Handling the Low-frequency Breakdown of the PMCHWT Integral Equation with the quasi-Helmholtz Projectors

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Abstract — **This contribution presents a quasi-Helmholtz projectors based regularization of the low frequency breakdown of the PMCHWT integral equation. The PMCHWT equation in the low-frequency regime shows an ill-conditioned behavior inherited from the Electric Field Integral Operators it contains. The stabilization via quasi-Helmholtz projectors, differently from the use of standard Loop-Star/Tree decompositions, does not introduce an additional mesh-size-related ill-conditioning and it applies smoothly to both simply and non-simply connected geometries. The presentation of the main formulation will be complemented by numerical results demonstrating the effectiveness and accuracy of the proposed scheme.**

1 INTRODUCTION

The Poggio-Miller-Chan-Harrington-Wu-Tsai (PMCHWT) integral equation is one of the main formulations for solving radiation and scattering problems for penetrable objects in the frequency domain [1].

The equation is of the first kind and, similarly to the electric field integral equation (EFIE) [2], it suffers from severe ill-conditioning problems when the simulation frequency is low. This is the well-known "low-frequency" breakdown phenomenon [3, 4]. Historically, the breakdown can be solved by using quasi-Helmholtz decompositions such as Loop-Star and Loop-Tree basis functions [5, 6, 7, 8]. These decompositions however, although solving the frequency-related ill-conditioning, introduce a parasitic and mesh-dependent form of ill-conditioning [9] since they can be interpreted as graph counterparts of differential operators [10].

Recently, a solution to this problem has been introduced for the EFIE based on newly introduced quasi-Helmholtz projectors [11]. These projectors realize a quasi-Helmholtz decomposition, but in an implicit and well–conditioned way, so that the undesired mesh-dependent ill-conditioning generated by standard approaches does not occur. The approach has been also successfully exploited in time domain [12].

This work generalizes this strategy to the treatment of the low-frequency breakdown of the PMCHWT integral equation. The generalization is challenging since, differently from the EFIE, the off-diagonal blocks of the PMCHWT requires a suitable analysis and treatment to ensure low-frequency stability.

Numerical results corroborates the theory and shows the practical applicability of the new approach.

2 BACKGROUND AND NOTATION

Consider a dielectric region Ω of intrinsic impedance $\eta_{k'}$ with its external region $\Omega/\mathbb{R}^3$ of intrinsic impedance η_k. We denote with $\Gamma = \partial\Omega$ its boundary surface with outward directed normal $\hat{n}$. The Electric Field Integral Operator (EFIO) and Magnetic Field Integral Operator (MFIO) read

$$\mathcal{T}_\alpha(\boldsymbol{f}(\boldsymbol{r})) = \hat{n} \times i\alpha \int_\Gamma \frac{e^{i\alpha|\boldsymbol{r}-\boldsymbol{r}'|}}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|}\boldsymbol{f}(\boldsymbol{r}')d\Gamma - \hat{n} \times \frac{1}{i\alpha}\nabla \int_\Gamma \frac{e^{i\alpha|\boldsymbol{r}-\boldsymbol{r}'|}}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|}\nabla' \cdot \boldsymbol{f}(\boldsymbol{r}')d\Gamma, \quad (1)$$

$$\mathcal{K}_\alpha(\boldsymbol{f}(\boldsymbol{r})) = -\hat{n} \times \int_\Gamma \nabla \frac{e^{i\alpha|\boldsymbol{r}-\boldsymbol{r}'|}}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|} \times \boldsymbol{f}(\boldsymbol{r}')d\Gamma, \quad (2)$$

where the subscript α represents the wave number k or k' for the region $\Omega/\mathbb{R}^3$ or Ω respectively. The PMCHWT equation models scattering from the penetrable region Ω when illuminated by an incident electromagnetic field $(\boldsymbol{E}_i(\boldsymbol{r}), \boldsymbol{H}_i(\boldsymbol{r}))$. It reads

$$\begin{bmatrix} \mathcal{T}_k/\eta_k + \mathcal{T}_{k'}/\eta_{k'} & -(\mathcal{K}_k + \mathcal{K}_{k'}) \\ (\mathcal{K}_k + \mathcal{K}_{k'}) & (\eta_k\mathcal{T}_k + \eta_{k'}\mathcal{T}_{k'}) \end{bmatrix} \begin{bmatrix} \boldsymbol{M}(\boldsymbol{r}) \\ \boldsymbol{J}(\boldsymbol{r}) \end{bmatrix} = \begin{bmatrix} -\hat{n} \times \boldsymbol{H}_i(\boldsymbol{r}) \\ -\hat{n} \times \boldsymbol{E}_i(\boldsymbol{r}) \end{bmatrix}, \quad (3)$$

where $\boldsymbol{J}(\boldsymbol{r})$ and $\boldsymbol{M}(\boldsymbol{r})$ represent the electric and magnetic currents, respectively.

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On each edge of the triangular surface mesh approximating the surface Γ, an RWG basis function $\boldsymbol{f}_n(\boldsymbol{r})$ is defined [2]. This set approximates the currents as $\boldsymbol{J}(\boldsymbol{r}) = \sum_{n=1}^{N} \alpha_n \boldsymbol{f}_n(\boldsymbol{r})$ and $\boldsymbol{M}(\boldsymbol{r}) = \sum_{n=1}^{N} \beta_n \boldsymbol{f}(\boldsymbol{r})$. The overall discretized PMCHWT equation is given by

$$\begin{bmatrix} \mathbf{T}_k/\eta_k + \mathbf{T}_{k'}/\eta_{k'} & -(\mathbf{K}_k + \mathbf{K}_{k'}) \\ (\mathbf{K}_k + \mathbf{K}_{k'}) & (\eta_k \mathbf{T}_k + \eta_{k'} \mathbf{T}_{k'}) \end{bmatrix} \begin{bmatrix} \beta \\ \alpha \end{bmatrix} = \begin{bmatrix} \mathbf{h} \\ \mathbf{e} \end{bmatrix} \tag{4}$$

where, element-wise, $(\mathbf{T}_k)_{mn} = \langle \hat{n} \times \boldsymbol{f}_m(\boldsymbol{r}), \mathcal{T}_k(\boldsymbol{f}_n(\boldsymbol{r})) \rangle_\Gamma$ using the notation $\langle \boldsymbol{a} \cdot \boldsymbol{b} \rangle_\chi = \int_\chi \boldsymbol{a} \cdot \boldsymbol{b} \; d\chi$ ($\mathbf{T}_{k'}$, $\mathbf{K}_k$ and $\mathbf{K}_k$ are similarly defined). The right hand side (RHS) vectors are given by $\mathbf{h}_m = \langle \hat{n} \times \boldsymbol{f}_m(\boldsymbol{r}), \boldsymbol{H}_i(\boldsymbol{r}) \rangle_\Gamma$ and $\mathbf{e}_m = \langle \hat{n} \times \boldsymbol{f}_m(\boldsymbol{r}), \boldsymbol{E}_i(\boldsymbol{r}) \rangle_\Gamma$. The EFIO blocks on the diagonal of the system matrix in equation (4) suffer from the low-frequency breakdown, a problem which is consequently transferred to the entire PMCHWT equation.

3 PROPOSED REGULARIZATION

To properly regularize the effect of low frequency breakdown in equation (4), it is useful to separate the different contributions in the EFIO

$$\begin{aligned} \mathbf{T}^a_{\boldsymbol{A}} &= \frac{1}{k}(\mathbf{T}^{\boldsymbol{A}}_k/\eta_k + \mathbf{T}^{\boldsymbol{A}}_{k'}/\eta_{k'}) \\ \mathbf{T}^a_\phi &= k(\mathbf{T}^\phi_k/\eta_k + \mathbf{T}^\phi_{k'}/\eta_{k'}) \\ \mathbf{T}^b_{\boldsymbol{A}} &= \frac{1}{k}(\mathbf{T}^{\boldsymbol{A}}_k\eta_k + \mathbf{T}^{\boldsymbol{A}}_{k'}\eta_{k'}) \\ \mathbf{T}^b_\phi &= k(\mathbf{T}^\phi_k\eta_k + \mathbf{T}^\phi_{k'}\eta_{k'}) \end{aligned} \tag{5}$$

where the subscripts ϕ and $\boldsymbol{A}$ represent the scalar and vector potential term respectively (the first and second term on the RHS of equation (1)). The MFIO matrix should also be decomposed as

$$\begin{aligned} \mathbf{K}_z &= 2\mathbf{K}_0 \\ \mathbf{K}_e &= \mathbf{K}_k + \mathbf{K}_{k'} - \mathbf{K}_z \end{aligned} \tag{6}$$

where $\mathbf{K}_0$ is the static MFIO matrix corresponding to $k = 0$. Using equation (5) and (6), the system matrix in the equation (4) can be written as

$$\begin{bmatrix} k\mathbf{T}^a_{\boldsymbol{A}} + \frac{1}{k}\mathbf{T}^a_\phi & -(\mathbf{K}_e + \mathbf{K}_z) \\ \mathbf{K}_e + \mathbf{K}_z & k\mathbf{T}^b_{\boldsymbol{A}} + \frac{1}{k}\mathbf{T}^b_\phi \end{bmatrix}. \tag{7}$$

The objective behind these matrix decompositions is to have a low-frequency-stable behavior of the conditioning of the overall system matrix after applying the quasi-Helmholtz projectors. These projectors can be defined using the loop/star-to-RWGs transformation matrices $\boldsymbol{\Sigma}$ and $\boldsymbol{\Lambda}$ (we omit the definition of these matrices for space limitations, the reader is invited to refer to [11] for an exhaustive definition) as $\mathbf{P}^\Sigma = \boldsymbol{\Sigma}(\boldsymbol{\Sigma}\boldsymbol{\Sigma}^T)^+\boldsymbol{\Sigma}^T$ and $\mathbf{P}^{\Lambda H} = \mathbf{I} - \boldsymbol{\Sigma}(\boldsymbol{\Sigma}\boldsymbol{\Sigma}^T)^+\boldsymbol{\Sigma}^T$ on the surface mesh and its dual counterparts $\mathbb{P}^\Lambda = \boldsymbol{\Lambda}(\boldsymbol{\Lambda}\boldsymbol{\Lambda}^T)^+\boldsymbol{\Lambda}^T$ and $\mathbb{P}^{\Sigma H} = \mathbf{I} - \boldsymbol{\Lambda}(\boldsymbol{\Lambda}\boldsymbol{\Lambda}^T)^+\boldsymbol{\Lambda}^T$ on the dual mesh [11]. The decomposition operator matrix is defined as $\mathbf{M} = \mathbf{P}^{\Lambda H}\frac{1}{\sqrt{k}} + i\mathbf{P}^\Sigma\sqrt{k}$. Similarly, the associated dual decomposition operator matrix is defined as $\mathbb{M} = \mathbb{P}^\Lambda\frac{1}{\sqrt{k}} + i\mathbb{P}^{\Sigma H}\sqrt{k}$. The associated Gram matrix linking the basis functions on the primal and the dual mesh is defined as $[\mathbf{G}_{mix}]_{mn} = \left\langle \boldsymbol{f}_m(\boldsymbol{r}), \hat{\boldsymbol{n}}_{\boldsymbol{r}} \times \boldsymbol{f}^{BC}_n(\boldsymbol{r}) \right\rangle_\Gamma$ where $\{\boldsymbol{f}^{BC}_n(\boldsymbol{r})\}^N_{n=1}$ is the set of Buffa-Christiansen (BC) basis functions [13] defined on the dual mesh.

The proposed regularization based on the quasi-Helmholtz projectors for the EFIO blocks in the equation (7) reads

$$\begin{aligned} &\mathbb{M}^T\mathbf{G}^{-1}_{mix}\mathbf{T}^x\mathbf{M} = (\mathbb{P}^\Lambda\mathbf{G}^{-1}_{mix}\mathbf{T}^x_{\boldsymbol{A}}\mathbf{P}^{\Lambda H} - \mathbb{P}^{\Sigma H}\mathbf{G}^{-1}_{mix}\mathbf{T}^x_\phi\mathbf{P}^\Sigma) \\ &+ ik(\mathbb{P}^\Lambda\mathbf{G}^{-1}_{mix}\mathbf{T}^x_{\boldsymbol{A}}\mathbf{P}^\Sigma + \mathbb{P}^{\Sigma H}\mathbf{G}^{-1}_{mix}\mathbf{T}^x_{\boldsymbol{A}}\mathbf{P}^{\Lambda H}) \\ &- k^2\mathbb{P}^{\Sigma H}\mathbf{G}^{-1}_{mix}\mathbf{T}^x_{\boldsymbol{A}}\mathbf{P}^\Sigma \end{aligned} \tag{8}$$

where the superscript x can be either a or b depending on the diagonal block matrix. In order to simplify the above expression, we have used $\mathbf{T}^x_\phi\mathbf{P}^{\Lambda H} = \mathbf{0}$ and $\mathbb{P}^\Lambda\mathbf{G}^{-1}_{mix}\mathbf{T}^x_\phi = \mathbf{0}$. Along the same lines, the proposed regularization for the MFIO blocks is

$$\begin{aligned} &\mathbb{M}^T\mathbf{G}^{-1}_{mix}\mathbf{K}\mathbf{M} = \frac{1}{k}\mathbb{P}^\Lambda\mathbf{G}^{-1}_{mix}\mathbf{K}_e\mathbf{P}^{\Lambda H} + i(\mathbb{P}^\Lambda\mathbf{G}^{-1}_{mix}\mathbf{K}_z\mathbf{P}^\Sigma \\ &+ \mathbb{P}^\Lambda\mathbf{G}^{-1}_{mix}\mathbf{K}_e\mathbf{P}^\Sigma + \mathbb{P}^{\Sigma H}\mathbf{G}^{-1}_{mix}\mathbf{K}_z\mathbf{P}^{\Lambda H} \\ &+ \mathbb{P}^{\Sigma H}\mathbf{G}^{-1}_{mix}\mathbf{K}_e\mathbf{P}^{\Lambda H}) - k(\mathbb{P}^{\Sigma H}\mathbf{G}^{-1}_{mix}\mathbf{K}_e\mathbf{P}^\Sigma \\ &+ \mathbb{P}^{\Sigma H}\mathbf{G}^{-1}_{mix}\mathbf{K}_z\mathbf{P}^\Sigma) \end{aligned} \tag{9}$$

where we have used the property of static MFIE operator matrix as $\mathbb{P}^\Lambda\mathbf{G}^{-1}_{mix}\mathbf{K}_z\mathbf{P}^{\Lambda H} = 0$.

The proposed quasi-Helmholtz projected PMCHWT system, where the new unknown vectors are $\tilde{\alpha} = \mathbf{M}^{-1}\alpha$ and $\tilde{\beta} = \mathbf{M}^{-1}\beta$, reads

$$\begin{bmatrix} \mathbf{Z}_{11} & \mathbf{Z}_{12} \\ \mathbf{Z}_{21} & \mathbf{Z}_{22} \end{bmatrix} \begin{bmatrix} \tilde{\alpha} \\ \tilde{\beta} \end{bmatrix} = \begin{bmatrix} \mathbf{h}_p \\ \mathbf{e}_p \end{bmatrix} \tag{10}$$

where $\mathbf{Z}_{11} = \mathbb{M}^T\mathbf{G}^{-1}_{mix}(k\mathbf{T}^a_{\boldsymbol{A}} + \frac{1}{k}\mathbf{T}^a_\phi)\mathbf{M}$, $\mathbf{Z}_{12} = -\mathbb{M}^T\mathbf{G}^{-1}_{mix}(\mathbf{K}_e + \mathbf{K}_z))\mathbf{M}$, $\mathbf{Z}_{21} = \mathbb{M}^T\mathbf{G}^{-1}_{mix}(\mathbf{K}_e + \mathbf{K}_z)\mathbf{M}$ and $\mathbf{Z}_{22} = \mathbb{M}^T\mathbf{G}^{-1}_{mix}(k\mathbf{T}^b_{\boldsymbol{A}} + \frac{1}{k}\mathbf{T}^b_\phi)\mathbf{M}$.

4 NUMERICAL RESULTS

The proposed regularization is applied to a scenario involving scattering due to a homogeneous dielectric sphere of the radius 1m with relative permittivity as $\epsilon_r = 3$. The electromagnetic source is a plane wave travelling in the z direction and is polarized along the x axis with an electric field intensity of 1V/m. The condition number of the regularized system matrix stays constant when compared to the original PMCHWT matrix as shown in Figure 1 when the frequency decreases from 1MHz to 10^{-40}Hz. The far field computed using the proposed regularization shows a good agreement with the Mie series in Figure 2 and 3 for the frequency 1MHz and 10^{-40}Hz respectively.

In a second example, a torus model with the centerline radius of 3m and the cross-section radius of 0.5m with relative permittivity as $\epsilon_r = 3$ is chosen. The parameters for the electromagnetic source are kept equal to that of the sphere. The condition number of the regularized system matrix stays constant as seen in Figure 4 when the frequency is decreased till 10^{-40}Hz. The far field due to both the original and regularized formulation shows a perfect matching in Figure 5 for a frequency of 10MHz.

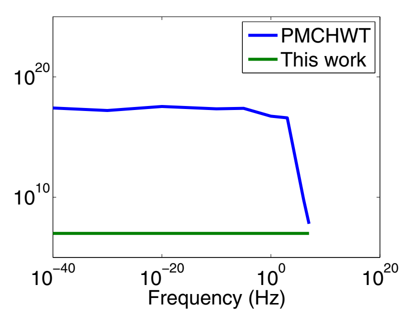

Figure 1: Sphere: Condition number w.r.t to frequency.

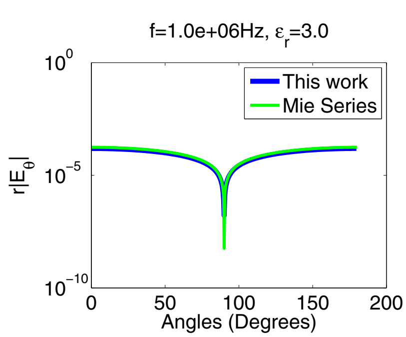

Figure 2: Sphere: Far field calculated at frequency of 1MHz.

5 CONCLUSION

We have presented a quasi-Helmholtz projectors based regularization for the system matrix arising

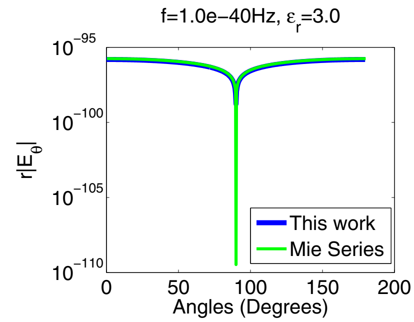

Figure 3: Sphere: Far field calculated at frequency of 10^{-40}Hz.

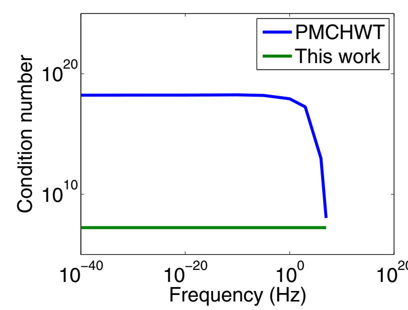

Figure 4: Torus: Condition number w.r.t to frequency.

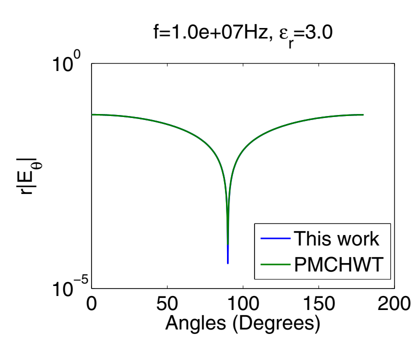

Figure 5: Torus: Far field calculated at frequency of 10MHz.

from the discretization of the PMCHWT integral equations. The proposed regularization ensures a frequency independent behavior for the condition number of the system matrix. Numerical results demonstrate the accuracy and applicability of the new scheme to both simply and non-simply connected geometries.

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