JOURNAL OF INDUSTRIAL AND
MANAGEMENT OPTIMIZATION
Volume **11**, Number **3**, July **2015**

doi:10.3934/jimo.2015.11.829

pp. **829–848**

PERFORMANCE ANALYSIS OF BUFFERS WITH TRAIN ARRIVALS AND CORRELATED OUTPUT INTERRUPTIONS

Bart Feyaerts[1], Stijn De Vuyst[2]
Herwig Bruneel[1] and Sabine Wittevrongel[1]

[1] SMACS Research Group
TELIN Department
Ghent University
Sint-Pietersnieuwstraat 41
9000 Gent, Belgium

[2] Supply Networks and Logistics Research Center
Department of Industrial Management
Ghent University
Technologiepark 903
9052 Zwijnaarde, Belgium

Abstract. In this paper, we study a discrete-time buffer system with a time-correlated packet arrival process and one unreliable output line. In particular, packets arrive to the buffer in the form of variable-length packet trains at a fixed rate of exactly one packet per slot. The packet trains are assumed to have a geometric length, such that each packet has a fixed probability of being the last of its corresponding train. The output line is governed by a Markovian process, such that the probability that the line is available during a slot depends on the state of the underlying J-state Markov process during that slot.

First, we provide a general analysis of the state of the buffer system based on a matrix generating functions approach. This also leads to an expression for the mean buffer content. Additionally, we take a closer look at the distributions of the packet delay and the train delay. In order to make matters more concrete, we next present a detailed and explicit analysis of the buffer system in case the output line is governed by a 2-state Markov process. Some numerical examples help to visualise the influence of the various model parameters.

1. **Introduction.** In packet-based telecommunication systems, buffers are essential for the temporary storage of information packets. A clear insight in these buffers and their behaviour is important in order to study the performance not only of the buffers themselves, but also of the entire system. This performance, expressed by means of performance measures such as buffer occupancy and packet delay, is greatly influenced by the characteristics of both the packet arrival process and the transmission process of packets from the buffer. Thus, a proper analysis requires these characteristics to be modelled as accurately as possible.

With respect to the arrival process, we assume in this paper that the packets are part of higher-level messages, referred to as (packet) trains. The notion of these trains reflects the layered structure of most packet-based communication networks

2010 *Mathematics Subject Classification.* Primary: 60K25, 68M20; Secondary: 90B22.
Key words and phrases. Queueing theory, performance analysis, server interruptions.
The reviewing process of the paper was handled by Wuyi Yue and Yutaka Takahashi as Guest Editors.

and introduces time correlation in the packet arrival process. In the discrete-time queueing model considered in this paper, packet trains span over multiple successive time slots, pushing packets into the system at a fixed rate of one packet per slot. The time correlation in the arrival process can be understood from the fact that a train of length n contributes to the packet arrival process during n consecutive slots; this makes the number of packet arrivals in a slot dependent on the number of packet arrivals in previous slots. Arrival processes involving multi-packet messages, such as the trains considered in this paper, have received considerable attention in literature. In [11], so-called dispersed messages consisting of a fixed number of packets are considered in case of an uncorrelated packet arrival process. Related continuous-time models using dispersed messages are studied in [10, 2]. Train arrival processes appear in [12, 6, 7, 8, 38, 39, 37]. A more complex model that introduces correlation is referred to as the session-based arrival process [20, 40, 21, 15], which is an extension of the train arrival process where messages generate a variable, yet strictly positive, number of packets per slot. Other types of arrival processes incorporate time correlation usually without the notion of higher-level messages. A simple class are on/off-type arrivals [36, 41, 22, 14, 24], where a finite number of users generate one packet per slot during on-periods and no packets during off-periods. Another important class of correlated arrival processes are the Markovian Arrival Processes (MAP), studied in [35, 29, 30, 3, 9].

With respect to the transmission process, we assume in this paper that the buffer has a single output line and packet transmission times are equal to one slot. The output line is however unreliable and prone to stochastic failures. Specifically, the accessibility of the output line is influenced by a Markov process that controls the state of the output line. The output line being available or not during a slot is then controlled by a Bernoulli process that is dependent on the output line state. Due to these failures, that are correlated by nature, the effective packet transmission times will be stochastic and correlated. This correlation allows real-life situations (e.g. loss of connection, failure in electrical components, ...) to be modelled in great detail, as they usually appear in a correlated fashion. Queueing models with server interruptions have been studied on many occasions in the literature, see e.g. [4, 5, 27, 28, 1, 31, 17, 23, 34, 25, 26].

The main contribution of our current paper, is that we focus on the analysis of a queueing model where the packet arrival process as well as the transmission process of packets may exhibit time correlation. This is different from most previous work, where at least one of these processes was described by a simple model without any time correlation. The considered packet arrival process moreover takes the layered structure of packet-based communication networks explicitly into account, as opposed to arbitrary Markovian arrival processes. The Markovian transmission process accounts for the typical time-dependent occurrence of output failures. Note however that in our queueing model the packet arrival process and the output interruption process are assumed to be two mutually independent processes.

The train arrival model we consider has both its merits and its limitations. Since the arrival process is explicitly described as induced by a two-layered structure, train arrivals are very suitable to model the common segmentation of data files into packets before their transmission through a communication network. In particular, files sent by a web server are usually too large to be sent as a single entity and are therefore fragmented in packets which can be sent individually; the arrival process in the output buffer of the server is therefore well modelled as a train arrival process.

However, once these packets leave the outgoing buffer of the server, they may travel to the client over different paths and experience different effects resulting in different delays. It can be seen quite intuitively that during transmission, these packets then behave less like consecutive fragments of a single entity, but more like individual packets. When focusing on a mid network buffer, one should therefore resort to other traffic models, such as general independent arrivals or correlated models that incorporate the *long range dependency* (LRD) property that is known to appear in Internet traffic.

This paper is structured as follows. In Section 2 we lay out a mathematical model for the buffer system described above that will allow us to study the behaviour and performance of the system. Section 3 presents a concise analysis of the packet arrival process. The behaviour of the system is analysed in Section 4 and Section 5 focuses on two performance measures, namely the mean system content and the system load. Other important performance measures, the packet delay and the train delay, are studied in Section 6 and Section 7. In Section 8 we derive some closed-form expressions for the special case where the Markov chain governing the output line has two possible states. Finally, the features of the queueing system are illustrated by some numerical examples in Section 9.

2. **Description of the buffer model under study.** We consider a discrete-time infinite-capacity buffer system for information packets with deterministic transmission times of one slot per packet and an unreliable output line. Information arrives to the system in the form of packet trains, i.e. a sequence of packets that arrive to the system over contiguous slots at a fixed rate of exactly one packet per slot. The packet trains are assumed to have a geometric length, such that each packet has a fixed probability of being the last of its corresponding train. The total number of packet arrivals a_k (which is equal to the number of active trains) during a random slot k can therefore be written as

$$a_k = b_k + \sum_{i=1}^{a_{k-1}} c_{i,k}, \tag{1}$$

where b_k denotes the number of new trains started during slot k and $c_{i,k}$ is a Bernoulli distributed random variable that indicates whether or not the ith train that was active during slot $k-1$ is continued in slot k. The numbers of new trains in consecutive slots are assumed to be independent and identically distributed (*iid*) random variables with common probability generating function (*pgf*) $B(z) \triangleq E\left[z^{b_k}\right]$. The pgf of the random variables $c_{i,k}$ is defined as

$$C(z) \triangleq 1-\gamma+\gamma z, \tag{2}$$

such that the pgf $L(z)$ of the geometric train length ℓ is given by

$$L(z) = \frac{(1-\gamma)\,z}{1-\gamma z}, \tag{3}$$

with mean

$$E[\ell] = L'(1) = \frac{1}{1-\gamma}. \tag{4}$$

As stated before, the transmission times of the packets from the buffer are equal to one slot per packet, but the packets are sent over an output line prone to probabilistic interruptions. The interruptions are governed by a Markovian process with J states S_j ($j \in \{1, 2, ..., J\}$); in what follows we will use the term 'output line state' to denote the state of this Markovian process. When the output line is in state S_j, the accessibility of the output line is governed by a Bernoulli distribution with success rate η_j, where success corresponds to the output line being accessible and failure corresponds to an interruption. The pgf of this Bernoulli distribution is given by

$$H_j(z) \triangleq 1 - \eta_j + \eta_j z. \tag{5}$$

The transition probabilities $\sigma_{j|j'}$, $j, j' \in \{1, 2, ..., J\}$ fully describe the Markovian interruption process according to the following definition:

$$\sigma_{j|j'} \triangleq \mathrm{Prob}[s_k = j | s_{k-1} = j'] \,, \tag{6}$$

where s_k and s_{k-1} denote the state index of the Markovian process during slots k and $k-1$, respectively. As a shorthand we finally introduce $\sigma_j \triangleq \sigma_{j|j}$.

In our analysis, we will frequently use matrix methods, therefore we also introduce some matrix notations. We can collect all transition probabilities in a matrix $\mathbf{H}$, such that $[\mathbf{H}]_{jj'} = \sigma_{j'|j}$. With matrix $\boldsymbol{\eta} \triangleq \mathrm{diag}\,(\eta_1, \ldots, \eta_J)$, we can decompose $\mathbf{H}$ into $\mathbf{H_0}$ and $\mathbf{H_1}$ as follows:

$$\mathbf{H_0} \triangleq \mathbf{H}\,(\mathbf{I_J} - \boldsymbol{\eta}) \qquad \text{and} \qquad \mathbf{H_1} \triangleq \mathbf{H}\boldsymbol{\eta}, \tag{7}$$

where $\mathbf{I_J}$ is the $J \times J$ identity matrix, such that $\mathbf{H_0} + \mathbf{H_1} = \mathbf{H}$. The matrices $\mathbf{H_0}$ and $\mathbf{H_1}$ can be combined to form the matrix generating function $\mathbf{H}^*(z) \triangleq \mathbf{H_0} + \mathbf{H_1} z$. Given that $\mathbf{H}$ is a stochastic matrix corresponding to an irreducible Markov chain, a stationary probability vector $\underline{\boldsymbol{\pi}}$ can be found, such that $\underline{\boldsymbol{\pi}}\mathbf{H} = \underline{\boldsymbol{\pi}}$ and $\underline{\boldsymbol{\pi}} \cdot \underline{\mathbf{e_J}} = 1$, where $\underline{\mathbf{e_J}}$ is a column vector of order J with all elements equal to 1. Note that due to its definition, $\underline{\boldsymbol{\pi}}$ must also be a left eigenvector of $\mathbf{H}$, corresponding to eigenvalue 1. The jth element of row vector $\underline{\boldsymbol{\pi}}$ corresponds to the probability that the output line resides in state S_j at the beginning of a random time slot.

3. **Packet arrival process and load.** Before attempting to tackle the system state distribution, we first focus on the packet arrival process. Note that the steady-state pgf $A(z)$ of the number of packet arrivals in a random slot satisfies the implicit equation

$$A(z) = B(z)A(C(z)) = B(z)A(1 - \gamma + \gamma z). \tag{8}$$

Recursive application of this equation results in the closed-form expression

$$A(z) = \prod_{i=0}^{\infty} B(1 - \gamma^i + \gamma^i z) = \prod_{i=0}^{\infty} B(C_i(z)), \tag{9}$$

where we introduced the shorthand $C_i(z) = 1 - \gamma^i + \gamma^i z$. The arrival rate λ can then be calculated as

$$\lambda \triangleq E[a] = A'(1) = \frac{B'(1)}{1-\gamma} = B'(1)L'(1). \tag{10}$$

The load ρ of the system can be found as the ratio of the mean number λ of arrivals per slot to the mean number of packets that can leave per slot, i.e.

$$\rho \triangleq \frac{\lambda}{E[\text{possible departure rate}]}. \tag{11}$$

Note that departures can only occur when the output line is in fact available, and that transmission times are equal to exactly one slot, such that

$$\begin{aligned} E[\text{possible departure rate}] &= E[\text{output line availability rate}] \\ &= \sum_{j=1}^{J} \pi_j \eta_j = \underline{\boldsymbol{\pi}}\ \boldsymbol{\eta}\ \underline{\mathbf{e_J}}. \end{aligned} \tag{12}$$

4. **Distribution of the system state.** Based on the considerations of the previous sections, we can express the system content u_{k+1} at the beginning of slot $k+1$ as

$$u_{k+1} = (u_k - t_k)^+ + a_k, \tag{13}$$

where $(x)^+$ is short for $\max(x, 0)$ and t_k is a random variable that is 0 if the output line is interrupted during slot k and 1 if the output line is accessible, such that

$$t_k = \begin{cases} 0, & \text{with probability } 1 - \eta_{s_k}, \\ 1, & \text{with probability } \eta_{s_k}. \end{cases} \tag{14}$$

From the equations (1), (2), (6), (13) and (14) it follows that the set of vectors $\{(a_{k-1}, s_{k-1}, u_k)\}$ forms a three-dimensional Markov chain. The state of our buffer system at the beginning of slot k can then be described by the system state vector (a_{k-1}, s_{k-1}, u_k). In the rest of this section, we present an analytical method to derive the steady-state distribution of the system state vector. This, in turn, will allow us to study such performance quantities as the system content, the packet delay and the train delay in Sections 5, 6 and 7. We start by defining the joint system state pgf $P_k(x, y, z)$ for a random slot k as

$$P_k(x, y, z) \triangleq E[x^{a_{k-1}} y^{s_{k-1}} z^{u_k}]\,. \tag{15}$$

For ease of calculation, we will split up this pgf into partial pgfs $P_{j,k}(x, z)$, which in turn can be combined into a row vector of order J, thus yielding the vector generating function $\underline{\mathbf{P_k^*}}(x, z)$, such that

$$\begin{aligned} \Big[\underline{\mathbf{P_k^*}}(x, z)\Big]_j &= P_{j,k}(x, z) \triangleq E[x^{a_{k-1}} z^{u_k}\, \{s_{k-1} = j\}] \\ &= \text{Prob}[s_{k-1} = j]\, E[\, x^{a_{k-1}} z^{u_k} \,|\, s_{k-1} = j] \end{aligned} \tag{16}$$

and

$$P_k(x, y, z) = \sum_{j=1}^{J} P_{j,k}(x, z) y^j. \tag{17}$$

Based on (1), (13) and (14), we will now determine the partial system state pgfs $P_{j,k+1}(x, z)$ for slot $k+1$ in terms of their slot k counterparts as

$$\begin{aligned} P_{j,k+1}(x, z) &= E[x^{a_k} z^{u_{k+1}}\, \{s_k = j\}] = E\Big[(xz)^{a_k} z^{(u_k - t_k)^+}\, \{s_k = j\}\Big] \\ &= B(xz) E\Big[(C(xz))^{a_{k-1}} z^{(u_k - t_k)^+}\, \{s_k = j\}\Big] \\ &= B(xz) \Big[(1 - \eta_j)\, E[(C(xz))^{a_{k-1}} z^{u_k}\, \{s_k = j\}] \\ &\quad + \eta_j E\Big[(C(xz))^{a_{k-1}} z^{(u_k - 1)^+}\, \{s_k = j\}\Big]\Big] \end{aligned}$$

$$= B(xz)\left[\hat{H}_j(z)E[(C(xz))^{a_{k-1}}z^{u_k}\{s_k=j\}] + \eta_j\frac{z-1}{z}\mathrm{Prob}[u_k=0,s_k=j]\right], \tag{18}$$

where $\hat{H}_j(z) \triangleq H_j(1/z)$. The partial mean in the right hand side of (18) can be calculated as

$$E[(C(xz))^{a_{k-1}}z^{u_k}\{s_k=j\}] = \sum_{j'=1}^{J}\sigma_{j|j'}P_{j',k}(C(xz),z) = \left[\underline{\mathbf{P}^*_\mathbf{k}}(C(xz),z)\mathbf{H}\right]_j. \tag{19}$$

Similarly, $\mathrm{Prob}[u_k=0,s_k=j]$ can be found as

$$\mathrm{Prob}[u_k=0,s_k=j] = \left[\underline{\mathbf{P}^*_\mathbf{k}}(0,0)\mathbf{H}\right]_j, \tag{20}$$

such that the partial system state pgf $P_{j,k+1}(x,z)$ for slot $k+1$ becomes

$$P_{j,k+1}(x,z) = B(xz)\left[\frac{z-1}{z}\underline{\boldsymbol{\nu}_\mathbf{k}} + \underline{\mathbf{P}^*_\mathbf{k}}(C(xz),z)\hat{\mathbf{H}}^*(z)\right]_j,$$

where $\hat{\mathbf{H}}^*(z) \triangleq \mathbf{H}^*(\frac{1}{z})$ and $\underline{\boldsymbol{\nu}_\mathbf{k}} \triangleq \underline{\mathbf{P}^*_\mathbf{k}}(0,0)\mathbf{H_1}$. Written in terms of vector generating functions, this becomes

$$\underline{\mathbf{P}^*_\mathbf{k+1}}(x,z) = B(xz)\left[\frac{z-1}{z}\underline{\boldsymbol{\nu}_\mathbf{k}} + \underline{\mathbf{P}^*_\mathbf{k}}(C(xz),z)\hat{\mathbf{H}}^*(z)\right].$$

Taking the limit of $\underline{\mathbf{P}^*_\mathbf{k+1}}(x,z)$ for $k\to\infty$ we then get the following equation for the steady-state counterpart $\underline{\mathbf{P}^*}(x,z)$:

$$\underline{\mathbf{P}^*}(x,z) = B(xz)\left[\frac{z-1}{z}\underline{\boldsymbol{\nu}} + \underline{\mathbf{P}^*}(C(xz),z)\hat{\mathbf{H}}^*(z)\right], \tag{21}$$

where $\underline{\boldsymbol{\nu}} \triangleq \underline{\mathbf{P}^*}(0,0)\mathbf{H_1}$ is the steady-state equivalent of $\underline{\boldsymbol{\nu}_\mathbf{k}}$.

Note that, independently of the state of the Markov process, for the system to be empty at the beginning of a random slot $k+1$, there must not have been any arrival in the previous slot k. From this consideration, we can find

$$\begin{aligned} P_{j,k+1}(x,0) &= E[x^{a_k}\{s_k=j,u_{k+1}=0\}] = E[x^{a_k}\{s_k=j,a_k=0,u_{k+1}=0\}] \\ &= P_{j,k+1}(0,0), \qquad \forall x. \end{aligned} \tag{22}$$

Using vector notations, we therefore have

$$\underline{\mathbf{P}^*_\mathbf{k}}(x,0) = \underline{\mathbf{P}^*_\mathbf{k}}(0,0), \qquad \text{and} \qquad \underline{\mathbf{P}^*}(x,0) = \underline{\mathbf{P}^*}(0,0), \forall x. \tag{23}$$

In order to determine the unknown vector $\underline{\boldsymbol{\nu}}$, we start by eliminating the recursiveness in (21). We do so by considering those values of x for which the first arguments of the vector functions $\underline{\mathbf{P}^*}(\cdot,z)$ on the left and right hand sides of (21) are equal to each other, i.e. for which

$$x = C(xz) = 1-\gamma+\gamma xz. \tag{24}$$

From the above relation, we get the following solution for x in terms of z:

$$x(z) = \frac{L(z)}{z}. \tag{25}$$

Equation (21) then gives

$$\underline{\mathbf{P}^*}(x(z),z)\left(\mathbf{I_J} - B(L(z))\hat{\mathbf{H}}^*(z)\right) = \frac{z-1}{z}B(L(z))\underline{\boldsymbol{\nu}}. \tag{26}$$

In general, $\underline{\mathbf{P}^*}(x(z), z)$ can be determined from (26) as

$$\underline{\mathbf{P}^*}(x(z), z) = \frac{(z-1)\, B(L(z))\, \underline{\boldsymbol{\nu}}\, \mathrm{adj}(\mathbf{M}^*(z))}{z\, \det\left(\mathbf{M}^*(z)\right)}, \tag{27}$$

where $\mathbf{M}^*(z) \triangleq \mathbf{I_J} - B(L(z))\hat{\mathbf{H}}^*(z)$ and $\mathrm{adj}(\mathbf{M}^*(z))$ is the adjugate matrix of matrix $\mathbf{M}^*(z)$. Note that every component of $\underline{\mathbf{P}^*}(x, z)$ is a partial pgf and thus is bounded for all arguments x, z on the closed unit disk (i.e. $|x|, |z| \le 1$). We can now exploit this boundedness to determine the unknown quantities $\nu_j \triangleq [\underline{\boldsymbol{\nu}}]_j$, as follows. First note that $z = 1$ is a zero of both the denominators and the numerators of the partial pgfs in the vector functions given by (27), irrespective of the values of the unknowns ν_j. On the other hand, if we choose $z = z^*$ where $|z^*| \le 1$, $z^* \ne 1$ and $\det\left(\mathbf{M}^*(z^*)\right) = 0$, the denominators of the partial pgfs in (27) become zero, and for the partial pgfs to remain bounded, the corresponding numerators should become zero as well. Expressing this condition, we find that

$$\underline{\boldsymbol{\nu}}\, \mathrm{adj}(\mathbf{M}^*(z^*)) = \underline{\mathbf{0_J}}, \tag{28}$$

where $\underline{\mathbf{0_J}}$ is a row vector of order J with all elements equal to 0. The matrix equation (28) corresponds to a homogeneous system of linear equations for the unknowns ν_j in which all the equations are linearly dependent, such that (28) yields only one useful equation for the unknowns. Assuming the system is stable, it follows from [19]) that $\det\left(z\mathbf{M}^*(z)\right) = z^J \det\left(\mathbf{M}^*(z)\right)$ has exactly $J-1$ roots z^* in the open unit disk, that may not all be distinct. We can therefore expand our method to make use of all these poles in order to construct $J-1$ homogeneous systems of linear equations as follows. For each root z^* of $\det\left(z\mathbf{M}^*(z)\right)$ located in the open unit disk and of multiplicity 1, we construct the system

$$z^{*\,J-1}\underline{\boldsymbol{\nu}}\, \mathrm{adj}\left(\mathbf{M}^*(z^*)\right) = \underline{\mathbf{0_J}}, \tag{29}$$

whereas for a root z^* of multiplicity $r > 1$, we obtain r systems

$$\left.\frac{\mathrm{d}^i}{\mathrm{d}z^i} z^{J-1}\underline{\boldsymbol{\nu}}\, \mathrm{adj}(\mathbf{M}^*(z))\right|_{z=z^*} = \underline{\mathbf{0_J}}, \qquad i = 0, \ldots, r-1. \tag{30}$$

One final equation for the unknowns ν_j is then still required in order to fully determine the vector $\underline{\boldsymbol{\nu}}$. Therefore we first evaluate the derivative of (26) for $z = 1$. In view of $\underline{\mathbf{P}^*}(1,1) = \underline{\boldsymbol{\pi}}$, this leads to

$$\underline{\boldsymbol{\nu}} = \underline{\boldsymbol{\pi}}\boldsymbol{\eta} - \lambda\underline{\boldsymbol{\pi}} + \left(\left.\frac{\mathrm{d}}{\mathrm{d}z}\underline{\mathbf{P}^*}(x(z), z)\right|_{z=1}\right)\left(\mathbf{I_J} - \mathbf{H}\right), \tag{31}$$

such that the sum of the components ν_j can be found from the product of (31) with $\underline{\mathbf{e_J}}$ as

$$\sum_{j=1}^{J} \nu_j = \underline{\boldsymbol{\nu}} \cdot \underline{\mathbf{e_J}} = \left(\sum_{j=1}^{J} \pi_j \eta_j\right) - \lambda. \tag{32}$$

Together with the $J-1$ equations that can be extracted from the $J-1$ systems (28), equation (32) concludes a system of J linear independent equations that allows us to determine the J unknowns ν_j.

In practice, each of the ν_j corresponds to the probability for the system to be empty at the beginning of a slot where the system is in state j and the output line is accessible, as can be seen from the definition $\underline{\boldsymbol{\nu}} \triangleq \underline{\mathbf{P}^*}(0,0)\mathbf{H_1}$. Therefore, the state vector $\underline{\mathbf{P}^*}(0,0)$ of an empty system can be found as

$$\underline{\mathbf{P}^*}(0,0) = \underline{\boldsymbol{\nu}}\ \mathbf{H_1}^{-1}. \tag{33}$$

5. **Distribution of the system content.** From the above analysis of the system state, the pgf $U(z)$ of the system content u at the beginning of an arbitrary slot in the steady state can be found directly as

$$\begin{aligned} U(z) \triangleq E[z^u] = P(1,1,z) &= \sum_{j=1}^{J} P_j(1,z) = \underline{\mathbf{P}^*}(1,z) \cdot \underline{\mathbf{e_J}} \\ &= B(z)\left[\frac{z-1}{z}\underline{\boldsymbol{\nu}} + \underline{\mathbf{P}^*}(C(z),z)\hat{\mathbf{H}}^*(z)\right]\underline{\mathbf{e_J}}. \end{aligned} \tag{34}$$

The empty system probability $U(0)$ especially unfolds as

$$U(0) = \underline{\mathbf{P}^*}(0,0) \cdot \underline{\mathbf{e_J}} = \underline{\boldsymbol{\nu}}\ \mathbf{H_1}^{-1}\ \underline{\mathbf{e_J}}. \tag{35}$$

In order to obtain the mean system content, we first focus on $\underline{\mathbf{P}^*}(x(z),z)$. More specifically, bearing in mind that the first argument $x(z)$ satisfies (25) and therefore $x(1) = 1$, we can calculate the first derivative of $\underline{\mathbf{P}^*}(x(z),z)$ with respect to z for $z = 1$ as

$$\left.\frac{\mathrm{d}}{\mathrm{d}z}\underline{\mathbf{P}^*}(x(z),z)\right|_{z=1} = (L'(1)-1)\,\frac{\partial}{\partial x}\underline{\mathbf{P}^*}(1,1) + \frac{\partial}{\partial z}\underline{\mathbf{P}^*}(1,1). \tag{36}$$

An expression for the mean system content then immediately follows by summation of the vector's elements on both sides of the above equation. Indeed, in view of the definition of $\underline{\mathbf{P}^*}(x,z)$ and the moment generating property of pgfs, we have that

$$\frac{\partial}{\partial x}\underline{\mathbf{P}^*}(1,1) \cdot \underline{\mathbf{e_J}} = \lambda, \tag{37}$$

i.e. the mean number of packet arrivals per slot, whereas

$$\frac{\partial}{\partial z}\underline{\mathbf{P}^*}(1,1) \cdot \underline{\mathbf{e_J}} = U'(1) \tag{38}$$

corresponds to the mean system content. Multiplying both sides of (36) by $\underline{\mathbf{e_J}}$, we then get the mean system content as

$$E[u] = U'(1) = \left(\left.\frac{\mathrm{d}}{\mathrm{d}z}\underline{\mathbf{P}^*}(x(z),z)\right|_{z=1}\right)\underline{\mathbf{e_J}} - (L'(1)-1)\,\lambda, \tag{39}$$

where the first term on the right hand side follows from (27) as

$$\begin{aligned} &\left(\left.\frac{\mathrm{d}}{\mathrm{d}z}\underline{\mathbf{P}^*}(x(z),z)\right|_{z=1}\right)\underline{\mathbf{e_J}} \\ &= \frac{1}{M'(1)}\left[\left(\lambda - 1 - \frac{M''(1)}{2M'(1)}\right)\underline{\boldsymbol{\nu}}\ \mathrm{adj}\left(\mathbf{I_J} - \mathbf{H}\right)\ \underline{\mathbf{e_J}} + \left.\frac{\mathrm{d}}{\mathrm{d}z}\underline{\boldsymbol{\nu}}\ \mathrm{adj}\left(\mathbf{M}^*(z)\right)\ \underline{\mathbf{e_J}}\right|_{z=1}\right], \end{aligned} \tag{40}$$

with $M(z) \triangleq \det\left(\mathbf{M}^*(z)\right)$.

Higher-order moments of the system content can be calculated in a similar way, by evaluation of higher-order derivatives of $\underline{\mathbf{P}^*}(x,z)$, although the mathematical calculations become more and more complicated. Tail probabilities of the system content (i.e. the probabilities to exceed a certain threshold) can be calculated, for a sufficiently large threshold, by means of a method presented e.g. in [38].

6. **Distribution of the packet delay.** In this section we analyse the delay experienced by the packets as they traverse through the system. The delay $d_{\mathcal{P}}$ of a random packet $\mathcal{P}$ is defined as the integer number of slots starting at the end of the arrival slot of $\mathcal{P}$ until the end of the departure slot of $\mathcal{P}$. This delay is not only influenced by the system state at the beginning of the arrival slot $\mathcal{S}$ of $\mathcal{P}$, but also by the number of arrivals during $\mathcal{S}$, as well as the condition of the output line in the subsequent slots. First we determine the number of packets $v_{\mathcal{P}}$ in the system just after slot $\mathcal{S}$, excluding the packets that have arrived during the same slot as $\mathcal{P}$, but will be transmitted later than $\mathcal{P}$ as

$$v_{\mathcal{P}} = (u_{\mathcal{S}} - t_{\mathcal{S}})^+ + f_{\mathcal{P}}, \tag{41}$$

where $f_{\mathcal{P}} \in \{1, \ldots, a_{\mathcal{S}}\}$ is the position of $\mathcal{P}$ within all slot $\mathcal{S}$ arrivals. Given that this position is uniformly distributed over the $a_{\mathcal{S}}$ arrivals in slot $\mathcal{S}$, we can determine the conditional probability that $\mathcal{P}$ has position n as $\mathrm{Prob}[\, f_{\mathcal{P}} = n |\, a_{\mathcal{S}} = i] = 1/i$, $n \in \{1, \ldots, i\}$. This allows us to find the following partial pgfs $V_{j,\mathcal{P}}(z)$ as

$$\begin{aligned} V_{j,\mathcal{P}}(z) &\triangleq E[z^{v_{\mathcal{P}}}\{s_{\mathcal{S}} = j\}] = E\Big[z^{(u_{\mathcal{S}}-t_{\mathcal{S}})^+ + f_{\mathcal{P}}}\{s_{\mathcal{S}} = j\}\Big] \\ &= \sum_{i=1}^{\infty}\sum_{n=1}^{i} z^n E\Big[z^{(u_{\mathcal{S}}-t_{\mathcal{S}})^+}\{s_{\mathcal{S}} = j, a_{\mathcal{S}} = i, f_{\mathcal{P}} = n\}\Big] \\ &= \sum_{i=1}^{\infty} \frac{1}{i} E\Big[z^{(u_{\mathcal{S}}-t_{\mathcal{S}})^+}\{s_{\mathcal{S}} = j, a_{\mathcal{S}} = i\}\Big] \sum_{n=1}^{i} z^n \\ &= \frac{z}{1-z}\sum_{i=1}^{\infty} \frac{1-z^i}{i} E\Big[z^{(u_{\mathcal{S}}-t_{\mathcal{S}})^+}\{s_{\mathcal{S}} = j, a_{\mathcal{S}} = i\}\Big]. \end{aligned} \tag{42}$$

Now note that based on arguments from renewal theory (see e.g. [33]), we can relate the partial mean on the right hand side of (42) to the system state variables for a random steady-state slot $k+1$ as

$$\begin{aligned} E\Big[z^{(u_{\mathcal{S}}-t_{\mathcal{S}})^+}\{s_{\mathcal{S}} = j, a_{\mathcal{S}} = i\}\Big] &= E\big[z^{u_{\mathcal{S}+1}-i}\{s_{\mathcal{S}} = j, a_{\mathcal{S}} = i\}\big] \\ &= z^{-i} E[z^{u_{\mathcal{S}+1}}\{s_{\mathcal{S}} = j, a_{\mathcal{S}} = i\}] \\ &= z^{-i}\frac{i}{\lambda} E[z^{u_{k+1}}\{s_k = j, a_k = i\}]. \end{aligned} \tag{43}$$

Hence, we can further transform equation (42) for $V_{j,\mathcal{P}}(z)$ into

$$\begin{aligned} V_{j,\mathcal{P}}(z) &= \frac{1}{\lambda}\frac{z}{1-z}\sum_{i=1}^{\infty}\left(z^{-i} - 1\right) E[z^{u_{k+1}}\{s_k = j, a_k = i\}] \\ &= \frac{1}{\lambda}\frac{z}{1-z}\left(P_j(z^{-1}, z) - P_j(1, z)\right), \end{aligned} \tag{44}$$

where $P_j(x, z)$ is the jth component of $\underline{\mathbf{P}^*}(x, z)$. We can combine all partial pgfs $V_{j,\mathcal{P}}(z)$ into a single partial vector generating function $\underline{\mathbf{V}^*_{\mathcal{P}}}(z)$ as

$$\underline{\mathbf{V}^*_{\mathcal{P}}}(z) = \frac{1}{\lambda}\frac{z}{1-z}\left(\underline{\mathbf{P}^*}(z^{-1}, z) - \underline{\mathbf{P}^*}(1, z)\right). \tag{45}$$

This row vector should be interpreted as a vector who's jth component is in fact the partial pgf of the number of packets $v_{\mathcal{P}}$ in the system at the start of slot $\mathcal{S}+1$, excluding the packets to be sent later than $\mathcal{P}$, in case the state of the output channel during slot $\mathcal{S}$ is j.

The delay $d_{\mathcal{P}}$ of a random packet $\mathcal{P}$ can then be found as the total number of slots needed by the system to complete the transmission of all these $v_{\mathcal{P}}$ packets. Note that, even though the transmission times are equal to 1 slot per packet, the *effective* transmission time of a packet can be greater due to output line interruptions. Introducing $s_{\text{eff},k}$ as the effective transmission time of a single packet, starting at slot k, we get

$$\text{Prob}\big[s_{\text{eff},k} = i, s_{k+i-1} = j' \,|\, s_{k-1} = j\big] = \big[\mathbf{H_0}^{i-1}\mathbf{H_1}\big]_{jj'}, \tag{46}$$

which yields the matrix generating function

$$\mathbf{S}^*_{\text{eff}}(z) = (\mathbf{I_J} - \mathbf{H_0}z)^{-1}\mathbf{H_1}z. \tag{47}$$

Assuming that during slot $\mathcal{S}$ the output channel is in state j, the delay $d_{\mathcal{P}}$ of $\mathcal{P}$ has partial pgf $\underline{\mathbf{1_j}}(\mathbf{S}^*_{\text{eff}}(z))^{v_{\mathcal{P}}}\underline{\mathbf{e_J}}$, where $\underline{\mathbf{1_j}}$ is a row vector with all zeroes except for the jth entry which is 1. The complete pgf $D_{\mathcal{P}}(z)$ can thus be found as

$$\begin{aligned} D_{\mathcal{P}}(z) \triangleq E\big[z^{d_{\mathcal{P}}}\big] &= \sum_{j=1}^{J} E\big[z^{d_{\mathcal{P}}}\{s_{\mathcal{S}} = j\}\big] = \sum_{j=1}^{J}\sum_{n=1}^{\infty} E\big[z^{d_{\mathcal{P}}}\{s_{\mathcal{S}} = j, v_{\mathcal{P}} = n\}\big] \\ &= \sum_{j=1}^{J}\sum_{n=1}^{\infty} \underline{\mathbf{1_j}}(\mathbf{S}^*_{\text{eff}}(z))^{n}\underline{\mathbf{e_J}}\text{Prob}[s_{\mathcal{S}} = j, v_{\mathcal{P}} = n] \\ &= \sum_{j=1}^{J} \underline{\mathbf{1_j}} V_{j,\mathcal{P}}(\mathbf{S}^*_{\text{eff}}(z))\underline{\mathbf{e_J}}. \end{aligned} \tag{48}$$

Note that the matrix functions $V_{j,\mathcal{P}}(\mathbf{S}^*_{\text{eff}}(z))$ are well-defined if and only if the eigenvalues of the matrix $\mathbf{S}^*_{\text{eff}}(z)$ are in the domain of the functions $V_{j,\mathcal{P}}(z)$. In that case, $D_{\mathcal{P}}(z)$ can be calculated using the spectral decomposition of $\mathbf{S}^*_{\text{eff}}(z)$ as

$$\begin{aligned} D_{\mathcal{P}}(z) &= \sum_{j=1}^{J} \underline{\mathbf{1_j}} \left(\sum_{i=1}^{\kappa(z)} V_{j,\mathcal{P}}(\lambda_i(z))\mathbf{S}^*_{i,\text{eff}}(z)\right)\underline{\mathbf{e_J}} \\ &= \sum_{j=1}^{J}\sum_{i=1}^{\kappa(z)} V_{j,\mathcal{P}}(\lambda_i(z)) \sum_{j'=1}^{J} \big[\mathbf{S}^*_{i,\text{eff}}(z)\big]_{jj'}, \end{aligned} \tag{49}$$

where the $\lambda_i(z)$, $i = 1, \ldots, \kappa(z)$ are the distinct eigenvalues of $\mathbf{S}^*_{\text{eff}}(z)$ for a particular value of z [32]. In case $\mathbf{S}^*_{\text{eff}}(z)$ is diagonalisable , the spectral projectors $\mathbf{S}^*_{i,\text{eff}}(z)$ can be obtained using the Lagrange interpolation formula as

$$\mathbf{S}^*_{i,\text{eff}}(z) = \frac{\prod_{i'=1,i\neq i'}^{\kappa(z)} (\mathbf{S}^*_{\text{eff}}(z) - \lambda_{i'}(z)\mathbf{I_J})}{\prod_{i'=1,i\neq i'}^{\kappa(z)} (\lambda_i(z) - \lambda_{i'}(z))}. \tag{50}$$

If $\mathbf{S}^*_{\text{eff}}(z)$ is not diagonalisable, both (49) and (50) require a more technical treatment [32]. Essential to our analysis however, is the requirement that the functions $\lambda_i(z)$ are analytic in at least a neighbourhood of $z = 1$ because we need to evaluate their derivatives in this point. In general, it is known that the eigenvalue functions may not be analytic or even continuous in the entire unit disk [18] but they are analytic in (a neighbourhood of) points where all eigenvalues are distinct. We therefore assume as a sufficient condition that $\mathbf{S}^*_{\text{eff}}(1)$ has $\kappa(1) = J$ distinct eigenvalues, so that ultimately, the mean packet delay $E[d_{\mathcal{P}}]$ can then be found as

$$E[d_{\mathcal{P}}] = D'_{\mathcal{P}}(1) = \sum_{j=1}^{J}\sum_{i=1}^{J}\left(\lambda'_i(1)V'_{j,\mathcal{P}}(\lambda_i(1))\sum_{j'=1}^{J}\left[\mathbf{S}^*_{i,\mathrm{eff}}(1)\right]_{jj'}\right.$$
$$\left. +V_{j,\mathcal{P}}(\lambda_i(1))\sum_{j'=1}^{J}\left[\mathbf{S}^{*}_{i,\mathrm{eff}}{}'(1)\right]_{jj'}\right). \tag{51}$$

The mean packet delay $E[d_{\mathcal{P}}]$ is also related to the mean system content $E[u]$ by Little's result [16]:

$$E[u] = \lambda E[d_{\mathcal{P}}]\,. \tag{52}$$

The analysis method for the packet delay established in this section constitutes a basic step to study the train delay, as we will explain next.

7. **Distribution of the train delay.** The delay $d_{\mathcal{T}}$ of an arbitrary train $\mathcal{T}$ is defined as the integer number of slots between the end of the arrival slot $\mathcal{S}$ of the first packet $\mathcal{P}_0$ of the train, and the end of the slot in which the final packet $\bar{\mathcal{P}} = \mathcal{P}_{\ell(\mathcal{T})-1}$ of $\mathcal{T}$ departs from the system, where $\ell(\mathcal{T})$ is the length of train $\mathcal{T}$. Note that the delay $d_{\mathcal{T}}$ in fact corresponds to the delay of the train's final packet, augmented with the number of slots between the arrival of the train's first and last packet, i.e.

$$d_{\mathcal{T}} = \ell(\mathcal{T}) - 1 + d_{\bar{\mathcal{P}}}. \tag{53}$$

Due to the fact that trains start independently from slot to slot, the first packet of a train starting in steady state perceives the system as in an arbitrary state, i.e. the distribution of the system state at the beginning of the train's starting slot $\mathcal{S}$ equals the system state distribution at the beginning of a random steady-state slot, as studied in Section 4.

The pgf of the total number of new trains $b_{\mathcal{S}}$ starting in slot $\mathcal{S}$ is expressed in terms of the pgf $B(z)$ of the number of starting trains in an arbitrary slot as (see e.g. [33])

$$E\left[z^{b_{\mathcal{S}}}\right] = z\frac{B'(z)}{B'(1)}. \tag{54}$$

From this and the correlated nature of the packet arrival process, it is then easily seen that the system-state distribution at the beginning of the arrival slot of a non-first packet of train $\mathcal{T}$ will be different from the system state distribution at the beginning of an arbitrary slot. In order to calculate the delay of the train's final packet, we therefore must take into account the system state at the beginning of slot $\mathcal{S}$, as well as the evolution of the packet arrival process and the Markov process governing the output line.

First, we calculate the partial pgf of the system state at the beginning of slot $\mathcal{S}+1$. With the above observations and using (21), we find

$$\begin{aligned} P_{j,\mathcal{S}+1}(x,z) &= E[x^{a_{\mathcal{S}}}z^{u_{\mathcal{S}+1}}\{s_{\mathcal{S}}=j\}] \\ &= E\left[(xz)^{a_{\mathcal{S}}}z^{(u_{\mathcal{S}}-t_{\mathcal{S}})^+}\{s_{\mathcal{S}}=j\}\right] \end{aligned}$$

$$
\begin{aligned}
&= xz\frac{B'(xz)}{B'(1)}E\Big[(C(xz))^{a_{\mathcal{S}-1}}z^{(u_{\mathcal{S}}-t_{\mathcal{S}})^+}\{s_{\mathcal{S}}=j\}\Big]\\
&= \frac{xz}{B(xz)}\frac{B'(xz)}{B'(1)}P_j(x,z)\\
&= xz\frac{B'(xz)}{B'(1)}\left[\frac{z-1}{z}\underline{\boldsymbol{\nu}}+\underline{\mathbf{P}^*}(C(xz),z)\ \hat{\mathbf{H}}^*(z)\right]_j, \qquad (55)
\end{aligned}
$$

which can be combined into the vector generating function

$$
\underline{\mathbf{P}^*_{\mathcal{S}+\mathbf{1}}}(x,z) = xz\frac{B'(xz)}{B'(1)}\left[\frac{z-1}{z}\underline{\boldsymbol{\nu}}+\underline{\mathbf{P}^*}(C(xz),z)\ \hat{\mathbf{H}}^*(z)\right]. \qquad (56)
$$

Next, note that packets of $\mathcal{T}$ arriving after slot $\mathcal{S}$ all arrive to a non-empty system, and are known not to be the first of the train $\mathcal{T}$. The partial pgfs of the system state at the beginning of slots $\mathcal{S}+k+1$ $(1 \le k < \ell(\mathcal{T}))$ can thus be found as

$$
\begin{aligned}
P_{j,\mathcal{S}+k+1}(x,z) &= E\big[(xz)^{a_{\mathcal{S}+k}}z^{u_{\mathcal{S}+k}-t_{\mathcal{S}+k}}\{s_{\mathcal{S}+k}=j\}\big]\\
&= xz\frac{B(xz)}{C(xz)}\hat{H}_j(z)E[(C(xz))^{a_{\mathcal{S}+k-1}}z^{u_{\mathcal{S}+k}}\{s_{\mathcal{S}+k}=j\}]\\
&= xz\frac{B(xz)}{C(xz)}\left[\underline{\mathbf{P}^*_{\mathcal{S}+\mathbf{k}}}(C(xz),z)\ \hat{\mathbf{H}}^*(z)\right]_j. \qquad (57)
\end{aligned}
$$

This yields the vector generating function

$$
\underline{\mathbf{P}^*_{\mathcal{S}+\mathbf{k}+\mathbf{1}}}(x,z) = xz\frac{B(xz)}{C(xz)}\ \underline{\mathbf{P}^*_{\mathcal{S}+\mathbf{k}}}(C(xz),z)\ \hat{\mathbf{H}}^*(z). \qquad (58)
$$

By iteration, we can then relate $\underline{\mathbf{P}^*_{\mathcal{S}+\mathbf{k}+\mathbf{1}}}(x,z)$ to $\underline{\mathbf{P}^*}(x,z)$ as

$$
\begin{aligned}
\underline{\mathbf{P}^*_{\mathcal{S}+\mathbf{k}+\mathbf{1}}}(x,z) &= xz^k\frac{\prod_{i=0}^{k-1}B(zG_i(x,z))}{G_k(x,z)}\ \underline{\mathbf{P}^*_{\mathcal{S}+\mathbf{1}}}(G_k(x,z),z)\ \hat{\mathbf{H}}^*(z)^k\\
&= \frac{xz^{k+1}}{B'(1)}B'(zG_k(x,z))\left(\prod_{i=0}^{k-1}B(zG_i(x,z))\right)\\
&\qquad \left[\frac{z-1}{z}\underline{\boldsymbol{\nu}}+\underline{\mathbf{P}^*}(G_{k+1}(x,z),z)\ \hat{\mathbf{H}}^*(z)\right]\ \hat{\mathbf{H}}^*(z)^k, \qquad (59)
\end{aligned}
$$

where $G_k(x,z)$ is defined iteratively as

$$
\begin{aligned}
&G_0(x,z) = x, && (60)\\
&G_k(x,z) = C(zG_{k-1}(x,z)), \qquad k>0. && (61)
\end{aligned}
$$

Note that (59) in fact yields (56) when substituting $k=0$.

With (59), we can calculate the vector generating function of the system state at the beginning of the arrival slot $\bar{\mathcal{S}} = \mathcal{S}+\ell(\mathcal{T})-1$ of $\mathcal{T}$'s final packet. The pgf of the delay of that final packet $\bar{\mathcal{P}}$ can then be obtained by means of the approach we discussed when calculating the delay of an arbitrary steady-state packet. In particular, we write

$$
\begin{aligned}
V_{j,\bar{\mathcal{P}}}(z) &\triangleq E[z^{v_{\bar{\mathcal{P}}}}\{s_{\bar{\mathcal{S}}}=j\}] = E\big[z^{u_{\bar{\mathcal{S}}}-t_{\bar{\mathcal{S}}}+f_{\bar{\mathcal{P}}}+1}\{s_{\bar{\mathcal{S}}}=j\}\big]\\
&= \frac{z}{1-z}\sum_{i=1}^{\infty}\frac{1-z^i}{i}E\big[z^{u_{\bar{\mathcal{S}}}-t_{\bar{\mathcal{S}}}}\{s_{\bar{\mathcal{S}}}=j,a_{\bar{\mathcal{S}}}=i\}\big]. \qquad (62)
\end{aligned}
$$

The partial mean in (62) cannot directly be related to values corresponding to a random steady-state slot, but rather we have that

$$E\left[z^{u_{\bar{S}}-t_{\bar{S}}}\{s_{\bar{\mathcal{S}}}=j,a_{\bar{\mathcal{S}}}=i\}\right]=E\left[z^{u_{\bar{S}+1}-i}\{s_{\bar{\mathcal{S}}}=j,a_{\bar{\mathcal{S}}}=i\}\right]$$
$$=z^{-i}E[z^{u_{\bar{S}+1}}\{s_{\bar{\mathcal{S}}}=j,a_{\bar{\mathcal{S}}}=i\}]\,. \tag{63}$$

Substitution in (62) then yields

$$\begin{aligned}V_{j,\bar{\mathcal{P}}}(z)&=\frac{z}{1-z}\sum_{i=1}^{\infty}\frac{z^{-i}-1}{i}E[z^{u_{\bar{S}+1}}\{s_{\bar{\mathcal{S}}}=j,a_{\bar{\mathcal{S}}}=i\}]\\&=\frac{z}{1-z}\int_1^{1/z}E\left[\alpha^{a_{\bar{S}}-1}z^{u_{\bar{S}+1}}\{s_{\bar{\mathcal{S}}}=j\}\right]\mathrm{d}\alpha\\&=\frac{z}{1-z}\int_1^{1/z}\frac{1}{\alpha}P_{j,\bar{\mathcal{S}}+1}(\alpha,z)\mathrm{d}\alpha,\end{aligned}\tag{64}$$

such that the pgf of the delay of the train's final packet $\bar{\mathcal{P}}$ can be determined as

$$D_{\bar{\mathcal{P}}}(z)=\sum_{j=1}^{J}\underline{\mathbf{1_j}}\;V_{j,\bar{\mathcal{P}}}(\mathbf{S}^*_{\text{eff}}(z))\;\underline{\mathbf{e_J}}. \tag{65}$$

Given that the position of the arrival slot $\bar{\mathcal{S}}$ of packet $\bar{\mathcal{P}}$ depends on the train length $\ell(\mathcal{T})$, the pgf $D_{\mathcal{T}}(z)$ of the train delay can then be determined as

$$D_{\mathcal{T}}(z)=\sum_{i=1}^{\infty}E\left[z^{i-1+d_{\bar{\mathcal{P}}}}\{\ell(\mathcal{T})=i\}\right]. \tag{66}$$

The mean train delay $E[d_{\mathcal{T}}]$ can then be found by substituting $z=1$ into the first derivative of (66).

8. **Special case: J = 2.** In this section, we perform a case study of a buffer system with train arrivals and an unreliable output line having $J=2$ states. For this special case, we derive more explicit results. For $J=2$, the transition probability matrix $\mathbf{H}$ is given by

$$\mathbf{H}=\begin{bmatrix}\sigma_1&\sigma_{2|1}\\\sigma_{1|2}&\sigma_2\end{bmatrix}=\begin{bmatrix}\sigma_1&1-\sigma_1\\1-\sigma_2&\sigma_2\end{bmatrix}. \tag{67}$$

The eigenvalues of this matrix $\mathbf{H}$ can be computed from the characteristic equation $\det(\mathbf{H}-\lambda\mathbf{I_2})=0$ as $\lambda_1=1$ and $\lambda_2=\phi\triangleq\sigma_1+\sigma_2-1$, with associated left eigenvectors

$$\underline{\boldsymbol{\pi}}=\frac{1}{2-\sigma_1-\sigma_2}\begin{bmatrix}1-\sigma_2\\1-\sigma_1\end{bmatrix}^T\qquad\text{and}\qquad\begin{bmatrix}1\\-1\end{bmatrix}^T, \tag{68}$$

respectively. Note that ϕ is in fact the correlation coefficient between the output line state in two subsequent slots in steady state, i.e.

$$\phi=\lim_{k\to\infty}\rho_{s_ks_{k-1}}=\lim_{k\to\infty}\frac{E[s_ks_{k-1}]-E[s_k]\,E[s_{k-1}]}{\sqrt{Var[s_k]\,Var[s_{k-1}]}}. \tag{69}$$

The matrix generating function $\mathbf{M}^*(z)$ is then given by

$$\mathbf{M}^*(z)=\begin{bmatrix}1-\sigma_1B(L(z))\hat{H}_1(z)&-(1-\sigma_1)\,B(L(z))\hat{H}_2(z)\\-(1-\sigma_2)\,B(L(z))\hat{H}_1(z)&1-\sigma_2B(L(z))\hat{H}_2(z)\end{bmatrix}, \tag{70}$$

with its adjugate

$$\operatorname{adj}\left(\mathbf{M}^*(z)\right) = \begin{bmatrix} 1-\sigma_2 B(L(z))\hat{H}_2(z) & (1-\sigma_1)\, B(L(z))\hat{H}_2(z) \\ (1-\sigma_2)\, B(L(z))\hat{H}_1(z) & 1-\sigma_1 B(L(z))\hat{H}_1(z) \end{bmatrix}. \tag{71}$$

The value of z^* can then be found as a zero of $\det\left(\mathbf{M}^*(z)\right)$, such that

$$0 = 1 - B(L(z^*))\left(\sigma_1 \hat{H}_1(z^*) + \sigma_2 \hat{H}_2(z^*)\right) + \phi B(L(z^*))^2 \hat{H}_1(z^*)\hat{H}_2(z^*). \tag{72}$$

The unknown vector $\underline{\boldsymbol{\nu}} = [\nu_1 \quad \nu_2]$ can be found from the system of equations composed of (32) and (28):

$$\begin{cases} \nu_1 + \nu_2 = \eta_1\pi_1 + \eta_2\pi_2 - \lambda, \\ \nu_1 - \left(\nu_1\sigma_2\hat{H}_2(z^*) + \nu_2\left(\sigma_2 - 1\right)\hat{H}_1(z^*)\right) B(L(z^*)) = 0, \\ \nu_2 - \left(\nu_2\sigma_1\hat{H}_1(z^*) + \nu_1\left(\sigma_1 - 1\right)\hat{H}_2(z^*)\right) B(L(z^*)) = 0, \end{cases} \tag{73}$$

where the last 2 equations are essentially equivalent. From this system, a solution for ν_j $(j \in \{1,2\})$ can be found as

$$\nu_j = \frac{(1-\sigma_{\hat{\jmath}})\left(\eta_1\pi_1 + \eta_2\pi_2 - \lambda\right)}{\phi\left(B(L(z^*))\hat{H}_{\hat{\jmath}}(z^*) - 1\right)}, \tag{74}$$

where $\hat{\jmath} = 3 - j$ denotes the state other than state j. This result follows from expressing ν_2 in terms of ν_1 in the second equation of (73) (or conversely expressing ν_1 in terms of ν_2 in the third equation) and adding $\det\left(\mathbf{M}^*(z^*)\right)$ to the left hand side of the first equation. The probabilities for the system to be empty can then be found from (33) as

$$P_j(0,0) = \frac{\eta_{\hat{\jmath}}\nu_j\sigma_{\hat{\jmath}} - \eta_j\nu_{\hat{\jmath}}(1-\sigma_{\hat{\jmath}})}{\phi\eta_1\eta_2}. \tag{75}$$

For the mean system content, we now determine the first and second derivatives of $M(z)$, for $z=1$ as

$$M'(1) = -\lambda\left(\sigma_1+\sigma_2\right) + \eta_1\sigma_1 + \eta_2\sigma_2 + \phi\left[2\lambda - \left(\eta_1+\eta_2\right)\right],$$

and

$$\begin{aligned} M''(1) = &-\lambda'\left(\sigma_1+\sigma_2\right) + 2\left(\lambda-1\right)\left(\eta_1\sigma_1+\eta_2\sigma_2\right) \\ &+ 2\phi\left[\lambda' + \lambda^2 - \left(2\lambda-1\right)\left(\eta_1+\eta_2\right) + \eta_1\eta_2\right], \end{aligned}$$

where we introduced $\lambda' \triangleq L''(1)B'(1) + L'(1)^2 B''(1)$. Note that, specifically for the case $J=2$,

$$\begin{aligned} \operatorname{adj}\left(\mathbf{M}^*(z)\right) &= \mathbf{I_2} - B(L(z))\operatorname{adj}\left(\hat{\mathbf{H}}^*(z)\right), \\ \operatorname{adj}\left(\mathbf{H}^*(z)\right) &= \operatorname{adj}\left(\mathbf{H_0}\right) + z\operatorname{adj}\left(\mathbf{H_1}\right), \end{aligned}$$

such that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}z}\,\underline{\boldsymbol{\nu}}\operatorname{adj}\left(\mathbf{M}^*(z)\right)\underline{\mathbf{e_2}}\bigg|_{z=1} &= \underline{\boldsymbol{\nu}}\left(\operatorname{adj}\left(\mathbf{H_1}\right) - \lambda\operatorname{adj}\left(\mathbf{H}\right)\right)\underline{\mathbf{e_2}} \\ &= \phi\left(\eta_1\nu_2 + \eta_2\nu_1 - \lambda\left(\nu_1+\nu_2\right)\right). \end{aligned}$$

TABLE 1. The mean system content and the average computation time for increasing values of J, in case of state independent transitions.

J	$E[u]$	computation time
1	2.604350382128157	6 ms
2	5.937683715461492	293 ms
3	5.810866367036650	383 ms
4	5.393539816183369	2909 ms
5	5.086202585347341	17566 ms

The mean system content then follows as

$$E[u] = \frac{1}{M'(1)}\left(\lambda - 1 - \frac{M''(1)}{2M'(1)}\right)(1-\phi)(\nu_1+\nu_2) + \frac{\phi}{M'(1)}(\eta_1\nu_2 + \eta_2\nu_1 - \lambda(\nu_1+\nu_2)) - \frac{\gamma\lambda}{1-\gamma}, \tag{76}$$

and the load ρ can be found as

$$\rho = \frac{\lambda}{\underline{\boldsymbol{\pi}}\ \boldsymbol{\eta}\ \underline{\mathbf{e_2}}} = \frac{(2-\sigma_1-\sigma_2)\lambda}{(1-\sigma_2)\eta_1 + (1-\sigma_1)\eta_2}. \tag{77}$$

9. **Numerical examples and discussion.** In this paper, we have obtained expressions for the main performance measures of a discrete-time buffer system with train arrivals and Markovian interruptions of the output line. In this section, we illustrate the impact of certain system parameters on the buffer performance by means of some numerical examples. Although most of the calculation steps are self-explanatory and can be implemented directly, we would like to clarify that in order to obtain these results, we have made use of the Illinois method described in [13] for finding the zeroes z^* of $\det(\mathbf{M}^*(z))$.

First, we focus on the number of states J of the Markovian process. Therefore, we assume that new packet trains originate according to a Poisson distribution, at a rate of $\alpha = 1/3$ trains per slot, such that $B(z) = e^{\alpha(z-1)}$. Trains that are active in one slot continue in the next slot with a fixed probability of $\gamma = 1/10$, such that the expected train length is $L'(1) = 10/9 \approx 1.11$ slots and the packet arrival rate is $\lambda = 10/27 \approx 0.37$. The output line is unreliable, in that it is accessible on average only half of the time. The accessibility of the output line is governed by a Markovian process of which the state remains the same from slot to slot with a fixed probability of $\sigma_{\text{stay}} = 0.85$. Transitions take place with a probability $\sigma_{j|j'} = \frac{1-\sigma_{\text{stay}}}{J-1}$ for $j, j' \in \{1, 2, \ldots, J\}$ and $j \neq j'$. This means that the mean sojourn times in the states are state independent. For each state S_j, the output line is accessible with probability $\eta_j = 1 - \frac{j-1}{J-1}$, such that the accessibility of the output line η_j decreases linearly from 1 to 0 as j increases. The number of states J can then be understood as a measure of the granularity of the channel model.

Table 1 shows how the granularity affects the value calculated for the mean system content $E[u]$. For $J = 1$, the accessibility of the output line is not correlated between slots, resulting in a moderate variance of the output line accessibility and therefore a relatively low mean system content. In the $J = 2$ case, the granularity of the channel model is at its smallest, since the output line will be always open in S_1 and always closed in S_2. Therefore the mean system content is also maximal for

TABLE 2. The mean system content and the average computation time for increasing values of J, in case transitions to neighbouring states are favored.

J	$E[u]$	computation time
1	2.604350382128157	5 ms
2	5.937683715461492	28 ms
3	6.483004754118678	411 ms
4	7.287010616773672	2881 ms
5	8.398168682998671	17503 ms

$J = 2$. Every additional output line state increases the granularity, thus explaining the decrease of the mean system content for increasing values of J. Note that the construction of the matrices $\mathbf{H}$ and $\boldsymbol{\eta}$ presented above is not valid for $J = 1$, in that case we have $\mathbf{H} = [1]$ and $\boldsymbol{\eta} = [0.5]$.

As can be expected, an increase in the number of states poses a non-linear increase in the processing power and computation time needed to obtain results of performance measures such as the mean system content. This is illustrated in Table 1 as well, where the third column shows the time taken by the author's computer to calculate the corresponding results, averaged out over 100 iterations. Note that this merely serves as a qualitative illustration, as better results can undoubtedly be obtained on more specialised computers or even from optimising the implementation.

Next, we slightly alter the previous system in order to make the transitions in the Markov chain state dependent, resulting in a more realistic model. We do so by changing the entries in $\mathbf{H}$ such that transitions between close neighbour states become more likely than transitions between far neighbour states. Note that the terms *close* and *far neighbours* reflect the difference in output line availability, such that two states S_j and $S_{j'}$ are close neighbours if $|\eta_j - \eta_{j'}|$ is small. Since we retain $\underline{\boldsymbol{\nu}}$ as it was defined in the previous example, the Markov chain will move more gracefully between *healthy* states, where the output line is more likely to be open, and *unhealthy* states, where the output line is less likely to be open. The entries of $\mathbf{H}$ are given by

$$[\mathbf{H}]_{ij} = \begin{cases} p_i, & i = j, \\ q^k, & |i - j| = k > 0, \end{cases}$$

where $p_i = 1 - \sum_{k=1}^{J-i} q^k - \sum_{k=1}^{i-1} q^k$. Note that the parameter q then represents the transition probability of the Markov chain between direct neighbour states. As was the case in the previous example, the output line is accessible on average half of the time.

Table 2 shows the mean system content for increasing values of J, with parameter $q = 0.15$ and the other parameters identical as in the previous example. From Table 2, we see that an increase of the number of states and therefore an increase of the correlation in the output line accessibility, results in an increase of the mean system content. This corresponds to the intuitive notion that increased correlation in transmission times results in an increased mean system content. Again, the average computation time on the author's computer was appended to the table as an indication of the complexity of the computations.

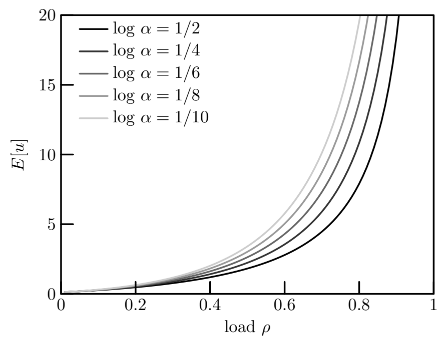

FIGURE 1. Mean system content vs the system load for $J = 3$, a mean possible departure rate of 0.5 and various values of α.

As the latter model represents a more realistic situation for the output line availability process, we now consider the Markovian interruption process of the previous example for the case of $J = 3$ states and Poisson arrivals of new trains. We focus on the effect of the system load ρ on the mean system content. For various values of $\alpha = B'(1)$, we vary the mean train length $E[\ell] = \frac{1}{1-\gamma}$ such that the system load ρ encompasses the entire spectrum $]0, 1[$. To this end, we refer to (10) and (11) and note that for matrices $\mathbf{H}$ and $\boldsymbol{\eta}$ as defined for the previous example, the mean possible departure rate equals 0.5. Figure 1 confirms the intuitive notion that under low to moderate load conditions, the mean system content increases only slightly, but for high load conditions, the mean system content grows unboundedly. Figure 1 moreover clearly illustrates the impact of the train-based packet generation on the system behavior. In particular, we see that for a given value of the load ρ, the mean system content is a decreasing function of α. This observation is intuitively clear since for a given load a lower value of α means that a smaller number of *longer* packet trains arrive in the buffer, which means that the packet arrival proces exhibits more time correlation, and thus higher system contents are expected. Moreover, for a given value of α, higher values of γ - and hence longer messages - lead to longer packet delays. Similar conclusions also hold for the mean packet delay in view of Little's law.

In our queueing model, there is time correlation at the incoming side due to the train arrival process and there is time correlation at the outgoing side due to the Markovian process controlling the output line state. In Figure 2, the influence of the output-line correlation on the mean packet delay is illustrated. We assume that new trains arrive according to a Poisson distribution with a mean of $\alpha = 1/6$ new trains per slot and consider various values of the train length parameter γ. For the output line, we consider a Markov process with $J = 2$ internal states with $\eta_1 = 0.75$ and $\eta_2 = 0.25$. The load is fixed at $\rho = 0.6$ and the probabilities σ_j of (67) are determined from (77) as $\sigma_j = (1-\phi)\frac{\lambda-\eta_j\rho}{(\eta_j-\eta_{\widehat{\jmath}})\rho} + 1$, where $\widehat{\jmath} = 3 - j$ and $\phi = \sigma_1 + \sigma_2 - 1$ is the correlation coefficient between consecutive output line states. Figure 2 shows the mean packet $E[d_{\mathcal{P}}]$ as a function of ϕ for various values of γ. Note that, in order to guarantee that $\sigma_j \in [0, 1]$, the value of ϕ must be greater than $\frac{\lambda-\eta_{\widehat{\jmath}}\rho}{\lambda-\eta_j\rho}$, such that the leftmost point differs for each curve. As expected intuitively we observe that an increase of the correlation coefficient ϕ leads to an increase of the mean packet delay.

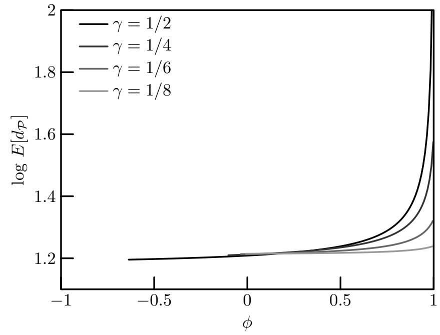

FIGURE 2. Mean packet delay vs the output-state correlation coefficient ϕ for $J = 2$, $\eta_1 = 0.75$, $\eta_2 = 0.25$, $\alpha = 1/6$ and various values of γ.

Acknowledgments. This research has been funded by the Interuniversity Attraction Poles Programme initiated by the Belgian Science Policy Office.

REFERENCES

[1] M. M. Ali, X. Zhang and J. F. Hayes, A performance analysis of a discrete-time queueing system with server interruption for modeling wireless ATM multiplexer, *Performance Evaluation*, **51** (2003), 1–31.

[2] E. Altman and A. Jean-Marie, The distribution of delays of dispersed messages in an $M/M/1$ queue, *Proceedings of IEEE INFOCOM '95* (Boston, 2-6 April 1995), (1995), 338–344.

[3] C. Blondia and O. Casals, Statistical multiplexing of VBR sources: A matrix-analytic approach, *Performance Evaluation*, **16** (1992), 5–20.

[4] H. Bruneel, On the behavior of buffers with random server interruptions, *Performance Evaluation*, **3** (1983), 165–175.

[5] H. Bruneel, Buffers with stochastic output interruptions, *Electronics Letters*, **19** (1983), 735–737.

[6] H. Bruneel, Packet delay and queue length for statistical multiplexers with low-speed access lines, *Computer Networks and ISDN Systems*, **25** (1993), 1267–1277.

[7] H. Bruneel, Calculation of message delays and message waiting times in switching elements with slow access lines, *IEEE Transactions on Communications*, **42** (1994), 255–259.

[8] B. D. Choi, D. I. Choi, Y. Lee and D. K. Sung, Priority queueing system with fixed-length packet-train arrivals, *IEE Proceedings-Communications*, **145** (1998), 331–336.

[9] A. Chydzinski, Time to reach buffer capacity in a BMAP queue, *Stochastic Models*, **23** (2007), 195–209.

[10] I. Cidon, A. Khamisy and M. Sidi, On queueing delays of dispersed messages, *Queueing Systems*, **15** (1994), 325–345.

[11] I. Cidon, A. Khamisy and M. Sidi, Delay, jitter and threshold crossing in ATM systems with dispersed messages, *Performance Evaluation*, **29** (1997), 85–104.

[12] J. Daigle, Message delays at packet-switching nodes serving multiple classes, *IEEE Transactions on Communications*, **38** (1990), 447–455.

[13] M. Dowell and P. Jarrat, A modified regula falsi method for computing the root of an equation, *BIT Numerical Mathematics*, **11** (1971), 168–174.

[14] K. Elsayed and H. Perros, The superposition of discrete-time Markov renewal processes with an application to statistical multiplexing of bursty traffic sources, *Applied Mathematics and Computation*, **115** (2000), 43–62.

[15] B. Feyaerts, S. De Vuyst, H. Bruneel and S. Wittevrongel, Analysis of discrete-time buffers with heterogeneous session-based arrivals and general session lengths, *Computers and Operations Research*, **39** (2012), 2905–2914.

[16] D. Fiems and H. Bruneel, A note on the discretization of Little's result, *Operations Research Letters*, **30** (2002), 17–18.

[17] D. Fiems, B. Steyaert and H. Bruneel, Discrete-time queues with generally distributed service times and renewal-type server interruptions, *Performance Evaluation*, **55** (2004), 277–298.

[18] H. R. Gail, S. L. Hantler, A. G. Konheim and B. A. Taylor, An analysis of a class of telecommunication models, *Performance Evaluation*, **21** (1994), 151–161.

[19] H. R. Gail, S. L. Hantler and B. A. Taylor, Spectral analysis of $M/G/1$ and $G/M/1$ type Markov chains, *Advances in Applied Probability*, **28** (1996), 114–165.

[20] L. Hoflack, S. De Vuyst, S. Wittevrongel and H. Bruneel, Analytics traffic model of a web server, *Electronics Letters*, **44** (2008), 61–63.

[21] L. Hoflack, S. De Vuyst, S. Wittevrongel and H. Bruneel, Discrete-time buffer systems with session-based arrival streams, *Performance Evaluation*, **67** (2010), 432–450.

[22] G. U. Hwang and B. D. Choi, Closed-form expressions on the geometric tail behavior of statistical multiplexers with heterogeneous traffic, *IEEE Transactions on Communications*, **46** (1998), 1575–1579.

[23] F. Ishizaki, Decomposition property in a discrete-time queue with multiple input streams and service interruptions, *Journal of Applied Probability*, **41** (2004), 524–534.

[24] F. Kamoun, Performance analysis of a discrete-time queueing system with a correlated train arrival process, *Performance Evaluation*, **63** (2006), 315–340.

[25] F. Kamoun, Performance analysis of a non-preemptive priority queuing system subjected to a correlated Markovian interruption process, *Computers & Operations Research*, **35** (2008), 3969–3988.

[26] F. Kamoun, Performance evaluation of a queueing system with correlated packet-trains and server interruption, *Telecommunication Systems*, **41** (2009), 267–277.

[27] K. Laevens and H. Bruneel, Delay analysis for discrete-time queueing systems with multiple randomly interrupted servers, *European Journal of Operations Research*, **85** (1995), 161–177.

[28] D. S. Lee, Analysis of a single server queue with semi-Markovian service interruption, *Queueing Systems*, **27** (1997), 153–178.

[29] D. M. Lucantoni, K. S. Meier-Hellstern and M. Neuts, A single-server queue with server vacations and a class of non-renewal arrival processes, *Advances in Applied Probability*, **22** (1990), 676–705.

[30] D. M. Lucantoni, New results on the single server queue with a batch Markovian arrival process, *Stochastic Models*, **7** (1991), 1–46.

[31] H. Masuyama and T. Takine, Stationary queue length in a FIFO single server queue with service interruptions and multiple batch Markovian arrival streams, *Journal of the Operations Research Society of Japan*, **46** (2003), 319–341.

[32] C. D. Meyer, *Matrix Analysis and Applied Linear Algebra*, SIAM, 2000.

[33] I. Mitrani, *Modelling of Computer and Communication Systems*, Cambridge University Press, Cambridge, UK, 1987.

[34] A. Mokhtar and M. Azizoglu, Analysis of state-dependent probabilistic server interruptions in discrete-time queues, *IEEE Communications Letters*, **8** (2004), 544–546.

[35] M. Neuts, *Structured Stochastic Matrices of $M/G/1$ type and Their Applications*, New York: Marcel Dekker, 1989.

[36] K. Sohraby, On the theory of general ON-OFF sources with applications in high-speed networks, *Proceedings of IEEE INFOCOM '93* (San Francisco, 28 March - 1 April 1993), (1993), 401–410.

[37] J. Walraevens, S. Wittevrongel and H. Bruneel, A discrete-time priority queue with train arrivals, *Stochastic Models*, **23** (2007), 489–512.

[38] S. Wittevrongel and H. Bruneel, Correlation effects in ATM queues due to data format conversions, *Performance Evaluation*, **32** (1998), 35–56.

[39] S. Wittevrongel, Discrete-time buffers with variable-length train arrivals, *Electronics Letters*, **34** (1998), 1719–1721.

[40] S. Wittevrongel, S. De Vuyst and H. Bruneel, Analysis of discrete-time buffers with general session-based arrivals, *16th International conference on analytical and stochastic modelling techniques and applications (ASMTA)* Madrid , Spain, *Lecture Notes in Computer Science*, **5513** (2009), 189–203.

[41] Y. Xiong, H. Bruneel, Buffer behavior of statistical multiplexers with correlated train arrivals, *International Journal of Electronics and Communications*, **51** (1997), 178–186.

Received September 2013; 1st revision February 2014; final revision May 2014.

E-mail address: bfeyaert@telin.ugent.be
E-mail address: stijn.devuyst@ugent.be
E-mail address: hb@telin.ugent.be
E-mail address: sw@telin.ugent.be