leq V$.*

Proof. Suppose there is a line $l \leq V$ with $\operatorname{cod}(x,l) = d$. Then by Lemma 4.2.2 there is a point z with $\operatorname{dist}(z,l) = 1$ and $\operatorname{cod}(x,z) < d$ such that $\operatorname{pr}_l(z) = l$. Since $\operatorname{cod}(x,z) < d$, we obtain $z \notin V$. Since $l \leq \operatorname{pr}_V(z)$, this is a contradiction to Proposition 4.1.4 ☐

By (A1) the codistance between two maximal connected rigid subspaces that have infinite diameter is always infinite. Hence by definition, two such subspaces can never be one-coparallel to each other. For this, we introduce the following terminology.

Definition 4.2.11. Let $d \in \mathbb{N}$ and let U and V be subspaces of an SPO space with $\operatorname{codm}(U \cup V) = d$. Furthermore, for every point $u \in U$, there is a point $v \in V$ with $\operatorname{cod}(u,v) = d$ and no line $l \leq V$ with $\operatorname{cod}(u,l) = d$. Then we call U *d-opposite* to V.

Let U and V be two metaplecta. Note that if U is one-coparallel to V with $\mathrm{cod}(U,V) = d$, then Proposition 2.1.12(ii) and Proposition 2.1.17(i) imply that U is $(d - \mathrm{diam}(V))$-opposite to V. By this definition 0-opposite is just the same as opposite.

Lemma 4.2.12. *Let g and h be one-coparallel lines of an SPO space. Further let U and V be maximal connected rigid subspaces with $h \le U$ and $g \le V$. Then U is d-opposite V for a natural number d.*

Proof. Since g and h are one-coparallel, we obtain $d := \mathrm{cod}(g,h) - 1 < \infty$. Hence, we may assume that $g \le U$ and $h \le V$ are one-coparallel lines with minimal possible codistance. Let w and x be two distinct points of h and let y and z be points on g such that $\mathrm{cod}(w,y) = \mathrm{cod}(x,z) = d+1$.
Suppose there is a point $q \in V$ with $\mathrm{cod}(w,q) < d$. Since $\mathrm{dist}(z,q) < \infty$, we may assume $q \perp z$ and $\mathrm{cod}(w,q) = d-1$ by Lemma 2.1.28. Hence, $\mathrm{dist}(q,y) = 2$ and by Proposition 2.1.12(iv) y is a cogate for w in $\langle q,y \rangle_{\mathrm{g}}$. Since $z \in \langle q,y \rangle_{\mathrm{g}}$ and $\mathrm{cod}(x,z) > \mathrm{cod}(x,y)$, we know $y \notin \mathrm{copr}_{\langle q,y \rangle_{\mathrm{g}}}(x)$. Thus by (A3), there is a point $p \in \langle q,y \rangle_{\mathrm{g}}$ with $\mathrm{cod}(x,p) < \mathrm{cod}(x,q) = d$. By Proposition 2.1.12(iv) this implies that z is a cogate for x in $\langle q,y \rangle_{\mathrm{g}}$. Since by Proposition 2.1.23 $\langle q,y \rangle_{\mathrm{g}}$ is an SPO space, there is a line g' in $\langle q,y \rangle_{\mathrm{g}}$ that is one-parallel to g with $\mathrm{dist}(g,g') = 1$. Then w and x have both a cogate in g' at codistance d in g' and these cogates are distinct. By Corollary 4.2.8 this implies that h and g' are one-coparallel to each other and $\mathrm{cod}(h,g')$. This is a contradiction to the minimality of d since $h' \le \langle q,y \rangle_{\mathrm{g}} \le V$. Thus, for every point $p \in h$ and every point $q \in V$, we obtain $\mathrm{cod}(p,q) \ge d$. Moreover, $\mathrm{cod}(p,l) > d$ for every line $l \le V$ by Lemma 4.2.10. By symmetric reasons $\mathrm{cod}(p,q) \ge d$ for every pair of points $(p,q) \in U \times g$ and $\mathrm{cod}(l,q) \ge d+1$ for every line $l \le U$ and every point $q \in V$. Since U is connected, it remains to show that for every point $p \in U \setminus h$ with $p \perp w$ there is a line $l \le V$ such that pw and l are one-coparallel to each other with $\mathrm{cod}(pw,l) = d+1$.
First assume $\mathrm{dist}(p,x) = 2$ and set $Y := \langle p,x \rangle_{\mathrm{g}}$. Suppose $x \in \mathrm{copr}_Y(z)$. Since $w \in Y$ and $\mathrm{cod}(p,z) \ge d$ for every point $p \in Y$, this implies $\mathrm{diam}(\mathrm{copr}_Y(z)) = 1$ by Propositions 2.1.12(ii). Thus, there is a line $k \le Y$ that is disjoint to $\mathrm{copr}_Y(z)$, a contradiction to $\mathrm{cod}(z,k) \ge d+1$. Hence, $x \notin \mathrm{copr}_Y(z)$ and therefore $\mathrm{cod}(z,Y) = d+2$. Analogously, $\mathrm{cod}(y,Y) = d+2$. By Proposition 2.1.12(iv) z has a cogate q in Y. This implies $q \perp x$. Hence, $\mathrm{cod}(q,y) = d+1$ since $\mathrm{cod}(x,y) = d$ and $z \perp y$. Now let r be a point with $r \leftrightarrow z$ and $\mathrm{dist}(r,w) = d$. Further let s be a point with $q \leftrightarrow s$ and $\mathrm{dist}(s,y) = d+1$. By Lemma 4.2.7 the metaplecta $\langle s,z \rangle_{\mathrm{g}}$ and $\langle q,r \rangle_{\mathrm{g}}$ are isomorphic via mapping every point of $\langle s,z \rangle_{\mathrm{g}}$ to the unique point at codistance $d+2$ in $\langle q,r \rangle_{\mathrm{g}}$. Thus, there is a symplecton $Z \le \langle s,z \rangle_{\mathrm{g}}$ with $Z \cong Y$ such that Y and Z are one-coparallel with $\mathrm{cod}(Y,Z) = d+2$. Since both y and z have distance $d+2$ to Y and $y \in \langle s,z \rangle_{\mathrm{g}}$, we conclude $g \le Z$. Since $Y \le U$, we know that both Y and Z are rigid and hence, $Z \le V$ by Proposition 4.1.2. Let $l' := \langle \mathrm{copr}_Z(p), \mathrm{copr}_Z(w) \rangle$ and

let $l \leq Z$ be a line that is one-parallel to l' with $\operatorname{dist}(l,l') = 1$. Then by Corollary 4.2.8 l and pw are one-coparallel to each other with $\operatorname{cod}(l, pw) = d+1$.
Now assume $\operatorname{dist}(p,x) = 1$. Then $S := \langle p,h\rangle$ is a singular subspace of rank 2. Assume $\operatorname{cod}(p,q) = d$ for a point $q \in g$. Let $p' \in h$ be the unique point of h with $\operatorname{cod}(p',q) = d+1$. Then p' is a cogate for q in pp'. Since every point of $g \setminus \{q\}$ has codistance $d+1$ to pp' and codistance d to p', Corollary 4.2.8 implies that pp' and g are one-coparallel to each other with $\operatorname{cod}(pp',g) = d+1$. Thus, we assume $\operatorname{cod}(p,g) = d+1$ and $\operatorname{copr}_l(p) = l$. Then Lemma 4.2.2 implies that there is a point q with $\operatorname{dist}(q,g) = 1$ and $g \leq q^\perp$ such that $\operatorname{cod}(p,q) = d$. Since $\langle q,g\rangle$ is rigid, Proposition 4.1.2 implies $q \in V$. Now y is a cogate for p in yq. Since $\operatorname{cod}(x,y) = d$ and $\operatorname{cod}(x,qy) = d+1$, Corollary 4.2.8 implies that px and qy are one-coparallel to each other with $\operatorname{cod}(pp',g) = d+1$. □

Corollary 4.2.13. *Let V and U be maximal connected rigid subspaces such that U is d-opposite V for $d \in \mathbb{N}$. Then V is d-opposite U.*

Proof. Assume that V is a singleton. Then there is no line containing V by the maximality of V. Hence, $\operatorname{cod}(u,V) = 0$ for every point $u \in U$. This implies that U is a singleton that is opposite V.
Now let $\operatorname{diam}(V) \geq 1$. Let $x \in U$ and let $y \in V$ such that $\operatorname{cod}(x,y) = d$. Let $z \in V$ be a point with $z \perp y$. By Lemma 4.2.10 we may assume $\operatorname{cod}(x,z) = d+1$. Hence by Corollary 4.2.8, there is a line l through x such that l and yz are one-coparallel to each other. Since l is rigid, we obtain $\operatorname{diam}(U) \geq 1$.
Let $w \leq U \setminus \{x\}$ be a point with $w \perp x$. Since U is d-opposite to V, we conclude $\operatorname{cod}(y,wx) = d+1$ by Lemma 4.2.10. Therefore we may assume $\operatorname{cod}(y,w) = d+1$. Since V is connected there is a point $z \in V$ with $y \perp z$ and $\operatorname{cod}(w,z) = d$ by Lemma 2.1.28. Since $\operatorname{cod}(x,yz) = d+1$ and $\operatorname{cod}(x,y) = d$, Corollary 4.2.8 implies that wx and yz are one-coparallel to each. With Lemma 4.2.12 this implies that V is c-opposite U for some $c \in \mathbb{N}$. Since $\operatorname{cod}(x,y) = d$ and $\operatorname{cod}(u,y) \geq d$ for every $u \in U$, we conclude $c = d$. □

Proposition 4.2.14. *Let V be a maximal connected rigid subspace. Further let x be a point such that* $\operatorname{cod}(x,y) < \infty$ *for a point $y \in V$. Then there is exactly one maximal rigid subspace U with $x \in U$ that is d-opposite V for some $d \in \mathbb{N}$.*

Proof. Set $d := \operatorname{cod}(x,y)$. Since $d < \infty$, we may assume that we chose $y \in V$ such that $\operatorname{cod}(x,v) \geq d$ for every point $v \in V$. If V is a singleton, then there is no line containing y by the maximality of V. Hence, $\operatorname{cod}(x,y) = 0$ and $\operatorname{cod}(w,y) = 0$ for every point $w \perp x$. By (A1) this implies that there is no line through x. Hence, $\{x\}$ is already a maximal connected rigid subspace.
Now let $\operatorname{diam}(V) \geq 1$. By Proposition 4.2.10, there is a point $z \in V$ with $z \perp y$ such that $\operatorname{cod}(x,z) = d+1$. Hence by Corollary 4.2.8, there is a point $w \perp x$ such that

wx and yz are one-coparallel to each other with $\operatorname{cod}(wx, yz) = d+1$. Let U be a maximal connected rigid subspace with $wx \leq U$. Then Lemma 4.2.12 implies that U is d-opposite to V since $\operatorname{cod}(x,y) = d$ and $\operatorname{cod}(x,v) \geq d$ for every point $v \in V$. Now let W be a maximal connected rigid subspace that is c-opposite V and contains x. Since $\operatorname{cod}(x,y) = d$ and $\operatorname{cod}(x,v) \geq d$ for every point $v \in V$, we obtain $c = d$. Suppose $W \neq U$. Then $U \cap W = \{x\}$ by Proposition 4.1.2. Since by Corollary 4.2.13 V is d-opposite W, there is a point $u \perp x$ with $\operatorname{cod}(u,z) = d$. Since $\langle u, xw\rangle_{\mathrm{g}}$ and V have the line xw in common and $u \notin V$, Proposition 4.1.2 implies that $\langle u, xw\rangle_{\mathrm{g}}$ is not rigid. Thus, $\operatorname{dist}(u,w) = 2$ and the only lines in $\langle u, w\rangle_{\mathrm{g}}$ through x are ux and uw. Hence, all points in $\langle u, w\rangle_{\mathrm{g}} \cap x^{\perp} \setminus \{x\}$ have codistance d to z. Thus by Propositions 2.1.16(i) and 2.1.16(ii), we obtain $\operatorname{copr}_{\langle u,w\rangle_{\mathrm{g}}}(z) = \{x\}$. Proposition 2.1.12(ii) implies that x is a cogate for z in $\langle u, w\rangle_{\mathrm{g}}$. On the other hand $\operatorname{cod}(y, ux) = \operatorname{cod}(y, wx) = d+1$ and $\operatorname{cod}(y,x) = d$ implies $\operatorname{cod}(y, \langle u, w\rangle_{\mathrm{g}}) = d+2$ by Proposition 2.1.16(i). Let $v \in \langle u, w\rangle_{\mathrm{g}}$ with $\operatorname{cod}(v,y) = d+2$. Then $\operatorname{dist}(x,v) = 2$ and hence, $\operatorname{cod}(z,v) = d-1$, a contradiction to $y \perp z$. □

4.3 Rigid twin SPO spaces

The aim of this section is to show that the equivalence relation for maximal rigid subspaces of one connected component we introduced in the first section of this chapter, can be extended to another connected component. More precisely, we are dealing with the two connected components of a twin SPO space and show that there is a canonical one-to-one correspondence between the equivalence classes of these two components.

Throughout this section let $\mathscr{S} = (\mathscr{S}^+, \mathscr{S}^-)$ be a twin SPO space. Further let $\mathfrak{M}$ be the set of maximal connected rigid subspaces of $\mathscr{S}$, i. e. of one of the components of $\mathscr{S}$. Let U and V be two elements of $\mathfrak{M}$. Then we write $U \parallel V$ if U and V are one-parallel to each other or if there is a natural number d such that U and V are d-opposite to each other. Otherwise we write $U \nparallel V$.

Proposition 4.3.1. *The relation $\parallel$ is an equivalence relation on $\mathfrak{M}$.*

Proof. Since every subspace is one-parallel to itself, $\parallel$ is reflexive. Thus, we may assume that both $\mathscr{S}^+$ and $\mathscr{S}^-$ are non-empty and hence, that every element of $\mathfrak{M}$ is non-empty. Assume that $\mathfrak{M}$ contains a singleton. Then this singleton is not contained in a line and hence, both connected components are singletons that are opposite. Thus, we may assume that every element of $\mathfrak{M}$ contains a line. By definition, the relation $\parallel$ is symmetric. Hence, it remains to show that $\parallel$ is transitive. Let U, V and W be distinct elements of $\mathfrak{M}$ such that $U \parallel V \parallel W$. First assume U and V are one-parallel to each other with $\operatorname{dist}(U,V) = 1$. Let $u \in U$ and let $v \in V$ be the point with $u \perp v$. Further let $p \in U \setminus \{u\}$ be a point with $p \perp u$ and

let $q \in V$ be the point with $p \perp q$. By Proposition 4.1.7 mapping every point of U onto its unique collinear point in V yields an isomorphism and hence pu and qv are one-parallel to each other at distance 1. This implies that $Y := \langle p, v \rangle_{\mathrm{g}}$ is a symplecton that contains q and u. Since Y and V have the line qv in common and $u \in Y \setminus V$, Proposition 4.1.2 implies that Y is not rigid.
Assume V is d-opposite W for some $d \in \mathbb{N}$. Let $r \in W$ be a point with $\mathrm{dist}(v, r) = d$. By Lemma 4.2.10 there is a point q' on qv at codistance $d+1$ to r. By Lemma 2.1.28 there is a point $w \in W$ collinear to r with $\mathrm{cod}(q', w) = d$. Since $\mathrm{cod}(w, qv) = d+1$ by Lemma 4.2.10 and r and w have distinct cogates in qv, Corollary 4.2.8 implies that qv and rw are one-coparallel to each other. Hence, we may assume $\mathrm{cod}(v, w) = \mathrm{cod}(q, r) = d+1$.
First let $\mathrm{cod}(u, w) = d+2$. Then Proposition 2.1.12(iv) implies that u is cogate for w in Y. Furthermore, $\mathrm{cod}(u, r) = d+1$ since $r \perp w$ and $u \perp v$. Since $\mathrm{cod}(q, r) = d+1$, $\mathrm{cod}(u, r) = d$ and $\langle r, u \rangle_{\mathrm{g}} = Y$, Proposition 2.1.16(i) implies $\mathrm{cod}(r, Y) \geq d+2$. Thus again by Proposition 2.1.12(iv), r has a cogate in Y. This cogate is collinear to both u and q. Since Y is not rigid, the only lines of Y through u are uv and pu and the only lines of Y through q are qv and pq. Hence, $\mathrm{cod}(p, r) = d+2$ and Corollary 4.2.8 implies that rw and pu are one-coparallel to each other. Thus, $U \parallel W$ by Lemma 4.2.12. Now let $\mathrm{cod}(u, w) = d$. Since uv and qv are the only lines of Y through v, we conclude $\mathrm{copr}_Y(w) = \{v\}$ by Proposition 2.1.16(ii). Hence, by Proposition 2.1.12(ii) v is a cogate for w in Y. This implies $\mathrm{cod}(p, w) = d-1$. Since $\mathrm{cod}(q, r) = d+1$, $q \perp p$ and $r \perp w$, we obtain $\mathrm{cod}(p, r) = d$. Hence by analogous reasons, q is a cogate for r in Y and $\mathrm{cod}(u, r) = d-1$. Again rw and pu are one-coparallel to each other and therefore $U \parallel W$. Finally let $\mathrm{cod}(u, w) = d+1$. If there is a point $u' \in uv$ with $\mathrm{cod}(u', w) = d+2$, then we obtain as for the case $\mathrm{cod}(u, w) = d+2$ that the unique line $l \leq Y$ through u' that is disjoint to uv is one-coparallel to rw with $\mathrm{cod}(l, rw) = d+2$. Moreover, both points r and w have a cogate in Y that is contained in l. Since uv is the only line through u' that intersects pu, the lines l and pu are one-parallel and hence, r and w have distinct cogates in pu that are at codistance $d+1$. Thus, rw and pu are one-coparallel at codistance $d+1$. Now consider the case $\mathrm{cod}(u, w) = d+1$ and $\mathrm{copr}_{uv}(w) = uv$. Since $\mathrm{cod}(w, q) = d$, there is no point in Y collinear to v at codistance $d+2$ to w. Hence, $\mathrm{cod}(w, Y) = d+1$ by Proposition 2.1.16(ii) and therefore $\mathrm{copr}_Y(w) = uv$ by Corollary 4.2.6. Since $\mathrm{cod}(r, v) = d$, we obtain $\mathrm{cod}(r, uv) \leq d+1$. Since $\mathrm{cod}(w, Y \setminus uv) = d$ and $w \perp r$, this implies $q \in \mathrm{copr}_Y(r)$. By Proposition 2.1.16(i), we conclude $\mathrm{cod}(r, u) < d+1$ since $u \notin \mathrm{copr}_Y(r)$. Hence, $\mathrm{cod}(u, w) = d+1$ implies $\mathrm{cod}(u, r) = d$. Thus, q is no cogate for r in Y and Corollary 4.2.6 implies that $\mathrm{copr}_Y(r)$ is a line. Since pq and qv are the only lines of Y through q and $\mathrm{cod}(r, v) = d$, we conclude $\mathrm{cod}(r, q) = d+1$. Again rw and pu are one-coparallel at codistance $d+1$. Thus, $U \parallel W$ by Lemma 4.2.12.

Now let U and V be one-parallel at distance $n > 1$ and let W be d-opposite V. Let p and u be distinct collinear points of U. By Proposition 4.1.7 the points q and v of V with $\mathrm{cod}(u,v) = \mathrm{cod}(p,q) = n$ are collinear and the lines pu and qv are one-parallel. Thus, the metaplecton $Z := \langle u,q\rangle_{\mathrm{g}}$ contains p and v. Let $(v_i)_{0\leq i\leq n}$ be a geodesic from v to u. Set $q_0 := r$. For $i < n$ let q_{i+1} be a point of $\langle p,v_{i+1}\rangle_{\mathrm{g}}$ that is collinear to q_i. Since $p \perp u$ and $\mathrm{dist}(p,v) = n+1$, we conclude $\mathrm{dist}(p,v_i) = n+1-i$. Since $q_i \in \langle p,v_i\rangle_{\mathrm{g}}$ and $\langle p,v_i\rangle_{\mathrm{g}} \leq \langle p,v_{i+1}\rangle_{\mathrm{g}}$, Proposition 2.1.17(i) implies that we always find such a point q_{i+1}. Now for every $i \leq n$ the sequences $(q_0,\dots,q_i,v_i,\dots v_n)$ and $(v_0,\dots,v_i,q_i,\dots q_n)$ are geodesics. As a direct consequence the lines q_iv_i are mutually one-parallel to each other. Let V_i be the maximal connected rigid subspace that contains q_iv_i. Then Lemma 4.1.5 implies that the subspaces V_i are mutually one-parallel to each other. Applying induction yields $U \parallel W$.
Let U and V are d-opposite and V and W are c-opposite for natural numbers c and d. Further let $w \in W$. By Proposition 4.1.6 there is a unique subspace $W' \in \mathfrak{M}$ with $w \in W'$ that is one-parallel to U. Now $W' \parallel U$ yields $W' \parallel V$ since U and V are d-opposite. By Proposition 4.2.14 W is the only element of $\mathfrak{M}$ containing w that is b-opposite to V for some $b \in \mathbb{N}$. Hence, we conclude $W = W'$ and therefore U and W are one-parallel.
Since $\parallel$ is symmetric, it remains the case that V is one-parallel to both U and W. By Proposition 4.2.14 there is subspace $V' \in \mathfrak{M}$ that is opposite V. By the above we obtain that there are natural numbers c and d such that U is c-opposite V' and W is d-opposite V'. Hence, U and W are one-parallel. ☐

The next proposition shows that every twin SPO space contains rigid subspaces that are again twin SPO spaces.

Proposition 4.3.2. *Let U and V be maximal connected rigid subspaces such that U is d-opposite V for some $d \in \mathbb{N}$. Then $U \cup V$ is a rigid twin SPO space with opposition relation $\leftrightarrow_d := \{(u,v) \in (U\cup V)\times(U\cup V) \mid \mathrm{cod}(u,v) = d\}$.*

Proof. Set $W := U \cup V$. By definition $\leftrightarrow_d$ is symmetric and by Corollary 4.2.13 it is total. Now Lemma 2.1.28 implies that for every two points x and y of W with $\mathrm{cod}(x,y) = n > d$, there is a point $z \in W$ with $y \perp z$ and $\mathrm{cod}(x,z) = n-1$. This implies $\mathrm{cod}(x,y) = \mathrm{cod}_d(x,y) + d$, where cod_d is the codistance with respect to $\leftrightarrow_d$. Thus, (A3) and (A4) are satisfied.
Let x, y and z be points of W with $x \leftrightarrow_d y$ and $\mathrm{dist}(y,z) = n$. Set $Y := \langle y,z\rangle_{\mathrm{g}}$. Let $Z \leq \mathrm{copr}_Y(x)$ be a metaplecton such that $\mathrm{diam}(Z) = \mathrm{diam}(\mathrm{copr}_Y(x))$. Since Y is an SPO space by Proposition 2.1.23, there is a metaplecton $Z' \leq Y$ that is opposite Z in Y. By Lemma 2.1.24 we conclude $\mathrm{cod}_d(x,Z') = 0$ since $y \in Y$. By Lemma 4.2.10 this implies $\mathrm{diam}(Z') = 0$ and consequently, $\mathrm{diam}(Z) = 0$. Thus,

$\mathrm{copr}_Y(x)$ is a singleton and (A2) holds. By Proposition 2.1.12(ii) x has a cogate in Y. Hence, (A1) follows from Proposition 2.1.3. □

Let I be an index set and let $(\mathscr{S}_i)_{i\in I}$ be a family of twin SPO spaces. For $i \in I$, we denote by $\mathscr{S}_i^+$ and $\mathscr{S}_i^-$ the two connected components of $\mathscr{S}_i$. Let $p_i \in \mathscr{S}_i^+$ and $q_i \in \mathscr{S}_i^-$ be points with $p_i \leftrightarrow q_i$ in $\mathscr{S}_i$. Then we define the *grid sum* of the twin SPO spaces $(\mathscr{S}_i)_{i\in I}$ with the *pair of origins* $((p_i)_{i\in I},(q_i)_{i\in I})$ as

$$\bigodot_{i\in I}(\mathscr{S}_i,(p_i,q_i)) := \left(\bigodot_{i\in I}(\mathscr{S}_i^+,p_i)\,,\,\bigodot_{i\in I}(\mathscr{S}_i^-,q_i)\right).$$

The opposition relation for $\bigodot_{i\in I}(\mathscr{S}_i,(p_i,q_i))$ is induced in the natural way, i. e. two points $(x_i)_{i\in I}$ and $(y_i)_{i\in I}$ are opposite if and only if $x_i \leftrightarrow y_i$ in $\mathscr{S}_i$ for every $i \in I$. For a point $x := (x_i)_{i\in I}$ of $\bigodot_{i\in I}(\mathscr{S}_i,(p_i,q_i))$, we define by $\mathrm{supp}(x) := \{i \in I \mid p_i \neq x_i \neq q_i\}$ the *support* of x. Let p and q be points of $\bigcap_{i\in I}\mathscr{S}_i$ such that $\mathscr{S}_i \cap \mathscr{S}_j = \{p,q\}$ for every two distinct indices i and j of I and $p \leftrightarrow q$ in $\mathscr{S}_i$. Then we write $\bigodot_{i\in I}\mathscr{S}_i$ instead of $\bigodot_{i\in I}(\mathscr{S}_i,(p,q))$.

Proposition 4.3.3. *Let I be an index set and let $(\mathscr{S}_i)_{i\in I}$ be a family of twin SPO spaces. For $i \in I$, let p_i and q_i be points of $\mathscr{S}_i$ that are opposite. Then $\bigodot_{i\in I}(\mathscr{S}_i,(p_i,q_i))$ is a twin SPO space.*

Proof. For $i \in I$, let $\mathscr{S}_i^+$ be the connected component that contains p_i and let $\mathscr{S}_i^-$ be the connected component that contains q_i. Set $\mathscr{S} := \bigodot_{i\in I}(\mathscr{S}_i,(p_i,q_i))$. Furthermore, for $\sigma \in \{+,-\}$, set $\mathscr{S}^\sigma := \bigodot_{i\in I}(\mathscr{S}_i^\sigma,r_i)$, where $r_i := p_i$ for $\sigma = +$ and $r_i := q_i$ otherwise. Let $x = (x_i)_{i\in I}$ and $y = (y_i)_{i\in I}$ be two points of $\mathscr{S}^+$. Then by definition $x \perp y$ if and only if there is an index $i \in I$ such that $x_i \perp y_i$ and $x_j = y_j$ for $j \in I \setminus \{i\}$. Furthermore, since the set $\{i \in I \mid x_i \neq y_j\} \leq \mathrm{supp}(x) \cup \mathrm{supp}(y)$ is finite, we obtain $\mathrm{dist}(x,y) = \sum_{i\in I}\mathrm{dist}(x_i,y_i)$, where the distance function always refers to the corresponding point-line space. As a direct consequence, for every point $v = (v_i)_{i\in I}$ on a geodesic from x to y, we obtain $\mathrm{supp}(v) \subseteq \mathrm{supp}(x) \cup \mathrm{supp}(y)$. Analogously to the distance, we obtain $\mathrm{cod}(x,y) = \sum_{i\in I}\mathrm{cod}(x_i,y_i)$ if $X \in \mathscr{S}^+$ and $y \in \mathscr{S}^-$. Consequently, (A4) holds in $\mathscr{S}$.
In all four axioms of Definition 2.1.1 we are dealing with finitely many points and the convex spans of two of them. Let J be the union of the supports of these points. Then we do not leave the subspace $\mathscr{S}' := \{v \in \mathscr{S} \mid \mathrm{supp}(v) \leq J\}$. Since $\mathscr{S}'$ is isomorphic to $\bigodot_{i\in J}(\mathscr{S}_i,(p_i,q_i))$ it suffices to prove the claim for a finite index set. Moreover, by induction we may restrain to the case $I = \{0,1\}$.
Let $y = (y_0,y_1)$ and $z = (z_0,z_1)$ be points of $\mathscr{S}^-$ and let $x = (x_0,x_1) \in \mathscr{S}^+$. Set $Y_i := \langle y_i,z_i\rangle_{\mathrm{g}}$ for $i \in \{0,1\}$. By the observation above, concerning the distance of two points of $\mathscr{S}^-$, we conclude that $\{(v_0,v_1) \mid v_0 \in Y_0 \wedge v_1 \in Y_1\}$ is a convex subspace. Hence, $Y := \langle y,z\rangle_{\mathrm{g}} = \{(v_0,v_1) \mid v_0 \in Y_0 \wedge v_1 \in Y_1\}$. Assume there is a

point $(v_0,v_1) \in Y$ with $x \leftrightarrow (v_0,v_1)$. Then there is a point $u = (u_0,u_1) \in Y$ such that for $i \in \{0,1\}$, the point u_i is a cogate for x_i in Y_i. This implies $\mathrm{cod}(x,Y) = \mathrm{cod}(x,u)$ and $\mathrm{copr}_Y(x) = \{u\}$. With

$$\begin{aligned}\mathrm{cod}(x,u) &= \mathrm{cod}(x_0,u_0) + \mathrm{cod}(x_1,u_1)\\ &= \mathrm{dist}(y_0,z_0) + \mathrm{dist}(y_1,z_1) = \mathrm{dist}(y,z)\end{aligned}$$

we conclude, that (A1) and (A2) hold.
Now assume $z \in \mathrm{copr}_Y(x)$ and Y does not necessarily contain a point that is opposite x. Since $\mathrm{copr}_Y(x) = \{(v_0,v_1) \mid v_0 \in \mathrm{copr}_{Y_0}(x_0) \ \wedge \ v_1 \in \mathrm{copr}_{Y_1}(x_1)\}$, we conclude that $z_i \in \mathrm{copr}_{Y_i}(x_i)$ for $i \in \{0,1\}$. Now let $w = (w_0,w_1)$ be a point with $w \perp x$ and $\mathrm{cod}(w,y) < \mathrm{cod}(x,y)$. We may assume $w_0 \perp x_0$ and $w_1 = x_1$ and hence, $\mathrm{cod}(w_0,y_0) = \mathrm{cod}(x_0,y_0) - 1$. Thus by (A3), $\mathrm{copr}_{Y_0}(w_0) \leq \mathrm{copr}_{Y_0}(x_0)$ and $\mathrm{cod}(w_0,Y_0) \geq \mathrm{cod}(x_0,Y_0)$, whereat equality does not hold in both cases. Since $\mathrm{copr}_{Y_1}(w_1) = \mathrm{copr}_{Y_1}(x_1)$ and $\mathrm{cod}(w_1,Y_1) = \mathrm{cod}(x_1,Y_1)$, we conclude that (A3) is fulfilled in $\mathscr{S}$. □

The corresponding assertion for grid products of twin SPO spaces does not hold since for an infinite index set I the point-line spaces $\bigotimes_{i\in I} \mathscr{S}_i^-$ is disconnected if for every $i \in I$, there at least two points in $\mathscr{S}_i$.

Lemma 4.3.4. *Let U and V be two maximal connected rigid subspaces of an SPO space such that $\mathrm{dist}(U,V) < \infty$ and $U \nparallel V$. Then there is a point $u \in U$ such that $\mathrm{pr}_U(v) = \{u\}$ for every point $v \in V$.*

Proof. Let $v \in V$ be a point. Then by Proposition 4.1.4 there is a point $u \in U$ such that $\mathrm{pr}_U(v) = \{u\}$. Now let $q \in V \setminus \{v\}$ with $q \perp v$ and let $p \in U$ with $\mathrm{pr}_U(q) = \{p\}$. For $\mathrm{dist}(v,U) < \mathrm{dist}(q,U)$, we obtain $u \in \mathrm{pr}_U(q)$ and hence, $u = p$. Analogously, $u = p$ for $\mathrm{dist}(v,U) > \mathrm{dist}(q,U)$. Hence, we may assume $\mathrm{dist}(v,U) = \mathrm{dist}(q,U) =: d$.
Suppose $p \neq u$. Then $\mathrm{dist}(q,u) = d+1$ since $q \perp v$. This implies $p \perp u$ since p is a gate for q in U by Proposition 4.1.4. Now Corollary 4.2.8 implies that pu and qv are one-parallel to each other, a contradiction to Lemma 4.1.5. The claim follows since V is connected. □

Remark 4.3.5. For a point $p \in \mathscr{S}$ we denote by $\mathfrak{M}_p := \{V \in \mathfrak{M} \mid p \in V\}$ the set of maximal connected rigid subspaces that contain p. By Proposition 4.1.2 every two distinct elements of $\mathfrak{M}_p$ intersects in the point p. Let q be another point of $\mathscr{S}$. By Propositions 4.1.6 and 4.2.14 there is a bijection $\varphi\colon \mathfrak{M}_p \to \mathfrak{M}_q$ such that $V \parallel V^\varphi$ for every $V \in \mathfrak{M}_p$ and $V \nparallel U$ for every $U \in \mathfrak{M}_q \setminus \{V^\varphi\}$.

Proposition 4.3.6. *Let $\mathscr{S}$ be a twin SPO space and let x and y be opposite points of $\mathscr{S}$. Further let $\varphi\colon \mathfrak{M}_x \to \mathfrak{M}_y$ be the bijection with $V \parallel V^\varphi$ for every $V \in \mathfrak{M}_x$. Then $\mathscr{S} \cong \bigodot_{V\in\mathfrak{M}_x}(V \cup V^\varphi)$.*

Proof. Set $\mathscr{S}' := \bigodot_{V\in\mathfrak{M}_x}(V\cup V^{\varphi})$. For $V\in\mathfrak{M}_x$ let $\pi(V)\colon \mathscr{S}\to V\cup V^{\varphi}$ be the map with $p^{\pi(V)}\in \mathrm{pr}_{V\cup V^{\varphi}}(p)$ for every point $p\in\mathscr{S}$. Since either $\mathrm{dist}(p,V)<\infty$ or $\mathrm{dist}(p,V^{\varphi})<\infty$, this map exists. Moreover, by Proposition 4.1.4 this map is uniquely defined. Now define $\psi\colon \mathscr{S}\to\mathscr{S}'\colon p\mapsto (p^{\pi(V)})_{V\in\mathfrak{M}_x}$.
Let p and q be points of $\mathscr{S}$ with $p^{\psi}=q^{\psi}$. Since for $\mathrm{dist}(p,x)<\infty$ we obtain $p^{\psi}\in\bigodot_{V\in\mathfrak{M}_x}V$ and for $\mathrm{dist}(p,y)<\infty$ we obtain $q^{\psi}\in\bigodot_{V\in\mathfrak{M}_y}V$, the points p and q are in the same connected component of $\mathscr{S}$. Thus, we may assume $\mathrm{dist}(p,x)<\infty$ and $\mathrm{dist}(q,x)<\infty$. Suppose $p\neq q$. Then there is a line $l\leq\langle p,q\rangle_{\mathrm{g}}$ through p. Let $Y\leq\langle p,q\rangle_{\mathrm{g}}$ be a maximal rigid subspace of $\langle p,q\rangle_{\mathrm{g}}$ with $l\leq Y$. Further let $U\in\mathfrak{M}_p$ with $Y\leq U$ and let $V\in\mathfrak{M}_x$ with $V\parallel U$. Since by Proposition 2.1.23 the metaplecton $\langle p,q\rangle_{\mathrm{g}}$ is an SPO space, we may apply Lemma 4.1.4 to conclude that q has a gate q' in Y. Now set $Z:=\langle q,q'\rangle_{\mathrm{g}}$. Since $\mathrm{dist}(q,Y)=\mathrm{dist}(q,q')$, we obtain $Z\cap Y=\{q'\}$ by Proposition 2.1.17(i). Let $(q_i)_{0\leq i\leq n}$ be a geodesic from q to q'.
Let $r\in Z$ be an arbitrary point. Then by Proposition 4.1.4 r has a gate r' in U. We conclude $r'\in\langle r,q'\rangle_{\mathrm{g}}\leq Z$. Since $U\cap\langle p,q\rangle_{\mathrm{g}}=Y$, this implies $r'=q'$. Hence, every line q_iq_{i+1} for $i<n$ is contained in a maximal connected rigid subspace that is not one-parallel U and consequently, not one-parallel V. By Lemma 4.3.4 this implies $\mathrm{pr}_V(q_i)=\mathrm{pr}_V(q_{i+1})$ for every $i<n$ and therefore $\mathrm{pr}_V(q)=\mathrm{pr}_V(q')$.
Let p_x be the gate of p in V and let q_x be the gate of q in V. Since $\mathrm{dist}(q,l)=\mathrm{dist}(p,q)-1$ by Proposition 2.1.17(i), we know $\mathrm{dist}(q,Y)<\mathrm{dist}(p,q)$ and hence, $q'\neq p$. Since q_x is the gate for q' in V and U and V are one-parallel to each other, this implies $p_x\neq q_x$ by Proposition 4.1.7. This is equivalent to $q^{\pi(V)}\neq p_x$, a contradiction to $p^{\psi}=q^{\psi}$. Thus, ψ is injective.
Let $l\leq\mathscr{S}$ be a line. We may assume $\mathrm{dist}(x,l)<\infty$. By Proposition 4.1.2 there is a unique subspace $W\in\mathfrak{M}$ with $l\leq W$. Let $V\in\mathfrak{M}_x$ with $V\parallel W$. By Proposition 4.1.7 there is a line $l'\leq V$ that is one-parallel to l with $\mathrm{dist}(l,l')=\mathrm{dist}(W,V)$. Then $l^{\pi(V)}=l'$. By Lemma 4.3.4 we obtain that $l^{\pi(U)}$ is a singleton for every $U\in\mathfrak{M}_x\setminus\{V\}$. Thus, l^{ψ} is a line of $\mathscr{S}'$ and therefore ψ is an injective morphism of point-line spaces.
Let $(p_V)_{V\in\mathfrak{M}_x}$ be a point of $\mathscr{S}'$. We may assume $(p_V)_{V\in\mathfrak{M}_x}\in\bigodot_{V\in\mathfrak{M}_x}V$. Let M be the support of $(p_V)_{V\in\mathfrak{M}_x}$. By definition M is finite. Set $n:=|M|$ and let V_i for $0\leq i<n$ such that $M=\{V_i\mid 0\leq i<n\}$. Now set $p_0:=p_{V_0}$. Further we recursively define points $p_i\in\mathscr{S}$ for $0<i<n$ and subspaces $W_i\in\mathfrak{M}$ for $0\leq i<n$ as follows: Let $W_i\in\mathfrak{M}$ with $p_{i-1}\in W_i$ and $W_i\parallel V_i$. By Proposition 4.1.6 W_i is uniquely defined. Let p_i be the gate of p_{V_i} in W_i. In other words $p_i=p_{V_i}^{\pi(W_i)}$ and since $W_i\parallel V_i$, this implies ${p_i}^{\pi(V_i)}=p_{V_i}$ by Proposition 4.1.7. By Lemma 4.3.4 we obtain ${p_i}^{\pi(V)}={p_{i-1}}^{\pi(V)}$ for every $V\in\mathfrak{M}_x\setminus\{V_i\}$. Now ${p_0}^{\pi(V_0)}=p_{V_0}$ and ${p_0}^{\pi(V)}=p_V=x$ for $V\in\mathfrak{M}_x\setminus M$ by Lemma 4.3.4. Thus, induction provides ${p_{n-1}}^{\pi(V)}=p_V$ for every $V\in\mathfrak{M}_x$ and hence, ${p_{n-1}}^{\psi}=(p_V)_{V\in\mathfrak{M}_x}$. We conclude that ψ is surjective.

Now let $(q_V)_{V\in\mathfrak{M}_x}$ be a point of $\mathscr{S}'$ that is collinear and distinct to $(p_V)_{V\in\mathfrak{M}_x}$. Then there is a subspace $U \in \mathfrak{M}_x$ such that $p_U \perp q_U$ and $p_V = q_V$ for $V \in \mathfrak{M}_x \setminus \{U\}$. Hence, ${p_{n-1}}^{\pi(V)} = q_V$ for every $V \in \mathfrak{M}_x \setminus \{U\}$. Let $W_n \in \mathfrak{M}$ with $W_n \parallel U$ and let p_n be the gate of q_U in W_n. Then ${p_n}^{\psi} = (q_V)_{V\in\mathfrak{M}_x}$ as above. Since $p_U \perp q_U$, Proposition 4.1.7 implies that p_{n-1} and p_n are collinear points. We conclude that $(p_{n-1}p_n)^{\psi}$ equals the line of $\mathscr{S}'$ through $(p_V)_{V\in\mathfrak{M}_x}$ and $(q_V)_{V\in\mathfrak{M}_x}$. Thus, ψ is an isomorphism. ☐

We conclude this chapter with a fundamental property of twin SPO spaces. As a consequence of this property, for the classification of twin SPO spaces, we may restrain ourselves to the rigid ones.

Theorem 4.3.7. *A point-line space $\mathscr{S}$ is a twin SPO space if and only if there is a family of rigid twin SPO spaces $(\mathscr{S}_i)_{i\in I}$ for an index set I such that $\mathscr{S} = \bigodot_{i\in I}\mathscr{S}_i$.*

Proof. By Proposition 4.3.6 there are opposite points x and y in $\mathscr{S}$ such that $\mathscr{S} \cong \bigodot_{V\in\mathfrak{M}_x}(V \cup V^{\varphi})$, where $\varphi\colon \mathfrak{M}_x \to \mathfrak{M}_y$ is the bijection with $V \parallel V^{\varphi}$ for every $V \in \mathfrak{M}_x$. By Proposition 4.3.2 $V \cup V^{\varphi}$ is a rigid twin SPO space. The claim follows since every grid sum of rigid twin SPO spaces is a twin SPO space by Proposition 4.3.3. ☐

5 Twin spaces

In this chapter we study some twin spaces that arise from connected point-line spaces with finite diameter. First we introduce a method how to construct for a point-line space $\mathscr{S}^+$ with finite diameter a second point-line space $\mathscr{S}^-$ such that $(\mathscr{S}^+, \mathscr{S}^-)$ is a twin space. In this case, $\mathscr{S}^-$ has always the same diameter as $\mathscr{S}^+$. In a second method, we construct out of a point-line space $\mathscr{S}$ with finite diameter two point-line spaces $\mathscr{S}^+$ and $\mathscr{S}^-$ such that $(\mathscr{S}^+, \mathscr{S}^-)$ is a twin space. In this second approach the two obtained point-line spaces have the same diameter which can be infinite. As we will show, all these twin spaces are twin SPO spaces.

5.1 Twin spaces with finite diameter

In this section we consider a point-line space $\mathscr{S}^+$ of finite diameter n. From this point-line space we construct a new point-line space $\mathscr{S}^-$ whose points are subspaces of $\mathscr{S}^-$. More precisely, we take maximal convex subspaces of $\mathscr{S}^+$ such that there exists a point in $\mathscr{S}^+$ that has distance n to this subspace. We ask the point-line space $\mathscr{S}^+$ to have a sufficient regularity, namely, for two points p and q of $\mathscr{S}^+$, those maximal convex subspaces that have distance n to p and those having distance n to q should be of the same type. Moreover, every point r that has distance n to p should be contained in such a maximal convex subspace of distance n to p.

5.1.1 Twin polar spaces

The most intuitive case is the situation where the distance between a point and a line is always smaller than the diameter of $\mathscr{S}^+$. In this case the maximal convex subspaces that have distance $\mathrm{diam}(\mathscr{S}^+)$ to a given point are just singletons. Thus, $\mathscr{S}^-$ will be canonically isomorphic to $\mathscr{S}^+$.

Definition 5.1.1. Let $\mathscr{S}$ be a non-degenerate polar space. Further let $\mathscr{S}'$ be a copy of $\mathscr{S}$ and let φ be an isomorphism from $\mathscr{S}$ onto $\mathscr{S}'$. Then we call the pair of point-line spaces $(\mathscr{S}, \mathscr{S}')$ with the opposition relation $\{(x, y^{\varphi}), (x^{\varphi}, y) \mid \{x, y\} \subseteq \mathscr{S} \wedge x \not\perp y\}$ a *twin polar space*.

Proposition 5.1.2. *Every twin polar space is a twin space.*

Proof. By Proposition A.2.7 each non-degenerate polar space is partially linear. Furthermore, (OP) follows directly from (BS). □

Theorem 5.1.3. *Every twin polar space is a twin SPO space.*

Proof. Let $\mathscr{S} = (\mathscr{S}^+, \mathscr{S}^-)$ be a twin polar space. Further let $\varphi\colon \mathscr{S}^+ \to \mathscr{S}^-$ be the isomorphism such that $x \leftrightarrow y \Leftrightarrow x^{\varphi} \not\perp y$ for a pair of points $(x, y) \in \mathscr{S}^+ \times \mathscr{S}^-$. Let y and z be two points of $\mathscr{S}^-$. Then $\langle y, z\rangle_{\mathrm{g}}$ equals the line yz if y and z are distinct collinear points. If y and z are not collinear, then $\langle y, z\rangle_{\mathrm{g}} = \mathscr{S}^-$ by Proposition A.2.6.
Let $x \in \mathscr{S}^+$. By Lemma A.2.3(i) there is for every point $p \in \mathscr{S}^- \setminus \{x^{\varphi}\}$ a point $q \in \mathscr{S}^-$ with $p \perp q$ and $x^{\varphi} \not\perp q$. This implies $x \leftrightarrow q$ and we conclude that x^{φ} is the unique point of $\mathscr{S}^-$ at codistance 2 to x. Moreover, the points at distance 1 to x^{φ} have all codistance 1 to x. Therefore, $\mathrm{cod}(p, q^{\varphi}) = 2 - \mathrm{dist}(p, q)$ for two points p and q of $\mathscr{S}^+$. Since φ is an isomorphism, the codistance is symmetric.
Now let y and z be points of $\mathscr{S}^-$. For $\mathrm{dist}(y, z) = 1$, (A1) and (A2) follow directly from (BS). For $y = z$, there is nothing to prove and for $\mathrm{dist}(y, z) = 2$, (A1) and (A2) are fulfilled since $x^{\varphi} \in \langle y, z\rangle_{\mathrm{g}}$. Now assume $\mathrm{cod}(x, \langle y, z\rangle_{\mathrm{g}}) = \mathrm{cod}(x, z)$. Further let $w \in \mathscr{S}^+$ with $\mathrm{dist}(w, x) = 1$ and $\mathrm{cod}(w, y) = \mathrm{cod}(x, y) - 1$. This implies $x \not\leftrightarrow y$ and hence, $x \not\leftrightarrow z$. For $y = z$, (A3) is always true. The case $\mathrm{dist}(y, z) = 2$ is not possible, since in this case we obtain $z = x^{\varphi}$ and hence, $x \leftrightarrow y$. Therefore we may assume that $\langle y, z\rangle_{\mathrm{g}}$ is a line. Since x^{φ} is the only point at codistance 2 to x, we obtain $\mathrm{cod}(x, y) = 1$ and hence, $w \leftrightarrow y$. Thus, (A3) holds for $\mathrm{cod}(x, z) = 1$. For $z = x^{\varphi}$, we conclude $\mathrm{cod}(w, z) = 1$ since φ is an isomorphism and therefore $\mathrm{dist}(w^{\varphi}, x^{\varphi}) = 1$. Hence, (A3) is always satisfied. Finally, (A4) follows from the symmetry of the codistance. □

5.1.2 Twin projective spaces

The next class of point-line spaces we consider is the most famous one, the class of projective spaces. Here, the maximal convex subspaces that are at maximal distance to a given point are hyperplanes.

Definition 5.1.4. Let $\mathscr{S}$ be a projective space. Further let $\mathfrak{M}$ be a non-empty set of hyperplanes of $\mathscr{S}$ such that $\bigcap \mathfrak{M} = \varnothing$ and every hyperplane H of $\mathscr{S}$ that contains the intersection of two elements of $\mathfrak{M}$ is contained in $\mathfrak{M}$. Let $\mathfrak{S}$ be the set of subspaces of $\mathscr{S}$ that arise from intersecting two distinct elements of $\mathfrak{M}$ and set $\mathscr{L} := \{\{M \in \mathfrak{M} \mid S \leq M\} \mid S \in \mathfrak{S}\}$. Then we call the pair $(\mathscr{S}, (\mathfrak{M}, \mathscr{L}))$ with the opposition relation $\{(p,M),(M,p) \mid (p,M) \in \mathscr{S} \times \mathfrak{M} \wedge p \notin M\}$ a *twin projective space* of $\mathscr{S}$.

By the definition of $\mathscr{L}$ it is clear that $(\mathfrak{M}, \mathscr{L})$ is a point-line space. Furthermore, since for every point of a projective space there is a hyperplane not containing this point, it follows that for every projective space there exists a twin projective space.

Later on, we will see that a twin projective space $(\mathscr{S}, \mathscr{D})$ of a projective space $\mathscr{S}$ is a twin space. Therefore, as usual, we will call every twin space that is isomorphic to $(\mathscr{S}^+, \mathscr{S}^-)$ a twin projective space.

Lemma 5.1.5. *Let $(\mathscr{S}, \mathscr{D})$ be a twin projective space of the projective space $\mathscr{S}$. Let $\mathfrak{F} \subseteq \mathscr{D}$ be a non-empty finite subset of hyperplanes of $\mathscr{S}$. Then every hyperplane of $\mathscr{S}$ that contains $\bigcap \mathfrak{F}$ is a point of $\mathscr{D}$.*

Proof. By $\mathfrak{M}$ we denote the set of hyperplanes of $\mathscr{S}$ that are points of $\mathscr{D}$. We proceed by induction over the size of $\mathfrak{F}$. For $|\mathfrak{F}| = 1$, there is nothing to prove and for $|\mathfrak{F}| = 2$, the claim follows from the definition of the lines of $\mathscr{D}$. Now let $|\mathfrak{F}| > 2$ and assume that the claim holds for every subset of $\mathfrak{M}$ that has less elements than $\mathfrak{F}$.

Let $M \in \mathfrak{F}$ and set $S := \bigcap(\mathfrak{F} \setminus \{M\})$. By the induction hypothesis every hyperplane that contains S is an element of $\mathfrak{M}$. If $S \leq M$, there is nothing to prove. Therefore we assume $S \not\leq M$. Let N be a hyperplane of $\mathscr{S}$ that contains $S \cap M$. We have to show $N \in \mathfrak{M}$ and therefore may assume $M \neq N$. Then $M \cap N$ is a common hyperplane of M and N and thus, $\mathrm{crk}_{\mathscr{S}} M \cap N = 2$. Since M intersects S in a hyperplane and $S \cap M \leq M \cap N$, we conclude that $N' := \langle S, M \cap N \rangle$ is a hyperplane of $\mathscr{S}$. Since $S \leq N'$, we know $N \in \mathfrak{M}$ and $N' \neq M$. Since $M \cap N$ is a hyperplane of both M and N', we conclude $M \cap N' = M \cap N$. Thus, $N \in \mathfrak{M}$ follows from $\{M, N'\} \subseteq \mathfrak{M}$. ☐

Proposition 5.1.6. *Let $(\mathscr{S}, \mathscr{D})$ be a twin projective space of the projective space $\mathscr{S}$. Then $\mathscr{D}$ is a projective space.*

Proof. Let $\mathfrak{M}$ be the set of hyperplanes of $\mathscr{S}$ that are points of $\mathscr{D}$ and let $\mathscr{L}_{\mathrm{m}}$ be the line set of $\mathscr{D}$. By definition of $\mathscr{L}_{\mathrm{m}}$ we know that $\mathscr{D}$ is linear. Hence, it remains to show that (VY) holds.

Let $P \in \mathfrak{M}$ and let h_0 and h_1 be two distinct lines of $\mathscr{D}$ with $P \notin h_0 \cup h_1$. Further let g_0 and g_1 be two distinct lines of $\mathscr{D}$ that contain P and intersect both h_0 and h_1. We have to show that h_0 and h_1 intersect. For $i \in \{0,1\}$, let S_i be the subspace of corank 2 in $\mathscr{S}$ that is contained in every element of g_i. Then S_0 and S_1 are distinct hyperplanes of P and hence, $S := S_0 \cap S_1$ is a subspace of corank 3 in $\mathscr{S}$. Since

$\mathscr{D}$ is linear, we obtain $g_0 \cap g_1 = \{P\}$. Moreover, since $P \notin h_0$, we conclude for $i \in \{0,1\}$ that there is a hyperplane $P_i \in \mathfrak{M} \setminus \{P\}$ of $\mathscr{S}$ such that $h_0 \cap g_i = \{P_i\}$. Since $g_0 \cap g_1 = \{P\}$ and $P \neq P_0$, we obtain $P_0 \neq P_1$ and hence, $T_0 := P_0 \cap P_1$ is the subspace of corank 2 in $\mathscr{S}$ contained in all elements of h_0. This implies that every hyperplane of $\mathscr{S}$ that is an element of h_0 contains S. Let T_1 be the subspace of corank 2 in $\mathscr{S}$ that is contained in every element of h_1. Then $S \leq T_1$ by analogous reasons. Since S is a hyperplane of both T_0 and T_1, we conclude that $Q := \langle T_0, T_1 \rangle$ is a hyperplane of $\mathscr{S}$. By Lemma 5.1.5 we obtain $Q \in \mathfrak{M}$ since $P \cap P_0 \cap P_1 = S_0 \cap S_1 = S \leq Q$. Thus, Q is a common point of h_0 and h_1. ☐

Let $\mathscr{S}$ be a projective space and let $(\mathscr{S}, \mathscr{D})$ be the twin projective space of $\mathscr{S}$ such that every hyperplane of $\mathscr{S}$ is a point of $\mathscr{D}$. Then we call $\mathscr{D}$ the *dual* of the projective space $\mathscr{S}$.

Proposition 5.1.7. *Every twin projective space is a twin space.*

Proof. Let $\mathscr{S}$ be a projective space and let $(\mathscr{S}, \mathscr{D})$ be a twin projective space of $\mathscr{S}$. Further let $\mathfrak{M}$ be the set of hyperplanes of $\mathscr{S}$ that are points of $\mathscr{D}$. Since both $\mathscr{S}$ and $\mathscr{D}$ are projective spaces, both point-line spaces are partially linear. Since $\bigcap \mathfrak{M}$ is empty, there is for every point $p \in \mathscr{S}$ a hyperplane $H \in \mathfrak{M}$ with $p \notin H$. Conversely, for every hyperplane H of $\mathscr{S}$, there is a point in $\mathscr{S}$ that is not contained in p. Thus, the opposition relation of $(\mathscr{S}, \mathscr{D})$ is total.
Every line of $\mathscr{S}$ is contained in a given hyperplane or intersects this hyperplane in a single point. Conversely, let l be a line of $\mathscr{D}$. Then the elements of l have a subspace S with $\operatorname{crk}_{\mathscr{S}}(S) = 2$ in common. For an arbitrary point $p \in \mathscr{S}$ we obtain either $p \in S$ and hence, p is contained in every element of l or $p \notin S$ and hence, $\langle p, S \rangle$ is the unique element of l that contains p. The claim follows. ☐

Proposition 5.1.8. *Let $(\mathscr{S}^+, \mathscr{S}^-)$ be a twin projective space. Then $(\mathscr{S}^-, \mathscr{S}^+)$ is a twin projective space.*

Proof. Set $\mathfrak{M} := \{\operatorname{copr}_{\mathscr{S}^-}(p) \mid p \in \mathscr{S}^+\}$. Since $\mathscr{S}^-$ is singular and $(\mathscr{S}^+, \mathscr{S}^-)$ is a twin space, we know that $\operatorname{copr}_{\mathscr{S}^-}(p)$ is a hyperplane of $\mathscr{S}^-$ for every point $p \in \mathscr{S}^+$. Moreover, for every point $q \in \mathscr{S}^-$ there is a point $p \in \mathscr{S}^+$ with $q \notin \operatorname{copr}_{\mathscr{S}^-}(p)$. Thus, it remains to prove that $\varphi \colon \mathscr{S}^+ \to \mathfrak{M} \colon p \mapsto \operatorname{copr}_{\mathscr{S}^-}(p)$ is a bijection that maps a line of $\mathscr{S}^+$ onto the set of all hyperplanes of $\mathscr{S}^-$ that contain a given subspace of corank 2 of $\mathscr{S}^-$.
Let p and q be two distinct points of $\mathscr{S}^+$ and let $x \in \mathscr{S}^-$ with $p \leftrightarrow x$. Since $(\mathscr{S}^+, \mathscr{S}^-)$, there is a point r on the line pq such that $r \not\leftrightarrow x$. Hence, there is a point $y \in \mathscr{S}^-$ with $y \leftrightarrow r$ and therefore $z \neq y$. By definition of twin projective spaces both $\operatorname{copr}_{\mathscr{S}^+}(x)$ and $\operatorname{copr}_{\mathscr{S}^+}(y)$ are hyperplanes of $\mathscr{S}^+$. Since $r \notin \operatorname{copr}_{\mathscr{S}^+}(y)$ and $pq \cap \operatorname{copr}_{\mathscr{S}^+}(x) = \{r\}$, the subspaces $\operatorname{copr}_{\mathscr{S}^+}(x)$ and $\operatorname{copr}_{\mathscr{S}^+}(y)$ intersect in a subspace S which is disjoint to pq and has corank 2 in $\mathscr{S}^+$. Hence, $\langle p, S \rangle$ is a

hyperplane of $\mathscr{S}^+$ and by the definition of a twin projective space, there is a point z in $\mathscr{S}^-$ such that $\mathrm{copr}_{\mathscr{S}^+}(z) = \langle p, S \rangle$. We obtain $z \leftrightarrow q$ and $z \not\leftrightarrow p$ and therefore φ is bijective.
Set $H := \mathrm{copr}_{\mathscr{S}^-}(p) \cap \mathrm{copr}_{\mathscr{S}^-}(q)$. Since φ is bijective, we obtain $\mathrm{crk}_{\mathscr{S}^-}(H) = 2$. For every point $x \in H$, we know $p \not\leftrightarrow x \not\leftrightarrow q$ and therefore, $pq \leq \mathrm{copr}_{\mathscr{S}^+}(x)$ by (OP). Thus, $H \leq \mathrm{copr}_{\mathscr{S}^-}(r)$ for every point $r \in pq$. Conversely, every hyperplane of $\mathscr{S}^-$ that contains H, is of the kind $\langle y, H \rangle$ for a point $y \in \mathscr{S}^- \setminus H$. Since $\mathrm{copr}_{\mathscr{S}^+}(y)$ is a hyperplane of $\mathscr{S}^+$, we find a point $r \in pq$ with $r \not\leftrightarrow y$. Since $\mathrm{copr}_{\mathscr{S}^-}(r)$ contains both y and H, we obtain $\mathrm{copr}_{\mathscr{S}^-}(r) = \langle y, H \rangle$. This concludes the proof. ☐

Example 5.1.9. Consider the vector space $\mathbb{Q}^{(\mathbb{N})}$ of all infinite sequences of rational numbers that contain a finite number of non-zero elements. Denote by $\mathrm{PG}(\mathbb{Q}^{(\mathbb{N})})$ the projective space whose points are the 1-dimensional subspaces and whose lines are the 2-dimensional subspaces of $\mathbb{Q}^{(\mathbb{N})}$. Then the set of points of $\mathrm{PG}(\mathbb{Q}^{(\mathbb{N})})$ is of smaller cardinality as the set of the hyperplanes and even of lower rank. Moreover, the dual of the dual of $\mathrm{PG}(\mathbb{Q}^{(\mathbb{N})})$ is not isomorphic to $\mathrm{PG}(\mathbb{Q}^{(\mathbb{N})})$. This fact justifies to ask in the definition of twin projective spaces that the constructed point-line space does not necessarily contain all hyperplanes of the underlying projective space. Otherwise Proposition 5.1.8 would not be true anymore.

Remark 5.1.10. As a matter of fact, the rank of the dual of any projective space $\mathscr{S}$ is $\geq \mathrm{rk}(\mathscr{S})$. Furthermore, $\mathrm{PG}(\mathbb{Q}^{(\mathbb{N})})$ is a projective space of lowest possible infinite rank and there is no projective space whose dual is isomorphic to $\mathrm{PG}(\mathbb{Q}^{(\mathbb{N})})$. Nevertheless, since for a twin projective space $(\mathscr{S}^+, \mathscr{S}^-)$ the projective space $\mathscr{S}^-$ is isomorphic to a subspace of $\mathscr{S}^+$ there are twin projective spaces such that the two components are both isomorphic to $\mathrm{PG}(\mathbb{Q}^{(\mathbb{N})})$.

Theorem 5.1.11. *Every twin projective space is a twin SPO space of diameter ≤ 1.*

Proof. Let $(\mathscr{S}^+, \mathscr{S}^-)$ be a twin projective space. Since both $\mathscr{S}^+$ and $\mathscr{S}^-$ are projective spaces, all convex spans of two points at finite distance are either singletons or lines. Moreover, the maximal possible finite codistance is 1. By Proposition 5.1.7 we know that $(\mathscr{S}^+, \mathscr{S}^-)$ is a twin space. Thus, all axioms of Definition 2.1.1 follow immediately from (OP). ☐

5.1.3 Exceptional strongly parapolar spaces

The last class of point-line spaces we consider in this section is a class of point-line spaces arising from weak buildings; see Appendix B. At this point, we are interested in only two types, namely $\mathrm{E}_{6,1}$ and $\mathrm{E}_{7,1}$.

Let $\mathscr{S}$ be a point-line space of type $E_{6,1}$. Further let $\mathscr{P}_\mathrm{m}$ be the set of symplecta of $\mathscr{S}$ and let $\mathscr{L}_\mathrm{m} \subseteq \mathfrak{P}(\mathscr{P}_\mathrm{m})$ contain all sets of symplecta that intersect in a common generator. We call $(\mathscr{P}_\mathrm{m}, \mathscr{L}_\mathrm{m})$ the *dual* of $\mathscr{S}$.

Definition 5.1.12. Let $\mathscr{S} = (\mathscr{P}, \mathscr{L})$ be a point-line space of type $E_{6,1}$. Further let $\mathscr{S}_\mathrm{m} = (\mathscr{P}_\mathrm{m}, \mathscr{L}_\mathrm{m})$ be the dual of $\mathscr{S}$. Then we call the pair $(\mathscr{S}, \mathscr{S}_\mathrm{m})$ with the opposition relation $\{(x,Y),(Y,x) \mid \{x,Y\} \in \mathscr{P} \times \mathscr{P}_\mathrm{m} \wedge \operatorname{dist}(x,Y) = 2\}$ a *twin* E_6*-space*.

As usual, isomorphic images of a twin E_6-space are also called twin E_6-spaces.

Remark 5.1.13. Let $\mathscr{S} = (\mathscr{P}, \mathscr{L})$ be the point-line space of a weak building of type $E_{6,1}$ and let $\mathscr{S}_\mathrm{m} = (\mathscr{P}_\mathrm{m}, \mathscr{L}_\mathrm{m})$ be the dual of $\mathscr{S}$. From Theorem B.3.5 and by the symmetry of the diagram E_6 we conclude that $(\mathscr{P}_\mathrm{m}, \mathscr{L}_\mathrm{m})$ is again the point-line space of a weak building of type $E_{6,1}$. Moreover, every point $p \in \mathscr{P}$ represents a symplecton of $\mathscr{S}_\mathrm{m}$ which is the set of symplecta of $\mathscr{S}$ containing p. Therefore, the dual of $\mathscr{S}_\mathrm{m}$, denoted by $\mathscr{D}$, is canonically isomorphic to $\mathscr{S}$.
By Propositions B.3.6(iv) and B.3.6(ii) we conclude that two distinct symplecta of $\mathscr{S}$ intersect either in a point or in a common generator. Hence by Proposition B.3.6(iii), two symplecta of $\mathscr{S}$ are collinear in $\mathscr{S}_\mathrm{m}$ if they have a generator in common and they have distance 2 in $\mathscr{S}_\mathrm{m}$ if they intersect in a single point.
Let $p \in \mathscr{P}$ and $Y \in \mathscr{P}_\mathrm{m}$ such that $\operatorname{dist}(p,Y) = 2$ in $\mathscr{S}$. Further let Z be a symplecton of $\mathscr{S}$ that contains p. Since every line of Z has distance ≤ 1 to p, the symplecta Z and Y have no line in common. Thus, every symplecton of $\mathscr{S}$ containing p is non-collinear to Y in $\mathscr{S}_\mathrm{m}$. This implies that the symplecton of $\mathscr{S}_\mathrm{m}$ which is represented by p has distance 2 to Y in $\mathscr{S}_\mathrm{m}$. Therefore we conclude that the twin E_6-space $(\mathscr{S}_\mathrm{m}, \mathscr{D})$ is canonically isomorphic to $(\mathscr{S}_\mathrm{m}, \mathscr{S})$ using as opposition relation for $(\mathscr{S}_\mathrm{m}, \mathscr{S})$ the opposition relation of the twin E_6-space $(\mathscr{S}, \mathscr{S}_\mathrm{m})$.
Thus, a pair of point-line spaces $(\mathscr{S}^+, \mathscr{S}^-)$ with an opposition relation $\leftrightarrow$ is a twin E_6-space if and only if $(\mathscr{S}^-, \mathscr{S}^+)$ with opposition relation $\leftrightarrow$ is a twin E_6-space.

Proposition 5.1.14. *Every twin* E_6*-space is a twin space.*

Proof. Let $\mathscr{S} = (\mathscr{P}, \mathscr{L})$ be a point-line space of type $E_{6,1}$ and denote the dual of $\mathscr{S}$ by $\mathscr{S}_\mathrm{m} = (\mathscr{P}_\mathrm{m}, \mathscr{L}_\mathrm{m})$. Since $\mathscr{S}$ is a parapolar space by Theorem B.3.5, it is partially linear. By Proposition B.3.6(vi), the opposition relation of $(\mathscr{S}, \mathscr{S}_\mathrm{m})$ is total. Hence by Remark 5.1.13 it suffices to show that for a point $p \in \mathscr{P}$ and a symplecton $Y \in \mathscr{P}_\mathrm{m}$ with $\operatorname{dist}(p,Y) = 2$ in $\mathscr{S}$, there is on every line $l \in \mathscr{L}$ through p exactly one point at distance 1 to Y.
Since $\mathscr{S}$ is a parapolar space, there is a symplecton $Z \in \mathscr{P}_\mathrm{m}$ containing l. By Proposition B.3.6(iv) the symplecta Z and Y intersect. Since $\operatorname{dist}(p,Y) = 2$, we conclude that Y and Z have no line in common and therefore Y and Z intersect

in a single point q. Since $\mathrm{dist}(q,p) = 2$ there is exactly one point p' on l that is collinear to q. Hence by Proposition B.3.6(v), p' is the only point on l at distance 1 to Y. $\square$

Theorem 5.1.15. *Every twin* E_6*-space is a twin SPO space.*

Proof. Let $\mathscr{S} = (\mathscr{P},\mathscr{L})$ be a point-line space of type $\mathrm{E}_{6,1}$ and denote the dual of $\mathscr{S}$ by $\mathscr{S}_\mathrm{m} = (\mathscr{P}_\mathrm{m},\mathscr{L}_\mathrm{m})$. We show that the twin E_6-space $(\mathscr{S},\mathscr{S}_\mathrm{m})$ fulfils the axioms of Definition 2.1.1.
Let $x \in \mathscr{P}$ and let $Y \in \mathscr{P}_\mathrm{m}$ such that $\mathrm{dist}(x,Y) = 1$ in $\mathscr{S}$. Then by Proposition B.3.6(vi) there is a point $y \in \mathscr{P}$ such that $\mathrm{dist}(y,Y) = 2$. Since $\mathscr{S}$ is a strongly parapolar space, we conclude by Proposition B.3.6(iii) that there is a symplecton $Z \in \mathscr{P}_\mathrm{m}$ that contains both x and y. Since $y \in Z$ and $\mathrm{dist}(y,Y) = 2$ there is no line of $\mathscr{S}$ contained in $Y \cap Z$. Thus, Proposition B.3.6(iv) implies that Y and Z intersect in a single point p. By Proposition B.3.6(v) we may assume that y is a point with $y \perp x$ and $\mathrm{dist}(p,y) = 2$. Hence, $y \leftrightarrow Y$ and consequently, $\mathrm{cod}(Y,x) = 1$. Since $x \notin Y$, there is a point $z \in Y$ with $z \perp p$ and $\mathrm{dist}(x,p) = 2$. Then $\langle x,z\rangle_\mathrm{g}$ is a symplecton that contains the line pz. Since $pz \leq Y$, Proposition B.3.6(ii) implies that $\langle x,z\rangle_\mathrm{g}$ and Y are collinear points of $\mathscr{S}_\mathrm{m}$. Thus, the symplecton of $\mathscr{S}_\mathrm{m}$ consisting of all elements of $\mathscr{P}_\mathrm{m}$ that contain x has distance 1 to Y. By Remark 5.1.13 this implies $\mathrm{cod}(x,Y) = 1$. Therefore (A4) holds and we conclude $\mathrm{cod}(x,Y) = 2 - \mathrm{cod}(x,Y)$ for any pair $(x,Y) \in \mathscr{P} \times \mathscr{P}_\mathrm{m}$.
Let y and z be points of $\mathscr{S}$ and set $V := \langle y,z\rangle_\mathrm{g}$. Further let $X \in \mathscr{P}_\mathrm{m}$. By Remark 5.1.13 it suffices to show that (A1), (A2) and (A3) hold for X, y and z. For $y = z$, we obtain $V = \{y\}$ and hence there is nothing to prove. By Proposition B.3.6(iii) this leaves the cases $\mathrm{dist}(y,z) = 1$ and $\mathrm{dist}(y,z) = 2$. Since $\mathscr{S}$ is a strongly parapolar space, we know that V is a line if $\mathrm{dist}(y,z) = 1$ and V is a symplecton if $\mathrm{dist}(y,z) = 2$. First assume that X contains a point $x \in \mathscr{P}$ with $x \leftrightarrow X$. Then there is a line l through x and a symplecton $Y \in \mathscr{P}_\mathrm{m}$ with $l \leq V \leq Y$. Since $\mathrm{dist}(x,X) = 2$ and every line of Y has at most distance 1 to x, we conclude by Proposition B.3.6(iv) that Y and X intersect in a single point p. Moreover, there is a unique point on l that is collinear to p and hence by Proposition B.3.6(v) there is a unique point on l at distance 1 to X. Since V equals either l or Y, we conclude that (A1) and (A2) are fulfilled.
Now V does not necessarily contain a point opposite to X. Assume $z \in \mathrm{copr}_V(X)$ and hence, $\mathrm{dist}(z,X) = \mathrm{dist}(V,X)$ in $\mathscr{S}$. Further let $W \in \mathscr{P}_\mathrm{m}$ be a symplecton of $\mathscr{S}$ such that $\mathrm{dist}(y,W) = \mathrm{dist}(y,X) + 1$. If $\mathrm{dist}(y,X) = 0$, then $\mathrm{dist}(z,X) = 0$ and hence $V \leq X$ since X is convex. Thus, $V = \mathrm{copr}_V(X)$ and (A3) holds. Therefore we may assume $\mathrm{dist}(y,X) \geq 1$. By Proposition B.3.6(iii) this implies $\mathrm{dist}(y,X) = 1$ and $\mathrm{dist}(y,W) = 2$. First assume $\mathrm{dist}(z,X) = 1$. Then $X \cap V = \varnothing$ and hence, V is a line by Proposition B.3.6(iv). Since (A2) holds, we know that there is no point on V that is opposite X. Thus, $V = \mathrm{copr}_V(X)$ and (A3) is fulfilled since $y \leftrightarrow W$.

It remains the case $z \in X$ and $\operatorname{dist}(y,X) = 1$. If V is a line, then there is a unique point on V that is not opposite W since (A2) holds. Since W and X are collinear points of $\mathscr{S}_{\mathrm{m}}$, we conclude that z is the unique point on V not opposite to W and hence, (A3) is fulfilled. If V is a symplecton, then V intersects both X and W by Proposition B.3.6(iv). Thus, $z \in X$. Moreover, since $\operatorname{dist}(y,W) = 2$ and every line of V has at most distance 1 to y, there is a point $p \in \mathscr{P}$ such that $V \cap W = \{p\}$. By Proposition B.3.6(v) we know $X \cap V > \{z\}$ since $\operatorname{dist}(y,X) < \operatorname{dist}(y,z)$. Thus, X and V intersect in a common generator G by Proposition B.3.6(ii). Since W and X are collinear points of $\mathscr{S}_{\mathrm{m}}$, we obtain $\operatorname{cod}(q,W) \geq 1$ and hence, $\operatorname{dist}(q,W) \leq 1$ for every point $q \in G$. By Proposition B.3.6(v) this implies $G \leq p^{\perp}$ and therefore $p \in G$ since G is a maximal singular subspace of V. We conclude that (A3) is satisfied. ☐

We conclude this section by considering point-line spaces of type $\mathrm{E}_{7,1}$.

Definition 5.1.16. Let $\mathscr{S}$ be the point-line space of type $\mathrm{E}_{7,1}$. Further let $\mathscr{S}'$ be a copy of $\mathscr{S}$ and let φ be an isomorphism from $\mathscr{S}$ onto $\mathscr{S}'$. Then we call the pair $(\mathscr{S},\mathscr{S}')$ with the opposition relation $\{(x^{\varphi},y),(x,y^{\varphi}) \mid \{x,y\} \subseteq \mathscr{S} \wedge \operatorname{dist}(x,y) = 3\}$ a *twin* E_7*-space*.

Proposition 5.1.17. *Every twin* E_7*-space is a twin space.*

Proof. Let $(\mathscr{S}^+,\mathscr{S}^-)$ be a twin E_7-space and let φ be the isomorphism from $\mathscr{S}^+$ onto $\mathscr{S}^-$ such that $p \leftrightarrow q$ if and only if $\operatorname{dist}(p^{\varphi},q) = 3$ for a pair of points $(p,q) \in \mathscr{S}^+ \times \mathscr{S}^-$. Let $(p,q) \in \mathscr{S}^+ \times \mathscr{S}^-$ be a pair of opposite points and let $l \leq \mathscr{S}^-$ be a line through q. By Proposition B.3.7(iv) we know $\operatorname{dist}(p^{\varphi},l) = 2$. Moreover, Proposition B.3.7(iii) implies that the opposition relation is total.
It remains to show that on a line $l \leq \mathscr{S}^-$ that contains two points at distance 2 to p^{φ} there is no point opposite to p. We may assume $\operatorname{dist}(p^{\varphi},l) = 2$ since otherwise we are done. Let q and q' be distinct points on l at distance 2 to p^{φ}. Then Proposition B.3.7(i) implies that $Y := \langle p^{\varphi},q\rangle_{\mathrm{g}}$ and $Z := \langle p^{\varphi},q'\rangle_{\mathrm{g}}$ are both symplecta. Hence by Proposition B.3.7(ii), there is a line $g \leq Y \cap Z$ through p^{φ}. Since Y is a polar space, there is a unique point s on g that is collinear to q. Analogously, there is a point s' on g collinear to q'. Suppose $s \neq s'$. Then $\operatorname{dist}(q,s') = 2$ and $\langle q,s'\rangle_{\mathrm{g}}$ is a symplecton that contains s and q'. Hence, g and l are both contained in $\langle q,s'\rangle_{\mathrm{g}}$, a contradiction since $\operatorname{dist}(p^{\varphi},l) = 2$. Thus, $s = s'$ and since $\mathscr{S}^-$ is a gamma space, we obtain $l \leq s^{\perp}$. Therefore, every point on l has distance 2 to p^{φ}. ☐

Theorem 5.1.18. *Every twin* E_7*-space is a twin SPO space.*

Proof. Let $(\mathscr{S}^+,\mathscr{S}^-)$ be a twin E_7-space and let φ be the isomorphism from $\mathscr{S}^+$ onto $\mathscr{S}^-$ such that $p \leftrightarrow q$ if and only if $\operatorname{dist}(p^{\varphi},q) = 3$ for a pair of points $(p,q) \in$

$\mathscr{S}^+ \times \mathscr{S}^-$. By Proposition B.3.7(iii) we conclude $\mathrm{cod}(p,q) = 3 - \mathrm{dist}(p^\varphi, q)$ for a pair of points $(p,q) \in \mathscr{S}^+ \times \mathscr{S}^-$. This implies that (A4) is fulfilled.
Let y and z be points of $\mathscr{S}^-$ and set $V := \langle y,z\rangle_{\mathrm{g}}$. Further let x be a point of $\mathscr{S}^+$. We have to check (A1), (A2) and (A3) for x, y and z. For $y = z$, we obtain $V = \{y\}$ and there is nothing to prove. Now assume $\mathrm{dist}(y,z) = 3$. By Proposition B.3.7(iv) we know that all lines through y and all lines through z are contained in V. Moreover, by Proposition B.3.7(iii) we know that for every point collinear to y there is a point at distance 3 that is collinear to z. Therefore we conclude that all points that are connected to y are contained in V and hence, $V = \mathscr{S}$. Since x^φ is the only point of $\mathscr{S}^-$ at codistance 3 to x, we conclude that (A1), (A2) and (A3) hold. Hence, we may restrain ourselves to the cases $\mathrm{dist}(y,z) = 1$ and $\mathrm{dist}(y,z) = 2$.
Since $\mathscr{S}^-$ is a strongly parapolar space, we know that V is a line if $\mathrm{dist}(y,z) = 1$ and V is a symplecton if $\mathrm{dist}(y,z) = 2$. If V is a line, then (A1) and (A2) are fulfilled since $(\mathscr{S}^+, \mathscr{S}^-)$ is a twin space. If V is a symplecton and contains a point that is opposite x, then by Proposition B.3.7(iv) there is a point $p \in V$ with $\mathrm{dist}(x^\varphi, p) = 2$. By Proposition B.3.7(ii) the symplecta V and $\langle x^\varphi, p\rangle_{\mathrm{g}}$ intersect in a line and hence, there is a point $x' \in V$ with $x^\varphi \perp x'$. Suppose there is a second point x'' in V that is collinear to x^φ. Then $x'' \perp x'$ since $x^\varphi \notin V$. Since $\mathscr{S}^-$ is a gamma space all points on $x'x''$ are collinear to x^φ. Since V is a polar space every point of V has at most distance 2 to x^φ, a contradiction. Thus, x' is the unique point of V collinear to x^φ and we conclude $\mathrm{copr}_V(x) = \{x'\}$. Thus, (A1) and (A2) are fulfilled.
Now assume $z \in \mathrm{copr}_V(x)$ and let $w \perp x$ be a point with $\mathrm{cod}(w,y) = \mathrm{cod}(x,y) - 1$. If $x^\varphi \in V$, then $z = x^\varphi$. If V is a line then $\mathrm{cod}(x,y) = 2$ and hence, $\mathrm{cod}(w,y) = 1$. Since $w \perp x$, we obtain $\mathrm{cod}(w,z) = 2$. Since $\mathscr{S}^-$ is a gamma space, z is the only point on V that is collinear to w^φ and therefore (A3) holds. If V is a symplecton, then $\mathrm{cod}(x,y) = 1$ and hence, $w \leftrightarrow y$. Thus, V contains a unique point at codistance 2 to w. Since $w \perp x$, we obtain $\mathrm{copr}_V(w) = \{z\}$ and (A3) is satisfied. Therefore we may assume $x^\varphi \notin V$. If $\mathrm{cod}(x,y) = 2$. Then both y and z are collinear to x^φ and we conclude that V is a line and all points of V are collinear to x^φ. Since $\mathrm{cod}(w,y) = 1$ and $w \perp x$, we know that all points of V have codistance 1 or 2 to w and hence, (A3) holds.
It remains the case $\mathrm{cod}(x,y) = 1$ and $x^\varphi \notin V$. This implies $w \leftrightarrow y$. First assume that V is a line. Then there is a unique point at codistance 1 to w in V. If $\mathrm{cod}(x,z) = 2$, this implies that z is the unique point on V not opposite to w since $w \perp x$. If $\mathrm{cod}(x,z) = 1$, then there is no point V opposite to x since $(\mathscr{S}^+, \mathscr{S}^-)$ is a twin space. Thus, (A3) holds in both cases. Now assume V is a symplecton. Since $\mathrm{dist}(x^\varphi, y) = 2$, we know that $\langle x^\varphi, y\rangle_{\mathrm{g}}$ is a symplecton. By Proposition B.3.7(ii) the symplecta V and $\langle x^\varphi, y\rangle_{\mathrm{g}}$ intersect in a line g through y and hence $\mathrm{dist}(x^\varphi, V) \leq 1$. Thus, there is a point $z' \in g$ with $\mathrm{cod}(x,z') = 2$ and consequently, $\mathrm{cod}(x,z) = 2$. Since $\mathrm{dist}(y,z) = 2$, we obtain $z \neq z'$. Since $x^\varphi \notin V$, we obtain $z \perp z'$.

Since $w \leftrightarrow y$ we know that there is a unique point in w' in V with $\mathrm{cod}(w,w') = 2$. If $\mathrm{cod}(x,w') = 2$, (A3) holds and we are done. Thus, we suppose $\mathrm{cod}(x,w') = 1$. This implies $\mathrm{cod}(w,z) = \mathrm{cod}(w,z') = 1$ since $w \perp x$. Hence, $\langle w^\varphi, z\rangle_{\mathrm{g}}$ is a symplecton and by Proposition B.3.7(ii) there is a line through z in $V \cap \langle w^\varphi, z\rangle_{\mathrm{g}}$. Since this line contains a point that is collinear to w^φ, we conclude that this line goes through w'. Thus, $w' \perp z$ and analogously, $w' \perp z'$. Since $\mathrm{dist}(x^\varphi, w') = 2$, this implies that both z and z' are contained in the symplecton $Z := \langle x^\varphi, w'\rangle_{\mathrm{g}}$. Moreover, since φ is an isomorphism, we obtain $w^\varphi \perp x^\varphi$. Since $\mathrm{cod}(w,w') = 2$, we know $w^\varphi \perp w'$ and therefore $w^\varphi \in Z$. Since $w^\varphi \notin V$, the intersection of Z and V is singular. Both $y^\perp$ and $(w^\varphi)^\perp$ contain a hyperplane of $Z \cap V$. Since $w \leftrightarrow y$, we conclude $y^\perp \cap (w^\varphi)^\perp = \varnothing$. Thus, $\mathrm{rk}(Z \cap V) \leq 1$. Since both z and z' are contained in $Z \cap V$, we conclude $Z \cap V = zz'$. Now $w' \in Z \cap V$ implies $w' \in zz'$. Since both z and z' are collinear to x^φ and $\mathrm{dist}(x^\varphi, w') = 2$, this is a contradiction to the fact that $\mathscr{S}^-$ is a gamma space. Thus, (A3) holds in all cases. □

5.2 Dual polar spaces

As a consequence of Proposition A.2.24 a dual polar space is disconnected whenever the underlying polar space $\mathscr{S}$ has infinite rank. Moreover, two generators M and M' of $\mathscr{S}$ are connected in the dual polar space if and only if they are commensurate. Thus, viewing just the dual polar space, we lose some information: For instance we cannot tell the rank of $M \cap M'$ if M and M' are contained in distinct connected components of the dual polar space. In this chapter we introduce a method how to construct out of polar space a twin space which is (viewed as the union of its components and without taking the opposition relation into account) a substructure of the dual polar space. Thereby we gain information compared to the dual polar space for the generators that are involved.

Throughout this section let $\mathscr{S}$ be a polar space and let $\mathscr{S}_{\mathrm{m}}$ be the dual polar space of $\mathscr{S}$. By $\mathrm{dist}_{\mathscr{D}}$ we denote the distance function in $\mathscr{S}_{\mathrm{m}}$.

The goal of this section is to show that the twin spaces that we construct out of $\mathscr{S}$ are twin SPO spaces. Since this twin spaces consist of subspaces of $\mathscr{S}_{\mathrm{m}}$ it is useful to know what the convex span of two points of $\mathscr{S}_{\mathrm{m}}$ at finite distance looks like.

Proposition 5.2.1. *Let M and N be generators of $\mathscr{S}$ with $\mathrm{crk}_M(M \cap N) < \infty$. Let G be the convex span of M and N in $\mathscr{S}_{\mathrm{m}}$.*

(i) *A generator $L \leq \mathscr{S}$ belongs to G if and only if $L \geq M \cap N$.*

(ii) *For every generator $L \leq \mathscr{S}$ with $M \cap N \leq L$, there is a generator L' such that $L \cap L' = M \cap N$.*

Proof. (i) Let H be the set of all generators of $\mathscr{S}$ containing $S := M \cap N$. Let K and L be distinct adjacent generators contained in H. Then $K \cap L \geq S$ and therefore all generators containing $K \cap L$ belong to H. Thus, H is a subspace of $\mathscr{S}_{\mathrm{m}}$. Now let K and L be two arbitrary generators of H with $\mathrm{dist}_{\mathscr{D}}(K,L) = k > 1$ and let $L' \in \mathscr{S}_{\mathrm{m}}$ with $\mathrm{dist}_{\mathscr{D}}(K,L') = k-1$ and $L \sim L'$. By Proposition A.2.20 there is a point $p \in K \cap L' \setminus L$. We obtain $L' = p \oplus L$ by Lemma A.2.16. Since $K \cap L \leq p^{\perp}$, we conclude $S \leq K \cap L \leq L'$. Hence, H is convex and therefore $G \leq H$.

To prove $H \leq G$ we apply induction over n. For $n = 0$ there is nothing to prove. For $n = 1$ we obtain $H = G$ by the definition of the lines in $\mathscr{S}_{\mathrm{m}}$. Now let $n > 1$ and let L be a generator of $\mathscr{S}$ with $S \leq L$. Assume there is a point $p \in L \cap M \setminus S$. Set $N' := p \oplus N$. Then $\mathrm{dist}_{\mathscr{D}}(M,N') = n-1$ and therefore $M \cap N' = \langle p,S \rangle$. Since $\langle p,S \rangle \leq L$, we may apply the induction hypothesis to conclude $L \in \langle M,N' \rangle_{\mathrm{g}}$ (here M and N' are treated as points of $\mathscr{S}_{\mathrm{m}}$ and hence the convex span of them is a subspace of $\mathscr{S}_{\mathrm{m}}$). Since $N' \in G$, this implies $L \in G$. Therefore we may assume $M \cap L = S$ and analogously $N \cap L = S$.

Let $p \in L \setminus S$. Set $M' := p \oplus M$ and $N' := p \oplus N$. Assume there is a point $q \in N \setminus S$ with $q \in M'$. Then $M' = q \oplus M$ and hence $\mathrm{dist}_{\mathscr{D}}(M',N) = n-1$. Thus, $M' \in G$ since $M \sim M'$. The line pq meets M in a point r since M intersects M' in a hyperplane. This implies $r \in M \cap N'$. With $pq \cap N = \{q\}$ we obtain $r \in M \setminus S$. Therefore we conclude $M' \cap N > S$ if and only if $M \cap N' > S$.

First let $M' \cap N = S$. Then $\mathrm{dist}_{\mathscr{D}}(M',N) = n$ and hence $\mathrm{dist}_{\mathscr{D}}(M',N') = n-1$ since $\langle p,S \rangle \leq M' \cap N'$. Since $\langle p,S \rangle \leq L$, the induction hypothesis provides $L \in \langle M',N' \rangle_{\mathrm{g}}$. Since $S \leq M' \cap M$ and $\mathrm{crk}_M(S) = n$, there is a singular subspace $U \leq M \cap M'$ with $\mathrm{rk}(U) = n-2$ and $S \cap U = \varnothing$. This implies $N \cap U = \varnothing$ and therefore $\mathrm{crk}_N(U^{\perp} \cap N) = n-1$ by Lemma A.2.22(i). Since $S \leq M \leq U^{\perp}$, we conclude that S is a hyperplane of $U^{\perp} \cap N$. Hence, there is a point $q \in N \setminus S$ with $U \leq q^{\perp}$. Set $M'' := q \oplus M$. Then $M \cap M' = \langle U,S \rangle \leq M''$ since $U \leq M \cap q^{\perp}$. Thus, M, M' and M'' lie on a common line in $\mathscr{S}_{\mathrm{m}}$. Since $M \sim M''$ and $\mathrm{dist}_{\mathscr{D}}(M'',N) = n-1$, we obtain $M'' \in G$. Hence, the line in $\mathscr{S}_{\mathrm{m}}$ that contains M and M'' is entirely contained in G and thus, $M' \in G$. Analogously, $N' \in G$ and we conclude $\langle M',N' \rangle_{\mathrm{g}} \leq G$. This implies $L \in G$.

It remains the case $M' \cap N > S$. Hence, we may assume $(r \oplus M) \cap N > S$ and $(r \oplus N) \cap M > S$ for every point $r \in L \setminus S$. Let $q \in M' \cap N \setminus S$. Then $M' = q \oplus M$ and therefore $\mathrm{dist}_{\mathscr{D}}(M',N) = n-1$. This implies $M' \in G$. Since $q \in N \setminus S$, we obtain $q \notin L$ and since L is a generator, there is a point $r \in L \setminus q^{\perp}$. Set $N'' := r \oplus N$. Then $M' \cap N'' < M' \cap N$ by Lemma A.2.17. Since $S \leq M' \cap N''$ and $M' \cap N = \langle q,S \rangle$, we conclude $M' \cap N'' = S$ and hence, $\mathrm{dist}_{\mathscr{D}}(M',N'') = n$. Since $p \in L \cap M'$ and $p \notin S$, we obtain $L \in \langle M',N'' \rangle_{\mathrm{g}}$ as above. Now $r \in L \setminus S$ implies $(r \oplus N) \cap M > S$. Hence, $\mathrm{dist}_{\mathscr{D}}(M,N'') = n-1$ and therefore $N'' \in G$. Thus, $L \in G$ and we conclude $H = G$.

(ii) By Proposition A.2.20 every two elements of H have finite distance in $\mathscr{S}_{\mathrm{m}}$.

Since G is a convex subspace of $\mathscr{S}_{\mathrm{m}}$ we may restrain ourselves to the case $L \sim M$. Furthermore, we may assume $L \cap N > S$ since otherwise there is nothing to prove. Since L and M have a hyperplane in common, S is a hyperplane of $L \cap N$. Let $p \in (L \cap N) \setminus S$. Then $p \notin M$ and by the maximality of M there is a point $q \in M$ that is not collinear p. By Lemma A.2.17 we conclude $(q \oplus N) \cap L < L \cap N$. Since $S \leq q^{\perp}$, this implies $(q \oplus N) \cap L = S$. □

5.2.1 Spanning pairs

An opposition relation in a twin space denotes the pairs of points that should be seen as points at maximal distance. By Proposition A.2.20 we know that for two generators M and N of $\mathscr{S}$ with $\mathrm{dist}_{\mathscr{D}}(M,N) < \infty$ the corank of $M \cap N$ in M equals $\mathrm{dist}_{\mathscr{D}}(M,N)$. In other words, the smaller the intersection of two generators the greater is their distance in the dual polar graph. The smallest intersection two generators can possibly have is if they intersect in the radical.

In polar spaces of arbitrary rank it might happen that there is a line in the dual polar space such that all generators of the polar space that are elements of this line intersect a given generator in the radical. By Definition 1.2.4 this implies that the pairs of generators that intersect in the radical do not always give rise to an opposition relation for a twin space. The aim of this subsection is to introduce an extra condition to resolve this problem:

Definition 5.2.2. Let M_+ and M_- be two generators of $\mathscr{S}$ such that for every point $p \in \mathscr{S}$ there are points $p_+ \in M_+$ and $p_- \in M_-$ with $p \in (M_+ \cup \{p_-\})^{\perp\perp} \cap (M_- \cup \{p_+\})^{\perp\perp}$. Then we call (M_+, M_-) a *spanning pair*.

Proposition 5.2.3. *Let (M_+, M_-) be a spanning pair of $\mathscr{S}$. Then $M_+ \cap M_- =$* $\mathrm{Rad}(\mathscr{S})$.

Proof. Since M_+ and M_- are both maximal, we obtain $\mathrm{Rad}(\mathscr{S}) \leq M_+ \cap M_-$. Now let $p \in \mathscr{S}$ and $q \in M_+ \cap M_-$. Then there is a point $p_+ \in M_+$ such that $p \in (M_- \cup \{p_+\})^{\perp\perp}$. Thus, $q \perp p$ since $q \in (M_- \cup \{p_+\})^{\perp}$ and therefore $q \in \mathrm{Rad}(\mathscr{S})$. □

A direct consequence of this proposition is that in a non-degenerate polar space the two generators of a spanning pair are always disjoint. The dual polar space of a polar space is isomorphic to the dual polar space of the associated non-degenerate polar space, see Theorem A.2.15. Moreover, as a consequence of Proposition A.2.10 and Lemma A.2.9(v) we know the intersection of two generators whenever we know the intersection of the corresponding generators in the associated non-degenerate polar space. In the following we consider subspaces of the dual polar space and generators as well as intersections of generators. Hence, we may restrain ourselves to non-degenerate polar spaces. Thus, in the remainder of this section $\mathscr{S}$ is always a non-degenerate polar space. Generalising the statements to the case of arbitrary polar spaces is straightforward and without any additional interest.

The following proposition gives two alternative conditions that characterise a spanning pair. Particularly condition (b) will be used quite often in the following to prove that a pair of generators is a spanning pair.

Proposition 5.2.4. *Let M_+ and M_1 be two generators of $\mathscr{S}$. Then the following statements are equivalent:*

(a) *(M_+, M_-) is a spanning pair.*

(b) *For $\sigma \in \{+,-\}$ and $p \in \mathscr{S} \setminus (M_+ \cup M_-)$, there is a point $p_\sigma \in M_\sigma$ with $p^\perp \cap M_{-\sigma} = {p_\sigma}^\perp \cap M_{-\sigma}$.*

(c) *For $\sigma \in \{+,-\}$ and $p \in \mathscr{S} \setminus (M_+ \cup M_-)$, there is a non-empty subspace $U_\sigma \leq M_\sigma$ of finite rank with $p^\perp \cap M_{-\sigma} \geq {U_\sigma}^\perp \cap M_{-\sigma}$.*

Proof. Note that for (b) and (c) the cases $\sigma = +$ and $\sigma = -$ are analogous.
(a) $\Rightarrow$ (b): Let $p \in \mathscr{S} \setminus (M_+ \cup M_-)$. Then there is a point $p_+ \in M_+$ with $p \in (M_- \cup \{p_+\})^{\perp\perp}$ and hence $(M_- \cup \{p_+\})^\perp \leq p^\perp$. Since M_- is a generator, we obtain ${M_-}^\perp = M_-$ and therefore $(M_- \cup \{p_+\})^\perp = {p_+}^\perp \cap M_-$. This implies ${p_+}^\perp \cap M_- \leq p^\perp \cap M_-$. The claim follows since $p^\perp \cap M_-$ and ${p_+}^\perp \cap M_-$ are both hyperplanes of M_-.
(b) $\Rightarrow$ (a): First let $p \in \mathscr{S} \setminus (M_+ \cup M_-)$. Then there is a point $p_+ \in M_+$ such that $p^\perp \cap M_- = {p_+}^\perp \cap M_-$. Since M_- is a generator, we conclude $(M_- \cup \{p_+\})^{\perp\perp} = (M_- \cap {p_+}^\perp)^\perp = (M_- \cap p^\perp)^\perp \geq (p^\perp)^\perp \ni p$. Now let $p \in M_-$. Then $p \in (M_- \cup \{p_+\})^{\perp\perp}$ for every choice of $p_+ \in M_+$ since $(M_- \cup \{p_+\})^\perp \leq M_-$. Finally, for $p \in M_+$, we obtain $p \in (M_- \cup p)^{\perp\perp}$. Hence, we set $p_+ := p$.
(b) $\Rightarrow$ (c): This follows with $U_\sigma := \{p_\sigma\}$.
(c) $\Rightarrow$ (b): Let $p \in \mathscr{S} \setminus (M_+ \cup M_-)$ and let $U_+ \leq M_+$ be a subspace of finite rank such that $p^\perp \cap M_- \geq {U_+}^\perp \cap M_-$. Lemma A.2.22(i) implies $\mathrm{crk}_{M_-}({U_+}^\perp \cap M_-) < \infty$. Hence, the corank k of ${U_+}^\perp \cap M_-$ in $p^\perp \cap M_-$ is finite. If $k > 0$, then there is a point $q \in (p^\perp \cap M_-) \setminus {U_+}^\perp$. Set $V_+ := q^\perp \cap U_+$. Then V_+ is a hyperplane of U_+ and hence, for a point $u \in U_+ \setminus V_+$, we obtain ${V_+}^\perp \cap u^\perp = {U_+}^\perp$. Thus, ${U_+}^\perp$ is a hyperplane of ${V_+}^\perp$ and therefore ${V_+}^\perp \cap M_- = \langle q, {U_+}^\perp \cap M_- \rangle$. Since $q \in p^\perp$, the corank ${V_+}^\perp \cap M_-$ in $p^\perp \cap M_-$ is $k-1$. After finitely many steps we end up with a non-empty subspace V_+ with $p^\perp \cap M_- = {V_+}^\perp \cap M_-$. Since $p^\perp \cap M_-$ is a hyperplane of M_-, we obtain $V_+ \not\leq M_-$ since otherwise ${V_+}^\perp \cap M_- = M_-$. Hence, there is a point $p_+ \in V_+ \setminus M_-$. Since ${p_+}^\perp \cap M_-$ is a hyperplane of M_- containing ${V_+}^\perp \cap M_-$, we conclude ${p_+}^\perp \cap M_- = p^\perp \cap M_-$. □

Remark 5.2.5. Let $\mathscr{S}$ be a non-degenerate polar space of finite rank. Then for an arbitrary generator M of $\mathscr{S}$ there is a generator $N \leq \mathscr{S}$ that is disjoint to M. Let p be a point of $\mathscr{S} \setminus (M \cup N)$. Then $H := N \cap p^{\perp}$ is a hyperplane of N. By Lemma A.2.22(i) we conclude that $H^{\perp}$ intersects M in exactly one point q. We obtain $q^{\perp}N = p^{\perp}N$. Therefore, in $\mathscr{S}$ every generator is part of a spanning pair. Furthermore, every pair of disjoint generators is a spanning pair.

A non-degenerate polar space with a spanning pair has, in fact, many spanning pairs. More precisely, we will show that for a given spanning pair (M_0, M_1) and a generator N_0 that is commensurate to M_0, there is a generator N_1 such that M_1 and N_1 are commensurate and (N_0, N_1) is a spanning pair. Hence, the set of spanning pairs induces a symmetric, total relation on the set of generators that are commensurate to M_0 or M_1. Since the symmetry of this relation is clear by Definition 5.2.2, we just show that it is total.

Lemma 5.2.6. *Let (M_0, M_1) be a spanning pair of $\mathscr{S}$. Further let M_2 be a generator with $M_1 \cap M_2 = \varnothing$ and $\operatorname{dist}_{\mathscr{D}}(M_0, M_2) = 1$. Then (M_1, M_2) is a spanning pair.*

Proof. Let $p \in \mathscr{S} \setminus (M_1 \cup M_2)$. We show that there are points $p_1 \in M_1$ and $p_2 \in M_2$ with ${p_1}^{\perp} \cap M_2 = p^{\perp} \cap M_2$ and ${p_2}^{\perp} \cap M_1 = p^{\perp} \cap M_1$.
Let $q \in M_2 \setminus M_0$. Since p and q are not contained in M_1, there are points p_0 and q_0 in M_0 with ${p_0}^{\perp} \cap M_1 = p^{\perp} \cap M_1 =: H_p$ and ${q_0}^{\perp} \cap M_1 = q^{\perp} \cap M_1 =: H_q$. If $H_p = H_q$ we set $p_2 := q$. Otherwise $p_0 \neq q_0$ and the line p_0q_0 meets M_2 in a point s since M_2 intersects M_0 in a hyperplane. We set $H := H_p \cap H_q$. Since $H \leq {p_0}^{\perp}$ and $H \leq {q_0}^{\perp}$, we conclude $H \leq s^{\perp}$. Since $H \leq q^{\perp}$, every point on sq is collinear to all points in H. Since H_q is a hyperplane in M_1, H is a hyperplane of H_p. Let $r \in H_p \setminus H$ and let $p_2 \in sq \cap r^{\perp}$. Then ${p_2}^{\perp}$ contains $\langle r, H \rangle = H_p$. Since ${p_2}^{\perp} \cap M_1$ is a hyperplane of M_1, we conclude ${p_2}^{\perp} \cap M_1 = H_p$.
Since $q \notin M_0 \cup M_1$, there is a point $q_1 \in M_1$ with ${q_1}^{\perp} \cap M_0 = q^{\perp} \cap M_0 = M_0 \cap M_2$. Since $q_1 \notin M_2$, the subspace ${q_1}^{\perp} \cap M_2$ is a hyperplane of M_2 and we conclude ${q_1}^{\perp} \cap M_2 = M_0 \cap M_2$. We may assume $p^{\perp} \cap M_2 \neq M_0 \cap M_2$ since otherwise we are done by setting $p_1 := q_1$. Hence, $p \notin M_0$ and there is point $r \in M_1$ with $r^{\perp} \cap M_0 = p^{\perp} \cap M_0$. Now ${q_1}^{\perp} \cap M_2 \leq M_0$ yields $\{r, q_1\}^{\perp} \cap M_2 \leq p^{\perp}$. Thus, Proposition 5.2.4 implies that there is a point $p_1 \in M_1$ with ${p_1}^{\perp} \cap M_2 = p^{\perp} \cap M_2$. □

Lemma 5.2.7. *Let (M_0, M_1) be a spanning pair of $\mathscr{S}$. Let $p_0 \in M_0$ and $p_1 \in M_1$ be two points that are not collinear. Then $(p_1 \oplus M_0, p_0 \oplus M_1)$ is a spanning pair.*

Proof. Set $M_0' := p_1 \oplus M_0$ and $M_1' := p_0 \oplus M_1$. Since $p_1 \not\perp p_0$, we obtain $p_1 \notin M_1'$. Since $M_1 \leq {p_1}^{\perp}$, the hyperplanes ${p_1}^{\perp} \cap M_1'$ and $M_1 \cap M_1'$ of M_1' are equal. With $p_1 \in M_0'$ we conclude $M_0' \cap M_1' \leq {p_1}^{\perp} \cap M_1' \leq M_1$. Hence, $M_0' \cap M_1 = \{p_1\}$ yields $M_0' \cap M_1' = \varnothing$. Let $p \in \mathscr{S} \setminus (M_0' \cup M_1')$. Because of symmetric reasons, we only have to show that there is a point $q \in M_1'$ with $p^{\perp} \cap M_0' = q^{\perp} \cap M_0'$. It suffices to

show $p^\perp \cap M_0' \leq q^\perp \cap M_0'$ since $q \notin M_0'$ and hence, $q^\perp \cap M_0'$ and $p^\perp \cap M_0'$ are both hyperplanes in M_0'.
Assume $p \in M_0$. Then $p^\perp \cap M_0' = M_0 \cap M_0' = {p_0}^\perp \cap M_0'$. Hence $q := p_0$ has the asked property. For $p \in M_1$, we obtain $p \neq p_1$ since $p_1 \in M_0'$. Hence, the line pp_1 intersects the hyperplane $M_1 \cap M_1'$ of M_1 in a point q. Since $p_1 \in M_0'$, we obtain $p^\perp \cap M_0' \leq {p_1}^\perp$. With $p^\perp \cap M_0' \leq p^\perp$ this implies $p^\perp \cap M_0' \leq q^\perp$.
It remains the case $p \notin M_0 \cap M_1$. Let $r \in M_1$ be the point with $p^\perp \cap M_0 = r^\perp \cap M_0$. If $r^\perp \cap M_0 = {p_1}^\perp \cap M_0$, then $p^\perp \cap M_0 = M_0 \cap M_0'$ and therefore $p^\perp \cap M_0' = M_0 \cap M_0' = {p_0}^\perp \cap M_0'$ and the claim follows with $q := p_0$. Hence, we may assume $r^\perp \cap M_0 = {p_1}^\perp \cap M_0$ and therefore $r \neq p_1$. The line $p_1 r$ meets M_1' in a point q_1 since M_1' intersects M_1 in a hyperplane. Since $p_1 \in M_0'$ and $q_1 \notin M_0'$, we obtain $r^\perp \cap M_0' = {q_1}^\perp \cap M_0'$. Since ${p_0}^\perp \cap M_0' = M_0 \cap M_0'$, we conclude $\{p_0, q_1\}^\perp \cap M_0' = {q_1}^\perp \cap M_0 \cap M_0' = r^\perp \cap M_0 \cap M_0' = p^\perp \cap M_0 \cap M_0' \leq p^\perp \cap M_0$. Thus, Proposition 5.2.4 implies that there is a point $q \in M_1'$ with $q^\perp \cap M_0' = p^\perp \cap M_0'$. □

Proposition 5.2.8. *Let (M_0, M_1) be a spanning pair of $\mathscr{S}$ and let (M_0', M_1') be a pair of disjoint generators with $\operatorname{dist}_{\mathscr{D}}(M_0, M_0') = n < \infty$ and $\operatorname{dist}_{\mathscr{D}}(M_1, M_1') = m < \infty$. Then (M_0', M_1') is a spanning pair.*

Proof. We proceed by induction over (n, m) using the strict total order $(n_0, m_0) \prec (n_1, m_1)$ if and only if $n_0 + m_0 < n_1 + m_1$ or $(n_0 + m_0 = n_1 + m_1 \ \wedge \ n_0 < n_1)$. If $n + m \leq 1$ the claim follows by Lemma 5.2.6. So from now on, we assume $n + m \geq 2$.
Assume there is a point $p \in M_i' \setminus M_i$ for $i = 0$ or $i = 1$ such that $(p \ominus M_i) \cap M_{1-i} = \varnothing$. Then $(p \ominus M_i, M_{1-1})$ is a spanning pair by Lemma 5.2.6. Since $\operatorname{dist}_{\mathscr{D}}(p \ominus M_i, M_i') = n - 1$, the claim follows from the induction hypothesis. Hence, we may from now on assume that there is no such point.
First assume $n \neq 0$. Let $p \in M_0' \setminus M_0$. Then there is a point $p_1 \in (p \ominus M_0) \cap M_1$. We obtain $p \ominus M_0 = p_1 \ominus M_0$. Since $M_0 \cap M_1 = \varnothing$, there is a point $p_0 \in M_0$ which is not collinear to p_1. By Lemma 5.2.7 the pair $(p_1 \ominus M_0, p_0 \ominus M_1)$ is a spanning pair. Since $\operatorname{dist}_{\mathscr{D}}(p_1 \ominus M_0, M_0') = n - 1$ and $\operatorname{dist}_{\mathscr{D}}(p_0 \ominus M_1, M_1') \leq m + 1$, the claim follows from the induction hypothesis.
Finally, assume $n = 0$ and $m \geq 2$. Then by Lemma A.2.19 there are generators N_i for $0 \leq i \leq m$ and points $s_i \in M_1'$ for $0 \leq i < m$ such that $N_{i+1} = s_i \ominus N_i$, $N_0 = M_1$ and $N_m = M_1'$. As assumed, there is a point $p_0 \in N_1 \cap M_0$. Since $M_0 = M_0'$ and $M_0' \cap M_1' = \varnothing$, there is a point s_j for $1 \leq j < m$ that is not collinear to p_0. Again there is a point $q_0 \in (s_j \ominus M_1) \cap M_0$. Since $q_0 \in M_0$, $s_j \in M_1'$ and $M_0 \cap M_1' = \varnothing$, we obtain $s_j \neq q_0$. Since M_1 intersects $s_j \ominus M_1$ in a hyperplane, the line $s_j q_0$ meets M_1 in a point q_1. Since $p_0 \not\perp s_j$, $p_0 \perp q_0$ and $q_0 \neq q_1$, we obtain $p_0 \not\perp q_1$. Now $(q_1 \ominus M_0, N_1)$ is a spanning pair by Lemma 5.2.7 since $N_1 = p_0 \ominus M_1$. With $s_j \in q_0 q_1 \leq q_1 \ominus M_0$ we use again Lemma 5.2.7 to conclude that $(p_0 \ominus (q_1 \ominus M_0), s_j \ominus N_1)$ is a spanning pair. Since $p_0 \in M_0 \setminus q_1 \ominus M_0$, we obtain $p_0 \ominus (q_1 \ominus$

$M_0) = M_0$ by Lemma A.2.19. With $s_j \in M_1' \setminus N_1$ we obtain $\mathrm{dist}_{\mathscr{D}}(s_j \oplus N_1, M_1') = \mathrm{dist}(N_1, M_1') - 1 = m - 2$ by Lemma A.2.19. Hence, the claim follows by the induction hypothesis. $\square$

Corollary 5.2.9. *Let (M_+, M_-) be a spanning pair of $\mathscr{S}$. For $\sigma \in \{+,-\}$, let $\mathscr{D}^\sigma$ be the connected component of $\mathscr{S}_\mathrm{m}$ that contains M_σ. Then for every generator $N_+ \in \mathscr{D}^+$, there is a disjoint generator $N_- \in \mathscr{D}^-$. Moreover, every pair $(N_+, N_-) \in \mathscr{D}^+ \times \mathscr{D}^-$ with $N_+ \cap N_- = \varnothing$ is a spanning pair.*

Proof. Let $\mathrm{dist}_{\mathscr{D}}(M_+, N_+) = 1$. If $N_+ \cap M_- = \varnothing$, we set $N_- := M_-$. Otherwise N_+ and M_- intersect in a point p. Let $q \in M_+ \setminus p^\perp$ and set $N_- := q \oplus M_-$. Then $N_- \cap N_+ = \varnothing$ since (N_+, N_-) is a spanning pair by Lemma 5.2.7. Thus, the first claim follows by induction. Applying Proposition 5.2.8 proves the second claim. $\square$

5.2.2 Twin dual polar spaces

In this subsection we show how to construct a twin space from a polar space using spanning pairs.

Definition 5.2.10. Let $\mathscr{S}$ be a polar space with spanning pair (M_+, M_-). For $\sigma \in \{+,-\}$, let $\mathscr{D}^\sigma$ be the connected component of the dual polar space of $\mathscr{S}$ that contains M_σ. Then the pair $(\mathscr{D}^+, \mathscr{D}^-)$ with the opposition relation $\{(M,N),(N,M) \mid (M,N) \in \mathscr{D}^+ \times \mathscr{D}^- \wedge M \cap N = \varnothing\}$ is called a *twin dual polar space* of $\mathscr{S}$.

Note that by Proposition 5.2.8 we know that the opposition relation consists of all spanning pairs that have one generator in $\mathscr{D}^+$ and one in $\mathscr{D}^-$. An isomorphic image of a twin dual polar space of $\mathscr{S}$ is simply called a twin dual polar space.

Note that if the polar space $\mathscr{S}$ has finite rank, then $\mathscr{D}^+$ and $\mathscr{D}^-$ are both identical to $\mathscr{S}_\mathrm{m}$ and hence, $(\mathscr{D}^+, \mathscr{D}^-)$ consists of two isomorphic point-line spaces. If $\mathscr{S}$ has infinite rank, then $\mathscr{D}^+ \cup \mathscr{D}^-$ is a proper subspace of the dual polar space of $\mathscr{S}$ by Proposition A.2.25.

By $\mathscr{S}$ we still denote a non-degenerate polar space. Furthermore, in the following (M_+, M_-) is always a spanning pair of $\mathscr{S}$ and $\mathscr{D} = (\mathscr{D}^+, \mathscr{D}^-)$ is a twin dual polar space with $M_+ \in \mathscr{D}^+$ and $M_- \in \mathscr{D}^-$. We denote the distance in $\mathscr{D}$ by $\mathrm{dist}_{\mathscr{D}}$. Since both $\mathscr{D}^+$ and $\mathscr{D}^-$ are connected components of $\mathscr{S}_\mathrm{m}$ and the distance of two elements of one of those halfs of the twin dual polar space is the same as their distance in $\mathscr{S}_\mathrm{m}$, we might still use $\mathrm{dist}_{\mathscr{D}}$ as well for the distance in $\mathscr{S}_\mathrm{m}$. Note that for $M \in \mathscr{D}^+$ and $N \in \mathscr{D}^-$ we always have $\mathrm{dist}_{\mathscr{D}}(M,N) = \infty$ in $\mathscr{D}$, whereas $\mathrm{dist}_{\mathscr{D}}(M,N)$ is finite if the rank of $\mathscr{S}$ is finite.

By Corollary 5.2.9 we know already that the spanning pairs form a symmetric, total relation on the points of $\mathscr{D}$. We now show that $\mathscr{D}$ is a twin space using this relation as an opposition relation.

Proposition 5.2.11. *Every twin dual polar space is a twin space.*

Proof. By the definition of the lines in the dual polar space it follows directly that $\mathscr{D}^+$ and $\mathscr{D}^-$ are partially linear spaces.
We know already that the spanning pairs of $\mathscr{S}$ form a symmetric, total relation on the points of $\mathscr{D}$. Now let $M \in \mathscr{D}^+$ and $N \in \mathscr{D}^-$ such that (M,N) is a spanning pair. Note that by Proposition 5.2.3 the generators of a spanning pair of $\mathscr{S}$ are always disjoint. Let G be a line of $\mathscr{D}^-$ that contains N. Further let $N' \in G \setminus \{N\}$. Then $H := N \cap N'$ is a hyperplane of both N and N'. Hence, there is a point p in N' such that $\langle p, H\rangle = N'$. By the maximality of N' we conclude $N \cap p^\perp = H$.
Since (M,N) is a spanning pair, we know by Proposition 5.2.4 that there is a point $q \in M$ such that $N \cap q^\perp = H$. Now $N_M := q \oplus N$ is a generator of $\mathscr{S}$ that contains H and thus, $N_M \in G$. By the maximality of N_M we obtain $H^\perp \cap M = \{q\}$. Hence, every element of G that intersects M, contains q. Therefore N_M is the only element of G that intersects M. Hence, $\mathscr{D}$ is a twin space by Proposition 5.2.8. □

In the following we denote the codistance function of the twin space $\mathscr{D}$ by $\mathrm{cod}_{\mathscr{D}}$.

Proposition 5.2.12. *Let $M \in \mathscr{D}^+$ and $N \in \mathscr{D}^-$. Then $\mathrm{cod}_{\mathscr{D}}(M,N) = \mathrm{rk}(M \cap N) + 1$.*

Proof. By Proposition 5.2.3 we obtain $\mathrm{cod}_{\mathscr{D}}(M,N) = 0$ if and only if $M \cap N = \varnothing$ for a pair of generators $(M,N) \in \mathscr{D}^+ \times \mathscr{D}^-$. Moreover, we have $\mathrm{rk}(M,N) < \infty$ since $\mathrm{dist}_{\mathscr{D}}(M,M_+) < \infty$ and $\mathrm{dist}_{\mathscr{D}}(N,M_-) < \infty$. Hence, Lemma A.2.17 together with induction implies $\mathrm{cod}_{\mathscr{D}}(M,N) = \mathrm{rk}(M \cap N) + 1$. □

Before checking whether a twin dual polar space satisfies the definition of a twin SPO space, we consider two special situations. First we show for a spanning pair (M_+,M_-) that the convex span of M_+ and a commensurate generator X contains a unique generator which has maximal possible intersection with M_-.

Lemma 5.2.13. *Let (M_+,M_-) be a spanning pair of $\mathscr{S}$. Further let X be a generator with $\mathrm{dist}_{\mathscr{D}}(M_+,X) = k < \infty$. Then there is a generator Y with $Y \cap M_+ = X \cap M_+$ and $\mathrm{rk}(Y \cap M_-) = k-1$. This generator is unique and satisfies $Y = (X \cap M_+) \oplus M_- = (Y \cap M_-) \oplus M_+$.*

Proof. Set $H := M_+ \cap X$. Since $\mathrm{crk}_X(H) = r$, there is an independent set of points $\{b_i \mid 0 \le i < k\}$ such that $\langle b_i \mid 0 \le i < k\rangle \cap H = \varnothing$ and $X = \langle b_0, \dots, b_{k-1}, H\rangle$. Then for every $j \le k$, Lemma A.2.22(i) implies $\mathrm{crk}_{M_+}(M_+ \cap \langle b_i \mid i < j\rangle^\perp) = j$. Since $H \le {b_i}^\perp$ for every $i < k$, this implies $M_+ \cap \{b_i \mid i < k\}^\perp = M_+ \cap \langle b_i \mid i < k\rangle^\perp = H$. Since (M_+,M_-) is a spanning pair and $b_i \notin M_+$, there is a point $p_i \in M_-$ with ${b_i}^\perp \cap M_+ = {p_i}^\perp \cap M_+$ for every $i < k$. We obtain $\{p_i \mid i < k\}^\perp \cap M_+ = H$ and

therefore $\mathrm{rk}(\langle p_i \mid i < k \rangle) = k-1$ by Lemma A.2.22(i). By Lemma A.2.22(ii) the subspace $Y := \{p_i \mid i \in k\} \oplus M_+$ is a generator with $\mathrm{dist}_{\mathscr{D}}(M_+, Y) = k$. Since $H = \{p_i \mid i < k\}^\perp \cap M_+ \leq Y$, we conclude $Y \cap M_+ = H$.
Since $Y = \langle H, p_i \mid i < k \rangle$, $H \leq M_+$ and $\langle p_i \mid i < k \rangle \leq M_-$, we obtain $Y \cap M_- = \langle p_i \mid i < k \rangle$ and therefore $Y = (Y \cap M_-) \oplus M_+$. Now let p be any point of $H^\perp \cap M_-$. Then p is collinear to p_i for $i < k$. Hence, $p^\perp \geq \langle H, p_i \mid i < k \rangle = Y$ and therefore $\langle p, Y \rangle$ is singular. Thus, $p \in Y$ since Y is a generator. Therefore, Y is uniquely determined and $Y = H \oplus M_-$. □

In the following lemma we show for a more general situation that whenever we have a convex span $G \leq \mathscr{S}_{\mathrm{m}}$ of two commensurate generators, we can choose two generators whose convex span is G such that one of them has maximal possible intersection and the other one has minimal possible intersection to a certain given generator.

Lemma 5.2.14. *Let (M_+, M_-) be a spanning pair of $\mathscr{S}$. Further let X, Y and Z be generators such that X and M_+ are commensurate and Y, Z and M_- are commensurate. Set $V := Y \cap Z$. Then there are generators Y' and Z' with $Y' \cap Z' = V$ such that $Y' \cap X = V \cap X$ and* $\mathrm{crk}_{Z' \cap X}(V \cap X) = \mathrm{dist}_{\mathscr{D}}(Y, Z)$.

Proof. By Corollary 5.2.9 there is a generator M with $\mathrm{dist}_{\mathscr{D}}(M_-, M) < \infty$ such that (X, M) is a spanning pair. Then $\mathrm{dist}_{\mathscr{D}}(M, Y)$ and $\mathrm{dist}_{\mathscr{D}}(M, Z)$ are finite. Hence, we may assume $X = M_+$ and $M_- = M$.
Set $Y_X := (M \cap Y) \oplus X$, $Z_X := (M \cap Z) \oplus X$ and $U := \langle X \cap Y_X, X \cap Z_X \rangle$. Then Y_X and Z_X are generators with $\mathrm{dist}_{\mathscr{D}}(M, Y_X) < \infty$ and $\mathrm{dist}_{\mathscr{D}}(M, Z_X) < \infty$ by Lemma 5.2.13. Hence, $X \cap Y_X$ and $X \cap Z_X$ have both finite rank and therefore $\mathrm{rk}(U) < \infty$. By Lemma 5.2.13 we obtain $Y_X \cap M = Y \cap M$ and $(X \cap Y_X)^\perp \cap M = (Y_X \cap M)$ and the corresponding for Z_X. We conclude

$$\begin{aligned} U^\perp \cap M &= ((X \cap Y_X) \cup (X \cap Z_X))^\perp \cap M \\ &= ((X \cap Y_X)^\perp \cap (X \cap Z_X)^\perp) \cap M \\ &= ((X \cap Y_X)^\perp \cap M) \cap ((X \cap Z_X)^\perp) \cap M) \\ &= (Y_X \cap M) \cap (Z_X \cap M) = (Y \cap M) \cap (Z \cap M) = V \cap M\,. \end{aligned}$$

Thus, $V_X := U \oplus M = \langle U, V \cap M \rangle$ is a generator by Lemma A.2.22(ii). Now let B be a basis of V containing a basis B_0 of $V \cap M$ and a basis B_1 of $V \cap V_X$. This is possible since $V \cap M \leq V_X$ and hence $B_0 \subseteq B_1$. Since $V_X = \langle U, V \cap M \rangle$, every subspace of V_X has a basis contained in $M \cup X$. Hence, we may assume that we chose B such that $B_1 \setminus B_0 \subseteq X$. Since V_X is a generator, we obtain $(V \cap M)^\perp \cap X = U$. With $V \leq (V \cap M)^\perp$ this implies $V \cap X \leq U$ and consequently, $\langle B_1 \setminus B_0 \rangle = V \cap X$ since $X \cap M = \varnothing$.
Set $B_V := B \setminus B_1$ and set $Z' := B_V \oplus V_X$. Then $\langle B_V \rangle$ is disjoint from V_X since

B_1 is a basis of $V \cap V_X$. Since $\mathrm{crk}_{V_X}(V \cap V_X) \le \mathrm{crk}_{V_X}(V \cap M) < \infty$, we obtain $\mathrm{crk}_Y(V \cap V_X) < \infty$ by Proposition A.2.20 and hence, $\mathrm{crk}_V(V \cap V_X) < \infty$. Thus, Z' is a generator with $\mathrm{dist}_{\mathscr{D}}(V_X, Z') = |B_V|$ by Lemma A.2.22(ii). Since $B_1 \subseteq B_V{}^{\perp} \cap V_X$, we obtain $B \subseteq Z'$ and hence, $V \le Z'$. This implies $Z' \cap X \le (V \cap M)^{\perp} \cap X = U$. Since $U = X \cap (V \cap M)^{\perp}$ and $V \cap M \le Z'$, we conclude $Z' \cap V_X = \langle V \cap M, Z' \cap U \rangle$. Hence,

$$
\begin{aligned}
\mathrm{crk}_{Z'\cap X}(V \cap X) &= \mathrm{crk}_U(V \cap X) - \mathrm{crk}_U(Z' \cap X) \\
&= \mathrm{crk}_{V_x}(\langle V \cap M, V \cap X \rangle) - \mathrm{crk}_{V_x}(\langle V \cap M, Z' \cap U \rangle) \\
&= \mathrm{crk}_{V_x}(\langle B_0 \rangle, \langle B_1 \setminus B_0 \rangle) - \mathrm{crk}_{V_x}(Z' \cap V_X) \\
&= \mathrm{crk}_{V_x}(\langle B_1 \rangle) - \mathrm{dist}_{\mathscr{D}}(V_X, Z') = \mathrm{crk}_Y(\langle B_1 \rangle) - |B_V| \\
&= \mathrm{crk}_Y(\langle B_1, B_V \rangle) = \mathrm{crk}_Y(V) = \mathrm{dist}_{\mathscr{D}}(Y, Z)\,.
\end{aligned}
$$

Set $Y_0 := Y$. Let $i < \mathrm{dist}_{\mathscr{D}}(Y,Z)$ be a natural number such that Y_i exists and $Y_i \cap X \not\le Z$. Then we choose a point $y_i \in Y_i \cap X \setminus Z$. Since Z is a generator, there is a point $z_i \in Z$ that is not collinear to y_i. Set $Y_{i+1} := z_i \oplus Y_i$. Since $Y_{i+1} \cap X \le y_i{}^{\perp}$ and $y_i{}^{\perp} \cap Y_{i+1} = Y_i \cap Y_{i+1}$, we conclude $Y_{i+1} \cap X \le Y_i \cap X$. Together with $y_i \in (Y_i \cap X) \setminus Y_{i+1}$ this implies $Y_i \cap X > Y_{i+1} \cap X$. Hence, after finitely many steps we obtain a generator Y_j for some $j \le \mathrm{dist}_{\mathscr{D}}(Y,Z)$ with $Y_j \cap X \le Z$. Set $Y' := Y_j$. Then $Y' \cap X \le V \cap X$ since $Y' \cap X \le Y_i$ for every $i \le j$ and hence $Y' \cap X \le Y$. On the other hand we obtain $V \le Y'$ since $V \le Y_0$ and $V \le z_i{}^{\perp}$ for every $i \le j$. Thus, $Y' \cap X = V \cap X$. Now $\mathrm{crk}_{Z'\cap X}(Y' \cap X) = \mathrm{dist}_{\mathscr{D}}(Y,Z)$ yields $\mathrm{dist}_{\mathscr{D}}(Y', Z') \ge \mathrm{dist}_{\mathscr{D}}(Y,Z)$. Since both, Y' and Z' contain V, this implies $\mathrm{dist}_{\mathscr{D}}(Y', Z') = \mathrm{dist}_{\mathscr{D}}(Y,Z)$ and $Y' \cap Z' = V$. □

Theorem 5.2.15. *Every twin dual polar space is a twin SPO space with singular rank* ≤ 1.

Proof. Let $\mathscr{S}$ be a non-degenerate polar space. Further let (M_+, M_-) be a spanning pair of $\mathscr{S}$ and denote by $(\mathscr{D}^+, \mathscr{D}^-)$ the twin dual polar space of $\mathscr{S}$ with $(M_+, M_-) \in \mathscr{D}^+ \times \mathscr{D}^-$. Since both $\mathscr{D}^+$ and $\mathscr{D}^-$ are subspaces of the dual polar space $\mathscr{S}_{\mathrm{m}}$ of $\mathscr{S}$, we conclude by the definition of the lines of $\mathscr{S}_{\mathrm{m}}$ that the singular rank of $(\mathscr{D}^+, \mathscr{D}^-)$ is at most 1.
To prove that $(\mathscr{D}^+, \mathscr{D}^-)$ is a twin SPO space it suffices to show that the conditions given in Definition 2.1.1 are fulfilled for a generator $X \in \mathscr{D}^+$ and generators Y and Z that are contained in $\mathscr{D}^-$. Set $n := \mathrm{dist}_{\mathscr{D}}(Y,Z)$ and let G be the convex span of Y and Z in $\mathscr{D}^-$.
Assume there is a generator $X' \in G$ that is opposite X. Then $Y \cap Z \cap X \varnothing$ since by Proposition 5.2.1 every element of G contains $Y \cap Z$ and $X \cap X' = \varnothing$. Since $\mathrm{crk}_Y(Y \cap Z) = n$, Proposition A.2.20 implies $\mathrm{crk}_N(Y \cap Z) = n$ for every generator $N \in G$ and hence, $\mathrm{cod}_{\mathscr{D}}(X,N) \le n$. Since by Lemma 5.2.14 there is a generator $Z' \in G$ such that the corank of $(Y \cap Z) \cap X$ in $Z' \cap X$ is n, we obtain $\mathrm{cod}_{\mathscr{D}}(X, Z') = n$

and hence (A1) holds. Since $X \cap X' = \varnothing$, we obtain $\mathrm{dist}_{\mathscr{D}}(Z', X') \geq n$ and therefore $Z' \cap X' = Y \cap Z$. Since (X, X') is a spanning pair, Lemma 5.2.13 implies that Z' is the unique generator with $\mathrm{rk}(X, Z') = n - 1$ that contains $Y \cap Z$. Thus, (A2) is satisfied.
Axiom (A4) is a direct consequence of Lemma A.2.17. Hence, it remains to check (A3). Therefore we assume $\mathrm{cod}_{\mathscr{D}}(X, Z) = \mathrm{cod}_{\mathscr{D}}(X, G)$. Since $\mathrm{crk}_N(Y \cap Z) = n$ for every $N \in G$, we know that $Y \cap Z \cap X$ has corank $\leq n$ in $N \cap X$. Hence, Lemma 5.2.14 implies that $Y \cap Z \cap X$ has corank n in $Z \cap X$. Since $\mathrm{crk}_Z(Y \cap Z) = n$, we conclude $Z = \langle X \cap Z, Y \cap Z \rangle$. Let $p \in Y \cap X$. Since $X \cup Y \subseteq p^\perp$, we obtain $Z = \langle X \cap Z, Y \cap Z \rangle \leq p^\perp$ and therefore $\langle p, Z \rangle$ is singular. By the maximality of Z we conclude $p \in Z$ and hence, $X \cap Y = X \cap (Y \cap Z)$. Now let W be a generator that is adjacent to X with $\mathrm{cod}_{\mathscr{D}}(W, Y) < \mathrm{cod}_{\mathscr{D}}(X, Y)$. Since W and X intersect in a common hyperplane, we conclude that $W \cap Y$ is a hyperplane of $X \cap Y$. Since $X \cap Y = X \cap (Y \cap Z)$, this implies $W \cap (Y \cap Z) = W \cap Z$. Since $\mathrm{crk}_{W \cap N}(W \cap Y \cap Z) \leq n$ for every $N \in G$ and $\mathrm{crk}_{X \cap Z}(X \cap Y \cap Z) = n$, this implies $\mathrm{cod}_{\mathscr{D}}(W, G) < \mathrm{cod}_{\mathscr{D}}(X, Z)$. Since W and X are adjacent, we obtain $\mathrm{cod}_{\mathscr{D}}(W, G) = \mathrm{cod}_{\mathscr{D}}(W, Z) = \mathrm{cod}_{\mathscr{D}}(X, Z) - 1$. Since $\mathrm{cod}_{\mathscr{D}}(X, Y) < \infty$, there is a generator $X' \in \mathscr{D}^-$ such that (X, X') is a spanning pair and $\mathrm{dist}_{\mathscr{D}}(Y, X') = \mathrm{cod}_{\mathscr{D}}(X, Y)$. Since $\mathrm{cod}_{\mathscr{D}}(X, Z) = \mathrm{cod}_{\mathscr{D}}(X, Y) + n$, we obtain $\mathrm{dist}_{\mathscr{D}}(Z, X') = \mathrm{dist}_{\mathscr{D}}(Z, Y) + \mathrm{dist}_{\mathscr{D}}(Y, X')$. Since

$$\begin{aligned} \mathrm{crk}_Z(Z \cap Y) + \mathrm{crk}_Y(Y \cap X') &\geq \mathrm{crk}_Z(Z \cap Y) + \mathrm{crk}_{Z \cap Y}(Z \cap Y \cap X') \\ &= \mathrm{crk}_Z(Z \cap Y \cap X') \geq \mathrm{crk}_Z(Z \cap X'), \end{aligned}$$

this implies $Z \cap Y \cap X' = Z \cap X'$ and hence, $Z \cap X' \leq Y \cap Z$. Now we may apply Lemma 5.2.13 to show that Z is the unique generator at codistance $\mathrm{cod}_{\mathscr{D}}(X, Z)$ to X that contains $Z \cap X'$. Hence, it is also the unique generator contained in G at this codistance to X. Analogously, Z is the only element of G at codistance $\mathrm{cod}_{\mathscr{D}}(W, G)$ to W. Thus, (A3) is satisfied. $\square$

5.3 Partial twin Grassmannians

A *Grassmannian* of a projective space $\mathscr{S}$ is a point-line space whose point set $\mathscr{P}$ consists of all subspaces of $\mathscr{S}$ of rank $k \in \mathbb{N}$ and whose lines are the maximal subsets of $\mathscr{P}$ whose elements intersect in a common subspace of rank $k-1$ and are contained in a common subspace of rank $k+1$. To be more specific, this point-line space is also called a Grassmannian of k-spaces. The Grassmannian of 0-spaces is canonically isomorphic to $\mathscr{S}$.

For a projective space of infinite rank $\mathscr{S}$, there is an analogous way to define a point-lines space whose points are the subspaces of corank k. The so obtained point-line space can be seen as the Grassmannian of corank-k-spaces. Thus, the Grassmannian of corank-1-spaces is just the dual of $\mathscr{S}$.

In the following we introduce a point-line space that is constructed out of a projective space and can be seen as a generalisation of a Grassmannian. This construction allows us for a projective space $\mathscr{S}$ of infinite rank to take as points of the new point-line space subspaces of $\mathscr{S}$ that have infinite rank and infinite corank.

Let U be a subspace of a projective space $\mathscr{S}$. Then we call a subspace $V \leq \mathscr{S}$ a *complement* of U if and only if U and V are disjoint and $\langle U, V\rangle = \mathscr{S}$. Since by this definition U is a complement to V, we call U and V *complementary* subspaces.

Definition 5.3.1. Let $\mathscr{S}$ be a projective space and let U_+ and U_- be non-trivial subspaces of $\mathscr{S}$ that are complementary. For $\sigma \in \{+,-\}$, let $\mathfrak{U}^\sigma$ be the set of subspaces of $\mathscr{S}$ that are commensurate to U_σ. Further set:

$$\begin{aligned}\mathscr{L}_{\mathrm{m}}^{\sigma} &:= \big\{\{Z \in \mathfrak{U}^\sigma \mid X \cap Y < Z < \langle X, Y\rangle\} \,\big|\, \{X,Y\} \subseteq \mathfrak{U}^\sigma \wedge \operatorname{crk}_X(X\cap Y) = 1\big\}\\ R &:= \{(M,N),(N,M) \mid (M,N) \in \mathfrak{U}^+ \times \mathfrak{U}^- \wedge M \cap N = \varnothing\}\end{aligned}$$

Then we call the pair $((\mathfrak{U}^+, \mathscr{L}_{\mathrm{m}}^+), (\mathfrak{U}^-, \mathscr{L}_{\mathrm{m}}^-))$ with the opposition relation R the *twin Grassmannian* of $\mathscr{S}$ with respect to (U_+, U_-).
For $\sigma \in \{+,-\}$, let $\mathscr{P}_{\mathrm{m}}^\sigma \subseteq \mathfrak{U}^\sigma$ be a subset that contains U_σ such that the following conditions are satisfied:

- **(TG1)** For every subspace $V \in \mathscr{P}_{\mathrm{m}}^\sigma$, there is a subspace $W \in \mathscr{P}_{\mathrm{m}}^{-\sigma}$ such that $V \cap W = \varnothing$.
- **(TG2)** $\langle V \mid V \in \mathscr{P}_{\mathrm{m}}^\sigma\rangle = \mathscr{S}$.
- **(TG3)** Let V and W be two elements of $\mathscr{P}_{\mathrm{m}}^\sigma$. Then $\{X \in \mathfrak{U}^\sigma \mid V \cap W \leq X \leq \langle V, W\rangle\} \subseteq \mathscr{P}_{\mathrm{m}}^\sigma$.

For $\sigma \in \{+,-\}$, set $\mathscr{L}_{\mathrm{m}}'^{\sigma} := \{L \in \mathscr{L}_{\mathrm{m}}^\sigma \mid L \subseteq \mathscr{P}_{\mathrm{m}}^\sigma\}$ and $R' := R \cap ((\mathscr{P}_{\mathrm{m}}^+ \cup \mathscr{P}_{\mathrm{m}}^-) \times (\mathscr{P}_{\mathrm{m}}^- \times \mathscr{P}_{\mathrm{m}}^+))$. Then $((\mathscr{P}_{\mathrm{m}}^+, \mathscr{L}_{\mathrm{m}}'^+), (\mathscr{P}_{\mathrm{m}}^-, \mathscr{L}_{\mathrm{m}}'^-))$ with the opposition relation R' is called a *partial twin Grassmannian* of $\mathscr{S}$ with respect to (U_+, U_-).

We will see later on that every (partial) twin Grassmannian is a twin space. Therefore we call a twin space a (partial) twin Grassmannian if it is isomorphic to a (partial) twin Grassmannian of a projective space. Throughout this section $\mathscr{S}$ is always a projective space and U_+ and U_- are non-trivial subspaces of $\mathscr{S}$ that are complementary. For $\sigma \in \{+,-\}$, we denote by $\mathfrak{U}^\sigma$ the set of subspaces of $\mathscr{S}$ that are commensurate to U_σ.

Note that if U_+ is a singleton and U_- is a hyperplane, then every partial twin Grassmannian is a twin projective space. This follows directly from (TG1) and (TG2). On the other hand, every twin projective space fulfils (TG1) and (TG2). Moreover, (TG3) follows in this case by the definition of the lines. Hence, every twin projective space is a partial twin Grassmannian.

The following remark concerns some immediate consequences that follow by the Axioms (TG1) and (TG3).

Remark 5.3.2. Every subspace of $\mathscr{S}$ has a basis that is contained in $U_+ \cup U_-$. Hence, for a subspace $V \in \mathfrak{U}^+$, there is a basis B of $\mathscr{S}$ such that $B \subseteq U_+ \cup U_-$ and $B \cap V$ is a basis of V. We conclude that $\langle B \setminus V \rangle$ is a complement to V that is commensurate to U_-. Moreover, for two points $b \in B \cap U_+$ and $c \in B \setminus U_+$, we obtain $\langle c, B \cap U_+ \setminus \{b\} \rangle \in \mathfrak{U}^-$. Therefore, every twin Grassmannian is a partial twin Grassmannian.
Since U_+ and V are commensurate, we obtain $\mathrm{crk}_{U_+}(U_+ \cap V) = |B \cap V \cap U_-|$. By symmetric reasons this implies that a subspace $W \in \mathfrak{U}^-$ is disjoint to V if and only if $\langle V, W \rangle = \mathscr{S}$. Therefore, the subspaces V and W of (TG1) are always complements.
From (TG3) and the definition of the lines of the twin Grassmannian it follows directly that every partial twin Grassmannian with respect to (U_+, U_-) is a subspace of the twin Grassmannian with respect to (U_+, U_-). A second consequence of (TG3) is that for a partial twin Grassmannian $((\mathscr{P}_{\mathrm{m}}^+, \mathscr{L}_{\mathrm{m}}^+), (\mathscr{P}_{\mathrm{m}}^-, \mathscr{L}_{\mathrm{m}}^-))$, both subspaces $(\mathscr{P}_{\mathrm{m}}^+, \mathscr{L}_{\mathrm{m}}^+)$ and $(\mathscr{P}_{\mathrm{m}}^-, \mathscr{L}_{\mathrm{m}}^-)$ are connected.

The Axiom (TG2) plays a special role. As we will see omitting it does not change anything about the definition of partial twin Grassmannians, but it would change the definition of a partial twin Grassmannian of a given projective space. Nevertheless, we cling to this axiom since it turns out to be useful.

Remark 5.3.3. For $\sigma \in \{+,-\}$, let $\mathscr{P}_1^\sigma \subseteq \mathfrak{U}^\sigma$ be a subset with $U_\sigma \in \mathscr{P}_1^\sigma$ such that (TG1) and (TG3) are fulfilled, but (TG2) is not. Set $\mathscr{S}' := \langle U \mid U \in \mathscr{P}_1^+ \rangle$. Further set $\mathscr{P}_0^- := \{V \cap \mathscr{S}' \mid V \in \mathscr{P}_1^-\}$. Let $U \in \mathscr{P}_1^+$ and $V \in \mathscr{P}_1^-$ such that U and V are complements. Then U and $V \cap \mathscr{S}'$ are complements in $\mathscr{S}'$. Thus, there is a twin Grassmannian $\mathscr{D}'$ of $\mathscr{S}'$ with respect to $(U_+, U_- \cap \mathscr{S}')$. We denote by $\mathfrak{U}_0^+$ and $\mathfrak{U}_0^-$ the point sets of this twin Grassmannian of $\mathscr{S}'$, where $U_+ \in \mathfrak{U}_0^+$ and $U_- \cap \mathscr{S}' \in \mathfrak{U}_0^-$.
For two elements V and W of $\mathscr{P}_1^-$ there are complements V' and W' in $\mathscr{P}_1^+$. Since V' and W' are complements of $V \cap \mathscr{S}'$ and $W \cap \mathscr{S}'$ in $\mathscr{S}'$ and furthermore V' and W' are commensurate, we conclude that $V \cap \mathscr{S}'$ and $W \cap \mathscr{S}'$ are commensurate. Therefore we obtain $\mathscr{P}_0^- \subseteq \mathfrak{U}_0^-$.
Assume that V and W are distinct. Then $\mathrm{rk}(U \cap \langle V, W \rangle) = \mathrm{crk}_{\langle V,W \rangle}(V) - 1$ since U is complementary to V. This implies $\mathrm{crk}_{\langle V,W \rangle \cap \mathscr{S}'}(V \cap \mathscr{S}') = \mathrm{crk}_{\langle V,W \rangle}(V)$ and consequently, $\mathrm{crk}_{V \cap \mathscr{S}'}(V \cap W \cap \mathscr{S}') = \mathrm{crk}_V(V \cap W)$. Hence, $\mathscr{P}_1^- \to \mathscr{P}_0^- : X \mapsto X \cap \mathscr{S}'$ is a bijection that maps lines of $(\mathscr{P}_{\mathrm{m}}^-, \mathscr{L}_{\mathrm{m}}^-)$ onto lines of $\mathscr{D}'$.
Let $X' \leq \mathscr{S}'$ be a subspace that is commensurate to $V \cap \mathscr{S}'$ such that $V \cap W \cap \mathscr{S}' \leq X' \leq \langle V, W \rangle \cap \mathscr{S}'$. Then $X := \langle X', V \cap W \rangle$ is commensurate to V with $V \cap W \leq X \leq \langle V, W \rangle$ and (TG3) implies $X \in \mathscr{P}_1^-$. Since $X' = X \cap \mathscr{S}'$, we conclude $X' \in \mathscr{P}_0^-$ and hence, (TG3) holds for $\mathscr{P}_0^-$. Therefore, restricting the elements of $\mathscr{P}_1^+$ and

$\mathscr{P}_1^-$ to the subspace $\mathscr{S}'$ leads to an isomorphic structure that still fulfils (TG1) and (TG3).
Suppose $\mathscr{S}'' := \langle U \mid U \in \mathscr{P}_0^- \rangle < \mathscr{S}'$. Then we set $\mathscr{P}_0^+ := \{V \cap \mathscr{S}'' \mid V \in \mathscr{P}_1^+\}$. By repeating the arguments we obtain that restricting the elements of $\mathscr{P}_1^+$ and $\mathscr{P}_1^-$ to the subspace $\mathscr{S}''$ leads to an isomorphic structure. Since now (TG1), (TG2) and (TG3) are all fulfilled, we conclude that the subspaces contained in $\mathscr{P}^+$ and $\mathscr{P}^-$ are the points of a partial twin Grassmannian. Therefore (TG2) can be seen as a condition that makes sure that $\mathscr{S}$ is "entirely utilised".

In the following $\mathscr{S}_\mathrm{m} := ((\mathscr{P}_\mathrm{m}^+, \mathscr{L}_\mathrm{m}^+), (\mathscr{P}_\mathrm{m}^-, \mathscr{L}_\mathrm{m}^-))$ is a partial twin Grassmannian of $\mathscr{S}$ with respect to (U_+, U_-). For $\sigma \in \{+,-\}$, we set $\mathscr{S}_\mathrm{m}^\sigma := (\mathscr{P}_\mathrm{m}^\sigma, \mathscr{L}_\mathrm{m}^\sigma)$. Moreover, we denote by $\mathscr{D} = (\mathscr{D}^+, \mathscr{D}^-)$ the twin Grassmannian of $\mathscr{S}$ with respect to (U_+, U_-).

Proposition 5.3.4. *Every partial twin Grassmannian is a twin space.*

Proof. Let R be the opposition relation of the partial twin Grassmannian $\mathscr{S}_\mathrm{m}$. By (TG1) R is a symmetric, total relation on $\mathscr{P}_\mathrm{m}^+ \cup \mathscr{P}_\mathrm{m}^-$. Let $U \in \mathscr{P}_\mathrm{m}^+$ and $V \in \mathscr{P}_\mathrm{m}^-$ such that U and V are complements. Further let $W \in \mathscr{P}_\mathrm{m}^-$ be a subspace such that V and W are distinct collinear points in $(\mathscr{P}_\mathrm{m}^-, \mathscr{L}_\mathrm{m}^-)$ and let $L \in \mathscr{L}_\mathrm{m}^-$ with $\{V, W\} \subseteq L$. Then V and W intersect in a common hyperplane. Hence, U and $\langle V, W \rangle$ intersect in a single point p since U is a complement to V. We conclude that $\langle p, V \cap W \rangle$ is the only element of L that is not disjoint to U. By the definition of the lines of a twin Grassmannian we conclude that $\mathscr{S}_\mathrm{m}$ is partially linear. $\square$

Let $\mathfrak{F}$ be the set of finite subsets of $\mathscr{P}_\mathrm{m}^+$ and set $\mathscr{S}' := \bigcup_{F \in \mathfrak{F}} \langle F \rangle$, where $\langle F \rangle$ is understood as the span in $\mathscr{S}$. Then $\mathscr{S}'$ is a subspace of $\mathscr{S}$ since the union of two finite sets is again finite. This implies $\langle U \mid U \in \mathscr{P}_\mathrm{m}^+ \rangle \leq \mathscr{S}'$ and hence, $\mathscr{S}' = \mathscr{S}$ by (TG2). We will make use of this fact for proving the following lemma.

Lemma 5.3.5. *Let p be a point of $\mathscr{S}$. Then there are elements $U \in \mathscr{P}_\mathrm{m}^+$ and $V \in \mathscr{P}_\mathrm{m}^-$ such that $p \in U \cap V$.*

Proof. Since $\langle U \mid U \in \mathscr{P}_\mathrm{m}^+ \rangle = \mathscr{S}$, there is a finite set $F := \{U_i \mid 0 \leq i < n\} \subseteq \mathscr{P}_\mathrm{m}^+$ where $n \in \mathbb{N}$ such that $p \in \langle F \rangle$. For $0 \leq j < n$, set $\mathscr{S}_j := \langle U_i \mid i \leq j \rangle$. We prove by induction that every point of $\mathscr{S}_j$ is contained in a subspace $U \in \mathscr{P}_\mathrm{m}^+$.
For $j = 0$ there is nothing to prove since $\mathscr{S}_0 = U_0$. Now assume the claim holds for $j < n-1$. We may assume $U_j \nleq \mathscr{S}_j$ since otherwise $\mathscr{S}_{j+1} = \mathscr{S}_j$. Let $q \in \mathscr{S}_{j+1} \setminus \mathscr{S}_j$. Then there are points $r \in U_j$ and $s \in \mathscr{S}_j$ such that q is on the line rs. By the induction hypothesis we know that there is a subspace $W \in \mathscr{P}_\mathrm{m}^+$ such that $s \in W$. Since $r \in \langle U_j, W \rangle$, (TG3) implies that there is a subspace $U \in \mathscr{P}_\mathrm{m}^+$ with $\langle q, U_j \cap W \rangle \leq U$. The claim follows by analogy. $\square$

Lemma 5.3.6. *Let $n \in \mathbb{N}$ and let $(U_i)_{0\leq i\leq n}$ be a family of elements of $\mathscr{P}^+_{\mathrm{m}}$. Then every subspace $V \in \mathfrak{U}^+$ with $\bigcap_{0\leq i\leq n} U_i \leq V \leq \langle U_i \mid 0 \leq i \leq n\rangle$ is an element of $\mathscr{P}^+_{\mathrm{m}}$.*

Proof. For $n = 0$ there is nothing to prove and for $n = 1$ this is just (TG3). We prove the claim by induction over n. Hence we may assume that every subspace $V \in \mathfrak{U}^+$ with $\bigcap_{0\leq i<n} U_i \leq V \leq \langle U_i \mid 0 \leq i < n\rangle$ is an element of $\mathscr{P}^+_{\mathrm{m}}$. Set $S := \bigcap_{0\leq i<n} U_i$ and $W := \langle U_i \mid 0 \leq i < n\rangle$. In the proof we distinguish three different situations:

(I) First consider the case $S \leq U_n$. Since U_0 and U_n are commensurate, there is a natural number $m \in \mathbb{N}$ and a family of points $(p_i)_{0\leq i<m}$ such that $\langle p_i, W \mid 0 \leq i < m\rangle = \langle U_n, W\rangle$. We proceed by another induction. Let $j < m$ such that every subspace $V \in \mathfrak{U}^+$ with $S \leq V \leq W' := \langle p_i, W \mid 0 \leq i < j\rangle$ is contained in $\mathscr{P}^+_{\mathrm{m}}$ and $p_j \notin W'$. Let $V \in \mathfrak{U}^+$ with $S \leq V \leq \langle p_j, W'\rangle$. We may assume $V \nleq W'$ since otherwise we know already $V \in \mathscr{P}^+_{\mathrm{m}}$. Since W' is a hyperplane of $\langle p_j, W'\rangle$, the subspace $V \cap W'$ is a hyperplane of V. Moreover, $S \leq V \cap W'$. Hence, for an arbitrary subspace $U' \leq W'$ that contains $V \cap W'$ as a hyperplane, we obtain $U' \in \mathscr{P}^+_{\mathrm{m}}$. Let H be a hyperplane of U' that contains $U' \cap U_n$. Then (TG3) implies $\langle p_j, H\rangle \in \mathscr{P}^+_{\mathrm{m}}$. If $H = V \cap W'$, we obtain $\langle p_j, H\rangle = U'$ and consequently, $V \in \mathscr{P}^+_{\mathrm{m}}$. Therefore we may assume $H \neq V \cap W'$. Then $U' = \langle H, V \cap W'\rangle$. Since $U_0 \neq U_1$, we conclude $U' < W'$ and hence, there is a point $p \in W' \setminus U'$. We know $V' := \langle p, V \cap W'\rangle \in \mathscr{P}^+_{\mathrm{m}}$. Now $V' \cap \langle p_j, H\rangle \leq W'$ and thus, $V' \cap \langle p_j, H\rangle \leq U'$. Since $V' \cap U' = V \cap W'$, we conclude $V' \cap \langle p_j, H\rangle \leq V = \langle p_j, V \cap W'\rangle \leq \langle V, \langle p_j, H\rangle\rangle$ and therefore $V \in \mathscr{P}^+_{\mathrm{m}}$ by (TG3). Thus, induction provides $V \in \mathscr{P}^+_{\mathrm{m}}$ for every $V \in \mathfrak{U}^+$ with $S \leq V \leq \langle U_n, W\rangle$.

(II) Now consider the case $U_n \leq W$. Let $\mathrm{crk}_S(S \cap U_n) \geq 2$. Then U_0 contains a hyperplane H with $U_0 \cap U_n \leq H$ and $S \nleq H$. Since $S \leq U_0$, we obtain $\langle S, H\rangle = U_0$ and $S \cap U_n \leq H$. Hence, for a point $p \in U_n \setminus U_0$, we conclude $S \nleq \langle p, H\rangle =: U'_n$ and therefore $U_0 \cap U'_n = H$. By (TG3) we conclude $U'_n \in \mathscr{P}^+_{\mathrm{m}}$. Since H intersects S in a hyperplane, we obtain $S \cap U_n < S \cap U'_n < S$. By the finiteness of $\mathrm{crk}_S(S \cap U_n)$ we may constrain ourselves to the case $\mathrm{crk}_S(S \cap U_n) = 1$. Now let $V \in \mathfrak{U}^+$ with $S \cap U_n \leq V \leq W$. We may assume $S \nleq V$ since otherwise we know already $V \in \mathscr{P}^+_{\mathrm{m}}$. Since $S \cap U_n$ is a hyperplane of S and $S \cap U_n \leq S \cap V < S$, we conclude $S \cap U_n = S \cap V$. Set $W' := \langle U_n, V\rangle$. Assume $S \leq W'$. Then there is a subspace $V' \in \mathfrak{U}^+$ with $S \leq V'$ and $U_n \cap V' = U_n \cap V$ and $\langle U_n, V'\rangle = W'$. Since $S \leq V' \leq W$, we obtain $V' \in \mathscr{P}^+_{\mathrm{m}}$ and consequently by (TG3) $V \in \mathscr{P}^+_{\mathrm{m}}$ since $U_n \cap V' \leq V \leq \langle U_n, V'\rangle$. Hence, we may assume $S \nleq W'$ and therefore $S \cap W' = S \cap V$. Let H be a hyperplane of V that contains $S \cap V$. Then $U' := \langle S, H\rangle \in \mathfrak{U}^+$. Moreover, $U' \in \mathscr{P}^+_{\mathrm{m}}$ since $S \leq U' \leq W$. Since $S \nleq W'$, we obtain $U' \cap W' = H$ and consequently, $U' \cap U_n \leq H$. Let $p \in U_n \setminus V$. Then $\langle p, H\rangle \in \mathfrak{U}^+$ and therefore $U' \cap U_n \leq \langle p, H\rangle \leq \langle U_n, U'\rangle$ yields $\langle p, H\rangle \in \mathscr{P}^+_{\mathrm{m}}$ by (TG3). Since $U_0 \neq U_1$, we know $S < U_0$ and thus,

$\mathrm{crk}_V(S \cap V) \geq 2$. Hence, there is a hyperplane H' of V that is distinct to H and contains $S \cap V$. As for U', we obtain $\langle S, H' \rangle \in \mathscr{P}_\mathrm{m}^+$. Since $S \nleq W'$, we conclude $\langle S, H' \rangle \cap \langle p, H \rangle \leq \langle S, H' \rangle \cap W' = H'$. Thus, $\langle p, H \rangle \cap \langle S, H' \rangle \leq V = \langle H, H' \rangle$ and (TG3) implies $V \in \mathscr{P}_\mathrm{m}^+$.
(III) Finally, consider $S \nleq U_n \nleq W$. Let $p \in U_n \setminus W$. Since $U_0 \neq U_n$, there is a hyperplane H of U_0 that contains $U_0 \cap U_n$. By (TG3) we conclude $U_n' := \langle p, H \rangle \in \mathscr{P}_\mathrm{m}^+$. Let $V \in \mathfrak{U}^+$ with $S \cap U_n' \leq V \leq \langle p, W \rangle$. If $S \leq U_n'$, then $V \in \mathscr{P}_\mathrm{m}^+$ follows from (I). If $\langle U_0, U_n' \rangle = \langle p, W \rangle$, then $V \in \mathscr{P}_\mathrm{m}^+$ follows from (TG3). Therefore we may assume $S \nleq U_n'$ and $\langle U_0, U_n' \rangle < \langle p, W \rangle$. Since H is a hyperplane of U_0 and $H \leq U_n'$, we conclude that $S \cap U_n' = S \cap H$ is a hyperplane of S. Since $S < U_0$, we obtain $S \cap H < H$. Hence there is a hyperplane H' of H such that $S \cap H \leq H'$. Then $\langle S, H' \rangle$ is a hyperplane of U_0. Now let $q \in W \setminus U_0$. Then $U' := \langle q, S, H' \rangle \in \mathfrak{U}^+$. Moreover, $U' \in \mathscr{P}_\mathrm{m}^+$ since $S \leq U' \leq W$. Since $U_n' \cap W \leq U_0$, we conclude $U' \cap U_n' \leq \langle S, H' \rangle$ and therefore $U' \cap U_n' = H'$. Thus, $V_0 := \langle p, S, H' \rangle \in \mathscr{P}_\mathrm{m}^+$ by (TG3). Since U_0 and U_n are commensurate, we obtain $\mathrm{crk}_{U_n}(W \cap U_n) < \infty$. Hence, there is a family $(V_i)_{0 \leq i < m}$ for a natural number m such that $S = (\bigcap_{0 \leq i < n} U_i) \cap (\bigcap_{0 \leq i < m} V_i)$ and $\langle W, U_n \rangle = \langle U_i, V_j \mid i < n \wedge j < m \rangle$. Thus, the claim follows from the two cases (I) and (II). □

The following proposition is Axiom (TG3) in a much stronger version.

Proposition 5.3.7. *Let $n \in \mathbb{N}$ and let $(U_i)_{0 \leq i \leq n}$ be a family of elements of $\mathscr{P}_\mathrm{m}^+$. Then every subspace $V \in \mathfrak{U}^+$ with $\bigcap_{0 \leq i \leq n} U_i \leq V$ is an element of $\mathscr{P}_\mathrm{m}^+$.*

Proof. Set $S := \bigcap_{0 \leq i \leq n} U_i$. Since U_0 is commensurate to every element of $\mathscr{P}_\mathrm{m}^+$, the intersection of U_0 and U_i has finite corank in U_0 for every $i \leq n$. Since n is finite, this implies $\mathrm{crk}_{U_0}(S) < \infty$. Now let $V \in \mathfrak{U}^+$ with $S \leq V$. Then $\mathrm{crk}_V(S) = \mathrm{crk}_{U_0}(S)$. Hence by Lemma 5.3.5, there is a family $(U_i)_{n < i \leq m}$ of elements of $\mathscr{P}_\mathrm{m}^+$ such that $V \leq \langle S, U_i \mid n < i \leq m \rangle$, where $m \in \mathbb{N}$ with $n \leq m$. Now $\bigcap_{0 \leq i \leq m} U_i \leq V \leq \langle U_i \mid 0 \leq i \leq m \rangle$ and the claim follows from Lemma 5.3.6. □

Corollary 5.3.8. *Let* $\mathrm{rk}(U_+) < \infty$. *Then* $\mathscr{S}_\mathrm{m}^+ = \mathscr{D}^+$.

Proof. Let $V \in \mathfrak{U}^+$ and take an arbitrary element W_0 of $\mathscr{P}_\mathrm{m}^+$. Set $S_0 := W_0$. By Lemma 5.3.5 there is for every point $p \in S_0 \setminus V$ a subspace $W \in \mathscr{P}_\mathrm{m}^-$ with $p \in W$. Hence by (TG1), there is a subspace $W_1 \in \mathscr{P}_\mathrm{m}^+$ with $p \notin W_1$ and we obtain $S_1 := S_0 \cap W_1 < S_0$. Since $\mathrm{rk}(V) < \infty$, we may repeat this argument to obtain a finite family $(W_i)_{0 \leq i \leq n}$ of elements of $\mathscr{P}_\mathrm{m}^+$ such that $\bigcap_{0 \leq i \leq n} U_i \leq V$. Now the claim follows from Proposition 5.3.7. □

The analogous of this corollary for finite corank does not hold as we know from the observations of twin projective spaces we made.

Remark 5.3.9. Partial twin Grassmannians can be seen as a generalisation of Grassmannians. This is because every subspace of a projective space has a complement and hence, every Grassmannian of k-spaces is together with the Grassmannian of corank-$(k+1)$-spaces a twin Grassmannian.
Conversely, Corollary 5.3.8 implies that whenever U_+ has finite rank then $\mathscr{D}^+$ is a Grassmannian in the usual sense. Hence, the two parts of a partial twin Grassmannian can be seen as Grassmannians of α-spaces, where α is an arbitrary cardinal. However, the reader should keep in mind that if α is infinite and equals the rank of $\mathscr{S}$, then the corank β of the considered subspaces can be of any possible cardinal between 1 and α. Hence, in this case, it does not suffice to mention the rank of the considered subspace. As long as β is smaller than α one can talk about a Grassmannian of corank-β-spaces. If β equals α one should mention both the rank and the corank.

Remark 5.3.10. The only case where $\mathfrak{U}^+$ and $\mathfrak{U}^-$ are not disjoint is $\mathrm{rk}(\mathscr{S}) < \infty$ and $\mathrm{rk}(U_+) = \mathrm{rk}(U_-)$. Moreover, by Corollary 5.3.8 this implies that $\mathfrak{U}^+$ and $\mathfrak{U}^-$ are disjoint or equal.

The following proposition characterises the singular subspaces of $\mathscr{S}_\mathrm{m}$.

Proposition 5.3.11. *Let U and V be two elements of $\mathscr{P}^+_\mathrm{m}$ that intersect in a common hyperplane H and let W be the span of U and V in $\mathscr{S}$. Set $M_H := \{X \in \mathscr{P}^+_\mathrm{m} \mid H < X\}$ and $M_W := \{X \in \mathscr{P}^+_\mathrm{m} \mid X < W\}$. Further let $L := \{X \in \mathscr{P}^+_\mathrm{m} \mid H < X < W\}$ be the element of $\mathscr{L}^+_\mathrm{m}$ that contains U and V.*

(i) *If U is a hyperplane of $\mathscr{S}$, then $M_H = L$. Otherwise M_H is a maximal singular subspace of $\mathscr{S}^+_\mathrm{m}$ with* $\mathrm{rk}(M_H) = \mathrm{crk}_{\mathscr{S}}(U)$.

(ii) *If U is a singleton, then $M_W = L$. Otherwise M_W is a maximal singular subspace of $\mathscr{S}^+_\mathrm{m}$.*

(iii) *Every subspace $Z \in \mathscr{P}^+_\mathrm{m}$ that intersects both U and V in a hyperplane is an element of M_H or of M_W.*

Proof. Let H' be a complement to H in $\mathscr{S}$. Further let X and Y be two distinct elements of M_H. Then $H = X \cap Y$ is a common hyperplane of X and Y since both are commensurate to U. Since H' is a complement to H and $X \neq Y$, there are distinct points x and y in H' such that $X \cap H' = \{x\}$ and $Y \cap H' = \{y\}$. We conclude that $G := \{Z \in \mathfrak{U}^+ \mid H \leq Z \leq \langle xy, H\rangle\}$ is the element of $\mathscr{L}^+_\mathrm{m}$ that contains both X and Y. Since $G \subseteq M_H$, we obtain that M_H is a singular subspace of $\mathscr{S}^+_\mathrm{m}$. Furthermore, a subspace $Z \in M_H$ is contained in G if and only if Z intersects xy. Since by Proposition 5.3.7 we know $\langle p, H\rangle \in \mathscr{S}^+_\mathrm{m}$ for every $p \in H'$, we conclude that $H' \to M_H :\ p \mapsto \langle p, H\rangle$ is an isomorphism from H' onto the singular subspace M_H of $\mathscr{S}^+_\mathrm{m}$. Therefore $\mathrm{rk}(M_H) = \mathrm{rk}(H') = \mathrm{crk}_{\mathscr{S}}(H) + 1 = \mathrm{crk}(U)$.

Now let X and Y be two distinct elements of M_W. Then both X and Y are hyperplanes of W and hence, we conclude that X and Y intersect in a common hyperplane. Thus, there is an element $G \in \mathscr{L}_{\mathrm{m}}^{+}$ that contains both X and Y and every element of G is contained in $\langle X,Y\rangle = W$. Thus, M_W is a singular subspace of $\mathscr{S}_{\mathrm{m}}^{+}$. If U is a singleton, then $H = \varnothing$ and hence, $M_W = L$. Otherwise H contains a point p. By Lemma 5.3.5 there is a subspace $Z \in \mathscr{P}_{\mathrm{m}}^{-}$ with $p \in Z$. Thus, (TG1) implies that there is a subspace $Y \in \mathscr{P}_{\mathrm{m}}^{+}$ with $p \notin Y$. Let X be a hyperplane of W that contains $H \cap Y$ and does not contain p. Then by Proposition 5.3.7 we obtain $X \in \mathscr{P}_{\mathrm{m}}^{+}$ and consequently, $X \in M_W$. Since $p \notin X$, we obtain $H \nleq X$ and therefore $M_W > L$.

To show that both M_H and M_W are maximal singular subspaces if they are not equal to L, it remains to prove (iii). Let $Z \in \mathscr{P}_{\mathrm{m}}^{+}$ be a subspace that intersects both U and V in a hyperplane. Assume $H \nleq Z$. Then $U \cap Z$ and $V \cap Z$ are distinct hyperplanes of Z. This implies $Z = \langle U \cap Z, V \cap Z\rangle$ and hence, $Z \le \langle U,V\rangle = W$. ☐

Our goal is to prove that $\mathscr{S}_{\mathrm{m}}$ is a twin SPO space. Therefore, we first show how the distance of two elements of $\mathscr{P}_{\mathrm{m}}^{+}$ and their convex span in $\mathscr{S}_{\mathrm{m}}$ can be expressed in terms of $\mathscr{S}$.

Proposition 5.3.12. *Let $\{U,V\} \subseteq \mathscr{P}_{\mathrm{m}}^{+}$. Then the following claims hold:*

(i) *The distance of U and V in $\mathscr{S}_{\mathrm{m}}^{+}$ is finite and equals* $\mathrm{crk}_U(U \cap V)$.

(ii) *The subspace $\langle U,V\rangle_{\mathrm{g}}$ of $\mathscr{S}_{\mathrm{m}}^{+}$ consists of all subspaces $W \in \mathfrak{U}^{+}$ with $U \cap V \le W \le \langle U,V\rangle$.*

Proof. (i) By definition of $\mathscr{L}_{\mathrm{m}}^{+}$ the distance between U and V is at least $\mathrm{crk}_U(U \cap V)$. Since U and V are commensurate, $\mathrm{crk}_U(U \cap V)$ is finite. For $U = V$, there is nothing to prove. Hence we may assume that there is a point $p \in V \setminus U$. Let H be a hyperplane of U containing $U \cap V$. Set $U' := \langle p,H\rangle$. Then U and U' are commensurate and hence, $U' \in \mathscr{P}_{\mathrm{m}}^{+}$ by (TG3). Since U and U' are collinear in $\mathscr{S}_{\mathrm{m}}^{+}$ and $\langle p, U \cap V\rangle \le U' \cap V$, the claim follows by induction.

(ii) By G we denote the convex span $\langle U,V\rangle_{\mathrm{g}}$ viewed as a subspace of $\mathscr{S}_{\mathrm{m}}^{+}$. Further we set $H := \{W \in \mathfrak{U}^{+} \mid U \cap V \le W \le \langle U,V\rangle\}$.

Let $W \in H$. Then $W \in \mathscr{P}_{\mathrm{m}}^{+}$ by (TG3). If U and V have a hyperplane in common, we obtain $W \in G$ by definition of $\mathscr{L}_{\mathrm{m}}^{+}$. Now let $\mathrm{crk}_U(U \cap V) = d > 1$. We prove $W \in G$ by induction and hence we assume that the claim holds for every two elements U' and V' of $\mathscr{P}_{\mathrm{m}}^{+}$ with $\mathrm{crk}_{U'}(U' \cap V') < d$. For $0 \le i < d$, there are points $p_i \in W$ such that $\langle p_i, U \cap V \mid 0 \le i < d\rangle = W$. If $W \cap (U \cup V) \ge U \cap V$, we may assume $p_{d-1} \in U \cup V$. Since $W \le \langle U,V\rangle$, for every $i < d$ there is a line through p_i that intersects both U and V. Hence for $0 \le i < d$, there are points $q_i \in U$ and $r_i \in V$ such that $\langle q_i, U \cap V \mid 0 \le i < d\rangle = U$, $\langle r_i, U \cap V \mid 0 \le i < d\rangle = V$ and $p_i \in \langle q_j, r_j, U \cap V \mid 0 \le j \le i\rangle$. Since $\mathrm{crk}_{\langle U,V\rangle}(U \cap V) = 2d$, the set $\{q_i, r_i \mid$

$0 \le i < d\}$ is independent.
First assume $p_{d-1} \in U$. Set $V' := \langle p_{d-1}, r_i, U \cap V \mid 0 \le i < d-1 \rangle$. Then V and V' are collinear in $\mathscr{S}_{\mathrm{m}}$ and the distance of U and V' is $d-1$. Hence, $V' \in G$. Then $U \cap V' = \langle p_{d-1}, U \cap V \rangle$ and therefore $U \cap V' \le W$. Since $p_{d-1} \in U$ and $\langle p_i \mid 0 \le i < d-1 \rangle \le \langle q_i, r_i, U \cap V \mid 0 \le i < d-1 \rangle \le \langle U, V' \rangle$, we conclude $W \in \langle U', V \rangle_{\mathrm{g}} \le G$ by the induction hypothesis.
The case $p_{d-1} \in V$ is analogous, therefore we may now assume $W \cap U = W \cap V = U \cap V$. Set $V' := \langle q_0, r_i, U \cap V \mid 1 \le i < d \rangle$. Then V and V' are collinear in $\mathscr{S}_{\mathrm{m}}$. Let $L \in \mathscr{L}_{\mathrm{m}}^+$ such that $\{V, V'\} \subseteq L$. Since the distance of U and V' is $d-1$ we obtain $V' \in G$. Now $V \cap V' = \langle r_i, U \cap V \mid 1 \le i < d \rangle$ and $\langle V, V' \rangle = \langle q_0, r_0, V \cap V' \rangle$ and therefore $W' := \langle p_0, r_i, U \cap V \mid 1 \le i < d \rangle \in L$. Since V is a hyperplane of $\langle V, V' \rangle$, we conclude $U \cap \langle V, V' \rangle = \langle q_0, U \cap V \rangle$. Thus, V' is the only element of L at distance $d-1$ to U. This implies $\mathrm{crk}_U(U \cap W') = d$. Since $W \cap W' = \langle p_0, U \cap V \rangle$, we obtain $W \in \langle U, W' \rangle_{\mathrm{g}}$ as above. Since $V' \in G$, we obtain $L \le G$ and hence $W \in \langle U, W' \rangle_{\mathrm{g}} \le G$.
Now as we know $H \subseteq G$ it remains to show that H is a convex subspace of $\mathscr{S}_{\mathrm{m}}^+$. Let U' and V' be two elements of H. Assume U' and V' are collinear in $\mathscr{S}_{\mathrm{m}}^+$ and let $L \in \mathscr{L}_{\mathrm{m}}^+$ such that $\{U', V'\} \subseteq L$. Then by definition every element of L is contained in $\langle U', V' \rangle \le \langle U, V \rangle$ and contains $U' \cap V' \ge U \cap V$. Now assume U' and V' are at distance $d \ge 1$ in $\mathscr{S}_{\mathrm{m}}^+$. Let $W \in \mathscr{P}_{\mathrm{m}}^+$ such that W is collinear to V' in $\mathscr{S}_{\mathrm{m}}^+$ and has distance $d-1$ to U'. Then there is a point $p \in U' \cap W \setminus V'$. Since V' and W have a hyperplane H in common, we obtain $W = \langle p, H \rangle \le \langle U', V' \rangle \le \langle U, V \rangle$. This implies $\mathrm{crk}_{U'}(U' \cap V') = \mathrm{crk}_{U'}(U' \cap H)$ and hence, $U' \cap V' = U' \cap H$. Thus, $W \ge U' \cap V' \ge U \cap V$. □

We now study the codistance of the twin space $\mathscr{S}_{\mathrm{m}}$. If we talk in the following of a codistance, we mean always the codistance in $\mathscr{S}_{\mathrm{m}}$ since the distance or the codistance in $\mathscr{S}$ it at most 2 and therefore can be expressed by collinearity and intersection.

Proposition 5.3.13. *Let $U \in \mathscr{P}_{\mathrm{m}}^+$ and $V \in \mathscr{P}_{\mathrm{m}}^-$. Then the codistance of U and V is finite and equals* $\mathrm{rk}(U \cap V) + 1$.

Proof. We obtain $r := \mathrm{rk}(U \cap V) < \infty$ since U has a complement U' in $\mathscr{P}_{\mathrm{m}}^-$ and every two elements of $\mathscr{P}_{\mathrm{m}}^-$ are commensurate in $\mathscr{S}$. Since $\mathrm{rk}(U \cap V) = r$ and V is commensurate to U', we obtain $\mathrm{crk}_{\mathscr{S}}(\langle U, V \rangle) = r+1$. Hence, there is a subspace $S \le U'$ with $\mathrm{rk}(S) = r$ such that $S \cap \langle U, V \rangle = \varnothing$. Moreover, S is a complement of $\langle U, V \rangle$. Since $U \cap U' = \varnothing$, there is a subspace $T \le V$ with $V \cap U' \le T$ such that T is a complement to U in $\langle U, V \rangle$. Since $\mathrm{rk}(U \cap V) = r$, we obtain $\mathrm{crk}_V(T) = r+1$. Hence, $V' := \langle S, T \rangle$ and V are commensurate and therefore $V' \in \mathfrak{U}^-$.
Since $\langle U, V' \rangle = \langle U, T, S \rangle = \langle U, V, S \rangle = \mathscr{S}$, we conclude that V' is a complement to U. Since $V \cap U' \le T \le V'$ and $V' = \langle S, T \rangle \le \langle U', V \rangle$, we conclude $V' \in \mathscr{P}_{\mathrm{m}}^-$ by

(TG3). Now V and V' have distance $r+1$ in $\mathscr{S}_{\mathrm{m}}^{-}$. Furthermore, every complement of U that is an element of $\mathscr{P}_{\mathrm{m}}^{-}$ is disjoint to $U \cap V$ and hence has distance $\geq r+1$ to V. Thus, $\operatorname{cod}(U,V) = r+1$. □

As preparation to show that $\mathscr{S}_{\mathrm{m}}$ satisfies the conditions given in Definition 2.1.1 we study the codistance of a given point of $\mathscr{S}_{\mathrm{m}}^{+}$ to the elements of the convex span of two points in $\mathscr{S}_{\mathrm{m}}^{-}$.

Lemma 5.3.14. *Let* $X \in \mathscr{P}_{\mathrm{m}}^{+}$ *and* $\{Y,Z\} \leq \mathscr{P}_{\mathrm{m}}^{-}$. *Set* $m := \operatorname{rk}(X \cap Y \cap Z) + 1$, $d := \operatorname{rk}(X \cap \langle Y,Z \rangle) + 1$ *and* $\operatorname{crk}_Y(Y \cap Z) = n$. *Then* $\max\{m, d-n\} - 1 \leq \operatorname{rk}(X \cap V) \leq \min\{m+n, d\} - 1$ *for every* $V \in \langle Y,Z \rangle_{\mathrm{g}}$. *Moreover, both bounds are sharp and* $\operatorname{rk}(X \cap Z) = \min\{m+n, d\} - 1$ *implies* $\operatorname{rk}(X \cap Y) = \max\{m, d-n\} - 1$.

Proof. We know that d, n and m are all finite. Since $\operatorname{crk}_V(Y \cap Z) = n$ for every $V \in \langle Y,Z \rangle_{\mathrm{g}}$, we obtain $m-1 \leq \operatorname{rk}(X \cap V) \leq m+n-1$. Since $\operatorname{crk}_{\langle Y,Z \rangle}(V) = n$ for every $V \in \langle Y,Z \rangle_{\mathrm{g}}$, we obtain $d-n-1 \leq \operatorname{rk}(X \cap V) \leq d-1$.
Let $\{p_i \mid 0 \leq i < d-m\}$ be a set of points such that $\langle p_i, X \cap Y \cap Z \mid 0 \leq i < d-m \rangle = X \cap \langle Y,Z \rangle$. If $d \leq m+n$, then the corank of $(Y \cap Z)$ in $\langle p_i, Y \cap Z \mid 0 \leq i < d-m \rangle$ is at most n. Thus there is a subspace $V \in \mathfrak{U}^{-}$ with $V \in \langle Y,Z \rangle_{\mathrm{g}}$ such that $V \geq \langle p_i, Y \cap Z \mid 0 \leq i < d-m \rangle$ and hence, $\operatorname{rk}(X \cap V) = d-1$. Assume $\operatorname{rk}(X \cap Z) = d-1$. Then $X \cap Z = X \cap \langle Y,Z \rangle$ and therefore $X \cap Y = X \cap Y \cap Z$. Thus, $\operatorname{rk}(X \cap Y) = m-1$. If $d > m+n$, then $V := \langle p_i, Y \cap Z \mid 0 \leq i < n \rangle \in \langle Y,Z \rangle_{\mathrm{g}}$ and $\operatorname{rk}(X \cap V) = m+n-1$. Assume $\operatorname{rk}(X \cap Z) = m+n-1$. Since $\operatorname{rk}(X \cap \langle Y,Z \rangle) \geq \operatorname{rk}(X \cap Y) + \operatorname{rk}(X \cap Z) - \operatorname{rk}(X \cap Y \cap Z)$, we conclude $\operatorname{rk}(X \cap Y) = d-n-1$. □

Theorem 5.3.15. *Every partial twin Grassmannian is a rigid twin SPO space whose symplecta are all of rank* 3 *and whose lines are contained in at most two maximal singular subspaces.*

Proof. Let $\mathscr{S}_{\mathrm{m}} = (\mathscr{S}_{\mathrm{m}}^{+}, \mathscr{S}_{\mathrm{m}}^{-})$ be a partial twin Grassmannian of a projective space $\mathscr{S}$, where $\mathscr{S}_{\mathrm{m}}^{\sigma} = (\mathscr{P}_{\mathrm{m}}^{\sigma}, \mathscr{L}_{\mathrm{m}}^{\sigma})$ for $\sigma \in \{+,-\}$. For elements $U \in \mathscr{P}_{\mathrm{m}}^{+}$ and $V \in \mathscr{P}_{\mathrm{m}}^{-}$, we write $U \leftrightarrow V$ if and only if U and V are complements in $\mathscr{S}$.
Let $X \in \mathscr{P}_{\mathrm{m}}^{+}$ and let Y and Z be elements of $\mathscr{P}_{\mathrm{m}}^{-}$. Then $n := \operatorname{crk}_Y(Y \cap Z) < \infty$ is the distance of Y and Z in $\mathscr{S}_{\mathrm{m}}^{-}$. In the following $\langle Y,Z \rangle_{\mathrm{g}}$ denotes the subspace of $\mathscr{S}_{\mathrm{m}}$ that is the convex span of the two points Y and Z of $\mathscr{P}_{\mathrm{m}}^{-}$. By $\langle Y,Z \rangle$ we always mean the subspace of $\mathscr{S}$ that is spanned by the two subspaces Y and Z of $\mathscr{S}$.
First assume that the subspace $\langle Y,Z \rangle_{\mathrm{g}}$ contains an element $U \leftrightarrow X$. By Lemma 5.3.12(ii) we obtain $Y \cap Z \leq U \leq \langle Y,Z \rangle$. Since $\operatorname{crk}_{\langle Y,Z \rangle}(Y) = n$ and U and Y are commensurate, we conclude $\operatorname{crk}_{\langle Y,Z \rangle}(U) = n$. Since U and X are complements, this implies $\operatorname{rk}(X \cap \langle Y,Z \rangle) = n-1$. Let $\{p_i \mid 0 \leq i < n\}$ be a set of points of $\mathscr{S}$ such that $\langle p_i \mid 0 \leq i < n \rangle = X \cap \langle Y,Z \rangle$. Set $X' := \langle p_i, Y \cap Z \mid 0 \leq i < n \rangle$. Since $X \cap U = \varnothing$, we know $X \cap Y \cap Z = \varnothing$. Thus, $\langle p_i \mid 0 \leq i < n \rangle \cap Y \cap Z = \varnothing$ and therefore $\operatorname{crk}_{X'}(Y \cap Z) = n$. Hence, Lemma 5.3.12(ii) implies $X' \in \langle Y,Z \rangle_{\mathrm{g}}$.

Since every element $V \in \langle Y,Z\rangle_{\mathrm{g}}$ contains $Y \cap Z$ and is contained in $\langle Y,Z\rangle$, we conclude $\mathrm{rk}(X \cap V) \geq n-1$ if and only if $\langle p_i \mid 0 \leq i < n\rangle \leq V$. We conclude $\mathrm{cod}(X,\langle Y,Z\rangle_{\mathrm{g}}) = n$ and X' is the only element in the coprojection of X' in $\langle Y,Z\rangle_{\mathrm{g}}$. Thus, (A1) and (A2) are fulfilled.

From now on $\langle Y,Z\rangle_{\mathrm{g}}$ does not necessarily contain an element that is opposite X. Set $m := \mathrm{rk}(X\cap Y\cap Z)+1$ and $d := \mathrm{rk}(X\cap\langle Y,Z\rangle)+1$. Then we obtain $\max\{m,d-n\}-1 \leq \mathrm{rk}(X\cap V) \leq \min\{m+n,d\}-1$ for every $V \in \langle Y,Z\rangle_{\mathrm{g}}$ by Lemma 5.3.14. Assume Z is in the coprojection of X in $\langle Y,Z\rangle_{\mathrm{g}}$. Then Lemma 5.3.14 implies $\mathrm{rk}(X\cap Z) = \min\{m+n,d\}-1$ and $\mathrm{rk}(X\cap Y) = \max\{m,d-n\}-1$. Let $W \in \mathscr{P}_{\mathrm{m}}^{+}$ such that X and W intersect in a common hyperplane and $\mathrm{rk}(W\cap Y) = \mathrm{rk}(X\cap Y)-1$. First assume $W\cap\langle Y,Z\rangle \leq X\cap\langle Y,Z\rangle$. Since $\mathrm{rk}(W\cap Y) = \mathrm{rk}(X\cap Y)-1$, we obtain $W\cap\langle Y,Z\rangle \neq X\cap\langle Y,Z\rangle$ and hence, $W\cap\langle Y,Z\rangle < X\cap\langle Y,Z\rangle$. Furthermore, $V\cap W \leq V\cap X$ for every $V \in \langle Y,Z\rangle_{\mathrm{g}}$. If $W\cap Y\cap Z < X\cap Y\cap Z$, then $V\cap W < V\cap X$ for every $V \in \langle Y,Z\rangle_{\mathrm{g}}$ and hence (A3) is fulfilled. Therefore we may assume $W\cap Y\cap Z = X\cap Y\cap Z$. Suppose $d \leq m+n$. Then $\mathrm{rk}(X\cap Z) = d-1$ and hence, $Z \geq X\cap\langle Y,Z\rangle$. This implies $X\cap Y = X\cap Y\cap Z = W\cap Y\cap Z$, a contradiction to $W\cap Y < X\cap Y$. Thus, $W\cap Y\cap Z = X\cap Y\cap Z$ yields $d > m+n$. Since $W\cap\langle Y,Z\rangle$ contains a hyperplane of $X\cap\langle Y,Z\rangle$, Lemma 5.3.14 implies $\mathrm{cod}(W,\langle Y,Z\rangle_{\mathrm{g}}) = \mathrm{cod}(X,\langle Y,Z\rangle_{\mathrm{g}}) = m+n$. Since $W\cap\langle Y,Z\rangle < X\cap\langle Y,Z\rangle$, there is a subspace $S \leq \langle Y,Z\rangle\cap X$ with $\mathrm{rk}(S) = n-1$ such that $S\cap Y\cap Z = \varnothing$ and $S \not\leq W$. Set $Z' := \langle S, Y\cap Z\rangle$. We conclude $Z' \in \mathscr{P}_{\mathrm{m}}^{-}$ by (TG3) and therefore $Z' \in \langle Y,Z\rangle_{\mathrm{g}}$ by Lemma 5.3.12(ii). Since $\mathrm{rk}(X\cap Z') = \langle S, X\cap Y\cap Z\rangle$, we obtain $\mathrm{cod}(X,Z') = n+m$ and consequently, Z' is in the coprojection of X in $\langle Y,Z\rangle_{\mathrm{g}}$. Since $S \not\leq W$, we obtain $X\cap Z' > W\cap Z'$ and hence, $\mathrm{cod}(W,Z') = n+m-1$. Thus, the coprojection of W in $\langle Y,Z\rangle_{\mathrm{g}}$ is properly contained in the coprojection of X in $\langle Y,Z\rangle_{\mathrm{g}}$. Therefore (A3) is fulfilled.

Now assume $W\cap\langle Y,Z\rangle \not\leq X\cap\langle Y,Z\rangle$. Since $W\cap Y < X\cap Y$, we conclude $W\cap\langle Y,Z\rangle \not\geq X\cap\langle Y,Z\rangle$ and consequently, $\mathrm{rk}(W\cap\langle Y,Z\rangle) = d-1$ since W and X have a hyperplane in common. Furthermore, $W\cap Y < X\cap Y$ yields $W\cap Y\cap Z \leq X\cap Y\cap Z$. Set $m' := \mathrm{rk}(W\cap Y\cap Z)+1$. By Lemma 5.3.14 we conclude $\mathrm{rk}(X\cap Y) = \max\{m,d-n\}$ and $\mathrm{rk}(W\cap Y) = \max\{m',d-n\}$. Since $W\cap Y$ is a hyperplane of $X\cap Y$, this implies $m' = m-1 \geq d-n$. Thus, $W\cap Y\cap Z$ is a hyperplane of $X\cap Y\cap Z$. Consequently, $W\cap V \not\geq X\cap V$ for every $V \in \langle Y,Z\rangle_{\mathrm{g}}$ and therefore $\mathrm{rk}(W\cap V) \leq \mathrm{rk}(X\cap V)$. Since $m'+n = m+n-1 \geq d$, we conclude $\mathrm{cod}(W,\langle Y,Z\rangle_{\mathrm{g}}) = \mathrm{cod}(X,\langle Y,Z\rangle_{\mathrm{g}}) = d$ by Lemma 5.3.14. This implies that the coprojection of W in $\langle Y,Z\rangle_{\mathrm{g}}$ is contained in the coprojection of X in $\langle Y,Z\rangle_{\mathrm{g}}$. Set $S := \langle X\cap\langle Y,Z\rangle, Y\cap Z\rangle$. Then $\mathrm{crk}_S(Y\cap Z) = d-m$ since $X\cap Y\cap Z$ has corank $d-m$ in $X\cap\langle Y,Z\rangle$. Since $\mathrm{crk}_{(W\cap\langle Y,Z\rangle)}(W\cap Y\cap Z) = d-m+1$, we conclude $W\cap\langle Y,Z\rangle \not\leq S$ and more precisely, S intersects $W\cap\langle Y,Z\rangle$ in a hyperplane. Let $p \in (W\cap\langle Y,Z\rangle)\setminus S$ and let $T \leq \langle Y,Z\rangle$ be a subspace of rank $n+m-d-1$ that is disjoint to $\langle p,S\rangle$. Then $\langle T,S\rangle \in \mathscr{P}_{\mathrm{m}}^{-}$ by (TG3) and hence, $\langle T,S\rangle \in \langle Y,Z\rangle_{\mathrm{g}}$. Since

$\langle T,S\rangle \geq X\cap\langle Y,Z\rangle$, we obtain $\mathrm{cod}(X,\langle T,S\rangle) = d$. Since $p \notin \langle T,S\rangle$, we obtain $\mathrm{cod}(W,\langle T,S\rangle) < d$ and therefore the coprojection of W in $\langle Y,Z\rangle_{\mathrm{g}}$ is properly contained in the coprojection of X in $\langle Y,Z\rangle_{\mathrm{g}}$. Thus, (A3) holds.
We already know that $\mathscr{S}_{\mathrm{m}}$ consists of two connected components. Moreover, by Proposition 5.3.11 we know that every line of $\mathscr{S}_{\mathrm{m}}$ is contained in at most two maximal singular subspaces. Therefore it remains to prove that all symplecta of $\mathscr{S}_{\mathrm{m}}$ have rank 3 if there are any. Let $\{Y,Z\} \subseteq \mathscr{P}^-_{\mathrm{m}}$ with $\mathrm{crk}_Y(Y\cap Z) = 2$. Further let U and V be elements of $\langle Y,Z\rangle_{\mathrm{g}}$ such that U and V intersect in a common hyperplane H. Since V is a hyperplane of $\langle U,V\rangle$ and $\mathrm{crk}_{\langle Y,Z\rangle}(V) = 2$, there is a point $p \in \langle Y,Z\rangle \setminus \langle U,V\rangle$. Set $W := \langle p,H\rangle$. We obtain $W \in \langle Y,Z\rangle_{\mathrm{g}}$. Since $H \leq W$, both U and V are collinear to W in $\mathscr{S}_{\mathrm{m}}$. Since $W \nleq \langle U,V\rangle$, there is no line of $\mathscr{S}_{\mathrm{m}}$ that contains U, V and W. Hence, $\mathrm{rk}(\langle Y,Z\rangle_{\mathrm{g}}) \geq 3$.
Now let $X \in \langle Y,Z\rangle_{\mathrm{g}}$ such that X has a common hyperplane with all U, V and W. Then $H \leq X$ by Proposition 5.3.11. Let $q \in U$ and $r \in V$ such that $\langle q,H\rangle = U$ and $\langle r,H\rangle = V$. Since $\langle p,q,r,H\rangle = \langle Y,Z\rangle$ and $\mathrm{crk}_{\langle Y,Z\rangle}(X) = 2$, we obtain $X\cap\langle p,q,r\rangle \neq \varnothing$. Thus, there is a point $s \in qr$ such that $X \cap ps \neq \varnothing$. Set $W' := \langle s,H\rangle$. Then $W' \in L$, where $L \in \mathscr{L}^-_{\mathrm{m}}$ such that $\{U,V\} \subseteq L$. Consequently, $X \in K$, where $K \in \mathscr{L}^-_{\mathrm{m}}$ such that $\{W,W'\} \subseteq K$. This implies that $\{\langle x,H\rangle \mid x \in \langle p,q,r\rangle\}$ is a generator of $\langle Y,Z\rangle_{\mathrm{g}}$ and hence, the symplecton $\langle Y,Z\rangle_{\mathrm{g}}$ has rank 3. ☐

Proposition 5.3.16. *Let U and V be two elements of $\mathscr{P}^+_{\mathrm{m}}$ that intersect in a common hyperplane. Let W be the span of U and V in $\mathscr{S}$ and set $M_W := \{X \in \mathscr{P}^+_{\mathrm{m}} \mid X < W\}$. Then $\mathrm{rk}(W) = \mathrm{rk}(M_W)$ or both $\mathrm{rk}(W)$ and $\mathrm{rk}(M_W)$ are infinite.*

Proof. Let $L \in \mathscr{L}^+_{\mathrm{m}}$ be the line that contains U and V. By Proposition 5.3.11 we know that M_W is a singular subspace of $\mathscr{S}^+_{\mathrm{m}}$. Moreover, it follows that the claim holds for the case $\mathrm{rk}(U) = 0$. Thus, we may assume $\mathrm{rk}(U) \geq 1$ and consequently, that M_W contains a subspace S of rank 2 with $L \leq S$.
By Theorem 5.3.15 we know that $(\mathscr{S}^+,\mathscr{S}^-)$ is a twin SPO space. Proposition 2.3.5 implies that there is a singular subspace $S' \leq \mathscr{S}^-_{\mathrm{m}}$ of rank 2 such that S' and S are opposite. Since every element of L is opposite to an element in S' and no element of S' is opposite to all elements of L, we conclude by Corollary 4.2.8 that there is a line $L' \leq S'$ that is opposite L. Let $U' \in L'$ with $\mathrm{cod}(U,U') = 1$ and $V' \in L'$ with $\mathrm{cod}(V,V') = 1$.
Set $H := U'\cap V'$ and $W' := \langle U',V'\rangle$. Since U and V' are complementary subspaces of $\mathscr{S}$, we know $H\cap U = \varnothing$. For an arbitrary point $p \in W \setminus U$, the subspace $\langle p, U\cap V\rangle$ is contained in L and hence, there is an element of L' that is disjoint to $\langle p,U\cap V\rangle$. Thus, $p \notin H$ and we conclude that H is a complement to W. Since H has corank 2 in W', the subspaces W and W' intersect in a line l of $\mathscr{S}$.
By Proposition 5.3.11 we know that S' is either a subspace of $M_H := \{X \in \mathscr{P}^-_{\mathrm{m}} \mid H < X\}$ or of $M_{W'} := \{X \in \mathscr{P}^-_{\mathrm{m}} \mid X < W'\}$. Let T be an arbitrary subspace of rank 2 of $M_{W'}$ that contains L' and let $X \in T \setminus L'$. Then $H \nleq X$ and since X

is a hyperplane of W', we conclude that $X \cap H$ has corank 3 in W'. Since l is disjoint to $X \cap H$, we know that $Y := \langle l, X \cap H \rangle$ is a hyperplane of W' and since $X \cap H = X \cap U' \cap V'$ we conclude $Y \in M_{W'}$ by Proposition 5.3.7. Every element of M_W is a hyperplane of W and hence, intersects l. Thus, Y intersects all elements of M_W. If $X \neq Y$, then $X \cap Y$ has corank 2 in W' and $\langle X, Y \rangle = W'$. Since $\langle l, H \rangle = W'$, we obtain $H \nleq X \cap Y$. Furthermore, since $X \cap H$ is a hyperplane of H, we conclude that $X' := \langle H, X \cap H \rangle$ is an element of the line of $\mathscr{S}_{\mathrm{m}}^{-}$ that contains X and Y. Since $X' \in L'$ and $X \in T \setminus L'$, this implies that Y is an element of T. Thus, T is not opposite to S and we conclude $S' \neq T$. Therefore $S' \leq M_H$.
Now let $R \leq M_W$ be a subspace of finite rank that contains S. If M_W has finite rank, we may assume $R = M_W$. Again by Proposition 2.3.5 there is a singular subspace $R' \leq \mathscr{S}_{\mathrm{m}}^{-}$ of rank $\mathrm{rk}(R)$ that is opposite R. Since this implies that R' contains a subspace that is opposite S, we may assume $S' \leq R'$ and hence, $R' \leq M_H$. Thus, if $\mathrm{rk}(M_W)$ is finite we conclude $\mathrm{rk}(M_W) \leq \mathrm{rk}(M_H)$ and if $\mathrm{rk}(M_W)$ is infinite, $\mathrm{rk}(M_H)$ is infinite, too. Assume $R' < M_H$. Then M_H is not opposite R by Proposition 2.3.5. Hence, there is an element $X \in M_H$ that intersects every element of R. By Lemma 2.1.21(ii) this implies that R is not a maximal singular subspace and consequently $R < M_W$ by Proposition 5.3.11. We conclude that either M_W and M_H have the same finite rank or both are of infinite rank. Now the claim follows from Proposition 5.3.11 since $\mathrm{crk}_{\mathscr{S}}(V) = \mathrm{rk}(U) + 1$. □

5.4 Half-spin spaces

In [Shu94, p. 441] half-spin spaces are introduced as geometries arising from a certain polar space: Let q be a quadratic form on a vector space of finite dimension $2r$ with $r > 1$ such that there exist totally singular subspaces of dimension r. Let $\mathscr{S}_q$ be the point-line space whose points are the 1-dimensional singular subspaces and whose lines are the totally singular 2-dimensional subspaces. Then $\mathscr{S}_q$ is a polar space of rank r with bipartite dual polar graph. Let $\{\mathfrak{M}_0, \mathfrak{M}_1\}$ be a partition of the generators of $\mathscr{S}_q$ such that every edge of the dual polar graph has exactly one vertex in $\mathfrak{M}_0$. Since $\mathscr{S}_q$ has finite rank r, every singular subspace of rank $r-2$ is the intersection of two generators of $\mathscr{S}_q$. Since the dual polar graph of $\mathscr{S}_q$ is bipartite, this implies that every singular subspace of rank $r-2$ is contained in unique elements of $\mathfrak{M}_0$ and $\mathfrak{M}_1$.

Let $\mathfrak{U}_{r-3}$ be the set of singular subspaces of $\mathscr{S}_q$ of rank $r-3$. The elements of $\mathfrak{U}_{r-3}$ correspond canonically to the totally singular subspaces of dimension $r-2$ with respect to q of the underlying vector space. Two distinct elements of $\mathfrak{M}_0$ that are adjacent to a common element of $\mathfrak{M}_1$ intersect in an element of $\mathfrak{U}_{r-3}$. Conversely, every element of $\mathfrak{U}_{r-3}$ is contained in a singular subspace of rank $r-2$ and hence, in an element of $\mathfrak{M}_0$. Since $\mathscr{S}_q$ has finite rank, this implies that every element of $\mathfrak{U}_{r-3}$ is the intersection of two generators where one is an element of $\mathfrak{M}_0$. By Proposition 2.2.8 we conclude that in this case, both generators are elements of $\mathfrak{M}_0$. Hence, there is a point-line space whose points are the elements of $\mathfrak{M}_0$ and whose lines are determined by the elements of $\mathfrak{U}_{r-2}$ in the canonical way. Such a point-line space is called a *half-spin space*.

5.4.1 Local half-spin spaces

Similar to the definition of half-spin spaces there is a way to define point-line spaces out of polar spaces of arbitrary rank:

Definition 5.4.1. Let $\mathscr{S}$ be a polar space and let $M \leq \mathscr{S}$ be a generator such that the connected component of the dual polar graph containing M is bipartite. Let $\mathfrak{M}_0$ be the set of generators of $\mathscr{S}$ that are commensurate with M and have even distance to M in the dual polar graph. Set $\mathfrak{U}^2 := \{N \cap L \mid \{N, L\} \subseteq \mathfrak{M}_0 \wedge \mathrm{crk}_N(N \cap L) = 2\}$. Then we call the point-line space $(\mathfrak{M}_0, \{\{N \in \mathfrak{M}_0 \mid S \leq N\} \mid S \in \mathfrak{U}^2\})$ a *local half-spin space of* $\mathscr{S}$.

Compared to half-spin spaces we consider for defining the lines in a local half-spin space only those subspaces of a generator with corank 2 which can be obtained as intersection of two generators. This is because for polar spaces of arbitrary rank it can happen that there are subspaces of corank 2 in a generator that are contained in no other generator. Hence, by the definition of $\mathfrak{U}^2$ we make sure that a local half-spin space is a point-line space. A point-line space that is isomorphic to a local half-spin space of a polar space is called a local half-spin space.

Note that a local half-spin space of a polar space is a structure that can be recovered out of the dual polar space. Hence, as for dual polar spaces, we may restrain ourselves to local half-spin spaces of non-degenerate polar spaces.

Remark 5.4.2. Since by Proposition A.2.20 the dual polar space of a polar space of finite rank is connected, we know that a half-spin space of a polar space of finite rank is a local half-spin space. Conversely, a local half-spin space of a polar space of finite rank is already a half-spin space.

Throughout this section let $\mathscr{S}$ be a non-degenerate polar space. Further let $\mathscr{D}$ be a local half-spin space of $\mathscr{S}$. The point set of $\mathscr{D}$ is denoted by $\mathfrak{M}_0$ and the line set by $\mathscr{L}$. Further let $\mathscr{S}_\mathrm{m}$ be the dual polar space of $\mathscr{S}$.

To avoid confusion, we denote the distance in $\mathscr{D}$ by $\mathrm{dist}_\mathscr{D}$ although the distance function of $\mathscr{S}$ will be not used since it always can be expressed in terms of collinearity or intersection. First we show how the distance function $\mathrm{dist}_\mathscr{D}$ of $\mathscr{D}$ can be expressed in terms of $\mathscr{S}$.

Proposition 5.4.3. *Let M and N be two generators of $\mathscr{S}$ that are both elements of $\mathscr{D}$. Then* $\frac{1}{2} \cdot \mathrm{crk}_M(M \cap N) = \mathrm{dist}_\mathscr{D}(M, N)$.

Proof. A geodesic from M to N in $\mathscr{S}_\mathrm{m}$ can be transformed into a path from M to N in $\mathscr{D}$ by erasing every second element. Conversely, two distinct generators that are collinear in $\mathscr{D}$ have distance 2 in $\mathscr{S}_\mathrm{m}$ by Proposition A.2.20. Hence, a geodesic from M to N in $\mathscr{D}$ can be transformed into a path in $\mathscr{S}_\mathrm{m}$ from M to N that has double length. Therefore, the distance of M and N in $\mathscr{D}$ is just the half of their distance in $\mathscr{S}_\mathrm{m}$. Thus, the claim follows from Proposition A.2.20. □

In the following two propositions we study subspaces of $\mathscr{D}$. First we show what kinds of maximal singular subspaces exist and give a correspondence to subspaces of $\mathscr{S}$.

Proposition 5.4.4. *Let M_0, M_1 and M_2 be elements of $\mathfrak{M}_0$ that are pairwise collinear in $\mathscr{D}$ but are not contained in a common element of $\mathscr{L}$. Set $U := M_0 \cap M_1 \cap M_2$ and $N := \langle M_i \cap M_j \mid 0 \leq i < j \leq 2\rangle$. Then the following claims hold for every $i \in \{0,1,2\}$.*

(i) $\mathrm{crk}_{M_i}(U) = 3$.

(ii) *N is a generator of $\mathscr{S}$ that intersects M_i in a hyperplane.*

(iii) *$S_U := \langle M \in \mathfrak{M}_0 \mid U \leq M\rangle$ is a maximal singular subspace of $\mathscr{D}$ and has rank 3.*

(iv) *$S_N := \langle M \in \mathfrak{M}_0 \mid \mathrm{crk}_M(N \cap M) = 1\rangle$ is a maximal singular subspace of $\mathscr{D}$ or equals $S_N \cap S_U$.*

(v) *$S_U \cap S_N$ is a singular subspace of rank 2 of $\mathscr{D}$.*

(vi) *Every element $M \in \mathfrak{M}_0$ that is in $\mathscr{D}$ collinear to all of M_0, M_1 and M_2 is an element of S_U or of S_N.*

Proof. For $\{i,j,k\} = \{0,1,2\}$, set $U_i = M_j \cap M_k$ and let $L_i \in \mathscr{L}$ be the line of $\mathscr{D}$ that contains M_j and M_k. Since $M_i \notin L_i$, we obtain $U_i \not\leq M_i$ for $i \in \{0,1,2\}$ and consequently, $U_i \neq U_j$ for $0 \leq i < j \leq 2$.
Let $p \in M_0 \setminus U_0$. Then by Lemma A.2.16 $p \ominus M_0$ is a generator of $\mathscr{S}$ that is adjacent to M_0. Hence, $p \ominus M_0$ is a point of $\mathscr{S}_\mathrm{m}$ that is not contained in $\mathfrak{M}$. Since $p \in M_1$, we obtain $U_2 \leq p^\perp$ and therefore $U_2 \leq p \ominus M_0$. Analogously, $U_1 \leq p \ominus M_0$. Since $U_1 \neq U_2$ this implies that $H_0 := \langle U_1, U_2\rangle$ is the common hyperplane of M_0 and $p \ominus M_0$. Moreover, U_1 and U_2 are both hyperplanes of H_0 and therefore $U = U_1 \cap U_2$ has corank 3 in M_0. Now (i) follows by symmetric reasons. Thus, U is a hyperplane of U_0 and we conclude $U_0 = \langle p, U\rangle$. Since $H_0 = \langle U_1, U_2\rangle = M_0 \cap p^\perp$, this implies $N = p \ominus M_0$ and hence, (ii) follows.
Let $M \in S_U \cap S_N$. Assume $M \notin L_0$ and $M \neq M_0$. Since $U \leq M \cap M_0$, we obtain by Proposition 5.4.3 that $\mathrm{crk}_M(M \cap M_0) = 2$. Since both M and M_0 intersect N in a hyperplane, we conclude $M \cap M_0 \leq N$. Take a point $q \in (M \cap M_0) \setminus U$. Since $q \in M_0 \setminus U$, we obtain $q \notin U_0$. By Proposition 5.2.1 we know that there is a

generator N' of $\mathscr{S}$ with $N \cap N' = U_0$. Hence, $q \notin N'$ and we obtain by Lemma A.2.16 that $M' := q \ominus N'$ and N' are adjacent generators. Since $q \in N$, we know $U_0 \leq q^{\perp}$ and therefore $\langle q, U_0 \rangle \leq M'$. Since N and N' are not adjacent, this implies that N and M' intersect in the common hyperplane $\langle q, U_0 \rangle$. Hence, $M' \in \mathfrak{M}_0$ and more precisely, $M' \in L_0$. Since $\langle q, U \rangle$ is contained in all M, M_0 and M', there is an element in $\mathscr{L}$ that contains $\{M, M_0, M'\}$ and (v) follows.
Let $r \in U_0 \setminus U$. Then by Lemma A.2.16 we know that $r \ominus M_0$ is the only generator adjacent to M_0 that contains r. Since $r \in N$, this implies $r \ominus M_0 = N$ and hence, N is the only generator adjacent to M_0 that contains U_0. Since N and N' have distance 2 in $\mathscr{S}_{\mathrm{m}}$ and $U_0 \leq N'$, we conclude that N' and M_0 have distance 3 in $\mathscr{S}_{\mathrm{m}}$. Thus, $N' \cap M_0 = U$ and $N' \notin \mathfrak{M}_0$. Now Proposition 5.2.1 implies that there is a generator $M_3 \in \mathscr{S}$ with $N \cap M_3 = U$ and hence, $M_3 \in \mathfrak{M}_0$.
By Proposition 5.4.3 we know that every two elements of S_U are collinear in $\mathscr{D}$. By definition of $\mathscr{L}$ this implies that S_U is a singular subspace of $\mathscr{D}$. Since $M_3 \in S_U \setminus S_N$, (v) implies that S_u has at least rank 3. Now let $M \in S_U \setminus S_N$ with $M \neq M_3$. Since $\mathrm{crk}_M(M \cap M_3) = 2$, there is a point $r \in (M \cap M_3) \setminus U$. Since $M_3 \cap N = U$, we conclude that $M' := r \ominus N$ is a generator that intersects N in a hyperplane. Hence, $U \leq r^{\perp}$ yields $M' \in S_U \cap S_N$. Now all M, M' and M_3 contain $\langle r, U \rangle$ and thus, they are contained in a common element of $\mathscr{L}$. This concludes claim (iii).
Let M and M' be two distinct elements of S_N. Since both M and M' intersect N in a hyperplane, we obtain $\mathrm{crk}_M(M \cap M' \cap N) \leq 2$. Thus, Proposition 5.4.3 implies that M and M' are collinear in $\mathscr{D}$ and furthermore, $M \cap M' \leq N$. Hence, every element of $\mathfrak{M}_0$ that contains $M \cap M'$ has a hyperplane with N in common since $N \notin \mathfrak{M}$. Thus, the line of $\mathscr{D}$ that contains M and M' is fully contained in S_N. Therefore S_N is a singular subspace of $\mathscr{D}$. Hence, it remains to show (vi) to prove (iv).
Let $M \in \mathfrak{M}$ such that M is collinear to all M_0, M_1 and M_2 in $\mathscr{D}$. We may assume $M \notin S_U$ since otherwise there is nothing left to show. Then by (i) we conclude that $M \cap M_1 \cap M_2$ has corank 3 in M. In other words M intersects U_0 in a hyperplane. Analogously, M intersects U_1 in a hyperplane. Since $M \notin S_U$, we know $M \cap U_0 \neq U$ and $M \cap U_1 \neq U$. Since $U = U_0 \cap U_1$, this implies $M \cap U_0 \neq M \cap U_1$ and therefore $M \cap N \geq \langle M \cap U_0, M \cap U_1 \rangle > M \cap U_0$. Since $\mathrm{crk}_M(M \cap N)$ has to be odd, we conclude that M and N intersect in a hyperplane. This finishes the proof. ☐

We now show what the convex span of two points of $\mathscr{D}$ looks like and how such a convex span can be expressed in terms concerning the polar space $\mathscr{S}$.

Proposition 5.4.5. *Let M and N be generators of $\mathscr{S}$ that are contained in $\mathscr{D}$ and let G be the convex span of M and N in $\mathscr{D}$. Then a generator $L \leq \mathscr{S}$ is contained in G if and only if $L \geq M \cap N$.*

Proof. Set $S := M \cap N$ and let H be the set of all generators of $\mathscr{S}$ that are elements of $\mathscr{D}$ and contain S. Let K and L be distinct generators contained in H with $\mathrm{crk}_K(K \cap L) = 2$. Then $K \cap L \geq S$ and therefore all generators containing $K \cap L$ belong to H. Thus, H is a subspace of $\mathscr{D}$. Now let K and L be two arbitrary generators of H with $\mathrm{crk}_K(K \cap L) = 2m > 2$ and let $L' \in \mathscr{D}$ with $\mathrm{crk}_K(K \cap L') = 2m-2$ and $\mathrm{crk}_L(L \cap L') = 2$. Then

$$2m = \mathrm{crk}_K(K \cap L) \leq \mathrm{crk}_K(K \cap L \cap L') \leq \mathrm{crk}_K(K \cap L') + 2 = 2m$$

and therefore $K \cap L = K \cap L \cap L'$. This implies $S \leq L'$. Hence, H is convex and consequently, $G \leq H$.
To prove $H \leq G$ we apply induction over $n := \mathrm{dist}_{\mathscr{D}}(M,N)$. For $n = 0$, there is nothing to prove. For $n = 1$, we obtain $H = G$ by the definition of the lines in $\mathscr{D}$. Now let $n > 1$ and assume that the claim holds for two generators at distance $< n$ in $\mathscr{D}$. Let K be a generator of $\mathscr{S}$ with $S \leq K$. Assume there is a point $p \in K \cap M \setminus S$. Then $\mathrm{crk}_M(K \cap M) \leq 2n-2$ since $\mathrm{crk}_M(\langle p,S \rangle) = 2n-1$. Thus, $M \cap K$ contains a line l that is disjoint to S. This implies that l is disjoint to N. Set $N' := l \ominus N$. By Lemma A.2.22(ii) we conclude that N and N' have distance 2 in $\mathscr{S}_{\mathrm{m}}$ and therefore N and N' are collinear in $\mathscr{D}$. This implies $\mathrm{dist}_{\mathscr{D}}(M,N') = n-1$ and moreover, $M \cap N' = \langle l,S \rangle$ since $S \leq Y \cap l^{\perp}$. Since $\langle l,S \rangle \leq K$, we may apply the induction hypothesis to conclude $K \in \langle M,N' \rangle_{\mathrm{g}}$. Since $N' \in G$, this implies $K \in G$.
Now assume $M \cap K = S$. Since $\mathrm{crk}_K(S) = 2n$, there is a point $p \leq K \setminus S$. Set $\tilde{M} := p \ominus M$. Then $\tilde{M}$ is a generator of $\mathscr{S}$ that has distance 1 to M in $\mathscr{S}_{\mathrm{m}}$. Since $S \leq p^{\perp}$, we conclude $S \leq \tilde{M}$ and hence, $\mathrm{crk}_N(\tilde{M} \cap N) \leq 2n$. Since M and $\tilde{M}$ are adjacent and hence $\tilde{M} \notin \mathfrak{M}_0$, we conclude $\mathrm{crk}_N(\tilde{M} \cap N) = 2n-1$ by Proposition A.2.20. Thus, there is a point $q \in \tilde{M} \cap N \setminus S$. Since $\mathrm{crk}_K(S) \geq 4$, there is a point $p' \in K \setminus \langle p,S \rangle$ with $p' \perp q$. Set $M' := p' \ominus \tilde{M}$. Since $p' \notin \tilde{M}$, we conclude $M' \in \mathscr{D}$ since M' and $\tilde{M}$ are adjacent and $\tilde{M}$ is not contained in $\mathscr{D}$. Moreover, M and M' are collinear in $\mathscr{D}$. Since $\langle q,S \rangle \leq K \cap \tilde{M}$, we obtain $\langle q,S \rangle \leq M'$ and hence, $\mathrm{crk}_N(N \cap M') \leq 2n-1$. This implies $\mathrm{dist}_{\mathscr{D}}(M,M') = 1$ and $\mathrm{dist}_{\mathscr{D}}(N,M') = n-1$ and hence, $M' \in G$. Since $q \notin M$, there is a point $r \in M$ with $r \not\perp q$. Now Lemma A.2.17 implies that $r \ominus N \cap M'$ is a hyperplane of $N \cap M'$. Hence, there is a point $q' \in r \ominus N \cap M' \setminus S$. Let $r' \in M$ with $r' \not\perp q'$ and set $N' := r' \ominus (r \ominus N)$. Applying Lemma A.2.17 again yields $N' \cap M' = S$ since $S \leq \{r,r'\}^{\perp}$. Thus, M' and N' have distance $2n$ in $\mathscr{S}_{\mathrm{m}}$ and we conclude $M' \in \mathscr{D}$. Since $r \in N' \cap M$ and $r \notin S$, we obtain $\mathrm{dist}_{\mathscr{D}}(M,N') \leq n-1$. Since $r \ominus N$ is adjacent to both N and N', we conclude $\mathrm{dist}_{\mathscr{D}}(M,N') = n-1$ and $\mathrm{dist}_{\mathscr{D}}(N,N') = 1$ and hence, $N' \in G$. Since $M' \cap K > M' \cap N' = S$, we obtain as above that K is contained in the convex span of M' and N' in $\mathscr{D}$. Since this convex span is contained in G, the claim follows. $\square$

5.4.2 Twin half-spin spaces

The goal of this section is to give a generalisation of half-spin spaces which yields a class of twin SPO spaces. Therefore we now introduce a method how to construct a twin space out of a polar space such that both halfs of the twin space are local half-spin spaces.

Definition 5.4.6. Let $\mathscr{S}$ be a polar space with a spanning pair (M_+, M_-). Further let Γ be the dual polar graph of $\mathscr{S}$ and for $\sigma \in \{+,-\}$, denote by Γ^σ the connected component of Γ that contains M_σ. Let Γ^+ be bipartite and let $\mathfrak{M}^\sigma$ be the set of vertices of Γ^σ that have even distance to M_σ. Further define the following sets:

$$\mathscr{L}^\sigma := \big\{\{L \in \mathfrak{M}^\sigma \mid M \cap N < L\} \;\big|\; \{M,N\} \subseteq \mathfrak{M}^\sigma \;\wedge\; \mathrm{crk}_M(M \cap N) = 2\big\}$$
$$R := \{(M,N),(N,M) \mid (M,N) \in \mathfrak{M}^+ \times \mathfrak{M}^- \;\wedge\; M \cap N = \varnothing\}$$

Then we call $((\mathfrak{M}^+, \mathscr{L}^+), (\mathfrak{M}^-, \mathscr{L}^-))$ with the opposition relation R the *twin half-spin space of* $\mathscr{S}$ with respect to (M_+, M_-).

Later on we will see that a twin half-spin space is a twin space. Therefore, a twin space that is isomorphic to a twin half-spin space of a polar space is called a twin half-spin space.

Remark 5.4.7. We know that in a polar space of finite rank every generator is part of a spanning pair and the dual polar graph is connected. Thus, if $\mathscr{D}^+$ is a half-spin space of a polar space $\mathscr{S}$ of finite rank, then there is a half-spin space $\mathscr{D}^-$ of $\mathscr{S}$ such that $(\mathscr{D}^+, \mathscr{D}^-)$ is a twin half-spin space. Conversely, a twin half-spin space of a polar space of finite rank consists of two half-spin spaces.
Let $(\mathscr{D}^+, \mathscr{D}^-)$ be a twin half-spin space of a polar space $\mathscr{S}$ of finite rank. Since two generators of $\mathscr{S}$ form a spanning pair if and only if they are disjoint, we conclude by Proposition A.2.20 that the two half-spin spaces $\mathscr{D}^+$ and $\mathscr{D}^-$ are identical if the rank of $\mathscr{S}$ is even. If the rank of $\mathscr{S}$ is odd, then every generator of $\mathscr{S}$ is either a point of $\mathscr{D}^+$ or of $\mathscr{D}^-$.

Remark 5.4.8. Let $\mathscr{S}$, M_+ and M_- be as in the definition above. Further let $(\mathscr{S}_\mathrm{m}^+, \mathscr{S}_\mathrm{m}^-)$ be the twin dual polar space of $\mathscr{S}$ with $(M_+, M_-) \in \mathscr{S}_\mathrm{m}^+ \times \mathscr{S}_\mathrm{m}^-$. By Theorem 5.2.15 we know that $(\mathscr{S}_\mathrm{m}^+, \mathscr{S}_\mathrm{m}^-)$ is a twin SPO space of singular rank ≤ 1. Since the collinearity graph of $\mathscr{S}_\mathrm{m}^+$ is bipartite, all lines of $\mathscr{S}_\mathrm{m}^+$ have cardinality 2. Since for every line in $\mathscr{S}_\mathrm{m}^-$ there is an opposite line in $\mathscr{S}_\mathrm{m}^+$, we conclude that every line of $\mathscr{S}_\mathrm{m}^-$ has cardinality 2. Let p be a point of $\mathscr{S}_\mathrm{m}^-$. By Lemma 3.2.1 p has a gate in every line of $\mathscr{S}_\mathrm{m}^-$ and we conclude that the collinearity graph of $\mathscr{S}_\mathrm{m}^-$ is bipartite, too, where the set of points at even distance to p and the set of points at odd distance to p form a partition. We conclude that both components of a twin half-spin space are local half-spin spaces.

From now on let (M_+, M_-) be a spanning pair of a non-degenerate polar space $\mathscr{S}$. Further let $\mathscr{D} = (\mathscr{D}^+, \mathscr{D}^-)$ be a twin half-spin space of $\mathscr{S}$ with respect to (M_+, M_-). For $\sigma \in \{+,-\}$ we denote the point set of $\mathscr{D}^\sigma$ by $\mathfrak{M}^\sigma$ and the line set by $\mathscr{L}^\sigma$. To avoid confusion, we denote the distance in $\mathscr{D}$ by $\mathrm{dist}_{\mathscr{D}}$ although the distance function of $\mathscr{S}$ will be not used since it always can be expressed in terms of collinearity or intersection. Since both $\mathscr{D}^+$ and $\mathscr{D}^-$ are local half-spin spaces, we may restrain ourselves to the case that $\mathscr{S}$ is a non-degenerate polar space.

The dual polar graph of $\mathscr{S}$ will be denoted by Γ and for $\sigma \in \{+,-\}$ we denote the connected component of Γ that contains M_σ by Γ^σ. Further let $\mathscr{S}_{\mathrm{m}}$ be the dual polar space of $\mathscr{S}$ and let $(\mathscr{S}_{\mathrm{m}}^+, \mathscr{S}_{\mathrm{m}}^-)$ be the twin dual polar space of $\mathscr{S}$ with respect to (M_+, M_-).

Lemma 5.4.9. *Let $M \in \mathfrak{M}^+$ be a generator of $\mathscr{S}$. Then there is a generator $N \in \mathfrak{M}^-$ such that (M,N) is a spanning pair. Moreover, every generator $K \leq \mathscr{S}$ that is commensurate to N and forms together with M a spanning pair is an element of $\mathfrak{M}^-$.*

Proof. By Corollary 5.2.9 we know already that there is a generator $N \in \Gamma^-$ such that (M,N) is a spanning pair of $\mathscr{S}$. Thus, it remains to show $N \in \mathfrak{M}^-$.
By the definition of a twin half-spin space there is a spanning pair $(M_+, M_-) \in \mathfrak{M}^+ \times \mathfrak{M}^-$ of $\mathscr{S}$. Let $(M_i)_{0 \leq i \leq m}$ be a geodesic from M_+ to M in Γ^+. Then m is even since Γ^+ is bipartite and M and M_+ are both contained in $\mathfrak{M}^+$. We know that all lines of the component of the dual polar space of $\mathscr{S}$ that contains M_+ have cardinality 2. Thus, for $i < m$, the set $\{M_i, M_{i+1}\}$ is a line of the dual polar space of $\mathscr{S}$. By Lemma 5.2.14 we conclude that $\mathrm{rk}(M_i \cap M_-)$ and $\mathrm{rk}(M_{i+1} \cap M_-)$ differ by 1. This implies that $\mathrm{rk}(M_i \cap M_-)$ is odd if and only if i is even and consequently, $\mathrm{rk}(M \cap M_-)$ is odd. Now let $(N_i)_{0 \leq i \leq n}$ be a geodesic from M_- to N in Γ^-. By analogous reasons we conclude for $i \leq n$ that $\mathrm{rk}(M \cap N_i)$ is odd if and only if i is even. Thus, n is even and the claim follows. ☐

Corollary 5.4.10. *The opposition relation of $\mathscr{D}$ consists of all spanning pairs that are contained in $(\mathfrak{M}^+ \times \mathfrak{M}^-) \cup (\mathfrak{M}^- \times \mathfrak{M}^+)$.*

Proof. By Proposition 5.2.8 we know that each pair of the opposition relation of $\mathscr{D}$ is a spanning pair. The converse follows since generators of a spanning pair are disjoint. ☐

Proposition 5.4.11. *Every twin half-spin space is a twin space.*

Proof. By the definition of $\mathscr{L}^+$ and $\mathscr{L}^-$ it follows directly that $\mathscr{D}^+$ and $\mathscr{D}^-$ are partially linear spaces.
By Corollary 5.2.9 and Lemma 5.4.9 we conclude that the opposition relation of $\mathscr{D}$ is total since it consists of the spanning pairs of $(\mathfrak{M}^+ \times \mathfrak{M}^-) \cup (\mathfrak{M}^- \times \mathfrak{M}^+)$.

Let $M \in \mathfrak{M}^+$ and $N \in \mathfrak{M}^-$ such that (M,N) is a spanning pair. Let $G \in \mathscr{L}^-$ such that G contains N. Further let $N' \in G \setminus \{N\}$. Then $S := N \cap N'$ has corank 2 in both N and N'. Hence, there are points p and q in N' such that $\langle p,q,S\rangle = N'$. By the maximality of N' we conclude $N \cap p^\perp \cap q^\perp = S$.
Since (M,N) is a spanning pair, we know by Proposition 5.2.4 that there are points p' and q' in M such that $N \cap \{p',q'\}^\perp = S$. Since the points of N that are collinear to p' form a hyperplane of N, we obtain $p' \neq q'$ and hence S has corank 2 in $N_M := \langle p',q',S\rangle$. By Proposition A.2.20 we conclude that N_M is a generator of $\mathscr{S}$ and thus, $N_M \in G$. By the maximality of N_M we obtain $S^\perp \cap M = p'q'$. Hence, every element of G that intersects M, contains a point of the line $p'q'$ and consequently, has a common hyperplane with N_M. By Proposition 5.4.3 this implies that N_M is the only element of G that intersects M. Hence, $\mathscr{D}$ is a twin space by Proposition 5.2.8. □

In the following we denote by $\mathrm{cod}_{\mathscr{D}}$ the codistance function of the twin space $\mathscr{D}$.

Proposition 5.4.12. *Let $M \in \mathscr{D}^+$ and $N \in \mathscr{D}^-$. Then* $\mathrm{cod}_{\mathscr{D}}(M,N) = \frac{1}{2}(\mathrm{rk}(M \cap N)+1)$.

Proof. By Proposition 5.4.3 we conclude $\mathrm{cod}_{\mathscr{D}}(M,N) \geq \frac{1}{2}(\mathrm{rk}(M\cap N)+1)$. By Proposition 5.2.12 there is a generator $M' \in \Gamma^-$ with $\mathrm{crk}_N(N \cap M') = \mathrm{rk}(M \cap N) + 1$ such that (M,M') is a spanning pair. Lemma 5.4.9 implies $M' \in \mathfrak{M}^-$. Hence, $\mathrm{dist}_{\mathscr{D}}(N,M') = \frac{1}{2}(\mathrm{rk}(M\cap N)+1)$ by Proposition 5.4.3. □

Theorem 5.4.13. *Every twin half-spin space is a rigid twin SPO space whose symplecta are all of rank* 4 *and whose singular subspaces of rank* 2 *are contained in at most two maximal singular subspaces one of which has rank* 3.

Proof. Let $\mathscr{S}$ be a polar space. Further let (M_+,M_-) be a spanning pair of $\mathscr{S}$ and let $\mathscr{D} = (\mathscr{D}^+,\mathscr{D}^-)$ be the twin half-spin space of $\mathscr{S}$ with $(M_+,M_-) \in \mathscr{D}^+ \times \mathscr{D}^-$. By $\mathscr{S}_\mathrm{m} = (\mathscr{S}_\mathrm{m}^+,\mathscr{S}_\mathrm{m}^-)$ we denote the twin dual polar space with $(M_+,M_-) \in \mathscr{S}_\mathrm{m}^+ \times \mathscr{S}_\mathrm{m}^-$. For $\sigma \in \{+,-\}$, we denote the point set of $\mathscr{D}^\sigma$ by $\mathfrak{M}^\sigma$ and the point set of $\mathscr{S}_\mathrm{m}^\sigma$ by $\mathscr{P}_\mathrm{m}^\sigma$. The distance function in $\mathscr{D}$ is denoted by $\mathrm{dist}_{\mathscr{D}}$. By Lemma 5.4.9 the set of spanning pairs of $\mathscr{S}$ induces a symmetric, total point-relation on the twin point-line space $\mathscr{D}$. The thereby induced codistance function is denoted by $\mathrm{cod}_{\mathscr{D}}$. We prove that $\mathscr{D}$ is a twin SPO space by showing that the four axioms of Definition 2.1.1 hold if we use the codistance $\mathrm{cod}_{\mathscr{D}}$. The axiom (A4) follows directly from Proposition 5.4.12.
Let $X \in \mathfrak{M}^+$ and let Y and Z be elements of $\mathfrak{M}^-$. By G we denote the convex span of Y and Z in $\mathscr{D}$. By $\tilde{G}$ we denote the convex span of Y and Z in $\mathscr{S}_\mathrm{m}$. Comparing Proposition 5.2.1 with Proposition 5.4.5 yields $G = \tilde{G} \cap \mathfrak{M}^-$. Let n be the distance of Y and Z in $\mathscr{D}$. Comparing Proposition A.2.20 with Proposition 5.4.3 implies

that Y and Z have distance $2n$ in $\mathscr{S}_{\mathrm{m}}$.
First consider that G contains a generator Y' of $\mathscr{S}$ such that (X,Y') is a spanning pair. Since by Theorem 5.2.15 $\mathscr{S}_{\mathrm{m}}$ is a twin SPO space, we conclude by (A12) that X has codistance $2n$ to $\tilde{G}$ in $\mathscr{S}_{\mathrm{m}}$ and there is exactly one element $Z' \in \tilde{G}$ that has codistance $2n$ to X. Since by Proposition 2.1.3 this implies that Z' and Y' have distance $2n$ in $\mathscr{S}_{\mathrm{m}}$, we conclude $Z' \in \mathfrak{M}^-$. By Proposition 5.2.12 we obtain $\mathrm{rk}(X \cap Z') = 2n-1$. Hence, Proposition 5.4.12 implies $\mathrm{cod}_{\mathscr{D}}(X,Z') = n$. Analogously, we conclude that all other elements of G have codistance $< n$ to X in $\mathscr{D}$. Hence, (A1) and (A2) are fulfilled.
In the following G does not necessarily contain an element that forms a spanning pair with X. Assume $\mathrm{cod}_{\mathscr{D}}(X,Z) = \mathrm{cod}_{\mathscr{D}}(X,G)$ and let $W \in \mathfrak{M}^+$ with $\mathrm{dist}_{\mathscr{D}}(X,W) = 1$ such that $\mathrm{cod}_{\mathscr{D}}(W,Y) = \mathrm{cod}_{\mathscr{D}}(X,Y) - 1$. Set $d := \mathrm{cod}_{\mathscr{D}}(X,Z)$. Then X and Z have codistance $2d$ in $\mathscr{S}_{\mathrm{m}}$. By Lemma 5.2.14 we conclude that there are elements $Y' \in \tilde{G}$ and $Z' \in \tilde{G}$ such that the codistance of Z' to X exceeds the codistance of Y' to X by $2n$. Hence, Proposition 2.1.12(iv) implies that Z' is a cogate of X in $\tilde{G}$. Analogously, W has in $\mathscr{S}_{\mathrm{m}}$ a cogate in $\tilde{G}$.
First assume $Z' \in \mathfrak{M}^-$. Then Z' is in $\mathscr{D}$ a cogate for X in G since the distance and the codistance in $\mathscr{S}_{\mathrm{m}}$ of two points of $\mathscr{D}$ is just the double as in $\mathscr{D}$. Hence, $Z' = Z$ and since $\mathrm{dist}_{\mathscr{D}}(Z,Y) = n$, we obtain $\mathrm{cod}_{\mathscr{D}}(X,Y) = d-n$. With $\mathrm{dist}_{\mathscr{D}}(X,W) = 1$ and $\mathrm{cod}_{\mathscr{D}}(W,Y) = d-n-1$ we conclude $\mathrm{cod}_{\mathscr{D}}(W,Z) = d-1$. Thus, Proposition 2.1.12(iv) implies for $\mathscr{S}_{\mathrm{m}}$ that Z is a cogate for X in $\tilde{G}$. Again this implies that Z is in $\mathscr{D}$ a cogate for X in G and (A3) is fulfilled in $\mathscr{D}$ for this case.
Now assume $Z' \notin \mathfrak{M}^-$. Then all elements of $\tilde{G}$ that intersect Z' in a hyperplane are contained in G and are precisely the elements of G that have codistance $\mathrm{cod}_{\mathscr{D}}(X,G)$ to X in $\mathscr{D}$. Hence, Z and Z' are collinear in $\mathscr{S}_{\mathrm{m}}$. We obtain $\mathrm{rk}(X \cap Z') = 2d$. Furthermore, since $\{Z,Z'\}$ is a line of $\tilde{G}$, Proposition 2.1.17(i) implies that Z' and Y have distance $2n-1$ in $\mathscr{S}_{\mathrm{m}}$. Thus, $\mathrm{rk}(X \cap Y) = 2d-2n+1$ and consequently, $\mathrm{rk}(W \cap Y) = 2d-2n-1$. Assume there is an element $V \in \tilde{G}$ with $\mathrm{rk}(W \cap V) = 2d-2n-2$. Then the codistance of X to $\tilde{G}$ in $\mathscr{S}_{\mathrm{m}}$ is at most $2d-1$. Since W and X have distance 2 in $\mathscr{S}_{\mathrm{m}}$, this implies that $\mathrm{rk}(X \cap Z') = 2d-2$ and Z' is the cogate of W in $\tilde{G}$. We conclude that an element of G has codistance $\mathrm{cod}_{\mathscr{D}}(W,G)$ to W if and only if it has codistance $\mathrm{cod}_{\mathscr{D}}(X,G)$ to X. Hence, (A3) is fulfilled in $\mathscr{D}$ for this case. Now assume $\mathrm{rk}(W \cap V) \geq 2d-2n-1$ for every element $V \in \tilde{G}$. Then we obtain $\mathrm{rk}(W \cap W') = 2d-1$ for the cogate W' of W in $\tilde{G}$. This implies that W' and Y have distance $2n$ in $\mathscr{S}_{\mathrm{m}}$ and therefore $W' \in \mathfrak{M}$. Thus, W' is a cogate for W in G regarding the point-line space $\mathscr{D}$. Since W and X have distance 2 in $\mathscr{S}_{\mathrm{m}}$, we obtain $\mathrm{rk}(W \cap Z') \geq 2d-2$. Since $W' \neq Z'$ this implies that W' and Z' are collinear in $\mathscr{S}_{\mathrm{m}}$ and we conclude $\mathrm{cod}_{\mathscr{D}}(W,W') = \mathrm{cod}_{\mathscr{D}}(X,W') = 2d$. This concludes that $\mathscr{D}$ is a twin SPO space.
By Proposition 5.4.4 every singular subspace of $\mathscr{D}^+$ that has rank 2 is contained in a maximal singular subspace of rank 3 and in at most one other maximal sin-

gular subspace. By symmetric reasons it remains to show that every symplecton of $\mathscr{D}^-$ has rank 4. Let Y and Z be elements of $\mathfrak{M}^-$ with $\mathrm{dist}_{\mathscr{D}}(Y,Z) = 2$. Then by Proposition 5.4.5 the symplecton G of $\mathscr{D}^-$ which is the convex span of Y and Z consists of all elements of $\mathfrak{M}^-$ that contain $S := Y \cap Z$. Moreover, $\mathrm{crk}_Y(S) = 4$ by Proposition 5.4.3. Let $Y_0 \in \mathfrak{M}^-$ such that both Y and Z are collinear to Y_0 in $\mathscr{D}^-$. Then $S \leq Y_0$ since $Y_0 \in G$ and furthermore, $Y_0 \cap Y$ and $Y_0 \cap Z$ have both corank 2 in Y_0. Let $p \in (Y_0 \cap Z) \setminus S$ and set $N := p \ominus Y$. Since $p \notin Y$, we conclude that N and Y are distinct adjacent generators and hence, $N \notin \mathfrak{M}^-$. Since $p \in Y_0$, we obtain $Y_0 \cap Y \leq p^\perp$ and hence, $\langle p, Y \cap Y_0 \rangle$ is a common hyperplane of Y_0 and N. Let $p' \in (Y_0 \cap Y) \setminus S$. Since $p' \notin Z$, there is a point $q \in Z$ with $q \not\perp p'$. Set $Y_1 := q \ominus N$. Since Y_1 and N are adjacent generators, we know that Y_1 is collinear to both Y and Y_0 in $\mathscr{D}^-$. Since $q \in Y_1$, we obtain $p' \notin Y_1$ and therefore $Y \cap Y_0 \not\leq Y_1$. Hence, Y, Y_0 and Y_1 are not on a common line in $\mathscr{D}^-$. Thus, we may apply Proposition 5.4.4 to conclude that $\{V \in \mathfrak{M}^- \mid Y \cap Y_0 \cap Y_1 \leq V\}$ is a maximal singular subspace of $\mathscr{D}^-$ that has rank 3. Since $S \leq Y \cap Y_0 \cap Y_1$, we conclude that this maximal singular subspace is a generator of G and consequently, $\mathrm{rk}(G) = 4$. $\square$

6 Twin SPO spaces

This chapter states the main result of the present work. We show that every twin SPO space is a grid sum of the twin spaces we studied in Chapter 5, that are twin polar spaces, twin E_6-spaces, twin E_7-spaces, twin dual polar spaces, partial twin Grassmannians and twin half-spin spaces (note that twin projective spaces are a subclass of partial twin Grassmannians). By Theorem 4.3.7 it suffices to show that every rigid twin SPO space is a grid sum of the mentioned twin spaces. Nevertheless, we include in two cases the non-rigid twin SPO spaces, this is because twin polar spaces and twin dual polar spaces are not always rigid.

As in Chapter 3 we proceed by a case differentiation with respect to the symplectic rank of the twin SPO space. Before we do so, we discuss two cases of twin SPO spaces with small diameter, which are the ones that match the twin projective and the twin polar spaces. The twin SPO spaces of symplectic rank ≥ 5 will be considered in the end of this chapter and are the ones that match the twin spaces that come from exceptional parapolar spaces. The twin SPO spaces of symplectic rank 2, 3 and 4 match the twin dual polar spaces, the partial twin Grassmannians and the twin half-spin spaces, respectively. Each of these twin spaces is constructed out of a point-line space $\mathscr{S}_{\mathrm{m}}$ of finite diameter that is either a polar space or a projective space. Our strategy is to construct this point-line space $\mathscr{S}_{\mathrm{m}}$ out of the twin SPO space $\mathscr{S}$ and in the next step, to show that $\mathscr{S}$ is isomorphic to a twin dual polar space, a partial twin Grassmannian or a twin half-spin space of $\mathscr{S}_{\mathrm{m}}$. Thereby, the points of $\mathscr{S}_{\mathrm{m}}$ are always coconvex spans of a point and a maximal singular subspace of $\mathscr{S}$ that are at almost minimal codistance.

From Section 2.3 we know already that for a twin SPO space $(\mathscr{S}^+, \mathscr{S}^-)$, the diameter of $\mathscr{S}^+$ and $\mathscr{S}^-$ are equal and furthermore, $\mathrm{srk}(\mathscr{S}^+)$ and $\mathrm{srk}(\mathscr{S}^-)$ are equal or both are infinite. Moreover, by Corollary 4.2.8 we conclude that if $\mathscr{S}^+$ has an finite symplectic rank, then $\mathrm{yrk}(\mathscr{S}^+) = \mathrm{yrk}(\mathscr{S}^-)$.

6.1 General properties

Before we start with the classification of rigid twin SPO spaces by giving a case differentiation, we prove some facts that are true for any rigid twin SPO space.

Lemma 6.1.1. *Let $\mathscr{S}$ be a rigid twin SPO space. Further let x be a point and let l be a line with* $\mathrm{cod}(x,l) < \infty$ *and* $\mathrm{copr}_l(x) = l$*. Then there is a point y with* $l \le y^\perp$ *and* $\mathrm{cod}(x,y) = \mathrm{cod}(x,l) - 1$.

(a) *For* $\mathrm{diam}(\mathscr{S}^+) > \mathrm{cod}(x,l)$*, there is a point z with* $l \le z^\perp$ *and* $\mathrm{cod}(x,z) = \mathrm{cod}(x,l) + 1$.

(b) *For* $\mathrm{diam}(\mathscr{S}^+) = \mathrm{cod}(x,l) \ge 2$*, there is a maximal singular subspace* $M \le \mathscr{S}$ *with* $l \le M$ *and* $\mathrm{copr}_M(x) = M$.

Proof. Set $d := \mathrm{cod}(x,l)$. By Lemma 4.2.1 there is a point $x' \leftrightarrow x$ with $\mathrm{dist}(x',l) = d$ and $\mathrm{pr}_l(x') = l$. Hence by Lemma 3.2.1, there is a point y with $\mathrm{dist}(x',y) = d-1$ and $l \le y^\perp$. This implies $\mathrm{cod}(x,y) = d-1$. Let p be point of l.
Assume $\mathrm{diam}(\mathscr{S}^+) > d$. Since there is a point at finite distance to p and at codistance $d+1$ to x, there is a point z' with $\mathrm{dist}(z',p) = 1$ and $\mathrm{cod}(x,z') = d+1$ by Proposition 2.1.16(ii). We may assume that there is a point $q \in l$ with $\mathrm{dist}(z',q) = 2$ since otherwise we are done. Hence $Y := \langle z',q\rangle_{\mathrm{g}}$ is a symplecton. Since $l \le Y$, we obtain $\mathrm{cod}(x,Y) \le d+1$. Since z' is not a cogate for x in Y, we conclude by Propositions 2.1.12(ii) and 4.2.5 that $\mathrm{copr}_Y(x)$ is a generator of Y. This implies $y \notin Y$. Since $l \le \mathrm{pr}_Y(y)$ and Y is rigid, we conclude $\mathrm{rk}(Y) > 2$ by Proposition 2.2.9(i). Thus, there is a point $z \in \mathrm{copr}_Y(x)$ with $l \le z^\perp$.
Now assume $\mathrm{diam}(\mathscr{S}^+) = d \ge 2$. Then there is a point $y' \perp y$ with $\mathrm{dist}(y',x') = d-2$. Then $\mathrm{cod}(x,y') = d-2$ and hence by Proposition 2.1.12(iv), p is a cogate for x in the symplecton $\langle p,y'\rangle_{\mathrm{g}}$. Thus for a point $q \in l \setminus \{p\}$, we obtain $q \notin \langle p,y'\rangle_{\mathrm{g}}$. Since $q \perp p$ and $\mathrm{dist}(q,y') = 2$, Proposition 2.1.12(ii) implies that $\langle p,y'\rangle_{\mathrm{g}} \cap q^\perp$ contains a line. Since $\langle p,y'\rangle_{\mathrm{g}}$ is rigid, Proposition 2.2.9(i) implies $\mathrm{yrk}(\mathscr{S}^+) \ge 3$. Hence by Lemma 3.1.1(i), there is a symplecton Y that contains $\langle y,l\rangle$. Since $l \le \mathrm{copr}_Y(x)$, Proposition 4.2.5 implies that $\mathrm{copr}_Y(x)$ is a generator of Y. We may assume that $\mathrm{copr}_Y(x)$ is properly contained in a maximal singular subspace M since otherwise we are done. Then Proposition 2.2.4(ii) implies that $y^\perp \cap \mathrm{copr}_Y(x) = y^\perp \cap M =: H$ is contained in exactly two maximal singular subspaces of $\mathscr{S}$. Suppose M contains a point z with $\mathrm{cod}(x,z) = d-1$. Then $z \notin H$ and hence, $\langle y,z\rangle_{\mathrm{g}}$ is a symplecton that contains H. Thus, Proposition 4.2.5 implies that $\mathrm{copr}_{\langle y,z\rangle_{\mathrm{g}}}(x)$ is a generator of $\langle y,z\rangle_{\mathrm{g}}$. This is a contradiction, since $\langle z,H\rangle$ and $\langle y,H\rangle$ are the only generators of $\langle y,z\rangle_{\mathrm{g}}$ that contain H. Thus, $\mathrm{cod}(x,z) = d$ for every point $z \in M$. □

Lemma 6.1.2. *Let $\mathscr{S}$ be a rigid twin SPO space with* $\mathrm{yrk}(\mathscr{S}^+) \in \{3,4\}$. *Further let M and N be two maximal singular subspaces with* $\mathrm{rk}(M \cap N) = \mathrm{yrk}(\mathscr{S}) - 2$. *Let $G \le M$ and $H \le N$. Then $\langle M \cap N, G, H\rangle_{\mathrm{g}} \cap M = \langle M \cap N, G\rangle$.*

Proof. Set $S := M \cap N$ and $V := \langle S, G, H\rangle_{\mathrm{g}}$. For $G \le M \cap N$, we obtain $V \le N$ and hence, $V \cap M = S$. For $H \le M \cap N$, we obtain $V = \langle S, G\rangle_{\mathrm{g}} = \langle S, G\rangle$. Therefore we assume that there are points $x \in G \setminus N$ and $y \in H \setminus M$.
Set $\mathfrak{F} := \{P \subseteq G \cup H \mid |P| < \infty \wedge \{x,y\} \le P\}$ and $U := \bigcup_{P \in \mathfrak{F}} \langle P\rangle_{\mathrm{g}}$. Then $U \subseteq V$. Let u and v be points of U. Further let P and Q be elements of $\mathfrak{F}$ such that $u \in \langle P\rangle_{\mathrm{g}}$ and $v \in \langle Q\rangle_{\mathrm{g}}$. Then $\langle u, v\rangle_{\mathrm{g}} \le \langle P, Q\rangle_{\mathrm{g}}$. Since $P \cup Q \in \mathfrak{F}$, this implies that U is a convex subspace. Since $S \le \langle x, y\rangle_{\mathrm{g}}$ and for every point $p \in G \cup H$, we obtain $\{p, x, y\} \in \mathfrak{F}$, this implies $V \le U$ and hence, $V = U$.
The subspace $\langle P\rangle_{\mathrm{g}}$ contains the symplecton $\langle x, y\rangle_{\mathrm{g}}$. Since $\mathrm{rk}(S) = \mathrm{rk}(\langle x, y\rangle_{\mathrm{g}}) - 2$, we obtain $\langle x, y\rangle_{\mathrm{g}} \cap M = \langle x, S\rangle$. Since P is finite, we conclude $\langle P\rangle_{\mathrm{g}} \cap M = \langle P \cap M, S\rangle$ by Propositions 3.4.5 and 3.5.5 together with induction. Thus, $U \cap M \le \langle G, S\rangle$. Since $\langle G, S\rangle \le V$, the claim follows. ☐

6.2 Twin SPO spaces with small diameter

In this section we consider the twin SPO spaces of diameter at most 2. Throughout this section let $\mathscr{S} = (\mathscr{S}^+, \mathscr{S}^-)$ be a twin SPO space. In this section we do not have to demand $\mathscr{S}$ to be rigid since there is only one case where $\mathscr{S}$ can be non-rigid. However in this case, the non-rigid case is just analogous to the rigid one. More restrictively, we consider twin SPO spaces where each component contains at most one symplecton. We start with the case without any symplecta.

Lemma 6.2.1. *Let $\mathscr{S}^+$ and $\mathscr{S}^-$ both be singular. Further let $U \le \mathscr{S}^+$ be a subspace of rank k. Then $\bigcap_{p \in U} \mathrm{copr}_{\mathscr{S}^-}(p)$ has corank $k+1$ in $\mathscr{S}^-$. Moreover, $q \in U$ if and only if $\bigcap_{p \in U} \mathrm{copr}_{\mathscr{S}^-}(p) \le \mathrm{copr}_{\mathscr{S}^-}(q)$ for every point $q \in \mathscr{S}^+$.*

Proof. By Theorem 2.1.22 both $\mathscr{S}^+$ and $\mathscr{S}^-$ are projective spaces. Let $\{p_i \mid 0 \le i \le k\}$ be a basis of U and set $V := \bigcap_{p \in U} \mathrm{copr}_{\mathscr{S}^-}(p)$. Then $V = \bigcap_{i \le k} \mathrm{copr}_{\mathscr{S}^-}(p_i)$. For every point $p \in \mathscr{P}^+$, the subspace $\mathrm{copr}_{\mathscr{S}^-}(p)$ is a hyperplane of $\mathscr{S}^-$. This implies $\mathrm{crk}_{\mathscr{S}^-}(V) \le k+1$. By Proposition 2.3.5 there is a subspace $U' \le \mathscr{S}^-$ with $\mathrm{rk}(U') = k$ that is opposite U. This implies $V \cap U' = \varnothing$ and consequently, V has corank $k+1$ in $\mathscr{S}^-$.
By Lemma 2.1.21(i) we obtain $U \le \mathrm{copr}_{\mathscr{S}^+}(q)$ for every point $q \in V$ and hence, $\bigcap_{p \in U} \mathrm{copr}_{\mathscr{S}^-}(p) = V$. Now let $q \in \mathscr{S}^+ \setminus U$. Then $\langle q, U\rangle$ has rank $k+1$. This implies that $\bigcap_{p \in \langle q, U\rangle} \mathrm{copr}_{\mathscr{S}^-}(p) = \bigcap_{i \le k} \mathrm{copr}_{\mathscr{S}^-}(p_i) \cap \mathrm{copr}_{\mathscr{S}^-}(q)$ has corank $k+2$ in $\mathscr{S}^-$ and hence, $V \not\le \mathrm{copr}_{\mathscr{S}^-}(q)$. ☐

This lemma already enables us to classify the twin SPO spaces without symplecta. Note that in this case there are no non-rigid examples.

Theorem 6.2.2. *Every twin SPO space of diameter* ≤ 1 *is a twin projective space.*

Proof. Let $(\mathscr{S},\mathscr{D})$ be a twin SPO space such that $\mathscr{S}$ is singular. By Theorem 2.1.22 we know that $\mathscr{S}$ and $\mathscr{D}-$ are both projective spaces. Let $\mathfrak{M}$ be the set of hyperplanes of $\mathscr{S}$. We know $\mathrm{copr}_{\mathscr{S}}(p) \in \mathfrak{M}$ for every point $p \in \mathscr{D}$. Set $\mathfrak{M}' := \{\mathrm{copr}_{\mathscr{S}}(p) \mid p \in \mathscr{D}\}$ and let $\mathfrak{S}$ be the set of subspaces of $\mathscr{S}$ that arise from intersecting two distinct elements of $\mathfrak{M}'$. We now prove that

$$\big(\mathscr{S},\big(\mathfrak{M}',\big\{\{M\in\mathfrak{M}' \mid S\leq M\}\;\big|\;S\in\mathfrak{S}\big\}\big)\big)$$

is a twin projective space of $\mathscr{S}$ that is isomorphic to $(\mathscr{S},\mathscr{D})$.
By Lemma 2.1.13 we conclude that for two distinct points p and q of $\mathscr{D}$, the hyperplanes $\mathrm{copr}_{\mathscr{S}}(p)$ and $\mathrm{copr}_{\mathscr{S}}(q)$ are distinct. This implies that the map $\mathscr{D} \to \mathfrak{M}'\colon p \mapsto \mathrm{copr}_{\mathscr{S}}(p)$ is bijective. Now take two distinct point p and q of $\mathscr{D}$. Then Lemma 6.2.1 implies $S := \mathrm{copr}_{\mathscr{S}}(p) \cap \mathrm{copr}_{\mathscr{S}}(q) \in \mathfrak{S}$ and every point $r \in \mathscr{D}$ is on the line pq if and only if $S \leq \mathrm{copr}_{\mathscr{S}}(r)$. Hence,

$$\mathscr{D} \to \big(\mathscr{S},\big(\mathfrak{M}',\big\{\{M\in\mathfrak{M}' \mid S\leq M\}\;\big|\;S\in\mathfrak{S}\big\}\big) : \quad p \mapsto \mathrm{copr}_{\mathscr{S}}(p)$$

is an isomorphism. Let $H \in \mathfrak{M}$ with $S \leq H$ and let $s \in H \setminus S$. Then there is a point $r \in pq$ that is non-opposite s. We obtain $\mathrm{copr}_{\mathscr{S}}(r) = H$ and consequently, $H \in \mathfrak{M}'$. ☐

Before we go on with the other cases, we give a method how to construct a Grassmannian out of a singular twin SPO space.

Proposition 6.2.3. *Let* $\mathscr{S}^+$ *be singular. For a natural number* $k \in \mathbb{N}$, *let* $\mathfrak{U}^+$ *be the set of subspaces of rank* k *of* $\mathscr{S}^+$ *and let* $\mathfrak{U}^-$ *be the set of subspaces of corank* $k+1$ *of* $\mathscr{S}^-$. *Set*

$$\psi\colon \quad \mathfrak{U}^+ \to \mathfrak{U}^-\colon \quad U \mapsto \bigcap_{p\in U} \mathrm{copr}_{\mathscr{S}^-}(p)$$

$$\mathscr{P}_{\mathrm{m}} := \big\{U \cup U^{\psi} \mid U \in \mathfrak{U}^+\big\}$$

$$\mathscr{L}_{\mathrm{m}} := \big\{\{R\in\mathscr{P}_{\mathrm{m}} \mid P\cap Q \leq R\}\;\big|\;\{P,Q\}\subseteq\mathscr{P}_{\mathrm{m}} \wedge \mathrm{rk}(P\cap Q) = k-1\big\}\,.$$

Then $(\mathscr{P}_{\mathrm{m}},\mathscr{L}_{\mathrm{m}})$ *is isomorphic to the Grassmannian of* k*-spaces of* $\mathscr{S}^+$ *via the map* $\varphi\colon \mathfrak{U}^+ \to \mathscr{P}_{\mathrm{m}}\colon U \mapsto U \cup U^{\psi}$.

Proof. By Lemma 2.3.3 $\mathscr{S}^-$ is singular. Hence, we may apply Lemma 6.2.1 to conclude $U^{\psi} \in \mathfrak{U}^-$ for every $U \in \mathfrak{U}^+$ and moreover, ψ is injective.
The map φ is a bijection by the definition of $\mathscr{P}_{\mathrm{m}}$. By the definition of $\mathscr{L}_{\mathrm{m}}$ the pair

$(\mathscr{P}_{\mathrm{m}}, \mathscr{L}_{\mathrm{m}})$ is a point-lines space and moreover, φ preserves collinearity. Now let U and V be two distinct elements of $\mathfrak{U}^+$ such that U and V have a hyperplane H in common. Set $W := \langle U, V\rangle$. Then W has rank $k+1$. By Lemma 6.2.1 we conclude that $S := \bigcap_{p\in W} \mathrm{copr}_{\mathscr{S}^-}(p)$ has corank $k+2$ in $\mathscr{S}^-$. Since U^Ψ and V^Ψ are distinct subspaces of corank $k+1$ in $\mathscr{S}^-$, this implies that U^Ψ and V^Ψ intersect in the common hyperplane S. Hence, for a subspace $X \in \mathfrak{U}^+$, we obtain by Lemma 6.2.1 $S \le X^\Psi$ if and only if $X \le W$. This implies $X \cup X^\Psi \in \{R \in \mathscr{P}_{\mathrm{m}} \mid H \cup S \le R\}$ if and only if $S < X < W$. Since $H \cup S = (U \cup U^\Psi) \cap (V \cup V^\Psi)$, we conclude that φ maps lines of the Grassmannian of k-spaces of $\mathscr{S}^+$ bijectively onto elements of $\mathscr{L}_{\mathrm{m}}$. □

The second case of this section is the case where $\mathscr{S}^+$ and $\mathscr{S}^-$ are opposite symplecta. We will see that we just have to ask that one of the components contains exactly one symplecton. Note that in this case we do not rely on the rigid case.

Theorem 6.2.4. *Let $(\mathscr{S}^+, \mathscr{S}^-)$ be a twin SPO space such that $\mathscr{S}^+$ contains exactly one symplecton. Then $(\mathscr{S}^+, \mathscr{S}^-)$ is twin polar space.*

Proof. By Lemma 2.3.3 we know that every point of $\mathscr{S}^+$ is contained in a symplecton and hence, $\mathscr{S}^+$ is a symplecton. Let $Y \le \mathscr{S}^-$ be a symplecton that is opposite $\mathscr{S}^+$. Further let q be a point of $\mathscr{S}^-$. Since there is a point in $\mathscr{S}^+$ opposite to q, (A12) implies that q has a gate p in $\mathscr{S}^+$ with $\mathrm{cod}(p,q) = 2$. Let p' be the cogate of p in Y. Then $\mathrm{cod}(p, p') = 2$. By Lemma 2.3.3 we obtain $\mathrm{dist}(p', q) \le 2$ and moreover there is a symplecton $Z \le \mathscr{S}^-$ containing both p' and q. Since $\mathscr{S}^+$ has to be opposite to Z, we conclude $q = p'$ and therefore $Y = Z = \mathscr{S}^-$.
Therefore $\mathscr{S}^+$ and $\mathscr{S}^-$ are isomorphic via mapping every point onto its cogate by Corollary 4.2.8. This concludes that $\mathscr{S}$ is a twin polar space. □

6.3 Twin SPO spaces of symplectic rank 2

Throughout this section let $\mathscr{S} = (\mathscr{S}^+, \mathscr{S}^-)$ be a twin SPO space such that all symplecta have rank 2. We do not ask the twin SPO space to be rigid. Furthermore, $\mathscr{S}$ does not necessarily contain a symplecton. But we demand as an extra condition that $\mathscr{S}$ contains no triangle or equivalently, $\mathrm{srk}(\mathscr{S}) \le 1$. The following proposition shows that every rigid twin SPO space of symplectic rank 2 satisfies this condition. Hence, the class of twin SPO spaces that we consider in this section includes the class of rigid twin SPO spaces of symplectic rank 2.

Proposition 6.3.1. *A rigid subspace of an SPO space with symplectic rank* 2 *contains no triangles.*

Proof. Let V be a connected rigid subspace of an SPO space with symplectic rank 2 and let $l \leq V$ be a line. It suffices to show that there is no triangle containing l. By Lemma 3.1.1(i) there is a symplecton $Y \leq V$ containing l. Since Y is a rigid symplecton of rank 2, there is a point on l that is contained in three lines of Y. By Proposition 2.2.4(i) this implies that l is maximal singular subspace of V. □

We consider twin SPO spaces of singular rank ≤ 1 because they behave somehow similar to rigid twin SPO spaces of symplectic rank 2. The reason for this is that rigid twin SPO spaces of symplectic rank 2 are, as we will see in this section, twin dual polar spaces. The grid sum of at least two non-trivial twin dual polar spaces is again a twin dual polar space that is not rigid anymore but has singular rank 1, as one can see by the definition of a grid sum.

By Proposition 4.1.2 both connected components of $\mathscr{S}$ are either singletons or every maximal rigid subspace of $\mathscr{S}$ is either a single line or has symplectic rank 2 and diameter ≥ 2.

Proposition 6.3.2. *Let $U \leq \mathscr{S}$ be a non-empty coconvex subspace and let $p \in \mathscr{S}$ be a point.*

(i) *If $\operatorname{dist}(p,U) < \infty$, then p has a gate in U.*

(ii) *If $\operatorname{dist}(p,U) = \infty$, then p has a cogate in U.*

Proof. If $\operatorname{dist}(p,U) < \infty$, set $V := \operatorname{pr}_U(p)$. Otherwise, set $V := \operatorname{copr}_U(p)$. Since $\mathscr{S}$ is a twin SPO space and $U \neq \varnothing$, we obtain $V \neq \varnothing$. Moreover, V is contained in one of the two connected components. By Propositions 2.1.16(i) and 2.1.25(ii) V is convex since the convex span of every two points of V is contained in V.
Suppose V contains a line l. If $\operatorname{dist}(p,l) = \infty$, then by Lemma 4.2.1 there is a point $p' \in \mathscr{S}$ with $\operatorname{dist}(p',l) < \infty$ and $\operatorname{pr}_l(p') = l$. Now Lemma 3.2.1 implies that $\mathscr{S}$ contains a triangle, a contradiction. Thus, V contains a single point v.
Now let $q \in U$ be an arbitrary point. First assume $\operatorname{dist}(p,U) = \infty$. Then U is connected and hence $\operatorname{dist}(q,v) < \infty$. Applying Proposition 2.1.12(ii) to p and the metaplecton $\langle q,v\rangle_{\mathrm{g}}$ implies $\operatorname{cod}(p,v) = \operatorname{cod}(p,q) + \operatorname{dist}(q,v)$. Thus, v is a cogate for p in U. Now assume $\operatorname{dist}(p,U) < \infty$. If $\operatorname{dist}(q,v) < \infty$, we apply Proposition 2.1.25(i) to p and $\langle q,v\rangle_{\mathrm{g}}$ to conclude $\operatorname{dist}(p,v)+\operatorname{dist}(v,q) = \operatorname{dist}(p,q)$. If $\operatorname{dist}(q,v) = \infty$, then $\langle p,v\rangle_{\mathrm{g}}$ is a metaplecton with diameter $\operatorname{dist}(x,U)$. Since $\operatorname{dist}(q,\langle p,v\rangle_{\mathrm{g}}) = \infty$, we know that q has a cogate q' in $\langle p,v\rangle_{\mathrm{g}}$. Since U is coconvex and v and q are contained in U, we obtain $q' \in U$ and consequently, $\langle q',v\rangle_{\mathrm{g}} \leq U$. This implies $\operatorname{dist}(p,\langle q',v\rangle_{\mathrm{g}}) = \operatorname{dist}(p,v)$ and therefore $v = q'$ by Proposition 2.1.17(i). Thus, $\operatorname{dist}(p,v) = \operatorname{cod}(q,v) - \operatorname{cod}(q,p)$. □

Every line of $\mathscr{S}$ is already a maximal singular subspace of $\mathscr{S}$. Moreover, each point has a cogate in every line at finite codistance. Hence, the coconvex span of a point and a maximal singular subspace at finite codistance is always a coconvex span of two points at finite codistance. Therefore we consider in this section the coconvex spans of two points.

We show that the nice property for rigid subspaces of symplectic rank 2 with finite diameter that we stated in Theorem 3.2.3 also holds in the present situation.

Proposition 6.3.3. *Let $U \leq \mathscr{S}$ be a metaplecton and let $p \in \mathscr{S}$ be a point with* $\mathrm{dist}(p,U) < \infty$*. Then $\langle p,U\rangle_{\mathrm{g}}$ is a metaplecton with diameter* $\mathrm{dist}(p,U) + \mathrm{diam}(U)$.

Proof. By Proposition 6.3.2 p has a gate q in U. By Proposition 2.1.12(iii) there is a point $r \in U$ with $\langle q,r\rangle_{\mathrm{g}} = U$. This implies $\langle p,U\rangle_{\mathrm{g}} = \langle p,r\rangle_{\mathrm{g}}$. Since q is a gate for p in U, we obtain $\mathrm{dist}(p,r) = \mathrm{dist}(p,q) + \mathrm{dist}(q,r) = \mathrm{dist}(p,U) + \mathrm{diam}(U)$. □

Corollary 6.3.4. *Let $U \leq \mathscr{S}$ be a convex subspace of finite diameter. Then U is a metaplecton.*

Proof. Let u and v be points of U with $\mathrm{dist}(u,v) = \mathrm{diam}(U)$. Then $\langle u,v\rangle_{\mathrm{g}} \leq U$. For every point $p \in U$, we obtain $\langle p,u,v\rangle_{\mathrm{g}} \leq U$. Hence, Proposition 6.3.3 implies $p \in \langle u,v\rangle_{\mathrm{g}}$. □

For the classification of the twin SPO spaces of singular rank ≤ 1, we use coconvex subspaces that have non-empty intersection with both parts of $\mathscr{S}$. More precisely, we consider coconvex spans of two points at finite codistance. The following two rather technical lemmas are useful tools to discover the shape of such coconvex subspaces.

Lemma 6.3.5. *Let $U \leq \mathscr{S}^-$ be a metaplecton with* $\mathrm{diam}(U) = n$*. Further let $u \in U$, $v \in \mathscr{S}^-$ and $x \in \mathscr{S}^+$ be points such that* $\mathrm{cod}(x,v) = \mathrm{cod}(x,u) + \mathrm{dist}(u,v)$. *Then* $\mathrm{cod}(x,\langle v,U\rangle_{\mathrm{g}}) = \mathrm{cod}(x,U) + \mathrm{dist}(v,U)$.

Proof. By Proposition 6.3.2 v has a gate u' in U and x has a cogate x' in U. Since $\mathrm{cod}(x,v) = \mathrm{cod}(x,u) + \mathrm{dist}(u,v) = \mathrm{cod}(x,u) + \mathrm{dist}(v,u') + \mathrm{dist}(u',u)$, we conclude $\mathrm{cod}(x,v) = \mathrm{cod}(x,u') + \mathrm{dist}(u',v)$. Thus, we may assume $u = u'$. Set $k := \mathrm{dist}(v,u)$ and let $(v_i)_{0 \leq i \leq k}$ be a geodesic from u to v. Further set $m := \mathrm{dist}(u,x')$ and let $(u_i)_{0 \leq i \in m}$ be a geodesic from u to x'.
We recursively define points w_i for $0 \leq i \leq m$ with $w_i \perp u_i$, $w_i \notin U$ and $\mathrm{cod}(w_i,x) = \mathrm{cod}(u_i,x) + 1$. Set $w_0 := v_1$. Now let $i < m$ such that w_i is defined. Then $\langle w_i,u_{i+1}\rangle_{\mathrm{g}} \leq \langle w_i,U\rangle_{\mathrm{g}}$. Since $w_i \notin U$ and $w_i \perp u_i$, we obtain $\mathrm{pr}_U(w_i) = \{u_i\}$ and thus $\mathrm{dist}(w_i,u_{i+1}) = 2$ by Proposition 6.3.2. Let w_{i+1} be the cogate of x in $\langle w_i,u_{i+1}\rangle_{\mathrm{g}}$. Then $\mathrm{cod}(u_{i+1},x) = \mathrm{cod}(w_i,x) = \mathrm{cod}(u_i,x) + 1$ yields $\mathrm{cod}(w_{i+1},x) = \mathrm{cod}(u_{i+1},x) + 1$ and $\mathrm{dist}(w_{i+1},u_i) = 2$. Now $w_i \notin U$ yields $w_{i+1} \notin U$ since $u_i \in U$ and $\langle w_{i+1},u_i\rangle_{\mathrm{g}} = \langle w_i,u_{i+1}\rangle_{\mathrm{g}}$. Since w_{i+1} is the cogate of x in $\langle w_i,u_{i+1}\rangle_{\mathrm{g}}$, the points w_{i+1} and u_{i+1} are collinear.
Since $\langle u_i,w_{i+1}\rangle_{\mathrm{g}} = \langle w_i,u_{i+1}\rangle_{\mathrm{g}}$, we conclude $\langle w_i,U\rangle_{\mathrm{g}} = \langle w_{i+1},U\rangle_{\mathrm{g}}$ and therefore

$\langle v_1, U\rangle_g = \langle w_m, U\rangle_g =: V$. Proposition 6.3.3 implies $\mathrm{diam}(V) = \mathrm{diam}(U) + 1$. Together with $\mathrm{copr}_U(x) = \{u_m\}$ and $\mathrm{cod}(w_m, x) = \mathrm{cod}(u_m, x) + 1$ this leads to $\mathrm{copr}_V(x) = \{w_m\}$ by Proposition 6.3.2 and therefore $\mathrm{cod}(x,V) = \mathrm{cod}(x,U) + 1$. Since $u \in V$, the claim follows by repeating this procedure k times. □

Lemma 6.3.6. *Let $U \le \mathscr{S}^-$ and $V \le \mathscr{S}^+$ be two metaplecta such that U is one-coparallel to V. Further let $v \in V$, $u \in U$ and $w \in \mathscr{S}^+$ be points such that* $\mathrm{cod}(w,u) = \mathrm{cod}(v,u) + \mathrm{dist}(w,v)$. *Then U is one-coparallel to $\langle w,V\rangle_g$ and* $\mathrm{cod}(\langle w,V\rangle_g, U) = \mathrm{cod}(V,U) + \mathrm{dist}(w,V)$.

Proof. Set $d := \mathrm{dist}(w,V)$ and $W := \langle w,V\rangle_g$. We may assume $d > 0$ since otherwise we are done. By Lemma 6.3.5 we obtain $\mathrm{cod}(u,W) = \mathrm{cod}(u,V) + d = \mathrm{cod}(U,V) + d$. By Proposition 6.3.3 W is a metaplecton and hence by Proposition 6.3.2, u has a cogate w' in W. Let v' be the cogate of u in V. Then $\mathrm{dist}(w',v') = d$ since $\mathrm{cod}(u,v') = \mathrm{cod}(u,w') - d$.
Now let $p \in U$ be an arbitrary point and let p' be the cogate of p in V. Further let $q \in W$ be a point with $\mathrm{dist}(p',q) < d$. Since u is the cogate of v' in U, we obtain $\mathrm{cod}(v',p) = \mathrm{cod}(v',u) - \mathrm{dist}(u,p)$. Since $\mathrm{cod}(p,p') = \mathrm{cod}(u,v')$, this implies $\mathrm{dist}(v',p') = \mathrm{dist}(u,p)$. Since $\mathrm{cod}(u,p') = \mathrm{cod}(u,v') - \mathrm{dist}(v',p')$ we obtain $\mathrm{dist}(w',p') = \mathrm{dist}(v',p') + d$ and consequently,

$$\begin{aligned}\mathrm{cod}(w',p) \ge \mathrm{cod}(w',u) - \mathrm{dist}(u,p) &= (\mathrm{cod}(v',u) + d) - \mathrm{dist}(v',p') \\ &= (\mathrm{cod}(p',p) + d) - (\mathrm{dist}(w',p') - d) \\ &> \mathrm{cod}(q,p) - \mathrm{dist}(w',q)\,.\end{aligned}$$

Thus, q is not a cogate for p in W and hence, $\mathrm{cod}(p,W) \ge \mathrm{cod}(p,p') + d$ since p has a cogate in W by Proposition 6.3.2. Since by Proposition 2.1.17(i) $\mathrm{dist}(r,V) \le d$ for every point $r \in W$, we conclude $\mathrm{cod}(p,W) = \mathrm{cod}(p,V) + d$. □

In the following lemma provides a method how to decide whether a point belongs to the coconvex span of two points or not.

Lemma 6.3.7. *Let $x \in \mathscr{S}^+$ and $y \in \mathscr{S}^-$ be two points. Then* $\langle x,y\rangle_G \cap \mathscr{S}^+ = \bigcup\{\langle x,z\rangle_g \mid z \in \mathscr{S}^+ \wedge \mathrm{cod}(x,y) + \mathrm{dist}(z,x) = \mathrm{cod}(z,y)\}$.

Proof. First we define the following two sets:

$$\begin{aligned}U^+ &:= \bigcup\{\langle x,z\rangle_g \mid z \in \mathscr{S}^+ \wedge \mathrm{cod}(x,y) + \mathrm{dist}(z,x) = \mathrm{cod}(z,y)\}\\ U^- &:= \bigcup\{\langle y,z\rangle_g \mid z \in \mathscr{S}^- \wedge \mathrm{cod}(x,y) + \mathrm{dist}(z,y) = \mathrm{cod}(z,x)\}\end{aligned}$$

Now let $z \in \mathscr{S}^+$ be a point with $\mathrm{cod}(x,y) + \mathrm{dist}(z,x) = \mathrm{cod}(z,y)$. Then $z \in \langle x,y\rangle_G$ and hence $\langle x,z\rangle_g \le \langle x,y\rangle_G$. Thus, $U^+ \subseteq \langle x,y\rangle_G$ and analogously, $U^- \subseteq \langle x,y\rangle_G$.

Hence, it suffices to show that $U^+ \cup U^-$ is a coconvex subspace.
For $i \in \{0,1\}$, let $z_i \in \mathscr{S}^+$ be a point with $\mathrm{cod}(x,y)+\mathrm{dist}(z_i,x) = \mathrm{cod}(z_i,y)$. Then we obtain by Proposition 6.3.3 that $\langle x,z_0,z_1\rangle_\mathrm{g}$ is a metaplecton with diameter $\mathrm{dist}(x,z_0)+\mathrm{dist}(z_1,\langle x,z_0\rangle_\mathrm{g})$. Let z_2 be the cogate of y in $\langle x,z_0,z_1\rangle_\mathrm{g}$. By Lemma 6.3.5 we conclude $\mathrm{cod}(y,z_2) = \mathrm{dist}(z_1,\langle x,z_0\rangle_\mathrm{g})+\mathrm{cod}(y,\langle x,z_0\rangle_\mathrm{g})$. Thus,

$$\begin{aligned}\mathrm{cod}(y,z_2)-\mathrm{cod}(y,x) &= \mathrm{dist}(z_1,\langle x,z_0\rangle_\mathrm{g})+\mathrm{cod}(y,\langle x,z_0\rangle_\mathrm{g})-\mathrm{cod}(y,x)\\ &= \mathrm{dist}(z_1,\langle x,z_0\rangle_\mathrm{g})+\mathrm{dist}(z_0,x)\\ &= \mathrm{diam}(\langle x,z_0,z_1\rangle_\mathrm{g}).\end{aligned}$$

Therefore we obtain $\mathrm{cod}(x,y)+\mathrm{dist}(x,z_2) = \mathrm{cod}(z_2,y)$ and hence, $\langle x,z_0,z_1\rangle_\mathrm{g} = \langle x,z_2\rangle_\mathrm{g} \leq U^+$. Thus, U^+ is a convex subspace. Analogously, U^- is a convex subspace and we conclude that $U^+ \cup U^-$ is a convex subspace.
For symmetric reasons it remains to show that for arbitrary points $u \in U^+$ and $v \in U^-$, every point w with $w \perp u$ and $\mathrm{cod}(w,v) = \mathrm{cod}(u,v)+1$ is contained in U^+. Let $z_u \in U^+$ and $z_v \in U^-$ be points with $\mathrm{cod}(x,y) = \mathrm{cod}(z_u,y)-\mathrm{dist}(z_u,x) = \mathrm{cod}(z_v,x)-\mathrm{dist}(z_v,y)$ such that $u \in \langle x,z_u\rangle_\mathrm{g}$ and $v \in \langle y,z_v\rangle_\mathrm{g}$. By Corollary 4.2.8 there is a point w_v at distance $\mathrm{dist}(y,z_v)$ to x such that $\mathrm{cod}(w_v,y) = \mathrm{cod}(x,y)+\mathrm{dist}(z_v,y)$ and hence $\langle x,w_v\rangle_\mathrm{g} \leq U^+$. Moreover, the metaplecta $V^+ := \langle x,w_v\rangle_\mathrm{g}$ and $V^- := \langle y,z_v\rangle_\mathrm{g}$ are one-coparallel to each other with $\mathrm{cod}(V^+,V^-) = \mathrm{cod}(x,z_v)$. By Proposition 6.3.3 $\langle z_u,V^+\rangle_\mathrm{g}$ is a metaplecton with $\mathrm{diam}(\langle z_u,V^+\rangle_\mathrm{g}) = \mathrm{diam}(V^+)+\mathrm{dist}(z_u,V^+)$. Moreover, Lemma 6.3.6 implie that V^- is one-coparallel to $\langle z_u,V^+\rangle_\mathrm{g}$ with $\mathrm{cod}(V^-,\langle z_u,V^+\rangle_\mathrm{g}) = \mathrm{cod}(V^-,V^+)+\mathrm{dist}(z_u,V^+)$. Analogously, $\langle w,z_u,V^+\rangle_\mathrm{g}$ is a metaplecton with diameter $\mathrm{diam}(V^+)+\mathrm{dist}(z_u,V^+)+\mathrm{dist}(w,\langle z_u,V^+\rangle_\mathrm{g})$ and V^- is one-coparallel to $\langle w,z_u,V^+\rangle_\mathrm{g}$ with

$$\begin{aligned}\mathrm{cod}(V^-,\langle w,z_u,V^+\rangle_\mathrm{g}) &= \mathrm{cod}(V^-,V^+)+\mathrm{dist}(z_u,V^+)+\mathrm{dist}(w,\langle z_u,V^+\rangle_\mathrm{g})\\ &= \mathrm{cod}(V^-,V^+)+\mathrm{diam}(\langle w,z_u,V^+\rangle_\mathrm{g})-\mathrm{diam}(V^+)\\ &= \mathrm{cod}(y,x)+\mathrm{diam}(\langle w,z_u,V^+\rangle_\mathrm{g})\,.\end{aligned}$$

Thus, $\mathrm{diam}(\langle w,z_u,V^+\rangle_\mathrm{g}) = \mathrm{dist}(x,z)$, where z is the cogate of y in $\langle w,z_u,V^+\rangle_\mathrm{g}$. This implies $\langle x,z\rangle_\mathrm{g} = \langle w,z_u,V^+\rangle_\mathrm{g}$ and moreover, $\langle x,z\rangle_\mathrm{g} \leq U^+$ since $\mathrm{cod}(x,y)+\mathrm{dist}(z,x) = \mathrm{cod}(z,y)$. □

For coconvex spans of two points at finite codistance in twin SPO spaces of singular rank ≤ 1, we obtain a regularity that corresponds to the property of metaplecta stated in Propositions 2.1.3, 2.1.12(i) and 2.1.12(iii).

Proposition 6.3.8. *Let x and y be two points of $\mathscr{S}$ with $\mathrm{cod}(x,y) = n$. Then $\mathrm{codm}(\langle x,y\rangle_\mathrm{G}) = n$ and for every point $u \in \langle x,y\rangle_\mathrm{G}$ there is a point $v \in \langle x,y\rangle_\mathrm{G}$ at codistance n. Moreover, $\langle x,y\rangle_\mathrm{G} = \langle u,v\rangle_\mathrm{G}$ for every two points u and v in $\langle x,y\rangle_\mathrm{G}$ with $\mathrm{cod}(u,v) = n$.*

Proof. We may assume $x \in \mathscr{S}^+$ and $y \in \mathscr{S}^-$. Set $U := \langle x,y\rangle_{\mathrm{G}}$. Further set $U^\sigma = \mathscr{S}^\sigma \cap U$ for $\sigma \in \{+,-\}$. Let $u \in U^+$. By Lemma 6.3.7 there is a point $z \in U^+$ such that $u \in \langle x,z\rangle_{\mathrm{g}} =: V^+$ and $\mathrm{cod}(y,z) - \mathrm{dist}(z,x) = n$. By Corollary 4.2.8 there is a point v' with $\mathrm{dist}(v',y) = \mathrm{dist}(x,z)$ such that $\mathrm{cod}(x,v') = \mathrm{cod}(x,y) + \mathrm{dist}(y,v')$ and $V^- := \langle v',y\rangle_{\mathrm{g}}$ and V^+ are one-coparallel to each other with $\mathrm{cod}(V^+,V^-) = n + \mathrm{dist}(x,z)$. By Lemma 6.3.7 we obtain $V^- \leq U^-$. Since V^+ is one-coparallel to V^- with $\mathrm{cod}(V^+,V^-) = n + \mathrm{diam}(V^-)$ and u has a cogate in V^- by Proposition 6.3.2, there is a point $v \in V^-$ with $\mathrm{cod}(u,v) = n$.
Now let $v \in U^-$ be an arbitrary point with $\mathrm{cod}(u,v) = n$. Then there exists a point $z' \in U^-$ such that $v \in \langle y,z'\rangle_{\mathrm{g}}$ and $\mathrm{cod}(x,z') - \mathrm{dist}(z',y) = n$. By Lemma 6.3.6 V^+ is one-coparallel to $\langle z',V^-\rangle_{\mathrm{g}}$ with $\mathrm{cod}(V^+,\langle z',V^-\rangle_{\mathrm{g}}) = \mathrm{cod}(V^+,V^-) + \mathrm{dist}(z',V^-)$. Moreover, $\mathrm{diam}(\langle z',V^-\rangle_{\mathrm{g}}) = \mathrm{diam}(V^-) + \mathrm{dist}(z',V^-)$ by Proposition 6.3.3. Thus, for every point $p \in V^+$ and every point $q \in \langle z',V^-\rangle_{\mathrm{g}}$, we obtain $\mathrm{cod}(p,q) \geq \mathrm{cod}(V^+,\langle z',V^-\rangle_{\mathrm{g}}) - \mathrm{diam}(\langle z',V^-\rangle_{\mathrm{g}}) = \mathrm{cod}(V^+,V^-) - \mathrm{diam}(V^-) = n$. Since $\mathrm{cod}(u,v) = n$, Proposition 6.3.2 implies $\mathrm{cod}(V^+,v) = n + \mathrm{diam}(V^+)$ and analogously, $\mathrm{cod}(u,\langle z',V^-\rangle_{\mathrm{g}}) = n + \mathrm{diam}(\langle z',V^-\rangle_{\mathrm{g}})$. Thus, $V^+ = \langle u,\mathrm{copr}_{V^+}(v)\rangle_{\mathrm{g}}$ and $\mathrm{cod}(v,\mathrm{copr}_{V^+}(v)) = \mathrm{cod}(u,v) + \mathrm{diam}(V^+)$. Therefore, $\mathrm{copr}_{V^+}(v) \in \langle u,v\rangle_{\mathrm{G}}$ and hence, $V^+ \leq \langle u,v\rangle_{\mathrm{G}}$. Analogously, $\langle z',V^-\rangle_{\mathrm{g}} \leq \langle u,v\rangle_{\mathrm{G}}$ and we conclude that x and y are contained in $\langle u,v\rangle_{\mathrm{G}}$. Thus, $\langle x,y\rangle_{\mathrm{G}} = \langle u,v\rangle_{\mathrm{G}}$.
By Lemma 6.3.7 we obtain $\mathrm{cod}(u,p) \geq n$ and $\mathrm{cod}(p,v) \geq n$ for all $p \in U$ and hence, $\mathrm{codm}(U) = n$. □

Proposition 6.3.9. *Let $U \leq \mathscr{S}$ be the coconvex span of two points at finite codistance. Further let $p \in \mathscr{S}$ be a point. Then* $\mathrm{codm}(\langle x,U\rangle_{\mathrm{G}}) = \mathrm{codm}(U) - \mathrm{dist}(p,U)$.

Proof. Since $\mathrm{codm}(U) < \infty$, we obtain $U \cap \mathscr{S}^+ \neq \varnothing$ and $U \cap \mathscr{S}^- \neq \varnothing$. Hence, $n := \mathrm{dist}(p,U) < \infty$. By Proposition 6.3.2 p has a gate q in U. By Proposition 6.3.8 there is a point $r \in U$ such that $\langle q,r\rangle_{\mathrm{G}} = U$ and $\mathrm{cod}(q,r) = \mathrm{codm}(U)$. Then $\mathrm{cod}(p,r) = \mathrm{cod}(q,r) - \mathrm{dist}(p,q)$ and hence, $q \in \langle p,r\rangle_{\mathrm{G}}$. Thus, $\langle q,r\rangle_{\mathrm{G}} \leq \langle p,r\rangle_{\mathrm{G}}$ and therefore $\langle p,U\rangle_{\mathrm{G}} = \langle p,r\rangle_{\mathrm{G}}$. The claim follows from Proposition 6.3.8. □

Proposition 6.3.10. *Every coconvex subspace $U \leq \mathscr{S}$ with* $\mathrm{codm}(U) < \infty$ *is the coconvex span of two points at finite codistance.*

Proof. Let u and v be points of U with $\mathrm{cod}(u,v) = \mathrm{codm}(U)$. Then $\langle u,v\rangle_{\mathrm{G}} \leq U$. For every point $p \in U$, we obtain $\langle p,u,v\rangle_{\mathrm{G}} \leq U$. Hence, Proposition 6.3.9 implies $p \in \langle u,v\rangle_{\mathrm{G}}$. □

Our goal in this section is to prove that $\mathscr{S}$ is a twin dual polar space. For this we construct a polar space from the twin SPO space $\mathscr{S}$ and show that this polar space has a twin dual polar space that is isomorphic to $\mathscr{S}$. We define the following sets:

$$\mathscr{P}_{\mathrm{m}} := \{\langle p,q\rangle_{\mathrm{G}} \mid (p,q) \in \mathscr{S}^+ \times \mathscr{S}^- \wedge \operatorname{cod}(p,q) = 1\}$$
$$\mathscr{L}_{\mathrm{m}} := \big\{\{P \in \mathscr{P}_{\mathrm{m}} \mid P > \langle p,q\rangle_{\mathrm{G}}\} \bigm| (p,q) \in \mathscr{S}^+ \times \mathscr{S}^- \wedge \operatorname{cod}(p,q) = 2\big\}$$

The set $\mathscr{P}_{\mathrm{m}}$ will be the point set and $\mathscr{L}_{\mathrm{m}}$ will be the line set of the polar space we construct.

Lemma 6.3.11. *Let $P \in \mathscr{P}_{\mathrm{m}}$ and let V be the coconvex span of two points at codistance n. Then $V \leq P$ or $V \cap P = \varnothing$ or* $\operatorname{codm}(V \cap P) = n+1$.

Proof. Assume $V \cap P \neq \varnothing$ and $V \not\leq P$. Let $u \in V \cap P$. Then by Proposition 6.3.8 there is a point $v \in V$ such that $V = \langle u,v\rangle_{\mathrm{G}}$. Since $V \not\leq P$, we obtain $v \notin P$. By Proposition 6.3.2 v has a gate w in P. Since there is a point $w' \in P$ with $\operatorname{cod}(w,w') = 1$, we obtain $v \perp w$. Since $u \in P$, we obtain $\operatorname{cod}(u,w) = \operatorname{cod}(u,v) + \operatorname{dist}(v,w) = n+1$ and therefore $w \in V$. This implies $\operatorname{codm}(V \cap P) \geq n+1$. Since $V \cap P$ is coconvex and $V \cap P < V$, the claim follows from Lemma 6.3.8. ☐

Lemma 6.3.12. *Let $P \in \mathscr{P}_{\mathrm{m}}$. Then for every point $p \in \mathscr{S} \setminus P$ there is a subspace $Q \in \mathscr{P}_{\mathrm{m}}$ with $p \in Q$ and $P \cap Q = \varnothing$.*

Proof. By Proposition 6.3.2 p has a gate p' in P. Let $q \in \mathscr{S}$ be a point that is opposite p' and let q' be the gate of q in P. Since $\operatorname{codm}(P) = 1$, we obtain $q \notin P$. Since P contains a point at codistance 1 to p', we obtain $p \perp p'$ and analogously $q \perp q'$. Hence, Proposition 6.3.2 implies $\operatorname{cod}(p',q') = 1$ and consequently $p \leftrightarrow q'$. Thus, the line qq' contains exactly one point r at codistance 1 to p. Since $r \neq q'$ and $q \notin P$, we conclude $r \notin P$. Hence, q' is the gate of r in P. Set $Q := \langle p,r\rangle_{\mathrm{G}}$.
Let $s \in P$. Assume that s is in the same connected component as p. Then $p' \in \langle p,s\rangle_{\mathrm{g}}$ since p' is the gate for p in P. Since $p' \leftrightarrow p$, we obtain $p' \notin Q$ and hence, $s \notin Q$. Analogously, $s \notin Q$ if $\operatorname{dist}(s,q) < \infty$. Thus, $P \cap Q = \varnothing$. ☐

In the following there are two cases that play a special role. The first is that $\mathscr{S}^+$ and $\mathscr{S}^-$ are both singletons. In this case there are no two points in $\mathscr{S}$ at codistance 1 to each other. Hence, $\mathscr{P}_{\mathrm{m}}$ is the empty set and so is $\mathscr{L}_{\mathrm{m}}$. In the second case $\mathscr{S}^+$ and $\mathscr{S}^-$ are both lines. Then for every point in $\mathscr{S}^+$ there is precisely one point in $\mathscr{S}^-$ that is not opposite. Hence, $\mathscr{P}_{\mathrm{m}}$ contains the unordered pairs of points that are at codistance 1. Furthermore, $\mathscr{L}_{\mathrm{m}}$ is empty.

Proposition 6.3.13. *The pair $(\mathscr{P}_{\mathrm{m}},\mathscr{L}_{\mathrm{m}})$ is a non-degenerate polar space.*

Proof. For $\operatorname{diam}(\mathscr{S}^+) \leq 1$, then $\mathscr{L}_{\mathrm{m}}$ is empty and hence, (BS) is vacuously true. Moreover, if $\operatorname{diam}(\mathscr{S}^+) = 0$, then $\mathscr{P}_{\mathrm{m}}$ is empty and consequently, the radical of $(\mathscr{P}_{\mathrm{m}},\mathscr{L}_{\mathrm{m}})$ is empty. If $\operatorname{diam}(\mathscr{S}^+) = 1$, then $\mathscr{P}_{\mathrm{m}}$ contains more than one point

and the radical is again empty.
Now let $\mathrm{diam}(\mathscr{S}^+) \geq 2$. Set $\mathscr{D} := (\mathscr{P}_\mathrm{m}, \mathscr{L}_\mathrm{m})$. To avoid ambiguity, we use in this proof the character $\perp$ only to denote collinear points in $\mathscr{D}$. For $\mathscr{S}$, we do not use any character to denote collinearity. Furthermore, dist, cod and $\leftrightarrow$ always refer to $\mathscr{S}$.
Let $G \leq \mathscr{S}$ be a coconvex subspace with $\mathrm{codm}(G) = 2$. Then there is a point $p \in \mathscr{S}$ with $\mathrm{dist}(p, G) = 1$. By Proposition 6.3.9 we obtain $P := \langle p, G\rangle_\mathrm{G} \in \mathscr{P}_\mathrm{m}$. Now let $q \in \mathscr{S} \setminus P$. Then by Lemma 6.3.12 there is a subspace $Q \in \mathscr{P}_\mathrm{m}$ with $q \in Q$ and $Q \cap P = \varnothing$. Let $r' \in G$ and let r be the gate of r' in Q. Then Proposition 6.3.9 implies $\mathrm{dist}(r, r') = 1$ since $r' \notin Q$. Again by Proposition 6.3.9 we obtain $R := \langle r, G\rangle_\mathrm{G} \in \mathscr{P}_\mathrm{m}$. Thus, $|\{S \in \mathscr{P}_\mathrm{m} \mid S > G\}| \geq 2$ and we conclude that $\mathscr{D}$ is a point-line space.
Now let G as before and let P be an arbitrary element of $\mathscr{P}_\mathrm{m}$. We prove that $\mathscr{D}$ is a polar space by showing that either all or exactly one element of $\{R \in \mathscr{P}_\mathrm{m} \mid R > G\}$ is contained in $P^\perp$. First assume $G \cap P \neq \varnothing$. Then every subspace $Q \in \mathscr{P}_\mathrm{m}$ with $G < Q$ intersects P and hence $\{R \in \mathscr{P}_\mathrm{m} \mid R > G\} \subseteq P^\perp$ by Lemma 6.3.11. Now assume $G \cap P = \varnothing$ and let v and u be points of G with $\mathrm{cod}(u, v) = 2$. Let u' be the gate of u in P. Then $\mathrm{dist}(u, u') = 1$ by Proposition 6.3.9 since $u \notin P$. Since $u' \notin G$, we conclude by Proposition 6.3.2 that u is the gate of u' in G and therefore $\mathrm{cod}(v, u') = 1$. Hence, $Q := \langle v, u'\rangle_\mathrm{G} \in \mathscr{P}_\mathrm{m}$. Moreover, by Lemma 6.3.11 $u' \in Q \cap P$ yields $Q \in P^\perp$ and $u \in \langle v, u'\rangle_\mathrm{G}$ implies $G < Q$. Conversely, let $R \in P^\perp$ with $G < R$. Then there is a point $w \in P \cap R$. Since u is the gate of u' in G and $u' \notin G$, we obtain $u' \in \langle u, w\rangle_\mathrm{G} \leq R$. By Proposition 6.3.9 we obtain $\langle u', G\rangle_\mathrm{G} \in \mathscr{P}_\mathrm{m}$. With $u' \in Q \cap R$ and $G \leq Q \cap R$ we conclude $Q = R$ by Proposition 6.3.8. Thus, Q is the unique element of $P^\perp$ that contains G. Therefore $\mathscr{D}$ is a polar space. Since for every $P \in \mathscr{P}_\mathrm{m}$, we find a point $p \in \mathscr{S} \setminus P$, the polar space $(\mathscr{P}_\mathrm{m}, \mathscr{L}_\mathrm{m})$ is non-degenerate by Lemma 6.3.12. □

We determine some objects of the polar space $(\mathscr{P}_\mathrm{m}, \mathscr{L}_\mathrm{m})$ by using terms of the twin SPO space $\mathscr{S}$. This provides some correspondences between objects of $\mathscr{S}$ and those of $(\mathscr{P}_\mathrm{m}, \mathscr{L}_\mathrm{m})$.

Lemma 6.3.14. *Let $x \in \mathscr{S}^+$ and let y be a point opposite to x. Set $M := \{S \in \mathscr{P}_\mathrm{m} \mid x \in S\}$ and $N := \{S \in \mathscr{P}_\mathrm{m} \mid y \in S\}$.*

(i) *M is a generator of the polar space $(\mathscr{P}_\mathrm{m}, \mathscr{L}_\mathrm{m})$.*

(ii) *Let $P \in \mathscr{P}_\mathrm{m} \setminus M$ and let z be the gate of x in P. Then $z \in S$ for every $S \in M$ with $S \cap P \neq \varnothing$.*

(iii) *(M, N) is a spanning pair of the polar space $(\mathscr{P}_\mathrm{m}, \mathscr{L}_\mathrm{m})$.*

Proof. Set $\mathscr{D} := (\mathscr{P}_\mathrm{m}, \mathscr{L}_\mathrm{m})$. If $\mathscr{S}^+$ is a singleton, then $(\mathscr{P}_\mathrm{m}, \mathscr{L}_\mathrm{m})$ is the empty space. This implies that M is empty and hence, M is a generator of $\mathscr{D}$. For (ii) there is nothing to prove. Since $\mathscr{P}_{\mathrm{m}} = \varnothing$, the condition in Definition 5.2.2 is vacuously fulfilled. If $\mathscr{S}^+$ is a line, then $M = \{\{x, x'\}\}$, where x' is the unique point in $\mathscr{S}^-$ that is not opposite x. Since $\mathscr{D}$ contains no lines, the singleton M is a generator of $\mathscr{D}$. Since every two distinct elements of $\mathscr{P}_{\mathrm{m}}$ are disjoint, (ii) holds. By the same reason and since M and N are disjoint, (M,N) is a spanning pair. Hence, from now on let $\operatorname{diam}(\mathscr{S}^+) \geq 2$.

(i) By the definition of $\mathscr{L}_{\mathrm{m}}$, the set M is a subspace of $\mathscr{D}$. Moreover, Lemma 6.3.11 implies that every two elements of M are collinear in $\mathscr{D}$ and hence, M is singular. Finally, Lemma 6.3.12 implies that for every $P \in \mathscr{P}_{\mathrm{m}} \setminus M$, there is a $Q \in M$ with $Q \cap P = \varnothing$. Therefore M is a generator of $\mathscr{D}$.

(ii) Since $x \notin P$, Proposition 6.3.9 provides $\operatorname{dist}(x,P) = 1$. The gate z of x in P exists by Proposition 6.3.2. Further let $S \in M$ with $P \cap S \neq \varnothing$ and let $p \in P \cap S$. Since z is the gate for x in P, we obtain $z \in \langle p, x \rangle_{\mathrm{G}} \leq S$.

(iii) Now let $P \in \mathscr{P}_{\mathrm{m}} \setminus M \cup N$. By symmetric reasons and Proposition 5.2.4 we have to show that there is a $Q \in N$ such that $Q \perp S \Leftrightarrow P \perp S$ for every $S \in M$. Let z be the gate of x in P. Again x and z are collinear. Since $x \leftrightarrow y$, there is a unique point z' on the line xz with $z' \not\leftrightarrow y$. Set $Q := \langle y, z' \rangle_{\mathrm{G}}$. Then $Q \in N$ and z' is the gate for x in Q since $x \notin Q$. By (ii) every $S \in M$ with $S \cap P \neq \varnothing$ contains z and hence, $z' \in S \cap Q$. Analogously, $z \in S \cap Q$ for every $S \in M$ with $S \cap Q \neq \varnothing$. Thus, $S \cap P \neq \varnothing$ if and only if $S \cap Q \neq \varnothing$. Now the claim follows from Lemma 6.3.11. □

Lemma 6.3.15. *Let $\mathfrak{G}$ be the set of generators of $(\mathscr{P}_{\mathrm{m}}, \mathscr{L}_{\mathrm{m}})$. Set $\varphi \colon \mathscr{S} \to \mathfrak{G} \colon p \mapsto \{S \in \mathscr{P}_{\mathrm{m}} \mid p \in S\}$. Further let $x \in \mathscr{S}^+$ be a point.*

(i) *Let $y \in \mathscr{S}$ be a point distinct to x. Then $x^\varphi = y^\varphi$ if and only if $\operatorname{diam}(\mathscr{S}^+) < \infty$ and y is the cogate for x in $\mathscr{S}^-$.*

(ii) *Let M be a generator of $(\mathscr{P}_{\mathrm{m}}, \mathscr{L}_{\mathrm{m}})$ that is commensurate to x^φ. Then there is a point $y \in \mathscr{S}^+$ such that $M = y^\varphi$.*

(iii) *Let $y \in \mathscr{S}^+$. Then x^φ and y^φ have a common hyperplane if and only if x and y are collinear.*

Proof. (i) Assume $y \in \mathscr{S}^+$. Then there is a point $z \in \mathscr{S}$ with $z \leftrightarrow x$ and $\operatorname{cod}(z,y) = \operatorname{dist}(x,y)$. Let $x' \in \mathscr{S}$ with $\operatorname{dist}(x,x') = 1$ and $\operatorname{dist}(x',y) = \operatorname{dist}(x,y) - 1$. Then $\langle x', z \rangle_{\mathrm{G}} \in y^\varphi \setminus x^\varphi$ and hence, $y^\varphi \neq x^\varphi$.

Now assume $y \in \mathscr{S}^-$. Set $d := \operatorname{cod}(x,y)$. Assume there is a point $x' \in \mathscr{S}$ with $\operatorname{dist}(x,x') = 1$ and $\operatorname{cod}(x',y) = d+1$. Then for a point $z \leftrightarrow x$ with $\operatorname{dist}(z,y) = d$, we obtain $\langle x', z \rangle_{\mathrm{G}} \in y^\varphi \setminus x^\varphi$ and hence, $y^\varphi \neq x^\varphi$. Now assume that there is no point in $\mathscr{S}^+$ at codistance $d+1$ to y that is collinear to x. Then by Proposition 2.1.16(ii) there is no point in $\mathscr{S}^+$ at codistance $d+1$ to y. By (A1) this implies that $\mathscr{S}^+$ has diameter d. Thus by Corollary 6.3.4, $\mathscr{S}^+$ is a metaplecton of diameter d. Consequently, $\mathscr{S}^+$ and $\mathscr{S}^-$ are opposite metaplecta and y is a cogate for x in $\mathscr{S}^+$

by Proposition 6.3.2.
Conversely, let $\mathscr{S}^+$ and $\mathscr{S}^-$ be opposite metaplecta and let y be the cogate of x in $\mathscr{S}^-$. Further let $S \in \mathscr{P}_\mathrm{m}$ with $x \in S$. Since there is a point $z \in S \cap \mathscr{S}^-$, we obtain $y \in \langle x, z \rangle_\mathrm{G} \leq S$. Analogously, $x \in S$ for every $S \in y^\varphi$ and hence, $x^\varphi = y^\varphi$.
(ii)&(iii) Since for two commensurate generators there is a finite sequence of generators such that two consecutive generators are adjacent, we may restrain ourselves to the case that x^φ and M intersect in a common hyperplane H. Then there is a $P \in \mathscr{P}_\mathrm{m}$ with $P \in M$ and $x \notin P$. Let y be the gate of x in P. For every $S \in H$, we obtain $y \in S$ by Lemma 6.3.14(ii). Since for every $Q \in M$ there is $S \in H$ and a $L \in \mathscr{L}_\mathrm{m}$ such that $\{P, Q, S\} \subseteq L$, this leads to $y \in Q$ by the definition of $\mathscr{L}_\mathrm{m}$. Thus, $M = y^\varphi$ since both are generators. Proposition 6.3.9 provides $\mathrm{dist}(x, y) = 1$. This proves the forward direction of (iii).
Conversely, let y be a point collinear to x. Let $L \in \mathscr{L}_\mathrm{m}$ with $L \leq x^\varphi$. Let $V \leq \mathscr{S}$ be the coconvex subspace of codiameter 2 that is contained in all elements of L. If $y \in V$, then $L \leq y^\varphi$. If $y \notin V$, then $\langle y, V \rangle_\mathrm{G} \in \mathscr{P}_\mathrm{m}$ by Proposition 6.3.9. Hence, $L \cap y^\varphi \neq \varnothing$ and the claim follows. □

Theorem 6.3.16. *Every twin SPO space with singular rank* ≤ 1 *is a twin dual polar space.*

Proof. By Proposition 6.3.13 $\mathscr{D} := (\mathscr{P}_\mathrm{m}, \mathscr{L}_\mathrm{m})$ is a non-degenerate polar space. Let $\mathfrak{G}$ be the set of generators of $\mathscr{D}$ and set $\varphi\colon \mathscr{S} \to \mathfrak{G}\colon p \mapsto \{S \in \mathscr{P}_\mathrm{m} \mid p \in S\}$. Let $x \in \mathscr{S}^+$ and $y \in \mathscr{S}^-$ be opposite points of $\mathscr{S}$. By Lemma 6.3.14(iii) we know that (x^φ, y^φ) is a spanning pair. Let $\mathscr{B}$ be the dual polar space of $\mathscr{D}$. Further let $\mathscr{B}^+$ be the connected component of $\mathscr{B}$ that contains x^φ and let $\mathscr{B}^-$ be the connected component that contains y^φ. We claim that $\mathscr{S}$ is isomorphic to the twin dual polar space $(\mathscr{B}^+, \mathscr{B}^-)$.
We conclude by Lemma 6.3.15 that φ maps $\mathscr{S}^+$ bijectively onto $\mathscr{B}^+$. Moreover, $\varphi|_{\mathscr{S}^+}$ preserves collinearity. Since every set of mutually collinear points of $\mathscr{S}$ in contained in a line of $\mathscr{S}$ and every set of mutually adjacent generators of $\mathscr{D}$ is contained in a line of $\mathscr{B}$, we conclude that φ induces an isomorphism from $\mathscr{S}^+$ onto $\mathscr{B}^+$. Analogously, φ maps $\mathscr{S}^-$ isomorphically onto $\mathscr{B}^-$. □

6.4 Twin SPO spaces of symplectic rank 3

In this section we consider the rigid twin SPO spaces of symplectic rank 3. Therefore, throughout this section let $\mathscr{S}$ be a twin SPO space of symplectic rank 3. This implies that $\mathscr{S}$ is rigid and has diameter ≥ 2. By $\mathscr{S}^+$ and $\mathscr{S}^-$ we denote the connected components of $\mathscr{S}$. Further we denote by $\mathfrak{M}$ the set of maximal singular subspaces of $\mathscr{S}$.

Since we have have already covered the case where $\mathscr{S}^+$ is a symplecton, we may constrain ourselves to the case where $\mathscr{S}^+$ contains a symplecton properly. By Proposition 3.4.1 this implies that every line is contained in exactly two elements of $\mathfrak{M}$. For reasons of convenience, we include in this section the case where $\mathscr{S}^+$, and therefore also $\mathscr{S}^-$, is a symplecton whose lines are contained in exactly two elements of $\mathfrak{M}$.

The subspaces we are interested in are the coconvex spans of an element of $\mathfrak{M}$ and a point at finite codistance. Therefore we examine the coprojection of a point at finite codistance in an element of $\mathfrak{M}$.

Lemma 6.4.1. *Let $M \in \mathfrak{M}$ and let x be a point with* $\mathrm{cod}(x,M) < \infty$*. Then the corank of* $\mathrm{copr}_M(x)$ *in M equals* $\mathrm{cod}(x,M)$ *or M equals* $\mathrm{copr}_M(x)$ *and* $\mathrm{rk}(M) = \mathrm{cod}(x,M) = \mathrm{diam}(\mathscr{S}^+)$.

Proof. We may assume $x \in \mathscr{S}^+$ and $M \leq \mathscr{S}^-$. Set $d := \min\{\mathrm{cod}(x,p) \mid p \in M\}$ and let y be a point with $y \leftrightarrow x$ and $\mathrm{dist}(y,M) = d$. By Lemma 3.4.2 we obtain $\mathrm{rk}(\mathrm{pr}_M(y)) = d$ and therefore $\mathrm{rk}(M) \geq d$. If $\mathrm{rk}(M) = d$, then $\mathrm{copr}_M(x) = \mathrm{pr}_M(y) = M$ and $\mathrm{cod}(x,M) = d$. By Theorem 3.4.4 this implies $\mathrm{diam}(\mathscr{S}^-) = d$.
Now assume $\mathrm{rk}(M) > d$ and let $z \in M \setminus \mathrm{pr}_M(y)$. Then by (A12) x has a cogate $z' \in \langle y,z\rangle_{\mathrm{g}}$ with $\mathrm{cod}(x,z') = d+1$. Thus z' is collinear to all points of $M \cap \langle y,z\rangle_{\mathrm{g}}$. Since $\langle z,\mathrm{pr}_M(y)\rangle$ is a maximal singular subspace of $\langle y,z\rangle_{\mathrm{g}}$ by Theorem 3.4.4, this implies $z' \in \langle z,\mathrm{pr}_M(y)\rangle \leq M$. We conclude $\mathrm{cod}(x,M) = d+1$ and $\langle z,\mathrm{pr}_M(y)\rangle \cap \mathrm{copr}_M(x) \neq \varnothing$. Thus, $\mathrm{crk}_M(\mathrm{copr}_M(x)) = d+1$ since $\mathrm{pr}_M(y) \cap \mathrm{copr}_M(x) = \varnothing$. ☐

For $\mathrm{srk}(\mathscr{S}^-) \geq 3$, Lemma 3.4.3(i) implies that there is a subset $\mathfrak{M}^- \subset \mathfrak{M}$ such that every line of $\mathscr{S}^-$ is contained in exactly one element of $\mathfrak{M}^-$ and every element of $\mathfrak{M}^-$ is contained in $\mathscr{S}^-$. Assume $\mathscr{S}^-$ is a symplecton. Then Proposition 2.2.8 implies that there is a subset $\mathfrak{M}^-$ of the set of generators of $\mathscr{S}^-$ such that every line of $\mathscr{S}^-$ is contained in exactly one element of $\mathfrak{M}^-$. Note that there is no given distinction between $\mathscr{S}^+$ and $\mathscr{S}^-$. Furthermore, there is no given distinction between $\mathfrak{M}^-$ and the maximal singular subspaces of $\mathscr{S}^-$ that are not contained in $\mathfrak{M}^-$. Hence, we may carry over all the results for the three other possible choices of $\mathfrak{M}^-$.

There is a correspondence between the bipartition of the elements of $\mathfrak{M}$ contained in $\mathscr{S}^-$ and the bipartition of those contained in $\mathscr{S}^+$ as the following shows.

Lemma 6.4.2. *Let $M \in \mathfrak{M}$ and let l be a line with* $\mathrm{cod}(l,M) = 1$*. Then there is exactly one subspace $N \in \mathfrak{M}$ with $l \leq N$ and* $\mathrm{cod}(M,N) = 1$.

Proof. Let y and z be distinct points on l. Since $\mathrm{rk}(M) \geq 2$, Lemma 6.4.1 implies that there is a point $x \in M$ with $\mathrm{cod}(x,y) = \mathrm{cod}(x,z) = 1$. With $\mathrm{cod}(l,M) = 1$ this implies $\mathrm{copr}_l(x) = l$. By Lemma 4.2.1 there is a point $w \leftrightarrow x$ with $l \in w^\perp$. By Lemma 3.1.1(i) there is a symplecton Y that contains $\langle w,l\rangle$. Now (A12) implies

that x has a cogate at codistance 2 in Y. Hence, there is an element in $\mathfrak{M}$ that contains l and the cogate of x in Y. Hence, there is at most one subspace in $\mathfrak{M}$ that contains $l \leq N$ and has codistance 1 to M.
Since $\langle w, l\rangle$ is a generator of Y, Proposition 2.2.5 implies that there is a unique subspace $N \in \mathfrak{M}$ with $\langle w, l\rangle \leq N$. Let p be an arbitrary point of $\langle w, l\rangle$. Then $p \not\leftrightarrow x$ if and only if $p \in l$. Furthermore, if $p \in l$, then there is a point $q \in M$ with $p \leftrightarrow q$ by Lemma 6.4.1 since $\mathrm{cod}(l, M) = 1$. Thus, every point of $\langle w, l\rangle$ is opposite to a point of M and we obtain $\mathrm{cod}(q, \langle w, l\rangle) = 1$ for every point $q \in M$. With Lemma 6.4.1 this implies $\mathrm{cod}(p, N) = 1$ since $\mathrm{rk}(\langle w, l\rangle) = 2$ and consequently, $\mathrm{cod}(M, N) = 1$. □

Proposition 6.4.3. *Let $l \leq \mathscr{S}^+$ be a line. Then there is exactly one subspace $M \in \mathfrak{M}$ with $l \leq M$ such that $\mathrm{cod}(M, K) = 1$ for a subspace $K \in \mathfrak{M}^-$.*

Proof. Let w and x be distinct points of l. Further let g be a line that is opposite l. Let $K \in \mathfrak{M}^-$ with $g \leq K$. By Lemma 6.4.2 there is a unique subspace $M \in \mathfrak{M}$ with $l \leq M$ and $\mathrm{cod}(M, K) = 1$. Let $N \in \mathfrak{M} \setminus \{M\}$ with $l \leq N$.
Now let $L \in \mathfrak{M}^- \setminus \{K\}$ be a subspace with $\mathrm{cod}(l, L) = 1$. We show $\mathrm{cod}(M, L) = 1$ and $\mathrm{cod}(N, L) = 2$. By Lemma 6.4.1 there is a point $w' \in L$ with $w \leftrightarrow w'$. Let w_0 be the cogate of x in g. Then $w \leftrightarrow w_0$. Assume $\mathrm{dist}(K, L) \geq 3$. Set $W := \langle w_0, w'\rangle_{\mathrm{g}}$ and let $x' \in W$ with $\mathrm{dist}(x', w') = \mathrm{dist}(w_0, w') - 1$ and $w' \perp x'$. By (A2) w has a cogate in $\langle x', w'\rangle_{\mathrm{g}}$ at codistance $\mathrm{dist}(x', w')$. Since $\mathrm{cod}(w, x') \leq 1$ and $\langle x', w'\rangle_{\mathrm{g}}$ is an SPO space by Proposition 2.1.23, there is a point $w_1 \in \langle x', w'\rangle_{\mathrm{g}}$ that is collinear to x' and at distance $\mathrm{dist}(x', w')$ to the cogate of w in $\langle x', w'\rangle_{\mathrm{g}}$. This yields $w \leftrightarrow w_1$. Since $\mathrm{dist}(w_0, w_1) \leq 2$ and $\mathrm{dist}(w_1, w') < \mathrm{dist}(w_0, w')$, repeating this argument leads to a finite sequence of points $(w_i)_{0 \leq i \leq n}$ that are all opposite w such that $w_n = w'$ and $\mathrm{dist}(w_i, w_{i+1}) \leq 2$ for $i < n$. By Corollary 4.2.8 there is a line g_i through w_i that is opposite l. Let $K_i \in \mathfrak{M}^-$ with $g_i \leq K_i$. Then $\mathrm{cod}(l, K_i) = 1$. Thus, it suffices to consider the case $\mathrm{dist}(K, L) \leq 2$. Moreover, we may assume $\mathrm{dist}(w_0, L) \leq 2$.
Assume $d := \mathrm{dist}(K, L) \geq 1$. By Lemma 6.4.1 there is a point $x' \in L$ with $x \leftrightarrow x'$. Since $\mathrm{dist}(w_0, L) \leq 2$, we obtain $\mathrm{dist}(w_0, x') \leq 3$. As before, we find a point $x_0 \in \langle w_0, x'\rangle_{\mathrm{g}}$ with $x \leftrightarrow x_0$ and $w_0 \perp x_0$. Since $\mathrm{cod}(x, w_0) = 1$, we obtain $w_0 \neq x_0$. Let $L_0 \in \mathfrak{M}^-$ with $w_0 x_0 \leq L_0$. By Corollary 4.2.8 $w_0 x_0$ is opposite l and hence, $\mathrm{cod}(l, L_0) = 1$. If $\mathrm{dist}(w_0, x') = d + 1$, we obtain $\mathrm{pr}_L(w_0) \leq \langle w_0, x'\rangle_{\mathrm{g}}$ and hence L intersects $\langle w_0, x'\rangle_{\mathrm{g}}$ in a singular subspace of rank $d + 1$ by Lemma 3.1.1(iii) and Theorem 3.4.4. Since $w_0 x_0 \leq \langle w_0, x'\rangle_{\mathrm{g}}$, we conclude by the same reason that L_0 intersects $\langle w_0, x'\rangle_{\mathrm{g}}$ in a singular subspace of rank $d + 1$. Since by Proposition 2.1.17(i) $\mathrm{dist}(r, L \cap \langle w_0, x'\rangle_{\mathrm{g}}) \leq d$ for every point $r \in L_0 \cap \langle w_0, x'\rangle_{\mathrm{g}}$, we conclude $\mathrm{dist}(L, L_0) < d$ by Lemma 3.4.2. If $\mathrm{dist}(w_0, x') = d$, then $\mathrm{dist}(w_0 x_0, x') = d - 1$ by Proposition 2.1.17(i). In both cases $K \cap L_0 \neq \varnothing$ and $\mathrm{dist}(L, L_0) < d$ and therefore we may restrain ourselves to the case $K \cap L \neq \varnothing$.
Since $K \neq L$, Lemma 3.4.2 implies that K and L intersect in a single point s. Since

$\text{cod}(s,M)=1$ there is a point $s' \in M$ with $s \leftrightarrow s'$ by Lemma 6.4.1. Since $\text{cod}(s,l)=1$, we may assume $\text{cod}(s,x)=1$. Again by Lemma 6.4.1 there is a point $x' \in L$ with $x \leftrightarrow x'$ since $\text{cod}(x,L)=1$. Let $K' \in \mathfrak{M} \setminus \{L\}$ with $sx' \leq K'$. Then $K' \notin \mathfrak{M}^-$ and therefore, $K \cap K'$ is a line by Lemma 3.4.2. Hence, $\text{cod}(K',M)=2$ by Lemma 6.4.2. Since $\text{copr}_{sx'}(x)=\{s\}$ and $\text{cod}(s,M)=1$, we conclude $\text{cod}(sx',M)=1$. Again by Lemma 6.4.2 this implies $\text{cod}(L,M)=1$ and consequently, $\text{cod}(L,N)=2$. □

Motivated by this proposition, we set

$$\mathfrak{M}^+ := \{M \in \mathfrak{M} \mid \forall K \in \mathfrak{M}^- \colon 2 \leq \text{cod}(M,L) < \infty\}.$$

With this definition it follows from the proposition above, that every line of $\mathscr{S}$ is contained in exactly one element of $\mathfrak{M}^+ \cup \mathfrak{M}^-$.

Remark 6.4.4. Let M and N be two elements of $\mathfrak{M}$ with $\text{cod}(M,N)=1$. Then by the definition of $\mathfrak{M}^+$ it follows that $M \in \mathfrak{M}^-$ implies $N \notin \mathfrak{M}^+$. By symmetric reasons, $M \in \mathscr{S}^-$ and $M \notin \mathfrak{M}^-$ implies $N \in \mathfrak{M}^+$. Thus, exactly one of the subspaces M and N is an element of $\mathfrak{M}^+ \cup \mathfrak{M}^-$.
Let $V \leq \mathscr{S}^-$ be a metaplecton with diameter $n \geq 2$. By Proposition 2.1.23 V is an SPO space. Let S and T be maximal singular subspace of V with $\text{dist}(S,T)=n-1$, or in other words at codistance 1 with respect to the opposition relation in V. Since by Proposition 2.1.17(i) every point of S has distance $n-1$ to T, Lemma 3.4.2 implies that exactly one of the subspaces S and T is contained in an element of $\mathfrak{M}^-$. This confirms that we made the "right" choice when defining $\mathfrak{M}^+$.

Lemma 6.4.5. *Let $M \in \mathfrak{M}^-$ and $N \leq \mathscr{S}^+$ with $N \in \mathfrak{M} \setminus \mathfrak{M}^+$. Then $\text{rk}(M) < \infty$ implies $\text{rk}(M) = \text{rk}(N)$.*

Proof. Assume $r := \text{rk}(M) < \infty$. By Proposition 2.3.5 there is a singular subspace $S \leq \mathscr{S}^+$ with $\text{rk}(S) = r$ such that M and S are opposite. Let $K \in \mathfrak{M}$ be the subspace with $S \leq K$. Suppose $S < K$. Then by Proposition 2.3.5 M and K are not opposite and hence, there is a point $p \in K$ with $\text{cod}(q,p)=1$ for every point $q \in M$. Thus, Lemma 2.1.21(ii) implies that M is not maximal, a contradiction. This leads to $S=K$ and $\text{cod}(M,K)=1$. Therefore, $K \notin \mathfrak{M}^+$ and we conclude $\text{rk}(N)=\text{rk}(K)=r$ with Lemmas 3.4.3(i) and 3.4.3(iii). □

To study coconvex spans of a point and a maximal singular subspace at finite codistance, we need some more properties concerning coprojections in a maximal singular subspace.

Lemma 6.4.6. *Let M and N be elements of $\mathfrak{M}$ that intersect in a single point s. Further let x be a point with $\text{cod}(x,M) < \infty$ such that $\text{copr}_M(x) < M$ and $\text{copr}_N(x) < N$. Then $\pi_{M,N}(\langle s, \text{copr}_M(x)\rangle) = \langle s, \text{copr}_N(x)\rangle$.*

Proof. Set $d := \mathrm{cod}(x,M)$. Further set $S := \mathrm{copr}_M(x)$ and $T := \mathrm{copr}_N(x)$. First let $\mathrm{cod}(x,N) \neq d$. By Lemma 3.3.3(iii) we may assume $\mathrm{cod}(x,N) = d-1$. That implies $\mathrm{cod}(x,s) = d-1$. Thus, $\mathrm{crk}_M(S) = d$ by Lemma 6.4.1 and consequently, $\mathrm{rk}(M) \geq d$ since $s \in M \setminus S$. Hence by Lemma 3.3.3(iv), $\mathrm{rk}(N) \geq d$ and consequently, $\mathrm{crk}_N(T) = d-1$ by Lemma 6.4.1. For every point $p \in S$, we obtain $\mathrm{pr}_N(p) \leq T$. Thus $\pi_{M,N}(\langle s,S\rangle) \leq T$. Since $\mathrm{crk}_M(\langle s,S\rangle) = d-1$, Lemma 3.3.3(iii) implies $\mathrm{crk}_N(\pi_{M,N}(\langle s,S\rangle)) = d-1$ and therefore $\pi_{M,N}(\langle s,S\rangle) = T$.
Now let $\mathrm{cod}(x,N) = d$. First assume $\mathrm{cod}(x,s) = d-1$. Then $\mathrm{crk}_M(S) = \mathrm{crk}_N(T) = d$ by Lemma 6.4.1. For $d = 1$, this implies $M = \langle s,S\rangle$ and hence, $\pi_{M,N}(\langle s,S\rangle) = N = \langle s,T\rangle$. Therefore we may assume $d > 1$ and hence by Lemma 6.4.1, that there is a point $q \in N \setminus \{s\}$ such that $\mathrm{cod}(x,sq) = d-1$. Let $p \in M$ such that $\mathrm{pr}_N(p) \cap T \neq \varnothing$. Then $q \notin \mathrm{pr}_N(p)$ since by Lemma 3.4.2 $\mathrm{pr}_N(p)$ is a line. Thus, $Y := \langle p,q\rangle_{\mathrm{g}}$ is a symplecton. By Lemma 3.1.1(iii), both M and N contain a generator of Y. Since $sq \leq Y$, we obtain $\mathrm{cod}(x,Y) \leq d$ and since $\mathrm{cod}(x,\mathrm{pr}_N(p)) = d$, we conclude $\mathrm{cod}(x,Y) = d$. Suppose x has a cogate in Y. Then this cogate would be contained in $\mathrm{pr}_N(p) \setminus \{s\}$ and hence there is a point in $M \cap Y$ at codistance $d-2$ to x, a contradiction. Thus by Proposition 4.2.5, $\mathrm{copr}_Y(x)$ is a generator of Y. Since $\mathrm{cod}(x,\mathrm{pr}_N(p)) = d$ and $\mathrm{cod}(x,sq) = d-1$, the generators $\mathrm{copr}_Y(x)$ and $Y \cap N$ intersect in a single point q'. Applying Proposition 2.2.8 yields that the corank of $\mathrm{copr}_Y(x) \cap M$ in $M \cap Y$ is even. With $s \in Y \cap M$ this implies, that $\mathrm{copr}_Y(x)$ and $Y \cap M$ intersect in a single point p'. Since $s \in \mathrm{pr}_N(p)$ and $\mathrm{cod}(x,\mathrm{pr}_N(p)) = d$, we conclude $\mathrm{pr}_N(p) = sq'$. Since $q' \perp p'$, we obtain $\pi_{N,M}(sq') = sp'$ and hence, $p \in sp'$ by Lemma 3.3.3(iii). Thus, $\mathrm{pr}_N(p) \leq \langle s,T\rangle$ implies $p \in \langle s,S\rangle$ and therefore $\pi_{M,N}(\langle s,S\rangle) \geq \langle s,T\rangle$. Since $\mathrm{copr}_M(S) = \mathrm{copr}_N(T) = d$, the claim follows from Lemma 3.3.3(iii).
Finally assume $s \in S$. Since $M > S$, there is a point $r \in M$ with $\mathrm{cod}(x,r) = d-1$. Let $q \in N$ such that $\mathrm{pr}_N(r) = sq$. Let $p \in S \setminus \{s\}$. Since $sp \leq S$ and $\mathrm{pr}_M(q) = sr$, this implies that $Y := \langle p,q\rangle_{\mathrm{g}}$ is a symplecton. By Lemma 3.1.1(iii), both M and N contain a generator of Y. Assume $\mathrm{cod}(x,Y) = d+1$. Then by Proposition 2.1.12(iv) x has a cogate y in Y. Thus $\langle y,sp\rangle$ is a generator of Y. Since $\langle y,sp\rangle$ and $M \cap Y$ are the only generators that contain sp, we conclude $\langle p,\mathrm{pr}_N(p)\rangle = \langle y,sp\rangle$ and therefore $\mathrm{pr}_N(p) \leq T$. Now assume $\mathrm{cod}(x,Y) = d$. Then by Proposition 4.2.5 $\mathrm{copr}_Y(x)$ is a generator of Y since $r \in Y \setminus \mathrm{copr}_Y(x)$ and $sp \leq \mathrm{copr}_Y(x)$. Since $M \cap Y$ and $\mathrm{copr}_Y(x)$ intersect in a common line, we conclude by Proposition 2.2.8 that the corank of $\mathrm{copr}_Y(x) \cap N$ in $N \cap Y$ is odd. Thus, $s \in \mathrm{copr}_Y(x) \cap N$ yields that $\mathrm{copr}_Y(x) \cap N$ is a line l. This implies $\mathrm{copr}_Y(x) = \langle p,l\rangle$ and hence, $\mathrm{pr}_N(p) = l \leq T$. Again we conclude $\pi_{M,N}(\langle s,S\rangle) \leq \langle s,T\rangle$ and the claim follows by Lemma 3.3.3(iii). □

Corollary 6.4.7. *Let V be a connected convex subspace with* $\mathrm{diam}(V) \geq 2$ *and let $M \in \mathfrak{M}^-$ be a subspace that contains a line of V. Further let $x \in \mathscr{S}^+$ be a* *point with* $\mathrm{copr}_M(x) \leq V$*. Then* $\mathrm{copr}_N(x) \leq V$ *for every subspace* $N \in \mathfrak{M}^-$ *with* $\mathrm{rk}(N \cap V) \geq 1$ *and* $\mathrm{copr}_N(x) < N$*.*

Proof. By Lemma 3.1.1(iii) $M \cap V$ and $N \cap V$ are maximal singular subspaces of V. First assume there is a subspace $K \in \mathfrak{M}^-$ with $\mathrm{rk}(K \cap V) > 1$ and $\mathrm{copr}_K(x) = K$. Then $K \leq \mathrm{copr}_{\mathscr{S}^-}(x)$ by Lemma 6.4.1 and hence, $K \cap V \leq \mathrm{copr}_V(x)$. Moreover, $\mathrm{rk}(K) = \mathrm{diam}(\mathscr{S}^-) =: d$. By Proposition 2.1.16(ii) there is a point $p \in \mathrm{copr}_V(x)$ such that $\mathrm{dist}(p,N) = d - \mathrm{cod}(x, N \cap V)$. Let $l \leq \mathrm{copr}_V(x)$ be a line through p and let $K' \in \mathfrak{M}^-$ with $l \leq K'$. By Lemma 3.4.3(iii) we know $\mathrm{rk}(K') = d$ and hence, $\mathrm{crk}_{K'}(l) = d-1$. Thus, Lemma 6.4.1 implies $K' \leq \mathrm{copr}_{\mathscr{S}^-}(x)$. Therefore we may assume $p \in K$ and hence $\mathrm{dist}(K,N) = d - \mathrm{cod}(x, N \cap V)$.
For every point $q \in N \setminus V$, we obtain $\mathrm{pr}_N(p) \leq \langle p,q \rangle_{\mathrm{g}}$ by Lemma 3.4.3(iii). Since p is in the projection of x in $\langle p,q \rangle_{\mathrm{g}}$, we conclude $\mathrm{cod}(x,q) \leq \mathrm{cod}(x, \mathrm{pr}_N(p))$ by Lemma 2.1.24. Thus, $\mathrm{copr}_N(x) \cap \mathrm{pr}_N(p) \neq \varnothing$ and we obtain $\mathrm{cod}(x,N) = \mathrm{cod}(x, N \cap V)$. By Lemma 3.4.3(iii) we know $\mathrm{rk}(N) = d$. Moreover, $N \cap V$ contains a subspace S of rank $d - \mathrm{cod}(x,N)$ whose points are all at distance $\mathrm{dist}(K,N)$ to K. Hence, S corank $\mathrm{cod}(x,N)$ in N. Since $\mathrm{crk}_N(\mathrm{copr}_N(x)) = \mathrm{cod}(x,N)$ by Lemma 6.4.1, we conclude $S = \mathrm{copr}_N(x)$.
Now assume $\mathrm{copr}_K(x) < K$ for every subspace $K \in \mathfrak{M}^-$ with $\mathrm{rk}(K \cap V) \geq 1$. Then by Lemma 3.4.3(ii) and since V is connected, we may assume that N intersects M in a single point s. Applying Lemma 6.4.6 yields $\pi_{M,N}(\langle s, \mathrm{copr}_M(x) \rangle) = \langle s, \mathrm{copr}_N(x) \rangle$. By Lemma 3.4.2 this implies $\langle s, \mathrm{copr}_N(x) \rangle \leq V$. □

Lemma 6.4.8. *Let* $M \in \mathfrak{M}^+$ *and* $N \in \mathfrak{M}^-$ *be maximal singular subspaces. Set* $S := \{p \in M \mid \mathrm{cod}(p,N) = \mathrm{cod}(M,N)\}$*. Then one of the following holds:*

(a) *The diameter of* $\mathscr{S}^+$ *is equal to* $\mathrm{cod}(M,N)$*. Furthermore,* $S = M$ *and* $\mathrm{crk}_N(\bigcap_{p \in N} \mathrm{copr}_M(p)) = \mathrm{cod}(M,N) + 1$*.*

(b) *The corank of* S *in* M *equals* $\mathrm{cod}(M,N)$ *and* $\mathrm{copr}_M(p) = S$ *for every point* $p \in N$ *with* $\mathrm{cod}(p,M) = \mathrm{cod}(M,N)$*.*

Proof. Set $d := \mathrm{cod}(M,N)$. Since $\mathscr{S}^+$ is connected, we know $d < \infty$. Moreover, $d \geq 2$ since $M \in \mathfrak{M}^+$ and $N \in \mathfrak{M}^-$. Let $x \in N$ be a point with $\mathrm{cod}(x,M) = d$. If $\mathrm{copr}_M(x) = M$, then $\mathrm{rk}(M) = d$ and $\mathrm{diam}(\mathscr{S}^+) = d$ by Lemma 6.4.1. By (A12), we conclude that $\mathscr{S}^+$ is not a metaplecton of diameter d. With Theorem 3.4.4 this implies $\mathrm{srk}(\mathscr{S}) > d$ and hence, $\mathrm{rk}(N) > d$ by Lemma 6.4.5. Since $S = M$, we obtain $\mathrm{cod}(p,q) \geq d-1$ for every pair $(q,p) \in M \times N$. By Lemma 6.4.1 this implies $\mathrm{cod}(p,M) = d$ for every point $p \in N$ since $\mathrm{rk}(M) = d$. Another consequence of Lemma 6.4.1 is $\mathrm{copr}_N(q) < N$ for every $q \in M$ since $\mathrm{rk}(N) > d$. Thus, for every point $q \in M$ there is a point $p \in N$ with $q \notin \mathrm{copr}_N(q)$ and we obtain $\bigcap_{p \in N} \mathrm{copr}_M(p) = \varnothing$. Hence, (a) holds.
Now let $\mathrm{copr}_M(x) < M$ for every point $x \in N$ with $\mathrm{cod}(x,M) = d$. Assume there

are points x and y in N with $\mathrm{cod}(x,M) = \mathrm{cod}(y,M) = d$ and $\mathrm{copr}_M(x) \neq \mathrm{copr}_M(y)$. Since $\mathrm{crk}_M(\mathrm{copr}_M(x)) = \mathrm{crk}_M(\mathrm{copr}_M(y)) = d$ by Lemma 6.4.1, there are points $x' \in \mathrm{copr}_M(x) \setminus \mathrm{copr}_M(y)$ and $y' \in \mathrm{copr}_M(y) \setminus \mathrm{copr}_M(x)$. Let $z \leftrightarrow x'$ and $z' \leftrightarrow x$ be points such that $\mathrm{dist}(y,z) = \mathrm{dist}(y',z') = d-1$. Then $V := \langle x,z\rangle_{\mathrm{g}}$ and $U := \langle x',z'\rangle_{\mathrm{g}}$ are opposite by Lemma 4.2.7. By Theorem 3.4.4 and Lemma 3.1.1(iii) we obtain $\mathrm{rk}(M\cap U) = d$ and analogously, $\mathrm{rk}(N\cap V) = d$. By Lemma 4.2.7 there is a singular subspace $N' \in \mathfrak{M}$ with $\mathrm{rk}(N'\cap U) = d$ such that $\mathrm{copr}_U(p) \leq N'$ for every point $p \in N\cap V$. Since by Proposition 2.1.23 U is an SPO space, we may apply Proposition 2.3.5 to conclude that there is a subspace $M' \in \mathfrak{M}$ with $\mathrm{rk}(M'\cap U) = d$ and $\mathrm{dist}(N',M') = d-1$. By Lemma 3.4.2 every point of $N'\cap U$ has distance d to a point in $M'\cap U$. Since every point $p \in N\cap V$ has a cogate in U that is contained in N', this implies $p \leftrightarrow q$ for a point $q \in M'\cap U$. Thus, $\mathrm{cod}(r,N\cap V) = 1$ for every point $r \in M'$. Since $\mathrm{rk}(N\cap V) = d \geq 2$, this implies $\mathrm{cod}(r,N) = 1$ by Lemma 6.4.1 and therefore, $\mathrm{cod}(M',N) = 1$. Since $N \in \mathfrak{M}^-$, this implies $M' \notin \mathfrak{M}^+$. Since $\mathrm{cod}(M',N') = d-1$ and $\mathrm{cod}(p,N') = d-1$ for every point $p \in N'\cap U$, we conclude $N' \in \mathfrak{M}^+$ by Lemma 3.4.2. Since $y \in V$ and $y' \in U$, we know that y' is the cogate of y in V. Hence, $y \in N$ implies $y' \in N'$. Analogously, $x' \in N'$ and therefore $N' = M$.

Now for $M \leq U$, Theorem 3.4.4 implies $\mathrm{diam}(\mathscr{S}^+) = d$. Hence, (a) holds since $\bigcap_{p\in N\cap V} \mathrm{copr}_M(p) = \varnothing$. Therefore we may assume $\mathrm{rk}(M) > d$. Since $\mathrm{rk}(M) > d$ and $\mathrm{crk}_M(\mathrm{copr}_M(x)) = d$ there is a line $l \leq \mathrm{copr}_M(x)$ through x'. Let $v \in V$ with $\mathrm{dist}(x,v) = d$. Then $x' \leftrightarrow v$ since x is the cogate for x' in V. Let w be the cogate of v in l. Then x is not a cogate for w in V. Let $p \in V \setminus N$. Then there is a point $q \in N\cap V$ with $\mathrm{dist}(p,q) \geq 2$. Hence, $\mathrm{cod}(p,q') \leq d-2$, where q' is the cogate for q in U. Since $q' \in M$, this implies $\mathrm{cod}(w,p) < d$ and consequently, $x \in \mathrm{copr}_V(w) \leq N$. Since x is no cogate for w in V, Proposition 2.1.12(ii) implies $\mathrm{copr}_V(w) > \{x\}$ and hence, Proposition 4.2.5 implies $\mathrm{copr}_V(w) = N\cap V$. Therefore we conclude that $H := \bigcap_{p\in N\cap V} \mathrm{copr}_M(p)$ intersects every line of $\mathrm{copr}_M(x)$ through x'. Since $y \in N\cap V$ and $x' \notin \mathrm{copr}_M(y)$, we obtain $x' \notin H$ and therefore H is a hyperplane of $\mathrm{copr}_M(x)$. By Lemma 6.4.1 this implies $\mathrm{crk}_M(H)d+1$. Since $H\cap(M\cap U) = \varnothing$ and $\mathrm{rk}(M\cap U) = d$, we conclude $M = \langle H, M\cap U\rangle$. Thus, for every point $q \in M$, there is a point $p \in M\cap U$ such that $q \in \langle p,H\rangle$. Let p' be the cogate of p in V. Then $\mathrm{copr}_M(p') = \langle p,H\rangle$ and hence, $q \in S$. Since $\mathrm{cod}(w,N) = d$ and $N\cap V \leq \mathrm{copr}_N(w)$, we obtain by Lemma 6.4.1 $N\cap V = N$ since $\mathrm{rk}(N\cap V) = d$ and consequently, $\mathrm{diam}(\mathscr{S}^+) = d$. Again (a) holds.

Finally let $\mathrm{copr}_M(x) = \mathrm{copr}_M(y) < M$ for every two distinct points x and y of N with $\mathrm{cod}(x,M) = \mathrm{cod}(x,M) = d$. Then $S = \mathrm{copr}_M(x)$ and the claim follows with Lemma 6.4.1 □

Corollary 6.4.9. *Let $M \in \mathfrak{M}$. Further let h be a line that is one-coparallel to a line $g \leq M$. Then there is a subspace $H \leq M$ with $g\cap H = \varnothing$ such that* $\mathrm{copr}_M(p) =$

$\langle \mathrm{copr}_g(p), H\rangle$ *for every point* $p \in h$.

Proof. We may assume $M \in \mathfrak{M}^-$. Let x and y be distinct points of h. Since h is one-coparallel to a line $g \leq M$, we obtain $h \leq \mathscr{S}^+$. Let $N \in \mathfrak{M}^+$ such that $h \leq N$. Set $d := \mathrm{cod}(h,g)$ and $S := \{p \in M \mid \mathrm{cod}(p,N) = \mathrm{cod}(M,N)\}$. Since $\mathrm{copr}_M(x) \neq \mathrm{copr}_M(y)$, Lemma 6.4.8 implies $\mathrm{crk}_M(S) = \mathrm{cod}(M,N) = d+1$ or $S = M$. In the second case we obtain $\mathrm{cod}(M,N) = d$ since $g \leq S$ and g and h are one-coparallel at codistance d. Hence by Lemma 6.4.8, there is a subspace $H \leq M$ with $\mathrm{crk}_M(H) = d+1$ that is contained in $\mathrm{copr}_M(p)$ for every point $p \in h$ and therefore $g \cap H = \varnothing$. Since $\mathrm{copr}_M(x) \neq \mathrm{copr}_M(y)$, Lemma 6.4.1 implies $\mathrm{crk}_M(\mathrm{copr}_M(p)) = d$ and the claim follows. In the case $\mathrm{crk}_M(S) = \mathrm{cod}(M,N) = d+1$, we conclude by Lemma 6.4.1 that S is a hyperplane of $\mathrm{copr}_M(p)$ for every $p \in h$. □

A coconvex subspace of $\mathscr{S}$ of finite codiameter consists of two parts of infinite diameter as long as $\mathscr{S}^+$ and $\mathscr{S}^-$ have infinite diameter. The following lemma gives another possibility to make assertions about the size of convex subspaces of infinite diameter by taking the intersection with the maximal singular subspaces into account.

Lemma 6.4.10. *Let* U *and* V *be two convex subspaces with* $U \leq V \leq \mathscr{S}^-$. *Further let* $M \in \mathfrak{M}^-$ *and* $N \in \mathfrak{M} \setminus \mathfrak{M}^-$ *be two subspaces that contain a line of* V. *Let* $M \cap V \leq U$ *and* $N \cap V \leq U$. *Then* $U = V$.

Proof. Let $M' \in \mathfrak{M}^-$ such that M' and N intersect in a line of V. Since $N \cap V \leq U$, we know $M' \cap N \leq U$. By Lemma 3.4.3(ii) and since U is connected, there is a finite sequence $(M_i)_{0 \leq i \leq m} \in (\mathfrak{M}^-)^{m+1}$ with $M_0 = M$ and $M_m = M'$ such that $M_i \cap M_{i+1} \neq \varnothing$ and M_i contains a line of U for $i < m$. Then M_i intersects both U and V in maximal singular subspace by Lemma 3.1.1(iii) for $i \leq m$. Assume $M_i \cap V \leq U$. Then $M_i \cap V = M_i \cap U$. By Lemma 3.4.2 we obtain $\pi_{M_i,M_{i+1}}(M_i \cap V) \leq V$ and $\pi_{M_{i+1},M_i}(M_{i+1} \cap V) \leq V$. Thus, Lemma 3.3.3(iii) implies $\pi_{M_i,M_{i+1}}(M_i \cap V) = M_{i+1} \cap V$. Analogously, $\pi_{M_i,M_{i+1}}(M_i \cap U) = M_{i+1} \cap U$ and therefore $M_{i+1} \cap V = M_{i+1} \cap U$. Induction leads to $M' \cap V \leq U$ and hence, $M' \cap V = M' \cap U$.
Now let $p \in V$ be a point. By Lemmas 3.4.3(ii) and 3.1.1(v) there is a finite sequence $(N_i)_{0 \leq i \leq n} \in \mathfrak{M}^{n+1}$ with $N_0 = N$, $N_1 = M'$ and $p \in N_n$ such that $N_i \cap N_{i+1}$ is line of V for $i < n$ and $N_i \cap N_{i+2} \neq \varnothing$ for $i < n-1$. Assume $N_i \cap V \leq U$ and $N_{i+1} \cap V \leq U$ for $i < n-1$. Then $N_{i+1} \cap N_{i+2} \leq U$. Thus, N_{i+2} contains a line of U and we obtain $N_{i+2} \cap V \leq U$ as before. Induction leads to $N_n \cap V \leq U$ and hence, $p \in U$. □

The following proposition shows that the coconvex span of a point and of a maximal singular subspace at finite codistance has properties that correspond to the properties of metaplecta stated in the Propositions 2.1.3, 2.1.12(i) and 2.1.12(iii).

Proposition 6.4.11. *Let $M \leq \mathfrak{M}^-$ and let $x \in \mathscr{S}^+$ be a point with $d := \operatorname{codm}(M \cup \{x\}) < \operatorname{diam}(\mathscr{S}^+)$.*

(i) *Let $l \leq \langle x, M \rangle_{\mathrm{G}}$ be a line. Further let $L \in \mathfrak{M}^+ \cup \mathfrak{M}^-$ and $K \in \mathfrak{M} \setminus \{L\}$ such that $K \cap L = l$. Then $L \leq \langle x, M \rangle_{\mathrm{G}}$ and $\operatorname{crk}_K(K \cap \langle x, M \rangle_{\mathrm{G}}) = d$.*

(ii) *Let $K \in \mathfrak{M}$ such that $M \cap K$ is a line. Then $K \cap \langle x, M \rangle_{\mathrm{G}} = \langle M \cap K, \operatorname{copr}_K(x) \rangle$ or $\operatorname{copr}_K(x) = K$.*

(iii) $\operatorname{codm}(\langle x, M \rangle_{\mathrm{G}}) = d$.

(iv) *For every point $u \in \langle x, M \rangle_{\mathrm{G}}$, there is a subspace $K \in \mathfrak{M}^+ \cup \mathfrak{M}^-$ with $K \leq \langle x, M \rangle_{\mathrm{G}}$ and $\operatorname{codm}(K \cup \{u\}) = d$. Moreover, $\langle u, K \rangle_{\mathrm{G}} = \langle x, M \rangle_{\mathrm{G}}$ for every such subspace K.*

Proof. Set $V := \langle x, M \rangle_{\mathrm{G}}$. Since we demanded $d < \operatorname{diam}(\mathscr{S}^+)$, Lemma 6.4.1 implies $\operatorname{crk}_M(\operatorname{copr}_M(x)) = d + 1$. Let $x' \in M$ be a point with $\operatorname{cod}(x, x') = d$ and let $g \leq M$ be a line through x' that intersects $\operatorname{copr}_M(x)$ in a point y. Then by Corollary 4.2.8 there is a line g' through x that is one-coparallel to g with $\operatorname{cod}(g, g') = d + 1$. Let $M' \in \mathfrak{M}^+$ be the subspace that contains g'. By Lemma 6.4.8 we obtain $\operatorname{cod}(M, M') = d + 2$ or $\operatorname{cod}(p, M') = \operatorname{cod}(p', M) = d + 1$ for every pair of points $(p, p') \in M \times M'$. Since V is coconvex, we conclude $\operatorname{copr}_{M'}(x') \leq V$ and hence, $\langle x, \operatorname{copr}_{M'}(x') \rangle \leq V$. Let $p \in M' \setminus \langle x, \operatorname{copr}_{M'}(x') \rangle$. Then $p \notin \operatorname{copr}_{M'}(y)$ by Corollary 6.4.9 and hence, $\operatorname{cod}(p, M) = d + 1$. Thus, $\operatorname{copr}_M(p) \neq \operatorname{copr}_M(x)$ and hence by Lemma 6.4.1, there is a point $q \in \operatorname{copr}_M(p) \setminus \operatorname{copr}_M(x)$. This implies $p \in \langle q, x \rangle_{\mathrm{G}} \leq V$. We conclude $M' \leq V$.
Let $N \in \mathfrak{M} \setminus \mathfrak{M}^-$ such that $g \leq N$. Set $U^- := \langle M, \operatorname{copr}_N(x) \rangle_{\mathrm{g}}$. Let analogously $N' \in \mathfrak{M} \setminus \mathfrak{M}^+$ such that $g' \leq N'$ and set $U^+ := \langle M', \operatorname{copr}_{N'}(x') \rangle_{\mathrm{g}}$. We will show $V = U^+ \cup U^-$. Since $\operatorname{cod}(x, N) = d + 1$ and $x' \in N$, we obtain $\operatorname{copr}_N(x) \leq V$ by the coconvexity of V and therefore, $U^- \leq V$. Analogously, $U^+ \leq V$. Since $U^+ \cup U^-$ is a convex subspace and $M \cup \{x\} \subseteq U^+ \cup U^-$, it remains to show that $U^+ \cup U^-$ is coconvex to conclude $V = U^+ \cup U^-$. By symmetric reasons it suffices to show that for a pair of points $(u, v) \in U^+ \times U^-$ and a point w with $w \perp v$ and $\operatorname{cod}(u, w) = \operatorname{cod}(u, v) + 1$, we obtain $w \in U^-$.
Since $g \leq N$, we obtain $\operatorname{crk}_N(\operatorname{copr}_N(x)) = d + 1$ by Lemma 6.4.1 and hence, $\operatorname{crk}_N(N \cap U^-) = d$ by Lemma 6.1.2. Let $l \leq M$ be an arbitrary line and let $K \in \mathfrak{M} \setminus \mathfrak{M}^-$ be the subspace that contains l. If U^- is singular and hence $U^- = M$, we obtain $K \cap U^- = l$. Furthermore, $N \cap U^- = g$ and hence, $\operatorname{rk}(N) = d + 1$. This implies $\operatorname{rk}(K) = d + 1$ by Lemma 3.4.3(iii) and thus, $\operatorname{crk}_K(K \cap U^-) = d$. If $l \cap \operatorname{copr}_M(x) = \varnothing$, then Lemma 6.1.1 implies $\operatorname{cod}(x, K) = d$ and hence, $l = \operatorname{copr}_K(x)$ by Lemma 6.4.1. If l intersects $\operatorname{copr}_M(x)$ in a singleton, then this singleton equals $\operatorname{copr}_K(x)$ by Lemma 6.4.1. If $l \leq \operatorname{copr}_M(x)$, then $K = \operatorname{copr}_K(x)$ by Lemma 6.1.1. Hence, (ii) holds for U^- if it is singular.
Now let $\operatorname{diam}(U^-) \geq 2$. Then $\operatorname{crk}_K(K \cap U^-) = d$ by Lemma 3.4.3(iii). Assume

$l \cap \mathrm{copr}_M(x) = \varnothing$. Then Lemma 6.1.1 implies that K contains a point at codistance $d-1$ to x and therefore $\mathrm{crk}_K(\mathrm{copr}_K(x)) = d$ by Lemma 6.4.1. By Corollary 6.4.7 we obtain $\mathrm{copr}_K(x) \leq U^-$ since $\mathrm{copr}_N(x) \leq U^-$. Thus, $\mathrm{copr}_K(x) = \langle l, \mathrm{copr}_K(x)\rangle = K \cap U^-$. Assume l intersects $\mathrm{copr}_M(x)$ in a single point. Then again $\mathrm{copr}_K(x) \leq U^-$ by Corollary 6.4.7. Since $\mathrm{copr}_K(\langle l, \mathrm{copr}_K(x)\rangle) = d$ by Lemma 6.4.1, this implies $\langle l, \mathrm{copr}_K(x)\rangle = K \cap U^-$. Finally assume $l \leq \mathrm{copr}_M(x)$. For $\mathrm{diam}(\mathscr{S}^-) = d+1$, we obtain $K = \mathrm{copr}_K(x)$ by Lemma 6.1.1. For $\mathrm{diam}(\mathscr{S}^-) \geq d+2$, we obtain $\mathrm{cod}(x,K) = d+2$ by Lemma 6.1.1. Hence, $\mathrm{crk}_K(\mathrm{copr}_K(x)) = d+2$ by Lemma 6.4.1 and again $\langle l, \mathrm{copr}_K(x)\rangle = K \cap U^-$ since $\mathrm{copr}_K(x) \leq U^-$ by Corollary 6.4.7. Therefore we conclude that (ii) holds for U^-.

Now let $v \in U^-$. Further let $w \perp v$ with $\mathrm{cod}(x,w) = \mathrm{cod}(x,v)+1$. Suppose $w \notin U^-$. First assume that $\mathrm{pr}_{U^-}(w)$ contains a line l through v and let $K \leq \mathfrak{M}$ be the subspace that contains $\langle w,l\rangle$. Then $w \in \mathrm{copr}_K(x) < K$ since $v \in K$. If U^- is singular and hence equals M, this is a contradiction since (ii) holds for U^-. If $\mathrm{diam}(U^-) \geq 2$, we obtain a contradiction by Corollary 6.4.7 since $\mathrm{copr}_M(x) < M \leq U^-$ and $\mathrm{copr}_N(x) < N \cap U^-$. Thus, $\mathrm{pr}_{U^-}(w) = \{v\}$. Let $l \leq U^-$ be a line through v. Then $Y := \langle w,l\rangle_{\mathrm{g}}$ is a symplecton. Since $w^\perp$ contains a hyperplane of $U^- \cap Y$, we conclude $U^- \cap Y = l$. Let $G \leq Y$ be a generator with $l \leq G$. Then $l' := w^\perp \cap G$ is a line. Let $w' \in l' \setminus \{v\}$ and let $v' \in l \setminus \{v\}$. Then $\mathrm{cod}(x,w') \geq \mathrm{cod}(x,w)-1$. Since $l \leq \mathrm{pr}_{U^-}(w')$, we obtain $\mathrm{cod}(x,w') \leq \mathrm{cod}(x,v)$ and $\mathrm{cod}(x,w') \leq \mathrm{cod}(x,v')$, otherwise we would obtain a contradiction as before. Hence, $\mathrm{cod}(x,w') = \mathrm{cod}(x,v) \leq \mathrm{cod}(x,v')$. This implies $\mathrm{copr}_{l'}(x) = l'$ and therefore, $\mathrm{cod}(x,Y) = \mathrm{cod}(x,w)$. Since $\mathrm{cod}(x,v') \geq \mathrm{cod}(x,w)-1$, w is not a cogate of x in Y and hence by Proposition 4.2.5, $\mathrm{copr}_Y(x)$ is a generator of Y. Thus, $\mathrm{copr}_Y(x)$ contains a point w'' with $l \leq \mathrm{pr}_{U^-}(w'')$, a contradiction. We conclude $w \in U^-$.

To prove that for every point $u \in U^+$ and every point $w \perp v$ with $\mathrm{cod}(u,w) = \mathrm{cod}(u,v)+1$, we obtain $w \in U^-$, it suffices now to show that there are subspaces $M_u \in \mathfrak{M}^-$ and $N_u \in \mathfrak{M}$ such that $g_u := M_u \cap N_u$ is a line that intersects $\mathrm{copr}_{M_u}(u)$ in a single point and $U^- = \langle M_u, \mathrm{copr}_{N_u}(u)\rangle_{\mathrm{g}} =: U_u$. By Lemmas 6.4.1 and 6.1.2 we know $\mathrm{crk}_{N_u}(\langle g_u, \mathrm{copr}_{N_u}(u)\rangle) = \mathrm{cod}(u,g_u)-1$. Hence by Lemmas 6.4.10 and 3.4.3(iii), it suffices to show $\mathrm{cod}(u,g_u) = d+1$ and $\langle g_u, \mathrm{copr}_{N_u}(u)\rangle \leq U^-$. Since U^+ is connected, we may restrict ourselves to the case $u \perp x$.

Assume $u \notin M'$. Let $K \in \mathfrak{M}^+$ be the subspace that contains ux. Then $K \cap M' = \{x\}$ by Lemma 3.4.2 and hence, $K \leq U^+$ by Lemma 3.4.3(iii). Since $\mathrm{crk}_N(N \cap U^-) = d$, we conclude by symmetric reasons $\mathrm{crk}_{N'}(N' \cap U^+) = d$. Hence, the subspace $\mathrm{copr}_{N'}(x')$ is a hyperplane of $N' \cap U^-$ by Lemma 6.4.1. Since N' and K intersect in a line of U^+ by Lemma 3.4.2, we obtain $K \cap \mathrm{copr}_{N'}(x') \neq \varnothing$. This implies $\mathrm{cod}(x',K) = d+1$ and hence again by Lemma 6.4.1, we obtain $\mathrm{crk}_K(\mathrm{copr}_K(x')) = d+1$ since $\mathrm{cod}(x,x') = d$. Suppose $\mathrm{cod}(p,q) \geq d+1$ for every pair $(p,q) \in \mathrm{copr}_K(x') \times M$. Then $\mathrm{cod}(K,M) = d+2$ since otherwise every point of M would have codistance $d+1$ to K, which is case (a) of Lemma 6.4.8, but

$\mathrm{copr}_K(x') \leq \bigcup_{q\in M} \mathrm{copr}_K(q)$ contradicts this case. Since $\mathrm{cod}(x,M) = d+1$, case (b) of Lemma 6.4.8 holds and consequently, there is a point $p \in \mathrm{copr}_K(x')$ with $\mathrm{cod}(p,M) = d+1$. Hence, $\mathrm{copr}_M(p) = M$ by the supposition. Since $\mathrm{cod}(K,M) = d+2$, we know $\mathrm{diam}(\mathscr{S}^-) \geq d+2$ and hence, $\mathrm{rk}(M) \geq d+2$ by Theorem 3.4.4. This is a contradiction to Lemma 6.4.1. Thus, there is a point $z \in \mathrm{copr}_K(x')$ with $\mathrm{codm}(M \cup \{z\}) = d$. Since $x' \in \mathrm{copr}_M(z) \setminus \mathrm{copr}_M(x)$, Lemma 6.4.1 implies that there is a point $z' \in \mathrm{copr}_M(x) \setminus \mathrm{copr}_M(z)$. By Corollary 4.2.8 the lines xz and $x'z'$ are one-coparallel to each other with $\mathrm{cod}(xz, x'z') = d+1$. Thus, we could have chosen $x'z'$ instead of g and K instead of M'. Therefore may restrain ourselves to the case $u \in M'$.

First assume $\mathrm{cod}(u,y) = d$. Then $\mathrm{cod}(u,M) = d+1$ and $\mathrm{copr}_M(u) \neq \mathrm{copr}_M(x)$. By Lemma 6.4.1 there is a point $z \in \mathrm{copr}_M(u) \setminus \mathrm{copr}_M(x)$ and hence, ux and yz are one-coparallel to each other by Corollary 4.2.8. Let $N_u \in \mathfrak{M} \setminus \mathfrak{M}^-$ with $yz \leq N_u$. Then $\mathrm{copr}_{N_u}(x)$ and $\mathrm{copr}_{N_u}(u)$ intersect in a common hyperplane H by Corollary 6.4.9. Since $\mathrm{copr}_{N_u}(x) \leq U^-$, we obtain $\mathrm{copr}_{N_u}(u) = \langle z, H\rangle \leq U^-$. Thus, $U_u = U^-$ follows with $M_u := M$ and $g_u := yz$. We consider a special case for $\mathrm{cod}(u,y) = d$: Let y' be the cogate of x' in g'. Then $\mathrm{cod}(y',y) = d$. For $u = y'$, we can choose $z = x'$ and hence, $N_u = N$. Thus, the above implies $U^- = \langle M, \mathrm{copr}_N(y')\rangle_{\mathrm{g}}$. Therefore we may exchange the role of x and y'. As a consequence, the case $\mathrm{cod}(u,x') = d$ is also done. Therefore we may assume $u \in \mathrm{copr}_{M'}(y) \cap \mathrm{copr}_{M'}(x') =: H$.

By Corollary 6.4.9, we know that H is a hyperplane of $\mathrm{copr}_{M'}(x')$. By Lemma 6.4.1 this implies $\mathrm{crk}_{M'}(H) = d+2$. In the case $\mathrm{cod}(M,M') = d+1$ we obtain $\mathrm{cod}(p,M') = \mathrm{cod}(p',M) = d+1$ for every pair of points $(p,p') \in M \times M'$. Hence, Lemma 6.4.8 implies $H = \bigcup_{p\in M} \mathrm{copr}_{M'}(p)$ and therefore $\mathrm{copr}_M(u) = M$. For $\mathrm{cod}(M,M') = d+2$, Lemma 6.4.8 implies that H consists exactly of the points that have codistance $d+2$ to M and hence, $\mathrm{cod}(u,M) = d+2$. We conclude $\mathrm{codm}(M \cup \{u\}) = d+1$ for both cases. Since $g' \leq u^\perp$ and g' is one-coparallel to g with $\mathrm{cod}(g',g) = d+1$, we obtain $\mathrm{cod}(u,g) = d+1$. Thus, $\mathrm{codm}(N \cup \{u\}) = d$ by Lemma 6.1.1. Since $g \not\leq \mathrm{copr}_N(x)$, we obtain $\mathrm{copr}_N(x) \neq \mathrm{copr}_N(u)$. Hence by Lemma 6.4.1, there is a point $z \in \mathrm{copr}_N(x) \setminus \mathrm{copr}_N(u)$. By Corollary 4.2.8 the lines zx' and ux are one-coparallel to each other. By Corollary 6.4.9 this yields that $\mathrm{copr}_N(u) \leq \langle x', \mathrm{copr}_N(x)\rangle$. Hence, $U^- = U_u$ for $N_u := N$ and $g_u := zx'$. This concludes $V = U^+ \cup U^-$.

We obtain $\mathrm{crk}_N(N \cap V) = d$ since $N \cap V = N \cap U^-$. Thus, $\mathrm{crk}_{N'}(N' \cap V) = d$ by symmetric reasons. Hence, (i) follows from Lemma 3.4.3(iii). Claim (ii) holds since we know that it holds for U^-. Now suppose there are points u and v in V with $\mathrm{cod}(u,v) = d-1$. Since $\mathrm{diam}(\mathscr{S}^-) > d$, there is a point $w \perp v$ with $\mathrm{cod}(u,w) = d$. By the coconvexity of V this implies $w \in V$ and hence, the subspace $K \in \mathfrak{M}^+ \cup \mathfrak{M}^-$ with $wv \leq K$ is contained in V. Hence, $\langle u, K\rangle_{\mathrm{G}} \leq V$ and we obtain $\mathrm{crk}_N(N \cap V) = d-1$ by (i), a contradiction. Thus, $\mathrm{codm}(V) = d$ and (iii) follows from $\mathrm{cod}(x,x') = d$. Finally, we already showed that for every point $u \in U^+$ there is a subspaces $M_u \in \mathfrak{M}^-$ such that $M_u \leq V$ and $\mathrm{codm}(M_u \cup \{u\}) = d$. Now let $K \in \mathfrak{M}^-$ be an arbitrary subspace with $K \leq V$ and $\mathrm{codm}(K \cup \{u\}) = d$. Then $\langle u,K\rangle_\mathrm{G} \leq V$ and therefore we conclude $\langle u,K\rangle_\mathrm{G} \cap \mathscr{S}^- = U^-$ and $\langle u,K\rangle_\mathrm{G} \cap \mathscr{S}^+ = U^+$ by (i) and Lemma 6.4.10. Thus, $\langle u,K\rangle_\mathrm{G} = V$ and (iv) follows by symmetric reasons. □

Among the coconvex spans of a point and a maximal singular subspace at finite codistance the ones of codiameter 1 play a special role.

Lemma 6.4.12. *Let $M \in \mathfrak{M}^-$ and let $x \in \mathscr{S}^+$ such that* $\mathrm{codm}(\langle x,M\rangle_\mathrm{G}) = 1$. *Set $V := \langle x,M\rangle_\mathrm{G}$.*

(i) *Let $p \in \mathscr{S} \setminus V$. Then $\langle p, \mathrm{pr}_{\langle x,M\rangle_\mathrm{G}}(p)\rangle$ is an element of $\mathfrak{M} \setminus (\mathfrak{M}^+ \cup \mathfrak{M}^-)$.*

(ii) *Let $N \in \mathfrak{M}$ such that $N \cap V$ contains no line. Then $N \cap V \neq \varnothing$ if and only if $N \in \mathfrak{M}^+ \cup \mathfrak{M}^-$.*

Proof. (i) By Proposition 6.4.11(iv) we may assume $p \in \mathscr{S}^-$. By Lemma 6.4.1 there is line $g \leq M$ with $\mathrm{cod}(x,g) = 1$ and set $d := \mathrm{dist}(p,g)$. Further let $z \in g$ with $\mathrm{dist}(p,z) = d$. Then $\mathrm{cod}(x, \langle p,z\rangle_\mathrm{g}) \geq d$ by Proposition 2.1.17(ii). Since $\mathrm{cod}(x,z) = 1$, there is by Proposition 2.1.16(ii) a point $z' \in \langle p,z\rangle_\mathrm{g}$ with $\mathrm{cod}(x,z') = d$ and $\mathrm{dist}(z,z') = d-1$. Since V is coconvex, we obtain $z' \in V$ and hence, $\langle z,z'\rangle_\mathrm{g} \leq V$. By Proposition 2.1.17(i) this implies $\mathrm{dist}(p,V) = 1$.
Now let $l \leq V$ be a line with $\mathrm{dist}(p,l) = 1$. Let $L \in \mathfrak{M}^-$ with $l \leq L$. Then $L \leq V$ by Proposition 6.4.11(i). Thus, $\mathrm{dist}(p,L) = 1$ and Lemma 3.4.2 implies that $\mathrm{pr}_L(p)$ is a line. We may assume $\mathrm{pr}_L(p) = l$. Let $L' \in \mathfrak{M}$ with $\langle p,l\rangle \leq L'$. By Lemma 3.4.3(i) and Proposition 6.4.11(i) V intersects L' in a hyperplane. Since $p \notin V$ and $\mathrm{dist}(p,V) = 1$ this implies $\mathrm{pr}_V(p) = L' \cap V$ and therefore, $\langle p, \mathrm{pr}_V(p)\rangle = L'$.
(ii) By Proposition 6.4.11(iv) we assume $N \leq \mathscr{S}^-$. Let $p \in N \setminus V$. Then by (i) there is a subspace $K \in \mathfrak{M} \setminus \mathfrak{M}^-$ such that $K = \langle p, \mathrm{pr}_V(p)\rangle$. Assume $N \notin \mathfrak{M}^-$. Then $N \cap K = \{p\}$ by Lemma 3.4.3(i). Since $p^\perp \cap V \leq K$ this implies $N \cap V = \varnothing$. Now assume $N \in \mathfrak{M}^-$. Then $l = N \cap K$ is a line by Lemma 3.4.2. Thus, l intersects $K \cap V$ in a single point since $K \cap V = \mathrm{pr}_V(p)$ is a hyperplane of K. □

The following two claims show that the elements the coconvex spans of a point of $\mathscr{S}^+$ and an element of $\mathfrak{M}^-$ induce a lattice structure.

Proposition 6.4.13. *Let $N \in \mathfrak{M}^-$ and $z \in \mathscr{S}^+$ with* $\mathrm{codm}(\langle z,N\rangle_\mathrm{G}) = 1$. *Set $W := \langle z,N\rangle_\mathrm{G}$. Further let $M \in \mathfrak{M}^-$ and $y \in \mathscr{S}^+$ such that $V := \langle y,M\rangle_\mathrm{G} \nleq W$ and $d :=$* $\mathrm{codm}(V) < \mathrm{diam}(\mathscr{S}^+)$.

(a) *If* $\mathrm{diam}(\mathscr{S}^+) = d+1$, *then there is a point $x \in \mathscr{S}^{-\sigma}$ such that $V \cap W = \{x\} \cup \mathrm{copr}_{\mathscr{S}^\sigma}(x)$, where $\sigma \in \{+,-\}$, such that* $\mathrm{rk}(K) = d+1$ *for every $K \in \mathfrak{M}^\sigma$.*

(b) *If* $\operatorname{diam}(\mathscr{S}^+) > d+1$, *then there is a subspace* $L \in \mathfrak{M}^-$ *and a point* $x \in \mathscr{S}^+$ *with* $\operatorname{codm}(\langle x, L\rangle_G) = d+1$ *such that* $V \cap W = \langle x, L\rangle_G$.

Proof. Note that $\operatorname{cod}(y,M) = d+1$ by Lemma 6.4.1 since $\operatorname{rk}(M) \geq d+1$ by Theorem 3.4.4. Let $p \in M$ and assume $p \notin W$. By Lemma 6.4.12(i) we obtain $K := \langle p, \operatorname{pr}_W(p)\rangle \in \mathfrak{M} \setminus \mathfrak{M}^-$. With Lemma 3.4.2 this implies that $M \cap K$ is a line. Therefore $M \cap W$ contains a point s since $\operatorname{crk}_K(K \cap W) = 1$. By Proposition 6.4.11(iv) there is a subspace $M' \in \mathfrak{M}^+$ such that $\langle s, M'\rangle_G = V$. Then analogously, $M' \cap W \neq \varnothing$. Thus by Proposition 6.4.11(iv), we may assume $y \in W$. Since $V \nleq W$, this implies $M \nleq W$ and hence, $M \cap W = \{s\}$ by Proposition 6.4.11(i). Hence, there is indeed a point $p \in M \setminus W$ as assumed. By the coconvexity of W, we conclude $s \in \operatorname{copr}_M(y)$. Set $U := V \cap W$. With $ps \leq V$ we conclude $\operatorname{crk}_K(K \cap V) = d$ by Proposition 6.4.11(i). Hence, $\operatorname{crk}_K(K \cap U) = d+1$ since $p \in K \cap V \setminus W$ and $\operatorname{crk}_K(K \cap W) = 1$.

Suppose $U \cap \mathscr{S}^-$ contains a singular subspace S such that $\operatorname{crk}_{K'}(S) = d$ for a subspace $K' \in \mathfrak{M} \setminus \mathfrak{M}^-$. Then S contains a line l since $\operatorname{rk}(K') \geq d+1$ by Theorem 3.4.4. Let $L' \in \mathfrak{M}^-$ with $l \leq L'$. Then by Proposition 6.4.11(i) L' is contained in both V and W and thus, $L' \leq U$. Since $\operatorname{crk}_{K'}(K' \cap U) \leq d$ and U is convex we conclude $U \cap \mathscr{S}^- = V \cap \mathscr{S}^-$ by Lemma 6.4.10. This is a contradiction to $M \nleq W$.

Suppose $U \cap \mathscr{S}^-$ contains a point q with $\operatorname{cod}(y,q) = d$. Since U is convex $\langle q, s\rangle_g \leq U$ and therefore Proposition 2.1.16(ii) implies that there is a point $q' \in U$ with $q' \perp q$ and $\operatorname{cod}(y,q') = d+1$. Let $K' \in \mathfrak{M} \setminus \mathfrak{M}^-$ with $qq' \leq K'$. Then $\operatorname{crk}_{K'}(\langle q, \operatorname{copr}_{K'}(y)\rangle) = d$ by Lemma 6.4.1. Since $\langle q, \operatorname{copr}_{K'}(y)\rangle \leq U$ by the coconvexity of U, this is contradiction. Thus, $\operatorname{cod}(y,q) \geq d+1$ for every $q \in U$.

Assume $\operatorname{diam}(\mathscr{S}^-) = d+1$. Then by Theorem 3.4.4 we conclude $\operatorname{rk}(K) = d+1$ or $\operatorname{rk}(M) = d+1$. Assume $\operatorname{rk}(K) = d+1$. Then for every subspace $K' \in \mathfrak{M} \setminus \mathfrak{M}^-$ with $K' \leq \mathscr{S}^-$, we obtain $\operatorname{rk}(K' \cap U) < 1$. Thus by Lemma 3.1.1(iii), $U \cap \mathscr{S}^-$ does not contain any line. Since $U \cap \mathscr{S}^-$ is convex, this implies $U \cap \mathscr{S}^- = \{s\}$. By Lemma 6.4.5 we know $\operatorname{rk}(K) = d+1$ if and only if every element of $\mathfrak{M}^+$ has rank $d+1$. Thus, $\operatorname{rk}(M) = d+1$ implies $U \cap \mathscr{S}^+ = \{y\}$ by symmetric reasons. Consequently, for $\operatorname{rk}(K) = \operatorname{rk}(M) = d+1$, we obtain $U = \{s, y\}$. Furthermore, $\mathscr{S}^-$ is a metaplecton by Theorem 3.4.4. Hence, $\operatorname{copr}_{\mathscr{S}^-}(y) = \{s\}$ by (A12).

Now let $\operatorname{rk}(M) = d+1$ and $\operatorname{rk}(K) > d+1$. Then $K \cap U$ contains a line l through s, since $\operatorname{crk}_K(K \cap U) = d+1$. Let $L \in \mathfrak{M}^-$ with $l \leq L$. Then $L \leq U$ since by Proposition 6.4.11(i) L is contained in both V and W. Since $\operatorname{diam}(\mathscr{S}^-) = d+1$ and $\operatorname{cod}(y,q) \geq d+1$ for every $q \in U$ we obtain $U \cap \mathscr{S}^- \leq \operatorname{copr}_{\mathscr{S}^-}(y)$, this implies $L \leq \operatorname{copr}_{\mathscr{S}^-}(y)$. Thus, $K \nleq \operatorname{copr}_{\mathscr{S}^-}(y)$ by Lemma 6.1.1. Hence, $\operatorname{crk}_K(\operatorname{copr}_K(y)) = d+1$ by Lemma 6.4.1 and therefore, $K \cap U = \operatorname{copr}_K(y)$. Applying Lemma 6.4.10 we conclude $U \cap \mathscr{S}^- = \operatorname{copr}_{\mathscr{S}^-}(y)$ since both $\operatorname{copr}_{\mathscr{S}^-}(y)$ and U are convex. Analogously, we obtain $U = \{s\} \cup \operatorname{copr}_{\mathscr{S}^+}(s)$ for $\operatorname{rk}(M) > d+1$ and $\operatorname{rk}(K) = d+1$.

Finally let $\operatorname{diam}(\mathscr{S}) > d+1$. Then again $K \cap U$ contains a line l through s and hence, $L \leq U$ for the subspace $L \in \mathfrak{M}^-$ wit $l \leq L$. By symmetric reasons there is are subspaces $L' \in \mathfrak{M}^+$ and $K' \in \mathfrak{M} \setminus \mathfrak{M}^+$ that intersect in a line of U such that $L' \leq U$ and $\mathrm{crk}_{K'}(K' \cap U) = d+1$. Since $s \in L$, we obtain $\mathrm{codm}(\langle y, L\rangle_{\mathrm{G}}) \leq d+1$. Since U is coconvex, we obtain $\langle y, L\rangle_{\mathrm{G}} \leq U$. Thus, we conclude $U \cap \mathscr{S}^\sigma = \langle y, L\rangle_{\mathrm{G}} \cap \mathscr{S}^\sigma$ for $\sigma \in \{+,-\}$ by Proposition 6.4.11(i) and Lemma 6.4.10. ☐

Lemma 6.4.14. *Let $x \in \mathscr{S}^+$ and let $M \in \mathfrak{M}^-$ such that $\mathrm{codm}(M \cup \{x\}) > 0$. Set $V := \{x\} \cup \mathrm{copr}_{\mathscr{S}^-}(x)$ if $\mathrm{copr}_M(x) = M$ and $V := \langle x, M\rangle_{\mathrm{G}}$ otherwise. Further let $y \in \mathscr{S}^-$ with $\mathrm{dist}(y,V) = 1$ such that $\mathrm{pr}_V(y)$ contains a line. Then there is a subspace $M' \in \mathfrak{M}^-$ with $\mathrm{codm}(M' \cup \{x\}) = \mathrm{codm}(M \cup \{x\}) - 1$ such that $\langle x, M'\rangle_{\mathrm{G}} = \langle y, V\rangle_{\mathrm{G}}$.*

Proof. Let $l \leq \mathrm{pr}_V(y)$ be a line and let $K \in \mathfrak{M}$ such that $\langle y, l\rangle \leq K$. Since $l \leq V$ and $M \leq V$, we obtain $K \notin \mathfrak{M}^-$ by Lemmas 3.1.1(iii) and 3.4.3(iii). Set $d := \mathrm{codm}(M \cup \{x\})$. If $\mathrm{copr}_M(x) = M$, then Lemma 6.4.1 implies $\mathrm{rk}(M) = d$ and $\mathrm{diam}(\mathscr{S}^-) = d$. Thus, $M \leq \mathrm{copr}_{\mathscr{S}^-}(x)$ and we obtain $\mathrm{codm}(K \cup \{x\}) = d-1$ by Lemma 6.1.1. Since $V \cap \mathscr{S}^- = \mathrm{copr}_{\mathscr{S}^-}(x)$, we obtain $\mathrm{crk}_K(K \cap V) = d$ by Lemma 6.4.1. If $\mathrm{copr}_M(x) < M$, then $\mathrm{crk}_K(K \cap V) = d$ follows from Proposition 6.4.11(i).
By Lemma 6.4.1 there is a line $g \leq M$ with $\mathrm{cod}(x,g) = d$. Let $N \in \mathfrak{M} \setminus \mathfrak{M}^-$ with $g \leq N$. Then $\mathrm{codm}(N \cup \{x\}) = d-1$ by Lemma 6.1.1. Set $V^- := V \cap \mathscr{S}^-$. By Lemma 6.1.2 we obtain $K \cap \langle y, V^-\rangle_{\mathrm{g}} = \langle y, K \cap V\rangle$ and therefore $\mathrm{crk}_N(N \cap \langle y, V^-\rangle_{\mathrm{g}}) = d-1$ by Lemma 3.4.3(iii). Since $\mathrm{crk}_N(\mathrm{copr}_N(x)) = d$ by Lemma 6.4.1, there is a line $h \leq N \cap \langle y, V^-\rangle_{\mathrm{g}}$ that intersects $\mathrm{copr}_N(x)$ in a single point. Let $M' \in \mathfrak{M}^-$ with $h \in M'$. Then $M' \leq \langle y, V^-\rangle_{\mathrm{g}}$ by Lemma 3.4.3(iii). Since $\mathrm{codm}(V) = d$ by Proposition 6.4.11(iii) and $\mathrm{crk}_N(\mathrm{copr}_N(x)) = d$, we conclude $N \cap V = \mathrm{copr}_N(x)$ by the coconvexity of V. Thus, $N \cap \langle y, V^-\rangle_{\mathrm{g}} = \langle h, \mathrm{copr}_N(x)\rangle$. By Proposition 6.4.11(ii) this equals $N \cap \langle x, M'\rangle_{\mathrm{G}}$. Applying Lemma 6.1.2 leads to $\langle y, V^-\rangle_{\mathrm{g}} = \langle h, \mathrm{copr}_N(x), M'\rangle_{\mathrm{g}} = \langle x, M'\rangle_{\mathrm{G}} \cap \mathscr{S}^-$. Since y and M are both contained in $\langle x, M'\rangle_{\mathrm{G}}$, we obtain $V \cup \{y\} \subseteq \langle x, M'\rangle_{\mathrm{G}}$. Hence, $M' \cup \{x\} \subseteq \langle y, V\rangle_{\mathrm{G}}$ yields $\langle x, M'\rangle_{\mathrm{G}} = \langle y, V\rangle_{\mathrm{G}}$. ☐

As already mentioned, the coconvex spans of a point of $\mathscr{S}^+$ and an element of $\mathfrak{M}^-$ induce a lattice structure. If $\mathscr{S}^+$ has infinite diameter, this follows already from the last two claims. The same is true for the case where every point of $\mathscr{S}^+$ has a cogate in $\mathscr{S}^-$. For the remaining cases we do not prove this fact since it is an immediate consequence of the following.

Our goal is to prove that $\mathscr{S}$ is a partial twin Grassmannian. Therefore we show that there is projective space arising from $\mathscr{S}$. Moreover, $\mathscr{S}$ is isomorphic to a partial twin Grassmannian of this projective space. For this we define the following two sets:

$$\mathscr{P}_{\mathrm{m}} := \left\{ \langle x, M \rangle_{\mathrm{G}} \mid (x, M) \in \mathscr{S}^{+} \times \mathfrak{M}^{-} \wedge \operatorname{codm}(M \cup \{x\}) = 1 \right\}$$
$$\mathscr{L}_{\mathrm{m}} := \left\{ \{ P \in \mathscr{P}_{\mathrm{m}} \mid U \cap V \leq P \} \,\middle|\, U \in \mathscr{P}_{\mathrm{m}} \wedge V \in \mathscr{P}_{\mathrm{m}} \setminus \{U\} \right\}$$

Proposition 6.4.15. *The pair* $(\mathscr{P}_{\mathrm{m}}, \mathscr{L}_{\mathrm{m}})$ *is a projective space.*

Proof. From the definition we know that every element of $\mathscr{L}_{\mathrm{m}}$ has at least two elements. Thus, $(\mathscr{P}_{\mathrm{m}}, \mathscr{L}_{\mathrm{m}})$ is a point-line space. Moreover, by definition $(\mathscr{P}_{\mathrm{m}}, \mathscr{L}_{\mathrm{m}})$ is singular. By Lemma 6.4.5 and symmetric reasons we may assume $\operatorname{rk}(M_{-}) \leq \operatorname{rk}(M_{+})$ for $\operatorname{diam}(\mathscr{S}^{+}) < \infty$, where $M_\sigma \in \mathfrak{M}^\sigma$ for $\sigma \in \{+,-\}$. By Lemma 6.4.5 and Theorem 3.4.4 this implies $\operatorname{rk}(M_{-}) = \operatorname{diam}(\mathscr{S}^{+})$ in this case.
Let U and V be two distinct elements of $\mathscr{P}_{\mathrm{m}}$. Further let X and Y be two distinct elements of $\mathscr{P}_{\mathrm{m}}$ such that $U \cap V \leq X \cap Y$. First assume $\operatorname{diam}(\mathscr{S}^{+}) = 2$. Then every element of $\mathfrak{M}^{-}$ has rank 2 and hence, Proposition 6.4.13 implies that there is a point $x \in \mathscr{S}^{+}$ such that $U \cap V = \{x\} \cup \operatorname{copr}_{\mathscr{S}^{-}}(x)$. Since $x \in X \cap Y$, Proposition 6.4.13 implies $X \cap Y = \{x\} \cup \operatorname{copr}_{\mathscr{S}^{-}}(x)$ and therefore, $U \cap V = X \cap Y$. Now assume $\operatorname{diam}(\mathscr{S}^{+}) > 2$. Then by Proposition 6.4.13 there is a point $x \in \mathscr{S}^{+}$ and a subspace $M \in \mathfrak{M}^{-}$ with $\operatorname{codm}(M \cup \{x\}) = 2$ such that $U \cap V = \langle x, M \rangle_{\mathrm{G}}$. Analogously, there is a point y and a subspace $N \in \mathfrak{M}^{-}$ with $\operatorname{codm}(N \cup \{y\}) = 2$ such that $X \cap Y = \langle y, N \rangle_{\mathrm{G}}$. Since $\langle x, M \rangle_{\mathrm{G}} = U \cap V \leq \langle y, N \rangle_{\mathrm{G}}$, we conclude $\langle x, M \rangle_{\mathrm{G}} = \langle y, N \rangle_{\mathrm{G}}$ by Proposition 6.4.11(iv). Thus, $(\mathscr{P}_{\mathrm{m}}, \mathscr{L}_{\mathrm{m}})$ is linear.
We show that $(\mathscr{P}_{\mathrm{m}}, \mathscr{L}_{\mathrm{m}})$ satisfies (VY). Let G and H be two distinct elements of $\mathscr{L}_{\mathrm{m}}$ and let $P \in \mathscr{P}_{\mathrm{m}} \setminus (G \cup H)$ such that for $i \in \{0,1\}$ there exist $L_i \in \mathscr{L}_{\mathrm{m}}$, $A_i \in G$ and $B_i \in H$ with $\{P, A_i, B_i\} \leq L_i$ and $L_0 \neq L_1$. Since $L_0 \neq L_1$ and $P \nleq G$, we obtain $A_0 \neq A_1$ since $(\mathscr{P}_{\mathrm{m}}, \mathscr{L}_{\mathrm{m}})$ is linear. Analogously, $B_0 \neq B_1$. Since we want to show $G \cap H \neq \varnothing$, we may assume $B_i \notin G$ and $A_i \notin H$ for $i \in \{0,1\}$.
First assume $\operatorname{diam}(\mathscr{S}^{-}) \geq 3$. Then by Proposition 6.4.13 there is a point $x \in \mathscr{S}^{+}$ and a subspace $M \in \mathfrak{M}^{-}$ with $\operatorname{codm}(M \cup \{x\}) = 2$ such that $A_0 \cap A_1 = \langle x, M \rangle_{\mathrm{G}}$. Set $S := \langle x, M \rangle_{\mathrm{G}} \cap P$. Since $P \neq A_0$ and $S \leq A_0 \cap P$, we obtain that every element of L_0 contains S. Thus, $S \leq B_0$ and analogously, $S \leq B_1$. Since $P \notin G$, we obtain $\langle x, M \rangle_{\mathrm{G}} \nleq P$. Assume S contains a subspace $N \in \mathfrak{M}^{-}$ and let $K \in \mathfrak{M} \setminus \mathfrak{M}^{-}$ such that $K \cap N$ is a line. For $\operatorname{diam}(\mathscr{S}^{-}) \geq 4$ this is the case by Proposition 6.4.13 and moreover, $S = \langle y, N \rangle_{\mathrm{G}}$ for a point $y \in \mathscr{S}^{+}$ with $\operatorname{codm}(N \cap \{y\}) = 3$ by Proposition 6.4.11(iv). Hence by Proposition 6.4.11(i), $\operatorname{crk}_K(K \cap S) = 3$ in this case. For $\operatorname{diam}(\mathscr{S}^{-}) = 3$, we have $\operatorname{rk}(N) = 3$. Moreover, $S = \{y\} \cup \operatorname{copr}_{\mathscr{S}^{-}}(y)$ for a point $y \in \mathscr{S}^{+}$ by Proposition 6.4.13. Since $N \leq \operatorname{copr}_{\mathscr{S}^{-}}(y)$, we know by (A2) that $\mathscr{S}^{-}$ is no metaplecton and hence, $\operatorname{rk}(K) > 3$ by Theorem 3.4.4. By Lemma 6.1.1 we conclude $\operatorname{codm}(K \cup \{y\}) = 2$ and hence, $\operatorname{crk}_K(K \cap S) = 3$ by Lemma 6.4.1. Since $S \leq \langle x, M \rangle_{\mathrm{G}}$, we obtain $\operatorname{crk}_K(K \cap \langle x, M \rangle_{\mathrm{G}}) = 2$ by Proposition 6.4.11(i). Let $p \in K \cap \langle x, M \rangle_{\mathrm{G}} \setminus S$. Then $\langle p, S \rangle_{\mathrm{G}} \leq \langle x, M \rangle_{\mathrm{G}}$ and hence $\langle p, S \rangle_{\mathrm{G}} = \langle x, M \rangle_{\mathrm{G}}$ by Lemma 6.4.14. Analogously, there is a point $q \in K \cap B_0 \cap B_1$

such that $\langle q,S\rangle_G = B_0 \cap B_1$. Since $G \neq H$ this implies $q \notin \langle p,S\rangle_G$ and hence, $\langle p,q,S\rangle_G \in \mathscr{P}_m$ by Lemma 6.4.14. Moreover, $\langle p,q,S\rangle_G \in G \cap H$.
For $\operatorname{diam}(\mathscr{S}^-) \geq 3$ it remains the case where $\operatorname{diam}(\mathscr{S}^-) = 3$ and S contains no element of $\mathfrak{M}^-$. By Proposition 6.4.13 there is a point $y \in \mathscr{S}^+$ such that $S = \{y\} \cup \operatorname{copr}_{\mathscr{S}^-}(y)$. If $S \cap \mathscr{S}^-$ contains a line, then the unique element of $\mathfrak{M}^-$ that contains this line is contained in all A_0, A_1 and P and hence in S, a contradiction. Thus, $\operatorname{copr}_{\mathscr{S}^-}(y)$ contains a single point z and $S = \{y,z\}$. Since for every subspace of $\mathfrak{M} \setminus \mathfrak{M}^-$ that contains z, the coprojection of y is $\{z\}$, we conclude by Lemma 6.4.1 that all maximal subspaces of $\mathscr{S}^-$ have rank 3. Hence by Theorem 3.4.4, $\mathscr{S}^-$ is a metaplecton of diameter 3. This implies $\langle x,M\rangle_G \cap \mathscr{S}^- = M$ because of Proposition 6.4.11(i). Thus, Proposition 6.4.11(iv) implies $\langle x,M\rangle_G = \langle y,M\rangle_G$. Analogously, $B_0 \cap B_1 = \langle y,N\rangle_G$, where $N \in \mathfrak{M}^-$. We know $z \in M \cap N$ since z is the cogate of y in $\mathscr{S}^-$ and both M and N are coconvex. Since $M \nleq S$, we obtain $M \nleq P$. With $P \geq A_0 \cap B_0$ this implies $M \nleq B_0$ and therefore $M \cap N = \{z\}$ by Lemma 3.4.2. Let $K \in \mathfrak{M} \setminus \mathfrak{M}^-$ with $z \in K$. Then $K \cap M$ and $K \cap N$ are distinct lines by Lemma 3.4.2. Let $p \in K \cap M \setminus \{z\}$ and $q \in K \cap N \setminus \{z\}$ and let $M' \in \mathfrak{M}^-$ with $pq \leq M'$. Then $z \notin M'$ by Proposition 2.2.5 since $\langle z, pq\rangle$ has rank 2. Since $M' \in \mathfrak{M}$, there is point $p' \in M'$ with $z' \not\perp z$. Since z is the cogate for y in $\mathscr{S}^-$, this implies $\operatorname{codm}(M' \cup \{y\}) = 1$ and hence, $\langle y,M'\rangle_G \in \mathscr{P}_m$. We obtain $z \in \langle y,M'\rangle_G$ by the coconvexity of $\langle y,M'\rangle_G$. Thus by Proposition 6.4.11(i), $zp \leq \langle y,M'\rangle_G$ yields $M \in \langle y,M'\rangle_G$. Analogously, $N \in \langle y,M'\rangle_G$ and therefore, $\langle q,M'\rangle_G \in G \cap H$.
Now assume $\operatorname{diam}(\mathscr{S}^-) = 2$. Then every element of $\mathfrak{M}^-$ has rank 2. Thus, Lemma 6.4.5 implies $\operatorname{rk}(K) = 2$ for every $K \in \mathfrak{M} \setminus \mathfrak{M}^+$ with $K \leq \mathscr{S}^+$. Hence, $Q \cap \mathscr{S}^+ \in \mathfrak{M}^+$ for every $Q \in \mathscr{P}_m$ by Proposition 6.4.11(i). Furthermore by Proposition 6.4.13, $Q \cap Q' \cap \mathscr{S}^+$ contains a single point for every $Q' \in \mathscr{P}_m \setminus \{Q\}$. Now let $x \in Q \cap \mathscr{S}^+$. By Proposition 6.4.11(iv) there is a subspace $M \in \mathfrak{M}^-$ such that $\langle x,M\rangle_G = Q$ and $\operatorname{codm}(M \cup \{x\}) = 1$. By Lemma 6.4.1 there is a line $l \in M$ with $\operatorname{cod}(x,l) = 2$ such that x has a cogate in l. Let $K \in \mathfrak{M} \setminus \mathfrak{M}^-$ with $l \leq K$. By Proposition 6.4.11(i) Q intersects K in a hyperplane. Hence there is a line $l' \leq K$ such that $l' \cap Q$ is a singleton that is contained in $\operatorname{copr}_K(x)$. Since $\operatorname{copr}_K(x) \leq Q$ by Proposition 6.4.11(ii), we obtain $\operatorname{codm}(l' \cup \{x\}) = 1$. Let $M' \in \mathfrak{M}^-$ with $l' \leq M'$. Then $Q' := \langle x,M'\rangle_G \in \mathscr{P}_m$. Since $l' \leq Q'$, we know $Q \neq Q'$ and therefore, $Q \cap Q' \cap \mathscr{S}^+ = \{x\} \cup \operatorname{copr}_{\mathscr{S}^-}(x)$ by Proposition 6.4.13. This implies $\operatorname{copr}_{\mathscr{S}^-}(x) \leq Q$ for every point $q \in Q \cap \mathscr{S}^+$. By the coconvexity of Q together with Proposition 2.1.16(ii) there is for every point $p \in Q \cap \mathscr{S}^-$ a point $q \in Q \cap \mathscr{S}^+$ with $\operatorname{cod}(p,q) = 2$. Thus, $Q \cap \mathscr{S}^- = \bigcup_{q \in Q \cap \mathscr{S}^+} \operatorname{copr}_{\mathscr{S}^-}(q)$.
For $i \in \{0,1\}$ let x_i the unique point in $A_i \cap P \cap \mathscr{S}^+$. Since x_0 and x_1 are both contained in P and $P \cap \mathscr{S}^+$ is an element of $\mathfrak{M}^+$, we obtain $x_0 \perp x_1$. Furthermore, $L_0 \neq L_1$ yields $x_0 \neq x_1$. Let y be the unique point of $A_0 \cap A_1 \cap \mathscr{S}^+$. Then $y \perp x_i$ for $i \in \{0,1\}$ since $A_i \cap \mathscr{S}^+$ contains both x_i and y. Since $P \notin G$, we ob-

tain $y \notin P$ and therefore $y \notin x_0x_1$. Thus, there is a unique subspace $K \in \mathfrak{M}$ with $\{x_0, x_1, y\} \subseteq K$. Since $P \cap \mathscr{S}^-$ is the unique element of $\mathfrak{M}^+$ that contains x_0x_1 and $y \notin P$, we obtain $K \notin \mathfrak{M}^+$. Analogously, the unique point z of $B_0 \cap B_1 \cap \mathscr{S}^+$ lies in a subspace of $\mathfrak{M} \setminus \mathfrak{M}^+$ that contains x_0x_1. Hence, $z \in K$ by the uniqueness of such a space. This implies $y \perp z$. Moreover, $y \neq z$ since $G \neq H$. Thus, there is a subspace $N \in \mathfrak{M}^+$ with $yz \leq N$. Let $p \in \mathscr{S}^-$ with $\operatorname{codm}(N \cup \{p\}) = 1$. Then $\langle p, N\rangle_G \in \mathscr{P}_m$. Furthermore, $\langle p, N\rangle_G$ contains $\{y\} \cup \operatorname{copr}_{\mathscr{S}^-}(y) = A_0 \cup A_1$ and $\{z\} \cup \operatorname{copr}_{\mathscr{S}^-}(z) = B_0 \cup B_1$. Thus, $\langle t, N\rangle_G \in G \cap H$ and the claim follows. □

Our next aim is to study correspondences between the subspaces of $\mathscr{S}$ and the ones of $(\mathscr{P}_m, \mathscr{L}_m)$.

For a point $p \in \mathscr{S}$, we define $\Gamma(p) := \{P \in \mathscr{P}_m \mid p \in P\}$. Furthermore, for a set of points $M \subseteq \mathscr{S}$, we define $\Gamma(M) := \{P \in \mathscr{P}_m \mid M \subseteq P\} = \bigcap_{p \in M} \Gamma(p)$. For two points p and q, we write $\Gamma(p,q)$ rather than $\Gamma(\{p,q\})$.

Lemma 6.4.16. *For every set of points $M \subseteq \mathscr{S}$, the set $\Gamma(M)$ is a subspace of $(\mathscr{P}_m, \mathscr{L}_m)$.*

Proof. Let P and Q be two distinct elements of $\Gamma(M)$. Then for every $M \subseteq P \cap Q$ and hence, $\{R \in \mathscr{P}_m \mid R \geq P \cap Q\} \subseteq \Gamma(M)$. The claim follows by the definition of $\mathscr{L}_m$. □

Proposition 6.4.17. *Let p and q be two points of $\mathscr{S}^-$. Then* $\operatorname{crk}_{\Gamma(p)}(\Gamma(p,q)) = \operatorname{dist}(p,q)$.

Proof. Set $d := \operatorname{dist}(p,q)$. We proceed by induction over d. For $p = q$, there is nothing to prove. Now let $d > 0$ and assume that there is a point $r \perp q$ with $\operatorname{dist}(p,r) = d-1$ such that $\operatorname{crk}_{\Gamma(p)}(\Gamma(p,r)) = d-1$. Let $G \in \mathscr{L}_m$ with $G \leq \Gamma(r)$ and set $S := \bigcap_{P \in G} P$.
First assume $\operatorname{diam}(\mathscr{S}^-) \geq 3$ or $\operatorname{rk}(K) \geq 3$ for every $K \in \mathfrak{M}^+$. If $\operatorname{diam}(\mathscr{S}^-) \geq 3$, then Proposition 6.4.13 implies that S is the coconvex span of a point $x \in \mathscr{S}^+$ and an element of $N \in \mathfrak{M}^-$ with $\operatorname{codm}(N \cup \{x\}) = 2$. Hence, Proposition 6.4.11(i) implies that there is a line $l \leq S$ through r. If $\operatorname{diam}(\mathscr{S}^-) = 2$ and $\operatorname{rk}(K) \geq 3$ for every $K \in \mathfrak{M}^+$, then Proposition 6.4.13 implies $S = \{x\} \cup \operatorname{copr}_{\mathscr{S}^-}(x)$ for a point $x \in \mathscr{S}^+$. By Lemma 6.4.5 we know $\operatorname{rk}(N) \geq 3$ for every $N \in \mathfrak{M} \setminus \mathfrak{M}^-$ with $r \in N$. Hence, Lemma 6.4.1 implies that there is a line $l \leq S$ through r. Now let $M \in \mathfrak{M}^-$ with $l \leq M$. Since $l \leq P$ for every $P \in G$, we obtain $M \leq S$ by Proposition 6.4.11(i). Assume $G \nleq \Gamma(q)$. Then $q \notin S$ and therefore $\operatorname{dist}(q,M) = 1$. Thus, $\operatorname{pr}_M(q)$ is a line by Lemma 3.4.2. By Lemma 6.4.14 we conclude $\langle q, S\rangle_G \in \mathscr{P}_m$ and therefore $\langle q, S\rangle_G \in G \cap \Gamma(q)$. We conclude that $\Gamma(q)$ contains a hyperplane of $\Gamma(r)$.
Now assume $\operatorname{rk}(K) = 2$ for every $K \in \mathfrak{M}^+$. Then $S \cup \operatorname{copr}_{\mathscr{S}^+}(r)$ by Proposition

6.4.13. By the definition of $\mathscr{L}_{\mathrm{m}}$ this implies that G is uniquely defined and consequently, $G = \Gamma(r)$. By Lemma 3.1.1(i) there is a symplecton $Y \leq \mathscr{S}^-$ that contains rq. Let $s \in \mathscr{S}^+$ be the cogate of r in a symplecton that is opposite Y. Then $\mathrm{cod}(r,s) = 2$ and $\mathrm{cod}(q,s) = 1$. Let $M \in \mathfrak{M}^-$ with $rq \leq M$. Then $\langle s,M\rangle_{\mathrm{G}} \in \Gamma(r) \cap \Gamma(q)$. Since $G = \Gamma(r)$, we conclude that $\Gamma(q)$ contains a hyperplane of $\Gamma(r)$.
By Lemma 2.1.13 there is a point $s \in \mathscr{S}^+$ with $s \leftrightarrow q$ and $\mathrm{cod}(s,p) = d$. This implies $\mathrm{cod}(s,r) = 1$. We conclude by Proposition 2.1.16(ii) that there is a line l through s with $\mathrm{cod}(r,s) = 2$. Let $M \in \mathscr{S}^+$ with $l \leq M$. Then $\langle r,M\rangle_{\mathrm{G}} \in \mathscr{P}_{\mathrm{m}}$. Since $s \in \langle r,M\rangle_{\mathrm{G}}$ and $s \leftrightarrow p$q, we conclude $\langle r,M\rangle_{\mathrm{G}} \in \Gamma(r) \setminus \Gamma(q)$ by Proposition 6.4.11(iii). Thus, $\Gamma(q)$ intersects $\Gamma(r)$ in a proper hyperplane. Since $\mathrm{dist}(p,r) = \mathrm{cod}(s,p) - \mathrm{cod}(s,r)$, we obtain $p \in \langle r,M\rangle_{\mathrm{G}}$ by the coconvexity of $\langle r,M\rangle_{\mathrm{G}}$ and therefore $\Gamma(p,r) \nleq \Gamma(q)$. This leads to $\mathrm{crk}_{\Gamma(p)}(\Gamma(p,q) \cap \Gamma(r)) = d$. Since $r \in \langle p,q\rangle_{\mathrm{g}}$ and $\langle p,q\rangle_{\mathrm{g}} \leq P$ for every $P \in \Gamma(p,q)$, we conclude $\Gamma(p,q) \leq \Gamma(r)$ and consequently, $\mathrm{crk}_{\Gamma(p)}(\Gamma(p,q)) = d$. $\square$

Proposition 6.4.18. *Let $p \in \mathscr{S}^-$ and $q \in \mathscr{S}^+$. Then the following holds:*

(i) *Let $p \leftrightarrow q$. Then $\Gamma(p)$ and $\Gamma(q)$ are complementary subspaces of the projective space $(\mathscr{P}_{\mathrm{m}}, \mathscr{L}_{\mathrm{m}})$.*

(ii) $\mathrm{rk}(\Gamma(p,q)) = \mathrm{cod}(p,q) - 1$.

Proof. (i) From Proposition 6.4.11(iii) we deduce $\Gamma(p) \cap \Gamma(q) = \varnothing$. Let $P \in \mathscr{P}_{\mathrm{m}}$ with $P \notin \Gamma(p) \cup \Gamma(q)$. Then Lemma 6.4.12(i) implies $K_x := \langle x, \mathrm{pr}_P(x)\rangle \in \mathfrak{M} \setminus (\mathfrak{M}^+ \cup \mathfrak{M}^-)$, where $x \in \{p,q\}$. By the definition of $\mathfrak{M}^+$ and $\mathfrak{M}^-$ we know $\mathrm{cod}(K_p,K_q) = 2$ since $p \leftrightarrow q$. For $\{x,y\} = \{p,q\}$, Lemma 6.4.8 implies that the set $S_x := \{r \in K_x \mid \mathrm{cod}(r,K_y) = 2\}$ is a subspace of K_x with corank 2. Furthermore, by Lemma 6.4.1 we know that $\mathrm{copr}_{K_x}(y)$ is a hyperplane of K_x. Thus, there is a line $l_x \leq \mathrm{copr}_{K_x}(y)$. Let $M_x \in \mathfrak{M}^+ \cup \mathfrak{M}^-$ with $l_x \leq M_x$. By Lemma 6.1.1 we obtain $\mathrm{codm}(M_x \cup \{y\}) = 1$ and therefore $P_y := \langle y, M_x\rangle_{\mathrm{G}} \in \Gamma(y)$. Set $G := \{Q \in \mathscr{P}_{\mathrm{m}} \mid P_p \cap P_q \leq Q\}$.
By Proposition 6.4.11(ii) and since $l_p \leq \mathrm{copr}_{K_p}(q)$, we obtain $\mathrm{copr}_{K_p}(q) = K_p \cap P_q$ and therefore $S_p \leq P_q$. Now let $p' \in S_p$. Then Lemma 6.4.1 implies $\mathrm{copr}_{K_q}(p') = S_q$. Thus $S_q \leq \langle q, p'\rangle_{\mathrm{G}} \leq P_q$ and analogously, $S_p \cup S_q \leq P_p$. Since $\langle p, \mathrm{copr}_{K_p}(q)\rangle = K_p$, we conclude $\mathrm{copr}_{K_p}(q) \nleq P_p$ by Proposition 6.4.11(i). Since S_p is a hyperplane of $\mathrm{copr}_{K_p}(q)$, this implies $K_p \cap P_p \cap P_q = S_p$. Since $\mathrm{crk}_{K_q}(\mathrm{pr}_P(q)) = 1$, there is a point $q' \in \mathrm{pr}_P(q) \setminus S_q$. Since $q' \notin S_q$, we know $\mathrm{cod}(q',K_p) = 1$ and hence, $K_p \cap P \leq \mathrm{copr}_{K_p}(q')$ by Proposition 6.4.11(iii). By Lemma 6.4.1 and Proposition 6.4.11(i) this implies $K_p \cap P = \mathrm{copr}_{K_p}(q')$. Thus, $S_p \leq \mathrm{copr}_{K_p}(q') \leq P$ and analogously, $S_q \leq P$.
Assume S_p contains a line l. Let $M \in \mathfrak{M}^-$ with $l \leq M$. Then $M \leq P_p \cap P_q \cap P$

by Proposition 6.4.11(i). With Lemma 6.4.10 we conclude $\langle S_p, M\rangle_{\mathrm{g}} = P_p \cap P_q \cap \mathscr{S}^- \leq P$ since $K_p \cap P_p \cap P_q = S_p$. Now assume that S_p is a singleton. Then $\mathrm{rk}(K_p) = 2$ and hence, $\mathrm{rk}(M_q) = 2$ by Lemma 6.4.5. This implies $P_p \cap P_q \cap \mathscr{S}^- = S_p$ by Proposition 6.4.13 and therefore again $P_p \cap P_q \cap \mathscr{S}^- \leq P$. We conclude analogously $P_p \cap P_q \cap \mathscr{S}^+ \leq P$ and therefore $P \in G$.
(ii) Set $d := \mathrm{cod}(p,q)$. We proceed by induction over d. For $d = 0$, the claim follows by (i). Now we assume $d > 0$ and that there is a point $r \perp p$ with $\mathrm{cod}(r,q) = d-1 = \mathrm{rk}(\Gamma(r,q)) - 1$. Since $p \in \langle r,q\rangle_{\mathrm{G}}$ and hence, $p \in P$ for every $P \in \Gamma(r,q)$, we obtain $\Gamma(r,q) \leq \Gamma(p,q)$. Let $q' \leftrightarrow q$ with $\mathrm{dist}(r,q') = d-1$. Since by Proposition 2.1.23 $\langle p,q'\rangle_{\mathrm{g}}$ is an SPO space, Lemma 2.1.21(ii) implies that there is a point $r' \in \langle p,q'\rangle_{\mathrm{g}}$ with $r' \perp q'$ such that $\mathrm{dist}(r,r') = d$ and $\mathrm{dist}(p,r') = d-1$. We obtain $\mathrm{cod}(q,r') = 1$. Let l be a line through r' with $\mathrm{cod}(q,l) = 2$ and let $M \in \mathfrak{M}^-$ with $l \leq M$. Then $\langle q,M\rangle_{\mathrm{G}} \in \Gamma(q)$. With $r' \in M$ and $p \in \langle r',q\rangle_{\mathrm{G}}$ we conclude $\langle q,M\rangle_{\mathrm{G}} \in \Gamma(p,q)$. Suppose $r \in \langle q,M\rangle_{\mathrm{G}}$. Then $\langle r,r'\rangle_{\mathrm{g}} = \langle p,q'\rangle_{\mathrm{g}} \leq \langle q,M\rangle_{\mathrm{G}}$ and hence, $q' \in \langle q,M\rangle_{\mathrm{G}}$, a contradiction to Proposition 6.4.11(iii). Thus, $\Gamma(p,q) > \Gamma(r,q)$. By Proposition 6.4.17 we know that $\Gamma(r)$ contains a hyperplane of $\Gamma(p)$ and hence, $\mathrm{crk}_{\Gamma(p,q)}(\Gamma(r,q)) = 1$. The claim follows. □

Lemma 6.4.19. *Let x and y be two distinct collinear points of $\mathscr{S}^-$. Further let $P \in \mathscr{P}_{\mathrm{m}}$ with $P \notin \Gamma(x) \cup \Gamma(y)$. Then there is a unique point $z \in \mathscr{S}^-$ such that $\Gamma(x,y) \cup \{P\} \subseteq \Gamma(z)$. Moreover, $z \in xy$ if and only if $\Gamma(z) \leq \langle \Gamma(x), \Gamma(y)\rangle$.*

Proof. Let $M \in \mathfrak{M}^-$ with $xy \leq M$. By Lemma 6.4.12(i) there is a subspace $K_p \in \mathfrak{M} \setminus \mathfrak{M}^-$ such that $p \in K_p$ and $K_p \cap P = \mathrm{pr}_P(p)$ for $p \in \{x,y\}$. If $K_x = K_y$, then xy intersects P in a single point z since $K_x \cap P$ is a hyperplane of K_x by Lemma 6.4.12(i). We obtain $z \in M$ in this case. If $K_x \neq K_y$, then $y \notin K_x$ and hence, $\mathrm{pr}_{K_x}(y)$ is a line by Lemma 3.4.2. Since this line contains x, it intersects P in a single point z. By Proposition 2.2.5 there is a unique subspace $M' \in \mathfrak{M}$ that contains $\langle y, \mathrm{pr}_{K_x}(y)\rangle$. Since K_x is the only element of $\mathfrak{M} \setminus \mathfrak{M}^-$ that contains $\mathrm{pr}_{K_x}(y)$ and $y \notin K_x$, we conclude $M' \in \mathfrak{M}^-$. Now $xy \leq M'$ implies $M = M'$ and thus, $z \in M$.
Let $Q \in \Gamma(x,y)$. Then $xy \leq Q$ and hence, $M \leq Q$ by Proposition 6.4.11(i). With $z \in M$ and $z \in P$ this implies $\Gamma(x,y) \cup \{P\} \subseteq \Gamma(z)$. The uniqueness of z follows by Proposition 6.4.17 since $\Gamma(x,y)$ is a hyperplane of $\Gamma(z)$ that does not contain P.
Let $K \in \mathfrak{M} \setminus \mathfrak{M}^-$ such that $xy \leq K$. Assume $P \in \langle \Gamma(x), \Gamma(y)\rangle$. Then there are elements $P_x \in \Gamma(x)$ and $P_y \in \Gamma(y)$ such that $P \geq P_x \cap P_y =: S$. By Lemma 6.4.12(i) we obtain $\langle x, \mathrm{pr}_{P_y}(x)\rangle \in \mathfrak{M} \setminus \mathfrak{M}^-$ and hence, $\langle x, \mathrm{pr}_{P_y}(x)\rangle = K$ since both contain xy. Analogously, $\mathrm{pr}_{P_x}(y)$ is a hyperplane of K. Since $x \in \mathrm{pr}_{P_x}(y) \setminus \mathrm{pr}_{P_y}(x)$ we conclude $\mathrm{crk}_K(K \cap S) = 2$. Since $\mathrm{rk}(K) \geq 2$ this implies $K \cap S \neq \varnothing$ and hence, $K \cap P \neq \varnothing$. Therefore P intersects K in a hyperplane by Lemma 6.4.12(ii) and Proposition 6.4.11(i). This implies $K = K_x = K_y$ and hence $z \in xy$ as above.
Conversely, let $z \in xy$. Since $x \notin P$ and $y \notin P$ we know $x \neq z \neq y$. By Proposition

6.4.17 there is an element $P_x \in \Gamma(x) \setminus \Gamma(z)$. With $x \in P_x$ and $z \in P$ we conclude by Lemma 6.4.12(ii) and Proposition 6.4.11(i) that both P_x and P contain a hyperplane of K. Thus, $\mathrm{crk}_K(K \cap P \cap P_x) = 2$ since $z \in P \setminus P_x$. Set $S := P \cap P_x$. If $\mathrm{diam}(\mathscr{S}^-) \geq 3$, then $\langle y, S\rangle_\mathrm{G} \in \Gamma(y)$ by Proposition 6.4.13 and Lemma 6.4.14. If $\mathrm{diam}(\mathscr{S}^-) = 2$ and $\mathrm{rk}(K) > 2$, then $S = \{w\} \cup \mathrm{copr}_{\mathscr{S}^-}(w)$ for a point $w \in \mathscr{S}^+$ by Proposition 6.4.13. Furthermore $K \cap S$ contains a line and hence, $\langle y, S\rangle_\mathrm{G} \in \Gamma(y)$ by Lemma 6.4.14. It remains the case $\mathrm{diam}(\mathscr{S}^-) = \mathrm{rk}(K) = 2$. By Lemma 6.4.5 and Proposition 6.4.13 we obtain $S = \{w\} \cup \mathrm{copr}_{\mathscr{S}^+}(w)$ for a point $w \in \mathscr{S}^-$. Since $K \cap S \neq \varnothing$, we know $w \in K$ and hence, $y \perp w$. Moreover, $w \neq y$ since $y \notin P$. By Lemma 3.1.1(i) there is a symplecton Y containing K. Since there exists a symplecton that is opposite Y, there is a point $p \in \mathrm{copr}_{\mathscr{S}^+}(w) \setminus \mathrm{copr}_{\mathscr{S}^+}(y)$. Let $M \in \mathfrak{M}^-$ with $wy \leq M$. Then $\mathrm{codm}(M \cup \{p\}) = 1$ and therefore $P_y := \langle p, M\rangle_\mathrm{G} \in \Gamma(y)$. Since $z \notin P_x$, we know $y \notin P_x$ and hence, $P_x \neq P_y$. Hence, $w \in P_x \cap P_y$ implies $P_x \cap P_y = \{w\} \cup \mathrm{copr}_{\mathscr{S}^+}(w)$ by Proposition 6.4.13. Thus in the point-line $(\mathscr{P}_\mathrm{m}, \mathscr{L}_\mathrm{m})$, the point P lies on the line through P_x and P_y and therefore $P \in \langle \Gamma(x), \Gamma(y)\rangle$. Thus, $\Gamma(z) = \langle P, \Gamma(x,y)\rangle \leq \langle \Gamma(x), \Gamma(y)\rangle$, since by Proposition 6.4.17 $\Gamma(x,y)$ is a hyperplane of $\Gamma(z)$. □

Lemma 6.4.20. *Let x and y be two distinct points of $\mathscr{S}^-$. Further let Θ be a subspace of $(\mathscr{P}_\mathrm{m}, \mathscr{L}_\mathrm{m})$ with $\Gamma(x,y) \leq \Theta$ and $\mathrm{crk}_\Theta(\Gamma(x,y)) = \mathrm{crk}_{\Gamma(x)}(\Gamma(x,y))$. Then there is a unique point $z \in \mathscr{S}^-$ such that $\Gamma(z) = \Theta$. Moreover, $z \in \langle x, y\rangle_\mathrm{g}$ if and only if $\Theta \leq \langle \Gamma(x), \Gamma(y)\rangle$.*

Proof. By Proposition 6.4.17 there exists at most one such point z. Set $d := \mathrm{dist}(x,y)$. We may assume $d > 0$ since otherwise there is nothing to prove. By Proposition 6.4.17 we know $\mathrm{crk}_{\Gamma(x)}(\Gamma(x,y)) = d$ and hence, $\mathrm{crk}_\Theta(\Gamma(x,y)) = d$. Thus there is a set $\{P_i \mid 0 \leq i < d\} \subseteq \mathscr{P}_\mathrm{m}$ and a natural number $k \leq d$ such that $\Theta = \langle P_i, \Gamma(x,y) \mid 0 \leq i < d\rangle$ and $\Theta \cap \langle \Gamma(x), \Gamma(y)\rangle = \langle P_i, \Gamma(x,y) \mid 0 \leq i < k\rangle$.
Assume there are points x_j and y_j in $\mathscr{S}^-$ such that $\Gamma(x_j, y_j) = \langle P_i, \Gamma(x,y) \mid 0 \leq i < j\rangle$ for some $j < d$. We show that there are point x_{j+1} and y_{j+1} such that $\Gamma(x_{j+1}, y_{j+1}) = \langle P_i, \Gamma(x,y) \mid 0 \leq i < j+1\rangle$. By the definition of P_j we know $P_j \notin \Gamma(x_j, y_j)$ and $\mathrm{crk}_{\Gamma(x_j)}(\Gamma(x_j, y_j)) = d - j$. This implies $\mathrm{dist}(x_j, y_j) = d - j$ by Proposition 6.4.17.
First assume $P_j \in \Gamma(x_j)$. Since $d > j$, there is a point $w \in \mathscr{S}^-$ such that $w \perp y_j$ and $\mathrm{dist}(w, x_j) = d - j - 1$. By Lemma 6.4.19 there is a point y_{j+1} such that $\Gamma(y_{j+1}) \geq \Gamma(w, y_j) \cup \{P_j\}$. Since $\Gamma(y_j)$ and $\Gamma(w)$ intersect in common hyperplane we know $y_{j+1} \perp y_j$. Moreover, since $\mathrm{dist}(w, x_j) = \mathrm{dist}(x_j, y_j) - 1$, we conclude by Proposition 6.4.17 that $\Gamma(x_j, y_j)$ is a hyperplane of $\Gamma(x_j, w)$ and therefore $\langle P_i, \Gamma(x,y) \mid 0 \leq i < j+1\rangle = \Gamma(x_j, w) \leq \Gamma(y_{j+1})$. Since $\mathrm{dist}(x_j, y_{j+1}) \geq \mathrm{dist}(x_j, y_j) - 1$, this leads to $\mathrm{dist}(x_j, y_{j+1}) = d - j - 1$ and $\Gamma(x_j, y_{j+1}) = \langle P_i, \Gamma(x,y) \mid 0 \leq i < j+1\rangle$. Hence, the claim follows with $x_{j+1} := x_j$. Moreover, we obtain $x_{j+1} \in \langle x_j, y_j\rangle_\mathrm{g}$ and $y_{j+1} \in \langle x_j, y_j\rangle_\mathrm{g}$ for this case.

Now assume $P_j \in \langle \Gamma(x_j), \Gamma(y_j) \rangle$. We may assume $P_j \notin \Gamma(x_j)$ and analogously, $P_j \notin \Gamma(y_j)$ since this case is already done. Then there are subspaces $P_x \in \Gamma(x_j)$ and $P_y \in \Gamma(y_j)$ such that $P_x \cap P_y \leq P_j$. Assume $d > j+1$. Then by Lemma 6.4.10 we know that $\langle x_j, y_j \rangle_\mathrm{g}$ is the convex span of the singular subspaces of $\langle x_j, y_j \rangle_\mathrm{g}$ that contain x_j. Thus, $\langle x_j, y_j \rangle_\mathrm{g}$ is the convex span of all points $w \perp x_j$ with $\mathrm{dist}(w, y_j) = d - j - 1$. Suppose $P_x \in \Gamma(w)$ for every point $w \perp x_j$ with $\mathrm{dist}(w, y_j) = d - j - 1$ and hence, $w \in P_x$. Then $P_x \geq \langle x_j, y_j \rangle_\mathrm{g}$ and we conclude $y_j \in P_x \cap P_y$, a contradiction to $P_j \notin \Gamma(y_j)$. Thus, there is a point $w \perp x_j$ with $\mathrm{dist}(w, y_j) = d - j - 1$ such that $P_x \notin \Gamma(w)$. For $d = j+1$, this is still true since $y_j \notin P_j$ and hence, $y_j \notin P_x$. By Lemma 6.4.19 there is a point x_{j+1} such that $\Gamma(x_{j+1}) \supseteq \Gamma(w, x_j) \cup \{P_j\}$. Analogously, there is a point w' such that $\Gamma(w') \supseteq \Gamma(w, x_j) \cup \{P_y\}$. Since $\Gamma(w, x_j)$ is a hyperplane of all $\Gamma(x_j)$, $\Gamma(x_{j+1})$ and $\Gamma(w')$ and therefore $\Gamma(x_{j+1}) \leq \langle \Gamma(x_j), \Gamma(w') \rangle$, we conclude by Lemma 6.4.19 that x_j, x_{j+1} and w' are on a common line in $\mathscr{S}$. Since $w \in \langle x_j, y_j \rangle_\mathrm{g}$, we know $\Gamma(w) \geq \Gamma(x_j, y_j)$ and therefore $\Gamma(w, x_j) \geq \Gamma(x_j, y_j)$. Thus, $\Gamma(w', y_j) \geq \langle P_y, \Gamma(x_j, y_j) \rangle > \Gamma(x_j, y_j)$. By Proposition 6.4.17 this implies that w' is the unique point on the line $x_j x_{j+1}$ with $\mathrm{dist}(w', y_j) = d - j - 1$. Suppose $w' = x_{j+1}$. Then $\{P_y, P_j\} \subseteq \Gamma(w')$. Thus, $P_x \in \Gamma(w')$ since P_x, P_y and P_j are distinct points on a common line in $(\mathscr{P}_\mathrm{m}, \mathscr{L}_\mathrm{m})$. Since $P_x \notin \Gamma(w)$, Lemma 6.4.19 implies that x_j is the unique point with $\Gamma(x_j) \supseteq \Gamma(w, x_j) \cup \{P_x\}$ and hence, $x_j = w'$, a contradiction to $\mathrm{dist}(x_j, y_j) = d - j$. Therefore we conclude $\mathrm{dist}(x_{j+1}, y_j) = d - j$. Since $P_j \in \Gamma(x_{j+1})$ we are in the situation above and hence, we find a point $y_{i+1} \in \langle x_{j+1}, y_j \rangle_\mathrm{g}$ such that $\Gamma(x_{j+1}, y_{j+1}) = \langle P_i, \Gamma(x, y) \mid 0 \leq i < j+1 \rangle$. Since $w' \in \langle x_j, y_j \rangle_\mathrm{g}$, we obtain $x_{j+1} \in \langle x_j, y_j \rangle_\mathrm{g}$ and consequently, $y_{j+1} \in \langle x_j, y_j \rangle_\mathrm{g}$. Thus we conclude by induction that for $\Theta \leq \langle \Gamma(x), \Gamma(y) \rangle$, there are points $z = x_d = y_d$ in $\langle x, y \rangle_\mathrm{g}$ such that $\Gamma(z) = \langle P_i, \Gamma(x, y) \mid 0 \leq i < d \rangle = \Theta$.

Finally assume $P_j \notin \langle \Gamma(x_j), \Gamma(y_j) \rangle$. Let $w \in \mathscr{S}^-$ be a point with $w \perp x_j$ and $\mathrm{dist}(w, y_j) = d - j - 1$. Then $\Gamma(x_j, w)$ is a hyperplane of $\Gamma(x_j)$ by Proposition 6.4.17. Since $P_j \notin \Gamma(x_j)$, Lemma 6.4.19 implies that there is a point $x_{j+1} \in \mathscr{S}^-$ such that $\langle P_j, \Gamma(x_j, w) \rangle = \Gamma(x_{j+1})$. Since $w \in \langle x_j, y_j \rangle_\mathrm{g}$, we obtain $\Gamma(x_j, y_j) \leq \Gamma(w)$ and hence, $\Gamma(x_j, y_j) \leq \Gamma(x_{j+1})$. Suppose $\Gamma(x_{j+1}, y_j) > \Gamma(x_j, y_j)$. Then there is an element $P \in \mathscr{P}_\mathrm{m}$ with $P \in \Gamma(x_{j+1}, y_j) \setminus \Gamma(x_j)$ and we obtain $\Gamma(x_{j+1}) = \langle P, \Gamma(x_j, w) \rangle \leq \langle \Gamma(x_j), \Gamma(y_j) \rangle$. Since $P_j \in \Gamma(x_{j+1})$, this is a contradiction to $P_j \notin \langle \Gamma(x_j), \Gamma(y_j) \rangle$. Thus, $\Gamma(x_j, y_j) = \Gamma(x_{j+1}, y_j)$ and consequently, $\mathrm{dist}(x_{j+1}, y_j) = d - j$. Since $P_j \in \Gamma(x_{j+1})$, there is as above a point $y_{i+1} \in \langle x_{j+1}, y_j \rangle_\mathrm{g}$ such that $\Gamma(x_{j+1}, y_{j+1}) = \langle P_j, \Gamma(x_j, y_j) \rangle$.

It remains to show that $z \in \langle x, y \rangle_\mathrm{g}$ implies $\Gamma(z) \leq \langle \Gamma(x), \Gamma(y) \rangle$. Let u and v be distinct points of $\langle x, y \rangle_\mathrm{g}$ such that $\langle \Gamma(u), \Gamma(v) \rangle \leq \langle \Gamma(x), \Gamma(y) \rangle$. It suffices to show $\Gamma(w) \leq \langle \Gamma(u), \Gamma(v) \rangle$ for every point $w \in uv$ if $u \perp v$ and for every point $w \in \langle u, v \rangle_\mathrm{g}$ with $w \perp u$ and $\mathrm{dist}(w, v) = \mathrm{dist}(u, v) - 1$ otherwise. The first follows from Lemma 6.4.19. Hence, let w be a point with $w \perp u$ and $\mathrm{dist}(w, v) = \mathrm{dist}(u, v) - 1$. Then Proposition 6.4.17 implies that $\Gamma(u)$ and $\Gamma(w)$ have a hyperplane in common and

$\mathrm{crk}_{\Gamma(v)}(\Gamma(w,v)) = \mathrm{crk}_{\Gamma(v)}(\Gamma(u,v)) + 1$. Thus, $\Gamma(u,v)$ is a hyperplane of $\Gamma(w,v)$. Let $P \in \mathscr{P}_\mathrm{m}$ such that $P \in \Gamma(w,v) \setminus \Gamma(u)$. We obtain $\Gamma(w) = \langle P, \Gamma(u,w)\rangle \leq \langle \Gamma(u), \Gamma(v)\rangle$. □

We are no ready the prove the main result of this section.

Proposition 6.4.21. *Let x and y be opposite points of $\mathscr{S}$. Further let $\mathscr{D}$ be the Grassmannian of $(\mathscr{P}_\mathrm{m}, \mathscr{L}_\mathrm{m})$ with respect to $(\Gamma(x), \Gamma(y))$. Then $\mu\colon \mathscr{S} \to \mathscr{D}\colon p \mapsto \Gamma(p)$ is an injective homomorphism that maps every line of $\mathscr{S}$ bijectively onto a line of $\mathscr{D}$.*

Proof. By Proposition 6.4.18 the subspaces $\Gamma(x)$ and $\Gamma(y)$ are complementary in $(\mathscr{P}_\mathrm{m}, \mathscr{L}_\mathrm{m})$. Hence, the twin Grassmannian $\mathscr{D}$ exists. By Proposition 6.4.17 we obtain $\mathrm{crk}_{\Gamma(p)}(\Gamma(p) \cap \Gamma(q)) = \mathrm{crk}_{\Gamma(q)}(\Gamma(p) \cap \Gamma(q)) < \infty$ for two points p and q of $\mathscr{S}^+$. Analogously, $\Gamma(p)$ and $\Gamma(q)$ are commensurate if p and q are points of $\mathscr{S}^-$. Thus, the image of $\mathscr{S}$ is contained in $\mathscr{D}$. Moreover, the map μ is injective by Proposition 6.4.17.
Let l be a line of $\mathscr{S}$ and let p and q be two distinct points on l. Then $\Gamma(p)$ and $\Gamma(q)$ have a hyperplane H in common by Proposition 6.4.17. Thus, $L := \{S \leq (\mathscr{P}_\mathrm{m}, \mathscr{L}_\mathrm{m}) \mid H < S < \langle \Gamma(p), \Gamma(q)\rangle\}$ is a line in $\mathscr{D}$. By Lemma 6.4.20 every element of L has a preimage. Moreover, l is mapped bijectively onto L. □

Theorem 6.4.22. *A rigid twin SPO space whose symplecta have rank 3 and whose lines are contained in at most two maximal singular subspaces is a partial twin Grassmannian of a projective space.*

Proof. We denote the rigid twin SPO space by $\mathscr{S}$ and its two connected components by $\mathscr{S}^+$ and $\mathscr{S}^-$. Let $x \in \mathscr{S}^+$ and $y \in \mathscr{S}^-$ be opposite points of $\mathscr{S}$.
First assume $\mathrm{diam}(\mathscr{S}^+) < 2$ and hence, $\mathrm{diam}(\mathscr{S}^-) < 2$. Then $\mathscr{S}^+$ is a projective space by Theorem 2.1.22. If $\mathscr{S}^+$ is a singleton, then $\mathscr{S}^-$ is a singleton, too. Moreover, $\mathscr{S}$ is isomorphic to the unique twin Grassmannian of the projective space $\mathscr{S}^+$.
Now assume that $\mathscr{S}^+$ contains a line. Then the subspace $\mathrm{copr}_{\mathscr{S}^+}(y)$ is a complement of $\{x\}$ in $\mathscr{S}^+$. Let $\mathscr{D} = (\mathscr{D}^+, \mathscr{D}^-)$ be the twin Grassmannian of $\mathscr{S}^+$ with respect to $(\{x\}, \mathrm{copr}_{\mathscr{S}^+}(y))$. Define a map

$$\mu\colon \mathscr{S} \to \mathscr{D}\colon \begin{cases} p \mapsto \{p\} & \text{if } p \in \mathscr{S}^+ \\ p \mapsto \mathrm{copr}_{\mathscr{S}^+}(p) & \text{if } p \in \mathscr{S}^- \end{cases}.$$

By Corollary 5.3.8 we conclude that μ maps $\mathscr{S}^+$ bijectively onto $\mathscr{D}^+$. Moreover, by the definition of the lines in $\mathscr{D}^+$ we see directly that μ induces an isomorphism from $\mathscr{S}^+$ onto $\mathscr{D}^+$. Now let p and q be distinct points of $\mathscr{S}^-$. Then by Lemma 2.1.13 there is a point $r \in \mathscr{S}^+$ with $r \leftrightarrow p$ with $r \not\leftrightarrow q$. Since p^μ and q^μ are both

hyperplanes of $\mathscr{S}^+$ and therefore commensurate, μ maps $\mathscr{S}^-$ injectively into $\mathscr{D}^-$. Let $r \in pq$. Then $\langle p^\mu, q^\mu \rangle = \mathscr{S}^+$. By (A2) every point $s \in \mathscr{S}^+$ with $p \not\leftrightarrow s \not\leftrightarrow q$ is non-opposite r and hence, $p^\mu \cap q^\mu < \mathrm{copr}_{\mathscr{S}^+}(r) < \langle p^\mu, q^\mu \rangle$. Conversely, for a hyperplane H of $\mathscr{S}^+$ with $H > p^\mu \cap q^\mu$ there is a point $s \in H \setminus (p^\mu \cap q^\mu)$. We obtain $s \leftrightarrow p$ or $s \leftrightarrow q$ and hence, there is a unique point $r \in pq$ with $r \not\leftrightarrow s$ and hence $r^\mu = H$. Thus, μ maps lines of $\mathscr{S}^-$ bijectively onto lines of $\mathscr{D}^-$.
For $p \in \mathscr{S}^+$ and $q \in \mathscr{S}^-$, we obtain $p^\mu \cap q^\mu \varnothing$ if and only if $p \notin \mathrm{copr}_{\mathscr{S}^+}(q)$ and hence $p \leftrightarrow q$. Since $\leftrightarrow$ is total, (TG1) holds. Since μ maps $\mathscr{S}^+$ bijectively onto the singletons of $\mathscr{S}^+$, the image of $\mathscr{S}^+$ under μ fulfills the conditions (TG2) and (TG3). For every $p \in \mathscr{S}^+$, we obtain $p \in q^\mu$ for every $q \in \mathrm{copr}_{\mathscr{S}^-}(p)$, hence (TG2) holds for the image of $\mathscr{S}^-$ under μ. Let p and q be two distinct points of $\mathscr{S}^-$ and let H be a hyperplane of $\mathscr{S}^+$ that contains $p^\mu \cap q^\mu$. Then H, p^μ and q^μ are on a common line in $\mathscr{D}^-$. Since μ maps lines of $\mathscr{S}^-$ bijectively onto lines of $\mathscr{D}^-$, we conclude that there is a point $r \in pq$ such that $r^\mu = H$. Thus, (TG3) holds and the claim follows.
Now assume $\mathrm{diam}(\mathscr{S}^+) \geq 2$. Since every line of $\mathscr{S}$ is contained in at most two maximal singular subspaces of $\mathscr{S}$, we may use the notations of this section. Let $\mathscr{D}$ be the twin Grassmannian of $(\mathscr{P}_\mathrm{m}, \mathscr{L}_\mathrm{m})$ with respect to $(\Gamma(x), \Gamma(y))$ and set $\mu\colon \mathscr{S} \to \mathscr{D}\colon p \mapsto \Gamma(p)$. By Proposition 6.4.21 we know that $\mu\colon \mathscr{S} \to \mathscr{D}\colon p \mapsto \Gamma(p)$ is an injective homomorphism. Hence, $\mathscr{S}^\mu$ is isomorphic to $\mathscr{S}$. Since $\leftrightarrow$ is total in $\mathscr{S}$, Proposition 6.4.18 implies (TG1). Let $P \in \mathscr{P}_\mathrm{m}$. Then by 6.4.11(iii) there are points $p \in P \cap \mathscr{S}^+$ and $q \in P \cap \mathscr{S}^+$. Since $P \in p^\mu \cap q^\mu$, we conclude that (TG2) holds. Finally, (TG3) follows from Lemma 6.4.20. □

By Proposition 3.4.1 the restriction that every line of $\mathscr{S}$ is contained in at most two maximal singular spaces does not affect the case where $\mathscr{S}^+$ contains a symplecton properly. If the two connected components are singular subspaces, this condition is obviously true. Hence, the only case that is affected is the case where $\mathscr{S}^+$ and $\mathscr{S}^-$ are opposite symplecta.

Remark 6.4.23. Let $P \in \mathscr{P}_\mathrm{m}$. By Proposition 6.4.11(i) the elements of $\mathfrak{M}^+ \cup \mathfrak{M}^-$ that contain a line of P are entirely contained in P whereas no element of $\mathfrak{M} \setminus (\mathfrak{M}^+ \cup \mathfrak{M}^-)$ is a subspace of P. Interchanging the roles of $\mathfrak{M}^+ \cup \mathfrak{M}^-$ and $\mathfrak{M} \setminus (\mathfrak{M}^+ \cup \mathfrak{M}^-)$ leads to exactly the same results. This is because every partial twin Grassmannian of a projective space $\mathscr{D}$ is also a partial twin Grassmannian of a subspace of the dual of the projective space $\mathscr{D}$.

6.5 Twin SPO spaces of symplectic rank 4

In this section we consider the rigid twin SPO spaces of symplectic rank 4. Therefore, throughout this section let $\mathscr{S}$ be a twin SPO space of symplectic rank 4.

This implies that $\mathscr{S}$ is rigid and has diameter ≥ 2. By $\mathscr{S}^+ = (\mathscr{P}^+, \mathscr{L}^+)$ and $\mathscr{S}^- = (\mathscr{P}^-, \mathscr{L}^-)$ we denote the connected components of $\mathscr{S}$. Further we denote by $\mathfrak{M}$ the set of maximal singular subspaces of $\mathscr{S}$. Further we denote by $\mathfrak{M}$ the set of maximal singular subspaces of $\mathscr{S}$.

Since we have have already covered the case where $\mathscr{S}^+$ is a symplecton, we may constrain ourselves to the case where $\mathscr{S}^+$ contains a symplecton properly. By Proposition 3.5.2 this implies that every singular space of rank 2 is contained in exactly two elements of $\mathfrak{M}$. For reasons of convenience, we include in this section the case where $\mathscr{S}^+$, and therefore also $\mathscr{S}^-$, is a symplecton whose singular subspaces of rank 2 are contained in exactly two elements of $\mathfrak{M}$.

By Theorem 3.5.4 we know $\mathrm{srk}(\mathscr{S}) = 3$ if $\mathscr{S}^+$ is a symplecton and $\mathrm{srk}(\mathscr{S}) \geq 4$ otherwise. In the latter case we set $\mathfrak{M}_1 := \{M \in \mathfrak{M} \mid \mathrm{rk}(M) \geq 4\}$. Moreover, for $\sigma \in \{+,-\}$, we set $\mathfrak{M}_1^\sigma := \{M \in \mathfrak{M}_1 \mid M \leq \mathscr{S}^\sigma\}$. For the case where $\mathscr{S}^+$ is a symplecton, we demanded that every singular subspace of rank 2 is contained in exactly two elements of $\mathfrak{M}$. Hence, Proposition 2.2.8 implies that the dual polar graph of $\mathscr{S}^+$ is bipartite. Thus, there is a subset $\mathfrak{M}_1^+ \subseteq \mathfrak{M}$ such that every singular subspace of rank 2 of $\mathscr{S}^+$ is contained in exactly one element of $\mathfrak{M}_1^+$. We choose a subspace $M \in \mathfrak{M}_1^+$ and a singular subspace $N \leq \mathscr{S}^-$ that is opposite M which exists by Proposition 2.3.5. Then we define $\mathfrak{M}_1^-$ to be the subset of $\mathfrak{M}$ such that $N \in \mathfrak{M}_1^-$ and every singular subspace of rank 2 of $\mathscr{S}^-$ is contained in exactly one element of $\mathfrak{M}_1^-$. We set $\mathfrak{M}_1 := \mathfrak{M}_1^+ \cup \mathfrak{M}_1^-$.

The set $\mathfrak{M} \setminus \mathfrak{M}_1$ is denoted by $\mathfrak{M}_0$. For $\sigma \in \{+,-\}$, we set $\mathfrak{M}_0^\sigma := \{M \in \mathfrak{M}_0 \mid M \leq \mathscr{S}^\sigma\}$. The following lemma affirms that we made the right choice determining the set $\mathfrak{M}_1^-$ for the case $\mathrm{srk}(\mathscr{S}) = 3$.

Lemma 6.5.1. *Let M and N be two elements of $\mathfrak{M}$ with $\mathrm{cod}(M,N) = 1$. Then $M \in \mathfrak{M}_0$ if and only if $N \in \mathfrak{M}_0$.*

Proof. By symmetric reasons it suffices to show that $M \in \mathfrak{M}_1$ implies $N \in \mathfrak{M}_1$. Since $\mathrm{rk}(N) \geq 3$ there is an independent set of points $\{p_i \mid 0 \leq i \leq 3\}$ such that $S := \langle p_i \mid 0 \leq i \leq 3\rangle \leq N$ is a subspace of rank 3.
Let $\mathrm{srk}(\mathscr{S}) > 3$. Then $\mathrm{rk}(M) > 3$, since $M \in \mathfrak{M}_1$. Since $\mathrm{cod}(M,N) = 1$, we obtain $\mathrm{cod}(p_i, M) = 1$ for $i \leq 3$. Moreover, $\mathrm{copr}_M(p_i)$ contains a hyperplane of M and therefore $\bigcap_{i \leq 3} \mathrm{copr}_M(p_i) \neq \varnothing$. Thus, there is a point $q \in M$ with $\mathrm{cod}(q, p_i) = 1$ for $i \leq 3$ and $\mathrm{cod}(q,S) = 1$. Hence, Lemma 2.1.21(ii) implies that there is a point $p_4 \leftrightarrow q$ with $S \leq {p_4}^\perp$. Thus, $\langle p_4, S\rangle$ is singular. We conclude $N > S$ and consequently, $N \in \mathfrak{M}_1$ by Lemma 3.1.1(i) and Proposition 2.2.5.
Now let $\mathrm{srk}(\mathscr{S}) = 3$. Then $\mathscr{S}^+$ and $\mathscr{S}^-$ are opposite symplecta and there is an isomorphism $\varphi\colon \mathscr{S}^+ \to \mathscr{S}^-$ that maps every point of $\mathscr{S}^+$ onto its cogate in $\mathscr{S}^-$. By definition of $\mathfrak{M}_1^-$ there are subspaces $M' \in \mathfrak{M}_1^+$ and $N' \in \mathfrak{M}_1^-$ such that M' and N' are opposite. Since $\mathrm{cod}(M',N') = 1$, we conclude $(M')^\varphi \cap N' = \varnothing$ and hence, $(M')^\varphi \in \mathfrak{M}_1$ by Proposition 2.2.8. Since φ is an isomorphism, this

implies $M^\varphi \in \mathfrak{M}_1$. Now $\operatorname{cod}(M,N) = 1$ yields $M^\varphi \cap N = \varnothing$ and hence again by Proposition 2.2.8 $N \in \mathfrak{M}_1$. $\square$

Lemma 6.5.2. *Let $M \in \mathfrak{M}_0$ and $N \in \mathfrak{M}_1$ such that M and N have a line in common. Then* $\operatorname{rk}(M \cap N) = 2$.

Proof. For $\operatorname{srk}(\mathscr{S}) = 3$, this is a consequence of Proposition 2.2.8. Therefore we may assume $\operatorname{srk}(\mathscr{S}) \geq 4$. By Proposition 3.5.2 we know $\operatorname{rk}(M \cap N) \leq 2$. Hence, there is a point $p \in M \setminus N$. Since N is maximal, there is a point $q \in N$ with $\operatorname{dist}(p,q) = 2$. The symplecton $\langle p,q\rangle_{\mathrm{g}}$ contains $\langle p, M \cap N\rangle$ and thus, $M \leq \langle p,q\rangle_{\mathrm{g}}$ by Lemma 3.1.1(iii). Hence, $\operatorname{pr}_M(q)$ is a hyperplane of M. By Proposition 3.5.2 we conclude that $\langle q, \operatorname{pr}_M(q)\rangle$ is contained in an element of $\mathfrak{M}_1$ and consequently, $N \geq \langle q, \operatorname{pr}_M(q)\rangle$ since N is the unique element of $\mathfrak{M}_1$ that contains $\langle q, M \cap N\rangle$. Therefore $M \cap N = \operatorname{pr}_M(q)$. $\square$

In this section the subspaces we are interested in are the coconvex spans of an element of $\mathfrak{M}_0$ and a point at finite codistance. Therefore we examine in the following the coprojection of a point at finite codistance in an element of $\mathfrak{M}$ and furthermore, how elements of $\mathfrak{M}$ at finite codistance are related to each other.

Lemma 6.5.3. *Let $M \in \mathfrak{M}$ and let x be a point with* $\operatorname{cod}(x,M) < \infty$.

(a) *If $M \in \mathfrak{M}_0$, then* $\operatorname{copr}_M(x)$ *is a singleton or a hyperplane of M.*

(b) *If $M \in \mathfrak{M}_1$ and* $\operatorname{copr}_M(x) = M$, *then* $\operatorname{cod}(x,M) = \operatorname{diam}(\mathscr{S}^+)$ *and* $\operatorname{rk}(M) = 2 \cdot \operatorname{diam}(\mathscr{S}^+)$.

(c) *If $M \in \mathfrak{M}_1$ and* $\operatorname{copr}_M(x) < M$, *then* $\operatorname{crk}_M(\operatorname{copr}_M(x)) = 2 \cdot \operatorname{cod}(x,M) - 1$.

Proof. By symmetric reasons we may assume $x \in \mathscr{S}^+$. First let $\mathscr{S}^-$ be a symplecton. Then x has a cogate x' in $\mathscr{S}^-$ and M is a generator of $\mathscr{S}^-$. If $x' \in M$, then $\operatorname{copr}_M(x) = \{x'\}$ and $\operatorname{crk}_M(\{x'\}) = 3 = 2 \cdot \operatorname{cod}(x,x') - 1$. If $x' \notin M$, then M contains a point that is not collinear to x' and equivalently is opposite x. This implies that $\operatorname{copr}_M(x)$ is a hyperplane of M and hence, $\operatorname{crk}_M(\operatorname{copr}_M(x)) = 1 = 2 \cdot \operatorname{cod}(x,M) - 1$. Now let $\mathscr{S}^-$ be not a symplecton and hence, $\operatorname{srk}(\mathscr{S}^-) > 3$ by Theorem 3.5.4. First assume $M \in \mathfrak{M}_1$. Let $y \in M$ such that $\operatorname{cod}(x,p) \geq \operatorname{cod}(x,y)$ for every $p \in M$. Further let $z \leftrightarrow x$ with $\operatorname{cod}(x,y) = \operatorname{dist}(y,z)$. If $y = z$, then $\operatorname{copr}_M(x)$ is a hyperplane of M and the claim follows. Thus we may assume $\operatorname{cod}(x,y) \geq 1$. Since $\operatorname{cod}(x,p) \geq \operatorname{cod}(x,y)$ for every $p \in M$, we obtain $\operatorname{dist}(z,M) = \operatorname{cod}(x,y)$ and hence, $\operatorname{rk}(\operatorname{pr}_M(z)) = 2 \cdot \operatorname{cod}(x,y)$ by Lemma 3.5.3(ii). Suppose $\operatorname{pr}_M(z) = M$. Then $\operatorname{cod}(x,M) = \operatorname{cod}(x,y)$ and the claim follows since $\operatorname{copr}_M(y) = M$ and hence, $\operatorname{diam}(\mathscr{S}^-) = \operatorname{cod}(x,y)$ by Theorem 3.5.4. Therefore we may assume $\operatorname{pr}_M(z) < M$. Let $S \leq M$ be a subspace such that $\operatorname{pr}_M(z)$ is a proper hyperplane of S. Then there is a point $x' \in S$ with $\operatorname{dist}(x',z) = \operatorname{cod}(x,y) + 1$. Since $\operatorname{pr}_M(z) \leq \langle x',z\rangle_{\mathrm{g}}$ and $\operatorname{rk}(S) = 2 \cdot \operatorname{cod}(x,y) + 1$, Theorem 3.5.4 implies that S is a maximal singular subspace of

$\langle x', z\rangle_{\mathrm{g}}$. By (A12) x has a cogate in $\langle x', z\rangle_{\mathrm{g}}$ and $\mathrm{cod}(x, \langle x', z\rangle_{\mathrm{g}}) = \mathrm{cod}(x, y) + 1$. Since $\mathrm{cod}(x, p) \geq \mathrm{cod}(x, y)$ for every $p \in S$, all points of S are collinear to the cogate of x in $\langle x', z\rangle_{\mathrm{g}}$. By the maximality of S in $\langle x', z\rangle_{\mathrm{g}}$ this implies $\mathrm{copr}_{\langle x', z\rangle_{\mathrm{g}}}(x) < S$ and hence, $\mathrm{copr}_M(x) \cap S \neq \varnothing$. We conclude that $\mathrm{copr}_M(x)$ is a complement to $\mathrm{pr}_M(z)$ in M and therefore $\mathrm{crk}_M(\mathrm{copr}_M(x)) = \mathrm{rk}(\mathrm{pr}_M(z)) + 1 = 2 \cdot \mathrm{cod}(x, M) - 1$.
Now assume $M \in \mathfrak{M}_0$. Assume $\mathrm{copr}_M(x)$ is not a hyperplane of M. Then there is a line g such that $\mathrm{cod}(x, p) \geq \mathrm{cod}(x, g)$ for every point $p \in M$. Let $y \in g$. Since $\mathrm{copr}_g(x) = g$, Lemma 6.1.1 implies that there is a point z with $\mathrm{cod}(x, z) = \mathrm{cod}(x, y) - 1$ and $g \leq z^{\perp}$. If $z \in M$, then by Lemma 3.1.1(i) there is a symplecton Y that contains $\langle z, g\rangle$. If $z \notin M$, then there is a point $z' \in M$ with $z' \not\perp z$ we set $Y := \langle z, z'\rangle_{\mathrm{g}}$. For both cases Lemma 3.1.1(iii) implies $M \leq Y$. Suppose that x has no cogate in Y. Since $\mathrm{cod}(x, g) > \mathrm{cod}(x, z)$, Proposition 2.1.12(iv) implies $g \leq \mathrm{copr}_Y(x)$. Furthermore, by Propositions 4.2.5 we conclude that $\mathrm{copr}_Y(x)$ is a generator of Y. Hence, $\mathrm{copr}_Y(x) = M$ since $\mathrm{cod}(x, p) \geq \mathrm{cod}(x, g)$ for every $p \in M$. Let M' be a generator of Y with $M \cap M' = \varnothing$. Then $\mathrm{copr}_{M'}(x) = M'$ and Proposition 2.2.9(iii) implies $M' \in \mathfrak{M}_0$. Since $\mathrm{cod}(x, M') = \mathrm{cod}(x, M) - 1$ we may repeat this construction to obtain after finitely many steps a subspace $M'' \in \mathfrak{M}_0$ with $\mathrm{cod}(x, M'') = 0$, a contradiction. Thus, x has a cogate x' in Y. Then $\mathrm{copr}_M(x)$ is a hyperplane of M if $x' \notin M$, a contradiction. Hence, $x' \in M$ and $\mathrm{copr}_M(x)$ is a singleton. □

Lemma 6.5.4. *Let $M \in \mathfrak{M}$ and let x and y be distinct collinear points such that* $\mathrm{cod}(x, M) = \mathrm{cod}(y, M) < \infty$.

(i) *Let $M \in \mathfrak{M}_0$. Further let $\mathrm{copr}_M(x)$ be a hyperplane of M and let $\mathrm{copr}_M(y)$ be a singleton. Then $\mathrm{copr}_M(y) < \mathrm{copr}_M(x)$.*

(ii) *Let $M \in \mathfrak{M}_1$. Further let $\mathrm{copr}_M(x)$ and $\mathrm{copr}_M(y)$ be both proper subspaces of M. Then $\mathrm{copr}_M(x)$ and $\mathrm{copr}_M(y)$ have a hyperplane in common.*

Proof. Set $d := \mathrm{cod}(x, M)$.
(i) Since $\mathrm{rk}(\mathrm{copr}_M(x)) = 2$, there is a line $g \leq \mathrm{copr}_M(x)$ with $g \cap \mathrm{copr}_M(y) = \varnothing$. Thus by Lemma 6.1.1, there is a point z with $\mathrm{cod}(z, y) = d - 2$ and $g \leq z^{\perp}$. Hence, $Y := \langle z, \mathrm{copr}_M(y)\rangle_{\mathrm{g}}$ is a symplecton that contains $\langle \mathrm{copr}_M(y), g\rangle$. By Lemma 3.1.1(iii) this implies $M \leq Y$. By Proposition 2.1.12(iv) we conclude that the point in $\mathrm{copr}_M(y)$ is a cogate for y in Y. Since $\mathrm{cod}(x, \mathrm{copr}_M(y)) \leq d$ and $x \perp y$, we obtain $\mathrm{cod}(x, Y) = d$. Hence, $\mathrm{copr}_M(x) \leq \mathrm{copr}_Y(x)$ and consequently, $\mathrm{cod}(x, z) = d - 1$. We conclude by Proposition 4.2.5 that $\mathrm{copr}_Y(x)$ is a generator of Y. Since every point of $\mathrm{copr}_Y(x)$ has at least codistance $d - 1$ to y, we conclude $\mathrm{copr}_Y(y) \leq \mathrm{copr}_Y(x)$ and hence, $\mathrm{copr}_M(y) \leq \mathrm{copr}_M(x)$.
(ii) Suppose there is a line $l \leq \mathrm{copr}_M(x)$ that is disjoint to $\mathrm{copr}_M(y)$. Then by Lemma 6.5.3 there is a point $p \in \mathrm{copr}_M(y) \setminus \mathrm{copr}_M(x)$. By Proposition 3.5.2 there is a subspace $N \in \mathfrak{M}_0$ such that $\langle p, l\rangle \leq N$. Then $l \leq \mathrm{copr}_N(x)$ and Lemma 6.5.3

implies that $\operatorname{copr}_N(x)$ is a hyperplane if N. On the other hand $l \cap \operatorname{copr}_N(y) = \varnothing$ and hence, $\operatorname{copr}_N(y) = \{p\}$ by Lemma 6.5.3. Since $p \in \operatorname{copr}_N(y) \setminus \operatorname{copr}_N(x)$, this is a contradiction to (i). ☐

Lemma 6.5.5. *Let $M \in \mathfrak{M}_0$. Further let N be a singular subspace of rank 3 that is one-coparallel to M. Then $N \in \mathfrak{M}_0$.*

Proof. Set $d := \operatorname{cod}(M,N)$. Let $x \in M$ and $y \in N$ be points with $\operatorname{cod}(x,y) = d$. Let $g \leq M$ be a line with $x \notin g$ and let $h \leq N$ be a line with $y \notin h$. By Lemma 4.2.1 there are points $w \leftrightarrow y$ with $\operatorname{dist}(w,g) = d-1$ and $\operatorname{pr}_g(w) = g$ and $z \leftrightarrow x$ with $\operatorname{dist}(z,h) = d-1$ and $\operatorname{pr}_h(z) = h$. Since $\operatorname{dist}(w,x) = \operatorname{dist}(y,z) = d$, $w \leftrightarrow y$ and $x \leftrightarrow z$, we conclude by (A12) and Corollary 4.2.8 that the metaplecta $\langle w,x\rangle_{\mathrm{g}}$ and $\langle y,z\rangle_{\mathrm{g}}$ are opposite and there is an isomorphism $\varphi\colon \langle w,x\rangle_{\mathrm{g}} \to \langle y,z\rangle_{\mathrm{g}}$ that maps every point onto its cogate. By Lemma 3.1.1(iii) $\langle x,g\rangle \leq \langle w,x\rangle_{\mathrm{g}}$ implies $M \leq \langle w,x\rangle_{\mathrm{g}}$. Analogously $N \leq \langle y,z\rangle_{\mathrm{g}}$. Now M^{φ} is a maximal singular subspace of $\langle y,z\rangle_{\mathrm{g}}$ with $\operatorname{rk}(M^{\varphi}) = 3$. If $d > 2$, then $\operatorname{srk}(\langle y,z\rangle_{\mathrm{g}}) > 3$ by Theorem 3.5.4 and hence we conclude $M^{\varphi} \in \mathfrak{M}_0$ by Proposition 3.5.2. If $d = 2$, then there is a generator $M' \leq \langle y,z\rangle_{\mathrm{g}}$ disjoint to M^{φ}. Since for a point in $p \in M$, the cogate for p in $\langle y,z\rangle_{\mathrm{g}}$ is contained in M^{φ}, there is a point in M' that is opposite p. Thus, $\operatorname{cod}(M,M') = 1$ and we obtain $M' \in \mathfrak{M}_0$. By Proposition 2.2.9(iii) this implies $M^{\varphi} \in \mathfrak{M}_0$.
Since y and every point on h have codistance d to a point in M, we obtain $\langle y,h\rangle \leq M^{\varphi}$. Let $p \in M^{\varphi} \setminus \langle y,h\rangle$ and let p' be the preimage of p with respect to φ. Then $\operatorname{copr}_h(p') = h$ and $\operatorname{cod}(p',h) = d-1$. Thus by Lemma 6.1.1, there is a point q with $\operatorname{cod}(p',q) = d-2$ and $h \leq q^{\perp}$. Then $\langle p,q\rangle_{\mathrm{g}}$ is a symplecton and Lemma 3.1.1(iii) implies $M^{\varphi} \leq \langle p,q\rangle_{\mathrm{g}}$ since $\langle p,h\rangle \leq \langle p,q\rangle_{\mathrm{g}}$. Let $N' \in \mathfrak{M}_1$ such that $\langle y,h\rangle \leq N'$. Then $N \leq N'$ or $N = M^{\varphi}$ by Proposition 3.5.2. Since $p \notin \langle y,h\rangle$ and $N' \cap M^{\varphi} = \langle y,h\rangle$, we obtain $p \notin N'$. By Proposition 2.1.12(iv) p is the cogate for p' in $\langle p,q\rangle_{\mathrm{g}}$. By Proposition 3.5.2 N' intersects $\langle p,q\rangle_{\mathrm{g}}$ in a generator and hence there is a point $q' \in N' \cap \langle p,q\rangle_{\mathrm{g}}$ with $\operatorname{cod}(p',q') = d-2$. Therefore $\operatorname{cod}(p',N) = d-1$ and we conclude $N = M^{\varphi}$. ☐

A coconvex subspace of $\mathscr{S}$ of finite codiameter consists of two parts of infinite diameter as long as $\mathscr{S}^{+}$ and $\mathscr{S}^{-}$ have infinite diameter. Similarly to the last section, the following lemma gives a possibility to make assertions about the size of convex subspaces of infinite diameter by taking the intersection with the maximal singular subspaces into account.

Lemma 6.5.6. *Let U and V be two convex subspaces with $U \leq V \leq \mathscr{S}^{-}$. Further let $M \in \mathfrak{M}_0$ and $N \in \mathfrak{M}_1$ with $M \leq U$ and $\operatorname{rk}(N \cap U) \geq 2$. Then $N \cap U = N \cap V$ implies $U = V$.*

Proof. If U is singular, then $U = M$ and hence by Proposition 3.5.2, $N \cap U$ is a proper hyperplane of M. Thus, $\operatorname{rk}(N \cap V) = 2$ and Lemma 3.1.1(iii) implies that

V is singular, too. The claim follows. Therefore we may from now on assume $\mathrm{diam}(U) \geq 2$ and hence, $\mathrm{rk}(N \cap U) > 2$ by Lemma 3.1.1(iii).
Let $p \in V$ be a point. Since p is contained in an element of $\mathfrak{M}$ that intersects V in a maximal singular subspace, Lemma 3.1.1(v) implies that there is a finite sequence $(N_i)_{0 \leq i \leq n} \in \mathfrak{M}^{n+1}$ with $N_0 = N$ and $p \in N_n$ such that $N_i \cap N_{i+1} \leq V$ and $\mathrm{rk}(N_i \cap N_{i+1}) = 2$ for $i < n$. By Proposition 3.5.2 we conclude $N_i \in \mathfrak{M}_1$ if and only if i is even. Assume $N_i \cap V = N_i \leq U$ for $i \leq n-1$ and i even. Then $N_{i+1} \leq U$ by Lemma 3.1.1(iii). If $i \leq n-2$, this implies $\mathrm{rk}(N_{i+2} \cap U) \geq 2$ and hence, $\mathrm{rk}(N_{i+2} \cap U) \geq 3$ by Lemma 3.1.1(iii). For a point $q \in N_{i+2}$, we conclude by Lemma 3.3.3(i)

$$q \in V \;\Leftrightarrow\; \mathrm{pr}_{N_i}(q) \leq V \;\Leftrightarrow\; \mathrm{pr}_{N_i}(q) \leq U \;\Leftrightarrow\; q \in U\,.$$

Since $N \cap V = N \cap U$, induction provides $N_i \cap V = N_i \cap U$ for every $i \leq n$ and therefore $p \in U$. □

Lemma 6.5.7. *Let M and N be elements of $\mathfrak{M}_1$ such that $l := M \cap N$ is a line. Further let x be a point with $\mathrm{cod}(x,M) < \infty$ such that $\mathrm{copr}_M(x) < M$ and $\mathrm{copr}_N(x) < N$. Then $\pi_{M,N}(\langle l, \mathrm{copr}_M(x)\rangle) = \langle l, \mathrm{copr}_N(x)\rangle$.*

Proof. We may assume $x \in \mathscr{S}^+$. Set $d := \mathrm{cod}(x,M)$. Further set $S := \mathrm{copr}_M(x)$ and $T := \mathrm{copr}_N(x)$. Then Lemma 6.5.3 implies $\mathrm{crk}_M(S) = 2d-1$. First let $\mathrm{cod}(x,N) \neq d$. By Lemma 3.3.3(iii) we may assume $\mathrm{cod}(x,N) = d-1$. This implies $\mathrm{cod}(x,l) = d-1$ and $\mathrm{copr}_l(x) = l$. Furthermore, Lemma 6.5.3 implies $\mathrm{crk}_N(T) = 2d-3$. For every $p \in S$ we obtain $\mathrm{pr}_N(p) \leq T$. Thus, $\pi_{M,N}(\langle l,S\rangle) \leq T$. Since $\mathrm{crk}_M(\langle l,S\rangle) = 2d-3$, Lemma 3.3.3(iii) implies $\mathrm{crk}_N(\pi_{M,N}(\langle l,S\rangle)) = 2d-3$ and therefore $\pi_{M,N}(\langle l,S\rangle) = T$.
Now let $\mathrm{cod}(x,N) = d$. Then Lemma 6.5.3 implies $\mathrm{crk}_N(T) = 2d-1$. First suppose $\mathrm{cod}(x,l) = d-1$. Then $d > 1$ and hence, there is a point $q \in N \setminus l$ such that $\langle q,l\rangle$ is disjoint to T. By Lemma 3.3.3(iii) there is a point $p \in M$ such that $\mathrm{pr}_N(p) \cap T \neq \varnothing$. Then $q \notin \mathrm{pr}_N(p)$ since $\mathrm{rk}(\mathrm{pr}_N(p)) = 2$ by Lemma 3.5.3(ii) and $l \leq \mathrm{pr}_N(p)$. Thus, $Y := \langle p,q\rangle_{\mathrm{g}}$ is a symplecton. By Lemma 3.1.1(iii), both M and N contain a generator of Y. Since $l \leq Y$, we obtain $\mathrm{cod}(x,Y) \leq d$ and since $\mathrm{cod}(x,\mathrm{pr}_N(p)) = d$, we conclude $\mathrm{cod}(x,Y) = d$. Suppose x has a cogate in Y. Then this cogate would be contained in $\mathrm{pr}_N(p) \setminus l$ and hence there is a point in $M \cap Y$ at codistance $d-2$ to x, a contradiction. Thus by Proposition 4.2.5, $\mathrm{copr}_Y(x)$ is a generator of Y. As a consequence this implies $\mathrm{srk}(\mathscr{S}^-) > 3$, since otherwise $\mathscr{S}^- = Y$ by Theorem 3.5.4 and therefore x has a cogate in Y by (A12). Since $\mathrm{cod}(x,\mathrm{pr}_N(p)) = d$ and $\mathrm{cod}(x,\langle q,l\rangle) = d-1$, the generators $\mathrm{copr}_Y(x)$ and $Y \cap N$ intersect in a single point q'. Hence, Proposition 2.2.9(iv) implies $\mathrm{copr}_Y(x) \in \mathfrak{M}_0$, a contradiction to Lemma 6.5.3. Thus, $\mathrm{cod}(x,l) = d$.
Assume that l and S intersect in a single point s. For $d = 1$, we obtain $M = \langle l,S\rangle$

and $N = \langle l, T \rangle$ and hence the claim follows from Lemma 3.3.3(iii). Therefore we may assume $d > 1$. Then there is a point $q \in N \setminus l$ such that $\langle q, l \rangle \cap T = \{s\}$. Let $p \in M$ be a point such that $\mathrm{pr}_N(p) \cap T > \{s\}$. Since $\mathrm{rk}(\mathrm{pr}_N(p)) = 2$ by Lemma 3.5.3(ii), we obtain $q \notin \mathrm{pr}_N(p)$ and conclude that $h := \mathrm{pr}_N(p) \cap T$ is a line. Thus, $Y := \langle p, q \rangle_{\mathrm{g}}$ is a symplecton. By Lemma 3.1.1(iii), both M and N contain a generator of Y. Since $\langle q, l \rangle \leq Y$, we conclude $\mathrm{cod}(x, Y) = d$. We obtain $h \leq \mathrm{copr}_Y(x)$ and therefore Proposition 4.2.5 implies that $\mathrm{copr}_Y(x)$ is a generator of Y. Since $\langle q, l \rangle \cap T = \{s\}$, the generators $\mathrm{copr}_Y(x)$ and $Y \cap N$ intersect in the line h. Applying Proposition 2.2.8 provides that the corank of $(Y \cap M) \cap \mathrm{copr}_Y(x)$ in $\mathrm{copr}_Y(x)$ is even. With $l \leq Y \cap M$ we conclude that $\mathrm{copr}_Y(x)$ and $Y \cap M$ intersect in a line g. Since $\langle p, \mathrm{pr}_N(p) \rangle$ is a generator of Y that intersects $N \cap Y$ in a hyperplane and $h \leq \mathrm{pr}_N(p) \cap \mathrm{copr}_Y(x)$, Proposition 2.2.8 implies that $\langle p, \mathrm{pr}_N(p) \rangle$ and $\mathrm{copr}_Y(x)$ intersect in a common hyperplane. Hence, $\langle p, \mathrm{pr}_N(p) \rangle \cap M = \langle p, l \rangle$ contains a line of S. Therefore $g \leq \langle p, l \rangle$ and we obtain $p \in \langle l, S \rangle$. We conclude $\pi_{M,N}(\langle l, S \rangle) \geq \langle l, T \rangle$. Since $\mathrm{crk}_M(S) = \mathrm{crk}_N(T) = 2d - 1$ the claim follows from Lemma 3.3.3(iii).
Now assume $l \leq S$. Since $S < M$, there is a point $r \in M$ with $\mathrm{cod}(x, r) = d - 1$. Let $q \in N$ such that $\mathrm{pr}_N(r) = \langle q, l \rangle$. Further let $p \in S \setminus l$. Since $\langle p, l \rangle \leq S$ and $\mathrm{pr}_M(q) = \langle r, l \rangle$ by the collinearity of r and q, this implies that $Y := \langle p, q \rangle_{\mathrm{g}}$ is a symplecton. By Lemma 3.1.1(iii), both M and N contain a generator of Y. Assume $\mathrm{cod}(x, Y) = d + 1$. Then by Proposition 2.1.12(iv) x has a cogate y in Y. Thus, y is collinear to all points of $\langle p, l \rangle$ and hence, $\langle y, p, l \rangle$ is a generator of Y. Since $\langle y, p, l \rangle$ and $M \cap Y$ are the only generators that contain $\langle p, l \rangle$, we conclude $\langle p, \mathrm{pr}_N(p) \rangle = \langle y, p, l \rangle$ and therefore $\mathrm{pr}_N(p) \leq T$. Now assume $\mathrm{cod}(x, Y) = d$. Then by Proposition 4.2.5 $\mathrm{copr}_Y(x)$ is a generator of Y since $r \in Y$ and $l \leq \mathrm{copr}_Y(x)$. Since $r \in M \setminus \mathrm{copr}_Y(x)$, we obtain $\mathrm{copr}_Y(x) \cap M = \langle p, l \rangle$. Hence by Proposition 2.2.8 and since $l \leq \mathrm{copr}_Y(x) \cap N$ we conclude that $N \cap Y$ and $\mathrm{copr}_Y(x)$ intersect in a common hyperplane. Since $p \in \mathrm{copr}_Y(x)$ this implies $\mathrm{pr}_N(p) = \mathrm{copr}_Y(x) \cap N$ and hence again $\mathrm{pr}_N(p) \leq T$. We conclude $\pi_{M,N}(S) \leq T$ and the claim follows from Lemma 3.3.3(iii). ☐

Corollary 6.5.8. *Let V be connected convex subspace with* $\mathrm{diam}(V) \geq 2$ *and let $M \in \mathfrak{M}_1$ be a subspace with* $\mathrm{rk}(M \cap V) \geq 2$*. Further let x be a point with* $\mathrm{cod}(x, M) < \infty$ *and* $\mathrm{copr}_M(x) \leq V$*. Then* $\mathrm{copr}_N(x) \leq V$ *for every subspace $N \in \mathfrak{M}_1$ with* $\mathrm{rk}(N \cap V) \geq 2$ *and* $\mathrm{copr}_N(x) < N$*.*

Proof. Let $N \in \mathfrak{M}_1$ with $\mathrm{rk}(N \cap V) \geq 2$. By Lemma 3.1.1(iii) $M \cap V$ and $N \cap V$ are maximal singular subspaces of V. First assume there is a subspace $K \in \mathfrak{M}_1$ with $\mathrm{rk}(K \cap V) \geq 2$ and $\mathrm{copr}_K(x) = K$. Then $K \leq \mathrm{copr}_{\mathscr{S}^-}(x)$ by Lemma 6.5.3. Moreover, $\mathrm{cod}(x, K) = \mathrm{diam}(\mathscr{S}^-) =: d$ and $\mathrm{rk}(K) = 2d$. Since by Lemma 6.5.3 there is no element of $\mathfrak{M}_0$ contained in $\mathrm{copr}_{\mathscr{S}^-}(x)$, we conclude by Proposition 2.1.16(i) that $\mathrm{copr}_{\mathscr{S}^-}(x)$ is singular and hence equals K. By Proposition 2.1.16(ii) every point $p \in \mathrm{copr}_N(x)$ has distance $d - \mathrm{cod}(x, p)$ to K. This implies

$\operatorname{dist}(K,N) = d - \operatorname{cod}(x,p)$ and hence $\operatorname{copr}_N(x) \leq V$ by Lemma 3.5.3(ii).
Now assume $\operatorname{copr}_K(x) < K$ for every subspace $K \in \mathfrak{M}_1$ with $\operatorname{rk}(K \cap V) \geq 2$. Then by Lemma 3.1.1(v) and since V is connected, we may assume that there is a subspace $L \in \mathfrak{M}_0$ such that both N and M intersect L in a hyperplane. For $M = N$ there is nothing to prove. Hence by Proposition 3.5.2 we may assume that M and N intersect in a line l. Applying Lemma 6.5.7 yields $\pi_{M,N}(\langle l, \operatorname{copr}_M(x)\rangle) = \langle l, \operatorname{copr}_N(x)\rangle$. With Lemma 3.5.3(ii) this implies $\langle l, \operatorname{copr}_N(x)\rangle \leq V$. □

Lemma 6.5.9. *Let $x \in \mathscr{S}^+$. Further let $H \leq \mathscr{S}^-$ be a singular subspace with* $\operatorname{rk}(H) = 2$ *and* $\operatorname{copr}_H(x) = H$. *Set* $d := \operatorname{cod}(x,H)$. *Let $M \in \mathfrak{M}_0$ and $N \in \mathfrak{M}_1$ such that $H = M \cap N$ and set* $d := \operatorname{cod}(x,H)$. *Then either*

(a) $\operatorname{cod}(x,M) = d+1$ *and* $\operatorname{cod}(x,N) = d$ *or*

(b) $\operatorname{copr}_M(x) = H$ *and* $\operatorname{cod}(x,q) \geq d$ *for every point $q \in N$.*

Proof. Let $p \in M \setminus H$. Then for every point $q \in N \setminus H$, the subspace $\langle p,q\rangle_{\mathrm{g}}$ is a symplecton and the only generators of $\langle p,q\rangle_{\mathrm{g}}$ that contain H are $\langle p,H\rangle$ and $\langle q,H\rangle$. By Lemma 6.5.3 we conclude that either $\operatorname{cod}(x,M) = d$ and $\operatorname{copr}_M(x) = H$ or $\operatorname{cod}(x,M) = d+1$ and $\operatorname{copr}_M(x)$ is a singleton.
First consider the case $\operatorname{cod}(x,M) = d+1$. Then we may assume that p is the unique point of M at codistance $d+1$ to x. Suppose that is a point $q \in N \setminus H$ with $\operatorname{cod}(x,q) = d+1$. Set $Y := \langle p,q\rangle_{\mathrm{g}}$. Then $\operatorname{cod}(x,Y) = d+1$ since $H \leq Y$. Hence, p and q are both contained in $\operatorname{copr}_Y(x)$, a contradiction to Proposition 2.1.16(i). Thus, $\operatorname{cod}(x,N) = d$ and (a) holds.
Now consider the case $\operatorname{cod}(x,M) = d$ and $\operatorname{copr}_M(x) = H$. Let $q \in N \setminus H$ and $Y := \langle p,q\rangle_{\mathrm{g}}$. If $\operatorname{cod}(x,Y) = d+1$, then Proposition 2.1.12(iv) implies that x has a cogate x' in Y. Since this cogate is collinear to all point of H, we conclude that $\langle x',H\rangle$ is a generator of Y. Since $\operatorname{cod}(x,M) = d$, we conclude $\langle x',H\rangle = \langle q,H\rangle$ and therefore $\operatorname{cod}(x,q) \geq d$. If $\operatorname{cod}(x,Y) = d$, then Proposition 4.2.5 implies that $\operatorname{copr}_Y(x)$ is a generator of Y. Since $\operatorname{cod}(x,p) = d-1$, we conclude $\operatorname{copr}_Y(x) = \langle q,H\rangle$ and therefore $\operatorname{cod}(x,q) = d$. Thus, (b) holds. □

Lemma 6.5.10. *Let $x \in \mathscr{S}^+$ and $M \in \mathfrak{M}_0^-$. Then there is a subspace $N \in \mathfrak{M}_0^+$ with $x \in N$ such that* $\operatorname{cod}(p,M) = \operatorname{cod}(q,N)$ *and* $\operatorname{rk}(\operatorname{copr}_M(p)) = \operatorname{rk}(\operatorname{copr}_N(q))$ *for every pair of points $(p,q) \in N \times M$.*

Proof. Set $d := \operatorname{cod}(x,M)$ and $k := \operatorname{rk}(\operatorname{copr}_M(x))$. Then $k \in \{0,2\}$ by Lemma 6.5.3. Hence, there is a point $x' \in M$ with $\operatorname{cod}(x,x') = d$. Let y be a point of $\operatorname{copr}_M(x)$.
First let $k = 0$. Then by Lemma 4.2.1 there is a point $z \leftrightarrow x$ with $\operatorname{dist}(z,M) = d-1$ and $\operatorname{rk}(\operatorname{pr}_M(z)) \geq 1$. Thus, $\operatorname{rk}(M \cap \langle y,z\rangle_{\mathrm{g}}) \geq 2$ and therefore $M \leq \langle y,z\rangle_{\mathrm{g}}$ by Lemma 3.1.1(iii). By Corollary 4.2.8 there is a metaplecton containing x that is opposite $\langle y,z\rangle_{\mathrm{g}}$. Moreover, this metaplecton contains a singular subspace N of rank 3 such

that M and N are one-coparallel to each other with $\operatorname{cod}(M,N)=d$ and $x\in N$. By Lemma 6.5.5 we obtain $N\in\mathfrak{M}_0$.
Now let $k=2$. Further assume $\operatorname{diam}(\mathscr{S}^+)\geq d+1$, then Lemma 6.1.1 implies that there is a point $z\perp y$ with $\operatorname{cod}(x,z)=d+1$ and $\operatorname{pr}_M(z)>\{y\}$. Thus by Lemma 3.1.1(iii), $\langle z,x'\rangle_{\mathrm{g}}$ is a symplecton that contains M. By Corollary 4.2.8 there is a symplecton Y with $x\in Y$ such that $\langle z,x'\rangle_{\mathrm{g}}$ and Y are one-coparallel to each other at codistance $d+1$. Let M' be the generator of Y that is one-coparallel to M with $\operatorname{cod}(M,M')=d+1$. Then $M'\in\mathfrak{M}_0$ by Lemma 6.5.5. Since $x\notin M'$ there is a generator $N\in Y$ with $x\in N$ that is disjoint to M'. By Proposition 2.2.8 we conclude $N\in\mathfrak{M}_0$. Since every point $q\in M$ has a cogate at codistance $d+1$ in Y that is contained in M', we obtain $\operatorname{cod}(q,N)=d$ and $\operatorname{copr}_N(q)$ is a hyperplane of N. On the other hand, every point $p\in N$ has a cogate in $\langle z,x'\rangle_{\mathrm{g}}$ at codistance $d+1$ that is not contained in M since $p\notin M'$. This implies $\operatorname{cod}(p,M)=d$ and $\operatorname{rk}(\operatorname{copr}_M(p))=2$.
Finally, let $k=2$ and $\operatorname{diam}(\mathscr{S}^+)=d$, then $\operatorname{copr}_M(x)\leq\operatorname{copr}_{\mathscr{S}^-}(x)$ and hence by (A2), $\mathscr{S}^-$ is not a metaplecton. By Theorem 3.5.4 this implies $\operatorname{srk}(\mathscr{S})=2d$. By Proposition 2.1.16(ii) there is a point $y'\in\mathscr{S}^+$ with $x\perp y'$ and $\operatorname{cod}(y',x')=d$. By Proposition 3.5.2 y' is contained in a singular subspace of rank $2d$. Hence, Lemma 6.5.3 implies that this singular subspace contains a line l of $\operatorname{copr}_{\mathscr{S}^+}(x')$. We may assume $y'\in l$. If $l\leq x^{\perp}$, Lemma 3.1.1(i) implies that there is a symplecton Y that contains x and l. Otherwise, we set $Y:=\langle x,l\rangle_{\mathrm{g}}$. By Proposition 4.2.5 $\operatorname{copr}_Y(x')$ is a generator and hence there is a generator $N\leq Y$ such that $x\in N$ and N intersects $\operatorname{copr}_Y(x')$ in a hyperplane. By Lemma 6.5.3 the generator $\operatorname{copr}_Y(x')$ is contained in an element of $\mathfrak{M}_1$. Thus, $N\in\mathfrak{M}_0$ by Proposition 3.5.2. Since $\operatorname{copr}_N(x')$ is a hyperplane of N that does not contain x, we conclude by Lemma 6.5.4(i) that $\operatorname{copr}_N(p)$ is a hyperplane of N for every point $q\in\operatorname{copr}_M(x)$. Moreover, $\operatorname{cod}(q,N)=d$. Analogously for every point $p\in\operatorname{copr}_N(x')$, we conclude $\operatorname{cod}(p,M)=d$ and $\operatorname{rk}(\operatorname{copr}_M(p))=2$. For every point $q\in M\setminus\operatorname{copr}_M(x)$, there is a point $q'\in\operatorname{copr}_M(x)$ such that $q\in q'x'$. Since $\operatorname{copr}_N(q')\cap\operatorname{copr}_N(x')$ contains a line, Proposition 2.1.12(iv) implies that every point on this line has codistance d to p. Hence, $\operatorname{cod}(q,N)=d$. Moreover, Lemma 6.5.3 implies that $\operatorname{copr}_N(p)$ is a hyperplane of N. Analogously for every point $p\in N\setminus\operatorname{copr}_N(x')$, we obtain $\operatorname{cod}(p,M)=d$ and $\operatorname{rk}(\operatorname{copr}_M(p))=2$. □

The following proposition shows that the coconvex span of a point of $\mathscr{S}^+$ and an element of $\mathfrak{M}_0^-$ has properties that correspond to the properties of metaplecta stated in the Propositions 2.1.3, 2.1.12(i) and 2.1.12(iii).

Proposition 6.5.11. *Let $x\in\mathscr{S}^+$ and $M\in\mathfrak{M}_0^-$. Set $V:=\langle x,M\rangle_{\mathrm{G}}$ and $n:=2\cdot\operatorname{cod}(x,M)+\frac{1}{2}\cdot\operatorname{rk}(\operatorname{copr}_M(x))-3$.*

(i) *Let $S\leq V$ be a singular subspace with $\operatorname{rk}(S)=2$. Further let $L\in\mathfrak{M}_0$ and $K\in\mathfrak{M}_1$ such that $K\cap L=S$. Then $L\leq V$ and $\operatorname{crk}_K(K\cap V)=n$.*

(ii) *Let* $K \in \mathfrak{M}_1$ *such that* $\mathrm{rk}(M \cap K) = 2$*. Then* $K \cap V = \langle M \cap K, \mathrm{copr}_K(x)\rangle$ *or* $\mathrm{copr}_K(x) = K$*.*

(iii) $\mathrm{codm}(V) = \mathrm{cod}(x,M) - 1$.

(iv) *For every point* $u \in V$ *there is a subspace* $K \in \mathfrak{M}_0$ *with* $K \le V$ *and* $2 \cdot \mathrm{cod}(u,K) + \frac{1}{2} \cdot \mathrm{rk}(\mathrm{copr}_K(u)) - 3 = n$*. Moreover,* $\langle u,K\rangle_{\mathrm{G}} = V$ *for every such subspace* K*.*

Proof. Set $d := \mathrm{cod}(x,M) - 1$ and $k := \mathrm{rk}(\mathrm{copr}_M(x))$. Then $n = 2d + \frac{k}{2} - 1$. Let y be a point of $\mathrm{copr}_M(x)$. Since $k \in \{0,2\}$ by Lemma 6.5.3, there is a point $x' \in M$ with $\mathrm{cod}(x,x') = d$.
By Lemma 6.5.10 there is a subspace $M' \in \mathfrak{M}_0^+$ with $x \in M'$ such that $\mathrm{cod}(p,M) = \mathrm{cod}(q,M')$ and $\mathrm{rk}(\mathrm{copr}_M(p)) = \mathrm{rk}(\mathrm{copr}_{M'}(q))$ for every pair of points $(p,q) \in M' \times M$. This implies $2 \cdot \mathrm{cod}(x',M') + \frac{1}{2} \cdot \mathrm{rk}(\mathrm{copr}_{M'}(x')) - 3 = n$. If $k = 0$, then for every point $p \in M'$, there is a point $q \in M$ with $\mathrm{cod}(p,q) = d + 1$. Since $\mathrm{copr}_{M'}(q) = \{p\}$, we obtain $p \in \langle x,q\rangle_{\mathrm{G}} \le V$ and hence, $M' \le V$. If $k = 2$, then $\mathrm{copr}_{M'}(x')$ is a hyperplane of M' that does not contain x and therefore $M' = \langle x, \mathrm{copr}_{M'}(x')\rangle \le \langle x,x'\rangle_{\mathrm{G}} \le V$.
Let $H \le M$ be a hyperplane of M such that $x'y \le H$. Then $\mathrm{rk}(\mathrm{copr}_H(x)) = \frac{k}{2}$. By Proposition 3.5.2 there is a subspace $N \in \mathfrak{M}_1$ such that $H \le N$. Let analogously $N' \in \mathfrak{M}_1$ such that $H' := M' \cap N'$ is a hyperplane of M' with $x \in H'$ and $\mathrm{cod}(x',H') = d + 1$. Then $\mathrm{rk}(\mathrm{copr}_{H'}(x')) = \frac{k}{2}$. Lemma 6.1.2 implies $N \cap U^- = \langle H, \mathrm{copr}_N(x)\rangle$. Since $x'y \le N$, we obtain $\mathrm{crk}_N(\mathrm{copr}_N(x)) = 2d + 1$ by Lemma 6.5.3. Since $\mathrm{crk}_H(H \cap \mathrm{copr}_N(x)) = 2 - \frac{k}{2}$, we conclude $\mathrm{crk}_N(N \cap U^-) = 2d + 1 - (2 - \frac{k}{2}) = n$. By symmetric reasons $\mathrm{crk}_{N'}(N' \cap U^+) = n$.
Set $U^- := \langle M, \mathrm{copr}_N(x)\rangle_{\mathrm{g}}$ and $U^+ := \langle M', \mathrm{copr}_{N'}(x')\rangle_{\mathrm{g}}$. We will show $V = U^+ \cup U^-$. Since $\mathrm{cod}(x,N) = d + 1$ and $x' \in N$, we obtain $\mathrm{copr}_N(x) \le V$ by the coconvexity of V and therefore, $U^- \le V$. Analogously, $U^+ \le V$. Thus, $U^+ \cup U^-$ is a convex subspace of V. Since $x \in U^+$ and $M \le U^-$, it remains to show that $U^+ \cup U^-$ is coconvex. By symmetric reasons it suffices to show that for a pair of points $(u,v) \in U^+ \times U^-$ and a point w with $w \perp v$ and $\mathrm{cod}(u,w) = \mathrm{cod}(u,v) + 1$, we obtain $w \in U^-$.
First assume that U^- is singular. Then $U^- = M$ and $N \cap U^- = H$. Thus, $\mathrm{rk}(N) = n + 2$ and therefore $\mathrm{rk}(N) = 2d + 1$ if $k = 0$ and $\mathrm{rk}(N) = 2d + 2$ if $k = 2$. We conclude $\mathrm{diam}(\mathscr{S}^-) = d + 1$ by Theorem 3.5.4. Moreover, if $k = 0$, then $\mathscr{S}^+$ and $\mathscr{S}^-$ are opposite metaplecta and therefore M and M' are one-coparallel to each other at codistance $d + 1$. Hence, $M \cup M'$ is coconvex and we conclude $V = M \cup M'$. Since $\mathrm{srk}(\mathscr{S}) = 2d + 1$, claim (i) follows by Lemma 3.5.3(i). By Lemma 6.5.3 we obtain (ii). For (iii) and (iv), there is nothing to prove. Now let $k = 2$. Assume there are points $u \in M'$, $v \in M$ and $w \perp v$ such that $\mathrm{cod}(u,v) = d$ and $\mathrm{cod}(u,w) = d + 1$. Then $w \perp p$ for every point $p \in \mathrm{copr}_M(u)$ since other-

wise the symplecton $\langle p,w\rangle_{\mathrm{g}}$ would be contained in $\mathrm{copr}_{\mathscr{S}^-}(u)$ by Proposition 2.1.16(i), a contradiction to $\mathrm{cod}(u,v)=d$. Thus, w is collinear to all points of $\langle v,\mathrm{copr}_M(u)\rangle$. Since $\mathrm{copr}_M(u)$ is a hyperplane of M and $v\notin \mathrm{copr}_M(u)$, we obtain $\langle v,\mathrm{copr}_M(u)\rangle = M$ and therefore $w\in M$ by the maximality of M. By symmetric reasons this implies that $M\cup M'$ is coconvex and hence, $V=M\cup M'$. Since $\mathrm{srk}(\mathscr{S})=2d+2$, claim (i) follows by Lemma 3.5.3(i). Let $K\in\mathfrak{M}_1$ such that K intersects M in a hyperplane. Then $\mathrm{copr}_M(x)\cap K$ contains a line. Since by Lemma 6.5.3 we obtain that $\mathrm{copr}_K(x)$ is a line if K contains a point at codistance d to x, this implies (ii). For (iii) and (iv), there is nothing to prove. Thus, from now on we may assume $\mathrm{diam}(U^-)\geq 2$.

Let S be an arbitrary hyperplane of M with $S\neq H$ and let $K\in\mathfrak{M}_1$ be the subspace that contains S. Then $\mathrm{crk}_K(K\cap U^-)=n$ by Lemma 3.5.3(i). Since S and H have a line in common, we obtain $\mathrm{copr}_K(x)\leq U^-$ or $\mathrm{copr}_K(x)=K$ by Lemma 6.5.7. Assume $S\cap \mathrm{copr}_M(x)=\varnothing$. Then necessarily $k=0$. By Lemma 6.5.9 we obtain $\mathrm{cod}(x,K)=d$ and consequently, Lemma 6.5.3 implies $\mathrm{crk}_K(\mathrm{copr}_K(x))=2d-1=n$. Hence, $\mathrm{copr}_K(x)=\langle S,\mathrm{copr}_K(x)\rangle = K\cap U^-$. Assume $S\cap\mathrm{copr}_M(x)$ is a point or a line. Then $\mathrm{crk}_K(\langle S,\mathrm{copr}_K(x)\rangle)=n$ by the same reason as for N. This implies again $\langle S,\mathrm{copr}_K(x)\rangle = K\cap U^-$. Finally assume $S\leq \mathrm{copr}_M(x)$. Then necessarily $k=1$. By Lemma 6.5.9 implies $\mathrm{cod}(x,p)\geq d+1$ for every point $p\in K$ and we conclude that either $\mathrm{copr}_K(x)=K$ or $\mathrm{cod}(x,K)=d+2$ holds. In the latter case we obtain $\mathrm{crk}_K(\mathrm{copr}_K(x))=2d+3$ by Lemma 6.5.3. Thus, $\mathrm{crk}_K(\langle S,\mathrm{copr}_K(x)\rangle)=2d=n$ and again $\langle S,\mathrm{copr}_K(x)\rangle = K\cap U^-$. Therefore (ii) holds for U^-.

Now let $w\perp v$ for a point $v\in U^-$ such that $\mathrm{cod}(x,w)=\mathrm{cod}(x,v)+1$. Suppose $w\notin U^-$. First assume that $\mathrm{pr}_{U^-}(w)$ contains a line l through v. Since by Lemma 3.1.1(i) l is contained in a symplecton of U^-, we obtain $\mathrm{rk}(\mathrm{pr}_{U^-}(w))\geq 3$ by Proposition 2.1.27. Thus there is a subspace $K\leq\mathfrak{M}_1$ with $w\in K$ and $K\cap U^-=\mathrm{pr}_{U^-}(w)$. This implies $w\in\mathrm{copr}_K(x)<K$ since $v\in K$, a contradiction to Corollary 6.5.8 since $\mathrm{copr}_N(x)<N\cap U^-$. Thus, $\mathrm{pr}_{U^-}(w)=\{v\}$. Let $l\leq U^-$ be a line through v. Then $Y:=\langle w,l\rangle_{\mathrm{g}}$ is a symplecton. Since $w^\perp$ contains a hyperplane of $U^-\cap Y$, we conclude $U^-\cap Y=l$. Let $G\leq Y$ be a generator with $l\leq G$. Then there is a line $l'\leq w^\perp\cap G$ that is disjoint to l. For every point $w'\in l'$, we conclude $\mathrm{cod}(x,w')\leq\mathrm{cod}(x,v)$ since $\mathrm{pr}_{U^-}(w')$ contains l and hence, $\mathrm{cod}(x,w')=\mathrm{cod}(x,w)-1$ since $w\perp w'$. This implies $w\in\mathrm{copr}_Y(x)$. Let $v'\in l\setminus\{v\}$. Then $\mathrm{cod}(x,v')\geq\mathrm{cod}(x,w')$ for every point $w'\in l'$ since $v'\in l\leq\mathrm{pr}_{U^-}(w')$. Thus, w is not a cogate of x in Y and we conclude by Proposition 4.2.5 that $\mathrm{copr}_Y(x)$ is a generator. This generator contains a point w'' with $l\leq\mathrm{pr}_{U^-}(w'')$, a contradiction. Therefore, $w\in U^-$.

To prove that for every point $u\in U^+$ and every point $w\perp v$ with $\mathrm{cod}(u,w)=\mathrm{cod}(u,v)+1$, we obtain $w\in U^-$, it suffices now to show that there are subspaces $M_u\in\mathfrak{M}_0^-$ and $N_u\in\mathfrak{M}_1$ such that $H_u:=M_u\cap N_u$ is a hyperplane of M_u with

$\mathrm{copr}_{H_u}(u) < H_u$ and $U^- = \langle M_u, \mathrm{copr}_{N_u}(u)\rangle_{\mathrm{g}} =: U_u$. By Lemma 6.5.3 we know $\mathrm{crk}_{N_u}(\langle H_u, \mathrm{copr}_{N_u}(u)\rangle) = 2\cdot\mathrm{cod}(u,H_u) - 1 - \mathrm{crk}_{H_u}(\mathrm{copr}_{H_u})$. By Lemma 6.1.2 we obtain $\langle H_u, \mathrm{copr}_{N_u}(u)\rangle = N_u \cap U_u$. Since by Lemma 6.5.3 $\mathrm{copr}_{M_u}(u)$ is either a singleton or a hyperplane of M_u, this implies $\mathrm{crk}_{N_u}(N_u\cap U_u) = 2\cdot\mathrm{cod}(u,M_u) + \frac{1}{2}\cdot\mathrm{rk}(\mathrm{copr}_{M_u}(u)) - 3$. Hence with Lemmas 6.5.6 and 3.5.3(i), it suffices to show $\mathrm{cod}(u,H_u) = d+1$, $\mathrm{rk}(\mathrm{copr}_{M_u}) = k$ and $\langle H_u, \mathrm{copr}_{N_u}(u)\rangle \le U^-$ to prove $U^- = U_u$. Since U^+ is connected, we may restrict ourselves to the case $x \perp u$.
Assume $u \in M'$. Then $\mathrm{rk}(\mathrm{copr}_M(u)) = \mathrm{rk}(\mathrm{copr}_M(x))$ and $\mathrm{cod}(u,M) = d+1$. Assume $\mathrm{copr}_M(u) = \mathrm{copr}_M(x)$. In the case $k=0$ this implies $u = x$ since M and M' are one-coparallel. In the case $k=2$ we obtain $\mathrm{copr}_M(q) = \mathrm{copr}_M(x)$ for every point $q \in \langle u,x\rangle$. Hence, $\langle u,x\rangle \cap \mathrm{copr}_{M'}(p) = \varnothing$ for a point $p \in M \setminus \mathrm{copr}_M(x)$ since $\mathrm{cod}(p,M') = d+1$. Since $\mathrm{copr}_{M'}(p)$ is a hyperplane of M', we conclude again $u = x$. Thus we may assume $\mathrm{copr}_M(u) \neq \mathrm{copr}_M(x)$. Then there is a hyperplane H_u of M such that $\mathrm{copr}_{H_u}(x)$ and $\mathrm{copr}_{H_u}(u)$ are both properly contained in H_u and $\mathrm{copr}_{H_u}(x) \neq \mathrm{copr}_{H_u}(u)$. Let $N_u \in \mathfrak{M}_1$ such that $H_u \le N_u$. By Lemma 6.5.4(ii) $\mathrm{copr}_{N_u}(x)$ and $\mathrm{copr}_{N_u}(u)$ intersect in a common hyperplane H and therefore $\mathrm{copr}_{N_u}(u) \le \langle \mathrm{copr}_{N_u}(x), H_u\rangle$. Since $\mathrm{copr}_{N_u}(x) \le U^-$, this implies $\mathrm{copr}_{N_u}(u) \le U^-$. Thus for $M_u := M$, we conclude $U_u = U^-$. As a consequence, $K \cap U^- = \langle M\cap K, \mathrm{copr}_K(p)\rangle$ for every $p \in M'$ and $K \in \mathfrak{M}_1$ with $\mathrm{rk}(M\cap K) = 2$. By symmetric reasons $K\cap U^+ = \langle M'\cap K, \mathrm{copr}_K(q)\rangle$ for every $q \in M$ and $K \in \mathfrak{M}_1$ with $\mathrm{rk}(M'\cap K) = 2$.
Now let $\mathrm{dist}(u,M') = 1$. Assume $\mathrm{pr}_{M'}(u) = \{x\}$. Let $y' \in \mathrm{copr}_{M'}(x')$. Since $\mathrm{cod}(x,x') \neq \mathrm{cod}(x,y)$ we obtain $\mathrm{copr}_{M'}(x') \neq \mathrm{copr}_{M'}(y)$. Hence, we may assume $y' \in \mathrm{copr}_{M'}(x') \setminus \mathrm{copr}_{M'}(y)$. By Corollary 4.2.8 we know that xy' and $x'y$ are one-coparallel. Now $\langle u,y'\rangle_{\mathrm{g}}$ is a symplecton that contains the line xy'. By Proposition 3.5.2 there is a generator G' of $\langle u,y'\rangle_{\mathrm{g}}$ with $xy' \le G'$ and $G' \in \mathfrak{M}_0$. Since $u \in U^+$, we obtain $G' \le U^+$. We show that there is a subspace $G \in \mathfrak{M}_0$ contained in U^- with $\mathrm{cod}(p,G) = \mathrm{cod}(q,G') = d+1$ and $\mathrm{rk}(\mathrm{copr}_G(p)) = \mathrm{rk}(\mathrm{copr}_{G'}(q)) = k$ for every pair of points $(p,q) \in G'\times G$ such that $\langle G, \mathrm{copr}_L(x)\rangle_{\mathrm{g}} = U^-$ for a subspace $L \in \mathfrak{M}_1$ with $\mathrm{rk}(G\cap L) = 2$ and $\mathrm{copr}_{G\cap L}(x) \le G\cap L$. Since $\mathrm{pr}_{G'}(u)$ is a hyperplane of G', this allows us to constrain ourselves to the case $\mathrm{pr}_{M'}(u) > \{x\}$.
First consider the case $k=0$. Then x' is the cogate of y' in M. If G' is one-coparallel to M, we are done. Hence we may assume that G' and M are not one-coparallel. Let $p \in G' \setminus M'$. Then there is a subspace $K \in \mathfrak{M}_1$ such that $\langle p, xy'\rangle \le K$. By Lemma 6.5.2 K intersects M in a hyperplane and therefore $p \in \langle M'\cap K, \mathrm{copr}_K(q)\rangle$ for every $q \in M$. This implies $\mathrm{cod}(p,q) \ge d$ for every $q \in M$ and hence, $\mathrm{cod}(p,M) \ge d+1$ by Lemma 6.5.3. Since x and y' have distinct cogates in M, we obtain $\mathrm{cod}(p,M) = d+1$. Hence, if every point of G' has a cogate in M, we conclude by Lemma 6.5.3 that G' and M are one-coparallel to each other. Thus, we may assume $\mathrm{rk}(\mathrm{copr}_M(p)) = 2$. By Lemma 6.5.4(i) we obtain $x'y \le \mathrm{copr}_M(p)$. By Lemma 6.5.3 this implies that both $\mathrm{copr}_{G'}(x')$ and $\mathrm{copr}_{G'}(y)$

are hyperplanes of G'. Hence $l' := \mathrm{copr}_{G'}(x') \cap \mathrm{copr}_{G'}(y)$ is a line through p since $\mathrm{cod}(x,y) = d+1$ and $\mathrm{cod}(x,x') = d$. Let $l \leq M$ be a line disjoint to $x'y$. Now let p be an arbitrary point of l'. Since $\mathrm{cod}(p,M) = d+1$ and $x'y \leq \mathrm{copr}_M(p)$, Lemma 6.5.3 implies that there is a unique point $q \in l$ at codistance $d+1$. Conversely, since $\mathrm{cod}(q,xy') = d$, Lemma 6.5.3 implies that p is the unique point of G' at codistance $d+1$ to q. Thus l and l' are one-coparallel at codistance $d+1$.
Let $L \in \mathfrak{M}_1$ be the subspace that contains $\langle y,l \rangle$ and let p and p' be distinct point of l'. Then $\mathrm{cod}(p,L) = \mathrm{cod}(p',L) = d+1$ since $l \leq L$. Moreover, Lemma 6.5.3 implies $\mathrm{crk}_L(\mathrm{copr}_L(p)) = \mathrm{crk}_L(\mathrm{copr}_L(p')) = 2d+1$. Since $\mathrm{cod}(x,y) = d+1$ and $\mathrm{cod}(x,l) = d$, we obtain $\mathrm{cod}(x,L) = d+1$ and hence $\mathrm{crk}_L(\mathrm{copr}_L(x)) = 2d+1$. Moreover, $\mathrm{crk}_L(\langle l, \mathrm{copr}_L(x) \rangle) = 2d-1 = n$ and therefore $L \cap U^- = \langle l, \mathrm{copr}_L(x) \rangle$. By Lemma 6.5.4(ii) we conclude $\mathrm{copr}_L(p) \leq \langle l, \mathrm{copr}_L(x) \rangle$ since $\mathrm{cod}(p,l) = d+1$ and $\mathrm{cod}(x,l) = d$. Analogously, $\mathrm{copr}_L(p') \leq \langle l, \mathrm{copr}_L(x) \rangle$. Since l intersects $\mathrm{copr}_L(p)$ in a single point, we conclude that the subspace $\langle l, \mathrm{copr}_L(p) \rangle$ is a hyperplane of $L \cap U^-$. Since $y \in \mathrm{copr}_L(p)$, we obtain $\langle y,l \rangle = M \cap L \leq \langle l, \mathrm{copr}_L(p) \rangle$. Thus, there is a point $q \in L \cap U^- \setminus M$ such that p has a cogate in $\langle q,l \rangle$. Since $\mathrm{copr}_L(p') \leq \langle l, \mathrm{copr}_L(x) \rangle$ by Lemma 6.5.4(ii) and $\langle q,l \rangle \cap \langle l, \mathrm{copr}_L(x) \rangle = l$, we conclude that also p' and hence every point on l' has a cogate in $\langle q,l \rangle$. Let $G \in \mathfrak{M}_0$ such that $\langle q,l \rangle \leq G$. Then Lemma 6.5.3 implies that every point of l' has a cogate at codistance $d+1$ in G. Since $\langle q,l \rangle \leq U^-$, we obtain $G \leq U^-$ by Lemma 3.1.1(iii). For every point $r \in xy'$, we have $\mathrm{cod}(r,l) = d$ and $\mathrm{copr}_l(r) = l$. Let s be an arbitrary point of $G \setminus l$. Then by Lemma 6.5.2 the subspace L' that contains $\langle s,l \rangle$ intersects M in a hyperplane and hence, $L' \cap U^- = \langle L' \cap M, \mathrm{copr}_{L'}(r) \rangle$. This implies $\mathrm{cod}(r,s) \geq d$ and hence, r has a cogate at codistance $d+1$ in G by Lemma 6.5.3. Now let $r \in G' \setminus (xy' \cup l')$. Then there are points $p_0 \in xy'$ and $p_1 \in l'$ such that $p \in p_0p_1$. Let q_i be the cogate of p_i in G for $i \in \{0,1\}$. Since $\mathrm{cod}(l,xy') = d$ and $q_1 \in l$, we obtain $q_0 \neq q_1$. Thus by Corollary 4.2.8, the lines p_0p_1 and q_0q_1 are one-coparallel to each other. Since $\mathrm{cod}(r,l) = d$, this implies $\mathrm{cod}(r,G) = d+1$ and $\mathrm{rk}(\mathrm{copr}_G(p)) = 0$ by Lemma 6.5.3. Hence, every point of G' has a cogate at codistance $d+1$ in G. By Lemma 6.5.4(i) we conclude that G and G' are one-coparallel to each other. Since $\mathrm{crk}_{L\cap U^-}(\mathrm{copr}_L(x)) = 2$, the cogate of x in G has to be contained in L. Hence $\langle L \cap G, \mathrm{copr}_L(x) \rangle = \langle l, \mathrm{copr}_L(x) \rangle = L \cap U^-$ and we obtain $\langle G, \mathrm{copr}_L(x) \rangle_{\mathrm{g}} = U^-$ by Lemma 6.5.6.
Now consider the case $k = 2$. Let $p \in G' \setminus M'$. Then there is a point $q \in M'$ with $\mathrm{dist}(p,q) = 2$. Set $Y := \langle p,q \rangle_{\mathrm{g}}$. Then $Y \leq U^+$ and by Lemma 3.1.1(iii) M' and G' are generators of Y. Let $K \in \mathfrak{M}_1$ be the subspace that contains $\mathrm{copr}_{M'}(x')$. Then K contains a generator of Y by Lemma 3.1.1(iii). Suppose $\mathrm{cod}(x',K) = d+1$ and $\mathrm{copr}_K(x') < K$. Then $K \cap U^+ = \langle K \cap M', \mathrm{copr}_K(x') \rangle = \mathrm{copr}_K(x')$. Since $\mathrm{crk}_K(\mathrm{copr}_K(x')) = 2d+1$ by Lemma 6.5.3, this is a contradiction to $\mathrm{crk}_K(K \cap U^+) = n = 2d$. Now suppose $\mathrm{cod}(x',K) = d+1$ and $\mathrm{copr}_K(x') = K$. Then $\mathrm{rk}(K) = 2d+2$ by Lemma 6.5.3, a contradiction to $\mathrm{crk}_K(K \cap U^+) = 2d$ and

$\mathrm{rk}(K \cap U^+) \geq 3$. Thus, $\mathrm{cod}(x', K) = d+2$ and therefore $\mathrm{crk}_K(\mathrm{copr}_K(x')) = 2d+3$ by Lemma 6.5.3. Since $\mathrm{copr}_K(x') \leq U^+$ and $\mathrm{crk}_K(K \cap U^+) = 2d$, we obtain $\mathrm{crk}_{K \cap Y}(\mathrm{copr}_K(x') \cap Y) \leq 3$. Thus there is a point $z \in K \cap Y$ with $\mathrm{cod}(x', z) = d+2$. By Proposition 2.1.12(iv) we conclude that z is a cogate for x' in Y.
Now let $r \in M \setminus \{x'\}$. Suppose $\mathrm{copr}_{M'}(r) = \mathrm{copr}_{M'}(x')$. Then $\mathrm{copr}_{M'}(r') = \mathrm{copr}_{M'}(x')$ for every point point $r' \in rx'$ and hence $rx' \cap \mathrm{copr}_M(x) = \varnothing$, a contradiction to $\mathrm{rk}(\mathrm{copr}_M(x)) = 2$. Thus there is a point $s \in \mathrm{copr}_{M'}(x')$ with $\mathrm{cod}(s, r) = d$. Since $\mathrm{copr}_{M'}(x') = K \cap M' = \mathrm{pr}_{M'}(z)$, we conclude $\mathrm{cod}(z, r) = d+1$ and therefore $\mathrm{copr}_M(z) = \{x'\}$. Let $L \in \mathfrak{M}_1$ such that L intersects M in the hyperplane $\mathrm{copr}_M(x)$. Then $\mathrm{cod}(z, L) = d+1$ by Lemma 6.5.9. Hence by Lemma 6.5.3, this implies $\mathrm{crk}_L(\mathrm{copr}_L(z)) = 2d+1$. Since $\mathrm{crk}_L(L \cap U^-) = 2d$, there is a point $z' \in L \cap U^-$ with $\mathrm{cod}(z, z') = d$. We conclude $z' \not\perp x'$ and hence, $Y' := \langle x', z' \rangle_{\mathrm{g}}$ is a symplecton of U^-.
By Lemma 6.5.9 we know that $\mathrm{cod}(x, L) = d+2$ or $\mathrm{copr}_L(x) = L$ holds. Since $\mathrm{diam}(\mathscr{S}^-) > d+1$, Lemma 6.5.3 implies $\mathrm{cod}(x, L) = d+2$ and hence, $L \cap U^- = \langle L \cap M, \mathrm{copr}_L(x) \rangle$. Consequently, $\mathrm{copr}_M(x)$ and $\mathrm{copr}_L(x)$ are complements in $L \cap U^-$. Since $\langle z', \mathrm{copr}_M(x) \rangle$ is a generator of Y' that is contained in $L \cap U^-$, Proposition 2.1.12(iv) implies that x has a cogate at codistance $d+2$ in Y' that is contained in L. Since $Y = \langle x, z \rangle_{\mathrm{g}}$, we conclude by Corollary 4.2.8 that the symplecta Y and Y' are one-coparallel to each other at codistance $d+2$. Let $\tilde{G}$ be the generator of Y' that is one-coparallel to G' at codistance $d+2$. Since xy' and $x'y$ are one-coparallel at codistance $d+1$, we obtain $x'y \cap \tilde{G} = \varnothing$. Thus there is generator $G \leq \langle x', z' \rangle_{\mathrm{g}}$ with $x'y \leq G$ and $G \cap \tilde{G} = \varnothing$. We conclude that every point of G' has codistance $d+1$ to G and the its coprojection in G has rank 2. Since G' and $\tilde{G}$ are one-coparallel, we obtain $\tilde{G} \in \mathfrak{M}_0$ by Lemma 6.5.5. Thus, $G \in \mathfrak{M}_0$ by Proposition 2.2.8. Since $\mathrm{crk}_N(N \cap U^-) = 2d$ and $\mathrm{crk}_N(\mathrm{copr}_N(x)) = 2d+1$, we obtain $N \cap U^- = \langle x', \mathrm{copr}_N(x) \rangle$ and therefore $N \cap U^- = \langle G \cap N, \mathrm{copr}_N(x) \rangle$. Thus, $\langle G, \mathrm{copr}_N(x) \rangle_{\mathrm{g}} = U^-$ by Lemma 6.5.6. This concludes $V = U^+ \cup U^-$.
We know already $M \leq U^-$, $M' \leq U^+$ and $\mathrm{crk}_N(N \cap U^-) = \mathrm{crk}_{N'}(N' \cap U^+) = n$. Thus, (i) follows by Lemma 3.5.3(i). Claim (ii) holds since it holds for U^-. Now suppose there are points u and v in V with $\mathrm{cod}(u, v) = d-1$. Since $\mathrm{diam}(\mathscr{S}^-) \geq d+1$, there is a point $w \perp v$ with $\mathrm{cod}(u, w) = d$. This implies $w \in V$ by the coconvexity of V. By Lemma 3.1.1(i) and Proposition 3.5.2 there is a subspace $K \in \mathfrak{M}_0$ with $wv \leq K$ such that $K \leq V$. This implies $\mathrm{crk}_N(N \cap V) \in \{2d-3, 2d-2\}$ by (i), a contradiction. Thus, $\mathrm{codm}(V) = d$. Finally, we showed that for every point $u \in U^+$ there is a subspaces $M_u \in \mathfrak{M}_0$ with $M_u \leq V$ such that $\mathrm{cod}(u, M_u) = d+1$ and $\mathrm{rk}(\mathrm{copr}_{M_u}(u)) = k$. Now let $K \in \mathfrak{M}_0$ be an arbitrary subspace with $K \leq V$, $\mathrm{cod}(u, K) = d+1$ and $\mathrm{rk}(\mathrm{copr}_K(u)) = k$. Then $\langle u, K \rangle_{\mathrm{G}} \leq V$ and hence $\langle u, K \rangle_{\mathrm{G}} \cap \mathscr{S}^- = U^-$ by (i) and Lemma 6.5.6. Analogously, $\langle u, K \rangle_{\mathrm{G}} \cap \mathscr{S}^+ = U^+$. Thus, $\langle u, K \rangle_{\mathrm{G}} = V$ and (iv) follows by symmetric reasons. □

Lemma 6.5.12. *Let $M \in \mathfrak{M}_0$ and let $x \in \mathscr{S}$ such that* $\mathrm{cod}(x,M) < \infty$. *Then* $\mathrm{dist}(p, \langle x,M\rangle_{\mathrm{G}}) < \mathrm{cod}(x,M)$ *for every point $p \in \mathscr{S}$.*

Proof. Set $V := \langle x,M\rangle_{\mathrm{G}}$ and $d := \mathrm{cod}(x,M) - 1$. By symmetric reasons we may assume $p \in \mathscr{S}^-$. Moreover, by Proposition 6.5.11(iv) we may assume $M \leq \mathscr{S}^-$. By Lemma 6.5.3 there is a point $z \in M$ with $\mathrm{cod}(x,z) = d$. Set $n := \mathrm{dist}(p,z)$. We may assume $n > d$ since otherwise we are done. By Proposition 2.1.17(ii) we obtain $\mathrm{cod}(x, \langle p,z\rangle_{\mathrm{g}}) \geq n$. Thus by Proposition 2.1.16(ii) there is a point $z' \in \langle p,z\rangle_{\mathrm{g}}$ with $\mathrm{cod}(x,z') = n$ and $\mathrm{dist}(z,z') = n-d$. Since V is coconvex, we obtain $z' \in V$ and hence, $\langle z,z'\rangle_{\mathrm{g}} \leq V$. Now Proposition 2.1.17(i) yields $\mathrm{dist}(p,V) \leq d$. □

For a point $p \in \mathscr{P}^+$ and a subspace $M \in \mathfrak{M}_0$ the minimal codistance is 1. In this case, the coconvex span of them equals $\mathscr{S}$ as follows from Lemma 2.3.2. The next greater possible codistance $\mathrm{cod}(p,M) = 2$ and among the two possibilities the case $|\mathrm{copr}_M(p)| = 1$ can be seen as the lower codistance of p and M. The coconvex subspaces of such two objects play a special role. More precisely, the will be the points of a point-line space we construct out of $\mathscr{S}$.

Lemma 6.5.13. *Let $M \in \mathfrak{M}_0$ and let $x \in \mathscr{S}$ such that* $\mathrm{cod}(x,M) = 2$ *and* $\mathrm{copr}_M(x)$ *is a singleton. Set $V := \langle x,M\rangle_{\mathrm{G}}$.*

(i) *For every point $p \in \mathscr{S} \setminus V$ the subspace $\langle p, \mathrm{pr}_V(p)\rangle$ is an element of $\mathfrak{M}_1$.*

(ii) *Let $N \in \mathfrak{M}_1$. Then $N \cap V$ is either a singleton or a hyperplane of N.*

Proof. (i) By Lemma 6.5.12 we know $\mathrm{dist}(p,V) = 1$. Assume $V \cap \mathscr{S}^-$ is singular and hence equals M. We conclude $\mathrm{srk}(\mathscr{S}^-) = 3$ by Proposition 6.5.11(i). Thus, $\mathscr{S}^-$ is a symplecton by Theorem 3.5.4. The claim follows by Proposition 2.2.8 since $\langle p, \mathrm{pr}_M(p)\rangle$ is a generator of $\mathscr{S}^-$ that intersects M in a hyperplane.
Now assume $\mathrm{diam}(V \cap \mathscr{S}^-) \geq 2$. Then by Lemma 3.1.1(i) there is a singular subspace $S \leq V$ with $\mathrm{dist}(p,S) = 1$ and $\mathrm{rk}(S) = 2$. By Proposition 3.5.2 there is a subspace $L \in \mathfrak{M}_1$ with $S \leq L$. By Proposition 6.5.11(i) we obtain $\mathrm{crk}_L(L \cap V) = 1$. If $p \in L$ we are done. Thus, we may assume $p \notin L$ and hence $\mathrm{rk}(\mathrm{pr}_L(p)) = 2$ by Lemma 3.5.3(ii). This implies that there is a line $l \leq \mathrm{pr}_L(p) \cap V$. By Lemma 3.1.1(i) there is a symplecton $Y \leq V$ with $l \leq Y$. Hence, Proposition 2.1.27 provides that $\mathrm{pr}_Y(p)$ is a generator of Y. Let $K \in \mathfrak{M}$ be the subspace that contains $\langle p, \mathrm{pr}_Y(p)\rangle$. Then $K \in \mathfrak{M}_1$ since $\mathrm{rk}(\langle p, \mathrm{pr}_Y(p)\rangle) = 4$. By Proposition 6.5.11(i) we obtain $\mathrm{crk}_K(K \cap V) = 1$ since $\mathrm{pr}_Y(p) \leq K \cap V$. Since $\mathrm{dist}(p,V) = 1$, we conclude that $\mathrm{pr}_Y(p)$ is singular. Thus, $K \cap V = \mathrm{pr}_V(p)$.
(ii) By Proposition 6.5.11(i) we know that there is a point $p \in N \setminus V$. Then $L := \langle p, \mathrm{pr}_V(p)\rangle$ is an element of $\mathfrak{M}_1$ by (i). If $L = N$ we are done, hence we assume $L \neq N$. Then by Lemma 3.5.3(ii) $L \cap N$ is a line. Since $L \cap V = \mathrm{pr}_V(p)$ is a hyperplane of L, there is a point $q \in V$ such that $L \cap N = pq$. Since L is maximal, there is for every point $x \in N \setminus L$ a point $y \in L$ with $\mathrm{dist}(x,y) = 2$. Since $y \neq p$ and

py meets $L \cap V$, we may assume $y \in V$. Since $p \in \langle x,y \rangle_g \setminus V$, we conclude $x \notin V$ and therefore $N \cap V = \{q\}$. □

Together with Proposition 6.5.11(i) and Lemmas 6.4.10 and 2.3.2 we conclude that the coconvex subspace of a point $p \in \mathscr{P}^+$ and a subspace $M \in \mathfrak{M}_0^-$ such that p has a cogate at codistance 2 in M are maximal coconvex proper subspace of $\mathscr{S}$. In the following proposition we consider the next smaller coconvex subspaces. These subspaces will induce the lines of the point-line space we are going to construct.

Proposition 6.5.14. *For $i \in \{0,1\}$, let $M_i \in \mathfrak{M}_0$ be a subspace and let $x_i \in \mathscr{S}$ be a point such that* $\mathrm{cod}(x_i,M_i) = 2$ *and* $\mathrm{rk}(\mathrm{copr}_{M_i}(x_i)) = 0$. *Set $V_i := \langle x_i, M_i \rangle_G$. Let $V_0 \cap V_1 \neq \varnothing$ and $V_0 \neq V_1$. If* $\mathrm{srk}(\mathscr{S}) = 3$, *then $V_0 \cap V_1$ consists of two one-coparallel lines at codistance* 2. *Otherwise, there is a point x and a subspace $M \in \mathfrak{M}_0$ with* $\mathrm{cod}(x,M) = 2$ *and* $\mathrm{rk}(\mathrm{copr}_M(x)) = 2$ *such that* $V_0 \cap V_1 = \langle x, M \rangle_G$.

Proof. Let $x \in V_0 \cap V_1 \neq \varnothing$. By Proposition 6.5.11(iv) we may assume $x = x_0 = x_1$. Since $V_0 \neq V_1$, Proposition 6.5.11(iv) implies $M_0 \not\leq V_1$. Since V_1 is coconvex, there is a point $p \in M_0 \setminus V_1$ with $\mathrm{cod}(p,x) = 1$. By Lemma 6.5.13(i) there is a subspace $N \in \mathfrak{M}_1$ such that $\langle p, \mathrm{pr}_{V_1}(p) \rangle = N$. By Proposition 6.5.11(iii) there is no point in $N \cap V_1$ opposite x. Since $p \not\leftrightarrow x$, there is no point in N opposite x. By Lemma 6.5.3 this implies $\mathrm{cod}(x,N) = 2$ and $\mathrm{crk}_N(\mathrm{copr}_N(x)) = 3$. By the coconvexity of V_0 and V_1 we obtain $\mathrm{copr}_N(x) \leq V_0 \cap V_1$. Thus, $\mathscr{S}^- \cap V_0 \cap V_1$ and $\mathscr{S}^+ \cap V_0 \cap V_1$ are both non-empty and we may assume $x \in \mathscr{S}^+$.
Consider the case $\mathrm{srk}(\mathscr{S}) = 3$. Then $\mathscr{S}^+$ and $\mathscr{S}^-$ are both symplecta and x has a cogate x' in $\mathscr{S}^-$. Moreover, M_0 and M_1 are both generators of $\mathscr{S}^-$ and $\mathrm{copr}_N(x) = \{x'\}$. Since $V_0 \cap \mathscr{S}^-$ is convex, we know $V_0 \cap \mathscr{S}^- = M_0$. Analogously, $V_1 \cap \mathscr{S}^- = M_1$. Thus, we conclude by Proposition 2.2.8 that $M_0 \cap M_1$ is a line since $M_0 \not\leq V_1$ and $x' \in M_0 \cap M_1$. By symmetric reasons $\mathscr{S}^+ \cap V_0 \cap V_1$ is a line, too. For every point on $\mathscr{S}^+ \cap V_0 \cap V_1$, we conclude analogously to x that its cogate in $\mathscr{S}^-$ is contained in $V_0 \cap V_1$. Hence, $\mathscr{S}^+ \cap V_0 \cap V_1$ and $\mathscr{S}^- \cap V_0 \cap V_1$ are one-coparallel lines at codistance 2.
Now consider the case $\mathrm{srk}(\mathscr{S}) \geq 4$. Then $\mathrm{rk}(N) \geq 4$ by Lemma 3.5.3(i) and hence, $\mathrm{copr}_N(x)$ contains a line l. Since $\langle p, \mathrm{copr}_N(x) \rangle \leq V_0$, we obtain $\mathrm{copr}_N(N \cap V_0) = 1$ by Proposition 6.5.11(i). Since $p \in V_0 \setminus V_1$, this implies $\mathrm{crk}_N(N \cap V_0 \cap V_1) = 2$. Because of $\mathrm{crk}_N(\mathrm{copr}_N(x)) = 3$ there is a point $y \in N \cap V_0 \cap V_1$ with $\mathrm{cod}(x,y) = 1$. By Proposition 3.5.2 there is a subspace $M \in \mathfrak{M}_0$ with $\langle y, l \rangle \leq M$. Since $\langle y, l \rangle \leq V_0 \cap V_1$, Lemma 3.1.1(iii) implies $M \leq V_0 \cap V_1$ and hence, $\langle x, M \rangle_G \leq V_0 \cap V_1$.
Since $\mathrm{cod}(x,y) = 1$ and $l \leq \mathrm{copr}_N(x)$, we obtain $\mathrm{cod}(x,M) = 2$ and consequently, $\mathrm{rk}(\mathrm{copr}_M(x)) = 2$ by Lemma 6.5.3. Hence by Proposition 6.5.11(i), we obtain $\mathrm{crk}_N(N \cap \langle x, M \rangle_G) = 2$. This implies $N \cap \langle x, M \rangle_G = N \cap V_0 \cap V_1$. Together with

$M \leq \langle x, M\rangle_{\mathrm{G}}$ we conclude $\langle x,M\rangle_{\mathrm{G}} \cap \mathscr{S}^- = V_0 \cap V_1 \cap \mathscr{S}^-$ by Lemma 6.5.6. For every point $x' \in M$, there are subspace $M' \in \mathfrak{M}_0^+$ and $M_0' \in \mathfrak{M}_0^-$ such that $\langle x', M'\rangle_{\mathrm{G}} = \langle x, M\rangle_{\mathrm{G}}$ and $\langle x', M_0'\rangle_{\mathrm{G}} = V_0$ by Proposition 6.5.11(iv). Since $V_0 \neq V_1$, we conclude $M_0' \not\leq V_1$ by Proposition 6.5.11(iv) and hence, $V_0 \cap V_1 \cap \mathscr{S}^+ < V_0 \cap \mathscr{S}^+$. Let $N' \in \mathfrak{M}_1^+$ such that N' contains a hyperplane of M'. Since $M' \leq V_0 \cap V_1 \cap \mathscr{S}^+$, Proposition 6.5.11(i) implies $\operatorname{crk}_{N'}(N' \cap V_0) = 1$. Hence, we conclude $\operatorname{crk}_{N'}(N' \cap V_0 \cap V_1) \geq 2$ by Lemma 6.5.6. Since $\operatorname{crk}_{N'}(N' \cap \langle x, M\rangle_{\mathrm{G}}) = 2$ by Proposition 6.5.11(i) and $\langle x, M\rangle_{\mathrm{G}} \leq V_0 \cap V_1$, Lemma 6.5.6 $\langle x, M\rangle_{\mathrm{G}} \cap \mathscr{S}^+ = V_0 \cap V_1 \cap \mathscr{S}^+$. $\square$

Motivated by this proposition we define the following two sets:

$$\mathscr{P}_{\mathrm{m}} := \left\{ \langle x, M\rangle_{\mathrm{G}} \mid (x,M) \in \mathscr{S}^+ \times \mathfrak{M}_0^- \wedge \operatorname{cod}(x,M) = 2 \wedge |\operatorname{copr}_M(x)| = 1 \right\}$$
$$\mathscr{L}_{\mathrm{m}} := \left\{ \{P \in \mathscr{P}_{\mathrm{m}} \mid U \cap V \leq P\} \,\middle|\, \{U,V\} \subseteq \mathscr{P}_{\mathrm{m}} \wedge \varnothing \neq U \cap V < U \right\}$$

By the definition of $\mathscr{L}_{\mathrm{m}}$ the pair $(\mathscr{P}_{\mathrm{m}}, \mathscr{L}_{\mathrm{m}})$ is a point-line space which in the following will be denoted by $\mathscr{S}_{\mathrm{m}}$.

Lemma 6.5.15. *Let U and V be elements of $\mathscr{P}_{\mathrm{m}}$ with $U \cap V \neq \varnothing$. Further let $N \in \mathfrak{M}_1$. Then $\operatorname{rk}(N \cap U) \neq \operatorname{rk}(N \cap V)$ implies $N \cap U < V$ or $N \cap V < U$.*

Proof. By Lemma 6.5.13(ii) and symmetric reasons if suffices to consider the case $\operatorname{crk}_N(U \cap N) = 1$ and $\operatorname{rk}(V \cap N) = 0$. Let $p \in N \setminus U$. Then $N = \langle p, \operatorname{pr}_U(p)\rangle$ by Lemma 6.5.13(i). By Proposition 6.5.14 and Lemma 6.5.12 we obtain $\operatorname{dist}(p, U \cap V) = 1$. Thus, $\operatorname{pr}_U(p) \cap V \neq \varnothing$. Let $q \in \operatorname{pr}_U(p) \cap V$. Then $q \in N$ and hence, $V \cap N = \{q\}$. $\square$

Proposition 6.5.16. *Let $U \in \mathscr{P}_{\mathrm{m}}$ and let x be a point with $x \notin U$. Then there is a subspace $V \in \mathscr{P}_{\mathrm{m}}$ such that $x \in V$ and $U \cap V = \varnothing$.*

Proof. By Proposition 6.5.13(i) there exists a subspace $N \in \mathfrak{M}_1$ such that $N = \langle x, \operatorname{pr}_U(x)\rangle$. Let $G \leq N$ be a subspace with $\operatorname{rk}(G) = 3$ and $x \in G$. By Lemma 3.1.1(i) there is a symplecton $Y \leq \mathscr{S}$ such that G is a generator of Y. Let M be a generator of Y such that $G \cap M = \{x\}$. Then Proposition 2.2.8 implies $M \in \mathfrak{M}_0$. Let Z be a symplecton that is opposite Y and let $z \in Z$ such that $\operatorname{cod}(z, M) = 2$. Then $V := \langle z, M\rangle_{\mathrm{G}} \in \mathscr{P}_{\mathrm{m}}$. Since Y contains a point opposite z, we obtain $Y \not\leq V$. Thus, $Y \cap V = M$ and hence, $V \cap N = \{x\}$ by Lemma 6.5.13(ii). Therefore we conclude $U \cap V = \varnothing$ by Lemma 6.5.15. $\square$

Lemma 6.5.17. *Let V and W be distinct elements of $\mathscr{P}_{\mathrm{m}}$ with $S := V \cap W \neq \varnothing$. Let p be a point such that $\operatorname{pr}_S(p)$ contains a line. Then there is a unique element $U \in \mathscr{P}_{\mathrm{m}}$ that contains S and p. Moreover, if $\operatorname{srk}(\mathscr{S}) > 3$, then $\langle p, S\rangle_{\mathrm{G}} = U$.*

Proof. Let $l \leq \mathrm{pr}_S(p)$ be a line. By Proposition 6.5.14 and Lemma 6.5.12 we obtain $\mathrm{dist}(p,S) = 1$ and hence $\langle p,l\rangle$ is a singular space of rank 2. By symmetric reasons we may assume $l \leq \mathscr{S}^-$. First consider the case $\mathrm{srk}(\mathscr{S}) = 3$. Then $\mathscr{S}^+$ and $\mathscr{S}^-$ are opposite symplecta and every element of $\mathscr{P}_\mathrm{m}$ consists of two elements of $\mathfrak{M}_0$ that are one-coparallel to each other at codistance 2. Moreover, by Proposition 6.5.14 S consists of two one-coparallel lines at codistance 2. By Proposition 3.5.2 there is a unique subspace $M \in \mathfrak{M}_0$ with $\langle p,l\rangle \leq M$. For an arbitrary point $x \in \mathscr{S}^+$ with $\mathrm{cod}(x,M) = 2$, the subspace $\langle x,M\rangle_\mathrm{G}$ consists of M and the unique generator of $\mathscr{S}^+$ that is one-coparallel to M at codistance 2. Thus, $S \leq \langle x,M\rangle_\mathrm{G}$ and $\langle x,M\rangle_\mathrm{G}$ is unique.
Now consider the case $\mathrm{srk}(\mathscr{S}) > 3$. Let $y \in l$. Then by Propositions 6.5.14 and 6.5.11(iv) there is a subspace $N \in \mathfrak{M}_0$ with $\mathrm{cod}(y,N) = 2$ and $\mathrm{rk}(\mathrm{copr}_N(y)) = 2$ such that $\langle y,N\rangle_\mathrm{G} = S$. Let $x \in N \setminus \mathrm{copr}_N(y)$ and let $L \in \mathfrak{M}_1$ such that $\langle p,l\rangle \leq L$. By Lemma 6.5.13(ii) and Proposition 6.5.11(i) both V and W contain a hyperplane of L. Thus, $\mathrm{crk}_L(L\cap S) \leq 2$ and since $\mathrm{rk}(L) > 3$ this implies $\mathrm{rk}(L\cap S) \geq 2$. By Proposition 6.5.11(i) we conclude $\mathrm{crk}_L(L\cap S) = 2$. Let H be a subspace of $L\cap S$ with $\mathrm{rk}(H) = 2$ and $l \leq H$. Further let $M \in \mathfrak{M}_0$ with $H \leq M$. Then $M \leq S$ by Lemma 3.1.1(iii). Since $H \leq \langle x,M\rangle_\mathrm{G}$, Proposition 6.5.11(i) implies $\mathrm{crk}_L(L\cap\langle x,M\rangle_\mathrm{G}) = 2\cdot\mathrm{cod}(x,M) + \frac{1}{2}\mathrm{rk}(\mathrm{copr}_M(x)) - 3$. Since $\langle x,M\rangle_\mathrm{G} \leq S$ and $\mathrm{crk}_L(L\cap S) = 2$, this implies $\mathrm{cod}(x,M) > 1$. Moreover, since $\mathrm{cod}(x,y) = 1$, we conclude $\mathrm{cod}(x,M) = 2$ and $\mathrm{rk}(\mathrm{copr}_M(x)) = 2$. Hence, $S = \langle x,M\rangle_\mathrm{G}$ by Proposition 6.5.11(iv).
Since $\mathrm{rk}(\mathrm{copr}_M(x)) = 2$, we obtain $\mathrm{cod}(x,H) = 2$ and hence $\mathrm{cod}(x,L) \geq 2$. With $\mathrm{cod}(x,y) = 1$, we conclude $\mathrm{crk}_L(\mathrm{copr}_L(x)) = 3$ by Lemma 6.5.3. Since $\mathrm{crk}_L(L\cap S) = 2$ and $p \notin S$, we know that $\langle p,L\cap S\rangle$ is a hyperplane of L. Thus $\langle p,L\cap S\rangle$ contains a subspace H' with $\mathrm{rk}(H') = 2$ that intersects $\mathrm{copr}_L(x)$ is a singleton. Let $M' \in \mathfrak{M}_0$ with $H' \leq M'$. Then $\mathrm{cod}(x,M') = 2$ and $\mathrm{copr}_{M'}(x)$ is a singleton by Lemma 6.5.3. Therefore, $U := \langle x,M'\rangle_\mathrm{G} \in \mathscr{P}_\mathrm{m}$. By Proposition 6.5.11(ii) we obtain $\mathrm{copr}_L(x) \leq S$ and $\langle H',\mathrm{copr}_L(x)\rangle = L\cap U$. Since $H' \leq \langle p,L\cap S\rangle$ and $\mathrm{crk}_L(L\cap U) = 1$ by Proposition 6.5.11(i), this implies $\langle H',\mathrm{copr}_L(x)\rangle = \langle p,L\cap S\rangle$. Thus, $H \leq U$ and Lemma 3.1.1(iii) implies $M \leq U$. We conclude $S = \langle x,M\rangle_\mathrm{G} \leq U$ and consequently, $\langle p,S\rangle_\mathrm{G} \leq U$. Since $H' \leq \langle p,L\cap S\rangle \leq \langle p,S\rangle_\mathrm{G}$, Lemma 3.1.1(iii) implies $M' \leq \langle p,S\rangle_\mathrm{G}$ and therefore $\langle p,S\rangle_\mathrm{G} = U$. This proves the uniqueness of U. □

Theorem 6.5.18. *The point-line space $\mathscr{S}_\mathrm{m}$ is a non-degenerate polar space.*

Proof. We show that $\mathscr{S}_\mathrm{m}$ fulfils the Buekenhout-Shult Axiom (BS). Let $U \in \mathscr{P}_\mathrm{m}$ and let $\Lambda \in \mathscr{L}_\mathrm{m}$. Further let V and W be distinct but not disjoint elements of $\mathscr{P}_\mathrm{m}$ such that $\Lambda = \{P \in \mathscr{P}_\mathrm{m} \mid P \geq V\cap W\}$. Set $S := V\cap W$. By Proposition 6.5.14 there is a line $g \leq S$. Since g is not a maximal singular subspace of $\mathscr{S}$, Proposition 3.5.2 implies that there is subspace $N \in \mathfrak{M}_1$ with $g \leq N$. By Lemma 6.5.13(ii) N

contains a point p of U. We may assume $p \notin S$ since otherwise U has non-empty with every element of Λ and we are done. Thus, $g \leq \mathrm{pr}_S(p)$ and Lemma 6.5.17 implies that there is a unique element in Λ that has an non-empty intersection with U.
By Lemma 6.5.13(ii) both V and W contain a hyperplane of N. By Proposition 6.5.14 this implies $\mathrm{crk}_N(N \cap S) = 2$. Moreover, for every two distinct elements V' and W' of Λ, we obtain $N \cap V' \cap W' = N \cap S$. Assume V' and W' have both non-empty intersection with U. By Lemma 6.5.13(ii) $U \cap N$ is a hyperplane of N or a singleton. In the first case we obtain $U \cap g \neq \varnothing$. Hence, $U \cap S \neq \varnothing$ and U intersects every element of Λ. In the second case we conclude by Lemma 6.5.15 that $U \cap N$ is contained in both V' and W'. Hence, $U \cap N \leq S$ and again every element of Λ has non-empty intersection with U.
By Proposition 6.5.11(iii) we obtain $U < \mathscr{S}$. Hence, there is a point $q \in \mathscr{S} \setminus U$. Thus by Proposition 6.5.16, there is a subspace $U' \in \mathscr{P}_{\mathrm{m}}$ with $U \cap U' = \varnothing$. Therefore $\mathscr{S}_{\mathrm{m}}$ is non-degenerate. □

Our goal is to prove that $\mathscr{S}$ is a twin half-spin space of $\mathscr{S}_{\mathrm{m}}$. Therefore we show some correspondences between subspaces of $\mathscr{S}$ and subspaces of $\mathscr{S}_{\mathrm{m}}$. For a point $p \in \mathscr{S}$, we set $\Gamma(p) := \{U \in \mathscr{P}_{\mathrm{m}} \mid p \in U\}$. For a subspace $N \in \mathfrak{M}_1$, we set $\Gamma(N) := \{U \in \mathscr{P}_{\mathrm{m}} \mid \mathrm{rk}(N \cap U) \geq 2\}$.

Proposition 6.5.19. *Let p be a point of $\mathscr{S}$ and let N be a subspace with $N \in \mathfrak{M}_1$.*

(i) *If $\mathrm{cod}(p,N) = 1$, then $\Gamma(p) \cap \Gamma(N)$ contains a single element.*

(ii) *Both $\Gamma(p)$ and $\Gamma(N)$ are generators of $\mathscr{S}_{\mathrm{m}}$.*

(iii) *If $p \in N$, then the generators $\Gamma(N)$ and $\Gamma(p)$ of $\mathscr{S}_{\mathrm{m}}$ intersect in a common hyperplane. Moreover, $\Gamma(p)$ and $\Gamma(N)$ are the only generators of $\mathscr{S}_{\mathrm{m}}$ that contain $\Gamma(p) \cap \Gamma(N)$.*

Proof. (i) By Lemma 6.5.3 we know that $\mathrm{copr}_N(p)$ is a hyperplane of N. Let $H \leq \mathrm{copr}_N(p)$ and let $M \in \mathfrak{M}_0$ such that $H \leq M$. Then Lemma 6.5.9 implies $\mathrm{cod}(p,M) = 2$. We conclude that $\mathrm{copr}_M(p)$ is a singleton and thus, $\langle p,M\rangle_{\mathrm{G}} \in \mathscr{P}_{\mathrm{m}}$. Since p and H are contained in $\langle p,M\rangle_{\mathrm{G}}$, we obtain $\langle p,M\rangle_{\mathrm{G}} \in \Gamma(p) \cap \Gamma(N)$.
Now let $P \in \Gamma(p) \cap \Gamma(N)$. Proposition 6.5.11(i) implies that P contains a hyperplane of N. By Proposition 6.5.11(iii) this hyperplane has to be $\mathrm{copr}_N(p)$. Thus, $H \leq P$ and we obtain $M \leq P$ by Proposition 6.5.11(i). Therefore $R = \langle p,M\rangle_{\mathrm{G}}$ by Proposition 6.5.11(iv).
(ii) Let P and Q be two distinct elements of $\Gamma(p)$. Then $P \cap Q \neq \varnothing$ since both contain p. Moreover, every element of $\mathscr{P}_{\mathrm{m}}$ that contains $P \cap Q$ is an element of $\Gamma(p)$. Thus, $\Gamma(p)$ is a singular subspace of $\mathscr{S}_{\mathrm{m}}$. Now let $R \in \mathscr{P}_{\mathrm{m}} \setminus \Gamma(p)$. Then by Proposition 6.5.16 there is element of $\Gamma(p)$ that is disjoint to R. Hence, $\Gamma(p)$ is a maximal singular subspace.

Let $N \in \mathfrak{M}_1$. Further let P and Q be two distinct elements of $\Gamma(N)$. Then $\operatorname{crk}_N(N \cap P \cap Q) \leq 2$. Since $\operatorname{rk}(N) \geq 3$, this implies $P \cap Q \neq \varnothing$. Now let $R \in \mathscr{P}_\mathrm{m}$ such that $R \geq P \cap Q$. Since $N \cap P \cap Q$ contains a line, we conclude by Lemma 6.5.13(ii) that N contains a hyperplane of R. Thus, $R \in \Gamma(N)$ and consequently $\Gamma(N)$ is a singular subspace of $\mathscr{S}_\mathrm{m}$.
Now let $R \in \mathscr{P}_\mathrm{m} \setminus \Gamma(N)$. Then by Lemma 6.5.13(ii) there is a point $y \in N$ such that $R \cap N = \{y\}$. Let x be a point that is opposite y. By (i) there is an element $R' \in \Gamma(x) \cap \Gamma(N)$. By Proposition 6.5.11(iii) we obtain $y \notin R'$ and hence, $R' \cap R = \varnothing$ by Lemma 6.5.15. We conclude that $\Gamma(N)$ is a maximal singular subspace.
(iii) Let U and V be two distinct elements of $\Gamma(N)$. Set $S := U \cap V$. Since both U and V contain a hyperplane of N, we obtain $\operatorname{crk}_S(S \cap N) = 2$ by Proposition 6.5.14. By Lemma 6.5.17 this implies that there is an element of $\mathscr{P}_\mathrm{m}$ that contains S and p. Thus, $\Gamma(p)$ contains a hyperplane of $\Gamma(N)$.
Now let q be a point opposite p. By Proposition 6.5.11(iii) we conclude that $\Gamma(p)$ and $\Gamma(q)$ are disjoint. On the other hand, $\operatorname{cod}(q,N) = 1$ and hence, $\Gamma(q)$ and $\Gamma(N)$ are not disjoint. Hence, $\Gamma(p) \cap \Gamma(N)$ is a hyperplane of $\Gamma(N)$ and by Lemma A.2.13 it is also a hyperplane of $\Gamma(p)$
Let $W \in \mathscr{P}_\mathrm{m}$ such that W has non-empty intersection with every element of $\Gamma(p) \cap \Gamma(N)$. Suppose $W \notin \Gamma(p) \cup \Gamma(N)$. Then W intersects N in a single point, q say, that is distinct to p. By Lemma 2.1.13 there is a point x with $x \leftrightarrow q$ and $\operatorname{cod}(x,p) = 1$. By (i) there is an element $P \in \Gamma(x) \cup \Gamma(N)$. Since by Proposition 6.5.11(i) P contains a hyperplane of N, Proposition 6.5.11(iii) implies $p \in P$. Thus, $P \in \Gamma(p) \cap \Gamma(N)$. Again by Proposition 6.5.11(iii) we obtain $q \notin P$ and hence, Lemma 6.5.15 implies $W \cap P = \varnothing$, a contradiction. Therefore $W \in \Gamma(p)$ or $W \in \Gamma(N)$. ☐

Lemma 6.5.20. *Let p be a point of $\mathscr{S}$ and let $N \in \mathfrak{M}_1$. Every generator of $\mathscr{S}_\mathrm{m}$ that intersects $\Gamma(N)$ in a hyperplane is of the kind $\Gamma(q)$ for a point $q \in N$. Every generator of $\mathscr{S}_\mathrm{m}$ that intersects $\Gamma(p)$ in a hyperplane is of the kind $\Gamma(L)$ for a subspace $L \in \mathfrak{M}_1$ with $p \in L$.*

Proof. Let Θ be generator of $\mathscr{S}_\mathrm{m}$ that intersects $\Gamma(N)$ in a hyperplane. Then there is an element $U \in \mathscr{P}_\mathrm{m}$ with $U \in \Theta \notin \Gamma(N)$. By Lemma 6.5.13(ii) this implies that U intersects N in a single point q. By Lemma 6.5.15 we know that every element of $\Gamma(N)$ that has non-empty intersection with U contains q and hence $\Theta \cap \Gamma(N) \leq \Gamma(q)$. Since $\Theta \neq \Gamma(N)$, Proposition 6.5.19(iii) implies $\Theta = \Gamma(q)$.
Now let Θ be generator of $\mathscr{S}_\mathrm{m}$ that intersects $\Gamma(p)$ in a hyperplane. Then there is an element $V \in \mathscr{P}_\mathrm{m}$ with $V \in \Theta \notin \Gamma(p)$. Then by Lemma 6.5.13(i) there is a subspace $L \in \mathfrak{M}_1$ such that $L = \langle p, \operatorname{pr}_V(p) \rangle$. By Lemma 6.5.15 we know that every element of $\Gamma(p)$ that intersects L in the single point p is disjoint to V. Thus by Lemma 6.5.13(ii) every element of $\Gamma(p)$ that has non-empty intersection with V

is an element of $\Gamma(L)$. Therefore, $\Theta \cap \Gamma(p) \leq \Gamma(L)$. Since $\Theta \neq \Gamma(p)$, Proposition 6.5.19(iii) implies $\Theta = \Gamma(L)$. □

Corollary 6.5.21. *Let p be a point of $\mathscr{S}^+$. Let Δ be the connected component of the dual polar graph of $\mathscr{S}_{\mathrm{m}}$ that contains $\Gamma(p)$. Then Δ is bipartite. Moreover, every edge of Δ is of the form $\{\Gamma(q), \Gamma(N)\}$, where q is a point of $\mathscr{S}^+$ and $N \in \mathfrak{M}^+$.*

Proof. This is a direct consequence of Lemma 6.5.20. □

Proposition 6.5.22. *Let $(p,q) \in \mathscr{S}^+ \times \mathscr{S}^-$ be a pair of opposite points. Then $(\Gamma(p), \Gamma(q))$ is a spanning pair of $\mathscr{S}_{\mathrm{m}}$.*

Proof. Let $R \in \mathscr{P}_{\mathrm{m}} \setminus (\Gamma(p) \cup \Gamma(q))$. By Proposition 5.2.4 we have to show that there are elements $P \in \Gamma(p)$ and $Q \in \Gamma(q)$ with the following properties:

$$R \cap X \neq \varnothing \Leftrightarrow P \cap X \neq \varnothing \qquad \text{for every } X \in \Gamma(q)$$
$$R \cap X \neq \varnothing \Leftrightarrow Q \cap X \neq \varnothing \qquad \text{for every } X \in \Gamma(p)$$

By symmetric reasons it suffices to show that such a P exists. Since $q \notin R$, Lemma 6.5.13(i) implies that there is a subspace $N \in \mathfrak{M}_1$ such that $N = \langle q, \mathrm{pr}_R(q) \rangle$. By Proposition 6.5.19(i) there is an element $P \in \mathscr{P}_{\mathrm{m}}$ in $\Gamma(p) \cap \Gamma(N)$.
Let $X \in \Gamma(q)$ with $X \cap R \neq \varnothing$. We know that $R \cap N = \mathrm{pr}_R(q)$ is a hyperplane of N that does not contain q. Since $q \in X$, we conclude by Lemma 6.5.15 $\mathrm{rk}(X \cap N) = \mathrm{rk}(R \cap N)$ and hence, $X \in \Gamma(N)$. Since $\Gamma(N)$ is a singular subspace of $\mathscr{S}_{\mathrm{m}}$ that contains P, we conclude that $X \cap P \neq \varnothing$.
Now let $X \in \Gamma(q)$ with $X \cap R = \varnothing$. Then $X \cap N = \{q\}$ since $R \cap N$ is hyperplane of N. Since $p \leftrightarrow q$, we obtain $q \notin P$ by Proposition 6.5.11(iii). Since $P \in \Gamma(N)$, Lemma 6.5.15 implies $X \cap P = \varnothing$. □

We now prove the main result of this section.

Theorem 6.5.23. *Let $\mathscr{S}$ be a twin SPO space satisfying the following two properties:*

- **(T4a)** *Every symplecton of $\mathscr{S}$ is of rank 4.*
- **(T4b)** *Every singular subspace of rank 2 is contained in a maximal singular subspace of rank 3 and in at most one other maximal singular subspace.*

Then $\mathscr{S}$ is a twin half-spin space.

Proof. We denote the two connected components of $\mathscr{S}$ by $\mathscr{S}^+ = (\mathscr{P}^+, \mathscr{L}^+)$ and $\mathscr{S}^- = (\mathscr{P}^-, \mathscr{L}^-)$.
First assume $\mathrm{diam}(\mathscr{S}^+) < 2$ and hence, $\mathrm{diam}(\mathscr{S}^-) < 2$. Then $\mathscr{S}^+$ is a projective

space by Theorem 2.1.22. If $\mathscr{S}^+$ is a singleton, then $\mathscr{S}^-$ is a singleton, too. Moreover, $\mathscr{S}$ is isomorphic to the twin half-spin space of the polar space that consists of two points and no lines. If $\mathscr{S}^+$ is a line, then $\mathscr{S}^-$ is a line that is one-coparallel to $\mathscr{S}^+$. In this case $\mathscr{S}$ is isomorphic to every twin half-spin space of the polar space $\mathscr{S}^+ \otimes \mathscr{S}^+$.
Now let $\mathscr{S}^+$ be a projective space of rank ≥ 2. Then (T4b) yields $\mathrm{rk}(\mathscr{S}^+) = 3$. Hence, $\mathscr{S}^-$ is a singular space of rank 3, too. We set:

$$\psi\colon \quad \mathscr{L}^+ \to \mathscr{L}^-\colon \quad l \mapsto \bigcap_{p\in l} \mathrm{copr}_{\mathscr{S}^-}(p)$$

$$\mathscr{P}_{\mathrm{m}} := \left\{ l \cup l^{\psi} \mid l \in \mathscr{L}^+ \right\}$$

$$\mathscr{L}_{\mathrm{m}} := \left\{ \{P \in \mathscr{P}_{\mathrm{m}} \mid U \cap V \leq P\} \;\middle|\; \{U,V\} \subseteq \mathscr{P}_{\mathrm{m}} \wedge \varnothing \neq U \cap V < U \right\}$$

By Proposition 6.2.3 the pair $(\mathscr{P}_{\mathrm{m}}, \mathscr{L}_{\mathrm{m}})$ is isomorphic to the Grassmannian of lines of $\mathscr{S}^+$ that we denote by $\mathscr{S}_{\mathrm{m}}$. Let l be a line of $\mathscr{S}_{\mathrm{m}}$ and let g and h be two lines of $\mathscr{S}^+$ with $\{g,h\} \subseteq l$. By Propositions 5.3.11 and 5.3.16 we conclude that l is contained in exactly two maximal singular subspaces of $\mathscr{S}_{\mathrm{m}}$ that are both of rank 2. Moreover, one of these maximal singular subspaces consists of all lines of $\mathscr{S}^+$ through the intersection point of g and h. The other maximal singular subspaces consists of all lines of $\mathscr{S}^+$ contained in $\langle g,h\rangle$. By Theorem 5.3.15 and Lemma 3.3.1(i) we conclude that $\mathscr{S}_{\mathrm{m}}$ is a symplecton. More precisely, by Corollary 2.1.18 we know that $\mathscr{S}_{\mathrm{m}}$ is a non-degenerate polar space of rank 3 whose lines are contained in exactly two generators. Hence, by Proposition 2.2.8 the dual polar graph of $\mathscr{S}_{\mathrm{m}}$ is bipartite. More precisely, for two distinct adjacent generators of $\mathscr{S}_{\mathrm{m}}$, one of them consists of all the lines through a given point of $\mathscr{S}^+$ and the other one consists of all the lines in a given hyperplane of $\mathscr{S}^+$. Since $\mathscr{S}_{\mathrm{m}}$ have finite rank, $\mathscr{S}_{\mathrm{m}}$ contains a spanning pair and hence, there exists a twin half-spin space $(\mathscr{D}^+, \mathscr{D}^-)$ of $\mathscr{S}_{\mathrm{m}}$. Since by Proposition A.2.20 two disjoint generators of $\mathscr{S}_{\mathrm{m}}$ have distance 3 in the dual polar graph, we may assume that $\mathscr{D}^+$ contains all the generators of $\mathscr{S}_{\mathrm{m}}$ that contains all lines of $\mathscr{S}^+$ through a given point and $\mathscr{D}^-$ contains all the generators of $\mathscr{S}_{\mathrm{m}}$ that contains all lines of a given hyperplane of $\mathscr{S}^+$.
We define the following map:

$$\varphi\colon\ (\mathscr{S}^+, \mathscr{S}^-) \to (\mathscr{D}^+, \mathscr{D}^-)\colon\ p \mapsto \begin{cases} \{l \in \mathscr{L}^+ \mid p \in l\} & \text{if } p \in \mathscr{P}^+ \\ \{l \in \mathscr{L}^+ \mid l \leq \mathrm{copr}_{\mathscr{S}^+}(p)\} & \text{if } p \in \mathscr{P}^- \end{cases}$$

By Lemma 6.2.1 every hyperplane of $\mathscr{S}^+$ is the coprojection of a point of $\mathscr{S}^-$ and for two distinct points of $\mathscr{S}^-$ the coprojections in $\mathscr{S}^+$ are distinct, we conclude that φ is a bijection. Let $l \in \mathscr{L}^+$. Then for two distinct points p and q on l, we obtain $p^{\varphi} \cap q^{\varphi} = \{l\}$. Since for a point $r \in \mathscr{P}^+$, we obtain $l \leq r^{\varphi}$ if and only if $r \in l$, we conclude by the definition of the lines of $\mathscr{D}^+$ that φ maps $\mathscr{S}^+$ isomorphically onto $\mathscr{D}^+$. Now let $l \in \mathscr{L}^-$ and let p and q be two distinct points on l. Then $\operatorname{copr}_{\mathscr{S}^+}(p)$ and $\operatorname{copr}_{\mathscr{S}^+}(q)$ are distinct hyperplanes of $\mathscr{S}^+$ and hence, they intersect in a line $l' \in \mathscr{L}^+$. By Lemma 6.2.1 we know that every point $r \in \mathscr{P}^-$ with $l' \leq \operatorname{copr}_{\mathscr{S}^+}(r)$ lies on l and therefore, φ maps $\mathscr{S}^-$ isomorphically onto $\mathscr{D}^-$. Finally, for points $p \in \mathscr{P}^+$ and $q \in \mathscr{P}^-$, we have

$$p \leftrightarrow q \;\Leftrightarrow\; p \notin \operatorname{copr}_{\mathscr{S}^+}(q) \;\Leftrightarrow\; p^\varphi \cap q^\varphi = \varnothing .$$

Therefore φ is an isomorphism of twin spaces.

Now assume $\operatorname{diam}(\mathscr{S}^+) \geq 2$. Since this is precisely the situation we considered in the beginning of this section, we may use the notations and the results of this section. Let $x \in \mathscr{P}^+$ and $y \in \mathscr{P}^-$ be opposite points of $\mathscr{S}$. Then Proposition 6.5.22 is a spanning pair. By Corollary 6.5.21 the connected component of the dual polar graph of $\mathscr{S}_\mathrm{m}$ that contains $\Gamma(x)$ is bipartite. Thus, there is a twin half-spin space $(\mathscr{D}^+, \mathscr{D}^-)$ of $\mathscr{S}_\mathrm{m}$ with respect to $(\Gamma(x), \Gamma(y))$. Moreover, $\mathscr{D}^\sigma = \{\Gamma(p) \mid p \in \mathscr{P}^\sigma\}$ for $\sigma \in \{+,-\}$.

Let $w \in \mathscr{P}^+$ be a point collinear and distinct to x. Then there is a subspace $N \in \mathfrak{M}_1^+$ such that $xw \leq N$. By Proposition 6.5.19(iii) both generators $\Gamma(x)$ and $\Gamma(w)$ are adjacent to $\Gamma(N)$ and hence, $\Gamma(x)$ and $\Gamma(w)$ are collinear in $\mathscr{D}^+$. By Lemma 2.1.13 we may assume $\operatorname{cod}(w,y) = 1$. By Proposition 6.5.19(i) we know that the subspaces $\Gamma(y)$ and $\Gamma(N)$ of $\mathscr{S}_\mathrm{m}$ intersect in a single element $U \in \mathscr{P}_\mathrm{m}$. By Proposition 6.5.11(i) we conclude that U intersects N in a hyperplane. Hence, Proposition 6.5.11(iii) yields $x \notin U$ and $w \in U$ and therefore $\Gamma(x) \neq \Gamma(w)$. Conversely, let $w \in \mathscr{P}^+$ such that $\Gamma(x)$ and $\Gamma(w)$ are collinear in $\mathscr{D}^+$. Then there is a generator Θ of $\mathscr{S}_\mathrm{m}$ that is adjacent to both $\Gamma(x)$ and $\Gamma(w)$. By Lemma 6.5.20 this implies that there is a subspace $N \in \mathfrak{M}_1^+$ such that $\Gamma(N) = \Theta$. Moreover, x and w are both points of N and therefore x and w are collinear in $\mathscr{S}^+$. Thus, $\mathscr{S}^+ \to \mathscr{D}^+ \colon p \mapsto \Gamma(p)$ is a bijection that preserves collinearity.

Let l be the line joining x and w and let z be a point collinear to both x and w. Since every element of $\Gamma(x) \cap \Gamma(w)$ is a subspace of $\mathscr{S}$, it contains l and hence, $z \in l$ implies $\Gamma(z) \geq \Gamma(x) \cap \Gamma(w)$. Now assume $z \notin l$. Then by Lemma 2.1.21(iii) we may assume $y \leftrightarrow z$ and $\operatorname{cod}(y,x) = \operatorname{cod}(y,w) = 1$. By Proposition 3.5.2 there is a subspace $N \in \mathfrak{M}_1^+$ of $\mathscr{S}$ that contains $\langle w,x,z \rangle$. By Proposition 6.5.19(i) we know that $\Gamma(y)$ and $\Gamma(N)$ intersect in a single element $U \in \mathscr{P}_\mathrm{m}$. Since $N \cap U$ is a hyperplane of N by Proposition 6.5.11(i), we conclude $\operatorname{copr}_N(x) = N \cap U$ by Proposition 6.5.11(iii). Thus, $U \in \Gamma(x) \cap \Gamma(w)$ and $U \notin \Gamma(z)$. Therefore $\{\Gamma(p) \mid p \in l\}$ is a line of $\mathscr{D}^+$. This concludes that $\mathscr{S}^+ \to \mathscr{D}^+ \colon p \mapsto \Gamma(p)$ is an isomorphism of point-line spaces.

Analogously, $\mathscr{S}^- \to \mathscr{D}^- \colon p \mapsto \Gamma(p)$ is an isomorphism of point-line spaces and it remains to prove that for a pair of point $(w,z) \in \mathscr{P}^+ \times \mathscr{P}^-$, the pair $(\Gamma(w), \Gamma(z))$ is a spanning pair if and only if $w \leftrightarrow z$. By Corollary 5.2.9 we just have to show that $\Gamma(w)$ and $\Gamma(z)$ are disjoint if and only if $w \leftrightarrow z$. By Proposition 6.5.11(iii) we conclude know that $w \leftrightarrow z$ implies $\Gamma(w) \cap \Gamma(z) = \varnothing$. Now assume $w \not\leftrightarrow z$. Then there is are points z' and w' in $\mathscr{P}^-$ with $w' \perp z'$ such that $w \leftrightarrow w'$ and $\operatorname{dist}(z,z') = \operatorname{cod}(w,z) - 1$. Hence, $\operatorname{cod}(w,z') = 1$. Let $N \in \mathfrak{M}_1$ with $w'z' \leq N$. Then there is an element U in $\Gamma(w) \cap \Gamma(N)$ by Proposition 6.5.19(i) and we obtain $z' \in U$. Since U is convex and w and z' are contained in U, we conclude $z \in U$ and hence, $\Gamma(w) \cap \Gamma(z) \neq \varnothing$. □

6.6 Twin SPO spaces of symplectic rank ≥ 5

Throughout this section let $\mathscr{S}$ be a twin SPO space of symplectic rank ≥ 5. By $\mathscr{S}^+$ and $\mathscr{S}^-$ we denote the connected components of $\mathscr{S}$. Further we denote by $\mathfrak{M}$ the set of maximal singular subspaces of $\mathscr{S}$.

By Theorem 6.2.4 we know that whenever $\mathscr{S}^+$ is a symplecton, $\mathscr{S}$ is a twin polar space. Thus, we may restrain ourselves in this section to the case where $\mathscr{S}^+$ contains a symplecton properly and analogously, $\mathscr{S}^-$ contains a symplecton properly. By Theorem 3.7.2 this leaves the two cases $\operatorname{yrk}(\mathscr{S}^+) = 5$ and $\operatorname{yrk}(\mathscr{S}^+) = 6$.

In the following k always denotes the symplectic rank of $\mathscr{S}$. We set $\mathfrak{M}_0 := \{M \in \mathfrak{M} \mid \operatorname{rk}(M) = k-1\}$ and $\mathfrak{M}_1 := \mathfrak{M} \setminus \mathfrak{M}_0$. Furthermore, we set $\mathfrak{M}^\sigma := \{M \in \mathfrak{M} \mid M \leq \mathscr{S}^\sigma\}$ and $\mathfrak{M}_i^\sigma := \mathfrak{M}_i \cap \mathfrak{M}^\sigma$ for $\sigma \in \{+,-\}$ and $i \in \{0,1\}$.

Proposition 6.6.1. *The sets $\mathfrak{M}_0^+$ and $\mathfrak{M}_1^+$ are non-empty. Moreover, every element of $\mathfrak{M}_1^+$ has rank k.*

Proof. Let $Y < \mathscr{S}^+$ be a symplecton. By Lemma 3.3.1(i) there is a point $x \in \mathscr{S}^+ \setminus Y$ such that $\operatorname{pr}_Y(x)$ contains a line. Then $\operatorname{pr}_Y(x)$ is a generator of Y by Proposition 2.1.27. Thus, $\operatorname{rk}(\langle x, \operatorname{pr}_Y(x)\rangle) = k$ and we conclude $\langle x, \operatorname{pr}_Y(x)\rangle \in \mathfrak{M}_1^+$. Moreover, Proposition 2.2.9(iv) implies that every generator of Y that intersects $\operatorname{pr}_Y(x)$ in a hyperplane is a maximal singular subspace of $\mathscr{S}^+$ and hence an element of $\mathfrak{M}_0^+$.
Now let $M \in \mathfrak{M}_1^+$ and let $S < M$ be a singular subspace of rank $k-1$. By Lemma 3.1.1(i) there is a symplecton $Z \leq \mathscr{S}^+$ that contains S as a generator. Let $z \in M \setminus S$. Then $\operatorname{pr}_Z(z) = S$ and we conclude $\langle z, S\rangle = M$ by Proposition 2.2.9(vii) and hence, $\operatorname{rk}(M) = k$. □

Theorem 6.6.2. *Let $\mathscr{S} = (\mathscr{S}^+, \mathscr{S}^-)$ a twin SPO space of symplectic rank ≥ 5 such that $\mathscr{S}^+$ contains a symplecton properly. Then one of the following cases holds:*

(a) *$\mathscr{S}$ is a twin* E_6*-space and* $\operatorname{yrk}(\mathscr{S}^+) = 5$.

(b) *$\mathscr{S}$ is a twin* E_7*-space and* $\operatorname{yrk}(\mathscr{S}^+) = 6$.

Proof. We first show that $\mathscr{S}^+$ is a strongly parapolar space. Since $\mathscr{S}^+$ is strongly parapolar by Theorem 2.1.20 and $\mathscr{S}^+$ has an symplectic rank $r \in \mathbb{N}$ by Corollary 2.2.7, it remains to check whether $\mathscr{S}^+$ is of spherical type.
The axiom (Bu4) is vacuously fulfilled since $\mathscr{S}^+$ is strongly parapolar. Let $S \leq \mathscr{S}^+$ be a singular subspace of rank $r-1$. Then S is contained in a symplecton of $\mathscr{S}^+$ by Lemma 3.1.1(i) and therefore it is a generator of this symplecton. Now Proposition 2.2.5 implies that (Sph1) is satisfied.
Now let V and W be singular subspaces of rank $r-1$ such that $\mathrm{rk}(V \cap W) = r-2$ and $V \nleq W^\perp$. Further let X be a subspace of rank r containing V. Since $V \nleq W^\perp$, there are points $v \in V$ and $w \in W$ that are not collinear. We conclude that $\langle v, w\rangle_{\mathrm{g}}$ is a symplecton containing $V \cap W$. Moreover, $V = \langle v, V \cap W\rangle$ and $W = \langle w, V \cap W\rangle$. Since $\mathrm{rk}(\langle v, w\rangle_{\mathrm{g}}) = r$, this implies that V and W are adjacent generators of $\langle v, w\rangle_{\mathrm{g}}$. Now Proposition 2.2.9(vii) implies that X is a maximal singular subspace and Proposition 2.2.9(iv) implies that W is a maximal singular subspace. Thus, (Sph2) holds.
Now let U, V and W be singular subspaces of rank $r-1$ with that $\mathrm{rk}(U \cap V) = \mathrm{rk}(V \cap W) = r-2$ such that $V \nleq W^\perp$ and U is maximal singular subspace. Let $v \in V \setminus U$. Then $U \nleq v^\perp$ since U is maximal. Hence, there is a point $u \in U$ such that $Y := \langle u, v\rangle_{\mathrm{g}}$ is a symplecton. The subspaces U and V are adjacent generators of Y. Since $\mathscr{S}^+$ contains a symplecton properly, we obtain $Y \neq \mathscr{S}^+$. Hence, Lemma 3.3.1(i) implies that there is a point $x \in \mathscr{S}^+ \setminus Y$ such that $X := x^\perp \cap Y$ is a generator of Y. By Proposition 2.2.9(ii) we conclude that $\mathrm{crk}_X(X \cap U)$ is odd. Hence, $\mathrm{crk}_X(X \cap V)$ is even by Proposition 2.2.9(iii). Using again Proposition 2.2.9(ii) implies that there is a singular subspace V' of rank r such that $V \leq V'$. Now (Sph2) implies that W is maximal and therefore (Sph3) holds.
The axioms (Sph4) follows from Lemma 2.2.3(i) since $r \geq 5$. By symmetric reasons this concludes that $\mathscr{S}^+$ and $\mathscr{S}^-$ are both exceptional strongly parapolar spaces. By Theorem 3.7.2 we know $\mathrm{yrk}(\mathscr{S}^+) \in \{5, 6\}$. Moreover, Theorem B.3.10 implies that both $\mathscr{S}^+$ and $\mathscr{S}^-$ are point-line spaces of type $\mathrm{E}_{r+1,1}$.
First assume $r = 5$. We denote by $\mathscr{D}^+$ the dual of the point-line space $\mathscr{S}^+$ of type $\mathrm{E}_{6,1}$. Let $x \in \mathscr{S}^-$. Then Proposition 4.2.4 implies that $X := \mathrm{copr}_{\mathscr{S}^+}(x)$ is a symplecton. Moreover, $\mathrm{cod}(x, X) = 2$ by Theorem 3.6.5. Hence, there exists a map $\varphi\colon \mathscr{S}^- \to \mathscr{D}^+$ such that $p^\varphi = \mathrm{copr}_{\mathscr{S}^+}(p)$.
Now let $y \in \mathscr{S}^-$ be a point distinct to x and set $Y := y^\varphi$. By Proposition 2.1.13 there is a point $z \in \mathscr{S}^+$ with $z \leftrightarrow x$ and $z \not\leftrightarrow y$. Let $Z \leq \mathscr{S}^+$ be a symplecton that contains z. Then (A12) implies that x is a cogate in Z. By Proposition 2.1.17(ii) we obtain $\mathrm{cod}(z, Y) = 2$. Moreover, $\mathrm{copr}_Z(y) \neq \mathrm{copr}_Z(x)$ since otherwise $z \leftrightarrow y$ by Proposition 2.1.12(ii). Thus, Y and X are distinct symplecta. This implies that φ is injective.
Let Z be an arbitrary symplecton of $\mathscr{S}^+$ and let x' and y' be non-collinear points of Z. By symmetric reasons $X' := \mathrm{copr}_{\mathscr{S}^-}(x')$ and $Y' := \mathrm{copr}_{\mathscr{S}^-}(y')$ are distinct

symplecta of $\mathscr{S}^-$. Since $\mathrm{cod}(y', X') = 2$ by Proposition 2.1.17(ii), we conclude that there is a point $z \in X' \cap Y'$. This implies $z^\varphi = Z$ by Proposition 2.1.16(i). Hence, φ is bijective.
Assume $x \not\perp y$. Then by Proposition 2.1.13 there is a point in X that is opposite y and hence by (A12), Y and X intersect in the cogate of y in X. Now assume $x \perp y$. Then there is no point in X that is opposite to y. Hence, Proposition 4.2.4 implies that $\mathrm{copr}_X(y)$ is a generator of X and we conclude that Y and X intersect in a common generator. Thus, φ preserves collinearity.
Again assume $x \perp y$ and let G be the common generator of X and Y. By symmetric reasons we obtain for two distinct points of G, the points of $\mathscr{S}^-$ that are at codistance 2 to both of them form a singular subspace. Hence, every point $z \in \mathscr{S}^-$ with $G \leq z^\varphi$ is contained in $\{x,y\}^\perp$. Let z be a point on the line xy. Then by Proposition 2.1.12(iv) every point of G has codistance 2 to z and therefore $G \leq z^\varphi$. Now let $z \in \{x,y\}^\perp \setminus xy$. Then by Lemma 2.1.21(iii) there is a point $w \in \mathscr{S}^+$ with $w \leftrightarrow z$ and $\mathrm{cod}(w,x) = \mathrm{cod}(w,y) = 1$. By Propositions 2.2.9(iv) and 2.2.9(v) we know that G is a maximal singular subspace. Hence, there is a point $w' \in G$ with $\mathrm{dist}(w,w') = 2$ and $W := \langle w, w' \rangle_\mathrm{g}$ is a symplecton. Since $\mathrm{cod}(x,w) = 1$ and $\mathrm{cod}(x,w') = 2$, Proposition 2.1.12(ii) yields $\mathrm{copr}_W(x) > \{w'\}$. Thus, W and X intersect in common generator by Proposition 4.2.4. Analogously, $W \cap Y$ is a generator of Y. Assume W does not contain a hyperplane of G. Then there is are lines $g \leq W \cap X$ and $h \leq W \cap Y$ with $g \cap G = h \cap G = \varnothing$. Let p be a point of g. Then $p^\perp$ intersects G in a hyperplane of G. Thus, $\mathrm{pr}_Y(p)$ is a generator of Y by Proposition 2.1.27. Since $\mathrm{pr}_Y(p)$ and G have a common hyperplane, we conclude that there is a point $q \in h$ with $\mathrm{dist}(p,q) = 2$. This implies $\langle p, q \rangle_\mathrm{g} = W$ and since both $p^\perp$ and $q^\perp$ contain a hyperplane of G, we obtain $\mathrm{rk}(G \cap W) \geq 2$. Hence, $G \cap w^\perp$ is not empty and therefore $G \not\leq z^\varphi$. This concludes that lines of $\mathscr{S}^-$ are mapped bijectively onto lines of $\mathscr{D}^+$ and thus, φ is an isomorphism.
To prove that $\mathscr{S}$ is twin E_6-space, it remains to show that for a pair a points $(x,y) \in \mathscr{S}^+ \times \mathscr{S}^-$ we obtain $x \leftrightarrow y$ if and only if $\mathrm{dist}(x, y^\varphi) = 2$. Let $Z \leq \mathscr{S}^+$ be a symplecton that contains x. Since Z and y^φ intersect by Proposition 2.1.17(ii), we conclude $\mathrm{dist}(x, y^\varphi) = 2 - \mathrm{cod}(x,y)$ by Proposition 2.1.16(ii). This proves the claim.
Now assume $r = 6$. Then $\mathscr{S}^+$ is a metaplecton of diameter 3 by Theorem 3.7.2. This implies that $\mathscr{S}^-$ is metaplecton that is opposite $\mathscr{S}^+$. Moreover, Corollary 4.2.8 implies that $\mathscr{S}^+$ and $\mathscr{S}^-$ are isomorphic and the map that sends every point of $\mathscr{S}^+$ onto its cogate in $\mathscr{S}^-$ is an isomorphism. Hence, $\mathscr{S}$ is a twin $\mathrm{E}_{7,1}$-space. $\square$

6.7 Final result

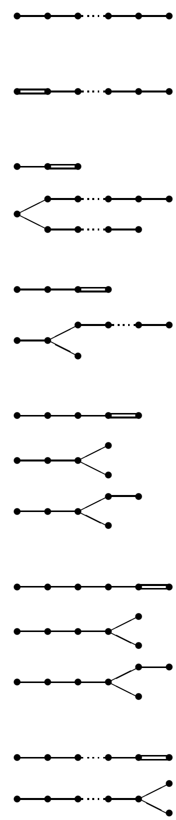
We summarize the results of this chapter and, in fact, the main result of the whole work in the following table. This table yields a complete classification of SPO spaces.

We discussed in Chapter 2 why we may restrain ourselves to partially linear twin SPO spaces. Moreover, in Theorem 4.3.7 we showed that each twin SPO space is exactly the same as a grid sum of rigid twin SPO spaces. Hence, a classification of rigid twin SPO spaces yields a classification of all twin SPO spaces.

Since every rigid twin SPO space consists either of two singular components or possesses a symplectic rank that is at least 2, we have in the following table a complete list of possible cases at the left hand side. Note that in the first case it is also possible that both components consist of a single point. In this case one may consider the twin SPO space as an empty grid sum or as the twin projective space of a projective space of rank 0. In the latter case, one may attach to this situation the empty diagram and call this diagram $\mathrm{A}_{0,1}$. The collection of the theorems in this chapter provides a proof for the correctness of this table.

Even though we mention diagrams at the right hand side, we do not claim that all rigid twin SPO spaces belong to a point-line of a (weak) building of this type since the twin spaces named in the middle column are generalisations of the $\mathrm{X}_{n,j}$-spaces. There are good reasons to avoid diagrams of infinite rank. Consider the projective space $\mathscr{S} := \mathrm{PG}(\mathbb{Q}^{(\mathbb{N})})$ of Example 5.1.9 and let B be a basis of $\mathscr{S}$. Then B has cardinality $\omega := |\mathbb{N}|$. Since $\mathfrak{P}(B)$ has cardinality 2^ω, there is a set $\mathfrak{S}$ of subspaces of $\mathscr{S}$ all of which are spanned by elements of B. For a given subspace S the set $\{T \in \mathfrak{S} \mid \mathrm{crk}_S(T \cap S)\}$ has cardinality ω. Thus, a diagram that possesses a vertex for each element of $\mathfrak{S}$ would consist of 2^ω vertices. Now let $(S_i)_{i\in\mathbb{N}}$ be a chain of subspaces that are all elements of $\mathfrak{S}$ such that $\mathrm{rk}(S_i) = i$ for $i \in \mathbb{N}$. Then each element of $\mathfrak{S} \setminus \{S_i \mid i \in \mathbb{N}\}$ is not incident with all elements of $\{S_i \mid i \in \mathbb{N}\}$. Hence, $(S_i)_{i\in\mathbb{N}}$ is a maximal flag that contains only ω elements. Let φ be a bijection from B onto $\mathbb{Q}$. Then $\{\langle b \mid b^\varphi < x\rangle \mid x \in \mathbb{R}\}$ is a chain of elements of $\mathfrak{S}$ which has cardinality 2^ω. Therefore one sees immediately that we do not obtain a chamber complex if we consider all subspaces of $\mathscr{S}$. A second approach is to consider only the subspaces of finite rank and of finite corank in $\mathscr{S}$ (note that if we would take only the ones of finite rank the diagrams of types $\mathrm{A}_{n,1}$, $\mathrm{C}_{n,1}$ and $\mathrm{D}_{n,1}$ would lead to the same infinite diagram). Again regarding the flag $\{S_i \mid i \in \mathbb{N}\}$ shows that there are maximal flags that do not have elements of each type. One can find maximal flags that contain only subspaces of finite corank as well as flags that contain subspaces of any given finite rank and finite corank. However, the more suitable approach is the second one as long as we forget about chambers and flag complexes. For further comments see [KS96]. In the diagrams below one may regard any occurrence of dots as a chain of vertices of possibly infinite length.

	twin space	**diagram**
diam: 1	twin projective space	$A_{n,1}$ $(n \geq 1)$
yrk: 2	twin dual polar space	$C_{n,n}$ $(n \geq 2)$
yrk: 3	twin polar space	$C_{3,1}$
	partial twin Grassmannian	$A_{n,j}$ $(n > j > 1)$
yrk: 4	twin polar space	$C_{4,1}$
	twin half-spin space	$D_{n,n}$ $(n \geq 4)$
yrk: 5	twin polar space	$C_{5,1}$
		$D_{5,1}$
	twin E_6-space	$E_{6,1}$
yrk: 6	twin polar space	$C_{6,1}$
		$D_{6,1}$
	twin E_7-space	$E_{7,1}$
yrk: ≥ 7	twin polar space	$C_{n,1}$ $(n \geq 7)$
		$D_{n,1}$ $(n \geq 7)$



Some of the theorems in this chapter are stated more generally than they occur in the table above. The reason for this is that the classes in the middle column are not always entirely used. Therefore we enlarged the subclasses of the considered twin SPO spaces in the sections of this chapter slightly to obtain a perfect match with one of the classes introduced in Chapter 5. The partial twin Grassmannians and twin half-spin spaces can be singular. For partial twin Grassmannians, one obtains an arbitrary twin projective space. For twin half-spin spaces, one obtains a twin projective space of rank 1 or 3. These cases are already covered by the first row. Twin dual polar spaces do not have to be rigid, but if they are not, they are still of symplectic rank 2 and a grid sum of rigid twin SPO spaces. Furthermore, they can be singular. In this case the connected components are both singletons or both lines and we are again in the first row. Hence, the case where both components of the twin SPO space are lines is a twin dual polar space, a partial twin Grassmannian and a twin half-spin space at the same time.

Conversely, as one can see, a case in the left column leads to more than one class in the middle column. The reason for this is that there are polar spaces of any given rank. In any case, if the point-line spaces are large enough (where large should be interpreted as “many vertices in the diagram”), the cases in the left column coincide with one class of the middle column.

The class of rigid twin SPO spaces is properly contained in the union of the classes of the middle column since twin dual polar spaces do not need to be rigid. The union of the classes of the middle column is properly contained in the class of twin SPO spaces. Finally, the class of twin SPO spaces coincides with the union of the classes of the middle column closed under taking grid sums. The classes in the middle column are precisely the generalisations of the point-line spaces that are related to those diagrams whose types match the list of Jordan pairs. Therefore we achieved the aim of this work.

A Famous point-line spaces

We introduce two well-known classes of point-line spaces, i. e. projective and polar spaces. Both of them are strongly related to algebraic structures and thus, it is not surprising that they appear as subspaces of the point-line spaces we consider in the present work. Both classes of point-line spaces are well studied. The aim of this chapter is to give a short introduction of projective and polar spaces. Furthermore, we give a list of results that we use in the main part of this work.

A.1 Projective spaces

Projective spaces are, besides the affine spaces, certainly the most famous point-line spaces and are studied in several fields of mathematics. A projective space can easily be obtained by taking the 1- and 2-dimensional subspaces of a K-vector space, where K is a division ring. Moreover, every projective space is a composition of projective spaces of this kind, projective planes and lines; see [VY65].

Definition A.1.1. A *possibly degenerate projective space* is a linear space satisfying the following property:

(VY) For every pair (l,k) of disjoint lines and every point $p \in \mathscr{P} \setminus (l \cup k)$ there is at most one line through p meeting both l and k.

The characterisation (VY) of O. Veblen and J. Young given in this definition is based on Pasch's Axiom. A projective space is called degenerate if it contains at most one line or at least one short line, i. e. a line of cardinality 2. Usually, projective spaces are required to be non-degenerate. However, if we talk about projective spaces, we always allow them to be degenerate. It is obvious by the definition that every subspace of a projective space is again a projective space.

We first show how degenerate projective spaces are composed of non-degenerate ones.

Lemma A.1.2. *Let $\mathscr{S} = (\mathscr{P}, \mathscr{L})$ be a projective space. For two points p and q we write $p \approx q$ if and only if they are joined by a thick line or $p = q$. Then $\approx$ is an equivalence relation.*

Proof. We only have to check the transitivity of $\approx$. Let $\{p,q,r\} \subseteq \mathscr{P}$ with $q \approx p \approx r$. If these three points are on a common line, we obtain $q \approx r$. Hence we assume they are not collinear. Let l denote the line joining p and q and let k denote the line joining p and r. Since both lines are thick, we find $q' \in l \setminus \{p,q\}$ and $r' \in k \setminus \{p,r\}$. Let h be the line joining q and r and let h' be the line joining q' and r'. Since $r \in h \cap k$ but $r \notin l \cap k = \{p\}$, we conclude $h \neq l$. Since $q \in h \cap l$, this leads to $p \notin h$. Analogously we obtain $p \notin h'$. Since l and k are lines through p meeting both h and h', the two lines h and h' intersect in some point p' by Definition A.1.1. Since $p \notin h'$, we obtain $h' \neq l$ and since $p' \in h' \cap l$, we conclude $q \notin h'$. Analogously, $r \notin h'$ and therefore $q \neq p' \neq r$. Thus, h is a thick line and $q \approx r$. ☐

Corollary A.1.3. *Let $\mathscr{S}$ be a projective space and U an equivalence class of $\approx$ in $\mathscr{S}$. Then $U \leq \mathscr{S}$ and U is either a singleton, a thick line or a non-degenerate projective space.*

Proof. Let p and q be two points of U. Then the line joining p and q is thick. Hence, for every point r on this line, we obtain $r \approx p$. So the whole line is contained in U and U has to be a subspace.
If U contains at least one line, U is a singleton or a thick line. If U contains more than one line, it is a non-degenerate projective space since every line is thick. ☐

Let I be an index set and let $(\mathscr{S}_i)_{i\in I}$ be a family of projective spaces. Then we define the *direct sum* of the projective spaces $(\mathscr{S}_i)_{i\in I}$ as

$$\bigoplus_{i\in I}\mathscr{S}_i := \left(\bigcup_{i\in I}\mathscr{P}_i, \bigcup_{i\in I}\mathscr{L}_i \cup \big\{\{p,q\} \mid (p,q) \in \mathscr{P}_i \times \mathscr{P}_j \wedge i \neq j\big\}\right).$$

Proposition A.1.4. *Every projective space is a uniquely determined direct sum of projective spaces, thick lines and singletons.*

Proof. By definition of the direct sum, every projective space is just the direct sum of the equivalence classes of $\approx$. Thus, Corollary A.1.3 proves the claim except for the uniqueness.
Let I be an index set and let $(\mathscr{S}_i)_{i\in I}$ be a family of disjoint projective spaces. Further let $\mathscr{S}_i$ be a point, a thick line or a non-degenerate projective space for every $i \in I$. Then in the point-line space $\prod_{i\in I}\mathscr{S}_i$ every thick line l is contained in a

non-degenerate projective space $\mathscr{S}_i$ for some $i \in I$ or $l \in \{\mathscr{S}_i \mid i \in I\}$. Furthermore, every short line of $\prod_{i\in I} \mathscr{S}_i$ joins two points of two distinct members of $\{\mathscr{S}_i \mid i \in I\}$ since no element of $\{\mathscr{S}_i \mid i \in I\}$ contains short lines. This proves the uniqueness. □

As already mentioned, there is a strong connection between projective spaces and vector spaces. As consequence of this fact is that projective spaces are generated by subsets of their elements that are similar to the bases of vector spaces and therefore are also called bases. We will see that each of these bases is obtained by adding bases of the non-degenerate components.

Lemma A.1.5. *Let $\mathscr{S}$ be a projective space and X a set of points of $\mathscr{S}$. Further let U be an equivalence class of $\approx$. Then $\langle X \cap U \rangle = \langle X \rangle \cap U$.*

Proof. We denote by $\mathscr{S}/\!\approx$ the set of equivalence classes in $\mathscr{S}$ with respect to $\approx$. The direct sum $\mathscr{S}' := \prod_{V \in (\mathscr{S}/\approx)\setminus\{U\}} V$ is a subspace of $\mathscr{S}$ since all thick lines of $\mathscr{S}$ are completely contained in $\mathscr{S}'$ or disjoint to $\mathscr{S}'$. By the definition of the direct sum we see that $\mathscr{S}$ is the disjoint union of $\mathscr{S}'$ and U. Hence X is the disjoint union of $X_0 := X \cap \mathscr{S}'$ and $X_1 := X \cap U$. For every two distinct points p and q of $\langle X_0 \rangle \cup \langle X_1 \rangle$ the line joining them is contained in $\langle X_i \rangle$ if $\{p,q\} \subset \langle X_i \rangle$ for $i \in \{0,1\}$ and equals $\{p,q\}$ otherwise. Thus, the set $\langle X_0 \rangle \cup \langle X_1 \rangle$ is a subspace. It follows that $\langle X_0 \rangle \cup \langle X_1 \rangle = \langle X \rangle$ and since $\langle X_0 \rangle \cap U = \varnothing$ and $X_1 \subseteq U$, we conclude $\langle X_1 \rangle = \langle X \rangle \cap U$. □

Let $\mathscr{S}$ be a projective space and let $X \subseteq \mathscr{S}$ be a set of points. If $p \notin \langle X \setminus \{p\} \rangle$ for every point $p \in X$, we call X *independent*. An independent set of points $B \subseteq \mathscr{S}$ with $\langle B \rangle = \mathscr{S}$ is called a *basis* of $\mathscr{S}$. A set of points which is not independent will be called *dependent*.

Lemma A.1.6. *Let $\mathscr{S}$ be a projective space and let U be a non-empty subspace of $\mathscr{S}$. Then for every point $p \in \mathscr{S}$ with $U \neq \{p\}$ the subspace $\langle p, U \rangle$ is the union of lines through p that intersect U.*

Proof. Let U_p be the union of lines through p meeting U. Since $\mathscr{S}$ is singular, we only have to show that U_p is a subspace. Let l be a line containing two distinct points q and r of U_p. If $p \in l$ or $p \in U$ the claim becomes trivial, hence we may assume $p \notin U \cup l$.
Let q' be the intersection point of U and pq and let r' be the intersection point of U and pr. If $q' = r'$ we obtain $pq = pr = qr \subseteq U_p$. Thus, we may assume $q' \neq r'$. Set $l' := q'r'$. If $q = q'$ and $r = r'$, then $l \in U \subseteq U_p$. Hence, we may assume $r \neq r'$. This implies $l \neq l'$ since $l \leq U$ and $r \notin U$. Since the two lines pq and pr intersect both l and l' and $p \notin l \cup l'$, there is a point $s \in l \cap l'$ by (VY). For an arbitrary point $t \in l$ the two distinct lines pr and l contain the point r and intersect both l' and

pt. Since $l' \neq pt$ and $r \notin U$, (VY) implies $l' \cap pt \neq \varnothing$ and therefore $pt \subseteq U_p$. We conclude $t \in U_p$ and hence $l \subseteq U_p$. □

Lemma A.1.7. *Let $\mathscr{S}$ be projective space. Further let $X \subseteq \mathscr{S}$ be an independent set of points and let p be a point of $\mathscr{S} \setminus \langle X \rangle$. Then $X \cup \{p\}$ is independent.*

Proof. Suppose $X \cup \{p\}$ is dependent. Then there is a finite set $X_0 \subseteq X$ such that $p \in \langle X_0 \rangle$ or $x \in \langle X_0, p \rangle$ for some point $x \in X \setminus X_0$. The first leads to $p \in \langle X \rangle$, a contradiction. Hence, we may assume that the second case holds. Since X is independent, we obtain $x \notin \langle X_0 \rangle$. Thus Lemma A.1.6 implies that the point x has to be on a line joining p and a point $y \in \langle X_0 \rangle$. We conclude $p \in \langle x, y \rangle \leq \langle x, X_0 \rangle \leq \langle X \rangle$, a contradiction. □

Proposition A.1.8. *In a projective space every independent set of points is contained in a basis.*

Proof. Let $\mathscr{S}$ be a projective space. Further let I be an index set and let $(X_i)_{i \in I}$ be a chain of independent subsets of $\mathscr{S}$. Set $X := \bigcup_{i \in I} X_i$. Suppose X is dependent. Then there is a point $x \in X$ with $x \in \langle Y \rangle$ for some finite subset $Y \subseteq X \setminus \{x\}$. For every point $y \in Y \cup \{x\}$ there is an element $i_y \in I$ with $y \in X_{i_y}$. Since $Y \cup \{x\}$ is finite, the union $\bigcup_{y \in Y \cup \{x\}} X_{i_y}$ is contained in $\{X_{i_y} \mid y \in Y \cup \{x\}\}$ and therefore a member of the family $(X_i)_{i \in I}$. Let X_j denote this member. Then the dependent set of points $Y \cup \{x\}$ is contained in X_j, a contradiction to the independence of X_j. We apply Zorn's Lemma to conclude that there are maximal independent sets of points.
Let X be a maximal independent set of points. Further let $p \in \mathscr{S} \setminus X$ be a point. By the maximality of X we know that $X \cup \{p\}$ is dependent. Hence, there is a finite subset $X_0 \subseteq X$ such that $p \in \langle X_0 \rangle$ or $x \in \langle X_0, p \rangle$ for some point $x \in X \setminus X_0$. In the first case we obtain $p \in \langle X \rangle$. In the second case we obtain $x \notin \langle X_0 \rangle$ since X is independent. Thus, Lemma A.1.6 implies that the point x is on a line joining p an a point $y \in \langle X_0 \rangle$. This leads to $p \in \langle x, y \rangle \leq \langle x, X_0 \rangle \leq \langle X \rangle$. We conclude $\langle X \rangle = \mathscr{S}$. □

Corollary A.1.9. *Every projective space has a basis.*

Proof. Since the independent set $\varnothing$ is contained in every projective space the claim follows from Proposition A.1.8. □

Corollary A.1.10. *Let $\mathscr{S}$ be a projective space. Further let $(U_i)_{0 \leq i < n}$ be a chain of subspaces of $\mathscr{S}$, where $n \in \mathbb{N}$. Then there is a basis B of $\mathscr{S}$ such that $B \cap U_i$ is a basis of U_i for all $0 \leq i < n$.*

Proof. Set $U_n := \mathscr{S}$. Assume B_i is a basis B_i of U_i for $0 \leq i < n$. Then by Proposition A.1.8 there is a basis B_{i+1} of U_{i+1} which contains B_i. Since $B_{i+1} \cap U_i$ has to be independent and B_i is maximal under this condition, we conclude $B_{i+1} \cap U_i = B_i$. Since U_0 has a basis, the claim follows by induction. □

Let $\mathscr{S}$ be a projective space with a basis B. Then for every point $p \in \mathscr{S}$, there is a finite subset $B_0 \subseteq B$ with $p \in \langle B_0 \rangle$. Now let B_0 and B_1 be two finite subsets of B with $p \in \langle B_i \rangle$ for $i \in \{0,1\}$. Since they both are finite, we may assume that they are minimal under this condition. Suppose $B_0 \not\subseteq B_1$. Then there is a point $b \in B_0 \setminus B_1$. By the minimality of B_0 we know $p \notin \langle B_0 \setminus \{b\} \rangle$. Thus, Lemma A.1.6 implies that there is a line joining b, p and some point $q \in \langle B_0 \setminus \{b\} \rangle$. We conclude $\langle p, B_0 \setminus \{b\} \rangle = \langle B_0 \rangle$. By Lemma A.1.7 the set $\{p\} \cup B_0 \setminus \{b\}$ is independent and therefore a basis of $\langle B_0 \rangle$. Since $p \in \langle B_1 \rangle$, this implies $b \in \langle B_0 \setminus \{b\}, B_1 \rangle$, a contradiction to the independence of B. We conclude $B_0 \subseteq B_1$ and analogously, $B_1 \subseteq B_0$. Hence, there is a unique minimal subset $B_p \subseteq B$ with $p \in \langle B_p \rangle$. We call B_p the *support* of p with respect to the basis B.

Lemma A.1.11. *Let $\mathscr{S}$ be a projective space. For $i \in \{0,1\}$, let $U_i \leq \mathscr{S}$ be a non-empty subspace such that $U_0 \cup U_1$ contains more than one point. Then $\langle U_0, U_1 \rangle$ is the union of lines meeting both U_0 and U_1.*

Proof. Since $\mathscr{S}$ is singular, it suffices to show that the union of all lines meeting both U_0 and U_1 is a subspace. We may assume $U_0 \nleq U_1$ and $U_1 \nleq U_0$ since otherwise there is nothing to prove. Let p_i and q_i be points of U_i for $i \in \{0,1\}$. Further let p be a point on $g_p := p_0 p_1$ and let q be a point on $g_q := q_0 q_1$ with $p \neq q$. Finally let r be an arbitrary point of $g := pq$. We have to show that there are distinct points $r_0 \in U_0$ and $r_1 \in U_1$ such that $r \in r_0 r_1$.
We may assume $p \neq r \neq q$ and $r \notin U_0 \cap U_1$ since otherwise we are done. Further we assume $p_0 \neq q_0$ and $p_1 \neq q_1$ since otherwise the claim is a direct consequence of Lemma A.1.6. First let $p \in U_0$. Then p and q are both contained in $\langle q_1, U_0 \rangle$ and the claim follows from Lemma A.1.6. Hence, we may assume $p \notin U_0$ and analogously, $p \notin U_1$ and $q \notin U_0 \cup U_1$.
If $g = g_p$, the claim follows with $r_0 := p_0$ and $r_1 := p_1$. Thus, we may assume $g \neq g_p$ and analogously $g \neq g_q$. Now g and g_p both contain p and intersect the lines $p_1 q$ and $p_0 r$. Since $g \neq g_p$, we obtain $p \notin p_1 q \cup p_0 r$ and $p_1 q \neq p_0 r$. Thus, there is a point $r' \in p_1 q \cap p_0 r$ by (VY). We may assume $r' \notin U_1$ since otherwise we are done. Since $g_q \cap U_1 = \{q_1\}$ and $p_1 \neq q_1$, we obtain $g_q \neq p_1 q$. Furthermore, $r \notin U_0$ implies $p_0 r \cap U_0 = \{p_0\}$ and therefore $q_0 \neq r'$. Now g_q and $p_1 q$ both contain q and intersect the lines $q_0 r'$ and $g_1 := p_1 q_1$. Since $q \notin U_1$, we know $q \notin g_1$ and $q_0 \notin g_1$ and hence $q_0 r' \neq g_1$. If $q \in q_0 r'$, we obtain $q_0 r' = g_q$ and set $r_1 := q_1$. Otherwise, we apply (VY) to conclude that $q_0 r'$ and g_1 intersect in a point r_1.
It remains to show that $r r_1$ and g_0 intersect. If $r' \in r r_1$, then $r r_1 = r' r_1 = q_0 r_1$ and hence, we are done. Thus, we may assume $r' \notin r r_1$. If $r' \in g_0$, then $r' = p_0$ since $p_0 r \cap U_0 = \{p_0\}$. Again $r r_1$ and g_0 intersect. Thus, we may assume $r' \notin g_0$. Now the lines $q_0 r'$ and $p_0 r$ both contain r' and intersect $r r_1$ and g_0. Since $p_0 \neq q_0$ and $r \notin U_0$, we know $q_0 r' \neq p_0 r$. Hence, applying (VY) proves the claim. □

Lemma A.1.12. *Let $\mathscr{S}$ be projective space. Further let $X \subseteq \mathscr{S}$ and $Y \subseteq \mathscr{S}$ be independent sets of points with $X \cap Y = \varnothing$. Then $X \cup Y$ is independent if and only if $\langle X \rangle \cap \langle Y \rangle = \varnothing$.*

Proof. Assume $\langle X \rangle \cap \langle Y \rangle = \varnothing$. Let $X_0 \subseteq X$ be a finite subset and let $\{y_i \mid 0 \leq i < n\} \subseteq Y$ for some $n \in \mathbb{N}$. Set recursively $X_{i+1} := X_i \cup \{y_i\}$ for $0 \leq i < n$. By Lemma A.1.11 the subspace $\langle X_i \rangle$ consists of the lines joining a point of $\langle X_0 \rangle$ with a point of $\langle y_j \mid 0 \leq j < i \rangle$. We show $y_k \notin \langle X_i \rangle$ for all $i \leq k \leq n$. Suppose this is not the case. Then there is a line g through y_k that meets $\langle X_0 \rangle$ in some point x and $\langle y_j \mid 0 \leq j < i \rangle$ in some point y. Since $\{y_i \mid 0 \leq i < n\}$ is independent, we obtain $y \neq y_k$ and therefore $x \in yy_k \leq \langle Y_0 \rangle$, a contradiction to $\langle X \rangle \cap \langle Y \rangle = \varnothing$. Since X_0 is independent, the independence of X_n follows by induction using Lemma A.1.7. Hence, $X \cup Y$ contains no dependent finite subset and therefore $X \cup Y$ is independent.
Now let $X \cup Y$ be independent. Then $X \cup Y$ is a basis of $\langle X, Y \rangle$. Let p be a point in $\langle X \rangle$. Then the support of p in $\langle X, Y \rangle$ with respect to the basis $X \cup Y$ is contained in X. Since the support is unique we obtain $p \notin \langle Y \rangle$. ☐

Lemma A.1.13. *Let B and C be two bases of the same projective space. Further let $c \in C \setminus B$. Then there exists an element $b \in B \setminus C$, such that $\{c\} \cup B \setminus \{b\}$ is again a basis.*

Proof. Set $A := B \cap C$, $B' := B \setminus A$ and $C' := C \setminus A$. Then $c \in C'$. Let B_0 be the support of c with respect to B. Since C is a basis, $c \notin \langle A \rangle$ and hence $B_0 \nsubseteq A$. Let $b \in B_0 \setminus A$. By the minimality of B_0 we know $c \notin \langle B_0 \setminus \{b\} \rangle$. Hence by Lemma A.1.6, c is on a line joining b and a point $p \in \langle B_0 \setminus \{b\} \rangle$. Since $c \notin \langle B_0 \setminus \{b\} \rangle$, we obtain $c \neq p$. This implies $b \in cp \leq \langle c, B_0 \setminus \{b\} \rangle$ and therefore $\langle c, B_0 \setminus \{b\} \rangle = \langle B_0 \rangle$. Since b is contained in the support of c with respect to B, we conclude $c \notin \langle B \setminus \{b\} \rangle$. Hence by Lemma A.1.7, the set $\{c\} \cup B \setminus \{b\}$ has to be independent. On the other hand $b \in \langle c, B \setminus \{b\} \rangle$ and thus, $\langle c, B \setminus \{b\} \rangle \geq \langle B \rangle$. Hence, $\{c\} \cup B \setminus \{b\}$ is a basis. ☐

Proposition A.1.14. *Every two bases of a projective space have the same cardinality.*

Proof. Let B and C be two bases of a projective space $\mathscr{S}$. First let B be finite. As long as there is an element $b \in B \setminus C$, we find by Lemma A.1.13 a basis B' and an element $c \in C \setminus B$ with $B = (B \cap B') \cup \{b\}$ and $B' = (B \cap B') \cup \{c\}$. Hence, $|B| = |B'|$ and $|B' \cap C| = |B \cap C| + 1$. Induction leads to a basis with the same cardinality as B that is contained in C. Since there cannot be a basis properly contained in another one, this basis equals C. Thus, $|B| = |C|$.
Now let B and C both be infinite sets. For all $c \in C$ let B_c be the support of c with respect to B and set $B' := \bigcup_{c \in C} B_c$. Then $c \in \langle B' \rangle$ for all $c \in C$ and therefore

$\langle B\rangle = \langle C\rangle \leq \langle B'\rangle$. This implies $B' = B$ since B is a basis. Thus, $|B| \leq \sum_{c\in C}|B_c|$. Since $|B_c| < |\mathbb{N}|$ for every $c \in C$, this leads to $|B| \leq |C|\cdot|\mathbb{N}|$. Since $|C|$ is infinite, we obtain $|\mathbb{N}| \leq |C|$ (by [Bou68, §6.3, Lemma 1]) and therefore $|B| \leq |C|\cdot|C|$. Finally $|B| \leq |C|$ (by [Bou68, §6.3, Theorem 2]). Exchanging B and C finishes the proof. ☐

Corollary A.1.15. *Let $\mathscr{S}$ be a projective space and let $U \leq \mathscr{S}$ be a subspace. Further let B_U and C_U be bases of U and let B and C be bases of $\mathscr{S}$ with $B_U \subseteq B$ and $C_U \subseteq C$. Then $|B \setminus B_U| = |C \setminus C_U|$ and $(B\setminus B_U)\cup C_U$ is again a basis of $\mathscr{S}$.*

Proof. Since $\langle B_U\rangle = \langle C_U\rangle$, we obtain $\langle B\setminus B_U, C_U\rangle = \langle B\setminus B_U, B_U\rangle = \langle B\rangle$. Lemma A.1.12 implies $\langle B_U\rangle \cap \langle B\setminus B_U\rangle = \varnothing$ since B is independent. Moreover, $\langle B_U\rangle = \langle C_U\rangle$ implies that $(B\setminus B_U)\cup C_U$ is independent and therefore a basis.
If $|B|$ is finite, $|B\setminus B_U| = |C\setminus C_U|$ is a direct consequence of Proposition A.1.14. In the infinite case we define for every $c \in C\setminus C_U$ the set B_c to be the intersection of $B\setminus B_U$ and the support of c with respect to B. The rest is just the same as in the proof of Proposition A.1.14. ☐

Let $\mathscr{S}$ be a projective space and let $U \leq \mathscr{S}$ be a subspace. Further let B_U be a basis of U and let B be a basis of $\mathscr{S}$ containing B_U. We call $|B\setminus B_U|$ the *corank* of U in $\mathscr{S}$ and denote it by $\mathrm{crk}_{\mathscr{S}}(U)$. As a consequence of the previous corollary, the corank is well-defined and does not depend of the choice of the basis.

Proposition A.1.16. *Let $\mathscr{S}$ be a projective space with basis B. Then* $|B| = \mathrm{rk}(\mathscr{S})+1$.

Proof. Let $\prec$ be a well-order on B. For every $b \in B$, set $R_b := \langle c \in B \mid c \prec b\rangle$. Then $\{R_b \mid b \in B\}\cup\{\mathscr{S}\}$ is a chain of subspaces such that $R_b < R_c$ for every unordered pair $\{b,c\} \subseteq B$ with $b \prec c$ since $b \in R_c \setminus R_b$. Hence, $(B,\prec) \to (\{R_b \mid b \in B\},<)\colon b \mapsto R_b$ is an isomorphism of well-ordered chains. We conclude $\mathrm{rk}(\mathscr{S}) \geq |B|+1-2 = |B|-1$.
Now let C be some maximal chain of subspaces of $\mathscr{S}$ that is well-ordered under $<$. For every $R \in C\setminus\{\mathscr{S}\}$, let $\mathfrak{n}(R)$ be the successor of R in C and let b_R be a point of $\mathfrak{n}(R)\setminus R$. Let $\{R,S\} \subseteq C$ be an unordered pair with $R < S$. Then $b_R \in S$ and therefore $\langle b_Q \mid Q \in C \wedge Q < S\rangle \leq S$. Suppose $\{b_Q \mid Q \in C\setminus\{\mathscr{S}\}\}$ is dependent. Then there is a finite subchain C_0 of $C\setminus\{\mathscr{S}\}$ such that $\{b_R \mid R \in C_0\}$ is dependent. Since C_0 is finite, Lemma A.1.7 implies that there is an element $T \in C_0$ such that $b_T \in \langle b_R \mid R \in C_0 \wedge R < T\rangle$. Since $b_R \in T$ for all $R < T$, this is a contradiction to $b_T \notin T$. Thus, $\{b_R \mid R \in C\setminus\{\mathscr{S}\}\}$ is independent. We conclude $|B| \geq |C|-1 = \mathrm{rk}(\mathscr{S})+1$. ☐

Note that the proof of this proposition works for any maximal well-ordered chain of subspaces and not only for those of maximal possible cardinality. Hence, for projective spaces all maximal well-ordered chains of subspace have the same cardinality. This is not true for arbitrary linear spaces.

Corollary A.1.17. *Let $\mathscr{S}$ be a projective space and let $U \leq \mathscr{S}$ be a subspace. Then* $\mathrm{rk}(\mathscr{S}) = \mathrm{crk}_{\mathscr{S}}(U) + \mathrm{rk}(U)$.

Proof. This is an immediate consequence of Proposition A.1.16. □

A.2 Polar spaces

Polar spaces are point-line spaces with a surprisingly nice characterisation that yields rather strong properties. Polar spaces are also studied outside the field of incidence geometry. For instance, they appear in disguise as solutions of quadratic forms on a vector space.

The following definition goes back to F. Buekenhout and E. Shult; see [BS74].

Definition A.2.1. A *polar space* is a point-line space $\mathscr{S} = (\mathscr{P}, \mathscr{L})$ satisfying the following property:

(BS) Let $(p,l) \in \mathscr{P} \times \mathscr{L}$. Then p is collinear to either all or exactly one point of l.

An equivalent condition to (BS) is that for every point p the set $p^\perp$ is a hyperplane of $\mathscr{S}$ or equals $\mathscr{P}$, see [Coh95]. We mention both conditions since each of them has its advantages in certain situations.

Let $\mathscr{S} = (\mathscr{P}, \mathscr{L})$ be a polar space. The *radical* $\mathrm{Rad}(\mathscr{S}) := \{p \in \mathscr{P} \mid p^\perp = \mathscr{P}\} = \mathscr{P}^\perp$ consists of all points which are collinear to all others. By definition a polar space is a gamma space and therefore $M^\perp$ is a subspace for every set of points $M \subseteq \mathscr{P}$. Since $\mathrm{Rad}(\mathscr{S}) = \mathscr{P}^\perp$, the radical is a singular subspace of $\mathscr{S}$. A maximal singular subspace of a polar space is called a *generator*.

The *rank* of a non-degenerate polar space $\mathscr{S}$ is defined as $\mathrm{rk}(\mathscr{S}) := \mathrm{srk}(\mathscr{S}) + 1$. A more general definition, which includes degenerate polar spaces is given in [Joh90]: The rank of $\mathscr{S}$ is the largest integer n, such that there is a chain of length $n+1$ of singular subspaces all containing $\mathrm{Rad}(\mathscr{S})$. If there is no such integer, the rank is set to be ∞. In the finite rank case, the rank equals $\mathrm{srk}(\mathscr{S}) - \mathrm{rk}(\mathrm{Rad}(\mathscr{S}))$. To include the cases which have infinite rank, we take the largest cardinal α such that there is a well-ordered chain of length $\alpha + 1$ of singular subspaces all containing $\mathrm{Rad}(\mathscr{S})$, instead. Note that a singular polar space has always rank 0. Hence, this rank might differ from the rank we obtain if the space is treated as a singular space.

Using Zorn's Lemma one sees that every polar space has generators and that every singular subspace of a polar space is contained in some generator. Furthermore, by Lemma 1.1.3 above, every set of mutually collinear points is contained in a generator.

In a non-degenerate polar space of finite rank n, all lines of which have cardinality at least 3, every singular subspace is contained in some singular subspace of rank $n-1$; see [Tit74, 7.2.1]. In other words, every generator has rank $n-1$. The equivalence between the axioms used there ([Tit74, 7.1]) and the ones used here is shown in [BS74, Theorem 4].

We will see later on that even if we have short lines, all generators of a non-degenerate polar space of finite rank have the same rank. In polar spaces of arbitrary rank it may occur that there are generators of different rank[1]; see [Joh90] for an example. But there are some weaker conditions that still hold.

A.2.1 The associated non-degenerate polar space

Among the class of polar spaces the class of non-degenerate polar spaces plays a prominent role. One reason is that non-degenerate polar spaces have a structure that is much nicer. We will see later that they are partially linear and their generators are even projective spaces. Both facts do not hold for arbitrary polar spaces. A second reason is that there is a functor from the category of polar spaces onto the category of non-degenerate polar spaces, which means that one can associate each polar space with a uniquely determined non-degenerate polar space.

Definition A.2.2. Let $\mathscr{S} = (\mathscr{P}, \mathscr{L})$ be a polar space. For every point $p \in \mathscr{P}$, we set $p^\rho := \langle p, \mathrm{Rad}(\mathscr{S})\rangle$ and for every line $l \in \mathscr{L}$, we set $l^\rho := \{p^\rho \mid p \in l\}$. Define the following two sets:

$$\mathscr{P}^\rho := \{p^\rho \mid p \in \mathscr{P} \setminus \mathrm{Rad}(\mathscr{S})\}$$
$$\mathscr{L}^\rho := \{l^\rho \mid l \in \mathscr{L} \wedge l \cap \mathrm{Rad}(\mathscr{S}) = \varnothing\}$$

Then $\mathscr{S}^\rho := (\mathscr{P}^\rho, \mathscr{L}^\rho)$ is called the *associated non-degenerate polar space* of $\mathscr{S}$.

By construction, $\mathscr{S}^\rho$ is again a point-line space provided that $|l^\rho| \geq 2$ holds for every line $l \subseteq \mathscr{S} \setminus \mathrm{Rad}(\mathscr{S})$. The notation used here corresponds with the one in [Coh95, 2.4]. In [Joh90] a different notation is used. The first part of the following lemma will show that the both notations lead to isomorphic point-line

[1] In this case, of course, both ranks are ∞, but there is no bijection between maximal well-ordered chains of pairwise properly contained singular subspaces.

spaces since using $p^\rho \setminus \mathrm{Rad}(\mathscr{S})$ instead of p^ρ provides the notation of [Joh90]. Therefore we may use some of the results given in [Joh90].

Lemma A.2.3. *Let $\mathscr{S} = (\mathscr{P}, \mathscr{L})$ be a polar space.*

(i) *Let $l \in \mathscr{L}$ be disjoint from $\mathrm{Rad}(\mathscr{S})$. Then for every point $p \in l$, there is a point $q \in \mathscr{P}$ with $l \cap q^\perp = \{p\}$.*

(ii) $p^\rho = \{q \in \mathscr{P} \mid q^\perp = p^\perp\} \cup \mathrm{Rad}(\mathscr{S})$ *for every point $p \in \mathscr{P} \setminus \mathrm{Rad}(\mathscr{S})$.*

(iii) *Let $l \in \mathscr{L}$ be disjoint from $\mathrm{Rad}(\mathscr{S})$. Then for every point $p \in l$, the line l intersects p^ρ in exactly one point.*

(iv) *Let $l \in \mathscr{L}$ intersect $\mathrm{Rad}(\mathscr{S})$ in a single point p. Then $\langle l, \mathrm{Rad}(\mathscr{S})\rangle = q^\rho$ for all points $q \in l \setminus \{p\}$.*

(v) *Let p^ρ and q^ρ be points of $\mathscr{S}^\rho$ on a common line $l^\rho \in \mathscr{L}^\rho$. Further let $(p', q') \in p^\rho \times q^\rho$ be a pair of points in $\mathscr{S}$. Then there is a line in $\mathscr{S}$ joining p' and q'.*

Proof. (i) See [Joh90, Proposition 3.1(i)].
(ii) See [Joh90, Proposition 3.1(ii)].
(iii) Let p and q be two points on l. Then (i) implies $p^\perp = q^\perp$ if and only if $p = q$. Thus by (ii), $p^\rho = q^\rho$ if and only if $p = q$. Hence, $l \to l^\rho : p \mapsto p^\rho$ is a bijection. Since by (ii) we obtain $p^\rho = q^\rho$ if $q \in p^\rho$, the claim follows.
(iv) Let $q \in l \setminus \{p\}$. Since $\{q, p\} \subseteq q^\rho$ and $q^\rho \leq \mathscr{S}$ we get $l \subseteq q^\rho$ and hence $\langle l, \mathrm{Rad}(\mathscr{S})\rangle \leq q^\rho$. The other inclusion is trivial.
(v) If $p' \in \mathrm{Rad}(\mathscr{S})$ or $q' \in \mathrm{Rad}(\mathscr{S})$, this is clear, hence we assume $\{p', q'\} \subseteq \mathscr{P} \setminus \mathrm{Rad}(\mathscr{S})$. By (iii) we may assume $\{p, q\} \subseteq l$ without loss of generality. We get $q' \in q^\rho \setminus \mathrm{Rad}(\mathscr{S})$ and therefore by (ii) we get $q^\perp = q'^\perp$. Hence $p' \in p^\rho = \langle p, \mathrm{Rad}(\mathscr{S})\rangle \leq q^\perp = q'^\perp$ and thus there is a line joining p' and q'. □

By Lemma A.2.3(iii) we obtain $|l| = |l^\rho|$ for every line disjoint to the radical. Hence, $\mathscr{S}^\rho$ is indeed a point-line space. Moreover, the following proposition justifies the name associated non-degenerate polar space.

Proposition A.2.4. *Let $\mathscr{S} = (\mathscr{P}, \mathscr{L})$ be a polar space. Then $\mathscr{S}^\rho$ is a non-degenerate polar space.*

Proof. First we show that (BS) holds in $\mathscr{S}^\rho$. Therefore we choose an arbitrary pair $(p^\rho, l^\rho) \in \mathscr{P}^\rho \times \mathscr{L}^\rho$. Since (BS) holds in $\mathscr{S}$, we find a point $q \in l$ with $q \perp p$. Since $l \cap \mathrm{Rad}(\mathscr{S}) = \varnothing$, we obtain $q \notin \mathrm{Rad}(\mathscr{S})$ and therefore $q^\rho \in l^\rho$. Let k be the line joining p and q. If $k \cap \mathrm{Rad}(\mathscr{S}) = \varnothing$, then k^ρ contains p^ρ and q^ρ. Otherwise $p^\rho = q^\rho$ by Lemma A.2.3(iv). Thus, p^ρ and q^ρ are collinear.
Assume there is a second point $r^\rho \in l^\rho$ being collinear to p^ρ. Let $r' \in r^\rho$ be the point of $\mathscr{S}$ belonging to l. By Lemma A.2.3(v) we obtain $p \perp r' \neq q$ and hence

$l \subseteq p^\perp$ by (BS). Analogously to the first part of this proof, p^ρ is collinear to s^ρ for every $s \in l$. This implies that all points on l^ρ are collinear to p^ρ. Thus $\mathscr{S}^\rho$ satisfies (BS).
Now assume that there is a point $p^\rho \in \mathrm{Rad}(\mathscr{S}^\rho)$. Then Lemma A.2.3(v) implies that every point $p' \in p^\rho$ is collinear to every point $q' \in q^\rho$ for every point $q^\rho \in \mathscr{P}^\rho$. This implies $\bigcup_{q^\rho \in \mathscr{P}^\rho} q^\rho \subseteq p'^\perp$. By Lemma A.2.3(ii) the set $\{q^\rho \setminus \mathrm{Rad}(\mathscr{S}) \mid q^\rho \in \mathscr{P}^\rho\}$ is a partition of $\mathscr{P} \setminus \mathrm{Rad}(\mathscr{S})$. Thus, $p' \in \mathrm{Rad}(\mathscr{S})$ and $p^\rho \leq \mathrm{Rad}(\mathscr{S})$. We conclude $p^\rho \notin \mathscr{P}^\rho$ and therefore $\mathscr{S}^\rho$ has to be non-degenerate. ☐

Note that if a polar space is singular, then the associated non-degenerate polar space is just the empty space, which is of course also a non-degenerate polar space. As mentioned above, non-degenerate polar spaces have nice properties that do not hold in general for degenerate ones. These properties, stated in the following two propositions, makes studying non-degenerate polar spaces much easier than studying arbitrary ones:

Proposition A.2.5. *Let p and q be non-collinear points of a non-degenerate polar space $\mathscr{S}$. Then $\{p,q\}^\perp$ is a non-degenerate polar space. Moreover, if $\mathscr{S}$ has finite rank, then* $\mathrm{rk}(\mathscr{S}) = \mathrm{rk}(\{p,q\}^\perp) + 1$.

Proof. The first property is [Coh95, Theorem 3.1(iii)]. Now let G be a generator of $\mathscr{S}$ that contains p. Then $G \cap q^\perp$ is a hyperplane of G that is contained in $\{p,q\}^\perp$. Conversely, if H is a generator of $\{p,q\}^\perp$, then $\langle p,H\rangle$ is a singular subspace of $\mathscr{S}$. ☐

Proposition A.2.6. *Let p and q be non-collinear points of a non-degenerate polar space $\mathscr{S}$. Then $\langle p,q\rangle_{\mathrm{g}} = \mathscr{S}$.*

Proof. By (BS) we obtain that $\langle p,q\rangle_{\mathrm{g}}$ contains all lines through p and all lines through q. Let $p' \neq p$ be a point collinear to p. Then by Lemma A.2.3(i) there is a point q' such that p is the only point on pp' that is collinear to q'. Since p' and q' are not collinear and both are contained in $\langle p,q\rangle_{\mathrm{g}}$, we conclude that $\langle p,q\rangle_{\mathrm{g}}$ contains all points collinear to p' and consequently, all points at distance 2 to p. This proves the claim. ☐

Proposition A.2.7. *Every non-degenerate polar space is partially linear.*

Proof. See [Joh90, Proposition 3.1(vii)] . ☐

Proposition A.2.8. *Every singular subspace of a non-degenerate polar space is a possibly degenerate projective space.*

Proof. See [Joh90, Theorem 3.2]. ☐

This knowledge helps us to investigate the lattice of subspaces of the associated non-degenerate polar space.

Lemma A.2.9. *Let $\mathscr{S}$ be a polar space and $U \le \mathscr{S}$.*

(i) *The set $U^\rho := \{p^\rho \mid p \in U \setminus \mathrm{Rad}(\mathscr{S})\}$ is a subspace of $\mathscr{S}^\rho$.*

(ii) *Let $V \le \mathscr{S}$. Then $V \le U$ implies $V^\rho \le U^\rho$.*

(iii) *Let $U^\dagger \le \mathscr{S}^\rho$. Then $U' := \bigcup_{p^\rho \in U^\dagger} p^\rho \le \mathscr{S}$ and $U'^\rho = U^\dagger$.*

(iv) *Set $U' := (\bigcup_{p^\rho \in U^\rho} p^\rho) \cup \mathrm{Rad}(\mathscr{S})$. Then $U \le U'$, where $U = U'$ holds, if $\mathrm{Rad}(\mathscr{S}) \le U$.*

(v) *U is singular if and only if U^ρ is singular.*

Proof. (i) Let l^ρ be a line of $\mathscr{S}^\rho$ on which there are two distinct points, p^ρ and q^ρ say, which are contained in U^ρ. Then there are points $p' \in p^\rho$ and $q' \in q^\rho$ with $\{p', q'\} \subseteq U \setminus \mathrm{Rad}(\mathscr{S})$. By Lemma A.2.3(v) there is a line l' of $\mathscr{S}$ joining p' and q'. Since U is a subspace, l' is contained in U. Since l'^ρ and l^ρ intersect in two different points, we obtain $l'^\rho = l^\rho$ by Proposition A.2.7. Hence, l^ρ is contained in U^ρ and therefore $U^\rho \le \mathscr{S}^\rho$.
(ii) We obtain $V^\rho \subseteq U^\rho$ by definition. The rest follows with (i).
(iii) Let p' and q' be two collinear distinct points in U'. Further let l be the line joining p' and q'. We choose two points p^ρ and q^ρ in $U^\dagger$ with $p' \in p^\rho$ and $q' \in q^\rho$. If $p' \in \mathrm{Rad}(\mathscr{S})$, then l is contained in $q^\perp$. Hence we may assume that neither p' nor q' is contained in $\mathrm{Rad}(\mathscr{S})$. Then $p'^\rho = p^\rho$ and $q'^\rho = q^\rho$ and l^ρ is just the line joining p^ρ and q^ρ. Since $U^\dagger$ is a subspace, it contains l^ρ and therefore l is contained in $\bigcup_{p^\rho \in U^\dagger} p^\rho$. For every point $p' \in U'$, there is a point $p^\rho \in U^\dagger$ with $p' \in p^\rho$. Since $p'^\rho \le p^\rho$, we obtain $U'^\rho \le U^\dagger$ and since $p \in p^\rho \le U'$, equality holds.
(iv) If $U \le \mathrm{Rad}(\mathscr{S})$, then $U^\rho = \varnothing$ and there is nothing to prove. Hence we assume $U \not\le \mathrm{Rad}(\mathscr{S})$. Let $p \in U$ and $qinU \setminus \mathrm{Rad}(\mathscr{S})$. Then $p \in p^\rho \le U'$, if $p \notin \mathrm{Rad}(\mathscr{S})$ and $p \in q^\rho \le U'$, if $p \in \mathrm{Rad}(\mathscr{S})$. Hence, $U \le U'$. On the other hand, if $\mathrm{Rad}(\mathscr{S}) \le U$, we obtain $p^\rho \le U$ for every $p \in U$ and therefore $U' \le U$.
(v) If U is singular, then two points p^ρ and q^ρ in U^ρ are joined by the line l^ρ, where l is a line joining p and q in U. Hence, U^ρ is singular. Now let U^ρ be singular and set $U' := (\bigcup_{p^\rho \in U^\rho} p^\rho) \cup \mathrm{Rad}(\mathscr{S})$. Then $U \le U'$ by (iv). Since U' is singular by Lemma A.2.3(v), the subspace $U \le U'$ is singular, as well. ☐

Proposition A.2.10. *Let $\mathscr{S}$ be a polar space. Further let $\mathfrak{U}_0$ be the set of all subspaces $U \le \mathscr{S}$ containing $\mathrm{Rad}(\mathscr{S})$ and let $\mathfrak{U}_1$ be the set of all subspaces of $\mathscr{S}^\rho$. Then the lattices $(\mathfrak{U}_0, \le)$ and $(\mathfrak{U}_1, \le)$ are isomorphic via $\varphi\colon \mathfrak{U}_0 \to \mathfrak{U}_1\colon U \mapsto U^\rho$.*

Proof. Let $U^\dagger \in \mathfrak{U}_1$ and $U := (\bigcup_{p^\rho \in U^\dagger} p^\rho) \cup \mathrm{Rad}(\mathscr{S})$. By Lemma A.2.9(iii) $U \leq \mathscr{S}$ and hence $U \in \mathfrak{U}_0$. Again by Lemma A.2.9(iii) we obtain $U^\rho = U^\dagger$ and thus, φ is surjective. For every subspace $V \in \mathfrak{U}_0$ with $V^\rho = U^\dagger$, we obtain $V = U$ by Lemma A.2.9(iv). Hence, φ is bijective.
Let U and V be in $\mathfrak{U}_0$. Then $U \leq V$ implies $U^\rho \leq V^\rho$ by Lemma A.2.9(ii). If $U^\rho \leq V^\rho$, then $U = (\bigcup_{p^\rho \in U^\rho} p^\rho) \cup \mathrm{Rad}(\mathscr{S}) \subseteq (\bigcup_{p^\rho \in V^\rho} p^\rho) \cup \mathrm{Rad}(\mathscr{S}) = V$ and therefore $U \leq V$ by Lemma A.2.9(iii). Hence, φ is an isomorphism of lattices. □

Corollary A.2.11. *Every polar space has the same rank as its associated non-degenerate polar space.*

Proof. Let $\mathscr{S}$ be a polar space. By Proposition A.2.10 and Lemma A.2.9(v) a chain of singular subspaces in $\mathscr{S}$ all containing $\mathrm{Rad}(\mathscr{S})$ can be mapped isomorphically on a chain of singular subspaces of $\mathscr{S}^\rho$ and vice versa. The claim follows. □

In a non-degenerate polar space the maximal well-ordered chains of subspaces of a given generator are all of the same cardinality since this generator is a projective space. Like in the corollary above, this implies for a given generator M of an arbitrary polar spaces that all maximal well-ordered chains of subspaces of M all containing $\mathrm{Rad}(\mathscr{S})$ are of the same cardinality. Note that this is no longer true if the singular subspaces are not demanded to contain the radical.

A.2.2 Dual polar spaces

In a polar space $\mathscr{S}$, two generators M and N are called *adjacent* when they intersect in a common hyperplane, denoted by $M \sim N$. Let $\mathfrak{G}$ be the set of generators. The graph on $\mathfrak{G}$ induced by $\sim$ is called the *dual polar graph* of $\mathscr{S}$. Let $\mathscr{C}^*$ be the set of maximal cliques, i. e. sets of vertices of maximal complete subgraphs, of the dual polar graph. Set $\mathscr{C} := \{x \in \mathscr{C}^* \mid |x| \geq 2\}$. Then $(\mathfrak{G}, \mathscr{C})$ is a point-line space, called the *dual polar space*. Point-line spaces which are isomorphic to such a space, are also called dual polar spaces.

There are non-isomorphic polar spaces whose dual polar spaces are isomorphic. To study dual polar spaces it suffices to check only one representative of each class of polar spaces with isomorphic duals. In the following we will show that we can always pick a non-degenerate representative.

Lemma A.2.12. *Let U be a singular space with a hyperplane $H < U$. Further let p be a point of $U \setminus H$. Then $U = \langle p, H \rangle$. More precisely, U is the union of the lines joining a point of H with p.*

Proof. Let q be an arbitrary point of $U \setminus \{p\}$. Since U is singular, there is a line joining p and q. Since H is a hyperplane of U, this line intersects H. The claim follows. □

Lemma A.2.13. *Let $\mathscr{S}$ be a polar space and let M and N be two generators of $\mathscr{S}$. Then $M \cap N$ is a hyperplane of M if and only if $M \cap N$ is a hyperplane of N.*

Proof. Assume that $H := M \cap N$ is a hyperplane of M. If N does not contain a line, there is nothing to prove. Thus, let l be a line in N. Now we take a point $p \in M \setminus H$. Then by (BS) there is a point q on l which is collinear to p. Since $q^\perp \cap M$ contains H and p, we obtain $M \leq q^\perp$ by Lemma A.2.12. Hence, Lemma 1.1.3 implies that there is a singular subspace which contains M and q. Since M is a generator, we obtain $q \in M$. Thus, H intersects l. Since $N \neq H$ by maximality of N, we conclude that H is a hyperplane of N. □

Proposition A.2.14. *Let X, Y and Z be generators of a polar space which are pairwise adjacent. Then X, Y and Z have a hyperplane in common.*

Proof. We may assume that X, Y and Z are pairwise distinct since otherwise the claim becomes trivial. Since X and Y are adjacent, they have a hyperplane H in common. Assume $H \leq Z$. Since Z intersects X in a hyperplane, $Z \cap X = H$ by Lemma A.2.12 and therefore H is hyperplane of Z by Lemma A.2.13.
Now assume $H \nleq Z$. Then Z intersects X in a hyperplane which is by Lemma A.2.12 not contained in H. Hence, there is a point $x \in X \setminus H$, which is contained in Z. Analogously, there is a point $y \in (Y \setminus H) \cap Z$. Since x and y are contained in Z, they are collinear. Thus, x and H are contained in $y^\perp$ and therefore $X \leq y^\perp$ by Lemma A.2.12. By Lemma A.2.13 there is a singular subspace containing X and y. This is a contradiction to $y \notin X$ and the maximality of X. □

From this proposition it follows that every line of a dual polar space corresponds to a hyperplane of a generator of the underlying polar space. Conversely, hyperplanes of generators which are contained in two different generators correspond to lines of the dual polar space. Note that there might be hyperplanes of generators, which are contained in only one generator and therefore do not correspond to any line of the dual.

Theorem A.2.15. *Every dual polar space is isomorphic to the dual of a non-degenerate polar space. More precisely, for a polar space $\mathscr{S}$, the dual polar space of $\mathscr{S}$ and the dual polar space of $\mathscr{S}^\rho$ are isomorphic.*

Proof. Let $\mathscr{S}$ be a singular. Then the dual polar space of $\mathscr{S}$ is clearly a singleton. This is still true for the empty space which is of course singular, too. Since the empty space is also the associated non-degenerate polar space of any singular polar space, this case is done. Hence, we assume that $\mathscr{S}$ is non-singular.
Let $\mathfrak{U}_0$ be the set of singular subspaces of $\mathscr{S}$ containing $\mathrm{Rad}(\mathscr{S})$ and let $\mathfrak{U}_1$ be the set of singular subspaces of $\mathscr{S}^\rho$. By Proposition A.2.10 and Lemma A.2.9(v) there is an isomorphism φ between the posets $(\mathfrak{U}_0, \leq)$ and $(\mathfrak{U}_1, \leq)$. By Lemma

1.1.3 all generators of $\mathscr{S}$ contain $\mathrm{Rad}(\mathscr{S})$. Hence φ induces a bijection between the generators of $\mathscr{S}$ and the generators of $\mathscr{S}^\rho$.
Now let $M \leq \mathscr{S}$ be a generator. Since $\mathscr{S}$ is non-singular, we obtain $\mathrm{Rad}(\mathscr{S}) < M$ by Lemma 1.1.3. Let H be a hyperplane of M containing $\mathrm{Rad}(\mathscr{S})$. Then $H^\rho < M^\rho$ by Proposition A.2.10. Let l^ρ be a line of M^ρ. Then l is contained in M by Lemma A.2.9(iv). We choose a point $p \in H \cap l$. Since l is disjoint from $\mathrm{Rad}(\mathscr{S})$, we obtain $p^\rho \in H^\rho \cap l^\rho$. Hence, H^ρ is a hyperplane of M^ρ. Conversely, let H^ρ be a hyperplane of M^ρ and set $H := (\bigcup_{p^\rho \in H^\rho} p^\rho) \cup \mathrm{Rad}(\mathscr{S})$. By Proposition A.2.10 we obtain $H < M$. Let l be a line of M. If l intersects $\mathrm{Rad}(\mathscr{S})$, then it also intersect H. If l is disjoint from $\mathrm{Rad}(\mathscr{S})$, then l^ρ intersects H^ρ in some point p^ρ. By Lemma A.2.3(iii) and Lemma A.2.3(ii) we may assume that p is the point contained in l. Thus, $p \in l \cap H$ and H is a hyperplane of M.
Since φ is an isomorphism of the posets $(\mathfrak{U}_0, \leq)$ and $(\mathfrak{U}_1, \leq)$, it follows that a set of maximal singular subspaces in $\mathscr{S}$ intersect in a common hyperplane if and only if their images under φ do. Hence, lines of the dual of $\mathscr{S}$ are mapped bijectively onto lines of the dual of $\mathscr{S}^\rho$. We conclude that the two dual polar spaces are isomorphic. ☐

In the rest of this section we study generators of polar spaces and their distances in the dual polar space. All subspaces that occur in this context contain the radical, since they are intersections of generators. Taking Proposition A.2.10 and the theorem above in account, we may always consider the associated non-degenerate polar space. Generalising the following statements and proofs to the case of arbitrary polar spaces is straightforward and without any additionally interest.

Let $\mathscr{S}$ be a polar space. Further let $U \subseteq \mathscr{S}$ be a set of points and let $V \leq \mathscr{S}$ be a subspace. Then we set $U \ominus V := \langle U, U^\perp \cap V \rangle$. For a single point p, we will write $p \ominus V$ rather than $\{p\} \ominus V$.

Lemma A.2.16. *Let M be a generator of a non-degenerate polar space $\mathscr{S}$ and let p be a point. Then $N := p \ominus M$ is again a generator. Moreover, if $p \in M$, then $M = N$ and if $p \notin M$, then N is the unique generator being adjacent to M and containing p.*

Proof. If $p \in M$, then $M \leq p^\perp$ and hence $p \ominus M = \langle p, M \rangle = M$. Now let $p \notin M$. Since M is a generator, Lemma 1.1.3 implies that p is not collinear to all points of M. Hence, $H := M \cap p^\perp < M$. Since $p^\perp$ is a hyperplane of $\mathscr{S}$, H has to be a hyperplane of M. By Lemma 1.1.3 the subspace $N = \langle p, H \rangle$ is again singular. Since $p \notin M$ we obtain $H < N$. Let N' be a generator containing N. With $p \in N' \setminus M$, we obtain $H \leq M \cap N' < M$ and therefore $M \cap N' = H$ by Lemma A.2.12. Thus, Lemma A.2.13 implies that H is a hyperplane of N'. Applying Lemma A.2.12 again leads to $N' = \langle p, H \rangle = N$ and therefore $N \sim M$.

Now let L be a generator containing p and being adjacent to M. Since $L \leq p^\perp$, we obtain $L \cap M \leq p^\perp \cap M \leq N$. Since $L \cap M$ is a hyperplane of L, this implies $L = \langle p, L \cap M \rangle \leq \langle p, p^\perp \cap M \rangle = N$. Since L is a generator, the claim follows. □

Lemma A.2.17. *Let M and N be two generators of a non-degenerate polar space with $M \cap N \neq \varnothing$. Further let p be a point not collinear to all points of $M \cap N$. Set $N' := p \oplus N$. Then $M \cap N'$ is a hyperplane of $M \cap N$.*

Proof. Let q be a point of $M \cap N$ not collinear to p. By Lemma A.2.12 N' is the union of the lines through p that meet N. Since $p \notin q^\perp$, each of these lines contains exactly one point being collinear to q by (BS). Since $N \leq q^\perp$, this point has to be the intersection point with N. Hence, $q^\perp \cap N' = N \cap N' =: H$ and since $M \leq q^\perp$, we conclude $M \cap N' \leq H$. With $H \leq N'$, we obtain $M \cap N' = M \cap H \leq M \cap N$. Since $q \in M \cap N$ and $q \notin N'$, we obtain $M \cap N' < M \cap N$. Finally, the claim follows since H is a hyperplane of N. □

Remark A.2.18. Let M be a generator of a non-degenerate polar space of finite rank. As a consequence of Lemma A.2.17, there is for every generator N with $N \cap M \neq \varnothing$ a generator N' with $N' \cap M < N \cap M$. Since the rank of M is finite, this implies that there exists a generator that is disjoint to M.

Lemma A.2.19. *Let M and N be two distinct generators of a non-generate polar space. Further let p be a point of $M \setminus N$ and set $N' := p \oplus N$. Then $N \cap M$ is a hyperplane of $N' \cap M$.*

Proof. Take a point $q \in N \setminus N'$. Since $p^\perp \cap N \leq N'$, the point q is not collinear to p. Since $p \in M \cap N'$, we may apply Lemma A.2.17 to conclude that $(q \oplus N') \cap M$ is a hyperplane of $N' \cap M$. Finally, Lemma A.2.16 implies $q \oplus N' = N$. □

We call two singular subspaces M and N of a point-line space *commensurate* if $\operatorname{crk}_M(M \cap N) = \operatorname{crk}_N(M \cap N) \in \mathbb{N}$.

Proposition A.2.20. *Let M and N be two generators of a non-degenerate polar space. Further let d be the distance of M and N in the dual polar space. Then M and N are commensurate and $d = \operatorname{crk}_M(M \cap N)$ or d, $\operatorname{crk}_M(M \cap N)$ and $\operatorname{crk}_N(M \cap N)$ are all infinite.*

Proof. Set $H := M \cap N$. First let $\operatorname{crk}_M(H) =: r < \infty$. We prove $d \leq r$ by induction. If $r = 0$, then $M = N$ and therefore $d = 0$. For $r > 0$ let $\{b_i \mid 0 \leq i < r\}$ be a set of points such that $\langle H, b_i \mid 0 \leq i < r \rangle = M$. Set $N_0 := N$ and $N_{i+1} := b_i \oplus N_i$ for $0 \leq i < r$. Then N_i and N_{i+1} are adjacent by Lemma A.2.19. Moreover, $\langle H, b_j \mid 0 \leq j \leq i \rangle \leq N_{i+1}$ since $H \leq N_0$ and $\langle H, b_j \mid 0 \leq j < i \rangle \leq N_i \cap b_i{}^\perp$. We conclude $N_r = M$ and thus, $d \leq r$.
Now let $d < \infty$. Then there are generators N_i for $0 \leq i \leq d$ with $N_0 = N$ and

$N_d = M$ such that N_i and N_{i+1} are adjacent for $i < d$. Since $\mathrm{crk}_{N_i}(N_i \cap N_{i-1}) = 1$ for $i > 0$, we obtain $\mathrm{crk}_{N_i}(N_i \cap N_0) \leq i$ and hence $\mathrm{crk}_M(H) \leq d$. □

Remark A.2.21. A direct consequence of Proposition A.2.20 is that all generators that are contained in a common connected component of the dual polar space are commensurate. Together with Corollary A.1.17 this implies, that all these generators have the same rank provided that the polar space is non-degenerate. Moreover, it suffices that one generator has finite rank n to prove that all generators are of rank n.
In non-degenerate polar spaces of arbitrary rank it might happen that there are generators M and N such that $\mathrm{rk}(M) > \mathrm{rk}(N)$. Since $\mathrm{rk}(M \cap N) \leq \mathrm{rk}(N) < \mathrm{rk}(M)$ and both generators are of infinite rank, we obtain $\mathrm{crk}_M(M \cap N) = \mathrm{rk}(M)$; see [Bou68, §6.3, Corollary 4].

Lemma A.2.22. *Let U be a singular subspace of a non-degenerate polar space $\mathscr{S}$ with $\mathrm{rk}(U) < \infty$ and let $M \leq \mathscr{S}$ be a generator. Then*

(i) $\mathrm{rk}(U) = \mathrm{crk}_M(M \cap U^\perp) + \mathrm{rk}(M \cap U)$ *and*

(ii) *$U \oplus M$ is a generator with distance $\mathrm{crk}_U(U \cap M)$ to M in the dual polar space.*

Proof. Set $k := \mathrm{rk}(M \cap U)$ and $n := \mathrm{rk}(U)$. Let $(p_i)_{0 \leq i \leq n}$ be a basis of U such that $(p_i)_{0 \leq i \leq k}$ is a basis of $M \cap U$. Then $\langle p_i \mid k < i \leq n \rangle \cap M = \varnothing$ by Lemma A.1.12. Set $M_0 := M$ and $M_{i+1} := p_{k+i+1} \oplus M_i$ for $i < n-k$. Then Lemma A.2.16 $(M_i)_{0 \leq i \leq n-k}$ is a sequence of pairwise adjacent generators. Hence, $d \leq n-k$, where d is the distance of M and M_{n-k} in the dual polar space.
We know $\langle p_j \mid j \leq k \rangle \leq M_0$. Hence we obtain $\langle p_j \mid j \leq k+i+1 \rangle \leq M_{i+1}$ since $\langle p_j \mid j \leq k+i \rangle \leq {p_{k+i+1}}^\perp$ for $i < n-k$. Analogously, $M \cap U^\perp \leq M_{n-k}$ since $M \cap U^\perp \leq M_0$ and $M \cap U^\perp \leq p^\perp$ for every $p \in U$. Since $\langle p_i \mid k < i \leq n \rangle \subseteq M_{n-k} \setminus M$, we obtain $\mathrm{crk}_{M_{n-k}}(M_{n-k} \cap M) \geq n-k$. Since $d \leq n-q$, we conclude $\mathrm{crk}_{M_{n-k}}(M_{n-k} \cap M) = d = n-k$ by Proposition A.2.20. This implies $\mathrm{crk}_M(M \cap U^\perp) \geq n-k$ since $M \cap U^\perp \leq M_{n-k}$. On the other hand $M \cap p^\perp$ is a hyperplane of M for every $p \in U \setminus M$ and $U^\perp = \bigcap_{k<i\leq n} {p_i}^\perp$. Therefore $\mathrm{crk}_M(M \cap U^\perp) = n-k$. This implies (i) and $M \cap U^\perp = M \cap M_{n-k}$. Since $\mathrm{crk}_{M_{n-k}}(M \cap M_{n-k}) = n-k$, we obtain $M_{n-k} = \langle U, M \cap M_{n-k} \rangle$ and the claim follows. □

Remark A.2.23. Let S be a non-maximal singular subspace of a non-degenerate polar space $\mathscr{S}$ of finite rank. Then there is a generator M containing S. Since $\mathscr{S}$ has finite rank, there is a generator N that is disjoint to M. Now $S \oplus N$ is a generator that intersects M in S. We conclude that in $\mathscr{S}$ every non-maximal singular subspace is the intersection of two generators.

We conclude this section by considering generators of a polar spaces that have infinite distance in the dual polar graph.

Proposition A.2.24. *In a non-degenerate polar space of infinite rank there are two generators M and N that are not connected in the dual polar space.*

Proof. Let M be a generator and let $\mathfrak{S}$ be the set of all singular subspaces that are disjoint to M. We have to show that $\mathfrak{S}$ contains an element with infinite rank. By Zorn's Lemma it suffices to show that $H \in \mathfrak{S}$ with $\mathrm{rk}(H) < \infty$ is not a maximal element of $\mathfrak{S}$. Set $M_H := M \cap H^\perp$. Then $\mathrm{crk}_M(M_H) = \mathrm{rk}(H) + 1$ by Lemma A.2.22(i). Let p be a point that is not collinear to all points of M_H. If $H \leq p^\perp$, then $\mathrm{rk}(\langle p, H\rangle) = \mathrm{rk}(H) + 1$ and $M \cap \langle p, H\rangle^\perp < M_H$. Thus, $\langle p, H\rangle \cap M = \varnothing$ by Lemma A.2.22(i) and we are done. Hence, we may assume $H \not\leq p^\perp$.
Set $G := p \oplus H$ and $M_G := M \cap G^\perp$. Since $p^\perp \cap H$ is hyperplane of H, we know that $G \cap H$ is common hyperplane of G and H. Hence, Lemma A.2.22(i) implies that M_H is a hyperplane of $M \cap (G \cap H)^\perp$. Since $M_H \not\leq p^\perp$ and $G = \langle p, G \cap H\rangle$, we conclude that M_G is a hyperplane of $M \cap (G \cap H)^\perp$. Thus $\mathrm{crk}_M(M_H) = \mathrm{crk}_M(M_G)$ and Lemma A.2.22(i) implies $G \in \mathfrak{S}$ since $\mathrm{rk}(G) = \mathrm{rk}(H)$.
Since $M_H \neq M_G$, there is a point $q \in M_G \setminus M_H$. Let s be an arbitrary point of $H \setminus G$. Since $p^\perp$ and $q^\perp$ contain the hyperplane $G \cap H$ of H, there is a point $r \in pq$ with $H = \langle s, G \cap H\rangle \leq r^\perp$. Since $r \perp q$, we obtain $r \notin H$ and hence, $\langle r, H\rangle$ is a singular subspace containing H properly. This implies $r \neq q$ and hence $r^\perp M = p \perp M$. Thus, $M_H \not\leq r^\perp$ and consequently, $\langle r, H\rangle^\perp \cap M$ is a hyperplane of M_H. By Lemma A.2.22(i) this implies $\langle r, H\rangle \cap M = \varnothing$. ☐

Proposition A.2.25. *A dual polar space never consists of exactly two connected components.*

Proof. We consider the underlying non-degenerate polar space $\mathscr{S}$ of a dual polar space. If $\mathscr{S}$ has finite rank, then $\mathrm{crk}_M(M \cap N) < \infty$ for every two generators M and N of $\mathscr{S}$. Hence, the dual of $\mathscr{S}$ is connected by Proposition A.2.20.
Now let $\mathscr{S}$ be of infinite rank. Then by Proposition A.2.24 there are two generators M and N that have infinite distance in the dual polar space. Let $\mathscr{M}$ be the set of pairs (X, Y, φ) such that $X \subseteq M$ and $Y \subseteq N$ are independent sets of points with $\langle X\rangle \cap N = \varnothing$ and $\langle Y\rangle \cap M = \varnothing$ such that $X \subseteq Y^\perp$ and φ is a bijection from X to Y. Further let $\prec$ be a strict partial order on $\mathscr{M}$ with $(X, Y, \varphi) \prec (X', Y', \varphi') \Leftrightarrow (X < X' \wedge Y < Y' \wedge \varphi'|_X = \varphi)$. Now let $(X_i, Y_i, \varphi_i)_{i \in I}$ be a chain in $\mathscr{M}$ with respect to $\prec$ for an index set I. Then $X := \bigcup_{i \in I} X_i$ is again an independent set of points with $\langle X\rangle \cap N = \varnothing$. Analogously, $Y := \bigcup_{i \in I} Y_i$ is independent with $\langle Y\rangle \cap M = \varnothing$. Since for every $x \in X$ and every $y \in Y$ there is an index $i \in I$ with $x \in X_i$ and $y \in Y_i$, we obtain $x \perp y$ and hence, $X \subseteq Y^\perp$.
Set $\varphi\colon X \to Y$, such that $x^\varphi = x^{\varphi_i}$ for every $x \in X_i$ where $i \in I$. By the construction of $\prec$ this map is well-defined. Since for two points x and x' of X and a point $y \in Y$ there is a set X_i with $i \in I$ such that $\{x, x'\} \leq X_i$ and $y \in Y_i$, the map φ has to be bijective. Hence (X, Y, φ) is an upper bound for the chain $(X_i, Y_i, \varphi_i)_{i \in I}$. We

may apply Zorn's Lemma to conclude that there are maximal elements in $\mathscr{M}$ with respect to $\prec$.
Let $(X,Y,\varphi) \in \mathscr{M}$ be such a maximal element. Suppose X and Y are finite. Set $S := M \cap N$. Then $\mathrm{crk}_M(\langle X,S\rangle)$ is infinite since $\mathrm{crk}_M(S)$ is infinite. Since $S \leq N \leq Y^\perp$, we obtain $\langle X,S\rangle \leq Y^\perp \cap M$. Thus, $M \cap Y^\perp > \langle X,S\rangle$ since $\mathrm{crk}_M(M \cap Y^\perp) < \infty$ by Lemma A.2.22(i). Let $x \in (M \cap Y^\perp) \setminus \langle X,S\rangle$ and set $X' = X \cup \{x\}$. Then $\langle X'\rangle \cap S = \varnothing$ and therefore $\langle X'\rangle \cap N = \varnothing$. Since $X' \subseteq Y^\perp$, we obtain $N \cap X'^\perp > \langle Y,S\rangle$ by repeating the same arguments as above. Let $y \in N \cap X'^\perp \setminus \langle Y,S\rangle$ and set $Y' := Y \cup \{y\}$. Further let $\varphi' \colon X' \to Y'$ be the map with $\varphi'|_X = \varphi$ and $x^\varphi = y$. Then $(X',Y',\varphi') \in \mathscr{M}$ and $(X,Y,\varphi) \prec (X',Y',\varphi')$, a contradiction. Hence, X and Y have to be infinite sets. Let L be a generator containing $X \cup Y$. Then $\mathrm{crk}_L(L \cap M)$ is infinite since $\langle Y\rangle \leq L \setminus M$ and analogously $\mathrm{crk}_L(L \cap N)$ is infinite. Thus, L, M and N are contained in three different connected components of the dual polar space of $\mathscr{S}$. $\square$

B Point-line spaces arising from buildings

In this appendix we consider point-line spaces that are related to (Tits) buildings. Therefore we first introduce buildings in the way of [Tit74]. We know already some of the point-line space that arise from the buildings, namely the projective, the polar and the dual polar spaces. Besides these spaces we obtain lots of other point-line spaces. Some of them occur in the present work and hence, will be studied here.

B.1 Buildings

An abstract *simplicial complex* Δ is a collection of sets such that $B \in \Delta$ for any subset B with $B \subseteq A \in \Delta$. A partial ordered set of sets that is isomorphic to a simplicial complex is also called a simplicial complex. A simplicial complex possesses a smallest element that we denote by 0. An elements that only contains 0 properly is called a *vertex*. An arbitrary element A of a simplicial complex Δ is called a *simplex* or, more specifically, an *n-simplex*, where $n+1$ is the number of vertices that are contained in A. Hence, the vertices are that 0-simplices. A *subcomplex* Δ' of a simplicial complex Δ is a subset of Δ such that Δ' is again a simplicial complex. Let A be a simplex of a simplicial complex Δ. Then the set of all simplices of Δ containing A is again a simplicial complex called the *residue* of A in Δ and denoted by $\mathrm{res}_\Delta(A)$ or simply by $\mathrm{res}(A)$ if there is no confusion about the underlying simplicial complex.

A simplicial complex Δ is called a *chamber complex* if every element of Δ is contained in a maximal element of Δ and if for two maximal elements C and C' of Δ, there exists a finite sequence $(C_i)_{0\leq i\leq m}$ such that $|C_i \setminus C_{i+1}| = |C_{i+1} \setminus C_i| = 1$ for every $i < m$. The maximal elements of a chamber complex are called *chambers* . Two chambers C and C' are called *adjacent* if $|C \setminus C'| = 1$. A chamber complex is call *thick* (respectively *thin*) if for any two adjacent chambers C and C' the subset $C \cap C'$ is contained in at least three (respectively exactly two) chambers. It follows immediately that in a chamber complex every two chamber have the same cardinality. In other words, there is a natural number n such that the set of chambers of Δ is the set of n-simplices. We call n the *rank* of Δ.

A *morphism of simplicial complexes* is a map $\varphi\colon \Delta \to \Delta'$ from a simplicial complex into another such that the restriction of φ on the subsets of any simplex $A \in \Delta$ is an isomorphism onto $\mathfrak{P}(A^{\varphi})$. Note that φ induces a map from the set of vertices of Δ into the set of vertices of Δ' which determines φ uniquely. A *morphism of chamber complexes* is a morphism of simplicial complexes such that chambers are mapped onto chambers.

Proposition B.1.1. *An endomorphism of a thin chamber complex that is injective on the set of chambers and leaves all simplices contained in a given chamber invariant is the identity.*

Proof. [Tit74, Corollary 1.7]. □

Definition B.1.2. Let Δ be a simplicial complex and let $\mathfrak{A}$ be a set of subcomplexes of Δ. The pair $(\Delta, \mathfrak{A})$ is called a *building* of which the elements of $\mathfrak{A}$ are called *apartments* if the following conditions hold:

- **(B1)** Δ is thick.
- **(B2)** The elements of $\mathfrak{A}$ are thin chamber complexes.
- **(B3)** Any two elements of Δ belong to an apartment.
- **(B4)** If two apartments Σ and Σ' contain two common simplices A and A', there exists an isomorphism of Σ onto Σ' which leaves A, A' and all simplices contained in one of them invariant.

A pair $(\Delta, \mathfrak{A})$ is called a *weak building* if it satisfies the axioms (B2), (B3) and (B4).

Let $(\Delta, \mathfrak{A})$ be a weak building. From the axioms it follows directly that Δ is a chamber complex and that the apartments are isomorphic subcomplexes. Any representative of the isomorphism class of the thin chamber complexes to which belong the apartments will be called the *Weyl complex* of $(\Delta, \mathfrak{A})$.

An idempotent endomorphism $\varphi\colon \Delta \to \Delta$ of a thin chamber complex is called a *retraction*. A retraction is called a *folding* if every chamber that is contained in the image of φ has exactly two preimages. The image of a folding is called a *root*.

Proposition B.1.3. *Let φ be a folding of a chamber complex Δ. Then there is a pair (C, C') of adjacent chambers such that $C \in \varphi$ and $C' \notin \varphi$. Moreover, for every such a pair, φ is the unique folding of Δ mapping C' onto C.*

Proof. [Tit74, Proposition 1.10]. □

A *Coxeter complex* is a thin chamber complex Σ such that for every pair (C,C') of adjacent chambers, there exists a root containing C and not C'. Let φ be the unique folding of Σ mapping C' onto C and let φ' be the folding mapping C onto C'. Then every element of Σ is either contained in the image of φ or in the image of φ'. Moreover, for two distinct elements A and B of Σ, we obtain $B = A^{\varphi}$ if and only if $A = B^{\varphi'}$; see [Tit74, Corollary 1.11]. The map

$$\psi\colon \quad \Sigma \to \Sigma\colon \quad A \mapsto \begin{cases} A^{\varphi} & \text{if } A \notin \Sigma^{\varphi} \\ A^{\varphi'} & \text{if } A \in \Sigma^{\varphi} \end{cases}$$

is an involutoric automorphism of Σ which is called the *reflection associated with* φ. Set $B := C \cap C'$. Since Σ is thin, C and C' are the unique chambers containing B. Thus, B determines ψ uniquely and therefore ψ is also called the *reflection with respect to B*. The group that is generated by all reflections of Σ is called the *Weyl group* of Σ.

Proposition B.1.4. *Let Σ be a Coxeter complex and let C be a chamber of Σ. There exists a unique retraction ρ_C of Σ whose image equals $\mathfrak{P}(C)$.*

Proof. [Tit74, Proposition 2.4]. □

Motivated by this proposition we introduce a type function for the elements of a Coxeter complex Σ. Let C be a chamber of Σ. Then two elements of Σ are said to be of the *same type* if their images under ρ_C coincide. Note that this definition is independent of the choice of C. We denote by $\mathrm{I}(\Sigma)$ the partition of the vertices of Σ that consists of the preimages of the vertices under ρ_C. Now we define the map $\mathrm{typ}\colon \Sigma \to \mathfrak{P}(\mathrm{I}(\Sigma))$ such that $\mathrm{typ}(B) := \{i \in \mathrm{I}(\Sigma) \mid \exists A \subseteq B\colon A \in i\}$ for every $B \in \Sigma$. For $A \in \Sigma$, the image $\mathrm{typ}(A)$ is called the *type of A*. In other words, for a vertex $A \in \Sigma$, the type $\mathrm{typ}(A)$ of A is a singleton containing the unique element of $\mathrm{I}(\Sigma)$ that contains A. For an arbitrary simplex B of Σ the type of B is the union of the types of the vertices of B. Every chamber C contains a unique simplex of any given type. The type of C equals $\mathrm{I}(\Sigma)$. Hence, typ induces the simplex structure of C on the set $\mathrm{I}(\Sigma)$. Therefore we call $\mathrm{I}(\Sigma)$ the *fundamental simplex*.

Theorem B.1.5. *The Weyl complex of a building is a Coxeter complex.*

Proof. [Tit74, Theorem 3.7]. □

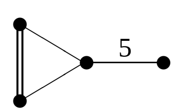
In the following we will also consider weak buildings but only the ones whose Weyl complex is a Coxeter complex.

Let Σ be an apartment of a weak building $(\Delta, \mathfrak{A})$ such that Σ is a Coxeter complex. Further let $C \in \Sigma$ be a chamber. For every simplex $A \in \Delta$, consider an apartment Σ' containing C and A. By Proposition B.1.1 there is a unique isomorphism from Σ' onto Σ which leaves all simplices contained in C invariant. Let $A' \in \Sigma$ be the image of A under this isomorphism. By (B4) it follows that A' does not depend on the choice of Σ'. Hence, from Proposition B.1.4 it follows that there exists a retraction λ_C from Δ onto $\mathfrak{P}(C)$. Since λ_C induces on every apartment that contains C the unique retraction onto $\mathfrak{P}(C)$, it follows from (B3) that λ_C is unique. Furthermore, for any chamber $D \in \Delta$, there is an automorphism α from $\mathfrak{P}(D)$ onto $\mathfrak{P}(C)$ which is induced by λ_C. The composition $\alpha^{-1} \circ \lambda_C$ is a retraction of Δ onto $\mathfrak{P}(C)$ and hence, it equals λ_D. This implies that the preimages of λ_C form a partition of Δ that does not depend on the choice of C. Therefore we denote as for Coxeter complexes the partition of the vertices of Δ that consists of the preimages of the vertices under λ_C by $\mathrm{I}(\Delta)$. The type function of Δ is the map $\mathrm{typ}\colon \Delta \to \mathfrak{P}(\mathrm{I}(\Delta))$ such that $\mathrm{typ}(B) := \{i \in \mathrm{I}(\Delta) \mid \exists A \subseteq B\colon A \in i\}$ for every $B \in \Sigma$. There is a canonical isomorphism from the fundamental simplex $\mathrm{I}(\Delta)$ of Δ and the one of Σ such that each image is a subset of its preimage. Hence, the fundamental simplices $\mathrm{I}(\Delta)$ and $\mathrm{I}(\Sigma)$ can be identified in a natural way.

A *Coxeter matrix* or a *diagram* over a set I or over a the simplex of all subsets of I is defined as a symmetric matrix $M = (m_{ij})_{(i,j)\in I\times I}$ whose entries are elements of $\mathbb{N} \cup \{\infty\}$ such that $m_{ii} = 1$ for all $i \in I$ and $m_{ij} \geq 2$ for $j \in I \setminus \{i\}$. The elements of I are represented by dots and called *vertices* of the diagram. The cardinality of I is called the *rank* of the diagram M.

We use the following pictorial representation of M: Every two vertices are joined by a stroke which is labelled with the number m_{ij}. For reasons of clearness, we omit the stroke if it is labelled with a 2. Furthermore, instead of a stroke with a 3, we draw a single stoke without any number and a stroke with a 4 is replaced by a double stroke. We give an example of a matrix and a diagram that belong to each other:

$$\begin{pmatrix} 1 & 4 & 3 & 2 \\ 4 & 1 & 3 & 2 \\ 3 & 3 & 1 & 5 \\ 2 & 2 & 5 & 1 \end{pmatrix}$$



Let Σ be a Coxeter complex of rank 2. Then there is an index set $I \subseteq \mathbb{Z}$ such there exists an bijection from I onto the set of vertices of Σ. For $i \in I$, we denote the image of i under this bijection by A_i. Moreover, I can be chosen in the way that for two elements i and j of I with $i < j$, the set $\{A_i, A_j\}$ is a chamber if and only if $j = i+1$ or $i = 0$ and $j = \sup(I)$. If I is infinite, then I equals $\mathbb{Z}$ and if I is finite, then $I = \{i \in \mathbb{N} \mid i \leq m\}$, where $m \in \mathbb{N}$ is odd with $m > 2$; see [Tit74, 2.2]. Two vertices A_i and A_j are of the same type if and only if $i + j$ is even. Calling the vertices with odd index “points” and those with even index “lines”, we obtain in a natural way the structure of a (possibly infinite) polygon.

Proposition B.1.6. *The residue of a simplex of a Coxeter complex is itself a Coxeter complex.*

Proof. [Tit74, Proposition 2.9]. □

Let C be a chamber of a Coxeter complex Σ. For every $i \in \mathrm{I}(\Sigma)$, let B_i be the simplex of type $I \setminus \{i\}$ with $B_i \subseteq C$ and let r_i denote the reflection of Σ with respect to B_i. For $\{i, j\} \subseteq \mathrm{I}(\Sigma)$, let m_{ij} be the order of the product $r_i r_j$ in the Weyl group. Then for any simplex $A \in \Sigma$ of type $\mathrm{I}(\Sigma) \setminus \{i, j\}$, the residue of A possesses $2m_{ij}$ chambers; see [Tit74, 2.11]. Moreover, if $i \neq j$, then $\mathrm{res}(A)$ carries the structure of an m_{ij}-gon. The matrix $(m_{ij})_{(i,j)\in \mathrm{I}(\Sigma)\times \mathrm{I}(\Sigma)}$ is a diagram over $\mathrm{I}(\Sigma)$, called the *diagram of* Σ. This diagram does not depend on the chamber C. For any simplex $A \in \Sigma$, the diagram of the Coxeter complex $\mathrm{res}(A)$ is the submatrix $(m_{ij})_{(i,j)\in J\times J}$, where $J = \mathrm{I}(\Sigma) \setminus \mathrm{typ}(A)$. Hence, the diagram of $\mathrm{res}(A)$ is deduced from the diagram of Σ by removing the vertices belonging to $\mathrm{typ}(A)$ and all affected strokes.

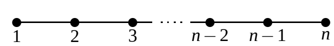
Let $(\Delta, \mathfrak{A})$ be a building and let $\Sigma \in \mathfrak{A}$ be an apartment. By the canonical identification of $\mathrm{I}(\Delta)$ with $\mathrm{I}(\Sigma)$, the diagram of Σ becomes a diagram over $\mathrm{I}(\Delta)$. This diagram does not depend on the choice of Σ and hence will be called the *diagram of* $(\Delta, \mathfrak{A})$.

Proposition B.1.7. *Let* $(\Delta, \mathfrak{A})$ *be a building and let* $A \in \Delta$ *be a simplex. Further set* $\mathfrak{A}(A) := \{\Sigma \cap \mathrm{res}(A) \mid A \in \Sigma \in \mathfrak{A}\}$. *Then* $(\mathrm{res}(A), \mathfrak{A}(A))$ *is a building whose diagram is obtained by removing from the diagram of* $(\Delta, \mathfrak{A})$ *all vertices which belong to* $\mathrm{typ}(A)$.

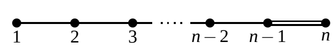
Proof. [Tit74, Proposition 3.12]. □

We give a list of diagrams that are well-known and will play a role in the following. Each of these diagrams is a diagram over a set $\{1, 2, \ldots, n\}$ where $n \in \mathbb{N}$ is the rank of the diagram. The vertices are labelled by the numbers they represents. All these diagrams have names that are listed at the left hand side.

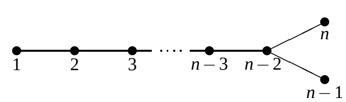
A_n, $n \geq 1$



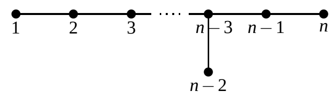
C_n, $n \geq 2$



D_n, $n \geq 3$



E_n, $n \geq 4$



For $X \in \{A, C, D, E\}$, we speak of weak building $(\Delta, \mathfrak{A})$ of type X_n, or simply of type X, if there is a bijection $\varphi\colon I(\Delta) \to \{1, \dots, n\}$ such that labelling the vertices of the diagram of Δ by its image under φ provides the diagram X_n. We call a simplex B of Δ to be of type $J \subseteq \{1, \dots, n\}$ if $(\mathrm{typ}(B))^{\varphi} = J$. A vertex $A \in \Delta$ of type $\{i\}$ with $1 \leq i \leq n$, is also said to be of type i.

B.2 Shadow spaces

Let V be a set endowed with a reflexive, symmetric relation called the *incidence relation*. Two elements that form a pair of the incidence relation are called *incident*. The subsets of V whose elements are pairwise incident form a simplicial complex $\mathrm{Flag}(V)$ whose vertices are the singletons of V. A simplicial complex that is isomorphic to $\mathrm{Flag}(V)$ is called a *flag complex*. The simplices of a flag complex are also called *flags*.

Motivated by this concept we call two vertices of a chamber complex incident if they are contained in a common chamber. Moreover, we call two simplices A and B of a chamber complex incident if they are contained in a common chamber or, equivalently, if every vertex contained in A is incident with every vertex contained in B.

Let $\mathscr{S} = (\mathscr{P}, \mathscr{L})$ be a projective space of finite rank r. For $0 \leq i < r$, let $\mathfrak{U}_i$ be the set of the subspaces of $\mathscr{S}$ that have rank i. Then we call $(\mathfrak{U}_i)_{i<r}$ a *projective geometry* of rank r. There is a natural identification between $\mathscr{P}$ and $\mathfrak{U}_0$. Moreover, $\mathfrak{U}_1$ coincides with $\mathscr{L}$. Thus, $(\mathfrak{U}_i)_{i<r}$ can be understood as an enrichment of the pair $(\mathscr{P}, \mathscr{L})$.

According to this, we define a *polar geometry* of rank r as a tuple $(\mathfrak{U}_i)_{i<r}$, where $\mathfrak{U}_i$ is the set of the singular subspaces of a non-degenerate polar space $\mathscr{S}$ that have rank i and $r \in \mathbb{N}$ is the rank of $\mathscr{S}$. Note that $\mathfrak{U}_{r-1}$ is the set of generators of $\mathscr{S}$.

Let $r \in \mathbb{N}$ and let $\mathscr{S} = (\mathfrak{U}_i)_{i<r}$ be a projective or polar geometry. Set $\mathfrak{U} := \bigcup_{i<r} \mathfrak{U}_i$. We define an incidence relation on $\mathfrak{U}$ such that two elements of $\mathfrak{U}$ are incident if and only if one is a subspace of the other. Then $\mathrm{Flag}(\mathscr{S})$ contains all chains of $\mathfrak{U}$ and is a chamber complex of rank r.

Theorem B.2.1. *Let $(\Delta, \mathfrak{A})$ be a weak building of type A_n. Then there exists a projective geometry $\mathscr{S}$ of rank n and an isomorphism $\varphi\colon \mathrm{Flag}(\mathscr{S}) \to \Delta$ sending the vertices that correspond with the subspaces of rank i of $\mathscr{S}$ onto the vertices of type $i+1$ of Δ.*

Proof. [Tit74, Theorem 6.3]. ☐

Theorem B.2.2. *Let $(\Delta,\mathfrak{A})$ be a weak building of type C_n with $n \geq 2$. Then there exists a polar geometry $\mathscr{S}$ of rank n and an isomorphism $\varphi\colon \mathrm{Flag}(\mathscr{S}) \to \Delta$ sending the vertices that correspond with the singular subspaces of rank i of $\mathscr{S}$ onto the vertices of type $i+1$ of Δ.*

Proof. [Tit74, Theorem 7.4]. ☐

Let $\mathscr{S} = (\mathfrak{U}_i)_{i<r}$ be a polar geometry of rank r. Set $\mathfrak{U} := \bigcup_{i<r} \mathfrak{U}_i$ and $\mathfrak{U}' := \mathfrak{U} \setminus \mathfrak{U}_{r-2}$. We call two elements of $\mathfrak{U}'$ incident if and only if either one is a subspace of the other or they intersect in an element of $\mathfrak{U}_{r-2}$. This gives raise to a flag complex which we call the *oriflamme complex* of $\mathscr{S}$, denoted by $\mathrm{Orifl}(\mathscr{S})$.

Theorem B.2.3. *Let $(\Delta,\mathfrak{A})$ be a weak building of type D_n with $n \geq 3$. Then there exists a polar geometry $\mathscr{S}$ of rank n and an isomorphism $\varphi\colon \mathrm{Orifl}(\mathscr{S}) \to \Delta$ such that for $i < n-2$ the vertices that correspond with the singular subspaces of rank i of $\mathscr{S}$ are sent onto the vertices of type $i+1$ of Δ and furthermore, the vertices that correspond with generators of $\mathscr{S}$ are sent onto the vertices of the types $n-1$ and n in such a way that two generators of $\mathscr{S}$ have even distance in the dual polar graph if and only if their images under φ are of the same type.*

Proof. [Tit74, Theorem 7.12]. ☐

Let $(\Delta,\mathfrak{A})$ be a weak building of type D_n and let $\mathscr{S}$ be a polar geometry such that the flag complex $\mathrm{Orifl}(\mathscr{S})$ is isomorphic to Δ in the way as in the theorem above. Then it follows directly that the dual polar graph of $\mathscr{S}$ is bipartite.

Let Δ be a chamber complex and let $M \subseteq \Delta$ be a set of simplices of Δ. For a simplex A of Δ, we call the set of all elements of M that are incident with A the *shadow of A on M*.

Now let $(\Delta,\mathfrak{A})$ be a building of type X_n, where $\mathrm{X} \in \{\mathrm{A},\mathrm{C},\mathrm{D},\mathrm{E}\}$ and $n \in \mathbb{N}$. Let $1 \leq i \leq n$ and define $\mathscr{P}$ as the set of all vertices of type i of Δ and $\mathfrak{M}$ as the set of all simplices of type $\{1,\ldots,n\} \setminus \{i\}$. Further set $\mathscr{L} := \{\{A \in \mathscr{P} \mid A \cup B \in \Delta\} \mid B \in \mathfrak{M}\}$. Note that $\mathscr{L}$ is the set of shadows of the elements of $\mathfrak{M}$ on the set $\mathscr{P}$. Then by (B2) and (B3) the pair $(\mathscr{P},\mathscr{L})$ is a point-line space which we call the ***i***-space of $(\Delta,\mathfrak{A})$. A point-line space that is isomorphic to $(\mathscr{S},\mathscr{L})$ is called a *point-line space of type* $\mathrm{X}_{n,i}$ or simply $\mathrm{X}_{n,i}$*-space*..

From Theorem B.2.1 it follows that spaces of type $\mathrm{A}_{n,1}$ or $\mathrm{A}_{n,n}$ are projective spaces of rank n. The spaces of types $\mathrm{C}_{n,1}$ for $n \geq 2$ and $\mathrm{D}_{n,1}$ for $n \geq 3$ are non-degenerate polar spaces of rank n which follows from Theorems B.2.2 and B.2.3. Furthermore, a space of type $\mathrm{C}_{n,n}$ is a dual polar space; see [Tit74, 12.1]. By definition, the lines of these point-line spaces are the shadows of a simplex of type J on the set of vertices of type i, where i is either 1 or n and J equals

$\{1,\ldots,n\} \setminus \{i\}$. Note that in all these cases the shadow of a simplex A of type J depends on only one vertex of A, namely the simplex of type 2 if $i = 1$ and the simplex of type $n-1$ if $i = n$.

B.3 Exceptional types

Before we consider some types of point-line spaces of weak buildings, we introduce some classes of point line spaces who are named after some subspaces they possess.

Definition B.3.1. A point-line space $\mathscr{S}$ is called *paraprojective* if every singular subspace of $\mathscr{S}$ is a projective space.

Definition B.3.2. Let $\mathscr{S}$ be a connected partial linear gamma-space possessing a collection of convex subspaces called *symplecta* each of which is a non-degenerate polar space of rank ≥ 2 such that the following two properties are satisfied:

(PP1) Every line of $\mathscr{S}$ is contained in a symplecton.

(PP2) Every pair of non-collinear points having at least two common neighbours is contained in a unique symplecton.

Then we call $\mathscr{S}$ a *parapolar space*.
A pair of points at distance 2 is called a *special pair* if they have exactly one common neighbour and a *symplectic pair* otherwise. A parapolar space that possesses no special pair is called a *strongly parapolar space*.

Since by Proposition A.2.6 a non-degenerate polar space equals the convex span of any pair of its points that are non-collinear, it follows directly from (PP2) that the convex span of a symplectic pair is always the unique symplecton containing it.

In [Bue82] F. Buekenhout defines two classes of point-line spaces that are quite similar to parapolar spaces. To adopt the results of [Bue82], we introduce these point-line spaces and compare them with parapolar spaces.

A *polarised space* is a point-line space $\mathscr{S}$ satisfying the following conditions:

(Bu1) $\mathscr{S}$ is a gamma space.

(Bu2) Let p and q be points at distance 2. Then $\{p,q\}^{\perp}$ is either a singleton or a non-degenerate polar space[1] of finite rank ≥ 2.

(Bu3) $\mathscr{S}$ is connected and non-singular.

[1] Note that in [Bue82] a polar space is non-degenerate by definition.

(Bu4) Let p and q be points at distance 2 such that $\{p,q\}^{\perp}$ contains a single point s. Then there are points p' and q' in $s^{\perp} \setminus \{s\}$ such that (p,p',q',q) is a path of length 3.

As for parapolar space, we call a pair of points at distance 2 a special pair if they have exactly one common neighbour and symplectic pair otherwise.

A polarised space $\mathscr{S}$ has the following properties; see [Bue82]:

(BuA) $\mathscr{S}$ is partial linear.

(BuB) $\mathscr{S}$ is paraprojective.

(BuC) Let p and q be points at distance 2. Then $\langle p,q\rangle_{\mathrm{g}}$ is a non-degenerate polar space of finite rank. Moreover, $\langle p,q\rangle_{\mathrm{g}} = \langle x,y\rangle_{\mathrm{g}}$ for every two non-collinear points x and y of $\langle p,q\rangle_{\mathrm{g}}$.

We determine the following correspondence between parapolar and polarised spaces.

Proposition B.3.3. *Let $\mathscr{S}$ be a point line space. Then the following two properties are equivalent:*

(a) *$\mathscr{S}$ is a parapolar space of symplectic rank ≥ 3 that fulfils (Bu4).*

(b) *$\mathscr{S}$ is a polarised space.*

Proof. (a) $\Rightarrow$ (b): A parapolar space satisfies by definition (Bu1) and (Bu3). Now let (p,q) be a symplectic pair of $\mathscr{S}$. Then p and q are contained in a unique symplecton Y which is a non-degenerate polar space. Since a symplecton is a convex subspace, we obtain $\{p,q\}^{\perp} \leq Y$. Now it follows from Proposition A.2.5 that $\{p,q\}^{\perp}$ is a non-degenerate polar space. This implies that a parapolar space fulfils (Bu2).
(b) $\Rightarrow$ (a): Let (p,q) be a symplectic pair of $\mathscr{S}$. Then $\langle p,q\rangle_{\mathrm{g}}$ is a non-degenerate polar space of finite rank by (BuC). Moreover, $\langle p,q\rangle_{\mathrm{g}} = \langle x,y\rangle_{\mathrm{g}}$ for every two non-collinear points x and y of $\langle p,q\rangle_{\mathrm{g}}$. Since $\{p,q\}^{\perp}$ has rank ≥ 2, we obtain $\mathrm{rk}(\langle p,q\rangle_{\mathrm{g}}) \geq 3$.
Now let l be a line of $\mathscr{S}$ and let p and q be distinct point on l. Assume there is a point $r \in p^{\perp} \setminus q^{\perp}$. Then $\mathrm{dist}(r,q) = 2$ and $l \leq \langle r,q\rangle_{\mathrm{g}}$. Now assume $p^{\perp} = q^{\perp}$ and $p^{\perp}$ is non-singular. Then there are point r and s in $p^{\perp}$ at distance 2 and we obtain $l \leq \langle r,s\rangle_{\mathrm{g}}$. Finally assume $p^{\perp} = q^{\perp}$ and $p^{\perp}$ is singular. Then by (Bu3) there is a point r such that $\mathrm{dist}(p,r) = 2$. Since $\langle p,r\rangle_{\mathrm{g}}$ is a non-degenerate polar space there are non-collinear points in $p^{\perp}$, a contradiction. Thus, $\mathscr{S}$ is a parapolar space. $\square$

Beside polarised spaces there is another kind of point-line spaces that occurs in [Bue82]: Let $r \in \mathbb{N}$ with $r \geq 2$. A *uniform polarised spaces of rank r* is a point-line spaces that satisfies (Bu1), (Bu4), (BuA), (BuB) and the following variations of (Bu2) and (Bu3):

(Bu2') Let p and q be points at distance 2. Then $\langle p,q\rangle_{\mathrm{g}}$ is either a polar space of rank r or consists of 2 lines.

(Bu3') $\mathscr{S}$ is connected.

Remark B.3.4. The axiom (Bu3') is weaker then (Bu3). With (BuA), (BuB) and (BuC) we conclude that a polarised space $\mathscr{S}$ is a uniform polarised spaces of rank r if and only if $\mathrm{rk}(\langle p,q\rangle_{\mathrm{g}}) = r$ for every two point p and q of $\mathscr{S}$ with $\mathrm{dist}(p,q) = 2$ and $|\{p,q\}^{\perp}| \geq 2$. Conversely, a uniform polarised spaces of rank r with $r \geq 3$ fulfils (Bu2) by Proposition A.2.5. Hence, a uniform polarised space of rank r is a polarised space if and only if it is not singular and $r \geq 3$.

A uniform polarised space $\mathscr{S}$ of rank r is said to be of *spherical type* if it satisfies the following properties:

(Sph1) Every singular subspace of rank $r-1$ of $\mathscr{S}$ is contained in a unique maximal singular subspace.

(Sph2) Let V and W be singular subspace of rank $r-1$ with $\mathrm{rk}(V \cap W) = r-2$ and $V \nleq W^{\perp}$ such that V is contained in a singular subspace X of rank r. Then X and W are maximal singular subspace.

(Sph3) Let U, V and W be singular subspaces of rank $r-1$ with $\mathrm{rk}(U \cap V) = \mathrm{rk}(V \cap W) = r-2$ such that $V \nleq W^{\perp}$ and U is a maximal singular subspace. Then W is a maximal singular subspace.

(Sph4) Let Y and Z be distinct symplecta that intersect in at least one singular subspace of rank $r-2$. Then $Y \cap Z$ is a singular subspace of rank $r-1$.

We merge two of the main results of [Bue82] and transfer this into the terminology of parapolar spaces.

Theorem B.3.5. *Let $\mathscr{S}$ be a point-line space. Then the following two condition are equivalent:*

(a) *$\mathscr{S}$ is a parapolar space with symplectic rank r, where $r \geq 5$, that contains more than one symplecton and satisfies (Bu4), (Sph1), (Sph2), (Sph3) and (Sph4).*

(b) *There is a weak building $(\Delta, \mathfrak{A})$ of type E_r, where $r \in \{6,7,8\}$, such that $\mathscr{S}$ is isomorphic to the 1-space of $(\Delta, \mathfrak{A})$. Moreover, let φ be an isomorphism from $\mathscr{S}$ onto the 1-space of $(\Delta, \mathfrak{A})$, denoted by $(\mathscr{P}, \mathscr{L})$. Then φ maps the singular subspaces of rank i of $\mathscr{S}$, where $i < r-3$, bijectively onto the shadows on $\mathscr{P}$ of a vertex of type $i+1$ and the symplecta of $\mathscr{S}$ are mapped bijectively onto the shadows on $\mathscr{P}$ of a vertex of type n.*

Proof. By Proposition B.3.3 and Remark B.3.4 the classes of polar spaces, polarised spaces and uniform polarised spaces of rank r coincide if we demand (Bu4) to be fulfilled and that every symplecton has to be of rank r, where $r \geq 3$. Hence, the claim follows from [Bue82, Theorems 2 and 3]. □

Motivated by this theorem we call a parapolar space to be of *spherical type* if it fulfils (Bu4), (Sph1), (Sph2), (Sph3) and (Sph4).

Proposition B.3.6. *Let $\mathscr{S}$ be a point-line space of type $\mathrm{E}_{6,1}$. Then $\mathscr{S}$ has the following properties:*

(i) *There is no special pair in $\mathscr{S}$.*

(ii) *Let Y and Z be symplecta of $\mathscr{S}$ that contain a common line. Then Y and Z contain a common generator.*

(iii) *The diameter of $\mathscr{S}$ equals* 2. *Moreover, for every point $p \in \mathscr{S}$ there is a point q with* $\mathrm{dist}(p,q) = 2$.

(iv) *Every two symplecta of $\mathscr{S}$ intersect.*

(v) *Let Y and Z be two symplecta that intersect in a single point p. Then* $\mathrm{dist}(q,p) = \mathrm{dist}(q,Z)$ *for every point $q \in Y$.*

(vi) *For every point $p \in \mathscr{S}$ there is a symplecton at distance* 2 *to p and for every symplecton $Y \leq \mathscr{S}$ there is a point at distance* 2 *to Y.*

Proof. By Theorem B.3.5 we know that $\mathscr{S}$ is a parapolar space of symplectic rank 5 that contains two symplecta.
(i) Let p and q be two points at distance 2 and let s be a point that is collinear to both p and q. Then the residue of $\{s\}$ is the geometry of a building of type D_5. By Theorem B.2.3, the symplecta of $\mathscr{S}$ that contain s are the points of a non-degenerate polar space $\mathscr{D}$ of rank 5. Moreover, the dual polar graph of $\mathscr{D}$ is bipartite such that for any two adjacent generators of $\mathscr{D}$, exactly one of them consists of all symplecta of $\mathscr{S}$ that contain a given line $l \leq \mathscr{S}$ through s. Hence by Proposition A.2.20, two generators of $\mathscr{D}$ that consist of the symplecta containing a given line through s cannot be disjoint. In other words, there is a symplecton containing the lines ps and qs and the claim follows.
(ii) For any point $p \in l$ the residue of the flag $\{p,l\}$ is the geometry of a building of type A_4. By Theorem B.2.1, the symplecta of $\mathscr{S}$ containing l are the points of a projective space $\mathscr{D}$ of rank 4 and every line of $\mathscr{D}$ consists of the symplecta containing a given subspace $S \leq \mathscr{S}$ with $\mathrm{rk}(S) = 4$ and $l \leq S$. Since projective spaces are linear, we conclude that every two symplecta of $\mathscr{S}$ that contain l have a singular subspace of rank 4 in common. Since $\mathrm{yrk}(\mathscr{S}) = 5$, this subspace is a common generator.
(iii) Since $\mathscr{S}$ contains a symplecton, we know $\mathrm{diam}(\mathscr{S}) \geq 2$. Now suppose there are points p and q in $\mathscr{S}$ with $\mathrm{dist}(p,q) = 3$. Then there is a line $l \leq \mathscr{S}$ such that $\mathrm{dist}(p,l) = \mathrm{dist}(q,l) = 1$. By (i) we know that both $\langle p,l \rangle_{\mathrm{g}}$ and $\langle q,l \rangle_{\mathrm{g}}$ are

symplecta. Since $p^\perp$ and $q^\perp$ are disjoint and both $p^\perp$ and $q^\perp$ contain a hyperplane of $\langle p,l\rangle_{\mathrm{g}} \cap \langle q,l\rangle_{\mathrm{g}}$, we conclude $\langle p,l\rangle_{\mathrm{g}} \cap \langle q,l\rangle_{\mathrm{g}} = l$. This is a contradiction to (ii) and therefore $\mathrm{diam}(\mathscr{S}) = 2$. Since $\mathscr{S}$ is a parapolar space, every line through a given point p is contained in a symplecton. Thus, there is a point at distance 2 to p.
(iv) By the symmetry of the diagram E_6 this is equivalent to the claim that every two distinct points of $\mathscr{S}$ are contained in a symplecton. For collinear points this follows from the fact that $\mathscr{S}$ is a parapolar space. For non-collinear points, the claim follows from (iii) together with (i).
(v) For $\mathrm{dist}(p,q) = 0$ there is nothing to prove. For $\mathrm{dist}(p,q) = 1$ the claim follows since $Y \cap Z = \{p\}$ and hence $q \notin Z$. It remains the case $\mathrm{dist}(p,q) = 2$. Suppose there is a point $q' \in Z$ with $q \perp q'$. Every point $p' \in Z \setminus \{p\}$ that is collinear to p is non-collinear to q since otherwise $p' \in \langle p,q\rangle_{\mathrm{g}} = Y$ and consequently, $p' \in Y \cap Z$. Thus, $\mathrm{dist}(q',p) = 2$ and for any point p' with $p \perp p' \perp q'$, we obtain $\mathrm{dist}(q,p') = 2$. By (i) $Z' := \langle q,p'\rangle_{\mathrm{g}}$ is a symplecton. Since $q' \in \langle q,p'\rangle_{\mathrm{g}}$, the symplecta Z' and Z have the line $p'q'$ in common. Thus, (ii) implies that Z' and Z have a generator G in common. Since $q \in Z'$, $p \in Z$ and $\mathrm{rk}(G) = 4$, we conclude that $S := G \cap p^\perp \cap q^\perp$ is a singular subspace of rank ≥ 2. Since $S \leq \langle p,q\rangle_{\mathrm{g}}$, we obtain $S \leq Y \cap Z$, a contradiction. Thus, $\mathrm{dist}(q,Z) = 2$.
(vi) By (i) and (iii) there is a symplecton Z such that $p \in Z$ for a given point p. Since Z is a non-degenerate polar space, we know that there is a point $q \in Z$ with $\mathrm{dist}(p,q) = 2$. In other words, for every point of Z, there exists a point such that Z is the only symplecton containing these two points. By the symmetry of the diagram E_6, we conclude equivalently that there is a symplecton Y such that q is the only point contained in both Y and Z. Thus, $\mathrm{dist}(p,Y) = 2$ by (v).
Conversely, for a given symplecton Y we choose a point $q \in Y$. As above there is a symplecton Z such that $Y \cap Z = \{q\}$ and a point $p \in Z$ with $\mathrm{dist}(p,q) = 2$. This is the same situation as above. Hence, it remains to prove $\mathrm{dist}(p,Y) = 2$. Again $\mathrm{dist}(p,Y) = 2$ by (v). ☐

Proposition B.3.7. *Let $\mathscr{S}$ be the point-lines space of type* $\mathrm{E}_{7,1}$. *Then $\mathscr{S}$ has the following properties:*

(i) *There is no special pair in $\mathscr{S}$.*

(ii) *Let Y and Z be symplecta of $\mathscr{S}$ that have a point p in common. Then Y and Z have a line through p in common.*

(iii) *The diameter of $\mathscr{S}$ equals* 3*. Moreover, for every two point p and p' in $\mathscr{S}$ there is a point q with* $\mathrm{dist}(p,q) = \mathrm{dist}(p,p') + \mathrm{dist}(p',q) = 3$.

(iv) *Let p be a point and let l be a line of $\mathscr{S}$. Then* $\mathrm{dist}(p,l) \leq 2$.

Proof. By Theorem B.3.5 we know that $\mathscr{S}$ is a parapolar space of symplectic

rank 6 that contains two symplecta. For any point $p \in \mathscr{S}$, the residue of $\{p\}$ is the geometry of a building of type E_6. More precisely, the lines of $\mathscr{S}$ through p are the points of a point-line space of type $E_{6,1}$ that we denote by $\mathscr{D}_p$. The lines of $\mathscr{D}_p$ consist of all lines through p that are contained in a given singular subspace $S \leq \mathscr{S}$ with $p \in S$ and $\mathrm{rk}(S) = 2$. The symplecta of $\mathscr{D}_p$ consist of all lines through p that are contained in a given symplecton of $\mathscr{S}$ containing p.
(i) Let p and q be points at distance 2 and let $s \in p^\perp \cap q^\perp$. Since $p \not\perp q$, there is no singular subspace in $\mathscr{S}$ that contains p and q. Thus, the lines sp and sq are non-collinear points in $\mathscr{D}_s$. By Proposition B.3.6(iii) this implies that sp and sq have distance 2 in $\mathscr{D}_s$. By Proposition B.3.6(i) we conclude that there is a symplecton in $\mathscr{D}_s$ containing sp and sq and thus, there is a symplecton in $\mathscr{S}$ containing p and q.
(ii) Let $p \in Y \cap Z$. Since every two symplecta of $\mathscr{D}_p$ intersect by Proposition B.3.6(iv), we conclude that Y and z have a line through p in common.
(iii) Suppose there are points p and q in $\mathscr{S}$ with $\mathrm{dist}(p,q) = 4$. Then there is a point s such that $\mathrm{dist}(p,s) = \mathrm{dist}(q,s) = 2$. By (i) both $Y := \langle p,s\rangle_{\mathrm{g}}$ and $Z := \langle q,s\rangle_{\mathrm{g}}$ are symplecta of $\mathscr{S}$. By (ii) there is a line l that is contained in both Y and Z. We obtain $\mathrm{dist}(p,l) = 1$ and $\mathrm{dist}(q,l) = 1$ and consequently, $\mathrm{dist}(p,q) \leq 3$, a contradiction.
For the second claim we may assume $\mathrm{dist}(p,p') < 3$ since otherwise there is nothing to show. Furthermore, we may restrain ourselves to the case $p \neq p'$ since the case $p = p'$ follows from any other case. Let l be a line through p with $\mathrm{dist}(p',l) = \mathrm{dist}(p,p') - 1$. Further let $Y \leq \mathscr{S}$ be a symplecton containing l. Since Y is a non-degenerate polar space, there is a point $s \in Y$ such that $\mathrm{dist}(s,p) = \mathrm{dist}(s,p') + \mathrm{dist}(p',p) = 2$. Hence, we have to find a point $q \perp s$ with $\mathrm{dist}(q,p) = 3$ to finish the proof.
By Proposition B.3.6(vi) there is for each symplecton in $\mathscr{D}_s$ a point in $\mathscr{D}_s$ at distance 2. Hence, there is a line g through s such that every line $h \leq Y$ through s is a point of $\mathscr{D}_s$ that is non-collinear to g. In other words, there is no singular subspace in $\mathscr{S}$ that contains g and h. Thus, s is the only point of $Y \cap s^\perp$ that is collinear to p. Suppose there is a point $p' \in (Y \setminus s^\perp) \cap p^\perp$. Then $p \in \langle s,p'\rangle_{\mathrm{g}} = Y$, a contradiction to $g \not\leq Y$. Thus, $Y \cap p^\perp = \{s\}$.
Let $h \leq Y$ be an arbitrary line through s. Since s is the only point on h that is collinear to p, we conclude by (i) that $Z := \langle p,h\rangle_{\mathrm{g}}$ is a symplecton of $\mathscr{S}$. Since $Y \neq Z$ and every symplecton is the convex span of any non-collinear pair of its points, we conclude that $Y \cap Z$ is singular. Since $p^\perp$ contains a hyperplane of $Y \cap Z$, we obtain $Y \cap Z = h$. Let $q \in Y$ be a point that is not collinear to s. Suppose $\mathrm{dist}(p,q) = 2$. Then $\langle p,q\rangle_{\mathrm{g}}$ is a symplecton of $\mathscr{S}$ by (i). Hence by (ii), $\langle p,q\rangle_{\mathrm{g}}$ and Y have a line l through q in common. We obtain $s \notin l$ since $\mathrm{dist}(s,q) = 2$. Since $p^\perp \cap l \neq \varnothing$, this is a contradiction to $Y \cap p^\perp = \{s\}$. Thus, $\mathrm{dist}(p,q) = 3$ since $p \perp s$.

(iv) Let $q \in l$. By (iii) we may assume $\operatorname{dist}(p,q) = 3$ since otherwise there is nothing to prove. Let s be a point with $s \perp q$ and $\operatorname{dist}(p,s) = 2$. Then $Y := \langle p,s\rangle_{\mathrm{g}}$ is a symplecton of $\mathscr{S}$ by (i). Assume $l \nleq s^{\perp}$. Then $\langle s,l\rangle_{\mathrm{g}}$ is a symplecton. By (ii) there is a line g of $\mathscr{S}$ through s that is contained in both Y and $\langle s,l\rangle_{\mathrm{g}}$. Hence, there is a point $p' \in g$ with $p' \perp p$. Since $p' \in \langle s,l\rangle_{\mathrm{g}}$, there is a point on l that is collinear to p' and we obtain $\operatorname{dist}(p,l) = 2$.
Now assume $l \leq s^{\perp}$. We may assume $s \notin l$ since otherwise we are done. Then $\langle s,l\rangle$ is a singular subspace of rank 2. in $\mathscr{D}_s$ every line is contained in a symplecton. This implies that there is a symplecton $Z \leq \mathscr{S}$ containing $\langle s,l\rangle$. Again there is a line $g \leq \mathscr{S}$ through s that is contained in $Y \cap Z$ since in $\mathscr{D}_s$ every two symplecta intersect. As before there is a point $p' \in g$ with $p' \perp p$ and $\operatorname{dist}(p',l) = 1$. □

Proposition B.3.8. *Let $\mathscr{S}$ be the point-lines space of a weak building of type* $\mathrm{E}_{8,1}$. *Then $\mathscr{S}$ has a special pair.*

Proof. Let p be a point of $\mathscr{S}$. The residue of $\{p\}$ is the geometry of a building of type E_7. More precisely, the lines of $\mathscr{S}$ through p are the points of the point-lines space of a weak building of type $\mathrm{E}_{7,1}$ that we denote by $\mathscr{D}$. Moreover, for a symplecton $Y \leq \mathscr{S}$ with $p \in Y$, the set of lines of Y through p is a symplecton of $\mathscr{D}$. For a singular subspace $S \leq \mathscr{S}$ with $p \in S$, the set of lines of S through p is a line of $\mathscr{D}$.
By Proposition B.3.7(iii) there are lines g and h of $\mathscr{S}$ through p such that g and h are points of $\mathscr{D}$ at distance 3. Let $q \in g \setminus \{p\}$ and $q' \in h \setminus \{p\}$. Suppose $q \perp q'$. Then $h \leq q^{\perp}$ and hence $\langle g,h\rangle$ is a singular subspace of $\mathscr{S}$ of rank 2. Hence, g and h are collinear in $\mathscr{D}$, a contradiction. Thus, $\operatorname{dist}(q,q') = 2$. Suppose (q,q') is a symplectic pair. Then $\langle q,q'\rangle_{\mathrm{g}}$ contains p. Thus, the lines g and h are points of the symplecton of $\mathscr{D}$ that consists of all lines of $\langle q,q'\rangle_{\mathrm{g}}$ through s. This leads to a contradiction since g and h have distance 3 in $\mathscr{D}$. □

Definition B.3.9. A strongly parapolar space of spherical type with symplectic rank r, where $r \geq 5$, that possesses at least two symplecta is called an *exceptional strongly parapolar space*.

Theorem B.3.10. *Let $\mathscr{S}$ be a point-line space and let $r \in \mathbb{N}$. Then the following two properties are equivalent:*

(a) *$\mathscr{S}$ is an exceptional strongly parapolar space with symplectic rank $r-1$.*

(b) *$\mathscr{S}$ is a point-line space of type* $\mathrm{E}_{r,1}$ *with* $r \in \{6,7\}$

Proof. Since a strongly parapolar space possesses no special pair, (Bu4) is vacuously fulfilled. By Propositions B.3.6(i) and B.3.7(i) we know that point-line spaces of types $\mathrm{E}_{6,1}$ and $\mathrm{E}_{7,1}$ are strongly parapolar. In contrast, point-line spaces of type $\mathrm{E}_{8,1}$ are not strongly parapolar by Proposition B.3.8. Thus, the claim follows immediately from Theorem B.3.5. □

C The independence of the axioms

In this chapter we prove that the axioms given in Definition 2.1.1 are independent by giving counterexamples that fulfil precisely three of the given axioms.

Example C.1. Set $\mathscr{P} := \mathbb{Z}/6\mathbb{Z}$ and $\mathscr{L} := \{\{v, v+1, v+2\} \mid v \in \mathscr{P}\}$. Call two points of the point-line space $\mathscr{S} := (\mathscr{P}, \mathscr{L})$ opposite if and only if they are distinct. The convex span of any two distinct points equals $\mathscr{S}$ since whenever a subspace contains the line $\{v, v+1, v+2\}$ for a point $v \in \mathscr{P}$, it also contains the line $\{v+1, v+2, v+3\}$. Hence, the convex span of two points of $\mathscr{S}$ is either a singleton or $\mathscr{S}$.
Since for any point $v \in \mathscr{P}$ the only point non-opposite v is v itself, (A2) is always fulfilled. Furthermore, for (A3) there is only case to check which is the case where x, y and z coincide since otherwise there is no way to decrease the codistance to y. Since in this case $\langle y, z\rangle_{\mathrm{g}} = \{y\}$ holds, (A3) is also fulfilled. Finally, (A4) is fulfilled since for any choice of the points x and z there is a point collinear to x and opposite z.
For any point $v \in \mathscr{P}$, we obtain $\operatorname{dist}(v, v+3) = 2$ and $\langle v, v+3\rangle_{\mathrm{g}} = \mathscr{S}$. Since v is the unique non-opposite point to v, we obtain $\operatorname{cod}(v, \mathscr{S}) = 1$. Thus, (A1) does not hold in $\mathscr{S}$.

Example C.2. Set $\mathscr{P} := \mathbb{Z}$ and $\mathscr{L} := \{\mathbb{Z}\}$. Call two points u and v of the point-line space $\mathscr{S} := (\mathscr{P}, \mathscr{L})$ opposite if and only if $u + v < 0$. Since $\mathbb{Z}$ is the only line of $\mathscr{S}$, we conclude that the convex span of any pair of distinct points equals $\mathscr{S}$.
Since for every point x we obtain $x \leftrightarrow -x-1$, the opposition relation of $\mathscr{S}$ is total. Moreover, since $x \not\leftrightarrow -x$, we conclude $\operatorname{cod}(x, \mathscr{S}) = 1$ for every point x and therefore (A1) is fulfilled. Now let w, x, y and z be points such that $\operatorname{cod}(w, y) < \operatorname{cod}(x, y)$ and $z \in \operatorname{copr}_{\langle y,z\rangle_{\mathrm{g}}}(x)$. Since $\operatorname{cod}(x, \mathscr{S}) = 1$, this implies $w \leftrightarrow y$ and $\operatorname{cod}(x, y) = 1$. Therefore $w + y < 0 \leq x + y$ and we conclude $w < x$. Now $w + v < x + v$ implies

that every point that is opposite x is opposite w and thus, $\mathrm{copr}_{\mathscr{S}}(w) < \mathrm{copr}_{\mathscr{S}}(x)$. Since $\langle y,z\rangle_{\mathrm{g}}$ is either $\{y\}$ or $\mathscr{S}$, (A3) holds. Since $\mathscr{S}$ is singular (A4) holds, too. For every point x, the points $-x$ and $1-x$ are contained in $\mathscr{S} = \langle x, x+1\rangle_{\mathrm{g}}$. Since $x \leftrightarrow -x-1 \in \mathscr{S}$ and $x \perp x+1$, (A2) is not fulfilled.

Example C.3. Set $\mathscr{P} := \mathbb{Z}/9\mathbb{Z}$ and $\mathscr{L} := \{\{v,v+1\},\{v,v+3\} \mid v \in \mathscr{P}\}$. Call two points u and v of the point-line space $\mathscr{S} := (\mathscr{P},\mathscr{L})$ opposite if $u = v+2$ or $v = u+2$. Let $v \in \mathscr{P}$, then $\{v+1,v+3,v+6,v+8\}$ is the set of points at distance 1 to v and every point of $\mathscr{P} \setminus v^{\perp}$ has distance 2 to v. Since $v+2$ and $v+7$ are the points that are opposite v, we conclude that for a point u, we obtain $u = v$ if and only if $\mathrm{cod}(v,u) = 2$ and $u \in \mathscr{P} \setminus \{v,v+2,v+7\}$ if and only if $\mathrm{cod}(v,u) = 1$. Since all lines of $\mathscr{S}$ are short, we obtain $\langle u,v\rangle_{\mathrm{g}} = \{u,v\}$ for any two collinear points u and v. Now let $\mathrm{dist}(u,v) = 2$. If $u = v+2$, then $v+1$ and $v+3$ are both contained in $\{u,v\}^{\perp}$ and hence in $\langle u,v\rangle_{\mathrm{g}}$. This implies $\langle v+1,v+3\rangle_{\mathrm{g}} \le \langle v,v+2\rangle_{\mathrm{g}}$ and by repeating this argument $\langle u,v\rangle_{\mathrm{g}} = \mathscr{S}$. Analogously, $\langle u,v\rangle_{\mathrm{g}} = \mathscr{S}$ for $v = u+2$ and hence $u = v+7$. If $u = v+4$, then $v+1$ and $v+3$ are both contained in $\{u,v\}^{\perp}$ and we obtain $\langle u,v\rangle_{\mathrm{g}} \ge \langle v+1,v+3\rangle_{\mathrm{g}} = \mathscr{S}$. Analogously, $\langle u,v\rangle_{\mathrm{g}} = \mathscr{S}$ for $u = v+5$.
For a point x, the two opposite points $x+2$ and $x+7$ are not collinear and hence, on every line there is at least one point non-opposite x. Since $\mathrm{copr}_{\mathscr{S}}(x) = \{x\}$ and $\mathrm{cod}(x,\mathscr{S}) = 2$, we conclude that (A1) and (A2) are both fulfilled. Since every point has codistance 2 to only itself, we conclude $\mathrm{cod}(u,v) = \mathrm{cod}(v,u)$ for any two points u and v of $\mathscr{S}$ and therefore (A4) is fulfilled.
Let $x \in \mathscr{P}$ and set $z := x$ and $y := x+4$. Then $\langle y,z\rangle_{\mathrm{g}} = \mathscr{S}$. Moreover, $z \in \mathrm{copr}_{\mathscr{S}}(x)$ and $\mathrm{cod}(x,y) = 1$. Now set $w := x+6$. Then $x \perp w$ and $w \leftrightarrow y$. Hence, $\mathrm{copr}_{\mathscr{S}}(w) = \{w\} \nsubseteq \mathrm{copr}_{\mathscr{S}}(x)$ and (A3) does not hold.

Example C.4. Set $\mathscr{P} := \mathbb{Z}/6\mathbb{Z}$ and $\mathscr{L} := \{\{v,v+1\} \mid v \in \mathscr{P} \wedge v \subseteq 2\mathbb{Z}\}$. Then the point-line space $\mathscr{S} := (\mathscr{P},\mathscr{L})$ has three connected components each of which consists of a single line that contains two points. Let $v \in \mathscr{P}$. If $v \subseteq 2\mathbb{Z}$, then $v+5$ is the only point opposite v, otherwise $v+1$ is the only point opposite v. Note that this opposition relation is symmetric.
Let y and z be two points at finite distance and set $V := \langle y,z\rangle_{\mathrm{g}}$. Then either $y = z$ and hence, $V = \{y\}$ or $y \perp z$ and $V = \{y,z\}$. Let $x \in \mathscr{P}$ such that $\mathrm{cod}(x,V)$ is finite. Then $x \leftrightarrow y$ or $x \leftrightarrow z$ since V is a connected component of $\mathscr{S}$. We may assume $y \leftrightarrow x$. Since y is the only point opposite to x, (A1) and (A2) are satisfied. Moreover, (A3) is vacuously fulfilled.
Now let $x \in \mathscr{P}$ with $x \subseteq 2\mathbb{Z}$. Then $\mathrm{cod}(x,x+2) = \infty$ and $\mathrm{cod}(x+2,x) = 1$. Thus, (A4) does not hold.

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List of Notations