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Authors: Todor Gramchev, Luigi Rodino, Stevan Pilipovic, and Jasson Vindas

In: Annali di Matematica Pura ed Applicata, Vol. 194(3), 823-841, (2015)

Optional: doi:10.1007/s10231-014-0400-z

WEYL ASYMPTOTICS FOR TENSOR PRODUCTS OF OPERATORS AND DIRICHLET DIVISORS

TODOR GRAMCHEV, STEVAN PILIPOVIĆ, LUIGI RODINO, AND JASSON VINDAS

Abstract. We study the counting function of the eigenvalues for tensor products of operators, and their perturbations, in the context of Shubin classes and closed manifolds. We emphasize connections with problems of analytic number theory, concerning in particular generalized Dirichlet divisor functions.

1. Introduction

As well known, there are deep connections between spectral theory and analytic number theory. One main topic is given by Weyl formula for self-adjoint partial differential operators or pseudo-differential operators. Namely, the leading term in the expansion of the counting function $N(\lambda)$ of the the eigenvalues $\leq \lambda$ is recognized to be proportional to the volume of the region defined by the λ-level surfaces of the symbol, and in turn to the number of the lattice points belonging to the region. Even, for relevant classes of operators, each point of the lattice corresponds exactly to one of eigenvalues, counted according to the multiplicity, and the computation of $N(\lambda)$ leads in a natural way to problems of number theory. Let us refer for example to [8, 9], [12]–[22], [28]–[32]. In this order of ideas, the attention will be fixed here on operators of the form of tensor products

$$P = P_1 \otimes ... \otimes P_p \tag{1.1}$$

where the operators P_j, $j = 1, \ldots, p$, are self-adjoint, say strictly positive (pseudo-differential) operators on corresponding Hilbert spaces with eigenvalues $\{\lambda_k^{(j)}\}_{k=1}^{\infty}$, $j = 1, ..., p$. Then, the eigenvalues of P are products of the form $\lambda_{k_1}^{(1)} ... \lambda_{k_p}^{(p)}$ and the eigenfunctions are tensor products of the corresponding eigenfunctions, cf. [8, 29] for the general functional analytic setting. Hence,

$$N_P(\lambda) = \# \left\{ (k_1, \ldots, k_p) \in \mathbb{N}^p : \ \lambda_{k_1}^{(1)} \lambda_{k_2}^{(2)} \ldots \lambda_{k_p}^{(p)} \leq \lambda \right\}. \tag{1.2}$$

The computation of $N_p(\lambda)$ meets then some classical divisor counting problems. To give a simple example, consider the Hermite operators

$$H_j = \frac{1}{2}(-\partial_{x_j}^2 + x_j^2) + \frac{1}{2}, \quad j = 1, 2. \tag{1.3}$$

Writing for short H_1 and H_2 for $H_1 \otimes I_2$ and $I_1 \otimes H_2$, we define the tensorized Hermite operator $H = H_1 \otimes H_2$. In applications, H is sometimes used as a substitute for the standard

2000 *Mathematics Subject Classification.* Primary 35P20; Secondary 35P15.

Key words and phrases. Weyl asymptotics; tensor producs of operators; polysingular operators; spectral theory; Dirichlet divisors.

two dimensional Hermite operator $H_1 + H_2$, producing the same eigenfunctions, i.e., two dimensional Hermite functions. The distribution of eigenvalues, counted with multiplicity, is however quite different, being related to the distribution of the prime numbers. In fact, the eigenvalues of the one-dimensional Hermite operator, normalized as above, are the positive integers; therefore, (1.2) reads in this case as

$$N_H(\lambda) = D(\lambda) = \sum_{n=1}^{[\lambda]} d(n), \quad \lambda \geq 1, \tag{1.4}$$

where $d(n)$ denotes the number of divisors of n and $[\lambda]$ stands for the integral part of λ. Dirichlet proved in 1849 that

$$D(\lambda) = \lambda \log \lambda + (2\tilde{\gamma} - 1)\lambda + E(\lambda), \tag{1.5}$$

where $\tilde{\gamma}$ is the Euler-Mascheroni constant and $E(\lambda) = O(\lambda^{1/2})$. The first term on the right hand side of (1.5) can be easily recognized as the volume of the hyperbolic region defined by the symbol of H, whereas the optimal growth order of the rest $E(\lambda)$ is a long-standing open problem in the analytic theory of numbers, see for example [1, 16, 20, 21, 33].

Natural generalizations of the Hermite operators H_j in (1.3) are the global pseudo-differential operators of Shubin [17, 7, 25, 32]. If P_j is globally elliptic self-adjoint in these classes, then the Weyl formula yields

$$N_{P_j}(\lambda) \sim A_j \lambda^{\alpha_j}, \tag{1.6}$$

where $\alpha_j = 2n_j/m_j$, with m_j the order of P_j and n_j the space dimension. The constant A_j depends on the symbol of P_j, according to the Weyl formula. Note that the tensorized product in (1.1) is not any longer globally elliptic on $\mathbb{R}^n, n = n_1 + ... + n_p$.

The first aim of the present paper will be to deduce from (1.6) an asymptotic expansion for the spectral counting function $N_P(\lambda)$ in (1.2). Particular attention will be devoted to lower order terms of the asymptotic expansion for some particular cases. As an example, define H_j as in (1.3) and consider now

$$H^{\vec{\beta}} = H_1^{\beta_1} H_2^{\beta_2}, \tag{1.7}$$

where $\vec{\beta} = (\beta_1, \beta_2)$ is a couple of positive integers with $\beta_2 \neq \beta_1$. Then, we shall prove that

$$N_{H^{\vec{\beta}}}(\lambda) = \zeta(\beta_2/\beta_1)\lambda^{1/\beta_1} + \zeta(\beta_1/\beta_2)\lambda^{1/\beta_2} + O(\lambda^{1/(\beta_1+\beta_2)}), \tag{1.8}$$

where $\zeta(z)$ is the Riemann zeta function, analytically continued in the complex plane for $z \neq 1$.

As we shall also detail in the paper, parallel results can be obtained when P_j in (1.1) are elliptic self-adjoint pseudo-differential operators on a closed manifold. In this case (1.6) is valid with $\alpha_j = n_j/m_j$, see [19].

Let us finally describe what, to the best of our knowledge, was already known about tensor products of pseudo-differential operators and their spectrum, as well as what is new in our paper. An algebra of "bisingular" pseudo-differential operators on the product of two manifolds $M_1 \times M_2$, containing $P_1 \otimes P_2$ with P_1 and P_2 being classical pseudo-differential operators on M_1, M_2, respectively, was studied by Rodino [28] in connection with the multiplicative property of the Atiyah-Singer index [4]. The spectral properties of this class were recently studied by Battisti [5]. The variant for the Shubin type operators has been considered in [6]. These results give a general framework for the study of the example (1.4) with the expansion (1.5), and provide as well the leading term in the expansion (1.8) for the example (1.7). Let us also mention the articles [14, 15], where starting from the twisted Laplacian of M. W. Wong [34], similar problems of Dirichlet divisor-type were met. The operators in [14, 15, 34] are not tensor products, but they can be reduced to the form (1.1) by conjugating with a Fourier integral operator, cf. [13].

From the point of view of Mathematical Physics, Kaplitskiĭ [22] has independently studied the spectral properties of operators on the torus $\mathbb{T}^2$ with principal part

$$P = P_x \otimes P_y = \partial^2_{x,y},$$

obtaining for the counting function estimates of type (1.5). Reference therein is made to Arnold [3], suggesting to transfer the Weyl formula to hyperbolic equations. The results in [22] can be essentially regarded as a particular case of those from [5]. Expansions of the type (1.5) appear also in the recent paper of Coriasco and Maniccia [10] concerning the spectrum of the so-called SG-operators.

Summing up, the results mentioned above cover the case of products of two operators, $P = P_1 \otimes P_2$, except for the computation of lower order terms in the expansions, cf. (1.8). Thus, our attention will be mainly focus on the case $p \geq 3$ of (1.1) and lower order terms.

In the present paper, the attention will be rather addressed to results of (elementary) analytic number theory, which we shall present in Section 2 in detail; they are new by themselves, we believe. The applications to spectral theory will be given in the conclusive Section 3. We shall not construct here an algebra of (polysingular) pseudo-differential operators containing $P_1 \otimes ... \otimes P_p$ for $p \geq 3$. Computations are cumbersome, involving a stratified calculus of the type of that from [24, 26, 30, 31], occurring in other contexts. We shall instead limit ourselves instead to consider perturbations of the type $P + Q$, where Q is a lower order pseudo-differential operator.

We would like to say some words about the motivation of this article. Our primary motivation is to exhibit the spectral asymptotic behavior of lower order perturbations of tensor products of partial differential equations (cf. Section 3.3). These operators appear in a natural way in several problems of mathematical analysis, but, in turn, their spectral asymptotics cannot be treated by the classical methods of Shubin [32] and Hörmander [19]. As shown here, their asymptotics can be obtained by combining techniques from elementary number theory for the analysis of the principal terms with functional-analytic methods (pseudo-differential techniques) for their perturbations. As a matter of fact, the estimate (3.24) is very rough, since only the asymptotic order of the leading term has been identified; nevertheless, we highlight that it is out of reach of the results from [17, 19, 32]. Although it is not the scope of this article, the opposite path of investigation, i.e. to deduce improvements in the remainder estimate for the Dirichlet divisor problem from subtle spectral theory methods involving FIOs, looks even more exciting. However, one quickly encounters highly nontrivial challenges, like what kind of non-classical phase function would be needed, the appearance of Hamilton-Jacobi equations with possible singularities, and so on.

2. Asymptotics of some counting functions

We study in this preparatory section the asymptotic behavior of some counting functions of "multi-divisor" type. They will be very helpful when applied to spectral asymptotics of various examples of "polysingular" operators.

2.1. **Counting functions of products of sequences.** We start by considering the following general question. Let $\{\lambda_k^{(j)}\}_{k=1}^{\infty}$, $j=1,\dots,p$, be non-decreasing sequences of positive real numbers. The sequences are rather arbitrary and they are not necessarily linked to any operator.

Assuming that we have some knowledge about each of the counting functions

$$N_j(\lambda) := \sum_{\lambda_k^{(j)}\leq\lambda} 1 = \#\left\{k\in\mathbb{N}:\ \lambda_k^{(j)}\leq\lambda\right\}, \quad j=1,\dots,p, \tag{2.1}$$

we would like to obtain asymptotic information about the counting function of the p products of the elements of the sequences, namely,

$$N(\lambda) := \sum_{\lambda_{k_1}^{(1)}\lambda_{k_2}^{(2)}\dots\lambda_{k_p}^{(p)}\leq\lambda} 1 = \#\left\{(k_1,\dots,k_p)\in\mathbb{N}^p:\ \lambda_{k_1}^{(1)}\lambda_{k_2}^{(2)}\dots\lambda_{k_p}^{(p)}\leq\lambda\right\}. \tag{2.2}$$

The next simple proposition tells us that it is always possible to find the asymptotic behavior of (2.2) whenever there is a block of counting functions (2.1) with dominating asymptotic behavior.

Proposition 2.1. *Suppose that there are non-negative numbers $\tau<\alpha$ and indices $j_1,\dots,j_\nu$, where $1\leq\nu\leq p$, such that*

$$N_{j_q}(\lambda)\sim A_{j_q}\lambda^{\alpha}, \quad \lambda\to\infty, \quad q=1,\dots,\nu, \tag{2.3}$$

with $A_{j_q}\neq 0$, and

$$N_j(\lambda)=O(\lambda^{\tau}), \quad \lambda\to\infty, \quad j\notin\{j_1,\dots,j_\nu\}. \tag{2.4}$$

Then, the counting function (2.2) *has asymptotic behavior*

$$N(\lambda)\sim A\lambda^{\alpha}\frac{(\alpha\log\lambda)^{\nu-1}}{(\nu-1)!}, \quad \lambda\to\infty, \tag{2.5}$$

where

$$A=\left(\prod_{q=1}^{\nu}A_{j_q}\right)\cdot\left(\prod_{j\notin\{j_1,\dots,j_\nu\}}\left(\sum_{k=1}^{\infty}\frac{1}{\left(\lambda_k^{(j)}\right)^{\alpha}}\right)\right). \tag{2.6}$$

We will divide the proof of Proposition 2.1 into two lemmas. The first lemma deals with the case in which all counting functions have asymptotic behavior of the same order.

Lemma 2.2. *If $N_j(\lambda)\sim A_j\lambda^{\alpha}$, with $\alpha>0$ and $A_j\neq 0$, for $j=1,2,\dots,p$, then* (2.2) *has asymptotics*

$$N(\lambda)\sim A\lambda^{\alpha}\frac{(\alpha\log\lambda)^{p-1}}{(p-1)!}, \quad \lambda\to\infty,$$

where $A=\prod_{j=1}^{p}A_j$.

Proof. We proceed by induction. Assume that

$$\tilde{N}(\lambda) = \sum_{\lambda^{(1)}_{k_1}\lambda^{(2)}_{k_2}\ldots\lambda^{(p-1)}_{k_{p-1}}\le\lambda} 1 \sim \tilde{A}\lambda^\alpha \frac{(\alpha\log\lambda)^{p-2}}{(p-2)!},$$

with $\tilde{A} = \prod_{j=1}^{p-1} A_j$. We then have,

$$\begin{aligned}
N(\lambda) &= \sum_{\lambda^{(p)}_k\le\lambda} \tilde{N}(\lambda/\lambda^{(p)}_k)\\
&= \frac{\tilde{A}\alpha^{p-2}}{(p-2)!}\sum_{\lambda^{(p)}_k\le\lambda}(\lambda/\lambda^{(p)}_k)^\alpha(\log(\lambda/\lambda^{(p)}_k))^{p-2} + \sum_{\lambda^{(p)}_k\le\sqrt{\lambda}} o((\lambda/\lambda^{(p)}_k)^\alpha\log^{p-2}\lambda)\\
&\quad + O(\lambda^\alpha\log^{p-2}\lambda)\cdot\sum_{\sqrt{\lambda}<\lambda^{(p)}_k\le\lambda}\frac{1}{(\lambda^{(p)}_k)^\alpha}\\
&= \frac{\tilde{A}\alpha^{p-2}}{(p-2)!}\sum_{\lambda^{(p)}_k\le\lambda}(\lambda/\lambda^{(p)}_k)^\alpha(\log(\lambda/\lambda^{(p)}_k))^{p-2} + o(\lambda^\alpha\log^{p-1}\lambda) + O(\lambda^\alpha\log^{p-2}\lambda)\\
&= \frac{\tilde{A}\alpha^{p-2}}{(p-2)!}\int_0^\lambda(\lambda/t)^\alpha(\log(\lambda/t))^{p-2}dN_p(t) + o(\lambda^\alpha\log^{p-1}\lambda)\\
&= \frac{\tilde{A}\alpha^{p-1}}{(p-2)!}\lambda^\alpha\int_0^\lambda(\log(\lambda/t))^{p-2}\frac{N_p(t)}{t^{\alpha+1}}dt + o(\lambda^\alpha\log^{p-1}\lambda)\\
&= \frac{A_p\tilde{A}\alpha^{p-1}}{(p-2)!}\lambda^\alpha\sum_{j=0}^{p-2}\binom{p-2}{j}(-1)^\nu(\log\lambda)^{p-2-\nu}\int_1^\lambda\frac{(\log t)^\nu}{t}dt + o(\lambda^\alpha\log^{p-1}\lambda)\\
&\sim A\lambda^\alpha\frac{(\alpha\log\lambda)^{p-1}}{(p-2)!}\sum_{j=0}^{p-2}\binom{p-2}{j}\frac{(-1)^\nu}{\nu+1}\\
&= A\lambda^\alpha\frac{(\alpha\log\lambda)^{p-1}}{(p-2)!}\int_0^1(1-t)^{p-2}dt = A\lambda^\alpha\frac{(\alpha\log\lambda)^{p-1}}{(p-1)!}.
\end{aligned}$$

□

We also have,

Lemma 2.3. *If*

$$M_1(\lambda) = \sum_{\mu^{(1)}_k\le\lambda} 1 = O(\lambda^\tau) \quad \textit{and} \quad M_2(\lambda) = \sum_{\mu^{(2)}_k\le\lambda} 1 \sim B\lambda^\alpha\log^b\lambda, \quad \lambda\to\infty$$

where $0\le\tau<\alpha$*,* $B\ne 0$*, and* $b\ge 0$*, then*

$$M(\lambda) = \sum_{\mu^{(1)}_k\mu^{(2)}_k\le\lambda} 1 \sim \tilde{B}\lambda^\alpha\log^b\lambda, \quad \lambda\to\infty,$$

where $\tilde{B} = B\sum_{k=1}^\infty(\mu^{(1)}_k)^{-\alpha}$*.*

Proof. Observe first that

$$\sum_{k=1}^{\infty}\frac{1}{\left(\mu_k^{(1)}\right)^{\alpha}}=\int_0^{\infty}t^{-\alpha}dM_1(t)=\alpha\int_0^{\infty}\frac{M_1(t)}{t^{1+\alpha}}dt$$

is convergent because $t^{-1-\alpha}M_1(t)=O(t^{-1-(\alpha-\tau)})$. Now,

$$\begin{aligned} M(\lambda)=\sum_{\mu_k^{(1)}\leq\lambda}M_2(\lambda/\mu_k^{(1)})&=\tilde{B}\lambda^{\alpha}\log^{b}\lambda-B\int_0^{\infty}\frac{\log^{b}t}{t^{\alpha}}dM_1(t)+o(\lambda^{\alpha}\log^{b}\lambda)\\ &=\tilde{B}\lambda^{\alpha}\log^{b}\lambda+O\left(\lambda^{\alpha}(\log^{b}\lambda)\int_1^{\lambda}\frac{dt}{t^{1+\alpha-t}}\right)+o(\lambda^{\alpha}\log^{b}\lambda)\\ &\sim\tilde{B}\lambda^{\alpha}\log^{b}\lambda, \end{aligned}$$

as claimed. □

We can now prove Proposition 2.1.

Proof of Proposition 2.1. Let $\{j_{\nu+1},\dots,j_p\}=\{1,2,\dots,p\}\setminus\{j_1,\dots,j_\nu\}$. We arrange the two sequences of products

$$\prod_{q=\nu+1}^{p}\lambda_{k_{j_q}}^{(j_q)}\quad\text{and}\quad\prod_{q=1}^{\nu}\lambda_{k_{j_q}}^{(j_q)}\tag{2.7}$$

in two non-decreasing sequences $\{\mu_k^{(1)}\}_{k=1}^{\infty}$ and $\{\mu_k^{(2)}\}_{k=1}^{\infty}$, respectively, where each element in these sequences is repeated as many times as it can be represented as in (2.7). The hypothesis (2.3) and Lemma 2.2 yield

$$M_2(\lambda)=\sum_{\mu_k^{(2)}\leq\lambda}1\sim B\lambda^{\alpha}\log^{\nu-1}\lambda,\quad\lambda\to\infty,$$

where $B=(\alpha^{\nu-1}/(\nu-1)!)\prod_{q=1}^{\nu}A_{j_q}$. On the other hand, using (2.4), one easily verifies that

$$M_1(\lambda)=\sum_{\mu_k^{(1)}\leq\lambda}1=O(\lambda^{\tau}\log^{p-\nu}\lambda),\quad\lambda\to\infty.$$

Applying Lemma 2.3 and noticing that

$$\sum_{k=1}^{\infty}\frac{1}{\left(\mu_k^{(1)}\right)^{\alpha}}=\left(\prod_{j\notin\{j_1,\dots,j_\nu\}}\left(\sum_{k=1}^{\infty}\frac{1}{\left(\lambda_k^{(j)}\right)^{\alpha}}\right)\right),$$

we obtain the asymptotic formula (2.5) with the constant (2.6). □

2.2. **Remainders.** We now study the remainder in (2.5). We impose stronger assumptions than (2.3) on the leading counting functions.

We start by looking at the case when a single counting function dominates all others.

Proposition 2.4. *Assume that there are non-negative numbers $\tau < \eta < \alpha$ and an index $l \in \{1, \dots, p\}$ such that $N_j(\lambda) = O(\lambda^\tau)$, for $j \neq l$, and N_l satisfies*

$$N_l(\lambda) = A_l \lambda^\alpha + O(\lambda^\eta), \quad \lambda \to \infty, \tag{2.8}$$

with $A_l \neq 0$. Then,

$$N(\lambda) = A\lambda^\alpha + O(\lambda^\eta), \quad \lambda \to \infty, \tag{2.9}$$

where

$$A = A_l \prod_{j \neq l} \left(\sum_{k=1}^{\infty} \frac{1}{\left(\lambda_k^{(j)}\right)^\alpha} \right). \tag{2.10}$$

Proof. By renaming the sequences, we may assume that $l = 1$. We use a recursive argument. Suppose that we have already established

$$\tilde{N}(\lambda) = \sum_{\lambda_{k_1}^{(1)} \lambda_{k_2}^{(2)} \dots \lambda_{k_{p-1}}^{(p-1)} \leq \lambda} 1 = \tilde{A}\lambda^\alpha + O(\lambda^\eta).$$

Since for any $b > \tau$

$$\sum_{\lambda \leq \lambda_k^{(p)}} \frac{1}{(\lambda_k^{(p)})^b} = \frac{N_p(\lambda)}{\lambda^b} + b \int_\lambda^\infty \frac{N_p(t)}{t^{b+1}}\, dt = O(\lambda^{\tau - b}), \quad \lambda \to \infty,$$

we have

$$\begin{aligned} N(\lambda) &= \sum_{\lambda_k^{(p)} \leq \lambda} \tilde{N}(\lambda/\lambda_k^{(p)}) \\ &= \tilde{A}\lambda^{\alpha_1} \left(\sum_{k=1}^{\infty} \frac{1}{(\lambda_k^{(p)})^\alpha} \right) + O(\lambda^\tau) + \sum_{\lambda_k^{(p)} \leq \lambda} O((\lambda/\lambda_k^{(p)})^\eta) \\ &= A\lambda^{\alpha_1} + O(\lambda^\tau) + O(\lambda^\eta), \end{aligned}$$

which shows (2.9). □

For the analysis of the remaining case, we will employ a complex Tauberian theorem of Aramaki [2].

Proposition 2.5. *Assume there are non-negative numbers $\tau < \alpha$ and indices $j_1, \dots, j_\nu$, where $1 \leq \nu \leq p$, such that*

$$N_{j_q}(\lambda) = A_{j_q} \lambda^\alpha + O(\lambda^\tau), \quad \lambda \to \infty, \quad q = 1, \dots, \nu, \tag{2.11}$$

with $A_{j_q} \neq 0$, and (2.4) holds. Then, there exists $\eta < \alpha$ such that (2.2) *has asymptotic expansion*

$$N(\lambda) = \sum_{q=1}^{\nu} \frac{B_q}{(q-1)!} \left(\frac{d}{dz}\right)^{q-1} \left(\frac{\lambda^z}{z}\right)\Bigg|_{z=\alpha} + O(\lambda^\eta) \tag{2.12}$$

$$= \lambda^\alpha \left(A\frac{(\alpha \log \lambda)^{\nu-1}}{(\nu-1)!} + \sum_{q=0}^{\nu-2} C_q \log^q \lambda \right) + O(\lambda^\eta)$$

where A is given by (2.6) *and the constants B_q can be computed as*

$$B_q = \frac{1}{(\nu-q)!} \left(\frac{d}{dz}\right)^{\nu-q} \left((z-\alpha)^\nu \prod_{j=1}^{p} \sum_{k=1}^{\infty} \frac{1}{\left(\lambda_k^{(j)}\right)^z} \right)\Bigg|_{z=\alpha} \tag{2.13}$$

Proof. Set

$$F_j(z) = \int_0^\infty t^{-z} dN_j(t) = \sum_{k=1}^{\infty} \frac{1}{\left(\lambda_k^{(j)}\right)^z}, \quad j = 1, \dots, p,$$

and

$$F(z) = \int_0^\infty t^{-z} dN(t).$$

It is easy to verify that $F(z)$ and the $F_j(z)$ are analytic on the half-plane $\Re e\ z > \alpha$ and they are connected by the formula

$$F(z) = \prod_{j=1}^{p} F_j(z).$$

Moreover, $F_j(z)$ is analytic on $\Re e\ z > \tau$ if $j \notin \{j_1, \dots, j_q\}$, whereas

$$F_{j_q}(z) - \frac{A_{j_q}\alpha}{z-\alpha}, \quad q = 1, \dots, \nu,$$

extend analytically to the same half-plane. Furthermore, these functions are of at most polynomial growth on any strip $\tau < a < \Re e\ z < b$. Thus, F has also at most polynomial growth on such strips and it is meromorphic on $\Re e\ z > \tau$, having a single pole at $z = \alpha$ of order ν. The hypotheses from Aramaki's Tauberian theorem are therefore satisfied, and the result follows at once from it. □

2.3. **Lower order terms in some special cases.** When the $\{\lambda_k^{(j)}\}_{k=1}^\infty$ arise as $\lambda_k^{(j)} = (c_j(k-1)+b_j)^{\beta_j}$, where c_j, b_j, β_j are positive constants, and one assumes $\beta_p > \beta_{p-1} > \dots > \beta_1 > 0$, it is possible to improve the asymptotic formula (2.9) by giving lower order terms in the asymptotic expansion. Given $\vec{c} = (c_1, c_2, \dots, c_p)$, $\vec{b} = (b_1, b_2, \dots, b_p)$, $\vec{\beta} = (\beta_1, \beta_2, \dots, \beta_p) \in \mathbb{R}_+^p$, we are interested in this subsection in the asymptotic behavior of the counting function

$$D_{\vec{c},\vec{b}}^{\vec{\beta}}(\lambda) = \# \left\{ (k_1, \dots, k_p) \in \mathbb{N}^p : \ (c_1k_1 + b_1)^{\beta_1} (c_2k_2 + b_2)^{\beta_2} \dots (c_pk_p + b_p)^{\beta_p} \leq \lambda \right\}. \tag{2.14}$$

For the constants in our expansions, we shall need the Hurwitz zeta function [1, p. 251]. It is defined for fixed $a > 0$ as

$$\zeta(z; a) = \sum_{k=0}^{\infty} \frac{1}{(k+a)^z}, \quad \Re e z > 1. \tag{2.15}$$

It is well-known that (2.15) admits meromorphic continuation to the whole complex plane, with a simple pole at $z = 1$ with residue 1 (cf. [1, p. 254] or [11, p. 348]). In particular, when $a = 1$ we recover $\zeta(z) = \zeta(z, 1)$, the Riemann zeta function. Using the Euler-Maclaurin summation formula [1, 11, 21], one easily deduces the following asymptotic formula

$$\sum_{0 \le k \le \lambda} \frac{1}{(k+a)^s} = \frac{(\lambda+a)^{1-s}}{1-s} + \zeta(s; a) + O(\lambda^{-s}), \quad \lambda \to \infty, \text{ when } 0 < s \text{ and } s \neq 1. \tag{2.16}$$

Observe that Proposition 2.1 immediately yields the dominant term in the asymptotic expansion of (2.14),

$$D^{\vec{\beta}}_{\vec{c},\vec{b}}(\lambda) \sim \frac{1}{c_1} \left(\prod_{j=2}^{p} \frac{\zeta(\beta_j/\beta_1; b_j/c_j)}{c_j^{\beta_j/\beta_1}} \right) \lambda^{\frac{1}{\beta_1}}, \quad \lambda \to \infty. \tag{2.17}$$

We begin with the analysis of the case $p = 2$. The proof of the following lemma is inspired by the classical Dirichlet hyperbola method [1, p. 57].

Lemma 2.6. *Let $\vec{\beta} = (\beta_1, \beta_2)$ be such that $0 < \beta_1 < \beta_2$. Then,*

$$D^{\vec{\beta}}_{\vec{c},\vec{b}}(\lambda) = \frac{\zeta(\beta_2/\beta_1; b_2/c_2)}{c_1 c_2^{\beta_2/\beta_1}} \lambda^{\frac{1}{\beta_1}} + \frac{\zeta(\beta_1/\beta_2; b_1/c_1)}{c_2 c_1^{\beta_1/\beta_2}} \lambda^{\frac{1}{\beta_2}} + O(\lambda^{\frac{1}{\beta_1+\beta_2}}). \tag{2.18}$$

Proof. Since $D^{\vec{\beta}}_{\vec{c},\vec{b}}(\lambda) = D^{\vec{\beta}}_{\vec{e},\vec{d}}(\lambda/(c_1^{\beta_1} c_2^{\beta_2}))$, where $\vec{e} = (1, 1)$ and $\vec{d} = (b_1/c_1, b_2/c_2)$, we may assume that $c_1 = c_2 = 1$. For ease of writing, we set $D^{\vec{\beta}}_{\vec{b}} = D^{\vec{\beta}}_{\vec{e},\vec{b}}$. We have that

$$\begin{aligned}
D^{\vec{\beta}}_{\vec{b}}(\lambda) &= \sum_{(k_1+b_1)^{\beta_1}(k_2+b_2)^{\beta_2} \le \lambda} 1 \\
&= \sum_{k_1+b_1 \le \lambda^{1/(\beta_1+\beta_2)}} \left(\frac{\lambda}{(k_1+b_1)^{\beta_1}} \right)^{\frac{1}{\beta_2}} - k_1 + \sum_{k_2+b_2 \le \lambda^{1/(\beta_1+\beta_2)}} \left(\frac{x}{(k_2+b_2)^{\beta_2}} \right)^{\frac{1}{\beta_1}} - k_2 + O(\lambda^{\frac{1}{\beta_1+\beta_2}}) \\
&= \lambda^{\frac{1}{\beta_2}} I_{1,\beta_1/\beta_2}(\lambda^{1/(\beta_1+\beta_2)} - b_1) + \lambda^{\frac{1}{\beta_1}} I_{2,\beta_2/\beta_1}(\lambda^{1/(\beta_1+\beta_2)} - b_2) - \lambda^{\frac{2}{\beta_1+\beta_2}} + O(\lambda^{\frac{1}{\beta_1+\beta_2}}),
\end{aligned}$$

where $I_{j,s}(x) = \sum_{k \le x} (k + b_j)^{-s}$. The asymptotic formula (2.16) gives

$$\begin{aligned}
\lambda^{\frac{1}{\beta_2}} I_{1,\beta_1/\beta_2}(\lambda^{1/(\beta_1+\beta_2)} - b_1) &= \lambda^{\frac{1}{\beta_2}} \left(\zeta(\beta_1/\beta_2; b_1) + \frac{\beta_2 \lambda^{\frac{\beta_2-\beta_1}{\beta_2(\beta_1+\beta_2)}}}{\beta_1 - \beta_2} + O(\lambda^{\frac{-\beta_1}{\beta_2(\beta_1+\beta_2)}}) \right) \\
&= \lambda^{\frac{1}{\beta_2}} \zeta(\beta_1/\beta_2; b_1) + \frac{\beta_2 \lambda^{\frac{2}{\beta_1+\beta_2}}}{\beta_2 - \beta_1} + O(\lambda^{\frac{1}{\beta_1+\beta_2}}),
\end{aligned}$$

and similarly

$$\lambda^{\frac{1}{\beta_1}} I_{2,\beta_2/\beta_1}(\lambda^{1/(\beta_1+\beta_2)} - b_2) = \lambda^{\frac{1}{\beta_1}} \zeta(\beta_2/\beta_1; b_2) + \frac{\beta_1 \lambda^{\frac{2}{\beta_1+\beta_2}}}{\beta_1 - \beta_2} + O(\lambda^{\frac{1}{\beta_1+\beta_2}}).$$

The relation (2.18) follows on combining the three previous asymptotic formulas. □

In general, we have:

Proposition 2.7. *Let $\vec{\beta} = (\beta_1, \dots, \beta_p) \in \mathbb{R}^p$ be such that $\beta_p > \beta_{p-1} > \cdots > \beta_1 > 0$. Then, the counting function (2.14) has asymptotics*

$$D^{\vec{\beta}}_{\vec{c},\vec{b}}(\lambda) = \sum_{j=1}^{p} A_j \lambda^{\frac{1}{\beta_j}} + O(\lambda^{\frac{p-1}{\beta_1+\cdots+\beta_p}}), \quad \lambda \to \infty, \tag{2.19}$$

where $A_j = A_{j,\vec{\beta},\vec{c},\vec{b}} = c_j^{-1} \prod_{\nu \neq j} c^{-\beta_\nu/\beta_j} \zeta(\beta_\nu/\beta_j; b_\nu/c_\nu)$.

Remark 2.8. In (2.19), some of the terms may be absorbed by the error term, only those j such that

$$(p-1)\beta_j \leq \beta_1 + \cdots + \beta_p$$

occur in the sum. Of course, this always holds for $j = 1, 2$; thus, at least, we always have two leading terms in (2.19).

Proof. The case $p = 2$ is Lemma 2.6. Assume the result is valid for $p - 1$, we proceed to show (2.19) by induction. As in Lemma 2.6, we may suppose that $c_1 = c_2 = \cdots = c_p = 1$. For simplicity, we write $D^{\vec{\beta}}_{\vec{b}} = D^{\vec{\beta}}_{\vec{c},\vec{b}}$. Set $\alpha = \sum_{j=2}^{p} \beta_j$, $\vec{d} = (b_2, b_3, \dots, b_p) \in \mathbb{R}^{p-1}_+$, and $\vec{\eta} = (\beta_2, \dots, \beta_p) \in \mathbb{R}^{p-1}_+$. Write

$$D^{\vec{\beta}}_{\vec{b}}(\lambda) = I_1(\lambda) + I_2(\lambda) + O(\lambda^{1/(\alpha+\beta_1)}),$$

where

$$I_1(\lambda) = \sum_{k_1+b_1 \leq \lambda^{1/(\alpha+\beta_1)}} D^{\vec{\eta}}_{\vec{d}}\left(\lambda/(k_1+b_1)^{\beta_1}\right),$$

$$I_2(\lambda) = \sum_{\prod_{j=2}^{p}(k_j+b_j)^{\beta_j} \leq \lambda^{\alpha/(\alpha+\beta_1)}} \left(\frac{\lambda}{(k_2+b_2)^{\beta_2}(k_3+b_3)^{\beta_3}\dots(k_p+b_p)^{\beta_p}}\right)^{1/\beta_1} - \lambda^{\frac{1}{\alpha+\beta_1}} D^{\vec{\eta}}_{\vec{d}}(\lambda^{\alpha/(\alpha+\beta_1)}),$$

and

$$\begin{aligned} D^{\vec{\eta}}_{\vec{d}}(\lambda) &= \# \left\{(k_2, \dots, k_p) \in \mathbb{N}^{p-1} : \ (k_2+b_2)^{\beta_2} \dots (k_p+b_p)^{\beta_p} \leq \lambda\right\} \\ &= \sum_{j=2}^{p} \tilde{A}_j \lambda^{\frac{1}{\beta_j}} + O(\lambda^{\frac{p-2}{\alpha}}), \quad \lambda \to \infty, \end{aligned}$$

with $\tilde{A}_j = \prod_{2\leq \nu, \nu\neq j} \zeta(\beta_\nu/\beta_j; b_\nu)$ for $j = 2, \dots, p$. If we combine the latter with (2.16), we conclude that the asymptotic behavior of $I_1(\lambda)$ is

$$\begin{aligned} I_1(\lambda) &= \sum_{j=2}^{p} \tilde{A}_j \sum_{k+b_1\leq \lambda^{1/(\alpha+\beta_1)}} \frac{\lambda^{1/\beta_j}}{(k+b_1)^{\beta_1/\beta_j}} + O(\lambda^{(p-1)/(\alpha+\beta_1)}) \\ &= \sum_{j=2}^{p} \tilde{A}_j \left(\zeta\left(\beta_1/\beta_j; b_1\right) \lambda^{1/\beta_j} + \frac{\beta_j \lambda^{(\alpha+\beta_j)/(\beta_j(\alpha+\beta_1))}}{\beta_j - \beta_1} \right) + O(\lambda^{(p-1)/(\alpha+\beta_1)}) \\ &= \sum_{j=2}^{p} A_j \lambda^{1/\beta_j} + \tilde{A}_j \frac{\beta_j \lambda^{(\alpha+\beta_j)/(\beta_j(\alpha+\beta_1))}}{\beta_j - \beta_1} + O(\lambda^{(p-1)/(\alpha+\beta_1)}). \end{aligned}$$

Observe that $C := \beta_1^{-1} \int_0^\infty t^{-1-1/\beta_1} D_{\vec{d}}^{\vec{\eta}}(t)dt$ is absolutely convergent. We then have

$$\begin{aligned} I_2(\lambda) &= \lambda^{1/\beta_1} \int_0^{\lambda^{\alpha/(\alpha+\beta_1)}} t^{-1/\beta_1} dD_{\vec{d}}^{\vec{\eta}}(t) - \lambda^{\frac{1}{\alpha+\beta_1}} D_{\vec{d}}^{\vec{\eta}}(\lambda^{\alpha/(\alpha+\beta_1)}) \\ &= \frac{\lambda^{1/\beta_1}}{\beta_1} \int_0^{\lambda^{\alpha/(\alpha+\beta_1)}} t^{-1-1/\beta_1} D_{\vec{d}}^{\vec{\eta}}(t)dt \\ &= C\lambda^{1/\beta_1} - \frac{\lambda^{1/\beta_1}}{\beta_1} \int_{\lambda^{\alpha/(\alpha+\beta_1)}}^{\infty} t^{-1-1/\beta_1} D_{\vec{d}}^{\vec{\eta}}(t)dt \\ &= C\lambda^{1/\beta_1} - \sum_{j=2}^{p} \frac{\tilde{A}_j \beta_j \lambda^{(\alpha+\beta_j)/(\beta_j(\alpha+\beta_1))}}{\beta_j - \beta_1} + O(\lambda^{(p-1)/(\alpha+\beta_1)}). \end{aligned}$$

Thus, we have shown (2.19) except for $C = \prod_{\nu=2}^{p} \zeta(\beta_\nu/\beta_1; b_\nu)$. But this fact follows by comparison with (2.17). The proof is complete. □

Remark 2.9. In connection with Proposition 2.7, Estrada and Kanwal have given an interesting distributional treatment of the asymptotic expansions of type (2.19), which often leads to improvements in the error term when interpreted in the distributional sense (cf. [11, Sec. 5.3]).

3. Counting functions for tensor products of pseudo-differential operators

We now apply results from Section 2 to the spectral asymptotics of the tensor products of pseudo-differential operators, and their perturbations. We shall mainly refer to operators in the Euclidean setting. Parallel results for operators on compact manifold will be outlined at the end. For the sake of completeness, we begin with a short survey of the classes of Shubin, cf. [7, 17, 25, 32].

3.1. **Globally elliptic pseudo-differential operators.** Write $z = (x, \xi) \in \mathbb{R}^{2n}$ and $<z>= (1 + |z|^2)^{1/2} = (1 + |x|^2 + |\xi|^2)^{1/2}$. One defines the class of symbols $\Gamma_\rho^m(\mathbb{R}^n)$, $m \in \mathbb{R}$, $0 < \rho \leq 1$, as the set of all functions $a \in C^\infty(\mathbb{R}^{2n})$ satisfying, for all γ,

$$|\partial_z^\gamma a(z)| \leq C_\gamma <z>^{m-\rho|\gamma|}, \quad z \in \mathbb{R}^{2n}, \tag{3.1}$$

with constants independent of z. The corresponding pseudo-differential operator is defined by Weyl quantization as

$$Pu(x) = a^w u(x) = \frac{1}{(2\pi)^n} \int e^{i(x-y)\xi} a\left(\frac{x+y}{2}, \xi\right) u(y) dy d\xi. \tag{3.2}$$

Note that if the symbol a is a polynomial in the ξ variables, i.e. P in (3.2) is a partial differential operator, then the estimates (3.1) force $a(z)$ to be a polynomial in the $x-$variables as well, i.e. P is a partial differential operator with polynomial coefficients.

Let us introduce the global Sobolev spaces $H^s(\mathbb{R}^n), s \in \mathbb{N}$, Hilbert spaces with the norm

$$||u||_s = \sum_{|\alpha|+|\beta| \leq s} ||x^\alpha D^\beta u|| < \infty. \tag{3.3}$$

By interpolation and duality the definition extends to $s \in \mathbb{R}$, and we have $\bigcap_s H^s(\mathbb{R}^n) = \mathcal{S}(\mathbb{R}^n)$, $\bigcup_s H^s(\mathbb{R}^n) = \mathcal{S}'(\mathbb{R}^n)$. The immersion $\iota :^s H^s \to H^t$ is compact for $s > t$. If $a \in \Gamma^m_\rho(\mathbb{R}^n)$, then $a^w : H^s(\mathbb{R}^n) \to H^{s-m}(\mathbb{R}^n)$ continuously for every $s \in \mathbb{R}$, hence $a^w : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$, $\mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$. In the following we shall assume that for large $|z|$,

$$a(z) = a_m(z) + a_{m-\rho}(z), \tag{3.4}$$

where $a_m(tz) = t^m a_m(z), t > 0$. We then say that a is globally elliptic if

$$a_m(z) \neq 0 \quad \text{for } z \neq 0. \tag{3.5}$$

Operators with globally elliptic symbol possess parametrix. Namely, there exists $b \in \Gamma^{-m}_\rho(\mathbb{R}^n)$ such that $a^w b^w = I + R_1$ and $b^w a^w = I + R_2$, where $R_1, R_2 : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$. It follows that $a^w : H^s(\mathbb{R}^n) \to H^{s-m}(\mathbb{R}^n)$ is a Fredholm operator and then eigenfunctions, i.e. solutions of $a^w u = 0$, do not depend on $s \in \mathbb{R}$ and belong to $\mathcal{S}(\mathbb{R}^n)$. Passing now to spectral theory, we assume that $a \in \Gamma^m_\rho(\mathbb{R}^n)$, $m > 0$, is real-valued and globally elliptic with $a_m(z) > 0$, for $z \neq 0$. Then $P = a^w u : H^m(\mathbb{R}^n) \mapsto L^2(\mathbb{R}^n)$ is self-adjoint. The resolvent is compact and the spectrum is given by a sequence of real eigenvalues $\lambda_k \to \infty$ with finite multiplicity; the eigenfunctions belong to $\mathcal{S}(\mathbb{R})$ and form an orthonormal basis. The spectral counting function $N_P(\lambda) = \#\{k : \lambda_k \leq \lambda\}$ behaves as

$$N_P(\lambda) = A\lambda^{2n/m} + O(\lambda^\sigma), \quad \lambda \to \infty, \tag{3.6}$$

for some $\sigma < 2n/m$, with

$$A = \frac{1}{(2\pi)^n} \int_{a_m(z) \leq 1} dz. \tag{3.7}$$

A sharp form of the remainder in (3.6) can be obtained when $a \in \Gamma^m(\mathbb{R}^n) = \Gamma^m_1(\mathbb{R}^n)$ admits an asymptotic expansion in homogeneous terms $a \sim \sum_{k\in\mathbb{N}} a_{m-2k}$. Then, with A as before,

$$N_P(\lambda) = A\lambda^{2n/m} + O(\lambda^{2(n-1)/m}), \tag{3.8}$$

see, for example, Helffer [17, p. 175]. In the sequel, we shall assume that P is strictly positive, so that $0 < \lambda_1 \leq \lambda_2 \leq \ldots$. For P as before, we may define the complex powers $P^z, z \in \mathbb{C}$.

They are trace class operators if $\Re ez < -2n/m$, and, by analytic continuation, we define the zeta function associated to P as

$$\zeta_P(z) = \mathrm{Tr}(P^{-z}) = \sum_{k=1}^{\infty} \lambda_k^{-z}. \tag{3.9}$$

3.2. **Spectral asymptotics for tensor products.** To give a precise functional frame to the results in the sequel, we shall introduce first the tensorized global Sobolev spaces. Write now $x_j, y_j \in \mathbb{R}^{n_j}$, $z_j = (x_j, y_j) \in \mathbb{R}^{2n_j}$, $j = 1, ..., p$, $n = n_1 + ... + n_p$, $x = (x_1, ..., x_p)$, $y = (y_1, ..., y_p) \in \mathbb{R}^p$, $z = (z_1, ..., z_p) = (x_1, y_1, ..., x_p, y_p)$. For $\vec{s} = (s_1, ..., s_p) \in \mathbb{R}^p$, we define the tensor product of Hilbert spaces

$$H^{\vec{s}}(\mathbb{R}^n) = \bigotimes_{j=1}^{p} H^{s_j}(\mathbb{R}^{n_j}). \tag{3.10}$$

When the components of $\vec{s}$ are non-negative integers, from (3.3) we recapture as norm

$$||u||_{\vec{s}} = \sum_{\substack{|\alpha_j|+|\beta_j|\le s_j \\ j=1,\dots,p}} ||x^{\alpha_1}...x^{\alpha_2} D_{x_1}^{\beta_1}...D_{x_p}^{\beta_p} u||. \tag{3.11}$$

We have $\bigcap_{\vec{s}} H^{\vec{s}}(\mathbb{R}^n) = \mathcal{S}(\mathbb{R}^n)$ and $\bigcup_{\vec{s}} H^{\vec{s}}(\mathbb{R}^n) = \mathcal{S}'(\mathbb{R}^n)$. The immersion $\iota : H^{\vec{s}}(\mathbb{R}^n) \to H^{\vec{t}}(\mathbb{R}^n)$ is compact if $\vec{s} > \vec{t}$, i.e. $s_j > t_j$ for $j = 1, ..., p$.

As announced at the Introduction, we consider now $P_j = a_j^w$ in $\mathbb{R}^{n_j}$, $j = 1, ..., p$, with real-valued symbol $a_j \in \Gamma^{m_j}(\mathbb{R}^{n_j})$, $m_j > 0$,, and $a_{m_j}(z) > 0$ for $z \neq 0$ in (3.5); we further define

$$P = P_1 \otimes ... \otimes P_p, \tag{3.12}$$

as operator $P : H^{\vec{s}}(\mathbb{R}^n) \to H^{\vec{s}-\vec{m}}(\mathbb{R}^n)$, $\vec{m} = (m_1, ..., m_p)$, for every $\vec{s} \in \mathbb{R}^p$. In particular, we have $P : H^{\vec{m}}(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ and $P : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$, $\mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$. Moreover, P is self-adjoint and strictly positive, if the factors P_j are assumed to be strictly positive.

If we denote by $\{\lambda_k^{(j)}\}_{k=1}^{\infty}$ the eigenvalues of P_j, according to the Introduction, the eigenvalues of P are of the form $\lambda_{k_1}^{(1)}...\lambda_{k_p}^{(p)}$ and the eigenfunctions are tensor products of the respective eigenfunctions, hence they belong to $\mathcal{S}(\mathbb{R}^n)$.

It is worth observing that P can be written in the pseudo-differential form (3.2) with symbol $a(z) = a_1(z_1)...a_p(z_p)$. However, the estimate (3.1) fails in general, and the considerations of Subsection 3.1 do not apply in this context. For the case $p = 2$, we address to [5], cf. [6, 26], where a calculus was achieved in terms of vector-valued symbols. Here, to find an asymptotic expansion for $N_P(\lambda)$, we shall use (1.2) in combination with the analysis of Section 2. In fact, from (3.6) and (3.7), we have

$$N_{P_j} \sim A_j \lambda^{2n_j/m_j} \quad \text{ with } A_j = \frac{1}{(2\pi)^{n_j}} \int_{a_{m_j}(z_j)\le 1} dz_j. \tag{3.13}$$

Writing ζ_{P_j} for the zeta function of P_j, we immediately obtain from Proposition 2.1:

Theorem 3.1. *Let $P = P_1 \otimes \cdots \otimes P_p$ be as above and let $\alpha = \max_j\{2n_j/m_j\}$. Let further $j_1, \ldots, j_\nu$ be the indices such that $\alpha = 2n_{j_q}/m_{j_q}$, $q = 1, \ldots, \nu$. Then, P has spectral asymptotics*

$$N_P(\lambda) = \sum_{\lambda^{(1)}_{k_1}\lambda^{(2)}_{k_2}\ldots\lambda^{(p)}_{k_p}\leq\lambda} 1 \sim \left(\prod_{q=1}^{\nu} A_{j_q} \cdot \prod_{j\notin\{j_1,\ldots,j_q\}} \zeta_{P_j}(\alpha)\right) \lambda^{\alpha} \frac{(\alpha \log\lambda)^{\nu-1}}{(\nu-1)!}, \quad \lambda \to \infty, \tag{3.14}$$

where A_j is given by (3.13).

We remark that the case $p = 2$, $\nu = 1$ or $\nu = 2$, of Theorem 3.1 also follows from the results of [6], see also [5]).

As far as the reminder in (3.14) concerns, from (3.6) and Proposition 2.5, we obtain

$$N_P(\lambda) = \lambda^{\alpha} \sum_{q=0}^{\nu-1} C_q \log^q \lambda + O(\lambda^{\eta}), \tag{3.15}$$

for some $\eta < \alpha$. The coefficient $C_{\nu-1}$ is given by (3.14) and the other constants C_q, $q = 0, ..., \nu-2$, are determined by (2.12), (2.13), and the values of the derivatives or poles of the zeta functions $\zeta_{P_j}(z)$ at $z = \alpha$, $j = 1, \ldots, p$.

Willing sharp values of η in the remainder, we further assume that $a_j \in \Gamma^{m_j}(\mathbb{R}^{n_j})$ with $a_j \sim \sum_{k\in\mathbb{N}} a_{m_j-2k}$ and we use (3.8). Proposition 2.4 yields,

Theorem 3.2. *Let $P = P_1 \otimes ... \otimes P_p$ be as above. Assume that there is an index $l \in \{1, ..., p\}$ such that $2n_l/m_l > \beta = \max_{j\neq l}\{2n_j/m_j\}$. Then*

$$N_P(\lambda) = \left(A_l \prod_{j\neq l} \zeta_{P_j}(\alpha)\right) \lambda^{\frac{2n_l}{m_l}} + O(\lambda^{\eta}), \tag{3.16}$$

for any η with $\eta > \max\{\beta, 2(n_l-1)/m_l\}$.

The following example shows that the exponent $\eta = \beta$ is sharp in (3.16).

Example 3.3 (Tensorized Hermite operators)**.** For tensor products of Hermite operators it is possible to detect lower order terms in the asymptotic expansion (3.16). Namely, let us fix $\vec{\beta} = (\beta_1, ..., \beta_p)$ with $\beta_1 < ... < \beta_p, \vec{c} = (c_1, ..., c_p), \vec{b} = (b_1, ..., b_p)$, p-tuples of positive real numbers, cf. Subsection 2.3, and consider

$$H_{j,c_j,b_j} = \frac{c_j}{2}(-\partial^2_{x_j} + x_j^2) - \frac{c_j}{2} + b_j, \quad j = 1, ..., p, \tag{3.17}$$

so that for $c_j = 1$, $b_j = 1$, we recapture H_j in (1.3) of the Introduction. The eigenvalues of H_{j,c_j,b_j}, as one dimensional operator, are $\lambda^{(j)}_k = c_j(k-1) + b_j$, $k = 1, 2, \ldots$. We then define the tensorized Hermite operator

$$H^{\vec{\beta}}_{\vec{c},\vec{b}} = \bigotimes_{j=1}^{p} H^{\beta_j}_{j,c_j,b_j}. \tag{3.18}$$

By Proposition 2.7, we have for the corresponding counting function

$$N(\lambda) = D^{\vec{\beta}}_{\vec{c},\vec{b}}(\lambda) = \sum_{j=1}^{p} A_j \lambda^{1/\beta_j} + O(\lambda^{\frac{p-1}{\beta_1+\ldots+\beta_p}}) \tag{3.19}$$

with A_j as in Proposition 2.7. In particular, for $p = 2, c_j = 1, b_j = 1, j = 1, 2$, we obtain (1.8) of the Introduction.

3.3. **Asymptotics for lower order perturbations.** For simplicity, we shall assume that the factors P_j in $P = P_1 \otimes \ldots \otimes P_p$ are partial differential operators with polynomial coefficients:

$$P_j = \sum_{|\alpha_j|+|\beta_j|\le m_j} c^{(j)}_{\alpha_j,\beta_j} x^{\alpha_j} D^{\beta_j}_{x_j}, \quad x_j \in \mathbb{R}^{n_j}. \tag{3.20}$$

As before, we assume that P_j is elliptic, with principal symbol

$$p^{(j)}_{m_j}(x,\xi) = \sum_{|\alpha_j|+|\beta_j|= m_j} c^{(j)}_{\alpha_j,\beta_j} x_j^{\alpha_j} \xi_j^{\beta_j} > 0 \quad \text{for } (x_j,\xi_j) \neq (0,0), \tag{3.21}$$

self-adjoint and strictly positive, $j = 1, \ldots, p$. We shall study

$$A = P + R, \tag{3.22}$$

where R is a partial differential operator with polynomial coefficients having lower order with respect to P, in the sense that, writing $\vec{\alpha} = (\alpha_1, \ldots, \alpha_p), \vec{\beta} = (\beta_1, \ldots, \beta_p) \in \mathbb{N}^n, n = n_1 + \ldots + n_p$,

$$R = \sum_{\substack{|\alpha_j|+|\beta_j|< m_j \\ j=1,\ldots,p}} c_{\alpha\beta} x^{\vec{\alpha}} D^{\vec{\beta}}. \tag{3.23}$$

Note that each term of the sum in (3.23) can be regarded as a tensor product:

$$x^{\vec{\alpha}} D^{\vec{\beta}} = x_1^{\alpha_1} D^{\beta_1}_{x_1} \otimes \ldots \otimes x_p^{\alpha_p} D^{\beta_p}_{x_p},$$

hence for every $\vec{s} \in \mathbb{R}^p$,

$$A = P + R : \; H^{\vec{s}}(\mathbb{R}^n) \to H^{\vec{s}-\vec{m}}(\mathbb{R}^n).$$

We shall first construct a parametrix for A. In absence of symbolic calculus, we shall use in the proof a direct argument.

Proposition 3.4. *For every fixed integer $M > 0$, we can find $B : H^{\vec{s}}(\mathbb{R}^n) \to H^{\vec{s}+\vec{m}}(\mathbb{R}^n)$ for every $\vec{s} = (s_1, \ldots, s_p) \in \mathbb{R}^p$, such that $BA = I + S', AB = I + S''$, where $S', S'' : H^{\vec{s}}(\mathbb{R}^n) \to H^{\vec{s}+\vec{M}}(\mathbb{R}^n)$, with $\vec{M} = (M, \ldots, M)$.*

Proof. Consider

$$P^{-1} = P_1^{-1} \otimes \ldots \otimes P_p^{-1} : H^{\vec{s}}(\mathbb{R}^n) \to H^{\vec{s}+\vec{m}}(\mathbb{R}^n).$$

We have $P^{-1}A = P^{-1}(P + R) = I - S$ with

$$S = -P^{-1}R : H^{\vec{s}}(\mathbb{R}^n) \to H^{\vec{s}+\vec{1}}(\mathbb{R}^n).$$

Define then

$$B = \sum_{j=0}^{M-1} S^j P^{-1} : H^{\vec{s}}(\mathbb{R}^n) \to H^{\vec{s}+\vec{M}}(\mathbb{R}^n).$$

We have

$$BA = \sum_{j=0}^{M-1} S_j(I-S) = I - S^M,$$

where $S' = -S^M : H^{\vec{s}}(\mathbb{R}^n) \to H^{\vec{s}+\vec{M}}(\mathbb{R}^n)$. It is easy to check that B is also a right parametrix. □

Corollary 3.5. The solution $u \in \mathcal{S}'(\mathbb{R}^n)$ of $Au = f \in \mathcal{S}(\mathbb{R}^n)$ belongs to $\mathcal{S}(\mathbb{R}^n)$.

Proof. If $u \in \mathcal{S}'(\mathbb{R}^n)$, then $u \in H^{\vec{s}}(\mathbb{R}^n)$ for some $\vec{s}$. Taking B as in Proposition 3.4, we obtain

$$BAu = (I + S')u = Bf,$$

hence, $u = Bf - S'u$. We have $Bf \in \mathcal{S}(\mathbb{R}^n)$ and $S'u \in H^{\vec{s}+\vec{M}}(\mathbb{R}^n)$. Since M in Proposition 3.4 can be fixed as large as we want, we conclude $u \in \mathcal{S}(\mathbb{R}^n)$. □

Corollary 3.6. The operator $A : H^{\vec{s}}(\mathbb{R}^n) \to H^{\vec{s}-\vec{m}}(\mathbb{R}^n)$ is Fredholm, for every fixed $\vec{s} \in \mathbb{R}^n$.

Proof. Let us apply Proposition 3.4 with $M = 1$. Since the inclusion $H^{\vec{s}+\vec{1}}(\mathbb{R}^n) \to H^{\vec{s}}(\mathbb{R}^n)$ is compact, the Fredholm property is proved. □

Let us assume now that the operator A in (3.22) is self-adjoint. It follows from the preceding arguments that the resolvent is compact and the eigenfunctions belong to $\mathcal{S}(\mathbb{R}^n)$. Assume further that A is strictly positive; we write $0 < \lambda_1 \leq \lambda_2 \leq ...$ for its eigenvalues and N_A for its spectral counting function. We give below an asymptotic formula for λ_k. In the sequel we write $f \asymp g$ to mean that $f = O(g)$ and $g = O(f)$ are both valid.

Theorem 3.7. *Let $A = P + R$ in* (3.22) *be as above. We use for P the notation of Theorem 3.1, namely we write $\alpha = \max_j\{2n_j/m_j\}$ and we assume that $\alpha = 2n_j/m_j$ for ν indices. We then have*

$$\lambda_k \asymp k^{1/\alpha}(\log k)^{-(\nu-1)/\alpha}, \quad k \to \infty \tag{3.24}$$

and

$$N_A(\lambda) \asymp \lambda^\alpha \log^{\nu-1} \lambda, \quad \lambda \to \infty. \tag{3.25}$$

Proof. We have

$$||Au||^2 = ||AP^{-1}Pu||^2 \leq C_1||Pu||^2 \text{ with } C_1 = ||AP^{-1}||^2_{\mathcal{L}(L^2)}.$$

On the other hand, using Proposition 3.4, we may write $I = BA - S'$ and thus

$$||Pu||^2 = ||P(BA - S')u||^2 \leq 2||PBAu||^2 + 2||PS'u||^2| \leq C_2(||Au||^2 + ||u||^2)$$

with

$$C_2 = 2\max\{||PB||^2_{\mathcal{L}(L^2)}, ||PS'||^2_{\mathcal{L}(L^2)}\},$$

where $||PS'||_{\mathcal{L}(L^2)} < \infty$ if M in Proposition 3.4 is chosen sufficiently large. We now rewrite the preceding estimates as

$$(A^2u, u) \leq (C_1P^2u, u) \text{ and } (P^2u, u) \leq (C_2(A^2 + I)u, u).$$

Using the classical max-min formula for the eigenvalues of A^2, P^2 and denoting here μ_k the eigenvalues of P, we deduce

$$\lambda_k^2 \leq C_1 \mu_k^2 \text{ and } \mu_k^2 \leq C_2(\lambda_k^2 + 1).$$

Hence $\lambda_k \asymp \mu_k$. As a final step in the proof, we apply the following lemma.

Lemma 3.8. *If the sequence* $0 < \mu_1 \leq \mu_2 \leq \ldots$, $\mu_k \to \infty$, *admits counting function*

$$N(\mu) \sim r\mu^\alpha \log^s \mu, \quad \mu \to \infty,$$

with $r, \alpha > 0$ *and* $s \geq 0$, *then*

$$\mu_k \sim \left(\frac{\alpha}{r}\right)^{1/\alpha} k^{1/\alpha} (\log k)^{-s/\alpha}, \quad k \to \infty. \tag{3.26}$$

The proof of this lemma is a simple combination of Proposition 4.6.4, page 198, and Lemma 5.2.9, page 219, from [25] and it is therefore omitted. Since for the counting function $N_P(\mu)$ we have from Theorem 3.1

$$N_P(\mu) \sim r\mu^\alpha (\log \mu)^{\nu - 1}$$

for a constant r, we deduce from Lemma 3.8 for the eigenvalues μ_k of P the asymptotics (3.26) with $s = \nu - 1$. Hence (3.24) follows. The asymptotic formula (3.25) can be easily deduced from (3.24), we leave details to the reader. $\square$

The rough asymptotics (3.25) can hopefully be improved, as suggested by the result from [6], which gives $N_A(\lambda) \sim N_P(\lambda)$ in the case $p = 2$. Furthermore, we expect formula (3.15) is invariant under lower order perturbations. On the contrary, the precise asymptotics (3.19) for tensorized Hermite operators should be lost, after addition of lower order terms.

3.4. **Pseudo-differential operators on closed manifolds.** We now look at pseudo-differential operators on closed manifolds. Let $M_1, M_2, \ldots, M_p$ be closed manifolds with $\dim M_j = n_j$. We consider elliptic self-adjoint pseudo-differential operators P_j on M_j of order m_j and principal symbol $a_{m_j}(x_j, \xi_j) > 0$ for $(x_j, \xi_j) \in T^*M_j \setminus (M_j \times \{0\}), j = 1, \ldots, p$. We denote by $dx_j d\xi_j$ the natural volume form on the cotangent bundle T^*M_j. Under these circumstances, Hörmander's theorem [32, Chap. III] gives us the asymptotic behavior of each counting function $N_{P_j}(\lambda)$ of the eigenvalues $\{\lambda_k^{(j)}\}_{k=1}^\infty$ of P_j. In fact,

$$N_{P_j}(\lambda) = \sum_{\lambda_k^{(j)} \leq \lambda} 1 = A_j \lambda^{n_j/m_j} + O(\lambda^{(n_j-1)/m_j}), \quad \lambda \to \infty,$$

where

$$A_j = \frac{1}{(2\pi)^{n_j}} \int_{a_{m_j}(x_j, \xi_j) < 1} dx_j d\xi_j, \quad j = 1, \ldots, p. \tag{3.27}$$

As usual, ζ_{P_j} denotes the zeta function of the operator P_j.

Proposition 2.4 directly gives the spectral asymptotics of the operator $P = P_1 \otimes P_2 \otimes \cdots \otimes P_p$ on the closed manifold $M = M_1 \times M_2 \times \cdots \times M_p$ of dimension $\dim M = n = n_1 + n_2 + \cdots + n_p$, whenever one of the counting functions N_{P_l} dominates all the others.

Theorem 3.9. *Let P_j be elliptic self-adjoint strictly positive pseudo-differential operator as above, $j = 1, \dots, p,$. Suppose that there is $l \in \{1, 2, \dots, p\}$ such that $n_l/m_l > n_j/m_j$ for all $j \neq l$. Then, the spectral counting function N_P of the operator $P = P_1 \otimes P_2 \otimes \cdots \otimes P_p$ has asymptotics*

$$N_P(\lambda) = \sum_{\lambda^{(1)}_{k_1}\lambda^{(2)}_{k_2}\dots\lambda^{(p)}_{k_p}\leq\lambda} 1 = \left(A_l \prod_{j\neq l} \zeta_{P_j}(n_l/m_l) \right) \lambda^{n_l/m_l} + O(\lambda^\tau), \quad \lambda \to \infty, \tag{3.28}$$

where A_l is given by (3.27) *and τ satisfies* $\max\{(n_l - 1)/m_l, \max_{j\neq l} n_j/m_j\} < \tau < n_l/m_l$.

For the special case of the tensor product of two elliptic operators with one counting function dominating the other one, the error term in (3.2) improves that from [5, Thrm. 3.2] for bisingular operators.

We leave to the reader statements and proofs for the counterparts of Theorem 3.1, (3.15) and Theorem 3.7 in the setting of closed manifolds.

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Dipartimento di Matematica e Informatica, Università di Cagliari, Via Ospedale 72, 09124 Cagliari, Italy
E-mail address: todor@unica.it

Department of Mathematics and Informatics, University of Novi Sad, trg. D. Obradovica 4, 21000 Novi Sad, Serbia
E-mail address: stevan.pilipovic@uns.dmi.ac.rs

Dipartimento di Matematica, Università di Torino, Via Carlo Alberto 10, 10123 Torino, Italy
E-mail address: luigi.rodino@unito.it

Department of Mathematics, Ghent University, Krijgslaan 281 Gebouw S22, B-9000 Gent, Belgium
E-mail address: jvindas@cage.Ugent.be