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The Generalized Clifford-Hermite Continuous Wavelet Transform

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ABSTRACT Specific wavelet kernel functions for a continuous wavelet transform in Euclidean space are constructed in the framework of Clifford analysis. Their relationship with the heat equation and a newly introduced wavelet differential equation is established.

1 Introduction

The continuous wavelet transform on the one hand and discrete orthonormal wavelets generated by multiresolution analysis on the other, are the two main themes in wavelet theory. They enjoy more or less opposite properties and both have their specific field of application. The discrete wavelet

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transform is a powerful technique for e.g. data compression, whereas the continuous wavelet transform is a succesful tool in signal analysis.

The one-dimensional continuous wavelet transform (CWT) offers a time-scale analysis suitable for non-stationary, inhomogeneous signals for which Fourier analysis is inadequate (see e.g. [5]). It is given by the integral transform

$$f(x) \mapsto F(a,b) = \int_{-\infty}^{+\infty} \overline{g^{a,b}}(x)\ f(x)\ dx.$$

The kernel function of this integral transform is the dilate translate of a so-called wavelet function g :

$$g^{a,b}(x) = \frac{1}{\sqrt{a}}\ g(\frac{x-b}{a}), \quad a > 0, \quad b \in \mathbb{R},$$

where the parameter b indicates the position of the wavelet, while the parameter a governs its frequency. The analyzing wavelet function g is a quite arbitrary L_2-function which is well localized both in the time domain and in the frequency domain. Moreover it has to satisfy the so-called admissibility condition:

$$\int_{-\infty}^{+\infty} \frac{|\widehat{g}(\xi)|^2}{|\xi|}\ d\xi < +\infty \quad ,$$

where $\widehat{g}$ denotes the Fourier transform of g . In the case where g is also L_1, this admissibility condition implies that g has mean value zero, is oscillating, and decays to zero at infinity; these properties explain the qualification as "wavelet" of this function g .

The CWT may be extended to higher dimensions while still enjoying the same properties as in the one-dimensional case. Many wavelets are available for practical applications, often linked to a specific problem. A typical example is the m-dimensional Mexican hat or Marr wavelet:

$$g(\underline{x}) = -\Delta\ \exp(-\frac{|\underline{x}|^2}{2})\,, \quad \underline{x} \in \mathbb{R}^m,$$

where Δ denotes the Laplacian. This wavelet was originally introduced by Marr [6] and is used in image processing; also higher order Laplacians of the Gaussian function are used as wavelets.

In a previous paper [3] specific wavelet kernel functions in $\mathbb{R}^m$ were constructed using the radial Hermite polynomials of Clifford analysis. The resulting Clifford-Hermite wavelets offer a refinement of the Marr wavelets; they are real and vector valued and show vanishing moments of any order.

In this paper a wider class of Clifford algebra valued basic wavelet functions is presented. The building blocks here are the generalized Hermite polynomials. The corresponding generalized Clifford-Hermite continuous wavelet transform (GCHCWT) in $\mathbb{R}^m$ offers the possibility of a pointwise and a directional analysis of signals. For directional wavelets in two dimensions we refer to [2]. It is also interesting noticing that Clifford algebra was already succesfully used for constructing higher dimensional discrete orthogonal wavelets (see [1]).

2 Generalized Clifford-Hermite Wavelets

2.1 Clifford analysis

Clifford analysis (see e.g.[4]) offers a function theory which is a higher dimensional analogue of the function theory of the holomorphic functions of one complex variable.
Consider functions defined in $\mathbb{R}^m$ $(m > 1)$ and taking values in the Clifford algebra $\mathbb{R}_m$ or its complexification $\mathbb{C}_m$. If $(e_1, \ldots, e_m)$ is an orthonormal basis of $\mathbb{R}^m$, the non-commutative multiplication in the Clifford algebra is governed by the rule:

$$e_j e_k + e_k e_j = -2\delta_{jk}, \quad j,k = 1, \ldots, m\,.$$

Two anti-involutions on the Clifford algebra are important.
Conjugation is defined as the anti-involution for which

$$\overline{e_j} = -e_j, \quad j = 1, \ldots, m$$

with the additional rule

$$\bar{i} = -i$$

in the case of $\mathbb{C}_m$.
Inversion is defined as the anti-involution for which

$$e_j^\dagger = e_j, \quad j = 1, \ldots, m\,.$$

The Euclidean space $\mathbb{R}^m$ is embedded in the Clifford algebras $\mathbb{R}_m$ and $\mathbb{C}_m$ by identifying $(x_1, \ldots, x_m)$ with the vector variable $\underline{x}$ given by

$$\underline{x} = \sum_{j=1}^{m} e_j x_j\,.$$

Notice that

$$\underline{x}^2 = - < \underline{x}, \underline{x} >= -|\underline{x}|^2.$$

If $\mathbb{R}_m^k$ denotes the subspace of k-vectors, i.e. the space spanned by the products of k different basis vectors, then the even subalgebra $\mathbb{R}_m^+$ of $\mathbb{R}_m$ is defined by

$$\mathbb{R}_m^+ = \sum_{k \text{ even}} \oplus \mathbb{R}_m^k .$$

The Clifford group $\Gamma(m)$ of $\mathbb{R}_m$ consists of those invertible elements m in $\mathbb{R}_m$ for which the action $\overline{m}\ \underline{x}\ m$ on a vector $\underline{x}$ is again a vector. Its subgroup Γ^+ is the intersection of Γ with the even subalgebra $\mathbb{R}_m^+$. The Spin-group $\text{Spin}(m)$ is the subgroup of Γ^+ of the elements $m \in \Gamma^+$ for which $m\ m^\dagger = 1$. The Spin-group is a two-fold covering group of the rotation group $\text{SO}(m)$.

An $\mathbb{R}_m$- or $\mathbb{C}_m$-valued function $F(x_1, \ldots, x_m)$ is called (left-) monogenic in an open region Ω of $\mathbb{R}^m$, if in Ω :

$$\partial_{\underline{x}} F \ = \ 0 .$$

Here $\partial_{\underline{x}}$ is the Dirac operator

$$\partial_{\underline{x}} \ = \ \sum_{j=1}^m e_j \partial_{x_j} ,$$

which splits the Laplacian in $\mathbb{R}^m$:

$$\Delta_{\underline{x}} \ = \ -\partial_{\underline{x}}^2 .$$

In the sequel the monogenic homogeneous polynomials will play an important rôle. We call $\mathcal{M}_k$ the (right) module of the (left) monogenic homogeneous polynomials of degree k. Its dimension is given by

$$dim \mathcal{M}_k = \frac{(m+k-2)!}{(m-2)!\ k!} .$$

2.2 The generalized Hermite polynomials

On the real line the Hermite polynomials are defined by

$$H_n(x) = (-1)^n \exp\left(\frac{x^2}{2}\right) d_x^n \exp\left(-\frac{x^2}{2}\right) \quad , \quad n = 0, 1, 2, \ldots$$

They constitute an orthogonal basis for $L_2(\mathbb{R}, \exp(-x^2/2))$ and satisfy the orthogonality relation

$$\int_{-\infty}^{\infty} \exp\left(-\frac{x^2}{2}\right) H_n(x) H_{n'}(x) dx = \sqrt{2\pi}\, n!\, \delta_{nn'} .$$

In [7] F Sommen introduced Hermite polynomials in Clifford analysis. The so-called generalized Hermite polynomials are defined for $n = 0, 1, 2, \ldots$ and $k = 0, 1, 2, \ldots$ by the relation

$$\exp\left(\frac{|\underline{x}|^2}{2}\right)(-\partial_{\underline{x}})^n\left(\exp\left(-\frac{|\underline{x}|^2}{2}\right)P_k(\underline{x})\right) = H_{n,k}(\underline{x})P_k(\underline{x})$$

where $P_k(\underline{x}) \in \mathcal{M}_k$ is a monogenic homogeneous polynomial of degree k. The functions $H_{n,k}(\underline{x})$ are polynomials of degree n with real coefficients depending on k, satisfying the recurrence relations

$$H_{2n+1,k}(\underline{x}) \;=\; (\underline{x} - \partial_{\underline{x}})\; H_{2n,k}(\underline{x})$$

and

$$H_{2n+2,k}(\underline{x}) \;=\; (\underline{x} - \partial_{\underline{x}})\; H_{2n+1,k}(\underline{x}) \;-\; 2k\,\frac{\underline{x}}{|\underline{x}|^2}\; H_{2n+1,k}(\underline{x}) \quad .$$

A straightforward calculation yields

$$\begin{aligned}
H_{0,k}(\underline{x}) &= 1 \\
H_{1,k}(\underline{x}) &= \underline{x} \\
H_{2,k}(\underline{x}) &= \underline{x}^2 + 2k + m \\
H_{3,k}(\underline{x}) &= \underline{x}^3 + (2k + m + 2)\underline{x} \\
H_{4,k}(\underline{x}) &= \underline{x}^4 + 2(2k + m + 2)\underline{x}^2 + (2k + m)(2k + m + 2) \\
&\text{etc.}
\end{aligned}$$

The generalized Hermite polynomials satisfy the orthogonality relation

$$\int_{\mathbb{R}^m} \exp\left(-\frac{|\underline{x}|^2}{2}\right)\overline{H_{n,k}(\underline{x})P_k(\underline{x})}H_{n',k'}(\underline{x})P_{k'}(\underline{x})d\underline{x} = 0\,,$$

whenever $n \neq n'$ or $k \neq k'$.
The set of products of the generalized Hermite polynomials and the monogenic homogeneous polynomials:

$$\{H_{n,k}(\underline{x})P_k^{(j)}(\underline{x}) : n \in \mathbb{N}, k \in \mathbb{N}, j \leq dim\mathcal{M}_k\}$$

constitutes an orthogonal basis for $L_2(\mathbb{R}^m, \exp\left(-\frac{|\underline{x}|^2}{2}\right))$.
Notice that for $k = 0$

$$H_{n,0}(\underline{x}) = \exp\left(\frac{|\underline{x}|^2}{2}\right)(-\partial_{\underline{x}})^n\left(\exp\left(-\frac{|\underline{x}|^2}{2}\right)\right) \qquad n = 0, 1, 2, \ldots$$

the radial Hermite polynomials are obtained; they were used in [3] for the construction of the so-called Clifford-Hermite wavelets.

2.3 The generalized Clifford-Hermite wavelets

For $(n,k) \neq (0,0)$ the above orthogonality relation implies that

$$\int_{\mathbb{R}^m} \exp\left(-\frac{|\underline{x}|^2}{2}\right) H_{n,k}(\underline{x}) P_k(\underline{x}) d\underline{x} = 0 \quad .$$

In terms of wavelet theory this means that the $L_1 \cap L_2-$ functions

$$\begin{aligned} \psi_{n,k}(\underline{x}) &= \exp\left(-\frac{|\underline{x}|^2}{2}\right) H_{n,k}(\underline{x}) P_k(\underline{x}) \\ &= (-1)^n \partial_{\underline{x}}^n \left(\exp\left(-\frac{|\underline{x}|^2}{2}\right) P_k(\underline{x})\right) \end{aligned}$$

have zero momentum and are good candidates for basic wavelet kernel functions in $\mathbb{R}^m$ if at least they satisfy an appropriate admissibility condition (see section 2.4). We call them the generalized Clifford-Hermite wavelets. Notice that for $k = 0$ they coincide with the already in [3] introduced Clifford-Hermite wavelets.

Their Fourier transform is given by

$$\begin{aligned} \widehat{\psi}_{n,k}(\underline{u}) &= \int_{\mathbb{R}^m} \exp(-i\langle \underline{u}, \underline{x}\rangle) \psi_{n,k}(\underline{x}) d\underline{x} \\ &= (2\pi)^{m/2} (-i)^n \underline{u}^n P_k(i\partial_{\underline{u}}) \exp\left(-\frac{|\underline{u}|^2}{2}\right) \end{aligned}$$

again a product of the Gaussian function with a polynomial of degree $(n+k)$.

2.4 The generalized Clifford-Hermite CWT

In order to introduce the corresponding generalized Clifford-Hermite continuous wavelet transform (GCHCWT), we consider, still for $(n,k) \neq (0,0)$, the continuous family of wavelets

$$\psi_{n,k}^{a,\underline{b},s}(\underline{x}) = \frac{1}{a^{m/2}} \, \overline{\psi}_{n,k}\left(\frac{\overline{s}(\underline{x}-\underline{b})s}{a}\right) \overline{s} \quad ,$$

where $a \in \mathbb{R}_+$, $\quad \underline{b} \in \mathbb{R}^m \quad$ and $\quad s \in \mathrm{Spin}(m)$.
They originate from the basic wavelet function $\psi_{n,k}$ by dilation, translation and spinor-rotation. Their Fourier transform is given by

$$\begin{aligned} \widehat{\psi}_{n,k}^{a,\underline{b},s}(\underline{u}) &= \exp(-i\langle \underline{b}, \underline{u}\rangle) \, \widehat{\psi}_{n,k}^{a,0,s}(\underline{u}) \\ &= \exp(-i\langle \underline{b}, \underline{u}\rangle) \, \widehat{\psi}_{n,k}^{a,0,1}(\overline{s}\underline{u}s) \, \overline{s} \\ &= a^{m/2} \exp(-i\langle \underline{b}, \underline{u}\rangle) \, \widehat{\overline{\psi}}_{n,k}(a\overline{s}\underline{u}s) \, \overline{s} \ . \end{aligned}$$

The reason why the family of wavelets is defined by means of the adjoint of the basic wavelet ψ rather than by the basic wavelet itself, becomes clear when looking at the definition of the corresponding CWT. The generalized GCHCWT applies to functions $f \in L_2(\mathbb{R}^m)$ by

$$\begin{aligned}
T_{n,k} f(a,\underline{b},s) &= F_{n,k}(a,\underline{b},s) \\
&= \langle \psi_{n,k}^{a,\underline{b},s} , f \rangle \\
&= \int_{\mathbb{R}^m} \overline{\psi}_{n,k}^{a,\underline{b},s}(\underline{x})\, f(\underline{x})\, d\underline{x} \\
&= \frac{1}{a^{m/2}} \int_{\mathbb{R}^m} s\, \psi_{n,k}\left(\frac{\overline{s}(\underline{x}-\underline{b})s}{a} \right) f(\underline{x})\, d\underline{x} \\
&= \frac{1}{a^{m/2}} \int_{\mathbb{R}^m} s \exp\left(-\frac{|\underline{x}-\underline{b}|^2}{2a^2} \right) H_{n,k}\left(\frac{\overline{s}(\underline{x}-\underline{b})s}{a} \right) \\
&\quad P_k\left(\frac{\overline{s}(\underline{x}-\underline{b})s}{a} \right) f(\underline{x})\, d\underline{x} \; .
\end{aligned}$$

This definition can be rewritten in terms of the Fourier transform as

$$\begin{aligned}
F_{n,k}(a,\underline{b},s) &= \frac{1}{(2\pi)^m} \langle\, \hat{\psi}_{n,k}^{a,\underline{b},s}(\underline{u}) \;,\; \hat{f}(\underline{u}) \,\rangle \\
&= \frac{1}{(2\pi)^m} \langle\, a^{m/2} \; \exp(-i\langle \underline{b}, \underline{u} \rangle)\; \hat{\overline{\psi}}_{n,k}(a\overline{s}\underline{u}s)\; \overline{s} \;,\; \hat{f}(\underline{u}) \,\rangle \\
&= \frac{a^{m/2}}{(2\pi)^m}\; s\; \widehat{\overline{\hat{\overline{\psi}}_{n,k}(a\overline{s}\underline{u}s)}\hat{f}(\underline{u})}(-\underline{b}) \, .
\end{aligned}$$

It is clear that the GCHCWT will map $L_2(\mathbb{R}^m)$ into a weighted L_2-space on $\mathbb{R}_+ \times \mathbb{R}^m \times \mathrm{Spin}(m)$ for some weight function still to be determined. This weight function has to be chosen in this way that the GCHCWT is an isometry, or in other words that the Parseval formula should hold. Introducing the inner product

$$[F_{n,k}, G_{n,k}] =$$

$$= \frac{1}{C_{n,k}} \int_{\mathrm{Spin}(m)} \int_{\mathbb{R}^m} \int_0^{+\infty} \overline{F}_{n,k}(a,\underline{b},s)\, G_{n,k}(a,\underline{b},s)\, \frac{da}{a^{m+1}}\; d\underline{b}\; ds \; ,$$

where ds stands for the Haar measure on $\mathrm{Spin}(m)$, we search for the constant $C_{n,k}$ in order to have the Parseval formula

$$\langle f, g \rangle = [F_{n,k}, G_{n,k}]$$

fulfilled.
We have consecutively

$$[F_{n,k}, G_{n,k}] =$$

$$\begin{aligned}
&= \frac{C_{n,k}^{-1}}{(2\pi)^{2m}} \int_{\mathrm{Spin}(m)} \int_{\mathbb{R}^m} \int_0^{+\infty} \overline{s\, \widehat{\overline{\widehat{\psi}}_{n,k}(a\bar{s}\underline{u}s)\hat{f}(\underline{u})}(-\underline{b})} \\
&\quad s\, \widehat{\overline{\widehat{\psi}}_{n,k}(a\bar{s}\underline{u}s)\hat{g}(\underline{u})}(-\underline{b})\ \frac{da}{a}\ d\underline{b}\ ds \\
&= \frac{C_{n,k}^{-1}}{(2\pi)^{m}} \int_{\mathrm{Spin}(m)} \int_{\mathbb{R}^m} \int_0^{+\infty} \overline{\hat{f}(\underline{u})}\ \widehat{\psi}_{n,k}(a\bar{s}\underline{u}s)\ \bar{s} \\
&\quad s\ \overline{\widehat{\psi}}_{n,k}(a\bar{s}\underline{u}s)\ \hat{g}(\underline{u})\ \frac{da}{a}\ d\underline{u}\ ds\,.
\end{aligned}$$

If we put

$$\int_{\mathrm{Spin}(m)} \int_0^{+\infty} \widehat{\psi}_{n,k}(a\bar{s}\underline{u}s)\ \overline{\widehat{\psi}}_{n,k}(a\bar{s}\underline{u}s)\ \frac{da}{a}\ ds\ =\ C_{n,k}\,,$$

taking into account that $\bar{s}\ s\ =\ 1$, we get the desired result:

$$\begin{aligned}
[F_{n,k}\ ,\ G_{n,k}] &= \frac{1}{(2\pi)^m} \int_{\mathbb{R}^m} \overline{\hat{f}(\underline{u})}\, \hat{g}(\underline{u})\, d\underline{u} \\
&= \frac{1}{(2\pi)^m} \langle \hat{f}, \hat{g} \rangle. \\
&= \langle f, g \rangle.
\end{aligned}$$

This means that if the functions $\psi_{n,k}$ satisfy the relation defining the constant $C_{n,k}$, the GCHCWT will be an isometry between two L_2-spaces. This relation is called the admissibility condition for the generalized CH-wavelets, and the constant $C_{n,k}$ involved is called the admissibility constant. By means of the substitution

$$\underline{u} = \frac{r}{a}\ \underline{\omega} \quad , \quad \underline{\omega} \in S^{m-1}$$

and taking into account that $\overline{s}\underline{\omega}s \in S^{m-1}$ for all $\underline{\omega} \in S^{m-1}$, the admissibility condition may be simplified to

$$\begin{aligned} C_{n,k} &= \int_0^{+\infty} \int_{S^{m-1}} \widehat{\overline{\psi}}_{n,k}(r\underline{\omega})\, \overline{\widehat{\overline{\psi}}}_{n,k}(r\underline{\omega})\, d\underline{\omega}\, \frac{dr}{r} \\ &= \int_{\mathbb{R}^m} \widehat{\overline{\psi}}_{n,k}(\underline{u})\, \overline{\widehat{\overline{\psi}}}_{n,k}(\underline{u})\, \frac{d\underline{u}}{|\underline{u}|^m} \\ &= \int_{\mathbb{R}^m} \widehat{\overline{\psi}}_{n,k}(\underline{u})\, \widehat{\psi}_{n,k}(-\underline{u})\, \frac{d\underline{u}}{|\underline{u}|^m} \\ &= \int_{\mathbb{R}^m} \overline{\widehat{\psi}}_{n,k}(-\underline{u})\, \widehat{\psi}_{n,k}(-\underline{u})\, \frac{d\underline{u}}{|\underline{u}|^m} \\ &= \int_{\mathbb{R}^m} \overline{\widehat{\psi}}_{n,k}(\underline{u})\, \widehat{\psi}_{n,k}(\underline{u})\, \frac{d\underline{u}}{|\underline{u}|^m}\,. \end{aligned}$$

Notice that in general the admissibility constant $C_{n,k}$ is a Clifford number. In order for $C_{n,k}$ to be a real constant it is sufficient to assume that

$$\overline{\widehat{\psi}}_{n,k}(\underline{u})\ \widehat{\psi}_{n,k}(\underline{u})$$

is real-valued. In that case it is also automatically positive and it does make sense to impose the condition

$$C_{n,k} < +\infty\,.$$

So we have proved

THEOREM

(i) A Clifford algebra valued function $\psi \in L_2 \cap L_1(\mathbb{R}^m)$ is a basic wavelet function if it satisfies in frequency space the admissibility condition:

$$\int_{\mathbb{R}^m} \overline{\widehat{\psi}}(\underline{u})\ \widehat{\psi}(\underline{u})\ \frac{d\underline{u}}{|\underline{u}|^m} \quad < \quad +\infty\,;$$

this condition implies it has zero momentum:

$$\int_{\mathbb{R}^m} \psi(\underline{x})\ d\underline{x} \quad = \quad 0 \quad .$$

(ii) The functions

$$\psi_{n,k}(\underline{x}) = \exp\left(-\frac{|\underline{x}|^2}{2}\right) H_{n,k}(\underline{x}) P_k(\underline{x})$$

satisfy the above condition provided that

$$\overline{P_k}(\underline{x})\ P_k(\underline{x})$$

be real-valued, which, by the Funk-Hecke Theorem implies that

$$\overline{\widehat{\psi}}_{n,k}(\underline{u})\ \widehat{\psi}_{n,k}(\underline{u})$$

is real-valued. In this case we call them "Generalized Clifford Hermite Wavelets of the first kind".

(iii) The GCHCWT $T_{n,k}$ maps $L_2(\mathbb{R}^m)$ isometrically into $L^2(\mathbb{R}_+ \times \mathbb{R}^m \times \text{Spin}(m), C_{n,k}^{-1}\ a^{-(m+1)}\ da\ d\underline{b}\ ds)$, where $C_{n,k}$ is the admissibility constant of $\psi_{n,k}$.

It should be emphasized that $T_{n,k}$ is no surjection onto $L^2(\mathbb{R}_+ \times \mathbb{R}^m \times \text{Spin}(m), C_{n,k}^{-1}\ a^{-(m+1)}\ da\ d\underline{b}\ ds)$. So there is a lot of freedom in constructing inversion formulae. However from the above Parseval formula it follows that if $f \in L_2(\mathbb{R}^m)$ and $F_{n,k}(a, \underline{b}, s) = T_{n,k} f(\underline{x})$ then:

$$\begin{aligned} f(\underline{x}) &= \frac{1}{C_{n,k}} \int_{\text{Spin}(m)} \int_{\mathbb{R}^m} \int_0^{+\infty} \psi_{n,k}^{a,\underline{b},s}(\underline{x})\, F_{n,k}(a, \underline{b}, s)\ \frac{da}{a^{m+1}}\ d\underline{b}\ ds \\ &= [\overline{\psi}_{n,k}^{a,\underline{b},s}(\underline{x})\ ,\ F_{n,k}(a, \underline{b}, s)] \end{aligned}$$

to hold weakly in $L_2(\mathbb{R}^m)$.
This means that the GCHCWT decomposes the signal $f(\underline{x})$ in terms of the analyzing wavelets $\psi_{n,k}^{a,\underline{b},s}(\underline{x})$ with coefficients $F_{n,k}(a, \underline{b}, s)$. The GCHCWT thus offers an analysis of a signal in higher dimensional space which is as well pointwise as directional.

3 The Generalized Wavelet Equation

3.1 The modified heat equation

For $n = k = 0$ the Gaussian function

$$\psi_{0,0}(\underline{x}) = \exp\left(-\frac{|\underline{x}|^2}{2}\right)$$

has a non-vanishing zero momentum and thus cannot be used as a basic wavelet function.
Nevertheless, introducing the time variable $t > 0$, the function

$$h(\underline{x}) = \frac{1}{(2\pi)^m}\ \psi_{0,0}(\underline{x})$$

generates the family of functions

$$h_t(\underline{x}) = \frac{1}{(2t)^{m/2}}\ h\left(\frac{\underline{x}}{\sqrt{2t}}\right)$$

which are used in a higher dimensional windowed Fourier or Gabor transform (see [3]).
Moreover the function $h_t(x)$ is the fundamental solution of the heat operator in $\mathbb{R}^m$:

$$\partial_t + \partial_{\underline{x}}^2 = \partial_t - \Delta_{\underline{x}} \,.$$

For $(n,k) \neq (0,0)$ the generalized Clifford-Hermite wavelets are no longer solutions of the heat equation. As was already done in [3] for the special case where $k=0$, we now construct the similar differential equation which is related to the GCHCWT in the same way the heat equation is linked to the windowed Fourier transform.

From the definition of the generalized CH-wavelets it follows at once that

$$\psi_{n,k}(\frac{\underline{x}}{\sqrt{2t}}) = (-1)^n \ (2t)^{n/2} \ \partial_{\underline{x}}^n \ \left(\exp\left(-\frac{|\underline{x}|^2}{4t}\right) \ P_k\left(\frac{\underline{x}}{\sqrt{2t}}\right)\right)$$

or still

$$\begin{aligned}(-1)^n \ 2^{-n/2} \ t^{-n/2} \ t^{-\frac{m-k}{2}} \ \psi_{n,k}(\tfrac{\underline{x}}{\sqrt{2t}}) &= \\ = \partial_{\underline{x}}^n \left(t^{-\frac{m-k}{2}} \ \exp\left(-\tfrac{|\underline{x}|^2}{4t}\right) P_k\left(\tfrac{\underline{x}}{\sqrt{2t}}\right)\right).\end{aligned}$$

Taking into account the homogeneous and monogenic, and thus harmonic, character of the inner spherical monogenics P_k , a direct calculation shows that

$$(\partial_t+\partial_{\underline{x}}^2) \left(\exp\left(-\frac{|\underline{x}|^2}{4t}\right) \ P_k\left(\frac{\underline{x}}{\sqrt{2t}}\right)\right) = \frac{m-k}{2t} \left(\exp\left(-\frac{|\underline{x}|^2}{4t}\right) \ P_k\left(\frac{\underline{x}}{\sqrt{2t}}\right)\right)$$

which leads to

$$(\partial_t + \partial_{\underline{x}}^2) \left(t^{-\frac{m-k}{2}} \ \exp\left(-\frac{|\underline{x}|^2}{4t}\right) \ P_k\left(\frac{\underline{x}}{\sqrt{2t}}\right)\right) = 0.$$

Hence

$$(\partial_t + \partial_{\underline{x}}^2) \left(t^{-\frac{m+n-k}{2}} \ \psi_{n,k}\left(\frac{\underline{x}}{\sqrt{2t}}\right)\right) = 0$$

or

$$(\partial_t + \partial_{\underline{x}}^2) \left(t^{-\frac{m+2(n-k)}{4}} \ t^{-m/4} \ \psi_{n,k}\left(\frac{\underline{x}}{\sqrt{2t}}\right)\right) = 0 \,.$$

which yields

$$(\partial_t + \partial_{\underline{x}}^2 - \frac{m+2(n-k)}{4t}) \ \left(t^{-\frac{m}{4}} \ \psi_{n,k}\left(\frac{\underline{x}}{\sqrt{2t}}\right)\right) = 0 \,.$$

In view of

$$\psi_{n,k}^{\sqrt{2t},\underline{b},s}(\underline{x}) = \frac{1}{(2t)^{m/4}} \ \overline{\psi}_{n,k}\left(\frac{\overline{s}(\underline{x}-\underline{b})s}{\sqrt{2t}}\right) \overline{s}$$

and the Spin(m)-invariance of the Dirac operator $\partial_{\underline{x}}$, we finally obtain the modified heat equation satisfied by the generalized CH-wavelets:

$$\left(\partial_t - \Delta_{\underline{b}} - \frac{m+2(n-k)}{4t}\right) \; \psi_{n,k}^{\sqrt{2t},\underline{b},s}(\underline{x}) = 0\,.$$

As

$$-\partial_{\underline{x}} \; \psi_{n,k}(\underline{x}) = \psi_{n+1,k}(\underline{x})$$

and hence

$$\sqrt{2t}\,\partial_{\underline{b}} \; \psi_{n,k}\left(\frac{\underline{x}-\underline{b}}{\sqrt{2t}}\right) = \psi_{n+1,k}\left(\frac{\underline{x}-\underline{b}}{\sqrt{2t}}\right)$$

this modified heat equation may at once be linearized, again taking into account the Spin(m)-invariance of the Dirac operator, into the first order system:

$$\left\{\begin{array}{lcl} \sqrt{2t}\;\left(\psi_{n,k}^{\sqrt{2t},\underline{b},s}\right)\partial_{\underline{b}} + \psi_{n+1,k}^{\sqrt{2t},\underline{b},s} & = & 0 \\ \sqrt{2t}\;\left(\partial_t - \frac{m+2(n-k)}{4t}\right)\psi_{n,k}^{\sqrt{2t},\underline{b},s} - \left(\psi_{n+1,k}^{\sqrt{2t},\underline{b},s}\right)\partial_{\underline{b}} & = & 0\,. \end{array}\right.$$

For the GCHW transforms $F_{n,k}(\sqrt{2t},\underline{b},s)$ themselves, this gives rise to the first order “special wavelet system”:

$$\left\{\begin{array}{lcl} \sqrt{2t}\;\partial_{\underline{b}}F_{n,k}^{\sqrt{2t},\underline{b},s} - F_{n+1,k}^{\sqrt{2t},\underline{b},s} & = & 0 \\ \sqrt{2t}\;\left(\partial_t - \frac{m+2(n-k)}{4t}\right)F_{n,k}^{\sqrt{2t},\underline{b},s} + \partial_{\underline{b}}F_{n+1,k}^{\sqrt{2t},\underline{b},s} & = & 0\,. \end{array}\right.$$

3.2 The wavelet differential equation

The above modified heat equation still depends on the degree of the polynomial factor in $\psi_{n,k}^{\sqrt{2t},\underline{b},1}$. In order to obtain one and the same differential equation which is satisfied by the whole sequence of generalized CH-wavelets, an extra variable is introduced such that

$$\begin{aligned} &(\partial_t - \Delta_{\underline{b}})\;\exp((2(n-k)+m)x_0)\;\overline{\psi}_{n,k}^{\sqrt{2t},\underline{b},s}(\underline{x}) \\ =\;& \frac{2(n-k)+m}{4t}\;\exp((2(n-k)+m)x_0)\;\overline{\psi}_{n,k}^{\sqrt{2t},\underline{b},s}(\underline{x})) \\ =\;& \frac{1}{4t}\partial_{x_0}\left(\exp((2(n-k)+m)x_0)\;\overline{\psi}_{n,k}^{\sqrt{2t},\underline{b},s}(\underline{x})\right)\,. \end{aligned}$$

In this way the so-called wavelet differential equation is obtained:

$$\left(\partial_t - \Delta_{\underline{b}} - \frac{1}{4t}\partial_{x_0}\right)\;\Psi_{n,k}^{\sqrt{2t},\underline{b},s}(x_0,\underline{x})$$

where we have put

$$\Psi_{n,k}^{\sqrt{2t},\underline{b},s}(x_0,\underline{x}) = \exp((2(n-k)+m)x_0)\ \overline{\psi}_{n,k}^{\sqrt{2t},\underline{b},s}(\underline{x})\,.$$

It should be noticed that the functions $\Psi_{n,k}^{\sqrt{2t},\underline{b},s}(x_0,\underline{x})$ are not the only solutions of this wavelet differential equation.

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