

# EXTENDED ISOGEOMETRIC ANALYSIS IN MODELLING CRACKED STRUCTURES

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**Abstract:** In this paper, we successfully extended eXtended IsoGeometric Analysis (XIGA) in simulation of stationary and propagating cracks. In this method, IsoGeometric Analysis (IGA) utilizing the Non-Uniform Rational B-Spline (NURBS) functions is incorporated with enrichment functions through the partition of unity. The Heaviside function is enriched to capture the discontinuous phenomenon at the crack faces while the asymptotic functions from analytical solution are incorporated with NURBS to perform the singular field at the crack tips. Based on the NURBS's characteristics, this approach allows us to achieve easily the higher approximation order and continuity of the basic functions. As a result, XIGA can improve accuracy and higher convergence rate as compared with traditional finite element.

**Keywords:** eXtended IsoGeometric Analysis; Linear Fracture Elastic Mechanics; crack propagation

## 1 INTRODUCTION

In service, the cracks generated and grown from the defects under a cyclic loading cause a reduction of the load carrying capacity of the structures. Therefore, study of fracture mechanics is virtually important in guarantee of the performance and safety of the structures. The prediction and analysis of the crack problems are the attractive topics for many scientists in general and for computational experts in particular. As a result, a numerous numerical methods have been developed including finite element method (FEM) [1], boundary method [2], meshfree method [3], extended finite element method (XFEM) [4], etc. In these methods, approximated geometries introduce some error in the solutions because different shape functions are utilized in describing geometry and analysis. To overcome this issue, Hughes et al. proposed isogeometric analysis (IGA) [5] which fulfils a seamless bridge link between computer aided design and finite element analysis based on using the same B-Spline or non-uniform rational B-Spline (NURBS) functions in describing the exact geometry of problem as well as constructing finite approximation for analysis. IGA provides a flexible way to make refinement and degree evaluation. It allows us to achieve easily the higher approximation order and continuity of the basic functions as compared with the traditional finite element method. Up-to-now, IGA has been extensively and successfully studied for various fields of engineering and science, including fracture mechanics. For the crack problems, the enrichment functions through the partition of unity method are incorporated into IGA to capture the discontinuous phenomenon at the crack faces and singular field at the crack tips. Recently, De Luycker et al. [6] has been used XFEM incorporated with IGA for linear fracture of mode I crack. He found that this method gains the greater accuracy and convergence rate. Then, Gorashi et al. [7] kept developing this approach (in short name XIGA) to perform the behaviour of the crack structures in 2D. XIGA also successfully applied in bi-material body with curved interface [8], assessment of collapse load of cracked plate [9], vibration of crack plate [10], cracked thin shell structures [11].

In this paper, we extended XIGA for simulation of stationary and propagating cracks. In this method, the Heaviside function is utilized for model the discontinuous phenomenon at the crack face while the asymptotic functions from analytical solution is used to capture the singular fields at the crack tips. Based on the NURBS, this approach allows us to achieve easily the higher approximation order and continuity of basic functions. As a result, XIGA can improve accuracy and higher convergence rate as compared with XFEM.

## 2 GOVERNING EQUATIONS OF CRACKED STRUCTURE

Consider a linear elastic solid defined in a domain  $\Omega$  with a boundary  $\Gamma$  such that  $\Gamma = \Gamma_u \cup \Gamma_t \cup \Gamma_c$ ,  $\Gamma_u \cap \Gamma_t \cap \Gamma_c = \emptyset$  where  $\Gamma_u$ ,  $\Gamma_t$ ,  $\Gamma_c$  are the Dirichlet and Neumann boundary and crack surfaces, respectively. The body subjected to body forces  $\mathbf{b}$  and to surface tractions  $\bar{\mathbf{t}}$  on the free portion  $\Gamma_t$ . The governing equations for this problem are

$$\nabla \cdot \boldsymbol{\sigma} + \mathbf{b} = 0 \text{ in } \Omega \quad \text{subjected to} \quad \begin{cases} \boldsymbol{\sigma} \cdot \mathbf{n} = \begin{cases} \bar{\mathbf{t}} & \text{on } \Gamma_t \\ 0 & \text{on } \Gamma_c \end{cases} \\ \mathbf{u} = \bar{\mathbf{u}} & \text{on } \Gamma_u \end{cases} \quad (1)$$

in which  $\mathbf{u}$  is the displacement field which satisfies the compatibility relation

$$\boldsymbol{\varepsilon} = \nabla_s \mathbf{u} \text{ in } \Omega \quad \text{where} \quad \nabla_s = \begin{bmatrix} \partial/\partial x & 0 & \partial/\partial y \\ 0 & \partial/\partial y & \partial/\partial x \end{bmatrix}^T \quad (2)$$

and the Cauchy stress tensor  $\boldsymbol{\sigma}$  is obtained from the constitutive relation based on Hooke's law

$$\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\varepsilon} \text{ in } \Omega \quad (3)$$

where  $\mathbf{D}$  is the elastic constant matrix.

The weak form of the equilibrium equations can be expressed as

$$\int_{\Omega} \boldsymbol{\varepsilon}^T \mathbf{D}\boldsymbol{\varepsilon} d\Omega = \int_{\Omega} \mathbf{b} \mathbf{u} d\Omega + \int_{\Gamma_t} \bar{\mathbf{t}} \mathbf{u} d\Gamma \quad (4)$$

## 3 DISCRETIZATION

### 3.1 A brief of B-spline/NURBS functions

A knot vector  $\Xi = \{\xi_1, \xi_2, \dots, \xi_{n+p+1}\}$  is a non-decreasing sequence of parameter values  $\xi_i$ ,  $i = 1, \dots, n+p$ , where  $\xi_i \in R$  called  $i^{\text{th}}$  knot lies in the parametric space,  $p$  is the order of the B-spline and  $n$  is number of the basis functions. Using Cox-de Boor algorithm, the univariate B-spline basis functions are defined recursively start with order  $p = 0$

$$N_{i,0}(\xi) = \begin{cases} 1 & \text{if } \xi_i < \xi < \xi_{i+1} \\ 0 & \text{otherwise} \end{cases} \quad (5)$$

as  $p \geq 1$  the basis functions are obtained from

$$N_{i,p}(\xi) = \frac{\xi - \xi_i}{\xi_{i+p} - \xi_i} N_{i,p-1}(\xi) + \frac{\xi_{i+p+1} - \xi}{\xi_{i+p+1} - \xi_{i+1}} N_{i+1,p-1}(\xi) \quad (6)$$

The multivariate B-spline basis functions are generated by a simple way - tensor product of the univariate B-splines

$$N_A(\boldsymbol{\xi}) = \prod_{\alpha=1}^d N_{i_\alpha, p_\alpha}(\xi^\alpha) \quad (7)$$

where  $d = 1, 2, 3$  is dimensional space. Fig. 1 illustrates an example of bivariate B-spline basis function from tensor product of two univariate B-splines  $\boldsymbol{\psi} = \{0, 0, 0, 0, \frac{1}{4}, \frac{1}{4}, \frac{3}{4}, 1, 1, 1, 1\}$  and  $\Xi = \{0, 0, 0, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{3}{5}, \frac{4}{5}, 1, 1, 1\}$  in  $\xi$  and  $\eta$  direction, respectively.

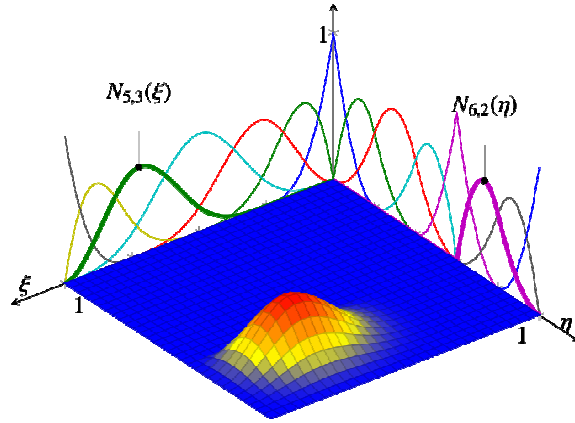


Fig. 1 B-splines basic functions.

To present exactly some conic sections, e.g., circles, cylinders, spheres, etc., non-uniform rational B-splines (NURBS) with an additional weight value  $\zeta_A > 0$  for each control point is used.

$$R_A(\xi, \eta) = N_A \zeta_A / \sum_A^{m \times n} N_A(\xi, \eta) \zeta_A \quad (8)$$

### 3.2 Extended isogeometric finite element method

In order to capture the local discontinuous and singular fields, the enriched displacement approximation is introduced according to idea of XFEM as follow:

$$\mathbf{u}^h(\mathbf{x}) = \sum_{I \in S} R_I(\xi) \mathbf{q}_I^{std} + \sum_{J \in S^{enr}} R_J^{enr}(\xi) \mathbf{q}_J^{enr} \quad (9)$$

Here, the NURBS basis functions are utilized instead of the Lagrange polynomials to create a new numerical procedure – so-called eXtended IsoGeometric Analysis (XIGA) [7].  $R_J^{enr}$  are the enrichment functions associated with node  $J$  located in enriched domain  $S^{enr}$  which is splitted up two parts including: a set  $S^c$  for crack faces enriched control points and a set  $S^f$  for crack tips enriched control points as shown in Fig. 2.

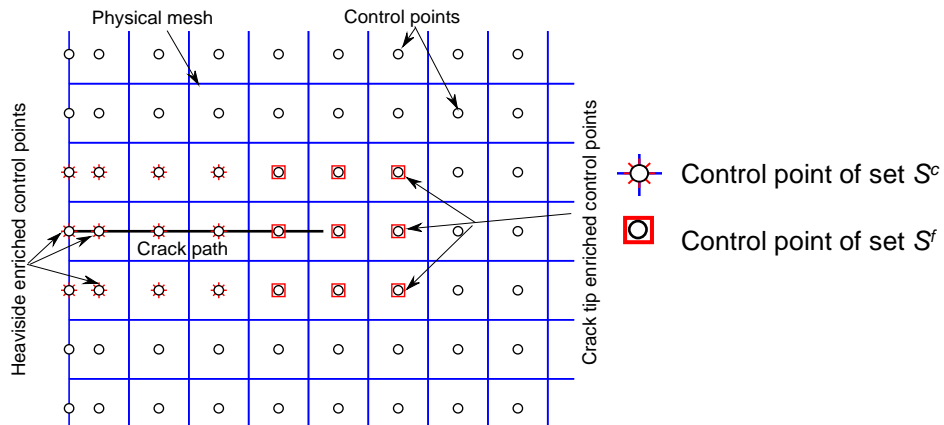


Fig. 2 Illustration of the nodal sets  $S^c$ ,  $S^f$  for a quadratic NURBS mesh.

To model the discontinuous displacement at crack faces, the Heaviside function, which otherwise becomes +1 if physical coordinate is above the crack and -1, is incorporated in the enriched functions:

$$R_J^{enr}(\xi) = R_J(\xi) (H(\mathbf{x}) - H(\mathbf{x}_J)) \quad (10)$$

The analytical displacement field of linear elastic fracture problem at crack tips in polar coordinate  $(r, \theta)$  is expressed as below [12]:

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \frac{K_I}{2\mu} \sqrt{\frac{r}{2\pi}} \begin{Bmatrix} \cos \frac{\theta}{2} (\kappa - 1 + 2 \sin^2 \theta) \\ \sin \frac{\theta}{2} (\kappa + 1 - 2 \cos^2 \theta) \end{Bmatrix} + \frac{K_{II}}{2\mu} \sqrt{\frac{r}{2\pi}} \begin{Bmatrix} \sin \frac{\theta}{2} (\kappa + 1 + 2 \cos^2 \frac{\theta}{2}) \\ \cos \frac{\theta}{2} (\kappa - 1 - 2 \sin^2 \frac{\theta}{2}) \end{Bmatrix} \quad (11)$$

where  $K_I$  and  $K_{II}$  are the stress intensity factors of mode I and mode II, respectively.

From the analytical displacement, the enriched functions for crack tips are chosen as [13]:

$$R_j^{enr}(\xi) = R_j(\xi) \left( \sum_{L=1}^4 (\psi(r, \theta) - \psi(r_j, \theta_j)) \right) \quad (12)$$

where

$$\psi(r, \theta) = \sqrt{r} \left\{ \sin \frac{\theta}{2} \quad \cos \frac{\theta}{2} \quad \sin \frac{\theta}{2} \sin \theta \quad \cos \frac{\theta}{2} \sin \theta \right\} \quad (13)$$

### 3.3 Discretization

Now, substituting Eq. (9) into Eq. (2), the strain can be expressed follow the nodal displacement as

$$\boldsymbol{\varepsilon} = \sum_{I=1}^{m \times n} \mathbf{B}_I^T \mathbf{q}_I \quad (14)$$

where the strain matrix  $\mathbf{B}$  is given by

$$\mathbf{B}_I = \left[ \mathbf{B}_I^{std} \mid \mathbf{B}_I^{enr} \right] \quad (15)$$

in which  $\mathbf{B}^{std}$  and  $\mathbf{B}^{enr}$  are the standard and enriched part of matrix  $\mathbf{B}$  defined in the following form

$$\mathbf{B} = \begin{bmatrix} \bar{R}_{,x} & 0 & \bar{R}_{,y} \\ 0 & \bar{R}_{,y} & \bar{R}_{,x} \end{bmatrix}^T \quad (16)$$

where  $\bar{R}$  can be either the NURBS basic functions  $R(\xi)$  or enriched functions  $R^{enr}$ .

Substituting Eq.(14) into Eq.(4), the linear static equation for crack problem is obtained

$$\mathbf{K}\mathbf{q} = \mathbf{F} \quad (17)$$

with the global stiffness and force vector

$$\mathbf{K} = \int_{\Omega} \mathbf{B}^T \mathbf{D} \mathbf{B} d\Omega \quad \text{and} \quad \mathbf{F} = \int_{\Omega} \bar{R} \mathbf{b} d\Omega + \int_{\Gamma_f} \bar{R} \mathbf{t} d\Gamma \quad (18)$$

## 4 NUMERICAL RESULTS

In this section, we study linear elastic fracture mechanics in some numerical examples with rectangular geometry such as: an infinite plate under in-plane tension, the static crack and crack propagation of an edge cracked plate subjected to shear stress. In all numerical examples, plane strain state is assumed. Herein, we illustrated the present method using meshing of cubic elements.

### 4.1 An infinite plate subjected to tension

Firstly, let us consider an isotropic infinite plate with material parameters of  $E = 10^7 \text{ N/mm}^2$ ,  $\nu = 0.3$  containing a centre crack of length  $2a$  subjected to a remote uniform stress  $\sigma = 10^2 \text{ N/mm}^2$ . A unit shaded domain, as shown in Fig. 3, with crack length of 0.5 mm is modelled. To evaluate the present method, Fig. 4 reveals the comparison between XFEM and XIGA in investigation of the relative error and convergence rate of displacement and energy which are given by

$$\|\mathbf{u}\| = \left[ \int_{\Omega} (\mathbf{u} - \mathbf{u}^h)^T (\mathbf{u} - \mathbf{u}^h) d\Omega / \int_{\Omega} \mathbf{u}^T \mathbf{u} d\Omega \right]^{1/2} \quad (19)$$

$$\|\mathbf{e}\| = \left[ \int_{\Omega} (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^h)^T (\boldsymbol{\sigma} - \boldsymbol{\sigma}^h) d\Omega / \int_{\Omega} \boldsymbol{\varepsilon}^T \boldsymbol{\sigma} d\Omega \right]^{1/2} \quad (20)$$

It is noted that as  $p = 1$ , XIGA formulation is identical to XFEM. As compared to XFEM, it is observed that present method archives supper accuracy and higher convergence rate in displacement error norm as well as energy error norm. Fig. 5 displays the contour plots of displacements and stress distributions in  $x$  and  $y$  directions, respectively.

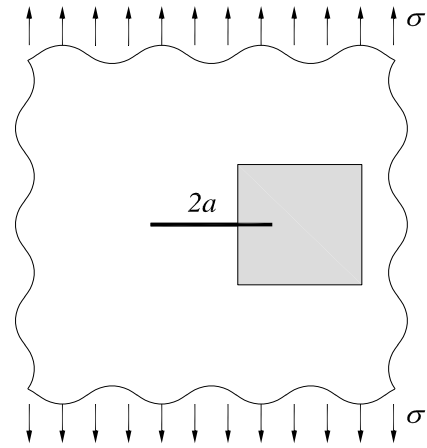


Fig. 3 Infinite crack plate in tension.

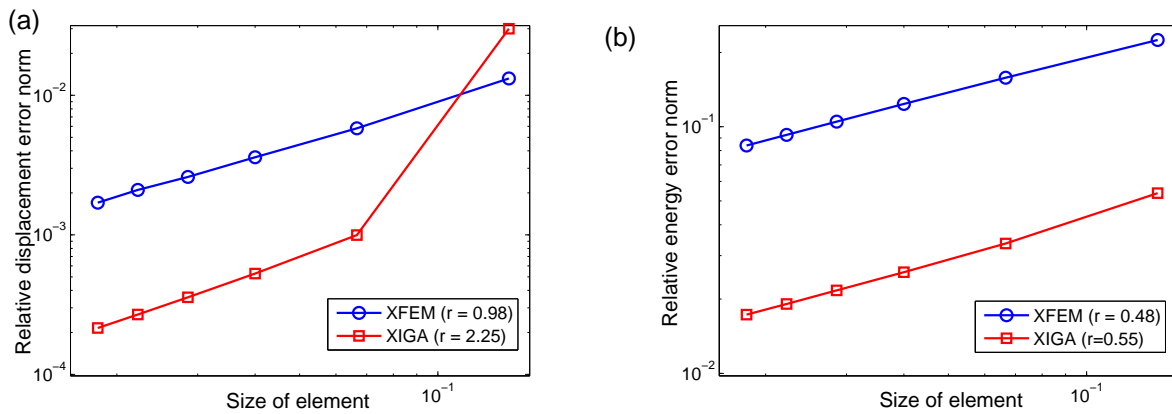


Fig. 4 Comparison of relative error norm of: (a) displacement, (b) energy.

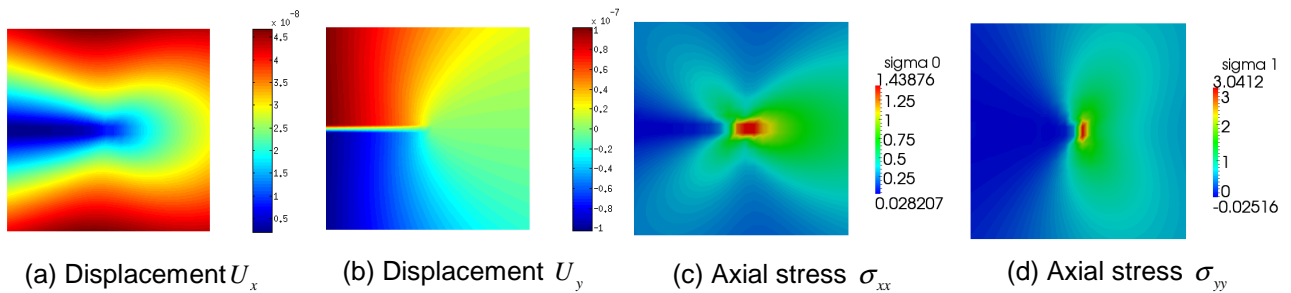


Fig. 5 Contour plot of the displacements and stresses distribution.

### 4.2 An edge cracked plate under shear stress

In next example, we consider a rectangular plate subjected to a shear stress  $\tau = 1 \text{ N/mm}^2$  as shown in Fig. 6 with material parameters of  $E = 3 \times 10^7 \text{ N/mm}^2, \nu = 0.25$ . Fig. 7a and b show the relations between relative error of stress intensity factor (SIF) via number of degrees of freedom (DOFs) and CPU time, respectively. It is again seen that present method archives more accurate than XFEM. Indeed, to get the accuracy of SIF  $K_I$  with error lower than 0.1%, XIGA needs approximate 4300 DOFs while XFEM uses more than 25000 variables with nearly two time computational cost. Fig. 8 plots the distribution of axial stress, shear stress and displacement along  $y$  direction. For comparison purpose of displacement  $U_y$ , 3D elastic solid is also modelled by XIGA with meshing of  $13 \times 27 \times 3$  elements. The same contour plot of displacement is observed in Fig. 8c and d.

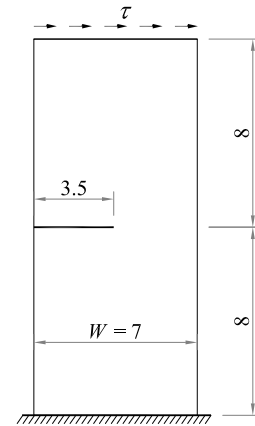


Fig. 6 Edge crack plate under shear

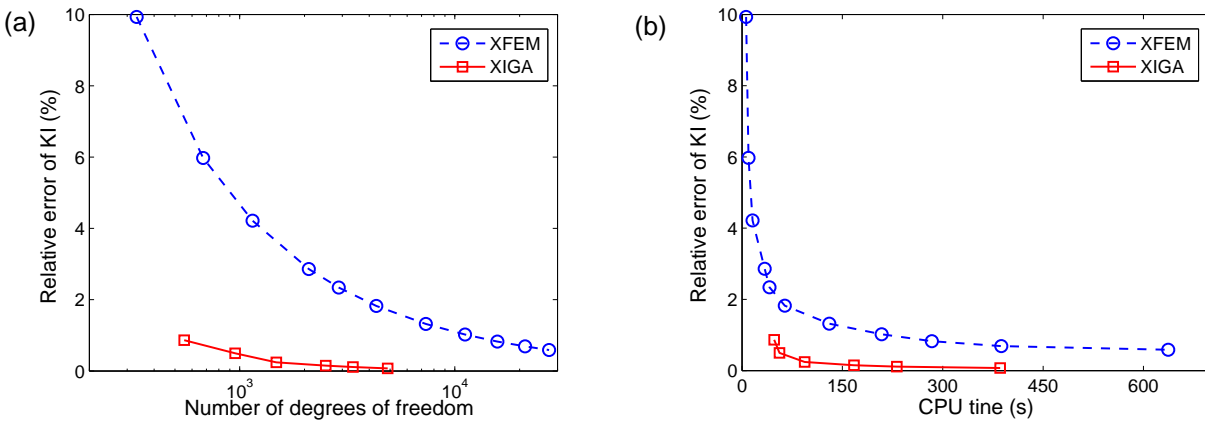


Fig. 7 Comparison between XIGA and XFEM.

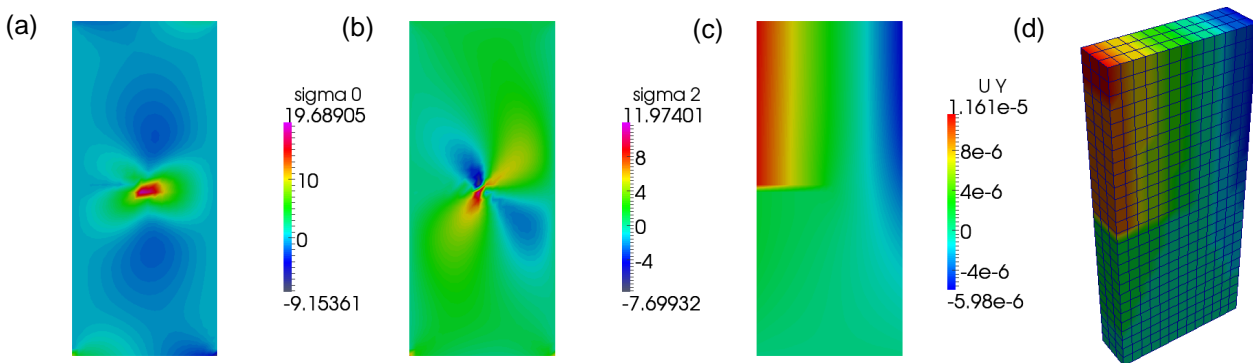


Fig. 8 Contour plot of the stresses and displacements: (a)  $\sigma_{xx}$ , (b)  $\sigma_{xy}$ ,  $U_y$  in 2D (c) and 3D (d).

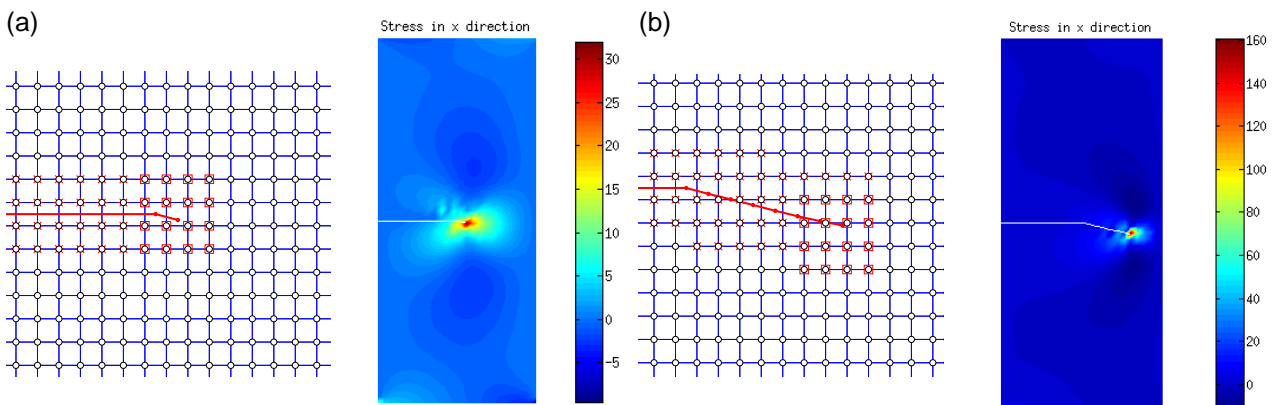
### 4.3 Crack propagation of an edge cracked plate under mixed mode loading

In this section, we want to show the capacity of present method in crack growth simulation without re-meshing. For this problem, two important parameters need to be specified during the crack growth

procedure: crack growth direction  $\theta_c$  and incremental crack length  $\Delta a$ . The crack growth angle can be identified follow to several available criteria such as: the critical plane approach, the maximum circumferential stress, the maximum energy release rate, the minimum strain energy density (see review in [14]). In this present work, we adopt the maximum circumferential stress to evaluate the crack growth direction [15]

$$\theta_c = 2\text{atan}\left(-2(K_{II}/K_I)/(1+\sqrt{1+8(K_{II}/K_I)^2})\right) \quad (21)$$

The incremental crack length is commonly determined from Paris's law. However, in this example, for simple, a constant crack increment length of  $\Delta a = 0.3$  mm is set for each step. Fig. 9 shows the meshes at the step number #1 and #8 with the axial stress distributions  $\sigma_{xx}$ , respectively. It is observed that, during the crack growth, the mesh is unchanged while the position of crack tip downwards as oblique path. It leads to change the enrichment status of the control points, for instance, from being tip-enriched to Heaviside enriched or some standard control points becomes the enriched ones.



**Fig. 9** Propagation of an edge crack under shear stress after step 1 (a) and step 8 (b).

## 5 CONCLUSIONS

In this paper, an eXtended IsoGeometric Analysis (XIGA) is extended to simulate the stationary and propagation of the crack problems. Based on idea of XFEM, the enrichment functions through the partition of unity method are integrated to capture the local discontinuous and singular fields. The advantage of present method is based on the NURBS which allows us to achieve easily the higher approximation and continuity in order of basic functions as compared with the traditional FEM. As a results, XIGA gains better accuracy and higher convergence rate than XFEM. Furthermore, the stresses and displacement are also plotted smoothly and continuous through the element boundaries.

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