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# $\mathfrak{osp}(4|2)$–monogenicity in Clifford analysis

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**Abstract**

A new concept of $\mathfrak{osp}(4|2)$–monogenicity is introduced as a refinement of quaternionic monogenicity; it is defined by means of four quaternionic Dirac operators, a scalar Euler operator $\mathcal{E}$ and a multiplicative Clifford algebra operator $P$.

*Key words:* $\mathfrak{osp}(4|2)$*–monogenicity, quaternionic structure*
*MSC 2000: 30G35*

## 1 Introduction and preliminaries

In a similar way as hermitian Clifford analysis in Euclidean space $\mathbb{R}^{2n}$ of even dimension arises as a refinement of euclidean Clifford analysis by introducing a complex structure on $\mathbb{R}^{2n}$, quaternionic Clifford analysis originates as a further refinement by the introduction of

## 2 Hermitian and quaternionic monogenicity

The central notion in Clifford analysis is that of a monogenic function, a continuously differentiable function defined in an open region of Euclidean space $\mathbb{R}^m$, taking its values in the Clifford algebra $\mathbb{R}_{0,m}$, or subspaces thereof, and vanishing under the action of the Dirac operator $\underline{\partial} = \sum_{\alpha=1}^{m} e_\alpha \, \partial_{X_\alpha}$, which is the Fourier dual of the Clifford variable $\underline{X}$. This notion is the higher dimensional counterpart of holomorphy in the complex plane. The Dirac operator factorizes the Laplacian: $\Delta_m = -\underline{\partial}^2$, and is invariant under the action of the Spin$(m)$–group which doubly covers the SO$(m)$–group, whence this framework is usually referred to as Euclidean (or orthogonal) Clifford analysis. Taking the dimension of $\mathbb{R}^m$ to be even: $m = 2n$, renaming the variables as $(X_1, \ldots, X_{2n}) = (x_1, y_1, x_2, y_2, \ldots, x_n, y_n)$ and considering the standard complex structure $\mathbb{I}_{2n}$, i.e. the complex linear real SO$(2n)$–matrix

$$\mathbb{I}_{2n} = \mathrm{diag}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

for which $\mathbb{I}_{2n}^2 = -E_{2n}$, $E_{2n}$ being the identity matrix, we define the rotated vector variable and the corresponding rotated Dirac operator

$$\begin{aligned} \underline{X}_{\mathbb{I}} &= \mathbb{I}_{2n}[\underline{X}] = \sum_{k=1}^{n} (-y_k e_{2k-1} + x_k e_{2k}) \\ \underline{\partial}_{\mathbb{I}} &= \mathbb{I}_{2n}[\underline{\partial}] = \sum_{k=1}^{n} (-\partial_{y_k} e_{2k-1} + \partial_{x_k} e_{2k}) \end{aligned}$$

A differentiable function $F$ then is called hermitian monogenic in some region $\Omega$ of $\mathbb{R}^{2n}$, if and only if in that region $F$ is a solution of the system

$$\underline{\partial} F = 0 = \underline{\partial}_{\mathbb{I}} F \tag{1}$$

An alternative approach to hermitian monogenicity uses the projection operators $\frac{1}{2}(\mathbf{1} \pm i\,\mathbb{I}_{2n})$, involving a complexification. In this approach we consider the Witt basis vectors $\mathfrak{f}_k = -\frac{1}{2}(\mathbf{1} - i\,\mathbb{I}_{2n})[e_{2k-1}]$ and $\mathfrak{f}_k^\dagger = \frac{1}{2}(\mathbf{1} + i\,\mathbb{I}_{2n})[e_{2k-1}]$, $k = 1, \ldots, n$, as well as the vector variables

$$\underline{z} = -\frac{1}{2}(\mathbf{1} - i\,\mathbb{I}_{2n})[\underline{X}] = \sum_{k=1}^{n} (x_k + i y_k)\mathfrak{f}_k = \sum_{k=1}^{n} z_k \mathfrak{f}_k, \qquad \underline{z}^\dagger = \frac{1}{2}(\mathbf{1} + i\,\mathbb{I}_{2n})[\underline{X}] = \sum_{k=1}^{n} \overline{z}_k \mathfrak{f}_k^\dagger$$

A refinement of hermitian Clifford analysis is obtained by considering the hypercomplex structure $\mathbb{Q} = (\mathbb{I}_{4p}, \mathbb{J}_{4p}, \mathbb{K}_{4p})$ on $\mathbb{R}^{4p} \simeq \mathbb{C}^{2p} \simeq \mathbb{H}^p$, the dimension $m = 2n = 4p$ now assumed to be a 4-fold. This hypercomplex structure arises by introducing, next to the first complex structure $\mathbb{I}_{4p}$, a second one, $\mathbb{J}_{4p}$, given by

$$\mathbb{J}_{4p} = \mathrm{diag}\begin{pmatrix} & & 1 & \\ & & & -1 \\ -1 & & & \\ & 1 & & \end{pmatrix}$$

Clearly $\mathbb{J}_{4p} \in \mathrm{SO}(4p)$, with $\mathbb{J}_{4p}^2 = -E_{4p}$, and it anti–commutes with $\mathbb{I}_{4p}$. Then there arises a third $\mathrm{SO}(4p)$–matrix $\mathbb{K}_{4p} = \mathbb{I}_{4p}\,\mathbb{J}_{4p} = -\mathbb{J}_{4p}\,\mathbb{I}_{4p}$, for which $\mathbb{K}_{4p}^2 = -E_{4p}$ and which anti–commutes with both $\mathbb{I}_{4p}$ and $\mathbb{J}_{4p}$. The introduction of a hypercomplex structure leads to a function theory for so-called quaternionic Clifford analysis, where the most genuine way to introduce the new concept of quaternionic monogenicity is to directly generalize the system (1) making use of the additional rotated Dirac operators $\underline{\partial}_{\mathbb{J}} = \mathbb{J}_{4p}[\underline{\partial}]$ and $\underline{\partial}_{\mathbb{K}} = \mathbb{K}_{4p}[\underline{\partial}]$, whence the following definition.

**Definition 1.** *A differentiable function $F : \mathbb{R}^{4p} \longrightarrow \mathbb{S}$ is called quaternionic monogenic in some region $\Omega$ of $\mathbb{R}^{4p}$, if and only if in that region $F$ is a solution of the system*

$$\underline{\partial} F = 0, \quad \underline{\partial}_{\mathbb{I}} F = 0, \quad \underline{\partial}_{\mathbb{J}} F = 0, \quad \underline{\partial}_{\mathbb{K}} F = 0 \tag{3}$$

Also here an alternative characterization is possible in terms of the hermitian Dirac operators, which in the actual dimension read:

$$\partial_{\underline{z}} = \sum_{j=1}^{p} (\partial_{z_{2j-1}} \mathfrak{f}_{2j-1}^\dagger + \partial_{z_{2j}} \mathfrak{f}_{2j}^\dagger), \qquad \partial_{\underline{z}}^\dagger = \sum_{j=1}^{p} (\partial_{\overline{z}_{2j-1}} \mathfrak{f}_{2j-1} + \partial_{\overline{z}_{2j}} \mathfrak{f}_{2j})$$

and their images under the action of $\mathbb{J}_{4p}$:

$$\partial_{\underline{z}}^J = \mathbb{J}_{4p}[\partial_{\underline{z}}] = \sum_{j=1}^{p} (\partial_{z_{2j}} \mathfrak{f}_{2j-1} - \partial_{z_{2j-1}} \mathfrak{f}_{2j}), \qquad \partial_{\underline{z}}^{\dagger J} = \mathbb{J}_{4p}[\partial_{\underline{z}}^\dagger] = \sum_{j=1}^{p} (\partial_{\overline{z}_{2j}} \mathfrak{f}_{2j-1}^\dagger - \partial_{\overline{z}_{2j-1}} \mathfrak{f}_{2j}^\dagger)$$

Now the original Dirac operator $\underline{\partial}$ and its rotated versions $\underline{\partial}_{\mathbb{I}}$, $\underline{\partial}_{\mathbb{J}}$ and $\underline{\partial}_{\mathbb{K}}$ may be expressed in terms of the hermitian Dirac operators $(\partial_{\underline{z}}, \partial_{\underline{z}}^\dagger)$ and their rotated versions $(\partial_{\underline{z}}^J, \partial_{\underline{z}}^{\dagger J})$. This

## 3 $\mathfrak{osp}(4|2)$–monogenicity

Our aim being the decomposition of the space $\mathcal{P}(\mathbb{R}^{4p};\mathbb{S})$ of spinor–valued polynomials into $\mathrm{Sp}(p)$–irreducibles, we should first take care of the irreducibility of the value space. Spinor space $\mathbb{S}$, already being decomposed into $\mathrm{U}(2p)$–irreducible homogeneous parts $\mathbb{S}^r$, should thus be further decomposed into $\mathrm{Sp}(p)$–irreducibles, which we will call symplectic cells. These symplectic cells are given by $\mathbb{S}^r_r = \mathrm{Ker}\, P|_{\mathbb{S}^r}$ and $\mathbb{S}^{2p-r}_r = \mathrm{Ker}\, Q|_{\mathbb{S}^{2p-r}}$ for $r = 0,\ldots,p$, and by $\mathbb{S}^{r+2k}_r = Q^k\, \mathbb{S}^r_r$ and $\mathbb{S}^{2p-r-2k}_r = P^k\, \mathbb{S}^{2p-r}_r$, for $k = 0,\ldots,p-r$, where the $\mathrm{Sp}(p)$–invariant left multiplication operators $P$ and $Q$ are defined as $P = \mathfrak{f}_2\mathfrak{f}_1 + \mathfrak{f}_4\mathfrak{f}_3 + \ldots + \mathfrak{f}_{2p}\mathfrak{f}_{2p-1}$ and $Q = \mathfrak{f}_1^\dagger\mathfrak{f}_2^\dagger + \mathfrak{f}_3^\dagger\mathfrak{f}_4^\dagger + \ldots + \mathfrak{f}_{2p-1}^\dagger\mathfrak{f}_{2p}^\dagger = P^\dagger$, their composition being a constant on each symplectic cell. One now could expect the building blocks of the Fischer decomposition to be the spaces of quaternionic monogenic bi–homogeneous polynomials with values in a symplectic cell: $\mathcal{Q}^{r,s}_{a,b} = \mathcal{P}_{a,b}(\mathbb{R}^{4p};\mathbb{S}^r_s) \cap \mathrm{Ker}\left(\partial_{\underline{z}}, \partial_{\underline{z}}^\dagger, \partial_{\underline{z}}^J, \partial_{\underline{z}}^{\dagger J}\right)$.

Unfortunately those spaces are reducible under $\mathrm{Sp}(p)$, and the notion of quaternionic monogenicity thus has to be refined. We first turn our attention to the Howe dual partner $\mathfrak{g}$ of $\mathrm{Sp}(p)$, where we want the already established differential operators $\partial_{\underline{z}}$, $\partial_{\underline{z}}^\dagger$, $\partial_{\underline{z}}^{\dagger J}$ and $\partial_{\underline{z}}^{\dagger J}$ and their algebraic counterparts $\underline{z}$, $\underline{z}^\dagger$, $\underline{z}^{\dagger J}$ and $\underline{z}^{\dagger J}$ to belong to (the odd part of) $\mathfrak{g}$, which has to be closed under the Lie bracket. Computing all anti–commutators of the involved operators, we then still find, next to the multiplication operators $P$ and $Q$, the new scalar differential operators $\mathcal{E} = \sum_{k=1}^{p} z_{2k-1}\, \partial_{\overline{z}_{2k}} - z_{2k}\, \partial_{\overline{z}_{2k-1}}$ and $\mathcal{E}^\dagger = \sum_{k=1}^{p} \overline{z}_{2k-1}\, \partial_{z_{2k}} - \overline{z}_{2k}\, \partial_{z_{2k-1}}$. The Howe dual partner now being found to be $\mathfrak{osp}(4|2)$, the refined definition below will lead to a well established Fischer decomposition in terms of $\mathrm{Sp}(p) \times \mathfrak{osp}(4|2)$ irreducibles.

**Definition 2.** *A differentiable function $F : \mathbb{R}^{4p} \longrightarrow \mathbb{S}$ is called $\mathfrak{osp}(4|2)$–monogenic in some region $\Omega$ of $\mathbb{R}^{4p}$, if and only if in $\Omega$ it is quaternionic monogenic and moreover a null–solution of both $\mathcal{E}$ and $P$.*

## References

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